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A Comprehensive Theory of Trichotomous Evaluative Linguistic Expressions*

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Abstract

In this paper, a logical theory of the, so called, trichotomous evaluative linguistic expressions (TEv-expressions) is presented. These are frequent expressions of natural language, such as "small, very small, roughly medium, extremely big", etc. The theory is developed using the formal system of higher-order fuzzy logic, namely the fuzzy type theory (generalization of classical type theory). First, we discuss informally what are properties of the meaning of TEv-expressions. Then we construct step by step axioms of a formal logical theory T^{Ev} of TEv-expressions and prove various properties of T^{Ev}. All the proofs are syntactical and so, our theory is very general. We also outline construction of a canonical model of T^{Ev}. The main elegancy of our theory consists in the fact that semantics of all kinds of evaluative expressions is modeled in a unified way. We also prove theorems demonstrating that essential properties of the vagueness phenomenon can be captured within our theory.

Keywords: Evaluative linguistic expression, intension, extension, fuzzy type theory, context, linguistic hedge, trichotomy.

1 Introduction

In many applications of fuzzy logic, namely in fuzzy control, but also in decision-making, classification and other ones based on fuzzy IF-THEN rules, the expressions such as "small, very small, medium, more or less big" etc. are considered. These expressions belong to the class of the, so called, *evaluative linguistic expressions* (in the sequel, we will usually omit the adjective "linguistic") which is a very important class encompassing also expressions such as "extremely deep, very intelligent, rather narrow, medium important, about one thousand, very

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tall, not very nice, not too expensive, rather low or medium high, weak but not very much" and many other ones. A subclass of evaluative linguistic expressions are *trichotomous evaluative expressions* (TEv-expressions for short) that are evaluative expressions containing evaluative trichotomy of the type *small — medium — big*. Let us stress that evaluative expressions are omnipresent in our language and we use them whenever we need some form of evaluation, judgement, estimation, and in many other situations. Hence, we are convinced that a working logical theory of their semantics can be very useful not only in applications of fuzzy logic and soft computing, but also in artificial intelligence, theory of commonsense reasoning, as well as in linguistics and philosophy.

Remark 1
In several older papers, the author has used the term "evaluating linguistic expressions" in the same meaning as presented in this paper. Since the latter term sometimes led to misunderstanding, it has been replaced by the term "evaluative linguistic expressions" introduced above and, henceforth, the older term will not be used anymore.

There are not many linguistic works dealing with our concept. The closest linguistic works are [2, 32] and partially also [10, 18]. However, these works deal mainly with *evaluative adjectives* rather than with expressions in our sense (see below). According to [18], evaluative adjectives "typically characterize a person's behavior or attitude in terms of the speakers subjective judgment". Their class is quite large and includes, for example, adjectives such as "rude, mean, clever, smart, nice, kind, silly, imprudent, generous, courteous, cruel, mad, mischievous, considerate, humane, pretentious, modest, charming, masochistic, intelligent, stupid, dumb, noble, cunning, farsighted, skillful, selfish, crazy, foolish". Not all these adjectives, however, can be taken as TEv-expressions. For example, the adjectives such as "mischievous, pretentious, masochistic" and other ones certainly do not belong among them. A more specific characterization of TEv-expressions is that they contain especially gradable adjectives and adjectives of manner (cf. [2]). But still we are not sure that all possibilities are covered. A crucial in this respect seems to be the necessity to form the basic evaluative trichotomy, i.e., nominal adjective, its antonym, and a middle member.

Linguistic studies of evaluative adjectives are done in relation with the objects whose characteristics are denoted by them. Thus, for example, the adjectives "good, superb, important" belong to the class of adjectives that are "attitude-based", while "big, heavy, forte, pricey, opulent, ripe, young", etc. are related to some numerical scale, such as size, weight, price, age, and other ones. Fleisher in [10] studies special "attributive-with-infinitive" constructions, e.g., "Middlemarch is an easy book to read in one sitting" and "Middlemarch is a long book to read in one sitting" in which "easy" relates to the whole construction while "long" relates only to "book". Of course, we cannot interchange in our speech the adjectives freely. For example, the adjectives such as *good, heavy, opulent, cold*, etc. are semantically related to some kind of a scale formed by values of a certain feature of the given object and so, the main difference between them is in their relation to the character of the scale together with the given object. Thus, we cannot say, e.g., "heavy performance" or "opulent temperature".

Still, we are convinced that this difference is semantically less significant. Nouns or more complex constructions considered above are bearers of certain features whose values fill some scale. For example, there is a certain (abstract) feature of quality of performance, feature of temperature (its values are degrees), etc. There is even an abstract feature of "complexity to read a book in one sitting". When emptying the nature of these features, an abstract scale remains on which the meaning of the corresponding adjective can be defined. We argue that this definition is more or less the same for all evaluative expressions, i.e., they behave semantically in a more or less the same way.

Consequently, we may state the following (somewhat vague) definition: *evaluative expressions are specific expressions of natural language, which characterize position on some bounded ordered scale.* Such a scale may consist of real measuring units such as meters, degrees, etc. but quite often, it is only an abstract scale consisting of some fictitious units.

In most applications of fuzzy logic, evaluative expressions are usually interpreted by simple fuzzy sets in the universe of real numbers having membership with triangular shape. This apparently does not suffice to capture their real meaning (although in engineering applications, in which a certain (imprecise) description of some function is at play, triangular shapes may be enough). Since evaluative linguistic expressions play an important role in all kinds of human thinking, a deeper working theory of them should be very useful so that advanced applications can be developed, for example in decision-making, classification, but also in control either of robots or various other technical devices (cf. [8]), in artificial intelligence which requires understanding natural language, and in many other fields. Another motivation for such a theory comes from the fact that evaluative expressions and especially the trichotomous ones can be taken as a principal bearer of the vagueness phenomenon. This means study of the meaning of TEv-expressions can help in deeper understanding to the latter.

Our goal in this paper is to analyze structure of evaluative expressions with the focus to TEv-expressions and especially, to provide a formal theory of their meaning. We will demonstrate that TEv-expressions are inherently vague and that their vagueness is always a manifestation of the, more or less hidden, phenomenon described as sorites paradox. It is thinkable to develop this theory using the standard means of fuzzy set theory. However, we want our theory to be logical, as general as possible, and also to have potential for further development in correspondence with the general theory of linguistic meaning. Hence, *formal fuzzy logic* seems to be appropriate means.

A question is raised, which kind of logical system should be used. A lot has been done using predicate first-order fuzzy logic. However, though the syntactic structure of evaluative expressions is not too complicated, the model of their semantics using first-order logic is not satisfactory. The reason is that any model of the meaning of words of natural language must be able to distinguish between intension and extension (cf. [5, 19]). When speaking about TEv-expressions, the latter concepts can be in a certain way included in the predicate first-order fuzzy logic with evaluated syntax (see [30]). However, we want our theory to have potential for inclusion in a theory of a wider part of natural language semantics. Therefore, we prefer the means of fuzzy type theory. Besides other advantages, it enables us to formulate explicitly their behavior in various contexts (in predicate logic, contexts are only implicit).

A concept of great significance, which makes the theory of semantics of evaluative expressions transparent and elegant, is that of *fuzzy equality* (fuzzy equivalence; fuzzy similarity). This is an imprecise equality using which we may characterize various degrees of similarity between objects. This idea spreads through the literature on fuzzy sets and fuzzy logic and is elaborated in a lot of works, e.g. [3, 14, 31, 34] and many others. The role of these relations in modeling of linguistic semantics has been raised already in [22] (recently also in [6]) where a related concept of the, so called, *indiscernibility relation* (see [35]) has been employed. The mathematical theory of the natural language semantics presented there relies on the following hypothesis.

Hypothesis 1
Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility phenomenon. Therefore, any extension of a natural-language expression can always be characterized by the some specific indiscernibility relation.

This hypothesis can be supported by many demonstrations but it is impossible to prove it. However, it justifies both our constructions below as well as the idea to tie all formulas of FTT characterizing semantics of evaluative expressions with some fuzzy equality.

In this paper, we will elaborate a logical theory of the meaning of TEv-expressions. We will define a special language of FTT, identify TEv-expressions with special formulas of FTT, and form axioms based on pondering of the main characteristics of the meaning of TEv-expressions. Because we are developing a logical theory, the proofs of most of the theorems are syntactic rather than semantic. At the end of this paper, however, we also outline semantics of our theory.

Let us remark that the constituent of our theory is also a semantic theory of *linguistic hedges* that are special adverbs allowed to modify the meaning of evaluative adjectives. The idea to model hedges as special operators modifying membership functions of fuzzy sets comes from L. A. Zadeh [38]. His proposal, however, had a problem that his operators did not modify kernel of the corresponding fuzzy set. This insufficiency has been first pointed out in a linguistic analysis of G. Lakoff in [17]. Therefore, a modification of Zadeh's approach which solves this problem has been proposed by the author in [20] and elaborated more extensively in the book [21]. The same problem has also been touched by B. Bouchon in [4]. The idea that linguistic modifiers, in fact, realize a shift of a certain horizon has been first introduced in [23] and elaborated within formal fuzzy logic in [30]. In this paper, we will elaborate the theory of linguistic hedges as an integral part of the semantics of TEv-expressions using the advanced formalism of fuzzy type theory.

Let us stress that our paper can be seen as a contribution to the development of the concept of Precisiated Natural Language (PNL), as proposed by L. A. Zadeh in [39] (cf. also [7]). At the same time it can be seen as a logical theory related to the concept of linguistic variable introduced by L. A. Zadeh in [38].

The paper is structured as follows. Section 2 contains a brief overview of fuzzy type theory and includes also some other main properties used further. Section 3 is the main part of the paper. We start with syntactic characterization of the evaluative expressions. Then we present informal discussion of their semantic properties. On the basis of them, we develop step by step a formal theory T^{Ev} and demonstrate that it fits the informal requirements. The section ends by explicit assignment of specific formulas that express meaning of the expressions in concern. Section 4 briefly describes construction of a canonical model of the theory T^{Ev}. This gives rules how the precise semantics of evaluative linguistic expressions can be constructed.

2 Preliminaries

2.1 Syntax of fuzzy type theory

The main tool for the logical analysis of evaluative linguistic expressions is fuzzy type theory (FTT) which is a higher order fuzzy logic. For the detailed presentation of general fuzzy logic see [9, 12, 30] and the citations therein.

In this section, we will very briefly overview some of the main points of FTT. The detailed explanation can be found in [25, 27]. The classical type theory is in details described in [1].

Let ϵ, o be distinct objects. The set of types is the smallest set *Types* satisfying:

(i) $\epsilon, o \in \mathit{Types}$,

(ii) If $\alpha, \beta \in \mathit{Types}$ then $(\alpha\beta) \in \mathit{Types}$.

The type ϵ represents elements and o truth values.

The *language* J of FTT consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ where $\alpha \in \mathit{Types}$, auxiliary symbol λ, and brackets.

Let a language J be given. A set of formulas of types $\alpha \in \mathit{Types}$ over the language J, denoted by Form_α, is a smallest set satisfying:

(i) If a variable $x_\alpha \in J$, $\alpha \in \mathit{Types}$, then $x_\alpha \in \mathit{Form}_\alpha$.

(ii) If a constant $c_\alpha \in J$, $\alpha \in \mathit{Types}$, then $c_\alpha \in \mathit{Form}_\alpha$.

(iii) If $B \in \mathit{Form}_{\beta\alpha}$ and $A \in \mathit{Form}_\alpha$ then $(BA) \in \mathit{Form}_\beta$.

(iv) If $A \in \mathit{Form}_\beta$ and $x_\alpha \in J$, $\alpha \in \mathit{Types}$, is a variable then $\lambda x_\alpha\, A \in \mathit{Form}_{\beta\alpha}$.

If $A \in Form_\alpha$ is a formula of the type $\alpha \in Types$ then we will often write A_α.

A specific constant that is always present in the language of FTT is $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in Types$. Then for each type we define the fuzzy equality by

$$\equiv \ := \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha.$$

This is a formula of type $(o\alpha)\alpha$. If A_α, B_α are formulas then $(A_\alpha \equiv B_\alpha)$ is a formula of type o. Note if $\alpha = o$ then $\equiv$ is the logical equivalence.

The formulas of type o (truth value) can be further joined by the following connectives (which are derived formulas): $\vee$ (disjunction), $\wedge$ (conjunction), $\&$ (strong conjunction), ∇ (strong disjunction), $\Rightarrow$ (implication). n-times strong conjunction of A_o will be denoted by A_o^n and similarly, n-times strong disjunction by nA_o.

There are also general ($\forall$) and existential ($\exists$) quantifiers defined again as special formulas. For the details about their definition and semantics — see [25]. Special derived formulas are also *truth* $\top$ and falsity $\bot$. If A_α is a formula substitutable into B for a free variable x_α then a formula resulting from B after replacing each free occurrence of x_α by A_α will be denoted by $B_{x_\alpha}[A_\alpha]$.

To simplify the notation as much as possible, which means especially to minimize the number of brackets, we will apply the following priority of the logical connectives:

1. $\neg, \boldsymbol{\Delta}$.
2. $\&, \nabla, \wedge, \vee$.
3. $\equiv$.
4. $\Rightarrow$.

Furthermore, we will also use the dot convention as follows: the formula $A \cdot B$ is equivalent to $A(B)$. For example, the formula $\lambda x \cdot Ax \wedge \cdot Bx \Rightarrow C$ is equivalent to $\lambda x\, (Ax \wedge (Bx \Rightarrow C))$.

Some common shorts will also be used, e.g., $A \not\equiv B$ is a short for $\neg(A \equiv B)$, and the like. If type of the formula in concern is clear from the context then we may omit it.

The symbol := should be read as "is" and it simply means a denotation of the expression on the right hand side by a symbol on the left hand side.

Finally, we will freely write or omit the type when no misunderstanding may occur. This means that we will write either of x_α, $x \in Form_\alpha$, or $x_\alpha \in Form_\alpha$ to stress that x is a formula (a variable) of type α.

If $A \in Form_{o\alpha}$ then A represents a property of elements of the type α. By abuse of language, we will often say "A is a property" (of elements of type α) and similarly, $A_{(o\alpha)\alpha}$ is a relation (between elements of type α).

2.2 Structure of truth values

The structure of truth values is supposed to form a complete IMTL$_\Delta$-algebra (see [9]) or a standard Łukasiewicz algebra extended by the delta operation.

Recall that IMTL-algebra is a complete residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

fulfilling the prelinearity condition

$$(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}, \qquad a, b \in L,$$

and, moreover, its negation function $\neg a = a \rightarrow \mathbf{0}$ is involutive, i.e. $\neg\neg a = a$ holds for all $a \in L$.

It is known that MTL-algebras are algebras (the negation needs not be involutive) of left-continuous t-norms (cf. [9, 16]). An example of left-continuous t-norm with involutive negation is *nilpotent minimum* defined by

$$a \otimes b = \begin{cases} a \wedge b, & \text{if } a + b > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

where the negation is standard strong negation $\neg a = 1 - a$, $a \in [0, 1]$.

We need to extend the IMTL algebra by the (Baaz) delta operation $\Delta : L \longrightarrow L$ which, in the case of linearly ordered algebra of truth values, is defined by

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{3}$$

(for the details about this operation, see [9, 25]).

A special case of IMTL$_\Delta$-algebra is also the standard Łukasiewicz$_\Delta$ algebra

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \oplus, \Delta, \rightarrow, 0, 1 \rangle \tag{4}$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum}, \\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow b &= 1 \wedge (1 - a + b), \\ \neg a &= a \rightarrow 0 = 1 - a, & a \oplus b &= 1 \wedge (a + b), \\ \Delta(a) &= \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Another special case is a finite Łukasiewicz$_\Delta$ algebra (cf. [30]).

It can be demonstrated on many examples that Łukasiewicz$_\Delta$ algebra is a very reasonable choice for applications when semantics of natural language is involved. In [26], a modified axiom systems for the Łukasiewicz style fuzzy type theory has been proposed.

2.3 Axiomatic system of FTT

The syntax of FTT consists of definitions of fundamental formulas, axioms and inference rules. A more detailed presentation would be too extensive and so, we will repeat only some of the main points.

The FTT has 17 axioms that may be divided into the following subsets: fundamental equality axioms, truth structure axioms, quantifier axioms and axioms of descriptions.

Fundamental equality axioms are the following:

(FT$_I$1) $\mathbf{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT$_I$2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(FT$_I$2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$

(FT$_I$3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$ where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

(FT$_I$4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

Further axioms characterize structure of truth values and are different for IMTL and Łukasiewicz FTT. Besides others, they assure that the corresponding predicate fuzzy logic with the $\mathbf{\Delta}$ connective is included in FTT (i.e., all theorems of the former are provable also in FTT).

Specific axiom important for proving in FTT is

(FT$_I$6) $(A_o \equiv \top) \equiv A_o$

The quantifier axiom is

(FT$_I$16) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$ where x_α is not free in A_o.

The substitution axiom is provable in FTT.

Finally, the axioms of descriptions are the following:

(FT$_I$17) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon.$

Recall that the *description operator* $\iota_{\alpha(o\alpha)}$ is interpreted by an operation assigning to each normal fuzzy set an element from its kernel (cf. [27]). In fuzzy set theory, such an operation is called *defuzzification*. We also define a special operator

$$\imath z_\alpha A_o := \iota_{\alpha(o\alpha)}(\lambda z_\alpha A_o)$$

that picks up an element of type α such that the formula A_o is true in the degree **1** for it.

There are two inference rules in FTT, namely

(R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then, infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

(N) *Let $A_o \in Form_o$. Then, from A_o infer $\mathbf{\Delta}A_o$.*

A theory T is a set of formulas of type o (determined by a subset of special axioms, as usual). Provability is defined as usual. If T is a theory and A_o a formula then $T \vdash A_o$ means that A_o is provable in T. The inference rules of modus ponens and generalization are derived rules of FTT.

Important role in FTT is played by the deduction theorem.

Theorem 1 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

In the sequel, we will often refer to axioms and various proved facts from the fuzzy type theory. Since it is not possible to list all of them here, we will simply write "by properties of FTT" and refer the reader to [25].

2.4 Semantics of FTT

Let J be a language of FTT. A *frame* for J is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{5}$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ is a structure of truth values (i.e., Łukasiewicz$_\delta$ algebra or IMTL$_\Delta$-algebra).

(ii) $=_\alpha$ is a fuzzy equality on M_α and $=_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

Recall that a fuzzy relation $=_\alpha$ is a fuzzy equality if it is *reflexive* ($[m =_\alpha m'] = \mathbf{1}$, $m \in M_\alpha$, where $[\cdot]$ denotes a truth value), *symmetric* ($[m =_\alpha m'] = [m' =_\alpha m]$, $m, m' \in M_\alpha$) and *transitive* ($[m =_\alpha m'] \otimes [m' =_\alpha m''] \leq [m =_\alpha m'']$, $m, m', m'' \in M_\alpha$). It is *separating* (or 1-faithful) if the following holds true: $[m =_\alpha m'] = \mathbf{1}$ iff $m = m'$.

Let us remark that, because of the prelinearity property, any IMTL$_\Delta$-algebra can be taken as a subdirect product of linearly ordered IMTL$_\Delta$-algebras. Therefore, we will suppose that the algebra of truth values $\mathcal{L}_\Delta$ is linearly ordered.

It is important to know that if $\beta\alpha$ is a type then the corresponding set $M_{\beta\alpha}$ contains functions $f : M_\alpha \longrightarrow M_\beta$ but in general, $M_{\beta\alpha} \neq M_\beta^{M_\alpha}$. We put $M_o = L$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.

Let p be an assignment of elements from $\mathcal{M}$ to variables. An interpretation $\mathcal{I}^{\mathcal{M}}$ is a function that assigns every formula A_α, $\alpha \in Types$ and every assignment p a corresponding element, that is, a function of the type α. As a special case, note that interpretation of a formula $A_{o\alpha}$ is a fuzzy set $\mathcal{I}^{\mathcal{M}}(A_{o\alpha}) \underset{\sim}{\subset} M_\alpha$.

A general model is a frame $\mathcal{M}$ such that

$$\mathcal{I}_p^{\mathcal{M}}(A_\alpha) \in M_\alpha \tag{6}$$

holds true. This means that each set M_α from the frame $\mathcal{M}$ has enough elements so that the interpretation of each formula $A_\alpha \in Form_\alpha$ is always defined in $\mathcal{M}$.

A frame $\mathcal{M}$ is a *model* of a theory T if all its axioms are true in the degree $\mathbf{1}$ in $\mathcal{M}$. If A_o is true in the degree $\mathbf{1}$ in all models of T then we write $T \models A_o$.

The following completeness theorem can be proved in FTT (for its proof and many other details — see [25]).

Theorem 2 (completeness)
(a) A theory T is consistent iff it has a general model $\mathcal{M}$.

(b) For every theory T and a formula A_o

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

2.5 Special properties of FTT

We say that a formula A_o is *crisp* if $\vdash A_o \vee \neg A_o$. This means that its interpretation is either $\mathbf{0}$ or $\mathbf{1}$. As a special case, each formula of the form $\mathbf{\Delta} A_o$ is crisp. We may use this fact also in the following characterization:

$$\vdash A_o \vee \neg A_o \quad \text{iff} \quad \vdash A_o \equiv \mathbf{\Delta} A_o. \tag{7}$$

for every formula $A_o \in \mathit{Form}_o$.

We will often use the equality theorem of FTT in the following form.

Lemma 1
Let $A_{\beta\alpha}, B_{\beta\alpha}$ be formulas of type $\beta\alpha$. Then

$$\vdash (\forall x_\alpha)(A_{\beta\alpha} x_\alpha \equiv B_{\beta\alpha} x_\alpha) \equiv (\lambda y_\alpha\, A_{\beta\alpha} y_\alpha \equiv \lambda y_\alpha\, B_{\beta\alpha} y_\alpha).$$

PROOF: Indeed, this follows from the lambda conversion axiom $\vdash C_{\beta\alpha} x_\alpha \equiv \lambda y_\alpha\, C_{\beta\alpha} y_\alpha \cdot x_\alpha$. □

It also follows from the properties of fuzzy equality that

$$\vdash (\forall x_\alpha)(\exists y_\beta)\mathbf{\Delta}(f_{\beta\alpha} x_\alpha \equiv y_\beta). \tag{8}$$

This means that each $x \in \mathit{Form}_\alpha$ is surely mapped via $f \in \mathit{Form}_{\beta\alpha}$ to some $y \in \mathit{Form}_\beta$ (namely, this y is just the fx).

Some further properties of FTT that will be used in the sequel are summarized in the following lemma.

Lemma 2
(a) $T \vdash (\exists x_\alpha)\mathbf{\Delta} B$ iff $T \cup B_{x_\alpha}[\mathbf{u}_\alpha]$ is a conservative extension of T where $\mathbf{u}_\alpha \notin J(T)$ (Rule C).

(b) Let $T \vdash (\exists x_\alpha)\mathbf{\Delta} B_{o\alpha} x_\alpha$. Then $T \vdash B_{o\alpha} \cdot \iota_{\alpha(o\alpha)} B_{o\alpha}$.

(c) Let $T \vdash (\exists x_\alpha) A_o$. Then $T \vdash (\exists x_\alpha) A_o^n$ for all $n \geq 1$.

(d) $\vdash (\exists x)\mathbf{\Delta} A_o \Rightarrow \mathbf{\Delta}(\exists x) A_o$ and $\vdash (\exists x)\mathbf{\Delta} A_o \Rightarrow (\exists x) A_o$.

(e) $\vdash (\exists x)(\exists y)\mathbf{\Delta} A \equiv (\exists x)\mathbf{\Delta}(\exists y)\mathbf{\Delta} A$.

(f) $\vdash \mathbf{\Delta}(x_o \,\&\, y_o) \equiv \mathbf{\Delta} x_o \,\&\, \mathbf{\Delta} y_o, \qquad \vdash \mathbf{\Delta}(x_o \vee y_o) \equiv \mathbf{\Delta} x_o \vee \mathbf{\Delta} y_o$.

(g) $\vdash (x_\alpha \equiv z_\alpha) \equiv (\exists y_\alpha)((x_\alpha \equiv y_\alpha) \,\&\, (y_\alpha \equiv z_\alpha))$ for all $\alpha \in$ Types.

(h) $(\forall x_\alpha)(A_{o\alpha}x_\alpha \equiv B_{o\alpha}x_\alpha) \Rightarrow \cdot(\forall x_\alpha)A_{o\alpha}x_\alpha \equiv (\forall x_\alpha)B_{o\alpha}x_\alpha$.

(i) $(\forall x_\alpha)(A_{o\alpha}x_\alpha \equiv B_{o\alpha}x_\alpha) \Rightarrow \cdot(\exists x_\alpha)A_{o\alpha}x_\alpha \equiv (\exists x_\alpha)B_{o\alpha}x_\alpha$.

(j) $\vdash (A \Rightarrow B)\,\&(C \Rightarrow D)\cdot \Rightarrow (A \vee C \Rightarrow \cdot B \vee D)$.

(k) $\vdash (\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow (\exists x)B)$.

(l) $\vdash (\exists x)(A\,\&\,B) \equiv A\,\&(\exists x)B$ *(x is not free in A).*

(m) $\vdash \mathbf{\Delta}(p_{o\alpha} \equiv q_{o\alpha}) \Rightarrow (\iota_{\alpha(o\alpha)}p_{o\alpha} \equiv \iota_{\alpha(o\alpha)}q_{o\alpha})$

PROOF: (a) Let $T \cup \{B_{x_\alpha}[\mathbf{u}_\alpha]\} \vdash A$ where A does not contain $\mathbf{u}_\alpha$. Analogously as in the classical proof of the similar property, we replace all occurrences of $\mathbf{u}_\alpha$ in the proof of A by a variable y_α not occurring in it. This means that $T \cup \{B_x[y_\alpha]\} \vdash A$. From this, using deduction theorem, we obtain $T \vdash \mathbf{\Delta}B_x[y_\alpha] \Rightarrow A$ and from it, using generalization and the properties of quantifiers to obtain $T \vdash (\exists x_\alpha)\mathbf{\Delta}B \Rightarrow A$. From this we obtain $T \vdash A$ using the assumption and modus ponens which proves conservativeness. The opposite implication is obvious.

The properties (b)–(e), (g)–(i), (k)–(m) have been proved in [25, 27]. The proof of the property (f) is the same as that of Lemma 2.4.11(4) from [12].

(j) We start with the provable tautologies $\vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow \cdot B \vee D)$, $\vdash (C \Rightarrow D) \Rightarrow (C \Rightarrow \cdot B \vee D)$. Then use the fact that $\vdash (E \Rightarrow F)$ and $\vdash (G \Rightarrow H)$ imply $\vdash E\,\&\,G \Rightarrow \cdot F\,\&\,H$. From this we obtain $\vdash (A \Rightarrow B)\,\&(C \Rightarrow D)\cdot \Rightarrow (A \Rightarrow \cdot B \vee D)\,\&(C \Rightarrow \cdot B \vee D)$ so that

$$\vdash (A \Rightarrow B)\,\&(C \Rightarrow D)\cdot \Rightarrow (A \vee C \Rightarrow \cdot B \vee D).$$

□

FTT provides also means for specification that the given formula of type o represents a nonzero truth value and a general (nontrivial) truth value different both from $\mathbf{0}$ as well as from $\mathbf{1}$. Namely, we will introduce the following higher-order formulas:

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg\mathbf{\Delta}(\neg z_o), \qquad \text{(nonzero truth value)}$$

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\mathbf{\Delta}(z_o \vee \neg z_o). \qquad \text{(general truth value)}$$

Both formulas Υ_{oo} as well as $\hat{\Upsilon}_{oo}$ are crisp.

Lemma 3
(a) Let $\mathcal{I}^{\mathcal{M}}$ be an interpretation and p an assignment. Then

$$\mathcal{I}^{\mathcal{M}}_p(\Upsilon z_o) = \mathbf{1} \quad \textit{iff} \quad p(z_0) > \mathbf{0},$$
$$\mathcal{I}^{\mathcal{M}}_p(\hat{\Upsilon} z_o) = \mathbf{1} \quad \textit{iff} \quad \mathbf{1} > p(z_0) > \mathbf{0}.$$

(b) Let $T \vdash \Upsilon z_o\,\&(z_o \Rightarrow y_o)$. Then $T \vdash \Upsilon y_o$.

(c) $\vdash \hat{\Upsilon} z_o \equiv \hat{\Upsilon}\neg z_o$.

(d) $\vdash \mathbf{\Delta} z_o \Rightarrow \neg\hat{\Upsilon} z_o, \qquad \vdash \mathbf{\Delta}\neg z_o \Rightarrow \neg\hat{\Upsilon} z_o$.

(e) $\vdash \hat{\Upsilon} z_o \Rightarrow \Upsilon z_o$

(f) Let $T \vdash \hat{\Upsilon} z_o \,\&\, \hat{\Upsilon} u_o \,\&(z_o \Rightarrow t_o)\,\&(t_o \Rightarrow u_o)$. *Then* $T \vdash \hat{\Upsilon} t_o$.

PROOF: (a) Put $p(z_0) = a$ and let $\mathcal{I}_p^{\mathcal{M}}(\Upsilon z_o) = \neg(\Delta(\neg a)) = \mathbf{1}$. Then $\Delta(\neg a) = \mathbf{0}$ which means that $\neg a \neq \mathbf{1}$ and so, a cannot be equal to $\mathbf{0}$. Conversely, let $a > \mathbf{0}$. If $a = \mathbf{1}$ then $\neg(\Delta(\neg a)) = \neg\mathbf{0} = \mathbf{1}$. Otherwise $\neg a < \mathbf{1}$ and so, $\Delta(\neg a) = \mathbf{0}$ so that $\neg(\Delta(\neg a)) = \neg\mathbf{0} = \mathbf{1}$.

For the second equation, if $\mathcal{I}_p^{\mathcal{M}}(\hat{\Upsilon} z_o) = \mathbf{1}$ then $\Delta(a \vee \neg a) = \mathbf{0}$ which means that $a \vee \neg a \neq \mathbf{1}$ and so, $a \neq \mathbf{1}$ as well as $a \neq \mathbf{0}$. The converse is obvious.

(b) From $T \vdash z_o \Rightarrow y_o$ we obtain $T \vdash \mathbf{\Delta}(\neg y_o \Rightarrow \neg z_o)$ using the properties of FTT, and further, $T \vdash (\mathbf{\Delta}\neg y_o \Rightarrow \mathbf{\Delta}\neg z_o)$. Finally, using contraposition and modus ponens, we obtain $T \vdash \neg\mathbf{\Delta}\neg y_o$ which implies $T \vdash \Upsilon y_o$.

(c) This follows immediately from $\vdash z_o \equiv \neg\neg z_o$ when realizing that $\vdash \hat{\Upsilon} z_o \equiv \neg\mathbf{\Delta}(\neg z_o \vee \neg\neg z_o)$.

(d) From $\vdash z_o \Rightarrow z_o \vee \neg z_o$ we prove $\vdash \mathbf{\Delta} z_o \Rightarrow \mathbf{\Delta}(z_o \vee \neg z_o)$. Then use definition of $\hat{\Upsilon}$ and the double negation. The second part is proved analogously.

(e) immediately follows from $\vdash \neg z_o \Rightarrow z_o \vee \neg z_o$ and the properties of FTT.

(f) Using (b) and (e) we prove $T \vdash \Upsilon t_o$ which is equivalent to $T \vdash \neg\mathbf{\Delta}\neg t_o$. Furthermore, from $T \vdash t_o \Rightarrow u_o$ we prove $T \vdash \neg\mathbf{\Delta} u_o \Rightarrow \neg\mathbf{\Delta} t_o$. From $T \vdash \hat{\Upsilon} u_o$ we obtain $T \vdash \neg\mathbf{\Delta} u_o \wedge \neg\mathbf{\Delta}\neg u_o$ using Lemma 2(f). From the last two formulas we derive $T \vdash \neg\mathbf{\Delta} t_o$ and so, we conclude that $T \vdash \neg\mathbf{\Delta} t_o \wedge \neg\mathbf{\Delta}\neg t_o$ which is equivalent to $T \vdash \hat{\Upsilon} t_o$. □

2.6 Extensionality

In FTT, we assume that all formulas are *weakly extensional* which means that they fulfil axiom ($\mathrm{FT}_I 1$). Recall that this means that if $\mathcal{I}_p^{\mathcal{M}}(x_\alpha \equiv y_\alpha) = \mathbf{1}$ (i.e., $p(x)$ and $p(y)$ are equal in the degree $\mathbf{1}$) then also the corresponding functional values of $\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha})$ are equal in the degree $\mathbf{1}$, i.e., $\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha) = \mathbf{1}$. Otherwise, the degree of equality of the latter can be arbitrary.

Quite often, however, we need a stronger property. We say that a formula $A_{o\alpha}$ is *strongly extensional* in a theory T, if

$$T \vdash x_\alpha \equiv y_\alpha \Rightarrow A_{o\alpha} x_\alpha \equiv A_{o\alpha} y_\alpha. \tag{9}$$

Strong extensionality requires "good behavior" already of all degrees, not only of $\mathbf{1}$.

Lemma 4
A formula $A_{o\alpha}$ is strongly extensional in T iff

$$T \vdash A_{o\alpha} y_\alpha \equiv (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} x_\alpha). \tag{10}$$

PROOF: By properties of FTT, we have $\vdash A_{o\alpha} y_\alpha \equiv \cdot y_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} y_\alpha$, i.e. $\vdash A_{o\alpha} y_\alpha \Rightarrow \cdot y_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} y_\alpha$ and, consequently,

$$\vdash A_{o\alpha} y_\alpha \Rightarrow (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} x_\alpha). \tag{11}$$

Let $A_{o\alpha}$ be strongly extensional. Then

$$T \vdash (x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} x_\alpha) \Rightarrow A_{o\alpha} y_\alpha.$$

Using rule of generalization and properties of quantifiers, we obtain

$$T \vdash (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} x_\alpha) \Rightarrow A_{o\alpha} y_\alpha$$

which together with (11) implies (10).

Conversely, let (10) hold. Then $T \vdash (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha} x_\alpha) \Rightarrow A_{o\alpha} y_\alpha$, and by properties of quantifiers, substitution and properties of FTT we obtain $T \vdash (x_\alpha \equiv y_\alpha) \Rightarrow (A_{o\alpha} x_\alpha \Rightarrow A_{o\alpha} y_\alpha)$. Similarly, we proceed with the variables x_α and y_α exchanged and so, obtain (9) by properties of FTT.

□

2.7 Transfer of properties

We will often need to transfer properties from one type to another one by means of a function. Let a function $f_{\beta\alpha}$ be given and let $A_{(o\alpha)\alpha}$ be a relation between elements of type α. Then it can be transferred to a relation $A^f_{(o\beta)\beta}$ between elements of type β that fall into the range of the function f. This will be explicitly stated by the following lemma.

First, we will introduce a special property $\mathrm{rng}_{(o\beta)(\beta\alpha)}$ stating that $y \in Form_\beta$ belongs to the *range* of $f \in Form_{\beta\alpha}$:

$$\mathrm{rng} \equiv \lambda f\, \lambda y_\beta\, (\exists x_\alpha) \boldsymbol{\Delta}(y_\beta \equiv f x_\alpha). \tag{12}$$

Note that it is a crisp property so that, by Rule C (Lemma 2(a)), if $\vdash \mathrm{rng}\, fy$ then there is an element x_α that can be denoted by some constant and that assures y_β to be a functional value of f at x_α. We will write $y \in \mathrm{rng}\, f$ instead of $\mathrm{rng}\, fy$. To find such an element x explicitly, we will use the description operator ι.

Let us denote

$$^{-1} \equiv \lambda f\, \lambda y \cdot \imath x \cdot y \equiv fx \tag{13}$$

(recall that $\imath x(y \equiv fx) \equiv \iota_{\alpha(o\alpha)}(\lambda x \cdot y \equiv fx)$). We will write $f^{-1}y$ instead of $(^{-1}f)y$. By the definition, $f^{-1}y$ is the x_α for which $y \equiv fx$ is true (provable) in the degree 1, and which is chosen using the description operator $\iota_{\alpha(o\alpha)}$.

Lemma 5
(a) Let $f \in Form_{\beta\alpha}$, $x \in Form_\alpha$, $y \in Form_\beta$. Then

$$\vdash (y \in \mathrm{rng}\, f) \equiv \boldsymbol{\Delta}(y \equiv f\,(f^{-1}y)).$$

(b) $\vdash (y \in \operatorname{rng} f)\,\&(y' \in \operatorname{rng} f) \Rightarrow (\mathbf{\Delta}(y \equiv y') \equiv \mathbf{\Delta}(f\,(f^{-1}y) \equiv f\,(f^{-1}y')))$.

PROOF: (a)

(L.1) $y \equiv fx \vdash y \equiv fx$ (assumption)

(L.2) $\vdash (\forall x)(y \equiv fx) \Rightarrow \cdot y \equiv f\,(f^{-1}y)$ (substitution axiom)

(L.3) $y \equiv fx \vdash y \equiv f\,(f^{-1}y)$ (L.1, L.2, generalization, modus ponens)

(L.4) $y \equiv fx \vdash \mathbf{\Delta}(y \equiv f\,(f^{-1}y))$ (L.3, rule (N))

(L.5) $\vdash (\exists x)\mathbf{\Delta}(y \equiv fx) \Rightarrow \mathbf{\Delta}(y \equiv f\,(f^{-1}y))$
(L.4, deduction theorem, properties of FTT)

(L.6) $\vdash y \in \operatorname{rng} f \Rightarrow \mathbf{\Delta}(y \equiv f\,(f^{-1}y))$ (L.5, (12), rule (R))

(L.7) $\vdash \mathbf{\Delta}(y \equiv f\,(f^{-1}y)) \Rightarrow (\exists x)\mathbf{\Delta}(y \equiv fx)$ (substitution)

The rest follows from L.6, L.7 by (12) and the properties of FTT.
(b)

(L.1) $y \equiv y' \vdash y \equiv y'$ (assumption)

(L.2) $\vdash (\lambda x \cdot y \equiv fx) \equiv (\lambda x \cdot y \equiv fx)$ (properties of FTT)

(L.3) $y \equiv y' \vdash (\lambda x \cdot y \equiv fx) \equiv (\lambda x \cdot y' \equiv fx)$ (L.1, L.2, rule (R))

(L.4) $y \equiv y' \vdash \mathbf{\Delta}((\lambda x \cdot y \equiv fx) \equiv (\lambda x \cdot y' \equiv fx))$ (L.3, rule (N))

(L.5) $\vdash \mathbf{\Delta}((\lambda x \cdot y \equiv fx) \equiv (\lambda x \cdot y' \equiv fx)) \Rightarrow ((f^{-1}y) \equiv (f^{-1}y'))$ (Lemma 2(m))

(L.6) $\vdash \mathbf{\Delta}(y \equiv y') \Rightarrow (f^{-1}y) \equiv (f^{-1}y')$
(L.4, L.5, properties of FTT, deduction theorem)

(L.7) $\vdash \mathbf{\Delta}((f^{-1}y) \equiv (f^{-1}y')) \Rightarrow (f(f^{-1}y) \equiv f(f^{-1}y'))$ (equality theorem)

(L.8) $y \equiv fx, y' \equiv fx' \vdash y \equiv f\,(f^{-1}y)\,\&\, y' \equiv f\,(f^{-1}y')$
(analogously to line L.3 of the proof of (a))

(L.9) $y \equiv fx, y' \equiv fx' \vdash \mathbf{\Delta}((f^{-1}y) \equiv (f^{-1}y')) \Rightarrow (y \equiv y')$
(L.7, L.8, properties of FTT)

(L.10) $\vdash (y \in \operatorname{rng} f)\,\&(y' \in \operatorname{rng} f) \Rightarrow (\mathbf{\Delta}(y \equiv y') \equiv \mathbf{\Delta}(f\,(f^{-1}y) \equiv f\,(f^{-1}y')))$.
(L.6, L.9, deduction theorem, properties of FTT)

□

By this lemma, if y (surely) belongs to the range of the function f then the element $f^{-1}y$ (of type α) is mapped to y via f. Moreover, if y and y' belong to the range of f then their boolean equality is equivalent with the boolean equality of $(f^{-1}y)$ and $(f^{-1}y')$ by the properties of the description operator ι. We may take $(f^{-1}y)$, in a certain sense, as "typical" element of type α corresponding to y via f.

Let $A \in Form_{(o\alpha)\alpha}$ be a relation between elements of type α and $f \in Form_{\beta\alpha}$ be a function. Then we put

$$A_f \equiv \lambda z\, \lambda z' \cdot A\,(f^{-1}z)\,(f^{-1}z'). \tag{14}$$

We obtain

$$A_f\, yy' \equiv A\,(f^{-1}y)\,(f^{-1}y') \tag{15}$$

by lambda conversion.

Lemma 6
Let $f \in Form_{\beta\alpha}$, $x, x' \in Form_\alpha$, $y, y' \in Form_\beta$.

(a) $\vdash y \in \mathrm{rng}\, f \,\&\, y' \in \mathrm{rng}\, f \Rightarrow$
$$(A\, xx' \,\&\, \boldsymbol{\Delta}(x \equiv (f^{-1}y)) \,\&\, \boldsymbol{\Delta}(x' \equiv (f^{-1}y')) \Rightarrow A_f\, yy').$$

(b) Let $f \in Form_{\beta\alpha}$ and $\vdash B \equiv B'_{z_1,\ldots,z_m}[A_1\, x_{j_1} x_{k_1}, \ldots, A_m\, x_{j_m} x_{k_m}] \in Form_o$ be a formula containing $A_i\, x_{j_i} x_{k_i} \in Form_o$, $i = 1, \ldots, m$ as subformulas so that the variables $\{x_{j_i}, x_{k_i} \mid i = 1, \ldots, m\} = \{x_1, \ldots, x_n\} \subset Form_\alpha$ are free in B. Then

$$\vdash y_1 \in \mathrm{rng}\, f \,\&\cdots\&\, y_n \in \mathrm{rng}\, f \,\&(\forall x_1, \ldots, x_n)B \Rightarrow B'_{z_1,\ldots,z_m}[A_{f,1}\, y_{j_1} y_{k_1}, \ldots, A_{f,m}\, y_{j_m} y_{k_m}]$$

PROOF: (a) From the equality theorem, we get

$$y \equiv fx, y' \equiv fx' \vdash Axx' \,\&\, \boldsymbol{\Delta}(x \equiv (f^{-1}y)) \,\&\, \boldsymbol{\Delta}(x' \equiv (f^{-1}y')) \Rightarrow A\,(f^{-1}y)\,(f^{-1}y') \tag{16}$$

Then (a) follows from (16), (15) and (12) using the deduction theorem and rule (R).

(b) By the definition, $f^{-1}y_j$, $j = 1, \ldots, n$ are formulas of type α. By the substitution axiom and (15), we obtain

$$y_1 \equiv fu_1, \ldots, y_n \equiv fu_n \vdash (\forall x_1, \ldots, x_n)B \Rightarrow B'_{z_1,\ldots,z_m}[A_{f,1}\, y_{j_1} y_{k_1}, \ldots, A_{f,m}\, y_{j_m} y_{k_m}].$$

Then (b) follows from the deduction theorem by the properties of FTT. □

By this lemma, we may extend a relation A among elements of type α to those elements of type β that belong to the range of f when using the "typical" elements of the form $(f^{-1}y)$. By the part (b), the general properties of A on elements of type α are inherited also to elements y from the range of f.

3 The Theory of Trichotomous Evaluative Linguistic Expressions

In this section, we will first describe briefly the grammatical structure of evaluative expressions. Then we focus on characterization the meaning of TEv-expressions. This will be done by means of a special formal language of FTT. Certain formulas of it will represent evaluative linguistic expressions which enables us to characterize their properties using logical means.

3.1 Grammatical structure of evaluative linguistic expressions

As already stated, evaluative expressions are special expressions that are used in natural language for characterization of a position on an ordered scale. Their base is formed by a specific evaluative adjective (cf. our discussion in the Introduction), or a numeral. The systematics presented below is needed to be able to introduce a clear definition of their semantics.

Definition 1
Evaluative linguistic expression *is either of the following:*

(i) Simple evaluative expression, *which is one of the linguistic expressions:*

  (a) $\langle$*trichotomous evaluative expression*$\rangle$:= $\langle$*linguistic hedge*$\rangle\langle$*TE-adjective*$\rangle$

  (b) $\langle$*fuzzy quantity*$\rangle$:= $\langle$*linguistic hedge*$\rangle\langle$*numeral*$\rangle$

(ii) Negative evaluative expression, *which is an expression*

not $\langle$*trichotomous evaluative expression*$\rangle$

(iii) Compound evaluative expression, *which is either of the following:*

  (a) $\langle$*trichotomous evaluative expression*$\rangle$ or $\langle$*trichotomous evaluative expression*$\rangle$

  (b) $\langle$*trichotomous evaluative expression*$\rangle$ and $\langle$*negative evaluative expression*$\rangle$

The connective "and" in the compound expression (iii)(b) can be replaced by the connective "but".

Further components of evaluative expressions are:

(iv) $\langle$*numeral*$\rangle$ *is a name of some element from the considered scale*[*)]

(v) $\langle$*linguistic hedge*$\rangle$ *is an intensifying adverb making the meaning of the trichotomous evaluative expression either more, or less specific:*

$\langle$*linguistic hedge*$\rangle$:= empty | $\langle$*narrowing adverb*$\rangle$ | $\langle$*widening adverb*$\rangle$ | $\langle$*specifying adverb*$\rangle$

[*)] Fuzzy quantities require a concrete semantics, i.e. a concrete scale. From the point of logic, it is a constant in a language expanded by names of all elements of the given model.

A precise definition of TE-adjectives is difficult. As mentioned, they include gradable adjectives, adjectives of manner, and possibly some other ones. The most important distinctive feature is that they must form pairs of antonyms, i.e., the pairs

$$\langle\text{nominal adjective}\rangle \text{ — } \langle\text{antonym}\rangle$$

which can be, furthermore, completed by a middle term so that the triple of expressions

$$\langle\text{linguistic hedge}\rangle\langle\text{nominal adjective}\rangle \text{ — } \langle\text{linguistic hedge}\rangle\langle\text{middle member}\rangle \text{ — } \langle\text{linguistic hedge}\rangle\langle\text{antonym}\rangle \quad (17)$$

forms the *evaluative linguistic trichotomy.* If all the linguistic hedges are empty then (17) is called the *fundamental evaluative trichotomy.*

Example 1
Typical trichotomous evaluative linguistic expressions are *small, medium, big*, and also *very small, more or less medium, very big*, etc. The pairs of antonyms are, e.g., "young — old", "ugly — nice", "stupid — clever", etc. The fundamental evaluative trichotomies, in these cases, are "young — medium age — old", "ugly — normal — nice", "stupid — medium intelligent — clever", etc.

Narrowing linguistic hedges are *very, extremely, significantly*, etc. Widening linguistic hedges are *more or less, roughly, very roughly*, etc. Specifying adverbs are *approximately, about, rather, precisely*, etc.

Fuzzy quantities are, e.g., *twenty five, about 150 thousand, roughly 100*, etc. Simple evaluative linguistic expressions are *very small, more or less medium, roughly big, about twenty five, approximately* x_0, etc. Negative evaluative expressions are *not small, not very big*, etc. Compound evaluative linguistic expressions are *roughly small or medium, quite roughly medium and/but not big* , etc.

Let us remark that the concept of empty hedge introduced in (v) enables us develop a uniform explication of all trichotomous evaluative expressions. The theory of their semantics thus becomes relatively simple and transparent.

The "fuzzy quantity" is a linguistic characterization of some element from a certain set. In a special case, this is a number from $\mathbb{R}$. This means that every linguistic characterization of a number is understood imprecisely. We will take the form "about x_0" (x_0 is a specific numeral) as canonical. Note, that e.g., colors can also be ranked among evaluative expressions because their interpretation leads to fuzzy sets in an ordered scale of wave lengths and thus, they can be understood as special names of fuzzy numbers.

The negative evaluative expressions are, in general, ambiguous since we must take the topic-focus articulation phenomenon into account (see [13, 33]). This means that each linguistic expression is divided into two parts: the *topic* – what is spoken about, which may be empty, and the *focus* – the new information. In sentential negation, the negated part is focus. For example "not very small" may have two readings: "not VERY small" and "not VERY SMALL". In the first case, only the hedge "very" is negated and so, we can take "not very" as a specific hedge. In the second case the whole expression "very small" is negated. In our theory, we will consider the second case only, i.e. the particle *not* acts on the whole linguistic expression following it.

Agreement: In the sequel, we will use the TE-adjectives *small, medium*, and *big* as canonical. Clearly, they can be replaced by arbitrary other ones — cf. the examples above.

3.2 Evaluative linguistic predications

Evaluative expressions usually occur in predicative position. The resulting expressions are called *evaluative (linguistic) predications*. These are special expressions of natural language that characterize features, such as sizes, volumes, magnitudes, intensities, etc. of specific objects characterized by nouns or more complex constructions.

Definition 2
(a) Let $\mathcal{A}$ be an evaluative linguistic expression. Then the linguistic expression

$$\langle noun \rangle \text{ is } \mathcal{A} \tag{18}$$

is an evaluative predication. *If $\mathcal{A}$ is a simple evaluative linguistic expression then (18) is a a simple evaluative predication.*

(b) The abstracted evaluative predication *is an expression of the form*

$$X \text{ is } \mathcal{A}$$

where X is a variable whose values can be arbitrary elements.

(c) If $\mathcal{A}$ and $\mathcal{B}$ are evaluative predications then expressions of the form '$\mathcal{A}$ and $\mathcal{B}$' and '$\mathcal{A}$ or $\mathcal{B}$' are compound predications.

(d) A fuzzy IF-THEN rule is an abstracted conditional linguistic clause consisting of abstracted evaluative linguistic predications, i.e. it is a linguistic expression of the form

$$\mathcal{R} := \text{IF } X \textit{ is } \mathcal{A} \text{ THEN } Y \textit{ is } \mathcal{B}. \tag{19}$$

Let us stress that in this definition, the verb "is" takes a specific role of a copula (copular verb), i.e., it simply assigns the property of $\mathcal{A}$ to the elements named by ⟨noun⟩ and so, it is not treated as a verb. From this point of view, the evaluative predication has the same meaning as a general relationship in the expression

$$\mathcal{A}\ \langle \text{noun} \rangle,$$

for example “small house”, “very tall man”, “more or less deep sea”, etc. Note, that objects named by ⟨noun⟩ may be quite complicated entities and so, the property of $\mathcal{A}$ may, in fact, concern only certain *feature* (or few features) of them that attain values from some ordered scale, and only the latter are evaluated by the evaluative expression $\mathcal{A}$. In the sequel we will disregard concrete elements that are normally denoted by the noun and so, we will mostly work with abstracted evaluative predications only.

3.3 Intuition about the meaning of TEv-expressions

In any model of the semantics of linguistic expressions, we must distinguish between the concepts of intension and extension in a possible world (see [5, 19]). A *possible world* is a state of the world at a given time moment and place (particular context in which the linguistic expression is used), or it can also be understood as a maximal set of consistent facts. Because of this very wide and ambiguous understanding, we will use a narrower term *context*, instead.

Intension of a linguistic expression, of a sentence, or of a concept, is identified with the property denoted by it. It leads to different truth values in various contexts but is invariant with respect to them. Expressions $\mathcal{A}$ of natural language are, in general, names of intensions.

Extension is a class of elements determined by an intension, which fall into the meaning of a linguistic expression in the given context. Thus, it depends on the particular context of use and changes whenever is the context changed. For example, the expression “high” is a name of an intension being a property of some feature of objects, i.e., of their height. In concrete case, its extension may cover values of about 30 cm when a beetle needs to climb a straw, of about 30 m for electrical pylon, but values of about 3 km or more for a mountain.

Evaluative expressions can be understood as linguistic characterization of the abstract concept of quantity. Quantities are in each context classes of elements taken from an ordered scale. According to *empirical observation*, scales considered in the linguistic meaning of evaluative expressions are *linearly ordered and bounded*. Since they can be very extensive, the number of necessary linguistic expressions would have to be very large; in limit case even infinite. Luckily, we have the concept of number at disposal and so, any value from an arbitrary scale may get its name. However, this is inconvenient or unnecessary in practical life. The power of natural language enables people to use only small (finite) number of expressions which, surprisingly, may characterize any element of any ordered set. The price we must pay is vagueness of the meaning of the used expressions. Natural language thus becomes an extremely powerful tool which makes it possible to characterize and ponder on various values (sizes, volumes, etc.) and, for example, to make relevant decisions on the basis of them.

A question arises, what is the source of vagueness of extensions of the evaluative expressions. The justification is based on ideas of P. Vopěnka (see [36]) concerning the concept of horizon. From now on, we will focus on trichotomous evaluative expressions (*TEv-expressions* for short) only.

Each context is represented by an ordered scale bounded by two limit points: a *left bound* and a *right bound*. These points are the "most typical" small value and the "most typical" big value, respectively. The properties of being "small" and "big" are the, so called, *primary recordable properties*. They are naturally vague since, though we can always point out some small (big) value (for example, the left or right limit), there does not exist the last small (big) value. The only fact we know is that small values run somewhere towards a certain point which is the *horizon of our seeing of all small values*. Everything beyond this point is *surely not small*. Note that this reasoning embraces the sorites paradox[†)]. The way how the sorites paradox is resolved in fuzzy logic (cf. [11]) consists in introduction of *degrees of truth* expressing that "we find ourselves still before the horizon".

Quite similarly, starting from the right bound and going in the opposite direction, we find a *horizon of big values* such that everything beyond it is surely not big. As a consequence, there exists a certain point which lays somewhere between the left and right bound, and such that both horizons vanish at it. This point will be called the *central limit point* [*)].

Human mind (and, consequently, natural language) enables us to distinguish parts of the horizon more subtly by modifying it. In other words, we may say that our mind *shifts horizon* along the world. We obtain new, either more, or less specific horizons that determine extensions of the evaluative expressions. Consequently, if an element of the scale falls in the extension of the more specific (i.e. "narrower") evaluative expression then it falls in the extensions of all less specific (i.e. "wider") ones (provided that they exist). For example, very small values cease to be "very small" sooner than to be "small". This means that each "very small" value is at the same time "small" but there exist small values, that are not "very small". Analogous reasoning can be made for the pair of evaluative expressions "small" and "roughly small", and also for all the other TEv-expressions with the same TE-adjective inside.

3.4 Informal characterization of the meaning of TEv-expressions

With respect to the above discussion, we will formulate the following global characteristics of the meaning of TEv-expressions:

(i) *Linguistic context* of TEv-expression is a nonempty, linearly ordered and bounded scale. In each context, three distinguished limit points can be determined: *left bound*, *right bound*, and a *central point* (laying somewhere in-between the former).

[†)] One grain does not form a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.

[*)] This is a distinguished inner element of the scale, which, provided that some metric is defined on the scale, needs not necessarily lay in its center.

(ii) *Intension* of TEv-expression is a function from the set of contexts into a set of fuzzy sets. This means that each context is assigned a fuzzy set inside it which forms an *extension* of TEv-expression in the given context.

(iii) Each of the limit points in (i) is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point. The latter is the point beyond which the horizon is vanished. Consequently, we may distinguish three horizons in each context:

   (a) a horizon from the left bound towards central point,

   (b) a horizon from the right bound back towards central point,

   (c) a horizon from the central point towards both left and right bounds (i.e., it is symmetrically spread around the central point).

(iv) Each horizon in (iii) is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

(v) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon* considered in items (iii) and (iv). The modification corresponds to a linguistic hedge and consists in *shifting* the horizon, i.e., moving it closer to, or farther from the limit point. This effect is accomplished by decreasing the truth values. Moreover, the decrease will be rather small for big truth values and, at the same time, big for small ones because big truth values express stronger certainty that the given element still lays inside the horizon than small ones.

   As a consequence, each element of the given context is contained in extensions of several simple evaluative expressions which differ from each other only in their hedges. Each limit point due to (i) which lays inside extension of an evaluative expression is *typical* for the latter.

(vi) Each scale is vaguely partitioned by the *fundamental evaluative trichotomy* consisting of a pair of antonyms, and a middle member. The antonyms characterize opposite sides of the context. There is no element of the context falling into extensions of both antonyms. However, there exist elements of the scale falling into extension of the middle member only. Consequently, any element of the scale is contained in the extension of at most two neighboring expressions from the fundamental evaluative trichotomy.

We will also suppose that extension of TEv-expression conforms with Hypothesis 1 (this is general requirement and so, we do not include it among the above items).

The aim of this paper is to formalize these characteristics explicitly using a formal means of of FTT. Actually, we will formulate special axioms on the basis of the items (i)–(vi) and thus develop a formal theory of the meaning of TEv-expressions. This theory denoted by T^{Ev} is explained below.

Note that the meaning of TEv-expressions does not depend on the elements forming the considered linguistic contexts. The only information contained in them is the information about limit points. Let us remark that fuzzy numbers, on the other hand, contain information about concrete elements of the context.

3.5 Formalization of meaning of TEv-expressions

3.5.1 Language and the formal theory $T^{\mathbf{Ev}}$

In this paper, we develop a formal theory T^{Ev} as a special formal theory of fuzzy type theory. The main ideas for its construction are contained in [24, 28, 30].

First, we define a formal language J^{Ev} of the theory T^{Ev}. Its special symbols are:

(i) A constant (formula) $F \in \mathit{Form}_{(oo)o}$ for additional fuzzy equality on truth values.

(ii) A special constant $\bar{\boldsymbol{\nu}}_{oo}$ for the standard (i.e. empty) hedge.

Recall that interpretation of any formula $A_{\alpha\beta}$ is a function $M_\beta \longrightarrow M_\alpha$ where M_α, M_β are sets assigned to the types α, β, respectively. The letters x, y (possibly with subscripts) will be usually considered as variables of some arbitrary type α, i.e. their interpretation are arbitrary objects (functions) from M_α. Similarly, the letters t, z (possibly with subscripts) will denote variables of type o; their interpretation are truth values.

Specific types introduced in intensional logic are also a type of possible world and a type of time. In the case of evaluative expressions, however, time plays a minor role and so, we will not consider a special type for it.

Furthermore, we have already noted that we prefer to speak about the *context* instead of possible world because this better corresponds to our idea. In our theory, we do not need to introduce a special elementary type for the context. Instead, we will assign it a formula $w_{\alpha o}$ of type αo, i.e., for arbitrary type $\alpha \in \mathit{Types}$ its interpretation is a function from the set of truth values to the set of objects of type α. This definition is motivated by the idea that people keep in mind a certain image of a bounded scale which they modify in accord with the concrete situation. Let us stress that the reasons for using the scale of truth values for this purpose are mostly technical since this trick frees us from the necessity to define explicitly special ordering in other contexts.

We will usually write w instead of the precise $w_{\alpha o}$ but we must always keep in mind that some concrete type α is at play when dealing with the context w. We will write $x \in w$ instead of $x \in \operatorname{rng} w$ to stress the interpretation "an element x belongs to the context w", i.e. (cf. (12))

$$x \in w := (\exists t)\boldsymbol{\Delta}(x \equiv wt).$$

We will also work with elements of the form $w^{-1}x$. By definition (13) we get $\vdash w^{-1}x \equiv \iota_{o(oo)}\lambda t \cdot x \equiv wt$. In words, this is a truth value t assigned to x for which the equality $x \equiv wt$ is true in the degree $\mathbf{1}$. This trivially holds for all truth values t just assigned to x via w. Note, however, that if there exist more elements x' surely equal to x (i.e. $\vdash x' \equiv x$) then $\iota_{o(oo)}$ must choose from all truth values t for which it surely holds that $\vdash x \equiv wt$ as well as $\vdash x' \equiv wt$.

Our definition of the context enables us to define important concepts of horizon and intension of evaluative expressions in a simple way and to minimize the number of special formulas and axioms. It also enables us to consider the context very generally so that the evaluative expressions may concern arbitrary objects — functions of arbitrary complexity. Thus, we may clearly distinguish the naked evaluative linguistic expressions from the evaluative linguistic predications (note that in the early models, this distinction has not been clear).

In the rest of this section, we will form step by step special axioms (EV1)–(EV11) of the theory T^{Ev}. Occasionally, we will introduce further special constants, if necessary.

3.5.2 Context and its properties

The first axiom of T^{Ev} is axiom assuring existence of a middle truth value:

(EV1) $(\exists z)\mathbf{\Delta}(\neg z \equiv z)$

On the basis (EV1) and Lemma 2(a), we will add a special constant † into J^{Ev} as the formula

$$\dagger := \iota_{o(oo)}\, \lambda z \cdot \neg z \equiv z.$$

Note that in the standard semantics, interpretation of † is the truth value 0.5.

Lemma 7
(a) $\vdash \neg(\top \equiv \bot)$,

(b) $T^{\mathrm{Ev}} \vdash \neg\mathbf{\Delta}\dagger$,

(c) $T^{\mathrm{Ev}} \vdash \neg\mathbf{\Delta}(\dagger \equiv \bot)$,

(d) $T^{\mathrm{Ev}} \vdash \neg\mathbf{\Delta}(\dagger \equiv \top)$,

(e) $T^{\mathrm{Ev}} \vdash \hat{\Upsilon}\dagger$.

PROOF: (a) is equivalent to the double negation $\vdash \neg\neg\top$ which is provable.
(b)

(L.1) $T^{\mathrm{Ev}} \vdash \dagger \Rightarrow (\dagger \Rightarrow \bot)$ (consequence of (EV1))

(L.2) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}(\dagger \Rightarrow \bot) \Rightarrow (\mathbf{\Delta}\dagger \Rightarrow \bot)$ (properties of $\mathbf{\Delta}$)

(L.3) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}\dagger \Rightarrow (\mathbf{\Delta}\dagger \Rightarrow \bot)$ (L.1, rule (N), L.2, properties of FTT)

(L.4) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}\dagger \Rightarrow (\mathbf{\Delta}\dagger \Rightarrow \bot) \Rightarrow \cdot(\mathbf{\Delta}\dagger \Rightarrow \mathbf{\Delta}\dagger) \Rightarrow (\mathbf{\Delta}\dagger \Rightarrow \bot)$ (properties of $\mathbf{\Delta}$)

(L.5) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}\dagger \Rightarrow \bot$ (L.3, L.4, properties of FTT)

(L.6) $T^{\mathrm{Ev}} \vdash \neg\mathbf{\Delta}\dagger$ (L.5, properties of FTT)

(c) From (EV1) and (b) we get $T^{\text{Ev}} \vdash \neg\boldsymbol{\Delta}\neg\dagger$ which is just (c).
(d) follows from (b), (EV1) and contraposition.
(e) This follows from (b), $\vdash \dagger \vee \dagger \equiv \dagger$ and $T^{\text{Ev}} \vdash \dagger \equiv \neg\dagger$ by rule (R). □

We must accomplish requirements of (i) from Subsection 3.4 which means, especially, existence of the leftmost, middle and rightmost elements in each context. Using Lemma 2(a) and (8) we will introduce special constants $\bot_w, \dagger_w, \top_w$ (for each type α) for every context $w \in \mathit{Form}_{\alpha o}$ so that the following should hold:

$$T^{\text{Ev}} \vdash (\bot_w \equiv w\bot) \wedge (\dagger_w \equiv w\dagger) \wedge (\top_w \equiv w\top)$$

At the same time, we introduce the axiom

(EV2) $(\bot \equiv w^{-1}\bot_w) \wedge (\dagger \equiv w^{-1}\dagger_w) \wedge (\top \equiv w^{-1}\top_w)$.

Since interpretation of $\equiv$ in (EV2) is a biresiduation which is separating fuzzy equality, this axiom assures that the assignment $\bot$ to $\bot_w$, $\dagger$ to $\dagger_w$ and $\top$ to $\top_w$ is one-to-one. Note that (EV2) is not a unique axiom but a scheme of axioms since we suppose it to hold for for arbitrary type $\alpha \in \mathit{Types}$ and arbitrary context $w \in \mathit{Form}_{\alpha o}$. The same holds also for all lemmas and theorems below that deal with the variable w.

The following is obvious.

Lemma 8
(a) $T^{Ev} \vdash (\bot_w \equiv w\bot) \wedge (\dagger_w \equiv w\dagger) \wedge (\top_w \equiv w\top)$,

(b) $T^{Ev} \vdash \bot_w \in w \wedge \dagger_w \in w \wedge \top_w \in w$.

We may see that $\bot_w, \dagger_w, \top_w \in \mathit{Form}_\alpha$ are elements, for which $\bot, \dagger$ and $\top$ are "typical" truth values with respect to the context w. We must further show that they indeed take the role of left bound, middle point and the right bound, respectively.

On the basis of (14), we can introduce formulas $\leq_w$, $=_w$ and $<_w$ for arbitrary context w by

$$\leq\, \equiv \lambda w\, \lambda y\, \lambda y' \cdot w^{-1}y \Rightarrow w^{-1}y', \tag{20}$$

$$=\, \equiv \lambda w\, \lambda y\, \lambda y' \cdot w^{-1}y \equiv w^{-1}y'. \tag{21}$$

$$<\, \equiv \lambda w\, \lambda y\, \lambda y' \cdot (w^{-1}y \Rightarrow w^{-1}y') \,\&\, \neg(w^{-1}y \equiv w^{-1}y'), \tag{22}$$

We will write $\leq_w$, $=_w$ and $<_w$ instead of $(\leq w)$, $(= w)$ and $(< w)$ in the sequel.

Because both $\Rightarrow$ and $\equiv$ (recall that on truth values, $\equiv$ is a logical equivalence) are reflexive (i.e. $\vdash (\forall t)(t \Rightarrow t)$; the same for $\equiv$) and transitive ($\vdash (\forall t)(\forall t')(\forall t'')((t \Rightarrow t' \,\&\, t' \Rightarrow t'') \Rightarrow t \Rightarrow t'')$; the same for $\equiv$), these properties transform also to elements from the range of w by Lemma 6(b). Furthermore, $\equiv$ is also symmetric, i.e., $\vdash (\forall t)(\forall t')(t \equiv t' \Rightarrow t' \equiv t)$, which means that $=_w$ is a fuzzy equality on objects (of type α) with respect to the context w.

Since we know that

$$\vdash (\forall t)(\forall t')((t \Rightarrow t' \wedge t' \Rightarrow t) \equiv (t \equiv t')),$$

we obtain

$$T^{\mathrm{Ev}} \vdash (x \leq_w y \wedge y \leq_w x) \equiv (x =_w y) \tag{23}$$

for all $x, y \in w$ and so, $\leq_w$ can be treated as ordering of w. Clearly, (22) is a sharp ordering.

The truth values have also the prelinearity property $\vdash (\forall t)(\forall t')(t \Rightarrow t' \vee t' \Rightarrow t)$ that transfers also to $\leq_w$. From the construction of the canonical model, however, we know that the set of truth values is linearly ordered. This property is then transferred also to the range of w. Finally, we know that

$$\vdash (\forall t)(\bot \Rightarrow t \wedge t \Rightarrow \top)$$

which means that $\bot$ is the left bound and $\top$ the right bound of the truth values. Lemma 7 then shows us that $\bot, \dagger, \top$ are distinguished points. Using Lemma 6(b), we may transfer all these properties also to $\leq_w$ and conclude that for each context w, $\leq_w$ fulfils the requirements of (i) of Subsection 3.4.

At the end of this subsection let us remark that we may consider also a special context $w_o \in \mathit{Form}_{oo}$ defined by

$$w_o \equiv \lambda t\, t$$

(i.e., it is the identity). It is easy to verify that it fulfils all the requirements on the context. Further requirements on the context are formulated in correspondence with other notions in the subsequent sections.

3.5.3 Horizon and its properties

Our further task is to define the horizon according to the requirements of items (iii) and (iv) of Subsection 3.4. As mentioned, a crucial role in its definition will be played by the sorites paradox. A model of the latter in fuzzy logic has been discussed in [11] and [30] and the model of horizon in predicate fuzzy logic with evaluated syntax $\mathrm{Ev_L}$ has been established in [24]. In this paper, we will introduce a special fuzzy equality and demonstrate that it can be used for modeling of the sorites paradox and, especially, for modeling of the horizon.

We will start with definition of a new formula for a specific fuzzy equality on truth values:

$$\sim_{(oo)o} := \lambda z\, \lambda t\, F_{(oo)o}\, tz. \tag{24}$$

Its properties are characterized by the following axioms:

(EV3) $t \sim t$,

(EV4) $t \sim u \equiv u \sim t$,

(EV5) $t \sim u \,\&\, u \sim z \cdot \Rightarrow t \sim z$,

(EV6) $\neg(\bot \sim \dagger)$,

(EV7) $\boldsymbol{\Delta}((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$, (EV8) $t \equiv t' \mathbin{\&} z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z'$,

(EV9) $(\exists u)\hat{\Upsilon}(\bot \sim u) \wedge (\exists u)\hat{\Upsilon}(\dagger \sim u) \wedge (\exists u)\hat{\Upsilon}(\top \sim u)$

Axioms (EV3)–(EV5) state that $\sim$ is a fuzzy equality. Axiom (EV6) expresses that falsity and medium truth are not equal in the sense of $\sim$. Axiom (EV7) expresses compatibility of $\sim$ with classical ordering of truth values. Axiom (EV8) expresses that $\sim$ is strongly extensional. Axiom (EV9) assures that $\sim$ is not crisp, i.e. we can find a truth value u that is not fully $\sim$-equal to either of $\bot$, $\dagger$ and $\top$.

Example 2
Let us consider the standard Łukasiewicz MV-algebra of truth values with the fuzzy equality interpreted by the biresiduation $a \leftrightarrow b = 1 - |a - b|$, $a, b \in [0, 1]$. *Then we may interpret* $\sim$ *by*

$$[a \sim b] = \frac{0 \vee (0.5 - |a - b|)}{0.5}. \tag{25}$$

Note that $[a \sim b] = (a \leftrightarrow b)^2$ *(the square is taken with respect to* $\otimes$*). It is easy to verify that axioms (EV3)–(EV9) are fulfilled; there is an infinite number of elements* b *such that* $0 < [0 \sim b] < 1$, $0 < [0.5 \sim b] < 1$, $0 < [1 \sim b] < 1$.

It is clear that we could simplify our theory when putting $\sim := (\equiv \mathbin{\&} \equiv)$. Introducing $\sim$, however, enables us to develop our theory in a more general way.

We will also introduce a fuzzy equality $(\approx w)$ in a context w that is induced by $\sim$ as follows:

$$\approx := \lambda w\, \lambda y\, \lambda y' \cdot (w^{-1}y \sim w^{-1}y'). \tag{26}$$

Using Lemma 6(b) we may show that axioms (EV3)–(EV9) transfer also to $(\approx w)$ (for elements belonging to the context w). For better readability we will write $\approx_w$ instead of $(\approx w)$.

We are now ready to define all three considered horizons using the above introduced fuzzy equality $\sim$ from (24) as follows:

$$LH_{oo} := \lambda z \cdot \bot \sim z, \tag{27}$$
$$MH_{oo} := \lambda z \cdot \dagger \sim z, \tag{28}$$
$$RH_{oo} := \lambda z \cdot \top \sim z. \tag{29}$$

The following properties are provable.

Lemma 9
(a) $T^{Ev} \vdash LH\, \bot$, $\quad T^{Ev} \vdash RH\, \top$, $\quad T^{Ev} \vdash MH\, \dagger$,

(b) $T^{Ev} \vdash \neg\, LH\, \dagger$, $\quad T^{Ev} \vdash \neg\, RH\, \dagger$, $\quad T^{Ev} \vdash \neg\, MH\, \bot \wedge \neg\, MH\, \top$,

(c) $T^{Ev} \vdash (\forall z)(\boldsymbol{\Delta}(z \Rightarrow z') \Rightarrow (LH\, z' \Rightarrow LH\, z))$,

(d) $T^{Ev} \vdash (\forall z)(\boldsymbol{\Delta}(z \Rightarrow z') \Rightarrow (RH\, z \Rightarrow RH\, z'))$,

(e) $T^{\mathrm{Ev}} \vdash (\forall z)(\mathbf{\Delta}(\dagger \Rightarrow z \,\&\, z \Rightarrow z') \Rightarrow (MH\, z' \Rightarrow MH\, z))$,

(f) $T^{\mathrm{Ev}} \vdash (\forall z)(\mathbf{\Delta}(z \Rightarrow z' \,\&\, z' \Rightarrow \dagger) \Rightarrow (MH\, z \Rightarrow MH\, z'))$,

(g) $T^{\mathrm{Ev}} \vdash (\forall z)(\mathbf{\Delta}(\dagger \Rightarrow z) \Rightarrow \neg\, LH\, z)$,

(h) $T^{\mathrm{Ev}} \vdash (\forall z)(\mathbf{\Delta}(z \Rightarrow \dagger) \Rightarrow \neg\, RH\, z)$,

(i) $T^{\mathrm{Ev}} \vdash \neg(\exists z)((LH\, z \,\&\, MH\, z) \vee (RH\, z \,\&\, MH\, z))$,

(j) $T^{\mathrm{Ev}} \vdash \neg(\exists z)(LH\, z \wedge RH\, z)$,

(k) $T^{\mathrm{Ev}} \vdash (\forall z)(LH\, z \Rightarrow \mathbf{\Delta}(z \Rightarrow \dagger))$.

(l) $T^{\mathrm{Ev}} \vdash (\forall z)(RH\, z \Rightarrow \mathbf{\Delta}(\dagger \Rightarrow z))$.

PROOF: (a) and (b) follow immediately from the definition and axioms (reflexivity) of $\sim$.

(c)–(h) follow from (EV7).

(i) From the definition and transitivity of $\sim$, we obtain $T^{\mathrm{Ev}} \vdash \bot \sim z \,\&\, z \sim \dagger \Rightarrow \bot \sim \dagger$ which is equivalent to $T^{\mathrm{Ev}} \vdash \neg(LH\, z \,\&\, MH\, z)$ by (EV6) and the definitions of LH and MH. Similarly the second conjunction and so, we get (i) by the properties of FTT.

(j) Using the properties (g), (h) and the properties of FTT, we prove that

$$T^{\mathrm{Ev}} \vdash \mathbf{\Delta}(\dagger \Rightarrow z) \vee \mathbf{\Delta}(z \Rightarrow \dagger) \Rightarrow (\neg\, LH\, z \vee \neg\, RH\, z).$$

Then, (j) follows from the prelinearity property of the truth values and properties of quantifiers.

(k) From $\vdash \mathbf{\Delta}(\dagger \Rightarrow z) \vee \mathbf{\Delta}(z \Rightarrow \dagger)$ (prelinearity) we obtain $\vdash \neg\mathbf{\Delta}(\dagger \Rightarrow z) \Rightarrow \mathbf{\Delta}(z \Rightarrow \dagger)$. From (g) we have $T^{\mathrm{Ev}} \vdash LH\, z \Rightarrow \neg\mathbf{\Delta}(\dagger \Rightarrow z)$ which implies (k). □

Lemma 10

(a) $T^{\mathrm{Ev}} \vdash (\exists u)\hat{\Upsilon}(LH\, u) \wedge (\exists u)\hat{\Upsilon}(MH\, u) \wedge (\exists u)\hat{\Upsilon}(RH\, u)$.

(b) Let $T^{\mathrm{Ev}} \vdash (\forall w)(\exists x)\hat{\Upsilon}\, LH(w^{-1}x)$ *(or* $T^{\mathrm{Ev}} \vdash (\forall w)(\exists x)\hat{\Upsilon}\, RH(w^{-1}x)$*). Let us fix some context* w *and choose a new constant* $\mathbf{r}_w \notin J^{\mathrm{Ev}}$ *so that* T *is a corresponding conservative extension of* T^{Ev} *(cf. Lemma 2(a)). Then* $T \vdash \mathbf{\Delta}(\bot_w <_w \mathbf{r}_w)$ *(or* $T' \vdash \mathbf{\Delta}(\mathbf{r}_w <_w \top_w)$*).*

PROOF: (a) follows immediately from Axiom (EV9).

(b) Using Lemma 9(c) we can prove $T \vdash \mathbf{\Delta}(\mathbf{r_w} \leq_w \bot_w) \Rightarrow (LH(w^{-1}\bot_w \Rightarrow LH(w^{-1}\mathbf{r}_w)$. From this, Lemma 3(d) and properties of LH we obtain $T \vdash \neg\mathbf{\Delta}(\mathbf{r}_w =_w \bot_w)$ which gives $T \vdash \mathbf{\Delta}(\bot_w <_w \mathbf{r}_w)$. The second part is proved analogously.

□

One may see from the above lemmas, that our definition of horizon conforms with the intuition. Let us now demonstrate that our concept of horizon can be used for solution of the sorites paradox.

We will first construct a theory T^H obtained from $T^{\rm Ev}$ as follows. All the Peano axioms are crisp and provable in T^H. Furthermore, we introduce in T^H numerals (formulas) $n \in Form_\epsilon$ representing natural numbers and a formula $\mathbb{N}_{o\epsilon}$ representing set of natural numbers; $\mathbf{0}$ represents zero. Then $T^H \vdash \mathbb{N}n$ means that n is a natural number. We will write $T^H \vdash n \in \mathbb{N}$ instead of the former. The N is a crisp formula, i.e. $T^H \vdash (\forall n)(n \in \mathbb{N} \vee n \notin \mathbb{N})$. The (crisp) linear ordering of natural numbers will be denoted by $\leq$ and the successor of n by $n+1$, as usual.

Furthermore, we will add the numerals $\mathbf{p}, \mathbf{q}$ as special constants into $J^{\rm Ev}(T^N)$ and consider a special context w_N (this is again a new constant of $J(T^{\rm Ev})$) with the following properties:

$$T^H \vdash (n \in w_N \Rightarrow n \in \mathbb{N}) \wedge (\bot \equiv w_N^{-1}\mathbf{0}) \wedge (\dagger \equiv w_N^{-1}\mathbf{p}) \wedge (\top \equiv w_N^{-1}\mathbf{q}).$$

The ordering $\mathbf{\Delta}(m \leq_{w_N} n)$ coincides with the classical ordering of natural numbers (in interpretation) and we will denote it by the ordinary symbol $\leq$ (and similarly also $<$).

Let $\approx_{w_N}$ be a formula given for the context w_N by (26) and define a formula $\mathbb{FN} \in Form_{(o\epsilon)(\epsilon o)}$ by

$$\mathbb{FN} := \lambda n \cdot \mathbf{0} \approx_{w_N} n. \qquad (30)$$

The intuitive meaning of this formula is "a fuzzy set of finite natural numbers". In other words, this formula is a many-valued model of the property "not being a heap".

Lemma 11
(a) $T^H \vdash \mathbb{FN}\,\mathbf{0}, \qquad T^H \vdash \mathbf{0} \not\equiv \mathbf{p}$,

(b) $T^H \vdash (\forall n)(n \in w_N \,\&\, \mathbf{\Delta}(p \leq n) \Rightarrow \neg\mathbb{FN}\,n)$,

(c) $T^H \vdash (\forall m)(\forall n)(m \in w_N \,\&\, n \in w_N \,\&\, \mathbf{\Delta}(m \leq n) \Rightarrow (\mathbb{FN}n \Rightarrow \mathbb{FN}m))$,

(d) $T^H \vdash (\exists m)(m \in w_N \,\&\, \hat{\Upsilon}(\mathbb{FN}m) \,\&\, \mathbf{0} < m)$.

PROOF: This lemma follows from Lemmas 6, 9 and 10. □

We will introduce a special constant $\mathbf{r}$ for the number m from (d) of the previous lemma (i.e., $\mathbf{r}$ represents a number for which $\mathbb{FN}\mathbf{r}$ has a general truth value).

Theorem 3

$$T^H \vdash \neg(\exists n)(n \in w_N \,\&\, \mathbf{\Delta}\mathbb{FN}n \,\&\, \mathbf{\Delta}\neg\mathbb{FN}(n+1)).$$

PROOF: Put $T = T^H \cup \{n \equiv w_N z \,\&\, \mathbb{FN}n \,\&\, \neg\mathbb{FN}(n+1)\}$.

(L.1) $T \vdash \mathbb{FN}n$ (axioms of T)

(L.2) $T \vdash \neg\mathbb{FN}(n+1)$ (axioms of T)

(L.3) $T \vdash (\forall m)(m \in w_N \mathbin{\&} \boldsymbol{\Delta}(m \leq n) \Rightarrow \mathbb{FN}m)$
(L.1, Lemma 11(c), properties of FTT)

(L.4) $T \vdash (\forall m)(m \in w_N \mathbin{\&} \boldsymbol{\Delta}(n + 1 \leq m) \Rightarrow \neg\mathbb{FN}m)$
(L.2, Lemma 11(c), properties of FTT)

(L.5) $T \cup \{m \equiv w_N t\} \vdash \boldsymbol{\Delta}(m \leq n) \Rightarrow \mathbb{FN}m$
(L.3, deduction theorem, properties of FTT)

(L.6) $T \cup \{m \equiv w_N t\} \vdash \boldsymbol{\Delta}(n + 1 \leq m) \Rightarrow \neg\mathbb{FN}m$
(L.4, deduction theorem, properties of FTT)

(L.7) $T \vdash (\forall m)(m \in w_N \Rightarrow \neg\hat{\Upsilon}(\mathbb{FN}m))$
(L.5, L.6, Lemma 3(d), properties of T^H and FTT)

(L.8) $T \vdash \mathbf{r} \in w_N \Rightarrow \neg\hat{\Upsilon}(\mathbb{FN}\mathbf{r})$ (L.7, substitution)

(L.9) $T^H, n \equiv w_N z \vdash \boldsymbol{\Delta}(\mathbb{FN}n \mathbin{\&} \neg\mathbb{FN}(n + 1)) \Rightarrow (\mathbf{r} \in w_N \Rightarrow \neg\hat{\Upsilon}(\mathbb{FN}\mathbf{r}))$
(L.8, deduction theorem)

(L.10) $T^H, n \equiv w_N z \vdash (\mathbf{r} \in w_N \mathbin{\&} \hat{\Upsilon}(\mathbb{FN}\mathbf{r})) \Rightarrow \neg(\boldsymbol{\Delta}\mathbb{FN}n \mathbin{\&} \boldsymbol{\Delta}\neg\mathbb{FN}(n + 1))$
(L.9, properties of FTT)

(L.11) $T^H, n \equiv w_N z \vdash \neg(\boldsymbol{\Delta}\mathbb{FN}n \mathbin{\&} \boldsymbol{\Delta}\neg\mathbb{FN}(n + 1))$ (L.10, Lemma 11(d))

(L.12) $T^H \vdash n \in w_N \Rightarrow \neg(\boldsymbol{\Delta}\mathbb{FN}n \mathbin{\&} \boldsymbol{\Delta}\neg\mathbb{FN}(n + 1))$
(L.11, deduction theorem, properties of FTT)

(L.13) $T^H \vdash \neg(\exists n)(n \in w_N \mathbin{\&} \boldsymbol{\Delta}\mathbb{FN}n \mathbin{\&} \boldsymbol{\Delta}\neg\mathbb{FN}(n + 1))$
(L.12, generalization, properties of FTT)

□

This theorem formally expresses essential property of the sorites paradox: there is no number n that would surely be inside the horizon and $n + 1$ surely outside it. In other words, no n "starts the heap", i.e., in our terms, no number strictly terminates the horizon of finite natural numbers. The same holds also for all modifications of the horizon corresponding to evaluative expressions such as "small", etc. — cf. Theorem 11. We see that the above characterization of the horizon using fuzzy equality well complies with the intuition.

We may also ask, where is the "mistake" of classical logic; recall that the paradox raises when assuming that $\vdash \mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$, i.e., taking as true the statement "if n does not form the heap then $n+1$ is also does not form it". The problem consists in the observation that the grouping of n stones very slightly changes when adding one stone to it and so, the new grouping is "nearer" to the state of being a heap, or, saying in degrees, that the new grouping is in greater degree a "heap" than the previous one. Classical logic has no means to express this. In FTT, however, we can easily prove the following.

Theorem 4

$$T^H \vdash (\forall n)(n \in w_N \Rightarrow (\mathbb{FN}n \Rightarrow \cdot (n \approx_{w_N} n+1) \Rightarrow \mathbb{FN}(n+1))).$$

PROOF: This immediately follows from (EV5) and the construction of $\approx_{w_N}$ by the properties of FTT. □

In accordance with [11], we can interpret the formula $(n \approx_{w_N} n+1) \Rightarrow \mathbb{FN}(n+1)$ in this theorem as "*it is almost true* that $\mathbb{FN}(n+1)$". The formula $n \approx_{w_N} n+1$ thus characterizes the *change of the grouping of stones* after adding one stone to it. Hence, we can also read the formula $\mathbb{FN}n \Rightarrow \cdot(n \approx_{w_N} n+1) \Rightarrow \mathbb{FN}(n+1)$ as "if n does not form a heap (in the given degree of truth) then *it is almost true* that $n+1$ also does not form it (in the same degree of truth)".

3.5.4 Horizon shift and hedges

In this subsection, we will formalize the requirements of items (v) and (vi) of Subsection 3.4. The symbols $z, t, u, v \in Form_o$ are variables of type o.

The horizon shift will be realized using special formulas of type oo that we will denote by $\boldsymbol{\nu}$ and call *abstract hedges* (or simply hedges). To characterize their properties, we will define the following auxiliary formulas of type $o(oo)$:

$$H^1 \equiv \lambda \boldsymbol{\nu} \cdot \boldsymbol{\nu}\, z \wedge \neg \boldsymbol{\nu}\, t, \tag{31}$$

$$H^2 \equiv \lambda \boldsymbol{\nu} \cdot ((t \Rightarrow z) \Rightarrow (\boldsymbol{\nu}\, t \Rightarrow t))\, \&((z \Rightarrow t) \Rightarrow (t \Rightarrow \boldsymbol{\nu}\, t)), \tag{32}$$

$$H^3 \equiv \lambda \boldsymbol{\nu} \cdot \boldsymbol{\Delta}(z \Rightarrow t) \Rightarrow (\boldsymbol{\nu}\, z \Rightarrow \boldsymbol{\nu}\, t). \tag{33}$$

Then we introduce a formula $Hedge \in Form_{o(oo)}$ saying that $\boldsymbol{\nu} \in Form_{oo}$ is a hedge:

$$Hedge \equiv \lambda \boldsymbol{\nu} \cdot (\exists t)(\exists z)(\boldsymbol{\Delta}((t \Rightarrow z) \wedge (t \not\equiv z) \wedge (H^1 \boldsymbol{\nu}))) \wedge (\exists z)\boldsymbol{\Delta}(\forall t)(H^2 \boldsymbol{\nu}) \wedge (\forall z)(\forall t)(H^3 \boldsymbol{\nu}). \tag{34}$$

The meaning of (34) is the following: formula H^1 expresses that the hedge $\boldsymbol{\nu}$ sends some truth value z to top and some truth value t to bottom. Using H^3 which expresses monotonicity it follows that also all bigger (smaller) truth values are mapped to top (bottom). Finally, formula H^2 requires existence of an "inner truth value" splitting behavior of the hedge $\boldsymbol{\nu}$ into two cases so that modification of truth values is "small" if they are "big", and "big" if they are "small". Hence, all three formulas assure that requirements of item (v) in Subsection 3.4 are fulfilled.

We say that a formula $\boldsymbol{\nu} \in Form_{oo}$ is a hedge if $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$.

Lemma 12
Let $\boldsymbol{\nu} \in Form_{oo}$ *so that* $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$. *Then*

(a) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(z \wedge t) \Rightarrow (\boldsymbol{\nu}\, z \wedge \boldsymbol{\nu}\, t)$, $\quad T^{\mathrm{Ev}} \vdash (\boldsymbol{\nu}\, z \vee \boldsymbol{\nu}\, t) \Rightarrow \boldsymbol{\nu}(z \vee t)$,

(b) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}(z \Rightarrow t) \Rightarrow (\boldsymbol{\nu}\, z \Rightarrow \boldsymbol{\nu}\, t)$,

(c) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}\neg z \Rightarrow \neg\, \boldsymbol{\nu}\, z, \qquad T^{\mathrm{Ev}} \vdash \mathbf{\Delta}\neg z \Rightarrow \mathbf{\Delta}\neg\, \boldsymbol{\nu}\, z$,

(d) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu} \top \equiv \top, \qquad T^{\mathrm{Ev}} \vdash \boldsymbol{\nu} \bot \equiv \bot$.

(e) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta} z \Rightarrow \mathbf{\Delta}\, \boldsymbol{\nu}\, z$.

(f) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}\, \mathbf{\Delta} z \equiv \mathbf{\Delta} z$

(g) *Put* $id \equiv \lambda t\, t$. *Then* $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\ \mathrm{id}$ *(*id *is a trivial hedge).*

PROOF: (a)–(d) are easy consequence of formula H^3 (monotonicity of $\boldsymbol{\nu}$ with respect to implication), formula H^1 and the basic properties of FTT.

(e) is obtained from formula H^3 (setting $z := \top$, $t := z$) and (d) using rule (N) and the deduction theorem.

(f) follows from (d) by Theorem 14 from [25] (rule of two cases).

(g) is obvious since $\vdash \mathrm{id}\, z_o \equiv z_o$.

□

Lemma 13
Let $\boldsymbol{\nu} \in \mathit{Form}_{oo}$ *so that* $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\ \boldsymbol{\nu}$, *and let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \notin J^{\mathrm{Ev}}$ *be new constants of type* o. *Furthermore, let*

$$T^* = T^{\mathrm{Ev}} \cup \{(\mathbf{a} \Rightarrow \mathbf{c} \wedge \mathbf{a} \not\equiv \mathbf{c}) \wedge H^1_{t,z}[\mathbf{a}, \mathbf{c}], H^2_z[\mathbf{b}]\}$$

be the conservative extension of T^{Ev} *(by adding constants). Then:*

(a) $T^* \vdash (\forall z)(\mathbf{\Delta}(z \Rightarrow \mathbf{a}) \Rightarrow \neg\, \boldsymbol{\nu}\, z)$,

(b) $T^* \vdash (\forall z)(\mathbf{\Delta}(\mathbf{c} \Rightarrow z) \Rightarrow (\boldsymbol{\nu}\, z \equiv \top))$,

(c) $T^* \vdash \mathbf{b} \equiv \boldsymbol{\nu}\, \mathbf{b}$,

(d) $T^* \vdash (\forall t)((\boldsymbol{\nu}\, t \Rightarrow t) \vee (t \Rightarrow \boldsymbol{\nu}\, t))$.

PROOF: (a)–(c) are easy consequence of the definition of hedge in (34).

(d) Use Lemma 2(j) where we put $A := \mathbf{b} \Rightarrow t$, $C := t \Rightarrow \mathbf{b}$, $B := \boldsymbol{\nu}\, t \Rightarrow t$ and $D := t \Rightarrow \boldsymbol{\nu}\, t$. Then (d) follows from the prelinearity property of the truth values. □

Lemma 14
Let $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\ \boldsymbol{\nu}$. *Then*

(a) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(t \sim t)$,

(b) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(t \sim z) \Rightarrow \boldsymbol{\nu}(z \sim t)$,

(c) *Let* $T^{\mathrm{Ev}} \vdash t \Rightarrow u$ *as well as* $T^{\mathrm{Ev}} \vdash u \Rightarrow z$. *Then*

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(t \sim z) \,\&\, \boldsymbol{\nu}(z \sim u) \Rightarrow \boldsymbol{\nu}(t \sim u).$$

PROOF: (a) and (b) are immediate.

(c)

(L.1) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}((t \Rightarrow u) \,\&\, (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$ (EV7)

(L.2) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}((t \Rightarrow u) \,\&\, (u \Rightarrow z))$ (assumption, properties of FTT)

(L.3) $T^{\mathrm{Ev}} \vdash t \sim z \Rightarrow t \sim u$ (L.1, L.2, modus ponens)

(L.4) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}(t \sim z \Rightarrow t \sim u)$ (L.3, rule (N))

(L.5) $T^{\mathrm{Ev}} \vdash \mathbf{\Delta}(t \sim z \Rightarrow t \sim u) \Rightarrow \cdot \boldsymbol{\nu}(t \sim z) \Rightarrow \boldsymbol{\nu}(t \sim u)$ (Lemma 12(b))

(L.6) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(t \sim z) \Rightarrow \boldsymbol{\nu}(t \sim u)$ (L.4, L.5, modus ponens)

(L.7) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(t \sim z) \,\&\, \boldsymbol{\nu}(z \sim u) \Rightarrow \boldsymbol{\nu}(t \sim u)$ (L.6, properties of FTT)

□

It follows from this lemma that the formula $\lambda t \, \lambda z \cdot \boldsymbol{\nu}(t \sim z)$ defines a sort of fuzzy equality which is reflexive and symmetric. The transitivity holds in a weaker sense.

Interpretation of hedges are functions $\nu : L \longrightarrow L$ on truth values having the following properties:

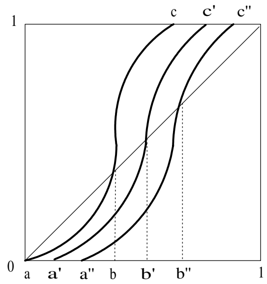
(i) There are $a, c \in L$ such that $a < c$, $\nu(a) = \mathbf{0}$ and $\nu(c) = \mathbf{1}$.

(ii) For all $x, y \in L$, $x \leq y$ implies $\nu(x) \leq \nu(y)$.

(iii) There is b, $a \leq b \leq c$ such that

$$x \leq b \text{ implies } \nu(x) \leq x,$$
$$b \leq x \text{ implies } x \leq \nu(x)$$

for all $x \in L$.



It is easy to see that if interpretation of $\boldsymbol{\nu}$ has these properties then the formulas H^1, H^2 and H^3 are true in the degree $\mathbf{1}$. A typical form of hedges ν is depicted on figure on the right hand side. The diagonal represents the trivial hedge.

Let us consider two hedges $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\, \boldsymbol{\nu}_1 \wedge \mathit{Hedge}\, \boldsymbol{\nu}_2$. We define a relation of partial ordering of hedges by

$$\preceq := \lambda p_{oo} \, \lambda q_{oo} \cdot (\forall z)(p_{oo} z \Rightarrow q_{oo} z). \tag{35}$$

We will write $\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ instead of $\preceq \boldsymbol{\nu}_1 \, \boldsymbol{\nu}_2$. If $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ then we say that the hedge $\boldsymbol{\nu}_1$ has a *narrowing effect* with respect to $\boldsymbol{\nu}_2$, and $\boldsymbol{\nu}_2$ has a *widening effect* with respect to $\boldsymbol{\nu}_1$.

The following lemma will later be used to prove that there are no elements that would be at the same time small as well as big.

Lemma 15
Let $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\,\boldsymbol{\nu}$*. Then*

(a) $T^{\mathrm{Ev}} \vdash \neg(\exists z)(\boldsymbol{\nu}\,LH\,z \wedge \boldsymbol{\nu}\,RH\,z)$,

(b) $T^{\mathrm{Ev}} \vdash (\forall z)(\boldsymbol{\nu}\,LH\,z \Rightarrow \neg\boldsymbol{\nu}\,RH\,z)$.

PROOF: (a) Use Lemmas 2(j) and 12(c) to prove that

$$T^{\mathrm{Ev}} \vdash (\boldsymbol{\Delta}\neg\,LH\,z \vee \boldsymbol{\Delta}\neg\,RH\,z) \Rightarrow (\neg\boldsymbol{\nu}\,LH\,z \vee \neg\boldsymbol{\nu}\,RH\,z).$$

Then use Lemma 9(j) (cf. its proof) and the properties of negation and quantifiers.

(b) By properties of quantifiers and FTT, it follows from (a) that

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}\,LH\,z \wedge \boldsymbol{\nu}\,RH\,z \Rightarrow \bot.$$

Then (b) is obtained from this by the properties of strong conjunction and the properties of implication. □

With respect to Lemma 10(a) we should also expect that

$$T^{\mathrm{Ev}} \vdash (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(LH\,u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(MH\,u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(RH\,u). \tag{36}$$

Our definition of hedge, however, is wider and covers hedges that are practically crisp. Therefore, (36) cannot be proved in general. A wider definition of hedge is useful for certain considerations and so, we will not restrict it too much. For example, the connective $\boldsymbol{\Delta}$ can also be considered as a hedge. On surface (linguistic) level it will be assigned the word "utmost".

We will, in addition, define a special class of hedges as follows. Let $\boldsymbol{\nu} \in \mathit{Form}_{oo}$ be a formula such that $T^{\mathrm{Ev}} \vdash \mathit{Hedge}\,\boldsymbol{\nu}$. We say that $\boldsymbol{\nu}$ is a *natural hedge* if (36) is provable and introduce a special formula

$$\mathit{NatHedge} \equiv \lambda\boldsymbol{\nu}\cdot \mathit{Hedge}\,\boldsymbol{\nu} \wedge ((\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(LH\,u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(MH\,u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(RH\,u)). \tag{37}$$

One of the hedges plays a central role in the theory of evaluative expressions. This hedge is on surface level *empty* (cf. Subsection 3.1) and its introduction makes possible to develop a uniform formal theory of evaluative expressions. We will denote it by a special constant $\bar{\boldsymbol{\nu}} \in \mathit{Form}_{oo}$ and call it a *standard hedge.* The following axioms characterize basic properties of it:

(EV10) $\mathit{NatHedge}\,\bar{\boldsymbol{\nu}}\,\&(\exists\boldsymbol{\nu})(\exists\boldsymbol{\nu}')(\mathit{Hedge}\,\boldsymbol{\nu}\,\&\,\mathit{Hedge}\,\boldsymbol{\nu}'\,\&(\boldsymbol{\nu}_1 \preceq \bar{\boldsymbol{\nu}} \wedge \bar{\boldsymbol{\nu}} \preceq \boldsymbol{\nu}_2))$.

(EV11) $(\forall z)((\Upsilon\bar{\boldsymbol{\nu}}(LH\,z)) \vee (\Upsilon\bar{\boldsymbol{\nu}}(MH\,z)) \vee (\Upsilon\bar{\boldsymbol{\nu}}(RH\,z)))$.

Axiom (EV10) expresses that $\bar{\boldsymbol{\nu}}$ is a natural hedge laying "among" other hedges so that some hedges have narrowing and some other ones have widening effect with respect to $\bar{\boldsymbol{\nu}}$. Axiom (EV11) will later be used to prove that the scale is covered by the fundamental evaluative trichotomy.

3.5.5 Formulas representing intension and extension of TEv-expressions

We will now introduce a special class of formulas that will be called TEv-formulas and that will represent intensions of evaluative linguistic expressions. We will also demonstrate that the requirements (ii), (iv)–(vi) of Subsection 3.4 are fulfilled by them.

However, the situation is a bit complicated. Recall that according to Subsections 3.1 and 3.2, we distinguish between evaluative linguistic expressions and predications. The former are rather abstract expressions characterizing position on some abstract scale while the latter characterize sizes (volumes, etc.) of concrete objects. This leads us to introduction of two classes of formulas. The first class is formed by formulas of type *oo* that represent the meaning of abstract evaluative expressions (recall that we have identified the mentioned abstract scale by the scale of truth values). These formulas will be referred to as $\widetilde{\text{TEv}}$-formulas.

The second class is formed by formulas that represent the meaning of evaluative predications. These require considering contexts that are formulas $w \in Form_{\alpha o}$ of type αo for some type $\alpha \in Types$. Hence, evaluative predications will be represented by formulas of type $(o\alpha)(\alpha o)$. These formulas will be referred to as TEv-formulas.

$\widetilde{\text{TEv}}$-formulas and TEv-formulas will further be divided into three subclasses whose members will correspond to evaluative expressions characterizing small, medium and big values.

Definition 3
Let $\boldsymbol{\nu} \in Form_{oo}$ so that $T^{\text{Ev}} \vdash Hedge\, \boldsymbol{\nu}$. The three subclasses of $\widetilde{\text{TEv}}$-formulas and TEv-formulas are the following:

(i) $\tilde{\text{S}}$-formula: $\widetilde{Sm} := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(LH\, z)$,
S-formula: $Sm := \lambda\boldsymbol{\nu}\ \lambda w\, \lambda x \cdot \boldsymbol{\nu}(LH(w^{-1}x))$.

(ii) $\tilde{\text{M}}$-formula: $\widetilde{Me} := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(MH\, z)$,
M-formula: $Me := \lambda\boldsymbol{\nu}\ \lambda w\, \lambda x \cdot \boldsymbol{\nu}(MH(w^{-1}x))$.

(iii) $\tilde{\text{B}}$-formula: $\widetilde{Bi} := \lambda\boldsymbol{\nu}\ \lambda z \cdot \boldsymbol{\nu}(RH\, z)$,
B-formula: $Bi := \lambda\boldsymbol{\nu}\ \lambda w\, \lambda x \cdot \boldsymbol{\nu}(RH(w^{-1}x))$.

Note that by λ-conversion, we also have $T^{\text{Ev}} \vdash Sm \equiv \lambda\boldsymbol{\nu}\ \lambda w\, \lambda x \cdot (\widetilde{Sm}\,\boldsymbol{\nu})(w^{-1}x)$ and similarly for *Me* and *Bi*.

To simplify the notation, we will use a general variable $(\widetilde{Ev}\,\boldsymbol{\nu}) \in Form_{oo}$ or $(Ev\,\boldsymbol{\nu}) \in Form_{(o\alpha)(\alpha o)}$ for an $\widetilde{\text{TEv}}$-formula or TEv-formula with a specific linguistic hedge, respectively.

Theorem 5
Let $T^{\text{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ and $w \in Form_{\alpha o}$ be a formula representing a context in the set of elements of type α. Then

(a) $T^{\text{Ev}} \vdash (\forall w)(Sm\,\boldsymbol{\nu})w\,\bot_w$,

(b) $T^{\text{Ev}} \vdash (\forall w)(Me\,\boldsymbol{\nu})w\,\dagger_w$,

(c) $T^{\text{Ev}} \vdash (\forall w)(Bi\,\boldsymbol{\nu})w\,\top_w$.

PROOF: We will prove only (a), the other proofs are similar.

(L.1) $T^{\text{Ev}} \vdash \bot \equiv w^{-1}\bot_w$ (EV2)

(L.2) $T^{\text{Ev}} \vdash \boldsymbol{\nu}(LH\,\bot)$ (Lemmas 12(d) and 9(a))

(L.3) $T^{\text{Ev}} \vdash \boldsymbol{\nu}(LH(w^{-1}\bot_w))$ (L.1, L.2, rule (R))

(L.4) $T^{\text{Ev}} \vdash (\forall w)(Sm\,\boldsymbol{\nu})w\bot_w$ (L.3, definition of Sm, λ-conversion)

□

Theorem 6

(a) $T^{\text{Ev}} \vdash (\forall z)((\widetilde{Sm\,\boldsymbol{\nu}})z \Rightarrow \neg\, RH\, z)$,

(b) $T^{\text{Ev}} \vdash (\forall z)((\widetilde{Bi\,\boldsymbol{\nu}})z \Rightarrow \neg\, LH\, z)$,

(c) $T^{\text{Ev}} \vdash (\forall z)((\widetilde{Sm\,\boldsymbol{\nu}})z \Rightarrow \neg(\widetilde{Bi\,\boldsymbol{\nu}})z)$.

PROOF: (a) By Lemma 9(k), rule (N), monotonicity of $\boldsymbol{\nu}$ (formula H^3) and Lemma 12(f), we obtain

$$T^{\text{Ev}} \vdash \boldsymbol{\nu}(LH\,z) \Rightarrow \boldsymbol{\Delta}(z \Rightarrow \dagger).$$

Then use Lemma 9(h). The proof of (b) is analogous.
(c) is reformulation of Lemma 15(b). □

By this theorem, no small value falls behind the right horizon and similarly, no big value falls before the left horizon. This property can be, of course, extended to all contexts and is very useful in the applications of vague reasoning.

Theorem 7
Let $w \in Form_{\alpha o}$ be a formula representing a context in the set of elements of type α.

(a) Let $T^{\text{Ev}} \vdash Hedge\,\boldsymbol{\nu}_1 \wedge Hedge\,\boldsymbol{\nu}_2$ and $(Ev\,\boldsymbol{\nu}_1), (Ev\,\boldsymbol{\nu}_2)$ differ only in the hedge. Then

$$T^{\text{Ev}} \vdash (\forall w)(\forall x)(\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2 \Rightarrow \cdot (Ev\,\boldsymbol{\nu}_1)wx \Rightarrow (Ev\,\boldsymbol{\nu}_2)wx).$$

(b) For all S-formulas

$$T^{\text{Ev}} \vdash (\forall w)(\forall x)(\forall y)(\boldsymbol{\Delta}(x \leq_w y) \Rightarrow \cdot (Sm\,\boldsymbol{\nu})\,wy \Rightarrow (Sm\,\boldsymbol{\nu})\,wx).$$

(c) For all M-formulas

$$T^{\mathrm{Ev}} \vdash (\forall w)(\forall x)(\forall y)(\boldsymbol{\Delta}(x \le_w y \,\&\, y \le_w \dagger_w) \Rightarrow \cdot (Me\,\boldsymbol{\nu})wx \Rightarrow (Me\,\boldsymbol{\nu})wy),$$
$$T^{\mathrm{Ev}} \vdash (\forall w)(\forall x)(\forall y)(\boldsymbol{\Delta}(\dagger_w \le_w x \,\&\, x \le_w y) \Rightarrow \cdot (Me\,\boldsymbol{\nu})\,wy \Rightarrow (Me\,\boldsymbol{\nu})wx).$$

(d) For all B-formulas,

$$T^{\mathrm{Ev}} \vdash (\forall w)(\forall x)(\forall y)(\boldsymbol{\Delta}(x \le_w y) \Rightarrow \cdot (Bi\,\boldsymbol{\nu})wx \Rightarrow (Bi\,\boldsymbol{\nu})wy).$$

PROOF: (a)

(L.1) $T^{\mathrm{Ev}} \vdash (\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2) \Rightarrow \cdot\, \boldsymbol{\nu}_1(LH(w^{-1}x)) \Rightarrow \boldsymbol{\nu}_2(LH(w^{-1}x))$
(substitution, monotonicity of hedges)

(L.2) $(\forall w)(\forall x)(\boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2 \Rightarrow \cdot (Ev\,\boldsymbol{\nu}_1)wx \Rightarrow (Ev\,\boldsymbol{\nu}_2)wx)$
(L.1, generalization, def. of TEv-formula)

(b)–(d) follow immediately from definition (20), Lemma 9(c)–(f) and monotonicity of hedges. □

On the basis of Theorem 7(a), we may extend the partial ordering $\preceq$ of hedges to TEv-formulas that differ only in the hedge.

Definition 4
Let Ev^1, Ev^2 be TEv-formulas. We say that Ev^1 is sharper than Ev^2 if $Ev^1 \preceq Ev^2$ or, if Ev^1 is S-formula and Ev^2 is not, or Ev^1 is M-formula and Ev^2 is B-formula.

The ordering of sharpness plays an important role in *perception-based* logical deduction (see [29]).

Lemma 16
Let $T^{\mathrm{Ev}} \vdash NatHedge\,\boldsymbol{\nu}$. Then

(a) $T^{\mathrm{Ev}} \vdash (\exists u)(\hat{\Upsilon}\,\boldsymbol{\nu}(LH\,u) \,\&\, \bot < u)$,

(b) $T^{\mathrm{Ev}} \vdash (\exists u)(\hat{\Upsilon}\,\boldsymbol{\nu}(RH\,u) \,\&\, u < \top)$.

PROOF: This is proved using Lemma 2(a) and (37) analogously as in the proof of Lemma 10(b). □

3.5.6 Global characterization of TEv-formulas

To accept the formulas $(Sm\,\boldsymbol{\nu}), (Me\,\boldsymbol{\nu})$ and $(Bi\,\boldsymbol{\nu})$ as intensions of natural language expressions, we require them to meet Hypothesis 1. This explicitly means that each extension $(Ev\,\boldsymbol{\nu})w$ should be extensional with respect to some fuzzy equality.

Theorem 8
Let $T^{\mathrm{Ev}} \vdash z \Rightarrow t$. Then

$$T^{\mathrm{Ev}} \vdash (\widetilde{Ev}\,\boldsymbol{\nu})\,t\,\&\, z \sim t \Rightarrow (\widetilde{Ev}\,\boldsymbol{\nu})z$$

holds for any $\widetilde{\mathrm{TEv}}$-*formula*$(\widetilde{Ev}\,\boldsymbol{\nu})$.

PROOF: We prove only the case when $(\widetilde{Ev}\,\boldsymbol{\nu}) := (\widetilde{Sm}\,\boldsymbol{\nu})$.

(L.1) $T^{\mathrm{Ev}} \vdash \boldsymbol{\Delta}(z \Rightarrow t)$ (rule (N))

(L.2) $\vdash \boldsymbol{\Delta}(\bot \Rightarrow z)$ (property of FTT)

(L.3) $T^{\mathrm{Ev}} \vdash \bot \sim t \Rightarrow \bot \sim z$ (L.1, L.2, (EV7))

(L.4) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(LH\,t) \Rightarrow \boldsymbol{\nu}(LH\,z)$ (L.3, definition of LH)

(L.5) $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}(LH\,t)\,\&(t \sim z) \Rightarrow \boldsymbol{\nu}(LH\,z)$ (properties of FTT)

(L.6) $T^{\mathrm{Ev}} \vdash (\widetilde{Sm}\,\boldsymbol{\nu})z\,\&(t \sim z) \Rightarrow (\widetilde{Sm}\,\boldsymbol{\nu})t$
(L.5, definition of $(\widetilde{Sm}\,\boldsymbol{\nu})$, λ-conversion)

The proof of the other cases is similar. □

Theorem 9
Let $w \in Form_{\alpha o}$ be a formula representing a context in the set of elements of type α and $T^{\mathrm{Ev}} \vdash y \leq_w x$. Then

$$T^{\mathrm{Ev}} \vdash x \in w\,\&\, y \in w \Rightarrow \cdot(Ev\,\boldsymbol{\nu})wx\,\&\, x \approx_w y \Rightarrow (Ev\,\boldsymbol{\nu})wy.$$

PROOF: This follows from Lemma 6 and Theorem 8 by the properties of FTT. □

By this theorem, extensions of evaluative expressions can be characterized using a fuzzy equality and so, they meet the requirement of Hypothesis 1.

The following theorem characterizes all $\widetilde{\mathrm{TEv}}$-formulas.

Theorem 10
Let $\boldsymbol{\nu}$ be a natural hedge. Then

$$T^{\mathrm{Ev}} \vdash (\exists u)(\hat{\Upsilon}(\widetilde{Ev}\,\boldsymbol{\nu})u) \tag{38}$$

PROOF: This is an immediate consequence of (37). □

Unfortunately, we cannot prove (38) for general TEv-formulas since the non-triviality depends also on the definition of the context w. Hence, if not stated otherwise, given a natural hedge $\boldsymbol{\nu}$, we will at the same time consider only *non-trivial* contexts w, i.e. those for which

$$T^{\mathrm{Ev}} \vdash (\exists x)(\hat{\Upsilon}(Ev\,\boldsymbol{\nu})wx). \tag{39}$$

Theorem 11
Let T^H be a theory from Theorem 3 and $\boldsymbol{\nu} \in Form_{oo}$ be a natural hedge, i.e. $T^{\mathrm{Ev}} \vdash NatHedge\, \boldsymbol{\nu}$ and, moreover, $T^{\mathrm{Ev}} \vdash (\exists m)\hat{\Upsilon}\, \boldsymbol{\nu}(LH(w_N^{-1}m))$. Then

$$T^H \vdash \neg(\exists n)(n \in w_N \,\&\, \boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})w_N n \,\&\, \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})w_N(n+1)).$$

PROOF: This follows from Lemma 12(c) and (e) and Theorem 3 using the definition of *Sm*, assumptions and the properties of FTT. □

It follows from this theorem, that if the context w_N assures also non-triviality then there is no natural number n that would surely be '⟨linguistic hedge⟩ small' and $n+1$ surely not '⟨linguistic hedge⟩ small'. Note that this property is in [15] taken as a crucial property of the vagueness phenomenon.

We can prove even a more general result stating that for all nontrivial S- and B-formulas and each non-trivial context there is *no last small (first big) value*. We say that $x \in w$ is the *last small* element if it is surely small and all y greater than x are surely not small, i.e. the following is provable:

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})wx \,\&\, (\forall y)(x <_w y \Rightarrow \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})wy). \tag{40}$$

Analogously, we define the first big element by

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\Delta}(Bi\,\boldsymbol{\nu})wx \,\&\, (\forall y)(y <_w x \Rightarrow \boldsymbol{\Delta}\neg(Bi\,\boldsymbol{\nu})wy). \tag{41}$$

The announced result is formulated in the following theorem.

Theorem 12
Let $\boldsymbol{\nu} \in Form_{oo}$ be a natural hedge and $w \in Form_{\alpha o}$ be a non-trivial context (i.e., (39) is provable) in the set of elements of type α. Then

$$T^{\mathrm{Ev}} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})wx \,\&\, (x <_w y \Rightarrow \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})wy)), \tag{42}$$

$$T^{\mathrm{Ev}} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Bi\,\boldsymbol{\nu})wx \,\&\, (y <_w x \Rightarrow \boldsymbol{\Delta}\neg(Bi\,\boldsymbol{\nu})wy)). \tag{43}$$

PROOF: We will prove only the first formula, the proof of the second is analogous. Note that (42) is equivalent to

$$T^{\mathrm{Ev}} \vdash (\forall x)(\exists y)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})wx \Rightarrow (x <_w y \,\&\, \neg\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})wy)). \tag{44}$$

Put $T = T^{\mathrm{Ev}} \cup \{(Sm\,\boldsymbol{\nu})wx\}$. By Lemma 16(a) and the assumption on non-triviality of w we get $T \vdash (\exists x)(\hat{\Upsilon}\,\boldsymbol{\nu}(LH\, w^{-1}x) \,\&\, \bot_w < w^{-1}x)$. Let T' be a conservative extension of T by a special constant $\mathbf{u}$. From this and the definition of *Sm*, we obtain $T' \vdash \hat{\Upsilon}(Sm\,\boldsymbol{\nu})w\mathbf{u} \,\&\, (\bot_w <_w w^{-1}\mathbf{u})$. Then, using Lemma 3(d) we get $T' \vdash \neg\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})w\mathbf{u}$ and from this $T' \vdash \neg\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})w\mathbf{u} \,\&\, (\bot_w <_w w^{-1}\mathbf{u})$. Finally, we obtain (44) using the deduction theorem, substitution axiom, rule of generalization and conservativeness of T'. □

The analogue of Theorem 4 is not direct since $\boldsymbol{\nu} \approx_{w_N}$ is not a fully-fledged fuzzy equality (cf. Lemma 14). Hence, we will formulate it as follows.

Theorem 13
Let T^H be a theory from Theorem 3. Then

$$T^H \vdash (\forall n)(n \in w_N \Rightarrow \cdot \\ (Sm\,\boldsymbol{\nu})w_N\, n \Rightarrow \cdot((Sm\,\boldsymbol{\nu})w_N\, n \Rightarrow (Sm\,\boldsymbol{\nu})w_N(n+1)) \Rightarrow (Sm\,\boldsymbol{\nu})w_N(n+1)).$$

PROOF: This follows the provable formula

$$T^{\mathrm{Ev}} \vdash (Sm\,\boldsymbol{\nu})w_N\, n\,\&((Sm\,\boldsymbol{\nu})w_N\, n \Rightarrow (Sm\,\boldsymbol{\nu})w_N(n+1)) \Rightarrow (Sm\,\boldsymbol{\nu})w_N(n+1)$$

by the properties of FTT. □

Thus, by this theorem, if the number of n stones is *small* then it is "almost true" that $n+1$ is also *small.* To be almost true means that the grouping of stones *imperceptibly changes* after adding one stone to it. On the other hand, the change is "measurable" and the measure is characterized by the formula $(Sm\,\boldsymbol{\nu})w_N n \Rightarrow (Sm\,\boldsymbol{\nu})w_N(n+1)$. Clearly, after adding a sufficiently large number of stones, the obtained grouping will no more be *small.*

An important conclusion follows from Theorems 3, 11, 12 and 13: notice that they are proved syntactically. Therefore, they demonstrate that the properties in concern *do hold in the formal system of FTT* and so, our system indeed models essential features of the vagueness phenomenon. On the other hand, the theory is fuzzy, i.e., models of T^{Ev} are many-valued and so, when dealing with specific interpretation of predicates, we cannot avoid assigning concrete truth values. *But concrete truth values occur only in models!* This fact explicitly demonstrates the often repeated (and not always understood) argument that concrete truth values are not important and *we use them only as specific precisiation* in the sense discussed by L. A. Zadeh (cf. [39]).

3.5.7 Fundamental evaluative trichotomy

Basic TEv-formulas are $(Sm\,\bar{\boldsymbol{\nu}})$, $(Me\,\bar{\boldsymbol{\nu}})$ and $(Bi\,\bar{\boldsymbol{\nu}})$ where $\bar{\boldsymbol{\nu}}$ is the standard hedge. These formulas are intensions of the fundamental evaluative trichotomy. Its properties are characterized by the following theorem.

Theorem 14
Let $T^{\mathrm{Ev}} \vdash Hedge\,\boldsymbol{\nu}$ and $w \in Form_{\alpha o}$ be a formula representing a context in the set of elements of type α. Then

(a) $T^{\mathrm{Ev}} \vdash \neg(\exists w)\neg(\exists x)((Sm\,\boldsymbol{\nu})wx \wedge (Bi\,\boldsymbol{\nu})wx)$,

(b) $T^{\mathrm{Ev}} \vdash (\forall w)(\forall x)((Sm\,\boldsymbol{\nu})wx \Rightarrow \neg(Bi\,\boldsymbol{\nu})wx)$,

(c) $T^{\mathrm{Ev}} \vdash (\forall w)(\forall x)(\Upsilon((Sm\,\bar{\boldsymbol{\nu}})\,wx) \vee \Upsilon((Me\,\bar{\boldsymbol{\nu}})\,wx) \vee \Upsilon((Bi\,\bar{\boldsymbol{\nu}})\,wx))$.

PROOF: (a) and (b) are immediate consequence of Lemma 15(a), (b) and the definition of *Sm* and *Bi*.

(c) is a consequence of the definition of TEv-formulas, Axiom (EV11) and Lemma 3(b). □

This theorem precisely states that the formulas $(Sm\,\boldsymbol{\nu})wx$, $(Bi\,\boldsymbol{\nu})wx$ for all hedges $\boldsymbol{\nu}$ have the basic property of antonyms and further, that $(Sm\,\bar{\boldsymbol{\nu}})wx$, $(Me\,\bar{\boldsymbol{\nu}})wx$ and $(Bi\,\bar{\boldsymbol{\nu}})wx$ form the fundamental evaluative trichotomy. Indeed, by (c), every x in every context w belongs to at least one of the evaluative expressions from the fundamental evaluative trichotomy with a non-zero truth degree. The property (b), similarly as stronger Theorem 6 is useful in the applications of vague reasoning.

3.5.8 Intension and extension of evaluative expressions and predications

Using the above developed theory we are now able to define *intension* of the trichotomous evaluative linguistic expressions and predications. Intensions are formulas $(\widetilde{Ev}\,\boldsymbol{\nu})$ and the latter are the formulas $(Ev\,\boldsymbol{\nu})$ (for various hedges $\boldsymbol{\nu}$). Note that interpretation of $(\widetilde{Ev}\,\boldsymbol{\nu})$ is a fuzzy set in the set of truth values L and so, intension turns out here to be identical with extension. This is not the case of evaluative predications because interpretation of $(Ev\,\boldsymbol{\nu})$ is a function which assigns to each context $w \in Form_{\alpha o}$ a fuzzy set of elements of type α having the property of the given evaluative expression. One can see that this well conforms with understanding to intension in the classical semantic theory. Recall that contexts take in our theory the role of possible worlds. The difference of our approach from the general understanding is only superficial since we have given concrete content to the notion of possible world.

Definition 5
Let us consider trichotomous evaluative expressions from Definition 1(i) and predications from Definition 2. Let ⟨linguistic hedge⟩ be assigned an abstract hedge $\boldsymbol{\nu} \in Form_{oo}$ and X be a variable representing objects of type α. Finally, let $t \in Form_o$, $x \in Form_\alpha$ and $w \in Form_{\alpha o}$. Then

(i)

$$\begin{aligned} \mathrm{Int}(\langle \textit{linguistic hedge}\rangle\ \mathsf{small}) &:= \lambda t \cdot (\widetilde{Sm}\,\boldsymbol{\nu})\,t, \\ \mathrm{Int}(X\ \mathsf{is}\ \langle \textit{linguistic hedge}\rangle\ \mathsf{small}) &:= \lambda w\,\lambda x \cdot (Sm\,\boldsymbol{\nu})wx. \end{aligned} \tag{45}$$

(ii)

$$\begin{aligned} \mathrm{Int}(\langle \textit{linguistic hedge}\rangle\ \mathsf{medium}) &:= \lambda t \cdot (\widetilde{Me}\,\boldsymbol{\nu})\,t, \\ \mathrm{Int}(X\ \mathsf{is}\ \langle \textit{linguistic hedge}\rangle\ \mathsf{medium}) &:= \lambda w\,\lambda x \cdot (Me\,\boldsymbol{\nu})wx. \end{aligned} \tag{46}$$

(iii)

$$\begin{aligned} \mathrm{Int}(\langle \textit{linguistic hedge}\rangle\ \mathsf{big}) &:= \lambda t \cdot (\widetilde{Bi}\,\boldsymbol{\nu})\,t, \\ \mathrm{Int}(X\ \mathsf{is}\ \langle \textit{linguistic hedge}\rangle\ \mathsf{big}) &:= \lambda w\,\lambda x \cdot (Bi\,\boldsymbol{\nu})wx. \end{aligned} \tag{47}$$

The above definition uses TEv-formulas introduced in Definition 3.

As a special case, the empty hedge is interpreted by the standard hedge $\bar{\boldsymbol{\nu}}$ and so, we put

$$\begin{aligned}
\mathrm{Int}(X \textsf{ is small}) &:= \lambda w\, \lambda x \cdot (Sm\, \bar{\boldsymbol{\nu}})\, wx, \\
\mathrm{Int}(X \textsf{ is medium}) &:= \lambda w\, \lambda x \cdot (Me\, \bar{\boldsymbol{\nu}})\, wx, \\
\mathrm{Int}(X \textsf{ is big}) &:= \lambda w\, \lambda x \cdot (Bi\, \bar{\boldsymbol{\nu}})\, wx.
\end{aligned}$$

It is easy to see that by λ-conversion, the following holds.

Lemma 17
Let $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ and $w \in Form_{\alpha o}$ be a formula representing a context in the set of elements of type α.

$$T^{\mathrm{Ev}} \vdash \mathrm{Int}(X \textsf{ is } \langle \textit{linguistic hedge} \rangle \textsf{ small}) \equiv \lambda w\, \lambda x \cdot \boldsymbol{\nu}(LH(w^{-1}x)), \tag{48}$$

$$T^{\mathrm{Ev}} \vdash \mathrm{Int}(X \textsf{ is } \langle \textit{linguistic hedge} \rangle \textsf{ medium}) \equiv \lambda w\, \lambda x \cdot \boldsymbol{\nu}(MH(w^{-1}x)), \tag{49}$$

$$T^{\mathrm{Ev}} \vdash \mathrm{Int}(X \textsf{ is } \langle \textit{linguistic hedge} \rangle \textsf{ big}) \equiv \lambda w\, \lambda x \cdot \boldsymbol{\nu}(RH(w^{-1}x)). \tag{50}$$

where $T^{\mathrm{Ev}} \vdash w^{-1}x \equiv \imath t \cdot x \equiv wt$ [†)]*.*

Extension of an evaluative expression or predication in a context $w \in Form_{\alpha o}$ is a fuzzy set of elements in the range of w. Since the meaning of trichotomous evaluative expressions is constructed in the abstract scale — the set of truth values — we conclude that its intension is equal to its extension. On the other hand, extension of evaluative predication depends on the context. Hence, we can distinguish extension of evaluative expression from that of evaluative predication, as can be seen from the following definition.

Definition 6
Let $\mathcal{A}$ be an evaluative expression and ⟨linguistic hedge⟩ *occurring in $\mathcal{A}$ be assigned an abstract hedge $\boldsymbol{\nu} \in Form_{oo}$. Let X be a variable representing objects of type α and $w \in Form_{\alpha o}$ a context. Then extension in w is defined as follows:*

(i) $\mathrm{Ext}(\mathcal{A}) \equiv \mathrm{Int}(\mathcal{A})$.

(ii) $\mathrm{Ext}_w(X \textsf{ is } \mathcal{A}) \equiv \mathrm{Int}(X \textsf{ is } \mathcal{A})w \equiv \lambda x \cdot (Ev\, \boldsymbol{\nu})\, wx$.

On the basis of this definition, we obtain

$$\begin{aligned}
\mathrm{Ext}_w(X \textsf{ is } \langle \text{linguistic hedge} \rangle \textsf{ small}) &\equiv \lambda x \cdot (Sm\, \boldsymbol{\nu})wx, \\
\mathrm{Ext}_w(X \textsf{ is } \langle \text{linguistic hedge} \rangle \textsf{ medium}) &\equiv \lambda x \cdot (Me\, \boldsymbol{\nu})wx, \\
\mathrm{Ext}_w(X \textsf{ is } \langle \text{linguistic hedge} \rangle \textsf{ big}) &\equiv \lambda x \cdot (Bi\, \boldsymbol{\nu})wx.
\end{aligned}$$

[†)] The way how this lemma is written has been chosen for the better readability. Since Int(. . .) does not belong to the language J^{Ev}, the correct way would be to use the right-hand side of (45)–(47) on the place of the corresponding Int(. . .) in (48)–(50), respectively.

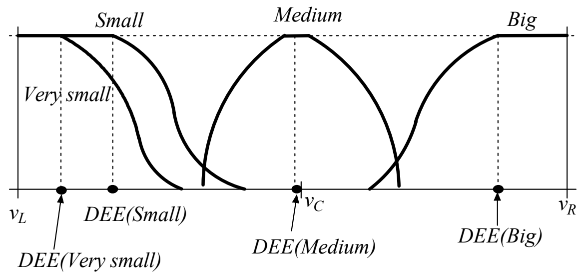


Figure 1: Scheme of the Defuzzification of Evaluative Expressions (DEE) method.

4 Canonical model of T^{Ev}

In this section we will briefly outline construction of a canonical model of the theory T^{Ev}. A detailed presentation will be a topic of a subsequent paper. For the details concerning interpretation of formulas of FTT — see [1, 25].

The *canonical model* of T^{Ev} is specified as follows. We construct a frame

$$\mathcal{M}^0 = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle.$$

The algebra $\mathcal{L}_\Delta$ is Łukasiewicz algebra with Δ (i.e. the set $M_o = [0,1]$). The special constant $F \in J^{\mathrm{Ev}}$ is interpreted by the fuzzy equality "$\sim$" from Example 2. The constant † is interpreted by 1/2.

The abstract hedges $\boldsymbol{\nu}$ can be interpreted by the functions

$$\nu_{a,h,c}(y) = \begin{cases} 1, & c \le y, \\ 1 - \frac{(c-y)^2}{(c-h)(c-a)}, & h \le y < c, \\ \frac{(y-a)^2}{(h-a)(c-a)}, & a \le y < h, \\ 0, & y < a \end{cases} \tag{51}$$

where $a, c \in [0,1]$ are parameters described in Subsection 3.5.4. The third parameter b is not equal to h from (51) but it can be easily computed. It can be verified that the functions (51) fulfil the requirements set for the natural hedges.

The set M_α of elements which may fall into the meaning of the TEv-expressions is assumed to be $M_\alpha = \mathbb{R}$. The fuzzy equality $\doteq$ is supposed to be separating, i.e., $[x \doteq y] = 1$ iff $x = y$.

The description operator is interpreted by a special defuzzification operation DEE (Defuzzification of Evaluative Expressions; see also [28]) whose action is clear from Figure 1. In general, the description operator $\iota_{\alpha(o\alpha)}$ can be understood as choosing a *prototype* (a *typical element*) of type α representing the given formula of type $o\alpha$. Then, DEE assigns the Last of Maxima to the fuzzy set of type "small", First of Maxima to the fuzzy set of type "big" and Mean of Maxima otherwise†). Note, that this works also for triangular fuzzy sets that are often used in the applications of fuzzy set theory.

The definition of the description operator influences definition of the context. For example, the context w_N discussed in Subsection 3.5.3 can be defined, for the case $q = 2p$, by

$$\mathcal{I}^{\mathcal{M}^0}(w_N)(x) = \begin{cases} m & \text{if } x = \frac{m}{q}, \quad m = 0, 2, 4, \dots, q \\ n & \text{if } x \in (\frac{n-1}{q}, \frac{n+1}{q}), \quad n = 1, 3, 5, \dots, q-1. \end{cases}$$

Then $\mathcal{I}^{\mathcal{M}^0}(w_N^{-1})(m) = \frac{m}{q}$, $m = 0, \dots, q$.

We may now formulate the following theorem.

Theorem 15
The frame $\mathcal{M}^0$ constructed above is a model of T^{Ev}.

PROOF: It can be verified that all axioms (EV1)–(EV11) of T^{Ev} are true in the degree 1 in $\mathcal{M}^0$. □

5 Conclusion

In this paper, we have developed a comprehensive logical theory of evaluative linguistic expressions. Our theory is very general and its formalism uses the formalism of fuzzy type theory. All the proofs of theorems are syntactical and so, they have wider validity with respect to arbitrary semantics. We have also successfully characterized the difference between evaluative linguistic expressions themselves, i.e., we can distinguish the meaning of simple expressions like *medium, very small, extremely big*, etc., from the meaning of evaluative predications such as *temperature is very high, pressure is rather small, curve is extremely steep*, etc. This difference has not been clearly understood yet and so, their meanings were quite often mixed.

We have focused mainly on the logical theory of trichotomous evaluative expressions, i.e., expressions consisting of the adjective *small, medium* or *big*, possibly preceded by the linguistic modifier. However, the class of evaluative expressions includes also fuzzy quantities and negation. But even more: our theory can be further extended to cover also various kinds of generalized (fuzzy) quantifiers (*most, a lot of, at least, at most, few* and many others). We will continue this theory in the subsequent papers.

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†) These types of fuzzy sets can be defined more generally than depicted on Figure 1. More details will be presented in a subsequent paper.

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