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Fuzzy Transforms

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Abstract

The technique of the direct and inverse fuzzy (F-)transforms of three different types is introduced and approximating properties of the inverse F-transforms are described. A number of theorems establishing best approximation properties have been proved. A method of lossy image compression and reconstruction on the basis of the F-transform is presented.

Key words: Universal approximation, fuzzy transform, basic functions

1 Introduction

In classical mathematics, various kinds of transforms (Fourier, Laplace, integral, wavelet) are used as powerful methods for construction of approximation models and for solution of differential or integral-differential equations. The main idea of them consists in transforming the original model into a special space where the computation is simpler. The transform back to the original space produces an approximate model or an approximate solution.

In this contribution, we put a bridge between these well known methods and methods for construction of fuzzy approximation models. We will develop a general method called *fuzzy transform* (or, shortly, *F-transform*) that encompasses both classical transforms as well as approximation methods based on elaboration of fuzzy IF-THEN rules studied in fuzzy modelling.

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In fact, any kind of transform is performed with the help of some kernel that can be understood as a "collection of local factors". Most of the (above mentioned) known classical transforms map functions to functions (only the discrete Fourier transform maps vectors to vectors, but the latter can be also taken as special kind of functions). The functions, forming the kernel, locally characterize a certain domain where an origin function is defined. This idea of local characterization, contained implicitly in classical methods, is made explicit in the F-transform. Namely, the precise values of independent and dependent variables are replaced by their approximate values related to some explicitly or implicitly known quantity, for example "close to 1.6", "approximately 3", or context-dependent expressions like "small", "large", etc. In mathematical terminology this means that values of the variables are taken up to a factorization with respect to some similarity relation. The factors, observed for an independent variable and combined either with precise or with factorized values of a function, form a local description of the considered function. Such description can be further aggregated in order to produce a global description of the function. As can be seen the F-transform has two phases: the *direct* one transforming imprecisely determined pieces of the original function into a special space (this can also be seen as a space of pairs of antecedents and consequents of fuzzy rules), and the *inverse* one which returns a "linear" combination of these pieces.

Note that the idea of local description is contained also in most formalizations of fuzzy IF-THEN rules originated by early works of Lotfi A. Zadeh [16, 17] and in the Takagi-Sugeno or singleton approximation models. We can show that all of them can be regarded as a special kind of transform of a space of original functions.

Our idea of fuzzy transform turned out to be very general and powerful. In classical mathematics, various kinds of transforms are used in a construction of approximation models and in a solution of differential or integral-differential equations. Similar results can be obtained also with the fuzzy transform (see [12, 14]). Moreover, it has nice filtering properties, it can be used for data compression (see the last part of this paper) and, as a special case, it is a part of the, so called, smooth perception-based deduction (see [10]) that is a method enabling to simulate "human-like understanding" to sets of fuzzy IF-THEN rules when interpreting them as special linguistic expressions.

In this paper, we will construct approximation models on the basis of three different fuzzy transforms and prove the best approximation properties in the corresponding approximation spaces. We will also show, how this technique can be applied to data compression and decompression.

The structure of the paper is the following. In Section 2, the concept of a fuzzy partition and a uniform fuzzy partition of the universe are introduced. In Section 3, the technique of the direct and inverse fuzzy transform (F-transform hereafter) is introduced and approximating properties of the inverse F-transform are established. In Section 4, we discuss the discrete F-transform, which means that an original function is known (or can be computed) only at some nodes. Two new fuzzy transforms, based on operations of a residuated lattice on $[0, 1]$, are introduced in Section 5. Their inverse transforms are discussed in Section 6. These new lattice F-transforms lead to new approximation models

(Section 7) which are expressed with help of weaker operations than the arithmetic ones used in the case of the F-transform from Section 3.

Section 8 contains comparison of all three types of fuzzy transforms. The case of functions of two and more variables and their fuzzy transforms is considered in Section 9. Finally, as an application, a method of lossy image compression and reconstruction on the basis of the F-transform is presented in Section 10.

2 Fuzzy Partition of the Universe

The core idea of the technique proposed in this paper is a fuzzy partition of the universe. We claim that for a sufficient representation of a function we may consider its average values within some intervals (or factors, or clusters). The latter may be regarded as similarity classes of some chosen nodes. Then, arbitrary function can be considered as a mapping from thus obtained set of factors to the set of function values.

We take an interval $[a, b]$ as a universe. That is, all (real-valued) functions considered in this chapter have this interval as a common domain. The *fuzzy partition* of the universe is given by fuzzy subsets of the universe $[a, b]$ (determined by their membership functions) which must have properties described in the following definition.

Definition 1
Let $x_1 < \ldots < x_n$ be fixed nodes within $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ defined on $[a, b]$, form a *fuzzy partition* of $[a, b]$ if they fulfil the following conditions for $k = 1, \ldots, n$:

(1) $A_k : [a, b] \longrightarrow [0, 1]$, $A_k(x_k) = 1$;
(2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$;
(3) $A_k(x)$ is continuous;
(4) $A_k(x)$, $k = 2, \ldots, n$, monotonically increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1, \ldots, n-1$, monotonically decreases on $[x_k, x_{k+1}]$;
(5) for all $x \in [a, b]$

$$\sum_{k=1}^{n} A_k(x) = 1. \tag{1}$$

The membership functions $A_1(x), \ldots, A_n(x)$ are called *basic functions*.

Let us remark that basic functions are specified by a set of nodes $x_1 < \ldots < x_n$ and the properties 1–5. The shape of basic functions is not predetermined and therefore, it can be chosen additionally.

Let us give some examples of basic functions. On Figure 1 a fuzzy partition of an interval by fuzzy sets with triangular shaped membership functions is shown. The following formulas give the formal representation of such triangular membership functions:

$$A_1(x) = \begin{cases} 1 - \frac{(x-x_1)}{h_1}, & x \in [x_1, x_2], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k(x) = \begin{cases} \frac{(x-x_{k-1})}{h_{k-1}}, & x \in [x_{k-1}, x_k], \\ 1 - \frac{(x-x_k)}{h_k}, & x \in [x_k, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_n(x) = \begin{cases} \frac{(x-x_{n-1})}{h_{n-1}}, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

where $k = 2, \dots n-1$, and $h_k = x_{k+1} - x_k$.

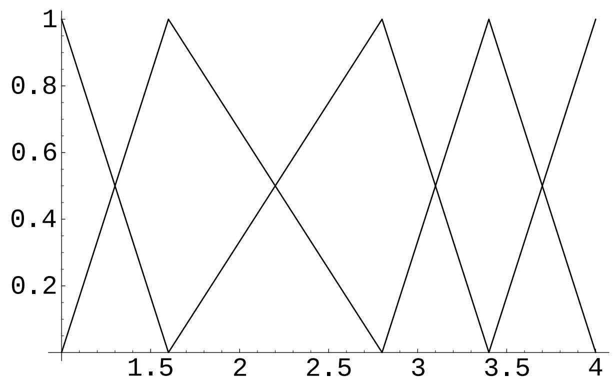


Fig. 1. An example of a fuzzy partition of $[1, 4]$ by triangular membership functions

Let a fuzzy partition of $[a, b]$ be given by fuzzy sets $A_1, \dots, A_n$ in the sense of Definition 1. We say that it is *uniform* if the nodes $x_1, \dots, x_n$, $n \geq 3$, are equidistant. This means that $x_k = a + h(k-1)$, $k = 1, \dots, n$, where $h = (b-a)/(n-1)$, and two additional properties are met:

(6) $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0, h]$, $k = 2, \dots, n-1$,
(7) $A_k(x) = A_{k-1}(x - h)$, for all $k = 2, \dots, n-1$ and $x \in [x_k, x_{k+1}]$, and
$A_{k+1}(x) = A_k(x - h)$, for all $k = 2, \dots, n-1$ and $x \in [x_k, x_{k+1}]$.

In the case of a uniform partition, h is a length of the support of A_1 or A_n, while $2h$ is a length of the support of other basic functions A_k, $k = 2, \dots, n-1$. Moreover, the value of h is unambiguously determined by the number n of the basic functions. Figure 2 shows a uniform partition by sinusoidal shaped basic functions. Their formal expressions are given below.

$$A_1(x) = \begin{cases} 0.5(\cos \frac{\pi}{h}(x - x_1) + 1), & x \in [x_1, x_2], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k(x) = \begin{cases} 0.5(\cos\frac{\pi}{h}(x - x_k) + 1), & x \in [x_{k-1}, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}$$

where $k = 2, \dots n-1$, and

$$A_n(x) = \begin{cases} 0.5(\cos\frac{\pi}{h}(x - x_n) + 1), & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

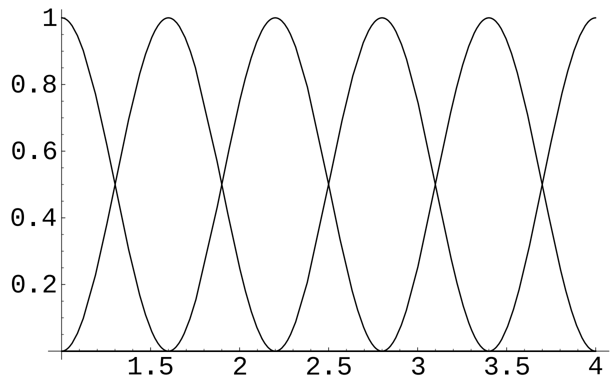


Fig. 2. An example of a uniform fuzzy partition of $[1, 4]$ by sinusoidal membership functions

The following lemma shows that, in the case of a uniform partition, the definite integral of a basic function does not depend on its concrete shape. This property will be further used to simplify the expression for F-transform components.

Lemma 1
Let the uniform partition of $[a, b]$ be given by basic functions $A_1(x), \dots, A_n(x)$, $n \geq 3$. Then

$$\int_{x_1}^{x_2} A_1(x)dx = \int_{x_{n-1}}^{x_n} A_n(x)dx = \frac{h}{2}, \tag{2}$$

and for $k = 2, \dots, n-1$

$$\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx = h \tag{3}$$

where h is the distance between each two neighboring nodes.

PROOF: Obviously,

$$\int_{x_1}^{x_3} A_2(x)dx = \dots = \int_{x_{n-2}}^{x_n} A_{n-1}(x)dx.$$

Therefore, to prove (3) it is sufficient to estimate

$$\int_{-h}^{h} A(x)dx$$

where $A(x) = A_2(x + a + h)$ and $x \in [-h, h]$. Based on properties 5 and 7 of basic functions, we can deduce that

$$1 - A(x) = A(x + h), x \in [-h, 0].$$

Then

$$\int_0^h A(x)dx = \int_{-h}^0 A(x + h)dx = h - \int_{-h}^0 A(x)dx$$

which implies (3). Equation (2) follows immediately from the symmetry of basic functions (property 6). □

3 Fuzzy transform

In this section, we introduce the fuzzy transform (shortly F-transform) which establishes a correspondence between the set of integrable functions on $[a, b]$ and the set of n-dimensional vectors. The formula to which we will refer as an inverse F-transform (inversion formula) converts an n-dimensional vector into a specially represented continuous function which approximates the original one. The advantage of the F-transform is that the inversion formula produces a simple and unique approximate representation of the original function. Thus, in complex computations we can use the inversion formula instead of a representation of the original function. Moreover, in a solution of many problems (e.g. computation of a definite integral, solution of differential equations, etc.) we may operate with images of the original functions obtained by applying the F-transform (see [12] for the details). By this trick, a problem can be transformed into a respective problem in the n-dimensional vector space and solved using methods of linear algebra. After finishing the computations, the result can be brought back to the space of continuous functions by the inversion formula. To be sure that this can be done we need to prove a number of propositions.

3.1 Direct fuzzy transform

Suppose that the universe is an interval $[a, b]$ and $x_1 < \ldots < x_n$ are fixed nodes from $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$. Let us formally extend the set of nodes by $x_0 = a$ and $x_{n+1} = b$. Assume that $A_1(x), \ldots, A_n(x)$ are basic functions which form a fuzzy partition of $[a, b]$.

Let $L_2([a, b], A_1, \ldots, A_n)$ be the set of functions which are weighted square integrable on the interval $[a, b]$ with the weights given by $A_k(x)$, $k = 1, \ldots, n$. The following definition (see also [12, 14]) introduces the fuzzy transform of a function $f \in L_2([a, b], A_1, \ldots, A_n)$.

Definition 2
Let $A_1, \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[a, b]$ and $f(x)$ be any function from $L_2([a, b], A_1, \ldots, A_n)$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$

given by

$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \qquad k = 1, \dots, n, \tag{4}$$

is the (integral) F-transform of f with respect to $A_1, \dots, A_n$.

Let us remark that this definition is correct because for each $k = 1, \dots n$, the product fA_k is an integrable function on $[a, b]$.

Denote the F-transform of a function f with respect to $A_1, \dots, A_n$ by $\mathbf{F}_n[f]$. Then according to Definitions 2, we can write

$$\mathbf{F}_n[f] = [F_1, \dots, F_n]. \tag{5}$$

The elements $F_1, \dots, F_n$ are called *components of the F-transform.*

If the partition $A_1, \dots, A_n$ of $[a, b]$ is uniform then the expression (4) for components of the F-transform may be simplified on the basis of Lemma 1 as follows:

$$F_1 = \frac{2}{h}\int_{x_1}^{x_2} f(x)A_1(x)dx, \tag{6}$$

$$F_n = \frac{2}{h}\int_{x_{n-1}}^{x_n} f(x)A_n(x)dx, \tag{7}$$

$$F_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx, \quad k = 2, \dots, n-1. \tag{8}$$

3.2 Properties of the F-transform

It is easy to see that if the fuzzy partition of $[a, b]$ (and therefore, basic functions) is fixed then the F-transform establishes a linear mapping from $L_2([a, b], A_1, \dots, A_n)$ to $\mathbb{R}^n$ so that

$$\mathbf{F}_n[\alpha f + \beta g] = \alpha \mathbf{F}_n[f] + \beta \mathbf{F}_n[g]$$

for $\alpha, \beta \in \mathbb{R}$ and functions $f, g \in L_2([a, b], A_1, \dots, A_n)$. This linear mapping is denoted by $\mathbf{F}_n$ where n is the dimension of the image space.

We will investigate the following problem: how well is the original function f represented by its F-transform? First of all, we will show that under certain assumptions on the original function, the components of its F-transform are the *weighted mean values* of the given function where the weights are given by the basic functions.

Theorem 1
Let $f(x)$ be a continuous function on $[a, b]$ and $A_1(x), \dots, A_n(x)$ be basic functions which form a fuzzy partition of $[a, b]$. Then the k-th component of the integral F-transform gives minimum to the function

$$\Phi(y) = \int_a^b (f(x) - y)^2 A_k(x)dx \tag{9}$$

with the domain $[f(a), f(b)]$.

PROOF: By the assumptions, the function $(f(x)-y)^2A_k(x)$ is continuously differentiable with respect to y in $(f(a), f(b))$, and we may write

$$\Phi'(y) = -2\int_a^b (f(x)-y)A_k(x)dx.$$

Moreover, it is easy to see that the function $\Phi(y)$ reaches its minimum at the point which gives a solution to the equation $\Phi'(y) = 0$, i.e.

$$y = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}.$$

This is the exact expression of the k-th F-transform component (cf. 4). □

Now, we will try to estimate each integral F-transform component F_k, $k = 1, \dots, n$, using different assumptions concerning the smoothness of f.

Lemma 2
Let $f(x)$ *be a continuous function on* $[a, b]$ *and* $A_1(x), \dots, A_n(x)$, $n \geq 3$, *be basic functions which form a uniform fuzzy partition of* $[a, b]$. *Let* F_k, $k = 1, \dots, n$, *be the integral F-transform components of* f *with respect to* $A_1, \dots, A_n$. *Then for each* $k = 1, \dots, n-1$, *and for each* $t \in [x_k, x_{k+1}]$ *the following estimations hold:*

$$|f(t) - F_k| \leq \omega(2h, f), \qquad |f(t) - F_{k+1}| \leq \omega(2h, f) \tag{10}$$

where $h = \frac{b-a}{n-1}$ *and*

$$\omega(2h, f) = \max_{|\delta|\leq 2h} \max_{x\in[a,b-\delta]} |f(x+\delta) - f(x)| \tag{11}$$

is the modulus of continuity of f *on* $[a, b]$.

PROOF: Let us choose a value of k in the range $1 \leq k \leq n-1$ and let $t \in [x_k, x_{k+1}]$. Then

$$|f(t) - F_k| = |f(t) - \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx| = |\frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} (f(t) - f(x))A_k(x)dx| \leq$$
$$\leq \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} |f(t) - f(x)|A_k(x)dx \leq \omega(2h, f).$$

For the second inequality in (10) the proof is analogous. □

A more sophisticated estimation of components F_k is given below.

Lemma 3
Let $f(x)$ *be a continuous function on* $[a, b]$ *and* $A_1(x), \dots, A_n(x)$, $n \leq 3$, *be basic functions which form a uniform fuzzy partition of* $[a, b]$. *Then for each* $k = 2, \dots, n-1$ *there*

exist constants $c_{k1} \in [x_{k-1}, x_k]$ *and* $c_{k2} \in [x_k, x_{k+1}]$ *such that the integral F-transform components fulfil the equality*

$$F_k = \frac{1}{h}\int_{c_{k1}}^{c_{k2}} f(x)dx.$$

For the case when $k = 1$ $(k = n)$ *there exists* $c \in [x_1, x_2]$ $(c \in [x_{n-1}, x_n])$ *such that*

$$F_1 = \frac{2}{h}\int_{x_1}^{c} f(x)dx \quad \left(F_n = \frac{2}{h}\int_{c}^{x_n} f(x)dx\right).$$

PROOF: The proof can be easily obtained from the second mean-value theorem. Indeed, let k lie between 2 and $n-1$. Then using the fact that $A_k(x)$ monotonically increases on $[x_{k-1}, x_k]$ and monotonically decreases on $[x_k, x_{k+1}]$, we obtain:

$$F_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx = \frac{1}{h}\int_{x_{k-1}}^{x_k} f(x)A_k(x)dx +$$
$$\frac{1}{h}\int_{x_k}^{x_{k+1}} f(x)A_k(x)dx = \frac{1}{h}\int_{c_{k1}}^{x_k} f(x)dx + \frac{1}{h}\int_{x_k}^{c_{k2}} f(x)dx = \frac{1}{h}\int_{c_{k1}}^{c_{k2}} f(x)dx$$

where $c_{k1} \in [x_{k-1}, x_k]$, $c_{k2} \in [x_k, x_{k+1}]$ are some constants.

The cases $k = 1$ and $k = n$ are considered analogously. □

Therefore, by Lemma 3, we can say that F_k is an *integral mean value* of f within the interval $[c_{k1}, c_{k2}]$ and thus, it accumulates the information about function f within this interval. However, this interval cannot be specified precisely for the given function and nodes of the partition. We may evaluate F_k more precisely under the assumption that function f is twice continuously differentiable.

Lemma 4
Let the conditions of Lemma 3 be fulfilled, but function f *be twice continuously differentiable in* (a, b). *Then for each* $k = 1, \ldots, n$

$$F_k = f(x_k) + O(h^2). \tag{12}$$

PROOF: The proof will be given for one fixed value of k which lies between 2 and $n-1$. The other two cases $k = 1$ and $k = n$ are considered analogously. We will apply the trapezium formula with nodes x_{k-1}, x_k, x_{k+1} to the computation of the integral

$$\frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx$$

and obtain

$$F_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx =$$

$$\frac{1}{h} \cdot \frac{h}{2} \left(f(x_{k-1}) A_k(x_{k-1}) + 2 f(x_k) A_k(x_k) + f(x_{k+1}) A_k(x_{k+1}) \right) + O(h^2) = f(x_k) + O(h^2).$$

□

3.3 Inverse F-transform

A reasonable question is the following: can we reconstruct the function by its F-transform? The answer is clear: not precisely in general, because we are losing information when passing to the F-transform. However, the function that can be reconstructed (by the inversion formula) approximates the original one in such a way that a universal convergence can be established. Moreover, the inverse F-transform fulfils the best approximation criterion which can be called the piecewise integral least square criterion.

Definition 3
Let $A_1(x), \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[a, b]$ and $f(x)$ be a function from $L_2([a, b], A_1, \ldots, A_n)$. Let $\mathbf{F}_n[f] = [F_1, \ldots, F_n]$ be the integral F-transform of f with respect to $A_1, \ldots, A_n$. Then the function

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x) \tag{13}$$

is called *the inverse F-transform*.

The theorem below shows that the inverse F-transform $f_{F,n}$ can approximate the original continuous function f with an arbitrary precision.

Theorem 2
Let $f(x)$ be a continuous function on $[a, b]$. Then for any $\varepsilon > 0$ there exist n_ε and a fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$ of $[a, b]$ such that for all $x \in [a, b]$

$$|f(x) - f_{F,n_\varepsilon}(x)| \leq \varepsilon \tag{14}$$

where f_{F,n_ε} is the inverse F-transform of f with respect to the fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$.

PROOF: Note that the function f is uniformly continuous on $[a, b]$, i.e. for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x', x'' \in [a, b]$, $|x' - x''| < \delta$ implies $|f(x') - f(x'')| < \varepsilon$. To prove our theorem we choose some $\varepsilon > 0$ and find the nodes $x_1, \ldots, x_n \in [a, b]$ such that $a = x_1 < \ldots < x_n = b$ and $|f(x') - f(x'')| < \varepsilon$ whenever $x', x'' \in [x_{k-1}, x_{k+1}]$, $k = 2, \ldots, n-1$. Let us put $n = n_\varepsilon$ and take a fuzzy partition determined by the chosen nodes and constituted by basic functions $A_1, \ldots, A_n$. To complete the proof it remains to verify (14).

Let $F_1, \dots, F_n$ be the components of the F-transform of f w.r.t. basic functions $A_1, \dots, A_n$. Then for all $t \in [x_k, x_{k+1}]$, $k = 1, \dots, n-1$, we evaluate

$$|f(t) - F_k| = |f(t) - \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx}| \leq \frac{\int_{x_{k-1}}^{x_{k+1}} |f(t) - f(x)|A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx} \leq \varepsilon$$

and analogously,

$$|f(t) - F_{k+1}| \leq \varepsilon$$

where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$. Therefore, having in mind (1), we obtain

$$|f(t) - \sum_{i=1}^{n} F_i A_i(t)| = |f(t) \sum_{i=1}^{n} A_i(t) - \sum_{i=1}^{n} F_i A_i(t)| \leq$$
$$\leq \sum_{i=1}^{n} A_i(t)|f(t) - F_i| = \sum_{i=k}^{k+1} A_i(t)|f(t) - F_i| \leq \varepsilon \sum_{i=k}^{k+1} A_i(t) = \varepsilon \sum_{i=1}^{n} A_i(t) = \varepsilon.$$

Because the argument t has been chosen arbitrary within the interval $[a, b]$, this proves the inequality (14). □

In the proof of Theorem 2 we have constructed the non-uniform partition of $[a, b]$. We can reformulate the result of Theorem 2 for the case of uniform fuzzy partitions of $[a, b]$ having in mind the fact that the number of nodes n determines the uniform fuzzy partition up to the shape of membership functions.

Corollary 1
Let $f(x)$ be any continuous function on $[a, b]$ and let $\{(A_1^{(n)}, \dots, A_n^{(n)})_n\}$ be a sequence of uniform fuzzy partitions of $[a, b]$, one for each n. Let $\{f_{F,n}(x)\}$ be the sequence of inverse F-transforms, each with respect to the given n-tuple $A_1^{(n)}, \dots, A_n^{(n)}$. Then for any $\varepsilon > 0$ there exists n_ε such that for each $n > n_\varepsilon$ and for all $x \in [a, b]$

$$|f(x) - f_{F,n}(x)| \leq \varepsilon. \tag{15}$$

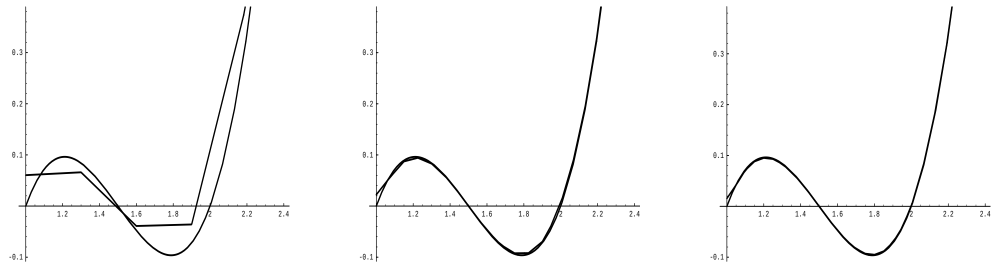
PROOF: The proof easily follows from the fact that for a chosen $\varepsilon > 0$ we can always find the respective value $n_\varepsilon > 2$ such that the corresponding value of $h = (b-a)/(n_\varepsilon - 1)$ guarantees that

$$|f(x') - f(x'')| < \varepsilon \quad \text{whenever} \quad |x' - x''| < h.$$

□

Corollary 2
Let the assumptions of Corollary 1 be fulfilled. Then the sequence of inverse F-transforms $\{f_{F,n}\}$ uniformly converges to f.

To illustrate the fact of uniform convergence we chose the original function $(x-1)(x-2)(2x-3)+6$ on the interval $[0, 2.5]$ (see Fig. 3) and consider approximations by their inverse F-transforms for different values of n. As we see below, the greater the value of n, the closer the approximating curve approaches the original function.



$n = 5$ $\quad$ $n = 20$ $\quad$ $n = 30$

Fig. 3. Function $(x-1)(x-2)(2x-3)+6$ and its inverse F-transforms based on n triangular shaped basic functions. Note that in the case $n = 30$, the inverse F-transform practically coincides with the original function.

It is worth noticing that so far, we have not specified any concrete shape of the basic functions. Thus, a natural question arises what is the influence of the shapes of basic functions on a quality of approximation. We can say the following: Theorem 2 guarantees the convergence of a sequence on inverse F-transforms which are based on arbitrary basic functions. This means that the convergence holds irrespective of shapes of basic functions. However, a speed of the convergence may be influenced by a concrete shape of basic functions.

The following theorem shows how the difference between any two approximations of a given function by the inverse F-transforms, based on different sets of basic functions, can be estimated. As can be seen, it depends on the character of smoothness of the original function expressed by its modulus of continuity.

Theorem 3
Let $f(x)$ be any continuous function on $[a, b]$ and $A'_1(x), \dots, A'_n(x)$ as well as $A''_1(x), \dots, A''_n(x)$, $n \geq 3$, be basic functions which form different uniform fuzzy partitions of $[a, b]$. Let $f'_{F,n}(x)$ and $f''_{F,n}(x)$ be two inverse F-transforms of f with respect to different sets of basic functions. Then for arbitrary $x \in [a, b]$

$$|f'_{F,n}(x) - f''_{F,n}(x)| \leq 2\omega(2h, f)$$

where $h = \frac{b-a}{n-1}$ and $\omega(2h, f)$ is the modulus of continuity of f on the interval $[a, b]$ (cf. (11)).

PROOF: Let us denote $[F'_1, \dots, F'_n]$ and $[F''_1, \dots, F''_n]$ components of the F-transforms of f with respect to the corresponding sets of basic functions $A'_1, \dots, A'_n$ and $A''_1, \dots, A''_n$. Then for arbitrary $x \in [a, b]$

$$|f'_{F,n}(x) - f''_{F,n}(x)| = |\sum_{i=1}^{n} F'_i A'_i(x) - \sum_{i=1}^{n} F''_i A''_i(x)| \leq$$

$$|\sum_{i=1}^{n}(F_i' - f(x))A_i'(x)| + |\sum_{i=1}^{n}(f(x) - F_i'')A_i''(x)| \leq$$
$$\sum_{i=1}^{n}|F_i' - f(x)|A_i'(x) + \sum_{i=1}^{n}|F_i'' - f(x)|A_i''(x).$$

Assume that $x \in [x_k, x_{k+1}]$ for some $k = 1, \ldots n-1$. Then by Lemma 2, $|F_i' - f(x)| \leq \omega(2h, f)$ as well as $|F_i'' - f(x)| \leq \omega(2h, f)$. Therefore,

$$|f'_{F,n}(x) - f''_{F,n}(x)| \leq \omega(2h, f)\sum_{i=1}^{n} A_i'(x) + \omega(2h, f)\sum_{i=1}^{n} A_i''(x) = 2\omega(2h, f).$$

□

We illustrate this theorem by considering two different inverse F-transforms of functions $\sin(1/x)$ and $\sin x$. One is based on triangular shaped basic functions and the other one is based on sinusoidal shaped basic functions (see Figs. 4, 5). Because $\sin(1/x)$ has a modulus of continuity greater than $\sin x$, the approximation of the latter with the same value of n looks smoother and therefore, a speed of convergence is greater.

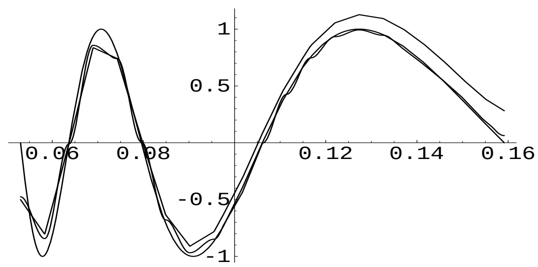


Fig. 4. Function $\sin(1/x)$ and two its inverse F-transforms based on triangular and sinusoidal shaped basic functions

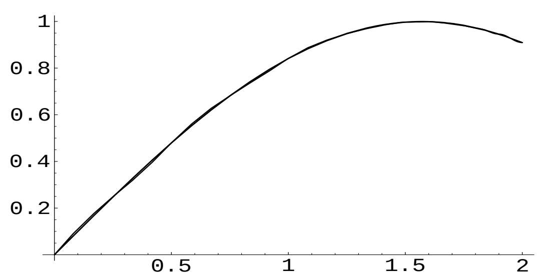


Fig. 5. Function $\sin x$ and two its inverse F-transforms based on triangular and sinusoidal shaped basic functions. Note that both approximations practically coincide with the original function.

3.4 Best approximation by the inverse F-transform

Assume that the basic functions $A_1(x), \ldots, A_n(x)$ are fixed and form a fuzzy partition of $[a, b]$ and $L_2([a, b], A_1, \ldots, A_n)$ is a space of original functions. Remember that the inverse F-transform of a function $f \in L_2([a, b], A_1, \ldots, A_n)$ is represented by the linear combination of basic functions with coefficients equal to the components of the F-transform.

We are going to prove that the inverse F-transform $f_{F,n}$ of f is a best approximation of f in the space of all linear combinations of $A_1(x), \ldots, A_n(x)$ with respect to a criterion explained below.

By $\mathrm{FT}(A_1, \ldots, A_n)$ we denote the set of continuous functions represented by linear combinations of $A_1(x), \ldots, A_n(x)$:

$$\mathrm{FT}(A_1, \ldots, A_n) = \{g(x) | g(x) = \sum_{i=1}^{n} c_i A_i(x)\} \tag{16}$$

where $c_1, \ldots, c_n$ are arbitrary real numbers. Due to the continuity of functions from $\mathrm{FT}(A_1, \ldots, A_n)$, we obtain that $\mathrm{FT}(A_1, \ldots, A_n) \subset L_2([a, b], A_1, \ldots, A_n)$. We will further refer to the functions from $\mathrm{FT}(A_1, \ldots, A_n)$ as approximating functions.

Note that the inverse F-transform of a function from $L_2([a, b], A_1, \ldots, A_n)$ belongs to $\mathrm{FT}(A_1, \ldots, A_n)$.

Lemma 5
The space of functions $L_2([a, b], A_1, \ldots, A_n)$ is a normed space with respect to each of the following norms:

$$\|f\|_k = \sqrt{\int_{x_{k-1}}^{x_{k+1}} f^2(x) A_k(x) dx}, \quad k = 1, \ldots, n. \tag{17}$$

PROOF: The proof easily follows from the equivalent representation of (17):

$$\|f\|_k = \sqrt{\int_{x_{k-1}}^{x_{k+1}} (f(x)\sqrt{A_k(x)})^2 dx}.$$

□

Let us remark that the norm $\|f\|_k$ turns $L_2([a, b], A_1, \ldots, A_n)$ into the metric space with the following metric:

$$d_k(f, g) = \|f - g\|_k.$$

Theorem 4
Let $f \in L_2([a, b], A_1, \ldots, A_n)$ and the set $FT(A_1, \ldots, A_n)$ of approximating functions be given by (16). Then components $F_1, \ldots, F_n$ of the F-transform minimize the following sum of squared distances

$$\sum_{k=1}^{n} d_k^2(f, c_k). \tag{18}$$

PROOF: The explicit expression for the sum of squared distances in (18) is given by the following formula

$$\Phi(c_1, \ldots, c_n) = \int_a^b \left(\sum_{i=k}^{n} (f(x) - c_k)^2 A_k(x) \right) dx \tag{19}$$

which represents the function defined on $[f(a), f(b)]^n$. The rest of the proof is analogous to the proof of Theorem 1. □

4 Discrete F-Transform

Let us specially consider the discrete case, when an original function $f(x)$ is known (may be computed) only at some nodes $p_1, \ldots, p_l \in [a, b]$. Then the (discrete) F-transform of f is introduced as follows.

Definition 4
Let a function $f(x)$ be given at nodes $p_1, \ldots, p_l \in [a, b]$ and $A_1(x), \ldots, A_n(x)$, $n \ll l$, be basic functions which form a fuzzy partition of $[a, b]$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$ is the discrete F-transform of f with respect to $A_1, \ldots, A_n$ if

$$F_k = \frac{\sum_{j=1}^{l} f(p_j)A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)}. \tag{20}$$

Similarly to the integral F-transform, we may show that the components of the discrete F-transform are the *weighted mean values* of the given function where the weights are given by the basic functions.

Lemma 6
Let function $f(x)$ be given at nodes $p_1, \ldots, p_l \in [a, b]$ and $A_1(x), \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[a, b]$. Then the k-th component of the discrete F-transform gives minimum of the function

$$\Phi(y) = \sum_{j=1}^{l} (f(p_j) - y)^2 A_k(p_j) \tag{21}$$

with the domain $[f(a), f(b)]$.

PROOF: The proof is similar to the proof of Theorem 1 and therefore, it is omitted. □

In the discrete case, we define the inverse F-transform only at nodes where the original function is given.

Definition 5
Let function $f(x)$ be given at nodes $p_1, \ldots, p_l \in [a, b]$ and $\mathbf{F}_n[f] = [F_1, \ldots, F_n]$ be the discrete F-transform of f w.r.t. $A_1, \ldots, A_n$. Then the function

$$f_{F,n}(p_j) = \sum_{k=1}^{n} F_k A_k(p_j), \tag{22}$$

defined at the same nodes, is *the inverse discrete F-transform.*

Analogously, the inverse discrete F-transform $f_{F,n}$ approximates the original function f at common nodes with an arbitrary precision.

Theorem 5
Let function $f(x)$ be given at nodes $p_1, \dots, p_l \in [a, b]$. Then, for any $\varepsilon > 0$ there exist n_ε and a fuzzy partition $A_1, \dots, A_{n_\varepsilon}$ of $[a, b]$ such that for all $p \in \{p_1, \dots, p_l\}$

$$|f(p) - f_{F,n_\varepsilon}(p)| \leq \varepsilon \tag{23}$$

where f_{F,n_ε} is the inverse discrete F-transform of f with respect to the fuzzy partition $A_1, \dots, A_n$ of $[a, b]$ and the chosen nodes.

PROOF: Let us take a continuous function $\tilde{f}$ such that $\tilde{f}(p) = f(p)$ for all $p \in \{p_1, \dots, p_l\}$. Then the rest of this proof is analogous to the proof of Theorem 2 where we replace integrals by the respective sums. □

Analogously to Corollary 1, we obtain the following proposition characterizing the convergence of the sequence of the inverse discrete F-transforms in the case of uniform fuzzy partitions of $[a, b]$.

Corollary 3
Let a function $f(x)$ be given at nodes $p_1, \dots, p_l \in [a, b]$ and let $\{(A_1^{(n)}, \dots, A_n^{(n)})_n\}$ be a sequence of uniform fuzzy partitions of $[a, b]$, one for each n. Let $\{f_{F,n}(x)\}$ be the sequence of the inverse discrete F-transforms, each with respect to the given n-tuple $A_1^{(n)}, \dots, A_n^{(n)}$. Then for any $\varepsilon > 0$ there exists n_ε such that for each $n > n_\varepsilon$ and for all $p \in \{p_1, \dots, p_l\}$

$$|f(p) - f_{F,n}(p)| \leq \varepsilon. \tag{24}$$

Remark 1
Let us remark that from the computational point of view, the discrete F-transform is of course, simpler than the integral one. Moreover, on the basis of Theorem 5 and its Corollary 3, a certain sequence of the inverse discrete F-transforms converges to an original function at all given nodes. Therefore, all computational algorithms may be based on the discrete type of the F-transform (and this is done in the paper).

5 F-Transforms Expressed By Residuated Lattice Operations

Our purpose is to introduce two new fuzzy transforms with the help of operations of a residuated lattice on $[0, 1]$. These new transforms lead to new approximation models which are expressed using weaker operations than the arithmetic ones used above in the case of the F-transform. However, these operations are successfully used in modelling dependencies expressed by natural language words (e.g. fuzzy IF–THEN rules) and also, in modelling continuous functions. Therefore, two new F-transforms which we are going to introduce in this section, extend and generalize the F-transform considered above.

Formally, both new F-transforms taken as transformations, map a space of the original functions onto the space of n-dimensional vectors. Moreover, two inverse F-transforms from different image spaces give lower and upper approximations of the original function.

Let us briefly introduce a residuated lattice which will be a basic algebra of operations in the sequel.

Definition 6
A residuated lattice is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle.$$

with four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are the least and the greatest elements, respectively;
- $\langle L, *, \mathbf{1} \rangle$ is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * \mathbf{1} = a$;
- the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.

$$a * b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The following operations of negation and biresiduation can be additionally defined:

$$\neg a = a \rightarrow 0$$
$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Let us remark that there are several other names for residuated lattice, namely (Höhle [5]) integral commutative, residuated l-monoid (or cl-monoid, if it is lattice complete); residuated commutative Abelian semigroup; complete lattice ordered semigroup. The term "residuated lattice" has been introduced by Dilworth and Ward in [2].

The well known examples of residuated lattices are boolean algebras, Gödel, Łukasiewicz and product algebras. In particular case $L = [0, 1]$, the multiplication $*$ is called a t-norm.

The following properties of operations will be used in the sequel:

$$x \leq y \quad \text{iff} \quad (x \rightarrow y) = \mathbf{1},$$
$$x \leftrightarrow y = \mathbf{1} \quad \text{iff} \quad x = y.$$

Moreover, we will use the fact that the operation "$\rightarrow$" is antitone with respect to its first argument and isotone with respect to the second one, and $*$ is isotone with respect to both arguments.

5.1 Direct $F^{\uparrow}$-transform

Let us fix some residuated lattice $\mathcal{L}$ on $[0, 1]$ and suppose that the universe is the interval $[0, 1]$, so that all functions in this section are defined on this interval and take values from it.

We redefine here the notion of *fuzzy partition* of $[0, 1]$ assuming that it is given by fuzzy sets $A_1, \ldots, A_n$ identified with their membership functions $A_1(x), \ldots, A_n(x)$ fulfilling the following (only one!) *covering property*

$$(\forall x)(\exists i) \quad A_i(x) > 0. \tag{25}$$

As above, the membership functions $A_1(x), \ldots, A_n(x)$ are called the *basic functions*. In the next definition we define a discrete $\mathrm{F}^{\uparrow}$-transform using operations of $\mathcal{L}$.

Definition 7
Let a function $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [0, 1]$ and $A_1(x), \ldots, A_n(x)$, $n \ll l$, be basic functions which form a fuzzy partition of $[a, b]$. We say that the n-tuple of real numbers $[F_1^{\uparrow}, \ldots, F_n^{\uparrow}]$ is a (discrete) $\mathrm{F}^{\uparrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ if

$$F_k^{\uparrow} = \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)). \tag{26}$$

We will not define another $\mathrm{F}^{\uparrow}$-transform different from the discrete one and therefore, the word "discrete" will be omitted in the sequel.

Suppose that the basic functions $A_1, \ldots, A_n$ are fixed. Denote the $\mathrm{F}^{\uparrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ by $\mathbf{F}_n^{\uparrow}[f]$. Then we may write:

$$\mathbf{F}_n^{\uparrow}[f] = [F_1^{\uparrow}, \ldots, F_n^{\uparrow}].$$

The elements $F_1^{\uparrow}, \ldots, F_n^{\uparrow}$ are called *components of the $F^{\uparrow}$-transform.*

5.2 Properties of the $F^{\uparrow}$-transform

We will show that the $\mathrm{F}^{\uparrow}$-transform being a mapping from the set of all functions on $[0, 1]$ to $\mathbb{R}^n$ has the property analogous to the property of linearity of the F-transform.

Lemma 7
Let functions f, g be defined at nodes $p_1, \ldots, p_l \in [0, 1]$ and $A_1(x), \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[0, 1]$. Then, for arbitrary $\alpha, \beta \in [0, 1]$, the following equality holds:

$$\mathbf{F}_n^{\uparrow}[\alpha * f \vee \beta * g] = \alpha * \mathbf{F}_n^{\uparrow}[f] \vee \beta * \mathbf{F}_n^{\uparrow}[g]. \tag{27}$$

where on the right-hand side, the operation $\vee$ is taken componentwise.

PROOF: Let us fix some $k : 1 \leq k \leq n$, and prove (27) for the k-th components. Denote by $[\alpha * f \vee \beta * g]^{\uparrow}_k$, $F^{\uparrow}_k$, $G^{\uparrow}_k$ the k-th components of $\mathbf{F}^{\uparrow}_n[\alpha * f \vee \beta * g]$, $\mathbf{F}^{\uparrow}_n[f]$ and $\mathbf{F}^{\uparrow}_n[g]$ respectively. Then, using properties of a residuated lattice, we obtain

$$[\alpha * f \vee \beta * g]^{\uparrow}_k = \bigvee_{j=1}^{l} (A_k(p_j) * (\alpha * f(p_j) \vee \beta * g(p_j))) =$$
$$\bigvee_{j=1}^{l} (A_k(p_j) * \alpha * f(p_j) \vee A_k(p_j) * \beta * g(p_j)) =$$
$$\alpha * \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)) \vee \beta * \bigvee_{j=1}^{l} (A_k(p_j) * g(p_j)) = \alpha * F^{\uparrow}_k \vee \beta * G^{\uparrow}_k.$$

□

Corollary 4
Let the condition of Lemma 7 be fulfilled. If $f \leq g$ then $\mathbf{F}^{\uparrow}_n[f] \leq \mathbf{F}^{\uparrow}_n[g]$ where the inequality $\leq$ between vectors is taken componentwise.

Below, we characterize components of the F$^{\uparrow}$-transform as optimal values of a certain criterion.

Lemma 8
Let a function $f(x)$ be defined at nodes $p_1, \dots, p_l \in [0,1]$ and $A_1(x), \dots, A_n(x)$ be basic functions which form a fuzzy partition of $[0,1]$. Then the k-th component of the F$^{\uparrow}$-transform is the least element of the following set:

$$S_k = \{a \in [0,1] | \quad f(p_j) \leq (A_k(p_j) \rightarrow a) \text{ for all } j = 1, \dots, l\} \tag{28}$$

where $k = 1, \dots, n$.

PROOF: It is easy to see that $F^{\uparrow}_k \in S_k$. We will show that $a \in S_k$ implies $F^{\uparrow}_k \leq a$. Indeed, from (28) we have

$$f(p_j) \leq (A_k(p_j) \rightarrow a) \text{ for all } j = 1, \dots, l$$

which implies (with help of the adjunction property)

$$a \geq A_k(p_j) * f(p_j) \text{ for all } j = 1, \dots, l$$

and therefore,

$$a \geq \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)) = F^{\uparrow}_k.$$

□

Corollary 5
Let the condition of Lemma 8 be fulfilled. Then the k-th component of the F$^{\uparrow}$*-transform is the least solution to the following equation with the unknown a:*

$$\bigwedge_{j=1}^{l} (f(p_j) \to (A_k(p_j) \to a)) = 1 \tag{29}$$

where $k = 1, \ldots, n$.

5.3 Direct F$^{\downarrow}$-transform

In this subsection, we define the F$^{\downarrow}$-transform which may be regarded as the dual to the F$^{\uparrow}$-transform.

Definition 8
Let $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [0,1]$ and $A_1(x), \ldots, A_n(x)$, $n \ll l$, be basic functions which form a fuzzy partition of $[a,b]$. We say that the n-tuple of real numbers $[F_1^{\downarrow}, \ldots, F_n^{\downarrow}]$ is the (discrete) F$^{\downarrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ if

$$F_k^{\downarrow} = \bigwedge_{j=1}^{l} (A_k(p_j) \to f(p_j)). \tag{30}$$

Suppose that the basic functions $A_1, \ldots, A_n$ are fixed. Denote the F$^{\downarrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ by $\mathbf{F}_n^{\downarrow}[f]$. Then we may write

$$\mathbf{F}_n^{\downarrow}[f] = [F_1^{\downarrow}, \ldots, F_n^{\downarrow}].$$

The elements $F_1^{\downarrow}, \ldots, F_n^{\downarrow}$ are called *components of the F*$^{\downarrow}$*-transform.*

5.4 Properties of the F$^{\downarrow}$-transform

The properties considered in this subsection, are in some sense dual to the properties of the F$^{\uparrow}$-transform. We start with a proof of the dual linearity property.

Lemma 9
Let functions f, g *be defined at nodes* $p_1, \ldots, p_l \in [0,1]$ *and* $A_1(x), \ldots, A_n(x)$ *be basic functions which form a fuzzy partition of* $[0,1]$. *Then for arbitrary* $\alpha, \beta \in [0,1]$ *the following equality holds:*

$$\mathbf{F}_n^{\downarrow}[(\alpha \to f) \wedge (\beta \to g)] = (\alpha \to \mathbf{F}_n^{\downarrow}[f]) \wedge (\beta \to \mathbf{F}_n^{\downarrow}[g]). \tag{31}$$

where on the right-hand side, the operation $\wedge$ *is taken componentwise.*

PROOF: Let us fix some $k : 1 \leq k \leq n$, and prove (31) for the k-th components. Denote by $[(\alpha \to f) \wedge (\beta \to g)]^{\downarrow}_k$, $F^{\downarrow}_k$, $G^{\downarrow}_k$ the k-th components of $\mathbf{F}^{\downarrow}_n[(\alpha \to f) \wedge (\beta \to g)]$, $\mathbf{F}^{\downarrow}_n[f]$ and $\mathbf{F}^{\downarrow}_n[g]$ respectively. Then using properties of a residuated lattice, we obtain

$$[(\alpha \to f) \wedge (\beta \to g)]^{\downarrow}_k = \bigwedge_{j=1}^{l} (A_k(p_j) \to (\alpha \to f) \wedge (\beta \to g)) =$$

$$\bigwedge_{j=1}^{l} (A_k(p_j) \to (\alpha \to f)) \wedge \bigwedge_{j=1}^{l} (A_k(p_j) \to (\beta \to g)) =$$

$$\bigwedge_{j=1}^{l} (\alpha \to (A_k(p_j) \to f)) \wedge \bigwedge_{j=1}^{l} (\beta \to (A_k(p_j) \to g)) =$$

$$(\alpha \to \bigwedge_{j=1}^{l} (A_k(p_j) \to f)) \wedge (\beta \to \bigwedge_{j=1}^{l} (A_k(p_j) \to g)) = (\alpha \to F^{\downarrow}_k) \wedge (\beta \to G^{\downarrow}_k).$$

□

Corollary 6
Let the condition of Lemma 9 be fulfilled. Then $f \leq g$ implies $\mathbf{F}^{\downarrow}_n[f] \leq \mathbf{F}^{\downarrow}_n[g]$ where the inequality $\leq$ between vectors is taken componentwise.

We characterize components of the $\mathrm{F}^{\downarrow}$-transform as optimal values of a certain criterion (cf. (28)).

Lemma 10
Let a function $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [0,1]$ and $A_1(x), \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[0,1]$. Then the k-th component of the $F^{\downarrow}$-transform is the greatest element of the set

$$T_k = \{a \in [0,1] | \quad f(p_j) \geq (A_k(p_j) * a) \text{ for all } j = 1, \ldots, l\} \tag{32}$$

where $k = 1, \ldots, n$.

PROOF: It follows from the definition of $F^{\downarrow}_k$ that $F^{\downarrow}_k \in T_k$. It remains to show that if $a \in T_k$ then $a \leq F^{\uparrow}_k$. Indeed, from (32) we have

$$f(p_j) \geq (A_k(p_j) * a) \text{ for all } j = 1, \ldots, l$$

which implies (with help of the adjunction property)

$$a \leq A_k(p_j) \to f(p_j) \text{ for all } j = 1, \ldots, l$$

and therefore,

$$a \leq \bigwedge_{j=1}^{l} (A_k(p_j) \to f(p_j)) = F^{\downarrow}_k.$$

□

Corollary 7
Let the condition of Lemma 10 be fulfilled. Then the k-th component of the F$^{\downarrow}$-transform is the greatest solution to the following equation with the unknown a:

$$\bigwedge_{j=1}^{l}((A_k(p_j) * a) \rightarrow f(p_j)) = 1 \quad (33)$$

where $k = 1, \ldots, n$.

6 Inverse F$^{\uparrow}$ (F$^{\downarrow}$)-Transforms

All F-transforms convert the space of functions into the space of n-dimensional real vectors. It is possible to define the inverse transform to be able to return to the original space of functions. We have defined the inverse F-transform in Subsection 3.3. In this section, we will define inverse F$^{\uparrow}$ and inverse F$^{\downarrow}$-transforms and prove their approximation properties.

6.1 Inverse F$^{\uparrow}$-transform

In the construction of the inverse F$^{\uparrow}$-transform we use the fact that the operations $*$ and $\rightarrow$ are mutually adjoint in a residuated lattice.

Definition 9
Let function $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [a, b]$ and let $\mathbf{F}_n^{\uparrow}[f] = [F_1^{\uparrow}, \ldots, F_n^{\uparrow}]$ be the F$^{\uparrow}$-transform of f w.r.t. basic functions $A_1, \ldots, A_n$. Then the function

$$f_{F,n}^{\uparrow}(p_j) = \bigwedge_{k=1}^{n}(A_k(p_j) \rightarrow F_k^{\uparrow}), \quad (34)$$

defined at the same nodes as f, is called the *inverse F$^{\uparrow}$-transform.*

Let us remark that the inverse F$^{\uparrow}$-transform has a similar construction as the conjunctive normal form for functions considered in this section (see [11, 13] for the details). This can be also seen from the following theorem which shows the relationship between the original function and its inverse F$^{\uparrow}$-transform.

Theorem 6
Let functions $f(x), f_{F,n}^{\uparrow}(x)$ *be defined at nodes* $p_1, \ldots, p_l \in [0, 1]$ *and the inverse F$^{\uparrow}$-transform* $f_{F,n}^{\uparrow}$ *of* f *is computed with respect to basic functions* $A_1(x), \ldots, A_n(x)$*. Then for all* $j = 1, \ldots, l$

$$f_{F,n}^{\uparrow}(p_j) \geq f(p_j). \quad (35)$$

PROOF: According to (34) and (26) we may write that

$$f^{\uparrow}_{F,n}(p_j) = \bigwedge_{k=1}^{n} (A_k(p_j) \to F^{\uparrow}_k) = \bigwedge_{k=1}^{n} (A_k(p_j) \to \bigvee_{i=1}^{l} (A_k(p_i) * f(p_i))).$$

Then the following inequalities obviously hold:

$$\bigwedge_{k=1}^{n} (A_k(p_j) \to \bigvee_{i=1}^{l} (A_k(p_i) * f(p_i)) \geq \bigwedge_{k=1}^{n} (A_k(p_j) \to (A_k(p_j) * f(p_j)) \geq f(p_j).$$

□

It is interesting that the functions $f(x), f^{\uparrow}_{F,n}(x)$ have the same F$^{\uparrow}$-transform. Therefore, the inverse F$^{\uparrow}$-transform of the function $f^{\uparrow}_{F,n}$ is again $f^{\uparrow}_{F,n}$. This easily follows from the theorem given below.

Theorem 7
Let functions $f(x), f^{\uparrow}_{F,n}(x)$ be defined at nodes $p_1, \dots, p_l \in [0,1]$ and the inverse F$^{\uparrow}$-transform $f^{\uparrow}_{F,n}$ of f be computed with respect to basic functions $A_1(x), \dots, A_n(x)$. Then for all $k = 1, \dots, n$

$$F^{\uparrow}_k = \bigvee_{j=1}^{l} (A_k(p_j) * f^{\uparrow}_{F,n}(p_j))$$

where $\bigvee_{j=1}^{l}(A_k(p_j) * f^{\uparrow}_{F,n}(p_j))$ is the k-th component of the F$^{\uparrow}$-transform of the function $f^{\uparrow}_{F,n}$.

PROOF: Let us fix some k, $1 \leq k \leq n$, and evaluate the value of $A_k(p_j) * f^{\uparrow}_{F,n}(p_j)$ for some j, $1 \leq j \leq l$:

$$A_k(p_j) * f^{\uparrow}_{F,n}(p_j) = A_k(p_j) * \bigwedge_{i=1}^{n} (A_i(p_j) \to F^{\uparrow}_i) \leq A_k(p_j) * (A_k(p_j) \to F^{\uparrow}_k) \leq F^{\uparrow}_k.$$

As a consequence, we will obtain

$$\bigvee_{j=1}^{l} (A_k(p_j) * f^{\uparrow}_{F,n}(p_j)) \leq F^{\uparrow}_k.$$

On the other side, with the help of (35), it is easy to show that

$$F^{\uparrow}_k = \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)) \leq \bigvee_{j=1}^{l} (A_k(p_j) * f^{\uparrow}_{F,n}(p_j)).$$

the last two inequalities prove the statement of the theorem. □

As follows from Theorem 7, different functions may have the same F$^{\uparrow}$-transform $[F^{\uparrow}_1, \dots, F^{\uparrow}_n]$. In the set of all such functions, the function $f^{\uparrow}_{F,n}$ is distinguished by the property formulated in the corollary given below.

Corollary 8
Let $\Phi^{\uparrow}_{F,n}$ be the set of functions which are defined at nodes $p_1, \dots, p_l \in [0, 1]$ and which have the same $F^{\uparrow}$-transform $[F^{\uparrow}_1, \dots, F^{\uparrow}_n]$ with respect to basic functions $A_1(x), \dots, A_n(x)$. Then $f^{\uparrow}_{F,n}(x)$ is the greatest element of $\Phi^{\uparrow}_{F,n}$.

Remark 2
Let us stress that all functions from the set $\Phi^{\uparrow}_{F,n}$ are restricted to the domain $\{p_1, \dots, p_l\}$. Therefore, the statement of Corollary 8 may be formalized as follows:

$$f \in \Phi^{\uparrow}_{F,n} \Rightarrow (\forall j)(f(p_j) \le f^{\uparrow}_{F,n}(p_j)).$$

Corollary 9
Let $[F^{\uparrow}_1, \dots, F^{\uparrow}_n]$ be an arbitrary vector of reals from $[0, 1]$ and $A_1(x), \dots, A_n(x)$ form a fuzzy partition of $[0, 1]$. Then for arbitrary set of nodes $p_1, \dots, p_l \in [0, 1]$, $n \ll l$, there is a function f defined at these nodes and such that $[F^{\uparrow}_1, \dots, F^{\uparrow}_n]$ is its $F^{\uparrow}$-transform with respect to $A_1, \dots, A_n$.

PROOF: According to Theorem 7, the function $f^{\uparrow}_{F,n}(x)$ fulfils the requirements of the corollary. □

6.2 Inverse F$^{\downarrow}$-transform

In the construction of the inverse F$^{\downarrow}$-transform we will use the fact that the operations $*$ and $\rightarrow$ are mutually adjoint. Therefore, this construction will be dual to that presented in Subsection 6.1.

Definition 10
Let a function $f(x)$ be defined at nodes $p_1, \dots, p_l \in [a, b]$ and let $\mathbf{F}^{\downarrow}_n[f] = [F^{\downarrow}_1, \dots, F^{\downarrow}_n]$ be the F$^{\downarrow}$-transform of f w.r.t. basic functions $A_1, \dots, A_n$. Then the function

$$f^{\downarrow}_{F,n}(p_j) = \bigvee_{k=1}^{n} (A_k(p_j) * F^{\downarrow}_k), \tag{36}$$

defined at the same nodes as f, is called the *inverse F$^{\downarrow}$-transform*.

Let us remark that the inverse F$^{\downarrow}$-transform has a similar construction as the disjunctive normal form for functions considered in this section (see [11, 13] for the details). This can be also seen from the following theorem which shows the relationship between the original function and its inverse F$^{\downarrow}$-transform.

Theorem 8
Let functions $f(x), f^{\downarrow}_{F,n}(x)$ be defined at nodes $p_1, \dots, p_l \in [0, 1]$ and the inverse F$^{\downarrow}$-transform $f^{\downarrow}_{F,n}$ of f is computed with respect to basic functions $A_1(x), \dots, A_n(x)$. Then for all $j = 1, \dots, l$

$$f^{\downarrow}_{F,n}(p_j) \le f(p_j). \tag{37}$$

PROOF: According to (36) and (30) we may write that

$$f^{\downarrow}_{F,n}(p_j) = \bigvee_{k=1}^{n} (A_k(p_j) * F^{\downarrow}_k) = \bigvee_{k=1}^{n} (A_k(p_j) * \bigwedge_{i=1}^{l} (A_k(p_i) \to f(p_i))).$$

Then the following inequalities obviously hold:

$$\bigvee_{k=1}^{n} (A_k(p_j) * \bigwedge_{i=1}^{l} (A_k(p_i) \to f(p_i)) \leq \bigvee_{k=1}^{n} (A_k(p_j) * (A_k(p_j) \to f(p_j)) \leq f(p_j).$$

□

As in the case of inverse F$^{\uparrow}$-transforms, the inverse F$^{\downarrow}$-transform of $f^{\downarrow}_{F,n}$ is again $f^{\downarrow}_{F,n}$. This fact easily follows from the theorem given below.

Theorem 9
Let functions $f(x), f^{\downarrow}_{F,n}(x)$ be defined at nodes $p_1, \ldots, p_l \in [0,1]$ and the inverse F$^{\downarrow}$-transform $f^{\downarrow}_{F,n}$ of f is computed with respect to basic functions $A_1(x), \ldots, A_n(x)$. Then for all $k = 1, \ldots, n$

$$F^{\downarrow}_k = \bigwedge_{j=1}^{l} (A_k(p_j) \to f^{\downarrow}_{F,n}(p_j))$$

where $\bigwedge_{j=1}^{l}(A_k(p_j) \to f^{\downarrow}_{F,n}(p_j))$ is the k-th component of the F$^{\downarrow}$-transform of the function $f^{\downarrow}_{F,n}$.

PROOF: Let us fix some k, $1 \leq k \leq n$, and evaluate the value of $A_k(p_j) \to f^{\downarrow}_{F,n}(p_j)$ for some j, $1 \leq j \leq l$:

$$A_k(p_j) \to f^{\downarrow}_{F,n}(p_j) = A_k(p_j) \to \bigvee_{i=1}^{n} (A_i(p_j) * F^{\downarrow}_i) \geq A_k(p_j) \to (A_k(p_j) * F^{\downarrow}_k) \geq F^{\downarrow}_k.$$

As a consequence, we will obtain

$$\bigwedge_{j=1}^{l} (A_k(p_j) \to f^{\downarrow}_{F,n}(p_j)) \geq F^{\downarrow}_k.$$

On the other side, with the help of (37), it is easy to show that

$$F^{\downarrow}_k = \bigwedge_{j=1}^{l} (A_k(p_j) \to f(p_j)) \geq \bigwedge_{j=1}^{l} (A_k(p_j \to f^{\downarrow}_{F,n}(p_j)).$$

Two last inequalities prove the statement of the theorem. □

As follows also from Theorem 9, different functions may have the same F$^{\downarrow}$-transform $[F^{\downarrow}_1, \ldots, F^{\downarrow}_n]$. In the set of all such functions, $f^{\downarrow}_{F,n}$ is distinguished by the property formulated in the corollary given below.

Corollary 10
Let $\Phi^{\downarrow}_{F,n}$ be the set of functions which are defined at nodes $p_1, \dots, p_l \in [0,1]$ and which have the same $F^{\downarrow}$-transform $[F^{\downarrow}_1, \dots, F^{\downarrow}_n]$ with respect to basic functions $A_1(x), \dots, A_n(x)$. Then $f^{\downarrow}_{F,n}(x)$ is the least element of $\Phi^{\downarrow}_{F,n}$.

Remark 3
The statement of Corollary 10 may be formalized as follows:

$$f \in \Phi^{\downarrow}_{F,n} \Rightarrow (\forall j)(f^{\downarrow}_{F,n}(p_j) \le f(p_j)).$$

Corollary 11
Let $[F^{\downarrow}_1, \dots, F^{\downarrow}_n]$ be an arbitrary vector of reals from $[0,1]$ and $A_1(x), \dots, A_n(x)$ form a fuzzy partition of $[0,1]$. Then for arbitrary set of nodes $p_1, \dots, p_l \in [0,1]$, $n \ll l$, there is a function f defined at these nodes and such that $[F^{\downarrow}_1, \dots, F^{\downarrow}_n]$ is its $F^{\downarrow}$-transform with respect to $A_1, \dots, A_n$.

PROOF: According to Theorem 9, the function $f^{\downarrow}_{F,n}(x)$ fulfils the requirements of the corollary. □

7 Approximation By Inverse F-Transforms

As we saw in the previous section, the inverse $F^{\uparrow}$-transform $f^{\uparrow}_{F,n}$ of f can be taken as its upper approximation. On the other hand, the inverse $F^{\downarrow}$-transform $f^{\downarrow}_{F,n}$ of f can be taken as its lower approximation. We may be interested in the estimation of proximity between f and its two mentioned approximations. This can be done with some additional assumptions on the residuated lattice, fuzzy partition of $[0,1]$ and functions on $[0,1]$ which we suppose to be valid throughout this section.

(1) Our first assumption reduces a residuated lattice on $[0,1]$ with which we worked before, to a more specific structure known as a BL-algebra ([4]), by adding two identities:

$$x * (x \to y) = x \wedge y,$$
$$(x \to y) \vee (y \to x) = 1.$$

We will write

$$x^n = \underbrace{x * \cdots * x}_{n \text{ times}}.$$

(2) Our second assumption concerns a fuzzy partition of $[0,1]$, given by fuzzy sets $A_1, \dots, A_n$ and fulfilling the covering property (25). We will additionally require

the fuzzy sets $A_1, \dots, A_n$ to be normal and fulfil the inequality

$$(\forall i)(\forall j) \qquad \left(\bigvee_{x\in[0,1]} (A_i(x) * A_j(x)) \leq \bigwedge_{x\in[0,1]} (A_i(x) \leftrightarrow A_j(x)) \right). \tag{38}$$

Let us remark that (38) means that the fuzzy sets $A_1, \dots, A_n$ establish a *semi-partition* of $[0, 1]$ (see e.g. [1]).

Due to the normality of fuzzy sets $A_1, \dots, A_n$, there exist points $x^1, \dots, x^n \in [0, 1]$ such that

$$(\forall i) \quad (A_i(x^i) = 1).$$

Due to the assumption (38), the fuzzy sets $A_1, \dots, A_n$ can be expressed as *fuzzy points* with respect to the fuzzy equivalence (similarity) E on $[0, 1]$ given by

$$E(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \leftrightarrow A_i(y)) \tag{39}$$

which means that

$$(\forall i) \quad A_i(x) = E(x^i, x) \tag{40}$$

(see [7]).

Recall that a fuzzy *similarity* relation or a *fuzzy equivalence* E on $[0, 1]$ fulfils the following three conditions for all $x, y, z \in [0, 1]$:

$$E(x, x) = 1, \qquad \textit{(reflexivity)}$$
$$E(x, y) = E(y, x), \qquad \textit{(symmetry)}$$
$$E(x, y) * E(y, z) \leq E(x, z). \qquad \textit{(transitivity)}$$

(3) The third assumption concerns functions which will be transformed into real vectors and then, inversely, into functions. We assume them to be defined at nodes $p_1, \dots, p_l$, $n \ll l$, to take values from $[0, 1]$ and to be *extensional* on the set $\{p_1, \dots, p_l\}$ with respect to the similarity E given by (39), i.e.

$$(\forall i)(\forall j) \quad (E(p_i, p_j) \leq f(p_i) \leftrightarrow f(p_j)). \tag{41}$$

Moreover, we assume that the points $x^1, \dots, x^n$ which witness the normality of fuzzy sets $A_1, \dots, A_n$, are among the nodes $p_1, \dots, p_l$.

7.1 Approximation by the inverse $F^{\uparrow}$-transform

The first result that we are going to prove, estimates the proximity between the functions f and $f^{\uparrow}_{F,n}$. To do it, we will use the biresiduation operator "$\leftrightarrow$". According to its properties, elements a and b are regarded to be close if $a \leftrightarrow b$ is close to 1. Therefore, the proximity between f and $f^{\uparrow}_{F,n}$ expressed by $f(p_j) \leftrightarrow f^{\uparrow}_{F,n}(p_j)$ is estimated in Theorem 10 below.

Theorem 10
Let normal fuzzy sets $A_1, \dots, A_n$ establish a semi-partition of $[0,1]$ and a function f, defined at nodes $p_1, \dots, p_l$, be extensional on the set $\{p_1, \dots, p_l\}$ with respect to the similarity E given by (39). Let moreover, the inverse $F^{\uparrow}$-transform $f^{\uparrow}_{F,n}$ of f be defined with respect to $A_1, \dots, A_n$ at the same nodes as f. Then

$$(\forall j) \quad (f(p_j) \leftrightarrow f^{\uparrow}_{F,n}(p_j)) \geq \bigvee_{i=1}^{n} A_i^2(p_j)). \tag{42}$$

PROOF: Let us fix some j, $1 \leq j \leq l$, and prove (42) for it. By (35),

$$f^{\uparrow}_{F,n}(p_j) \geq f(p_j)$$

and, therefore,

$$f(p_j) \leftrightarrow f^{\uparrow}_{F,n}(p_j) = f^{\uparrow}_{F,n}(p_j) \to f(p_j).$$

On the basis of the last equality, we will prove (42) in the simplified form:

$$f^{\uparrow}_{F,n}(p_j) \to f(p_j) \geq \bigvee_{k=1}^{n} A_k^2(p_j). \tag{43}$$

As an auxiliary inequality, we prove that for $i = 1, \dots, n$,

$$F_i^{\uparrow} \leq f(x^i) \tag{44}$$

where $F_i^{\uparrow}$ is the i-th component of the $F^{\uparrow}$-transform of f and x^i is the node which proves the normality of fuzzy set A_i. Indeed,

$$F_i^{\uparrow} = \bigvee_{s=1}^{l} (A_i(p_s) * f(p_s)) = \bigvee_{s=1}^{l} (E(x^i, p_s) * f(p_s))$$

and

$$E(x^i, p_s) \leq f(x^i) \leftrightarrow f(p_s) \leq f(p_s) \to f(x^i)$$

so that

$$E(x^i, p_s) * f(p_s) \leq f(x^i).$$

Therefore,

$$F_i^{\uparrow} = \bigvee_{s=1}^{l} (E(x^i, p_s) * f(p_s)) \geq f(x^i).$$

Here we used the fact that each fuzzy set A_i is a fuzzy point (40) with respect to the similarity E given by (39), and the function f is extensional on the set $\{p_1, \dots, p_l\}$ with respect to the same similarity.

With the help of (44) we may write the following chain of inequalities:

$$A_i^2(p_j) * f^{\uparrow}_{F,n}(p_j) =$$

$$A_i^2(p_j) * \bigwedge_{k=1}^{n}(A_k(p_j) \to F_k^{\uparrow}) \le A_i^2(p_j) * (A_i(p_j) \to F_i^{\uparrow}) \le$$
$$A_i(p_j) * \min((A_i(p_j), F_i^{\uparrow})) \le A_i(p_j) * \min(A_i(p_j), f(x^i)) \le$$
$$A_i(p_j) * f(x^i) \le f(p_j).$$

By the adjointness property,

$$A_i^2(p_j) \le f_{F,n}^{\uparrow}(p_j) \to f(p_j)$$

and moreover,

$$\bigvee_{i=1}^{n} A_i^2(p_j) \le f_{F,n}^{\uparrow}(p_j) \to f(p_j)$$

which coincides with (43). □

7.2 Approximation by the inverse $F^{\downarrow}$-transform

In this subsection, we are going to prove the second result about approximation by the inverse $F^{\downarrow}$-transform. We respect here all the assumptions given above. The proximity between f and the function $f_{F,n}^{\downarrow}$ approximating it will be expressed by $f(p_j) \leftrightarrow f_{F,n}^{\downarrow}(p_j)$ and estimated from below.

Theorem 11
Let normal fuzzy sets $A_1, \ldots, A_n$ establish a semi-partition of $[0,1]$ and a function f defined at nodes $p_1, \ldots, p_l$ be extensional on the set $\{p_1, \ldots, p_l\}$ with respect to the similarity E given by (39). Let moreover, the inverse $F^{\downarrow}$-transform $f_{F,n}^{\downarrow}$ of f be defined with respect to $A_1, \ldots, A_n$ at the same nodes as f. Then

$$(\forall j) \quad (f(p_j) \leftrightarrow f_{F,n}^{\downarrow}(p_j)) \ge \bigvee_{i=1}^{n} A_i^2(p_j)). \tag{45}$$

PROOF: Let us fix some j, $1 \le j \le l$, and prove (45) for it. By (37),

$$f_{F,n}^{\downarrow}(p_j) \le f(p_j)$$

and, therefore,

$$f(p_j) \leftrightarrow f_{F,n}^{\downarrow}(p_j) = f(p_j) \to f_{F,n}^{\downarrow}(p_j).$$

On the basis of the last equality, we will prove (45) in the simplified form:

$$f(p_j) \to f_{F,n}^{\downarrow}(p_j) \ge \bigvee_{k=1}^{n} A_k^2(p_j). \tag{46}$$

As an auxiliary inequality, we prove that for $i = 1, \ldots, n$,

$$F_i^{\downarrow} \ge f(x^i) \tag{47}$$

where $F_i^{\downarrow}$ is the i-th component of the F$^{\downarrow}$-transform of f and x^i is the node which proves the normality of fuzzy set A_i. Indeed,

$$F_i^{\downarrow} = \bigwedge_{s=1}^{l} (A_i(p_s) \to f(p_s)) = \bigwedge_{s=1}^{l} (E(x^i, p_s) \to f(p_s))$$

and

$$E(x^i, p_s) \le f(x^i) \leftrightarrow f(p_s) \le f(x^i) \to f(p_s)$$

so that

$$f(x^i) \le E(x^i, p_s) \to f(p_s).$$

Therefore,

$$F_i^{\downarrow} = \bigwedge_{s=1}^{l} (E(x^i, p_s) \to f(p_s)) \ge f(x^i).$$

Here we again used the assumptions that each fuzzy set A_i is a fuzzy point (40) with respect to the similarity E given by (39), and the function f is extensional on the set $\{p_1, \ldots, p_l\}$ with respect to the same similarity.

With the help of (47) we may write the following chain of inequalities:

$$\begin{aligned} A_i^2(p_j) \to f^{\downarrow}_{F,n}(p_j) = \\ A_i^2(p_j) \to \bigvee_{k=1}^{n} (A_k(p_j) * F_k^{\downarrow}) \ge A_i^2(p_j) \to (A_i(p_j) * F_i^{\downarrow}) \ge \\ A_i(p_j) \to F_i^{\downarrow} \ge A_i(p_j) \to f(x^i) \ge f(p_j). \end{aligned}$$

By the adjointness property,

$$A_i^2(p_j) \le f(p_j) \to f^{\downarrow}_{F,n}(p_j)$$

and moreover,

$$\bigvee_{i=1}^{n} A_i^2(p_j) \le f(p_j) \to f^{\downarrow}_{F,n}(p_j)$$

which coincides with (46). □

8 Comparison of Three F-Transforms

This section is aiming to compare three different F-transforms which have been introduced in this paper. For this purpose we will consider $[0, 1]$ as the common universe which enables us to use arithmetic operations as well as operations of a residuated lattice. Moreover, we will choose the Łukasiewicz MV-algebra

$$\mathcal{L}_{\text{Ł}} = \langle [0, 1], \vee, \wedge, \otimes, \to_{\text{Ł}}, 0, 1 \rangle$$

where

$$x \otimes y = \max(0, x + y - 1),$$
$$x \to_{\text{Ł}} y = \min(1, 1 - x + y).$$

as a basic residuated lattice. Note that $\mathcal{L}_{\text{Ł}}$ is also a special case of a BL-algebra on $[0, 1]$ which we extend by two further operations

$$x \oplus y = \min(1, x + y),$$
$$\neg x = 1 - x.$$

The following identity will be used in the sequel:

$$x \to_{\text{Ł}} b = \neg x \oplus y.$$

Let us consider a partition of $[0, 1]$ in the sense of Definition 1 which, moreover, fulfils the covering property (25). Then, for a given function $f : [0, 1] \longrightarrow [0, 1]$, we may construct all three different discrete F-transforms and also, the inverse F-transforms. We will see that components of the discrete F-transform are embraced by the respective component of the $\text{F}^{\uparrow}$-transform and the component of the $\text{F}^{\downarrow}$-transform. A similar property also holds for the three inverse F-transforms.

We say that a partition of $[0, 1]$ determined by the basic functions $A_1(x), \ldots, A_n(x)$, is *normalized* with respect to nodes $p_1, \ldots, p_l \in [0, 1]$ if for all $k = 1, \ldots, n$

$$\sum_{j=1}^{l} A_k(p_j) = 1.$$

Note that at given nodes, each partition can be easily transformed to the normalized one by dividing each basic function A_k by $\sum_{j=1}^{l} A_k(p_j)$.

Theorem 12
Let a function $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [0, 1]$ and $A_1(x), \ldots, A_n(x)$ be basic functions which form a normalized fuzzy partition of $[0, 1]$ with respect to the given nodes. Let moreover, Łukasiewicz algebra be chosen as the underlying residuated lattice on $[0, 1]$. Then the k-th components ($k = 1, \ldots, n$) of the discrete F-transform, $\text{F}^{\uparrow}$-transform and $\text{F}^{\downarrow}$-transform fulfil the inequalities

$$F_k^{\uparrow} \le F_k \le F_k^{\downarrow}.$$

PROOF: Let us fix some $k : 1 \le k \le n$, and denote $a_{kj} = A_k(p_j)$, $j = 1, \ldots, l$. Then we may write

$$F_k^{\uparrow} = \bigvee_{j=1}^{l} (a_{kj} \otimes f(p_j)), \quad F_k = \sum_{j=1}^{l} (a_{kj} \cdot f(p_j)), \quad F_k^{\downarrow} = \bigwedge_{j=1}^{l} (a_{kj} \to_{\text{Ł}} f(p_j)).$$

The following sequence of inequalities gives the proof to the left inequality $F_k^{\uparrow} \le F_k$:

$$\begin{aligned}
(a_{kj} \otimes f(p_j)) &\le (a_{kj} \cdot f(p_j)), \\
\bigvee_{j=1}^{l} (a_{kj} \otimes f(p_j)) &\le \bigvee_{j=1}^{l} (a_{kj} \cdot f(p_j)), \\
\bigvee_{j=1}^{l} (a_{kj} \cdot f(p_j)) &\le \sum_{j=1}^{l} (a_{kj} \cdot f(p_j)), \\
\bigvee_{j=1}^{l} (a_{kj} \otimes f(p_j)) &\le \sum_{j=1}^{l} (a_{kj} \cdot f(p_j)).
\end{aligned}$$

The following sequence of inequalities proves the right inequality $F_k \le F_k^{\downarrow}$:

$$\begin{aligned}
a_{kj} \cdot f(p_j) &\le a_{kj}, \\
a_{kj} \cdot f(p_j) &\le f(p_j), \\
\bigoplus_{\substack{j=1 \\ j \ne i}}^{l} (a_{kj} \cdot f(p_j)) &\le \bigoplus_{\substack{j=1 \\ j \ne i}}^{k} a_{kj}, \\
\bigoplus_{\substack{j=1 \\ j \ne i}}^{l} a_{kj} &= 1 - a_{ki} = \neg a_{ki}, \\
\bigoplus_{j=1}^{l} (a_{kj} \cdot f(p_j)) &\le \neg a_{ki} \oplus f(p_i), \\
\bigoplus_{j=1}^{l} (a_{kj} \cdot f(p_j)) &\le a_{ki} \rightarrow_{\text{L}} f(p_i), \\
\bigoplus_{j=1}^{l} (a_{kj} \cdot f(p_j)) &\le \bigwedge_{j=1}^{l} (a_{kj} \rightarrow_{\text{L}} f(p_j)), \\
\sum_{j=1}^{l} (a_{kj} \cdot f(p_j)) &\le \bigwedge_{j=1}^{l} (a_{kj} \rightarrow_{\text{L}} f(p_j)).
\end{aligned}$$

Here we used the following facts, valid under the normality assumption:

$$\begin{aligned}
\bigoplus_{j=1}^{l} a_{kj} &= \sum_{j=1}^{l} a_{kj} = 1, \\
\bigoplus_{j=1}^{l} (a_{kj} \cdot f(p_j)) &= \sum_{j=1}^{l} (a_{kj} \cdot f(p_j)).
\end{aligned}$$

□

The theorem given below shows that the inverse F-transform "lies in between" the two other inverse F-transforms, based on residuated lattice operations, which are proved to be a lower and an upper bound of the original function. This result is illustrated on Fig. 6.

Theorem 13
Let function $f(x)$ be defined at nodes $p_1, \ldots, p_l \in [0,1]$. Then for any $\varepsilon > 0$ there exist n_ε and a fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$ of $[0,1]$ such that for all $p_j, j = 1, \ldots, l$,

$$f^{\downarrow}_{F,n_\varepsilon}(p_j) - \varepsilon \le f_{F,n_\varepsilon}(p_j) \le f^{\uparrow}_{F,n_\varepsilon}(p_j) + \varepsilon \tag{48}$$

where the inverse F-transforms are computed with respect to the partition $A_1, \ldots, A_{n_\varepsilon}$ and the chosen nodes.

PROOF: Let us choose some $\varepsilon > 0$ and find a fuzzy partition of $A_1, \ldots, A_{n_\varepsilon}$ of $[0,1]$ such that for all $p_j, j = 1, \ldots, l$,

$$|f(p_j) - f_{F,n_\varepsilon}(p_j)| < \varepsilon$$

or

$$f(p_j) - \varepsilon \le f_{F,n_\varepsilon}(p_j) \le f(p_j) + \varepsilon.$$

The existence of such partition is guaranteed by Theorem 5. Furthermore, for this partition we construct the inverse F$^{\uparrow}$-transform $f^{\uparrow}_{F,n}$ and the inverse F$^{\downarrow}$-transform $f^{\downarrow}_{F,n}$ of f. Taking into account the inequalities $f^{\uparrow}_{F,n}(p_j) \ge f(p_j)$ (cf. (35)) and $f^{\downarrow}_{F,n}(p_j) \le f(p_j)$ (cf. 37)) we easily deduce (48). □

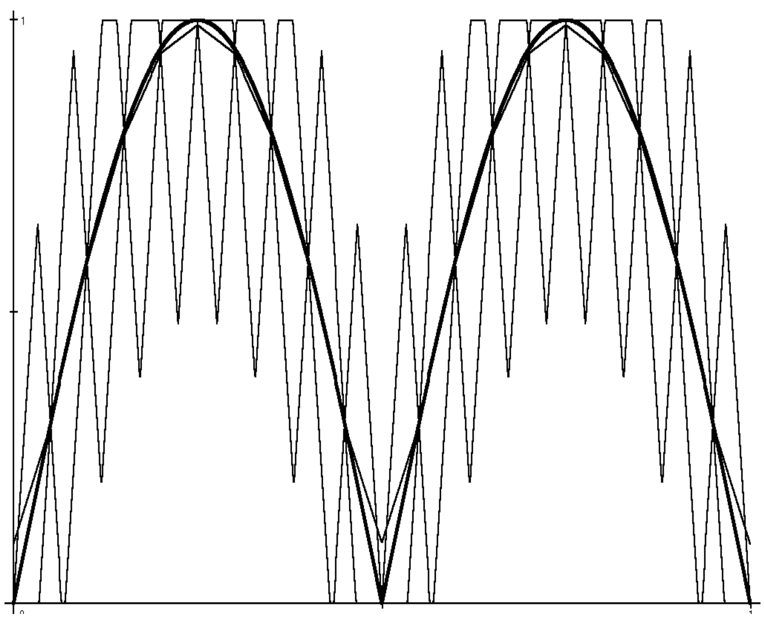


Fig. 6. Function $|\sin 2\pi x|$ and its three inverse F-transforms. The one, based on the arithmetic operations, is close to the original functions and lies in between the two other inverse F-transforms, based on residuated lattice operations.

The direct and inverse F-transforms of a function of two and more variables can be introduced as a direct generalization of the case of one variable. We will do it briefly and refer to [15] for more details.

Suppose that the universe is a rectangle $[a, b] \times [c, d]$ and $x_1 < \ldots < x_n$ are fixed nodes from $[a, b]$ and $y_1 < \ldots < y_m$ are fixed nodes from $[c, d]$, such that $x_1 = a$, $x_n = b$, $y_1 = c$, $x_m = d$ and $n, m \geq 2$. Let us formally extend the set of nodes by $x_0 = a$, $y_0 = c$ and $x_{n+1} = b$, $y_{m+1} = d$. Assume that $A_1(x), \ldots, A_n(x)$ are basic functions which form a fuzzy partition of $[a, b]$ and $B_1(x), \ldots, B_m(x)$ are basic functions which form a fuzzy partition of $[c, d]$.

Let $L_2([a, b] \times [c, d], A_1, \ldots, A_n, B_1, \ldots, B_m)$ be the set of functions of two variables $f(x, y)$ which are weighted square integrable on the domain $[a, b] \times [c, d]$ with the weights given by all combinations $A_k(x)B_l(y)$, $k = 1, \ldots, n$, $l = 1, \ldots, m$.

Definition 11
Let $A_1, \ldots, A_n(x)$ be basic functions which form a fuzzy partition of $[a, b]$ and $B_1(x), \ldots, B_m(x)$ be basic functions which form a fuzzy partition of $[c, d]$. Let $f(x, y)$ be any function from $L_2([a, b] \times [c, d], A_1, \ldots, A_n, B_1, \ldots, B_m)$. We say that the $n \times m$-matrix of real numbers $\mathbf{F}_{nm}[f] = (F_{kl})$ is the (integral) F-transform of f with respect to $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ if for each $k = 1, \ldots, n$, $l = 1, \ldots, m$,

$$F_{kl} = \frac{\int_c^d \int_a^b f(x, y)A_k(x)B_l(y)dxdy}{\int_c^d \int_a^b A_k(x)B_l(y)dxdy}. \tag{49}$$

In the discrete case, when an original function $f(x, y)$ is known only at some nodes $(p_i, q_j) \in [a, b] \times [c, d]$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, the (discrete) F-transform of f can be introduced analogously to the case of a function of one variable.

Definition 12
Let a function $f(x, y)$ be given at nodes $(p_i, q_j) \in [a, b] \times [c, d]$, $i = 1, \ldots, N$, $j = 1, \ldots, M$, and $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ where $n \ll N$, $m \ll M$, be basic functions which form fuzzy partitions of $[a, b]$ and $[c, d]$ respectively. We say that the $n \times m$-matrix of real numbers $\mathbf{F}_{nm}[f] = (F_{kl})$ is the discrete F-transform of f with respect to $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ if

$$F_{kl} = \frac{\sum_{j=1}^{M} \sum_{i=1}^{N} f(p_i, q_j)A_k(p_i)B_l(q_j)}{\sum_{j=1}^{M} \sum_{i=1}^{N} A_k(p_i)B_l(q_j)}. \tag{50}$$

holds for all $k = 1, \ldots, n$, $l = 1, \ldots, m$.

As in the case of functions of one variable, the elements F_{kl}, $k = 1, \ldots, n$, $l = 1, \ldots, m$, are called *components of the F-transform.*

If the partitions of $[a, b]$ and $[c, d]$ by $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ are uniform then the expression (49) for the components of the F-transform may be simplified on the basis of expressions which can be easily obtained from Lemma 1:

$$\begin{aligned}
F_{11} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_1(y) dx dy, \\
F_{1m} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_m(y) dx dy, \\
F_{n1} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_1(y) dx dy, \\
F_{nm} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_m(y) dx dy,
\end{aligned}$$

and for $k = 2, ..., n-1$ and $l = 2, ..., m-1$

$$\begin{aligned}
F_{k1} &= \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_1(y) dx dy, \\
F_{km} &= \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_m(y) dx dy, \\
F_{1l} &= \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_l(y) dx dy, \\
F_{nl} &= \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_l(y) dx dy, \\
F_{kl} &= \frac{1}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_l(y) dx dy.
\end{aligned}$$

Remark 4
The expression (49) can be rewritten with the help of a repeated integral

$$F_{kl} = \frac{\int_c^d B_l(y) dy \int_a^b f(x,y) A_k(x) dx}{\int_c^d B_l(y) dy \int_a^b A_k(x) dx}.$$

On the basis of this expression, all the properties (linearity etc.) which have been proved for the F-transform of a function of one variable (see subsection 3.2) can be easily generalized and proved for the considered case too.

Definition 13
Let $A_1, \ldots, A_n(x)$ and $B_1(x), \ldots, B_m(x)$ be basic functions which form fuzzy partitions of $[a, b]$ and $[c, d]$ respectively. Let $f(x, y)$ be a function from $L_2([a, b] \times [c, d], A_1, \ldots, A_n, B_1, \ldots, B_m)$ and $\mathbf{F}_{nm}[f]$ be the F-transform of f with respect to $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. Then the function

$$f^F_{nm}(x,y) = \sum_{k=1}^{n} \sum_{l=1}^{m} F_{kl} A_k(x) B_l(y) \tag{51}$$

is called the *the inverse F-transform*.

Similarly to the case of a function of one variable we can prove that the inverse F-transform $f^F_{n,m}$ can approximate the original continuous function f with an arbitrary precision.

Theorem 14
Let $f(x, y)$ be any continuous function on $[a, b] \times [c, d]$. Then for any $\varepsilon > 0$ there exist n_ε, m_ε and fuzzy partitions $A_1, \dots, A_{n_\varepsilon}$ and $B_1, \dots, B_{m_\varepsilon}$ of $[a, b]$ and $[c, d]$ respectively, such that for all $(x, y) \in [a, b] \times [c, d]$

$$|f(x, y) - f^F_{n_\varepsilon m_\varepsilon}(x, y)| \leq \varepsilon. \tag{52}$$

The function $f^F_{n_\varepsilon m_\varepsilon}$ in (52) is the inverse F-transform of f with respect to $A_1, \dots, A_{n_\varepsilon}$ and $B_1, \dots, B_{m_\varepsilon}$.

Remark 5
We can analogously generalize the F$^\uparrow$-transform and the F$^\downarrow$-transform to the case of a function of two and more variables.

10 Application of the F-transform to Image Compression and Reconstruction

A method of lossy image compression and reconstruction on the basis of fuzzy relations has been proposed in a number of papers (see e.g. [6, 8]). Let us briefly formulate this problem and show that the technique of the F-transform can be successfully applied. Moreover, we will compare a method proposed here with the known methods proposed in the papers cited above and prove the advantage of our proposal.

Let an image I of the size $N \times M$ pixels be represented by a function of two variables (a fuzzy relation) $f_I : \mathbb{N} \times \mathbb{N} \longrightarrow [0, 1]$ partially defined at nodes $(i, j) \in [1, N] \times [1, M]$. The value $f_I(i, j)$ represents an intensity range of each pixel. We propose to compress this image with the help of the discrete F-transform of a function of two variables (50) by the $n \times m$-matrix of real numbers $\mathbf{F}_{nm}[f_I] = (F_{kl})$ where

$$F_{kl} = \frac{\sum_{j=1}^{M} \sum_{i=1}^{N} f_I(i, j) A_k(i) B_l(j)}{\sum_{j=1}^{M} \sum_{i=1}^{N} A_k(i) B_l(j)}$$

and $A_1, \dots, A_n$, $B_1, \dots, B_m$ where $n \ll N$, $m \ll M$, be basic functions which form fuzzy partitions of $[1, N]$ and $[1, M]$, respectively.

A reconstruction of the image f_I, being compressed by $\mathbf{F}_{nm}[f_I] = (F_{kl})$ with respect to $A_1, \dots, A_n$ and $B_1, \dots, B_m$, is given by the inverse F-transform (51) adapted to the domain $[1, N] \times [1, M]$:

$$f^F_{nm}(i, j) = \sum_{k=1}^{n} \sum_{l=1}^{m} F_{kl} A_k(i) B_l(j).$$

On the basis of Theorem 14, we are convinced that the reconstructed image is close to the original one and moreover, it can be obtained with a prescribed level of accuracy. Let us illustrate the proposed method on Fig. 7.

To show the advantage of this method over other ones proposed in [6, 8], let us remark that the latter are based precisely on our concept of F$^\uparrow$-transform applied to a function of

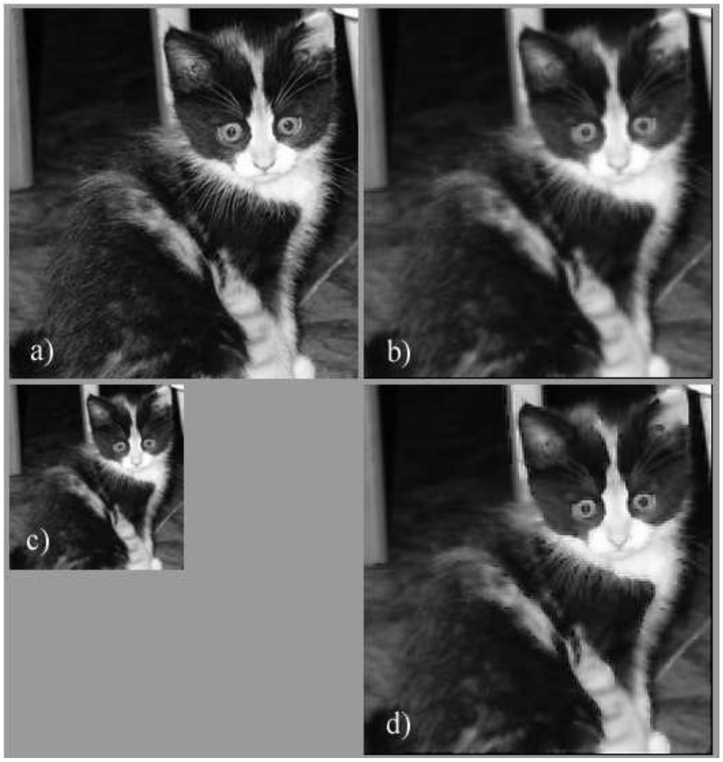


Fig. 7. The image a) is compressed with the ratio 0.25 and the compressed image is shown on c). The reconstructed image is shown on b) and the reconstructed image sharpened by a simple algorithm is shown on d).

two variables. Therefore, on the basis of Theorem 12 and Theorem 13, we may state that the compression which is based on the F-transform, leads to a better approximation of an original image than the compressions based either on the $F^{\uparrow}$-transform, or on the $F^{\downarrow}$-transform. Moreover, the computational complexity of the F-transform based compression is lower.

We will not go into further details in this paper since a detailed paper on the compression methods is in preparation. Let us only illustrate our conclusion by an example of the same image as in Fig. 7 which has been compressed and reconstructed with the help of the $F^{\uparrow}$-transform, based on Łukasiewicz algebra (cf. [6]).

11 Conclusion

We have introduced a new technique of direct and inverse fuzzy transforms (F-transforms for short) which enables us to construct various approximating models depending on the choice of basic functions. The best approximation property of the inverse F-transform is established within the respective approximating space.

Two fuzzy transforms are built with the help of operations of a residuated lattice on $[0, 1]$. They lead to new approximation models which are expressed with the help of weaker

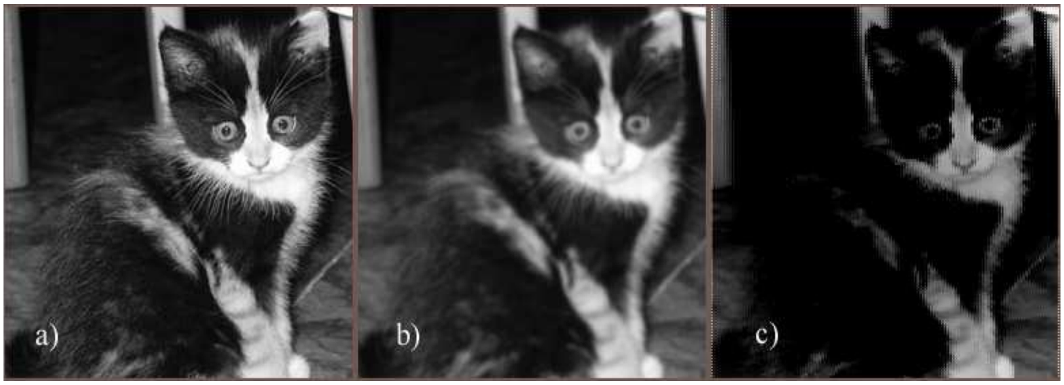


Fig. 8. The image a) is compressed with the ratio 0.25 and the reconstructed images are shown on b) (by the F-transform) and d) (by the $F^{\uparrow}$-transform).

operations than the arithmetic ones used in the case of the first fuzzy transform.

Different fuzzy transforms are compared and the central position of the fuzzy transform based on the arithmetic operations is proved. Fuzzy transforms of functions with two and more variables are introduced as a direct generalization of the fuzzy transform of functions of one variable.

A method of lossy image compression and reconstruction on the basis of fuzzy transforms has been proposed and its advantage over the similar method based on the $F^{\uparrow}$-transform is discussed.

Finally, let us remark that the fuzzy transforms are related to the concept of fuzzy rough sets [3] since both are based on special fuzzy partitions of a universe of discourse. We suppose to elaborate this aspect of fuzzy transforms in the forthcoming papers.

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