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Abstract

In this paper, we turn our attention to model theory of higher-order fuzzy logic (fuzzy type theory). This theory generalizes model theory of predicate logic but has some interesting specificities. We will introduce few basic concepts related to homomorphism, isomorphism, submodel, etc. and show some properties of them.

Keywords: fuzzy type theory; model theory; EQ-algebra

1. Introduction

Higher-order logic, usually called *type theory* (TT), has irreplaceable role in logic and has also many kinds of applications. It was introduced by B. Russell in [27] and since then developed in various directions by many authors (see [28] and citations therein).

The position of type theory in logic was very nicely characterized by Farmer in the paper [10] where he gave a detailed analysis of the following virtues: (1) type theory has a simple and highly uniform syntax, (2) its semantics is based on a small collection of well-established ideas, (3) it is a highly expressive logic, (4) it admits categorical theories of infinite structures, (5) there is a simple, elegant, and powerful proof system for TT, (6) techniques of first-order model theory can be applied to TT so that distinction between standard and nonstandard models is illuminated and, finally, (7) there are practical extensions of TT that can be effectively implemented. Various kinds of applications of type theory range from logical analysis of linguistic semantics to theoretical computer science. It should be noted from this list that type theory has much higher expressive power than first-order logic and its model theory has potential to give better understanding to various mathematical concepts (such as the mentioned distinction between standard and non-standard models, or infinite models of categorical theories).

In eighties and nineties, propositional and first-order fuzzy logics were successfully established (see, e.g., [5, 6, 11, 18, 25, 26] and elsewhere) as constituents of mathematical fuzzy logic. The picture was completed by developing higher-order fuzzy logic called *fuzzy type theory* (FTT) in [20] which naturally generalizes the classical type theory by replacing the two-valued boolean algebra of truth values for classical logic by a proper many-valued algebra. As a first step, the IMTL-structure of truth values was considered but later on, also other versions of FTT have been established [21, 22, 23]. This suggests and idea that each first-order fuzzy logic can be extended also to its higher-order version.

The crucial notion in TT as well as in FTT is that of *type*, usually denoted by greek letters $\alpha, \beta, \ldots$, which can be seen as an index used to denote a set of elements of certain kind, namely truth values, individuals, or functions. Then, semantics of type theory consists of a frame and interpretation of formulas in it. The frame is a set $(M_\alpha)_{\alpha \in Types}$ of sets constructed iteratively starting with the basic sets of truth values M_o and individuals M_ϵ and then each higher-order set M_α for $\alpha \neq o, \epsilon$ consists of functions between two lower order (and so, already constructed) sets. If all higher-order sets contain all possible functions then the model is *standard*, otherwise it is *general*.

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As has been proved, type theory is not complete with respect to standard models. However, Henkin proved in [13, 14] that it is complete with respect to general models. Referring to virtue (6) above, Andrews demonstrated in [1] that this is, in fact, quite natural since requiring standard completeness is tantamount to requiring that, e.g., all models of arithmetics must be standard which can hardly be defended.

Recall the other mentioned result about categorical theories. This requires clear concept of isomorphism of models. But the latter is not as straightforward as in first-order predicate logics. Namely, model theory of higher-order logics has certain specificities and difficulties which must be carefully carried off.

For these reasons, we think that developing model theory of fuzzy type theory will bring better understanding to the latter and will also shed other light on classical (as well as fuzzy) first-order logic and its model theory. Therefore, we will turn our attention to model theory of FTT in this paper. Though it is a generalization of the classical model theory of first-order logic (see [4, 15, 16]) and also of model theory of fuzzy logics [7, 8, 17, 19, 12], it has some specificities which must be respected and it seems to be richer than the former.

This paper should be considered as introductory to the topic. We will introduce the concepts of homomorphism and isomorphism of models, submodel, elementary equivalence and elementary submodel and few other variants of basic concepts. A more advanced theory will be the topic of future paper.

In the following section, we first briefly overview the main concepts and properties of fuzzy type theory. The considered FTT is based on the EQ-algebra (cf. [23, 24]) which is the most general structure of truth values suitable for FTT. Let us emphasize that each residuated lattice is also an EQ-algebra. Section 3 is the main part of the paper introducing the mentioned concepts and some of their properties. Of course, the concepts introduced here are valid also for the special cases of FTT, in which specific kinds of structures of truth values are taken into account.

2. Fuzzy type theory

In this section we will overview few main concepts of FTT. The details including precise definitions and proofs of all theorems can be found in [23] and also in [20].

2.1. Algebra of truth values

Truth values for FTT form a good, non-commutative, bounded, linearly ordered EQ-algebra with Δ operation, namely the algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}, \mathbf{0}, \Delta \rangle \tag{1}$$

of type (2, 2, 2, 0, 0, 1) fulfilling the following axioms for all $a, b, c \in E$:

- (E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a commutative idempotent monoid ($\wedge$-semilattice with the top element $\mathbf{1}$). We put $a \le b$ iff $a \wedge b = a$, as usual.
- (E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid such that $\otimes$ is isotone w.r.t. $\le$.
- (E3) $a \sim a = \mathbf{1}$, (reflexivity)
- (E4) $((a \wedge b) \sim c) \otimes (d \sim a) \le c \sim (d \wedge b)$, (substitution)
- (E5) $(a \sim b) \otimes (c \sim d) \le (a \sim c) \sim (b \sim d)$, (congruence)
- (E6) $(a \wedge b \wedge c) \sim a \le (a \wedge b) \sim a$, (isotonicity of implication)
- (E7) $a \otimes b \le a \sim b$, (boundedness)
- (E8) $a \sim \mathbf{1} = a$. (goodness)
- (E9) $\mathbf{0}$ is the bottom element and

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The *implication* operation $\to$ is defined by

$$a \to b = (a \wedge b) \sim a.$$

Many properties of EQ-algebras can be found in [9, 24].

Note that EQ-algebras are slightly more general than residuated lattices considered in fuzzy logics up to now. Namely, each residuated lattice $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ is an EQ-algebra where $a \sim b = (a \to b) \wedge (b \to a)$ but not vice-versa.

2.2. Syntax of FTT

2.2.1. Types and formulas

Basic syntactical objects of FTT are classical — see [2], namely *type* and *formula*. The atomic types are ϵ (elements) and o (truth values). General types are denoted by Greek letters $\alpha, \beta, \ldots$. The set of all types is denoted by *Types*.

The *language* of FTT, denoted by J, consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in \mathit{Types}$), the symbol λ, and brackets. We will consider the following concrete special constants: $\mathbf{E}_{(oo)o}, \mathbf{E}_{(o\alpha)\alpha}$, $\alpha \in \mathit{Types}$ (fuzzy equality), $\mathbf{G}_{o\epsilon}$ (generating function), $\mathbf{C}_{(oo)o}$ (conjunction), $\mathbf{S}_{(oo)o}$ (strong conjunction) $\mathbf{D}_{(oo)}$ (delta operation on truth values), and $\iota_{\epsilon(o\epsilon)}, \iota_{o(oo)}$ (description operators).

Formulas[1] are formed of variables and constants of various types, and the symbol λ. Each formula A is thus assigned a type and we write A_α.[2] A set of formulas of type α is denoted by Form_α. The set of all formulas is $\mathit{Form} = \bigcup_{\alpha \in \mathit{Types}} \mathit{Form}_\alpha$.

A variable x_α is *bound* in a formula A_δ if the latter has a well formed part $(\lambda x_\alpha B_\beta)$. Otherwise x_α is *free*. A formula A is *closed* if it does not contain free variables. A closed formula A_o of type o is called a *sentence*.

Recall that if $B \in \mathit{Form}_{\beta\alpha}$ and $A \in \mathit{Form}_\alpha$ then $(BA) \in \mathit{Form}_\beta$. Similarly, if $A \in \mathit{Form}_\beta$ and $x_\alpha \in J$, $\alpha \in \mathit{Types}$, is a variable then $(\lambda x_\alpha\, A) \in \mathit{Form}_{\beta\alpha}$. The set of all formulas is $\mathit{Form} = \bigcup_{\alpha \in \mathit{Types}} \mathit{Form}_\alpha$.

The following special formulas are defined:

(i) *Fuzzy equality:*

(a) $\equiv_{(oo)o} \;:=\; \lambda x_o\, \lambda y_o\, (\mathbf{E}_{(oo)o}\, y_o) x_o$,

(b) $\equiv_{(o\epsilon)\epsilon} \;:=\; \lambda x_\epsilon\, \lambda y_\epsilon\, (\mathbf{E}_{(oo)o}\, (\mathbf{G}_{o\epsilon} y_\epsilon))(\mathbf{G}_{o\epsilon} x_\epsilon)$,

(c) $\equiv_{(o(\beta\alpha))(\beta\alpha)} \;:=\; \lambda f_{\beta\alpha}\, \lambda g_{\beta\alpha}\, (\mathbf{E}_{(o(\beta\alpha))(\beta\alpha)}\, g_{\beta\alpha}) f_{\beta\alpha}$.

(ii) Representation of *truth* and *falsity*:

$$\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o), \qquad \bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top).$$

(iii) *Implication:* $\Rightarrow \;:=\; \lambda x_o \lambda y_o\, (x_o \wedge y_o) \equiv x_o$

(iv) *Negation:* $\neg := \lambda x_o (x_o \equiv \bot)$.

(v) *Strong conjunction* $\,\&\; := \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o}\, y_o) x_o$.

(vi) *Conjunction:* $\wedge := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o}\, y_o) x_o$.

(vii) *Disjunction:* $\vee := \lambda x_o \lambda y_o ((x_o \Rightarrow y_o) \Rightarrow y_o) \wedge ((y_o \Rightarrow x_o) \Rightarrow x_o)$.

(viii) *Delta connective:* $\boldsymbol{\Delta} := \lambda x_o \mathbf{D}_{oo} x_o$.

[1] In the up-to-date type theory employed in the theoretical computer science, "formulas" are often called "lambda-terms". We prefer the former in this paper because FTT is logic and so, it is more natural to use the term "formula" in it.

[2] The type is uniquely tied with the given formula and this is followed also in the notation. Thus, if α, β are different types then we understand the formulas A_α and A_β to be also different.

(ix) Quantifiers:

$$(\forall x_\alpha)A_o := (\lambda x_\alpha\, A_o \equiv \lambda x_\alpha\, \top),$$
$$(\exists x_\alpha)A_o := (\forall y_o)((\forall x_\alpha)\boldsymbol{\Delta}(A_o \Rightarrow y_o) \Rightarrow y_o).$$

As usual, we will write $x_\alpha \equiv y_\alpha$ instead of $(\equiv y_\alpha)x_\alpha$ and similarly for the other formulas defined above. Clearly, $(x_\alpha \equiv y_\alpha) \in \mathit{Form}_o$ for arbitrary type $\alpha \in \mathit{Types}$ and, therefore, we will usually omit the type at the formula (fuzzy equality) $\equiv$. Note that if $\alpha = o$ then $\equiv$ is the logical equivalence. Furthermore, the n-times strong conjunction of A_o is denoted by A_o^n and n-times strong disjunction (denoted by nA_o). To minimize the number of brackets, we sometimes write a dot before a subformula. This replaces the left bracket and the right bracket at the end of the given subformula is omitted.

2.2.2. Axioms and inference rules

Fundamental axioms.

(FT-fund1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$,

(FT-fund2) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$,

(FT-fund3) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$,

(FT-fund4) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$ (*lambda conversion*)

Axioms of truth values. Let $\bigcirc \in \{\wedge, \mathbin{\&}\}$.

(FT-tval1) $(x_o \wedge y_o) \equiv (y_o \wedge x_o)$,

(FT-tval2) $(x_o \bigcirc y_o) \bigcirc z_o \equiv x_o \bigcirc (y_o \bigcirc z_o)$,

(FT-tval3) $(x_o \equiv \top) \equiv x_o$,

(FT-tval4a) $(x_o \bigcirc \top) \equiv x_o$,

(FT-tval4b) $(\top \mathbin{\&} x_o) \equiv x_o$,

(FT-tval5) $(x_o \wedge x_o) \equiv x_o$,

(FT-tval6) $((x_o \wedge y_o) \equiv z_o) \mathbin{\&} (t_o \equiv x_o) \Rightarrow (z_o \equiv (t_o \wedge y_o))$,

(FT-tval7) $(x_o \equiv y_o) \mathbin{\&} (z_o \equiv t_o) \Rightarrow (x_o \equiv z_o) \equiv (y_o \equiv t_o)$

(FT-tval8) $(x_o \Rightarrow (y_o \wedge z_o)) \Rightarrow (x_o \Rightarrow y_o)$

(FT-tval10a) $\boldsymbol{\Delta}(x_o \Rightarrow y_o) \Rightarrow (x_o \mathbin{\&} z_o \Rightarrow y_o \mathbin{\&} z_o)$

(FT-tval10b) $\boldsymbol{\Delta}(x_o \Rightarrow y_o) \Rightarrow (z_o \mathbin{\&} x_o \Rightarrow z_o \mathbin{\&} x_o)$

(FT-tval10) $((x_o \Rightarrow y_o) \Rightarrow z_o) \Rightarrow ((y_o \Rightarrow x_o) \Rightarrow z_o) \Rightarrow z_o$

Axioms of delta.

(FT-delta1) $(g_{oo}(\boldsymbol{\Delta} x_o) \wedge g_{oo}(\neg\boldsymbol{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\boldsymbol{\Delta} y_o)$

(FT-delta2) $\boldsymbol{\Delta}(x_o \wedge y_o) \equiv \boldsymbol{\Delta} x_o \wedge \boldsymbol{\Delta} y_o$

(FT-delta3) $\boldsymbol{\Delta}(x_o \vee y_o) \Rightarrow \boldsymbol{\Delta} x_o \vee \boldsymbol{\Delta} y_o$

(FT-delta4) $\boldsymbol{\Delta} x_o \vee \neg\boldsymbol{\Delta} x_o$

Axioms of quantifiers.

(FT-quant1) $\mathbf{\Delta}(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$, x_α is not free in A_o

(FT-quant2) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha)A_o \Rightarrow B_o)$, x_α is not free in B_o

(FT-quant3) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha)A_o \vee B_o)$, x_α is not free in B_o

Axioms of descriptions.

(FT-descr1) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha$, $\alpha = o, \epsilon$

Inference rules.

(R) Infer B' from $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$ which comes from B by replacing one occurrence of A_α by A'_α (provided that A_α in B is not an occurrence of a variable immediately preceded by λ).

(N) Infer $\mathbf{\Delta} A_o$ from A_o.

The inference rules of modus ponens and generalization are derived rules in FTT. The concepts of *provability* and *proof* are defined in the same way as in classical logic. A *theory* T over FTT is a set of formulas of type o (i.e. $T \subset Form_o$). By $J(T)$ we denote the language of the theory T. By $T \vdash A_o$ we mean that A_o is provable in T, i.e. that there exists a proof of A_o from axioms of T.

2.3. Semantics of FTT

2.3.1. Fuzzy equality

The most important connective in FTT is the fuzzy equality connective $\equiv$ which is interpreted by a fuzzy equality on various kinds of sets.

Definition 1
Let $\mathcal{E}$ be an EQ-algebra of truth values and M be a set. A fuzzy equality $\doteq$ on M is a binary fuzzy relation on M $\doteq: M \times M \longrightarrow E$ such that the following holds for all $m, m', m'' \in M$:

(i) $\doteq(m, m) = \mathbf{1}$, *(reflexivity)*

(ii) $\doteq(m, m') = \doteq(m', m)$, *(symmetry)*

(iii) $\doteq(m, m') \otimes \doteq(m', m'') \leq \doteq(m, m'')$. *($\otimes$-transitivity)*

If $m, m' \in M$ then we will usually write $[m \doteq m']$ instead of $\doteq(m, m')$. We say that $\doteq$ is *separated*, provided that $[m \doteq m'] = \mathbf{1}$ iff $m = m'$ holds for all $m, m' \in M$.

A *fuzzy equality on truth values* is the operation $\sim$ from the EQ-algebra $\mathcal{E}$. This fuzzy equality is separated.

If M is a set of (arbitrary) individuals then it seems natural to generate the fuzzy equality $\doteq$ using $\sim$ and a function $G : M \longrightarrow E$ by

$$[m \doteq m'] = G(m) \sim G(m'), \qquad m, m' \in M. \tag{2}$$

The function G is called *generating function* for $\doteq$ and it was considered in [23]. The generated fuzzy equality $\doteq$ in (2) is separated iff the generating function G is an injection. Since we need the fuzzy equality to be separated, the latter property may not be desirable because if G is an injection then the cardinality of M must not exceed the cardinality of E. Therefore, we may sometimes omit the generating function from our considerations and introduce the fuzzy equality $\doteq$ on M explicitly.

If $M \subseteq M_\beta^{M_\alpha}$, i.e. objects of M are functions $M_\alpha \longrightarrow M_\beta$ endowed with the corresponding fuzzy equalities $\doteq_\alpha, \doteq_\beta$ then we introduce the fuzzy equality $\doteq: M \times M \longrightarrow E$ by

$$[h \doteq h'] = \bigwedge_{m \in M_\alpha} [h(m) \doteq_\beta h'(m)], \qquad h, h' \in M \tag{3}$$

where $\doteq_\beta$ is a fuzzy equality on a set M_β. Furthermore, a function $f : M_\alpha \longrightarrow M_\beta$ is *weakly extensional* if for all $m, m' \in M_\alpha$, $[m \doteq_\alpha m'] = \mathbf{1}$ implies $[f(m) \doteq_\beta f(m')] = \mathbf{1}$.

A triple $(\mathcal{E}, M, \doteq)$ where $\mathcal{E}$ is a good EQ-algebra, M a set and $\doteq: M \times M \longrightarrow E$ is a fuzzy equality is called a *set with fuzzy equality.* If $\mathcal{E}$ is known, then we may write only $(M, \doteq)$.

2.3.2. Frame

The above defined fuzzy equalities are used in the definition of interpretation of formulas of fuzzy type theory.

Definition 2
Let J be a language of FTT and $(M_\alpha)_{\alpha\in Types}$ be a system of sets called basic frame *such that M_o, M_ϵ are sets and for each $\alpha, \beta \in$ Types, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$, i.e. it is a set of weakly extensional functions*[3] *from M_α to M_β. The* frame *is a tuple*

$$\mathcal{M} = \langle (M_\alpha, \mathring{=}_\alpha)_{\alpha\in Types}, \mathcal{E}, G\rangle \tag{4}$$

such that the following holds:

(i) The $\mathcal{E}$ is a structure of truth values which is an EQ_Δ-algebra. We put $M_o = E$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{E}$. The fuzzy equality on truth values is $\mathring{=}_o := \sim$.

(ii) The set M_ϵ is the set of individuals. *The fuzzy equality $\mathring{=}_\epsilon$ on individuals is defined in either of the two ways:*

  (a) by formula (2) using the generating function $G \in M_o^{M_\epsilon}$;

  (b) explicitly as a separated fuzzy equality on individuals M_ϵ. In this case, the generating function G is omitted from (4) and the symbol **G** *is omitted from the language J.*

(iii) If $\alpha \neq o, \epsilon$ then $\mathring{=}_\alpha$ is a fuzzy equality on M_α defined by (3). We assume that $\mathring{=}_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in$ Types.

2.3.3. Interpretation of formulas

To define interpretation of formulas in a frame $\mathcal{M}$, we must consider an assignment p of elements from $\mathcal{M}$ to variables. Namely, p is a function from the set of all variables of the language J to elements from $\mathcal{M}$ in keeping with the corresponding types. Given an assignment p, we define a new assignment $p' = p\backslash x_\alpha$ which equals to p for all variables except for x_α. The set of all assignments over $\mathcal{M}$ is denoted by $\mathrm{Asg}(\mathcal{M})$.

Given a language J of FTT. Interpretation of formulas in a general frame $\mathcal{M}$ is a function $\mathcal{I}$ that assigns to every formula A_α, $\alpha \in$ *Types*, and to every assignment $p \in \mathrm{Asg}(\mathcal{M})$ a corresponding element from the set M_α. Interpretation of a formula A_α in a frame $\mathcal{M}$ under the assignment p is written as $\mathcal{M}_p(A_\alpha)$.

Let us remark that in [20], the interpretation was written as $\mathcal{I}_p^{\mathcal{M}}(A_\alpha)$. We can also use the notation $\|A_\alpha\|_p^{\mathcal{M}}$ adopted from [11]. Our notation minimizes overburden of symbols by subscripts. The definition of interpretation $\mathcal{I}$ is recursive:

Interpretation of constants. These are interpreted as follows (the assignment p plays no role): $\mathcal{M}(\mathbf{E}_{(oo)o}) := \sim$, $\mathcal{M}(\mathbf{E}_{(o(\beta\alpha))(\beta\alpha)}) := \mathring{=}_{\beta\alpha}$, $\mathcal{M}(\mathbf{G}_{o\epsilon}) := G$ (if G is considered), $\mathcal{M}(\mathbf{C}_{(oo)o}) := \wedge$, $\mathcal{M}(\mathbf{S}_{(oo)o}) := \otimes$, $\mathcal{M}(\mathbf{D}_{oo}) := \Delta$.

The basic description operators $\iota_{\epsilon(o\epsilon)}$ and $\iota_{o(oo)}$ are interpreted by functions $\mathcal{M}(\iota_{\epsilon(o\epsilon)}) : M_o^{M_\epsilon} \longrightarrow M_\epsilon$ and $\mathcal{M}_p(\iota_{o(oo)}) : M_o^{M_o} \longrightarrow M_o$ assigning to each non-empty normal[4] fuzzy set from $M_o^{M_\epsilon}$ or from $M_o^{M_o}$, respectively, an element from its kernel. These functions are undefined for subnormal fuzzy sets.

Interpretation of variables. Given an assignment $p \in \mathrm{Asg}(\mathcal{M})$ and a variable x_α, we define $\mathcal{M}_p(x_\alpha) = p(x_\alpha)$.

[3] By currying, we may confine only to unary functions.

[4] A fuzzy set $A : M \longrightarrow E$ is normal if $A(m) = \mathbf{1}$ for some $m \in M$. Otherwise it is subnormal. The set $\{m \mid A(m) = \mathbf{1}, m \in M\}$ is a *kernel* of the fuzzy set A.

Interpretation of complex formulas.

$$\mathcal{M}_p(B_{\beta\alpha}A_\alpha) = \mathcal{M}_p(B_{\beta\alpha})(\mathcal{M}_p(A_\alpha)),$$
$$\mathcal{M}_p(\lambda x_\alpha\, A_\beta) = F : M_\alpha \longrightarrow M_\beta$$

where F is a weakly extensional function assigning to each $m_\alpha \in M_\alpha$ an element $F(m_\alpha) = \mathcal{M}_{p'}(A_\beta)$ for an assignment $p' = p\backslash x_\alpha$ such that $p'(x_\alpha) = m_\alpha$.

Note that by (3), we obtain

$$\mathcal{M}_p(A_{\beta\alpha} \equiv B_{\beta\alpha}) = \bigwedge\{\mathcal{M}_p(A_{\beta\alpha})(\mathcal{M}_{p'}(x_\alpha)) \mathrel{\mathring{=}}_\beta \mathcal{M}_p(B_{\beta\alpha})(\mathcal{M}_{p'}(x_\alpha)) \mid p' = p \setminus x_\alpha, p \in \mathrm{Asg}(\mathcal{M})\} \quad (5)$$

General model.

Definition 3
Let $\mathcal{M}$ be a frame and $\mathcal{I}$ an interpretation such that for every formula A_α, $\alpha \in$ Types and every assignment $p \in \mathrm{Asg}(\mathcal{M})$, the interpretation $\mathcal{I}$ gives

$$\mathcal{M}_p(A_\alpha) \in M_\alpha.$$

Then the couple $(\mathcal{M}, \mathcal{I})$ is called a general model[5].

If the assignment plays no role (e.g. when A_α is a closed formula) then we will write $\mathcal{M}(A_\alpha)$ only. As a special case, if $M_{\beta\alpha} = M_\beta^{M_\alpha}$ for every $\alpha, \beta \in$ *Types* then the obtained model is called *standard*. If the interpretation $\mathcal{I}$ is known or clear from the context then we will simply say $\mathcal{M}$ is a (general or standard) model.

Given a theory T, we say that a model $(\mathcal{M}, \mathcal{I})$ is a *model of* T and write $\mathcal{M} \models T$ if all axioms of T are true in the degree **1** in $\mathcal{M}$. If a formula A_o is true in the degree **1** in all models of T then we write $T \models A_o$.

The following completeness theorem can be proved (the proof is analogous to the proof of completeness given in [20, 23]).

Theorem 1 (completeness)
(a) A theory T is consistent iff it has a general model $\mathcal{M}$.

(b) For every theory T and a formula A_o

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

3. Models in FTT

3.1. Relations among basic frames

Recall that a basic frame is a system of sets $(M_\alpha)_{\alpha\in Types}$ such that M_o, M_ϵ are sets and for each $\alpha, \beta \in$ *Types*, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$.

Below, we deal with various sets $M_\alpha^1, M_\alpha^2, \ldots$. For clarity, we will denote their elements by lowercase letters $m_\alpha^1, m_\alpha^2 \ldots$, respectively. To distinguish more elements from the same set, we may possibly use bars or apostrophes. Note that quite often, $M_{\beta\alpha}^i$ is a set of functions and so, its elements $m_{\beta\alpha}^i \in M_{\beta\alpha}^i$ are functions. For example, if $M_{\beta\alpha}^i \subseteq {M_\beta^i}^{M_\alpha^i}$ then $m_{\beta\alpha}^i \in M_{\beta\alpha}^i$ is a function $m_{\beta\alpha}^i : M_\alpha^i \longrightarrow M_\beta^i$.

To develop model theory of FTT, we first need to introduce the concept of commuting triple of functions. Similar concept called "isomorphism condition" was introduced already in classical type theory by Andrews in [2].

[5]Note that this definition of general model includes also the concept of *safe structure* introduced in [11] for first-order fuzzy logics.

Definition 4
Let $(M^1_\alpha)_{\alpha\in Types}, (M^2_\alpha)_{\alpha\in Types}$ be basic frames. We say that functions $f^\alpha : M^1_\alpha \longrightarrow M^2_\alpha$, $f^\beta : M^1_\beta \longrightarrow M^2_\beta$, $f^{\beta\alpha} : M^1_{\beta\alpha} \longrightarrow M^2_{\beta\alpha}$ for some $\alpha, \beta \in Types$ is a commuting triple and write $\langle f^\alpha, f^\beta, f^{\beta\alpha}\rangle$ if the following diagram commutes:

$$\begin{array}{ccc}
M^1_\alpha & \xrightarrow{f^\alpha} & M^2_\alpha \\
{\scriptstyle m^1_{\beta\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle m^2_{\beta\alpha}=f^{\beta\alpha}(m^1_{\beta\alpha})} \\
M^1_\beta & \xrightarrow{f^\beta} & M^2_\alpha
\end{array}$$

where $m^i_{\beta\alpha} \in M^i_{\beta\alpha}$, $i = 1, 2$ are functions.

Lemma 1
Let $f^\alpha : M^1_\alpha \longrightarrow M^2_\alpha$, $f^\beta : M^1_\beta \longrightarrow M^2_\beta$ be bijections. Then the function $f^{\beta\alpha}(m^1_{\beta\alpha}) = (f^\alpha)^{-1} \circ m^1_{\beta\alpha} \circ f^\beta$ is a bijection on $M_\beta^{M_\alpha}$ with the inverse $(f^{\beta\alpha})^{-1}(m^2_{\beta\alpha}) = f^\alpha \circ m^2_{\beta\alpha} \circ (f^\beta)^{-1}$ and $\langle f^\alpha, f^\beta, f^{\beta\alpha}\rangle$ is a commuting triple. Moreover, if $\langle f^\alpha, f^\beta, q\rangle$ is another commuting triple then $f^{\beta\alpha} = q$.

PROOF: To show that $f^{\beta\alpha}$ is an injection, consider two function $m^1_{\beta\alpha}, \bar{m}^1_{\beta\alpha}$ such that $(f^\alpha)^{-1} \circ m^1_{\beta\alpha} \circ f^\beta = (f^\alpha)^{-1} \circ \bar{m}^1_{\beta\alpha} \circ f^\beta$. By composition with f^α from left and f^β from right we conclude that $m^1_{\beta\alpha} = \bar{m}^1_{\beta\alpha}$. Similarly, we show that $f^{\beta\alpha}$ and $(f^{\beta\alpha})^{-1}$ are inverse to each other and, consequently, $f^{\beta\alpha}$ is a bijection.

The second part is obtained by

$$m^2_{\beta\alpha}(f^\alpha(m^1_\alpha)) = ((f^\alpha)^{-1} \circ m^1_{\beta\alpha} \circ f^\beta)(f^\alpha(m^1_\alpha)) = f^\beta(m^1_{\beta\alpha}((f^\alpha)^{-1}(f^\alpha(m^1_\alpha)))) \\ = f^\beta(m^1_{\beta\alpha}(m^1_\alpha)).$$

Finally, since $\langle f^\alpha, f^\beta, q\rangle$ is a commuting triple, we have $m^1_{\beta\alpha} \circ f^\beta = f^\alpha \circ q(m^1_{\beta\alpha})$. From it follows $(f^\alpha)^{-1} \circ m^1_{\beta\alpha} \circ f^\beta = f^{\beta\alpha}(m^1_{\beta\alpha}) = q(m^1_{\beta\alpha})$.

□

Lemma 2
Let $\langle f^\alpha, f^\beta, f^{\beta\alpha}\rangle$ be a commuting triple for the basic frames $(M^1_\alpha)_{\alpha\in Types}, (M^2_\alpha)_{\alpha\in Types}$ and $\langle g^\alpha, g^\beta, g^{\beta\alpha}\rangle$ a commuting triple for the basic frames $(M^2_\alpha)_{\alpha\in Types}, (M^3_\alpha)_{\alpha\in Types}$. Then $\langle f^\alpha \circ g^\alpha, f^\beta \circ g^\beta, f^{\beta\alpha} \circ g^{\beta\alpha}\rangle$ is a commuting triple for the basic frames $(M^1_\alpha)_{\alpha\in Types}, (M^3_\alpha)_{\alpha\in Types}$.

PROOF: We must show that

$$g^{\beta\alpha}(f^{\beta\alpha}(m^1_{\beta\alpha}))(g^\alpha(f^\alpha(m^\alpha))) = g^\beta(f^\beta(m^{\beta\alpha}(m^1\alpha))). \tag{6}$$

Realizing that, by the assumption

$$f^{\beta\alpha}(m^1_{\beta\alpha})(f^\alpha(m^1_\alpha)) = f^\beta(m^1_{\beta\alpha}(m^1_\alpha)),$$
$$g^{\beta\alpha}(f^{\beta\alpha}(m^1_{\beta\alpha}))(g^\beta(m^2_\alpha)) = g^\beta(f^{\beta\alpha}(m^1_{\beta\alpha}(m^2_\alpha)),$$

and $m^2_\alpha = f^\alpha(m^1_\alpha)$ we obtain (6).

□

3.2. Relations among frames

A frame is the tuple (4) where $(M_\alpha)_{\alpha\in Types}$ is a basic frame. One can see that each set M_α is endowed with a fuzzy equality $\doteq_\alpha$ taking values from a good EQ-algebra $\mathcal{E}$ whose support E is the set M_o. We suppose that all the fuzzy equalities $\doteq_\alpha$, $\alpha \in Types$ are separated.

Definition 5
Let $(\mathcal{E}^1, M^1, \doteq^1)$, $(\mathcal{E}^2, M^2, \doteq^2)$ be two sets with fuzzy equalities. We say that a couple of functions (s, f), $s : E^1 \longrightarrow E^2$ and $f : M^1 \longrightarrow M^2$ is a homomorphism between $(\mathcal{E}^1, M^1, \doteq^1)$, $(\mathcal{E}^2, M^2, \doteq^2)$ if the following holds:

(i) s is an algebra-homomorphism preserving all infima and such that $s(a) = \mathbf{1}^2$ iff $a = \mathbf{1}^1$.

(ii) $s([m \doteq^1 m']) = [f(m) \doteq^2 f(m')]$, for all $m, m' \in M^1$.

If (s, f) is a homomorphism between $(\mathcal{E}^1, M^1, \doteq^1)$ and $(\mathcal{E}^2, M^2, \doteq^2)$, then we will write $(s, f) : (\mathcal{E}^1, M^1, \doteq^1) \longrightarrow (\mathcal{E}^2, M^2, \doteq^2)$. If $\mathcal{E}^1 = \mathcal{E}^2$ then we may consider only $f : (M^1, \doteq^1) \longrightarrow (M^2, \doteq^2)$.

Lemma 3
Let $(s, t) : (\mathcal{E}^1, M^1, \doteq^1) \longrightarrow (\mathcal{E}^2, M^2, \doteq^2)$ and $\doteq^1, \doteq^2$ be separated. Then both s, f are injections.

PROOF: Since both $\mathcal{E}^1, \mathcal{E}^2$ are good, both $\sim^1, \sim^2$ are separated. Let $s(a) = s(b)$, $a, b \in E^1$. Then $s(a) \sim^2 s(b) = \mathbf{1}^2 = s(a \sim b)$. Since s fulfils Definition 5(i), $a \sim b = \mathbf{1}^1$ and so, $a = b$ by separateness.

In the same way, let $f(m) = f(m')$ where $m, m' \in M^1$. Then $[f(m) \doteq^2 f(m')] = \mathbf{1}^2 = s([m \doteq^1 m'])$. Since s fulfils Definition 5(i), $[m \doteq^1 m'] = \mathbf{1}^1$ and so, $m = m'$ by separateness. □

Thus, s is an embedding of the EQ-algebra $\mathcal{E}^1$ in $\mathcal{E}^2$. If $\doteq^1, \doteq^2$ are separated then we will call also (g, f) an *embedding of $(\mathcal{E}^1, M^1, \doteq^1)$ in $(\mathcal{E}^2, M^2, \doteq^2)$*.

3.3. Finite models

Let $\mathcal{M}$ (more precisely, $(\mathcal{M}, \mathcal{I})$) be a model. In [1], it was proved that finite models of type theory must be standard. Using the main idea adopted from this paper we show below that similar property holds also in FTT.

Theorem 2
Let $\mathcal{M}$ be a general model such that M_ϵ is finite. Let α be a type not containing o and β be a type. Then $M_{\beta\alpha} = M_\beta^{M_\alpha}$.

PROOF: Clearly, if α does not contain the type o (representing possibly infinite set of truth values) then M_α is finite. We will show that to each $f \in M_{\beta\alpha}$ there is a formula $A_{\beta\alpha}$ such that $\mathcal{M}_p(A_{\beta\alpha}) = f$ for some assignment p.

Let $M_\alpha = \{u_1, \ldots, u_m\}$. Then $f : M_\alpha \longrightarrow M_\beta$ is finite. Let us consider a formula

$$A_{\beta\alpha} := \lambda x_\alpha \cdot \iota_{\beta(o\beta)} \lambda y_\beta \cdot \mathbf{\Delta} \bigvee_{j=1}^{m} ((x \equiv w_j) \wedge (y \equiv z_j)).$$

where w_j are variables of type α and z_j variables of type β, $j = 1, \ldots, m$. Let p be an assignment such that $p(w_j) = u_j$ and $p(z_j) = f(u_j)$. Finally, let $p' = p \backslash \{x, y\}$. We argue that

$$\mathcal{M}_{p'}\left(\mathbf{\Delta} \bigvee_{j=1}^{m} ((x \equiv w_j) \wedge (y \equiv z_j))\right) = \mathbf{1} \tag{7}$$

iff $p'(x) = u_{j_0}$ and $p'(y) = f(u_{j_0})$ for some $j_0 \in \{1, \ldots, m\}$.

($\Rightarrow$): By the properties of $\mathbf{\Delta}$, $\mathcal{M}_{p'}(\mathbf{\Delta}(x \equiv w_{j_0}) \wedge \mathbf{\Delta}(y \equiv z_{j_0})) = \mathbf{1}$ for some $j_0 \in \{1, \ldots, m\}$. Because interpretation of any $\equiv$ is a separated fuzzy equality, this implies that $p'(x) = p'(w_{j_0}) = p(w_{j_0}) = u_{j_0}$ and $p'(y) = p'(z_{j_0}) = p(z_{j_0}) = f(u_{j_0})$.

($\Leftarrow$): Obviously, then $u_{j_0} = p'(x) = p(w_{j_0})$ and $f(u_{j_0}) = p'(y) = p(z_{j_0})$ which gives (7).

Now, (with reference to the definition of interpretation of formulas) the formula

$$B_{o\beta} := \lambda y_\beta \, \mathbf{\Delta} \bigvee_{j=1}^{m} ((x \equiv w_j) \wedge (y \equiv z_j))$$

has the free variable x_α. Thus, given assignments p and $p' = p \backslash x$, if $p'(x) = u_j$ then interpretation of $\mathcal{M}_{p'}(B_{0\alpha})$ is the fuzzy set

$$\mathcal{M}_{p'}(B_{0\alpha}) = \left\{ \mathbf{1}/f(u_j) \right\}.$$

The reason is that f is a function and therefore, if $p'' = p'\backslash y$ then

$$\mathcal{M}_{p''}\left(\boldsymbol{\Delta}\bigvee_{j=1}^{m}((x \equiv w_j) \wedge (y \equiv z_j))\right) = \mathbf{0}$$

holds for any $p''(y) = f(u_i)$, $u_i \neq u_j$. Consequently, $\mathcal{M}_{p'}(\iota_{\beta(o\beta)}\, B_{o\beta}) = f(u_j)$ and we conclude that $\mathcal{M}_p(A_{\beta\alpha}) = f$.

□

Corollary 1
Let $\mathcal{M}$ be a general model in which both sets M_ϵ as well as M_o are finite. Then it is standard.

Example 1
Let us construct a (standard) finite model $\mathcal{M}^1$ as follows. Put

$$M_o^1 = E^1 = \{\mathbf{0}^1 = a_0^1, \ldots, a_n^1 = \mathbf{1}^1\}, \qquad n < \aleph_0, n - \textit{even}$$
$$M_\epsilon^1 = E^1 = \{m_0^1, \ldots, m_r^1\}, \qquad r < \aleph_0$$

The truth values form an EQ-algebra

$$\mathcal{E}^1 = \{E^1, \wedge^1, \otimes^1, \sim^1, \mathbf{0}^1, \mathbf{1}^1, \Delta^1\}$$

where

$$a_i^1 \wedge^1 a_j^1 = a^1_{\min(i,j)}, \qquad \mathbf{0}^1 = a_0^1,$$
$$a_i^1 \otimes^1 a_j^1 = a_{\max(0,i+j-n)}, \qquad \mathbf{1}^1 = a_n^1,$$
$$a_i^1 \sim^1 a_j^1 = a_{n-|i-j|}, \qquad \Delta^1(a_i^1) = \begin{cases} a_n^1, & \textit{if } i = n, \\ a_0 & \textit{otherwise.} \end{cases}$$

The fuzzy equality $\doteq^1_\epsilon$ is defined by

$$[m_i^1 \doteq^1_\epsilon m_j^1] = a^1_{\max(0,n-|i-j|)}.$$

Note that it is separated. The other sets are $M^1_{\beta\alpha} = (M^1_\beta)^{M^1_\alpha}$. The fuzzy equalities $\doteq_{\beta\alpha}$ in $M^1_{\beta\alpha}$ are defined using (5).

Finally, we define $\mathcal{M}^1(\mathbf{E}_{(oo)o}) = \sim^1$, $\mathcal{M}^1(\mathbf{S}_{(oo)o}) = \otimes^1$, $\mathcal{M}^1(\mathbf{C}_{(oo)o}) = \wedge^1$, $\mathcal{M}^1(\mathbf{D}_{oo}) = \Delta^1$ and

$$\mathcal{M}^1(\iota_{o(oo)})(m^1_{oo}) = a^1_{\min\{i | m^1_{oo}(a^1_i) = \mathbf{1}^1\}}, \qquad a^1_i \in E^1 \tag{8}$$
$$\mathcal{M}^1(\iota_{\epsilon(o\epsilon)})(m^1_{o\epsilon}) = m^1_{\min\{i | m^1_{o\epsilon}(m^1_i) = \mathbf{1}^1\}}, \qquad m^1_i \in M^1_\epsilon \tag{9}$$

where $m^1_{oo} \in (E^1)^{E^1}$ and $m^1_{o\epsilon} \in (E^1)^{M^1_\epsilon}$. We do not consider the generating function G (and so, the constant $\mathbf{G}_o o$ is omitted from the language).

One can verify that all axioms of FTT are fulfiled in $\mathcal{M}^1$. We conclude that the latter is a model of FTT.

3.4. Homomorphism of models

Let us consider some language J of FTT.

Definition 6
Let

$$\mathcal{M}^1 = \langle (M^1_\alpha, \doteq^1_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{E}^1 \rangle$$
$$\mathcal{M}^2 = \langle (M^2_\alpha, \doteq^2_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{E}^2 \rangle$$

be two models[6]*. Let us consider a set of functions*

$$\mathfrak{f} = \{f^\alpha : M^1_\alpha \longrightarrow M^2_\alpha \mid \alpha \in \mathit{Types}\} \tag{10}$$

such that the following holds:

(i) For all $\alpha, \beta \in \mathit{Types}$*,* $(f^\alpha, f^\beta, f^{\beta\alpha})$ *forms a commuting triple.*

(ii) $f^o : E^1 \longrightarrow E^2$ *preserves all existing infima and, moreover,* $f^o(a) = \mathbf{1}^2$ *iff* $a = \mathbf{1}^1$*.*

(iii) For each constant $c_\alpha \in J$*,* $\alpha \in \mathit{Types}$*, it holds that* $f^\alpha(\mathcal{M}^1(c_\alpha)) = \mathcal{M}^2(c_\alpha)$*.*

Then the set (10) is a homomorphism *of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *and we will formally write*

$$\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2.$$

If all functions in (10) are injections then $\mathfrak{f}$ is an *embedding* of $\mathcal{M}^1$ in $\mathcal{M}^2$.

It follows from this definition that if $\alpha = \gamma\beta$ then

$$f^\beta \circ f^{\gamma\beta}(\mathcal{M}^1(c_{\gamma\beta})) = \mathcal{M}^1(c_{\gamma\beta}) \circ f^\gamma, \tag{11}$$

i.e.

$$(f^{\gamma\beta}(\mathcal{M}^1(c_{\gamma\beta})))(f^\beta(m^1_\beta)) = f^\gamma(\mathcal{M}^1(c_{\gamma\beta})(m^1_\beta)), \qquad m^1_\beta \in M^1_\beta.$$

Let $p \in \mathrm{Asg}(\mathcal{M}^1)$ be an assignment of elements from $\mathcal{M}^1$ to variables and let $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ be a homomorphism. We will define a related assignment $p \circ \mathfrak{f} \in \mathrm{Asg}(\mathcal{M}^2)$ as a set of compositions of the function p (defined on a set of all variables) and the functions f^α for all $\alpha \in \mathit{Types}$:

$$(p \circ \mathfrak{f})(x_\alpha) = f^\alpha(p(x_\alpha)). \tag{12}$$

It also follows from (12) and the definition of homomorphism that $(p \circ \mathfrak{f})(x_{\gamma\beta}) = f^{\gamma\beta}(p(x_{\gamma\beta}))$ such that

$$f^\beta \circ f^{\gamma\beta}(p(x_{\gamma\beta})) = p(x_{\gamma\beta}) \circ f^\gamma. \tag{13}$$

Lemma 4
Let $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ *be a homomorphism of models.*

(a) If $\alpha = o$ *then* f^o *is an embedding of the EQ-algebra* $\mathcal{E}^1$ *in* $\mathcal{E}^2$*.*

(b) For every $\alpha \in \mathit{Types}$*,* (f^0, f^α) *is an embedding* $(f^o, f^\alpha) : (\mathcal{E}^1, M^1_\alpha, \mathring{=}^1_\alpha) \longrightarrow (\mathcal{E}^2, M^2_\alpha, \mathring{=}^2_\alpha)$*, i.e.* f^α *is an injection and*

$$f^o([m^1_\alpha \mathring{=}^1 \bar{m}^1_\alpha]) = [f^\alpha(m^1_\alpha) \mathring{=}^2 f^\alpha(\bar{m}^1_\alpha)] \tag{14}$$

holds for all $m^1_\alpha, \bar{m}^1_\alpha \in M^1_\alpha$*.*

PROOF: Let $c_\alpha := \mathbf{E}_{(o\beta)\beta}$. Using (11) after tedious rewriting we finally obtain

$$f^o([m^1_\beta \doteq^1 \bar{m}^1_\beta]) = [f^\beta(m^1_\beta) \doteq^2 f^\beta(\bar{m}^1_\beta)], \qquad m^1, \bar{m}^1_\beta \in M^1_\beta. \tag{15}$$

As a special case, putting $\beta = o$, we obtain from (15)

$$f^o(a \sim^1 b) = f^o(a) \sim^2 f^o(b), \qquad a, b \in E^1. \tag{16}$$

Similarly, for $c_\alpha := \mathbf{C}_{(oo)o}, \mathbf{S}_{(oo)o}, \mathbf{D}_{oo}$ we show that (16) can be proved also for the operations $\wedge, \otimes, \Delta$.

[6]Recall that $M^i_o = E^i$, $i = 1, 2$ where E^i is a support of the EQ-algebra $\mathcal{E}^i$.

As for the top element $\mathbf{1}$, note that $\mathcal{M}^1_p(\lambda x_o\, x_o) = id_{E^1}$ and so, $\mathcal{M}^1(\top) = [id_{E^1} \doteq^1_{oo} id_{E^1}]$. Then using (15) we obtain

$$[f^{oo}(id_{E^1}) \doteq^2_{oo} f^{oo}(id_{E^1})] = \bigwedge_{a^2\in E^2} (f^{oo}(id_{E^1})(a^2) \sim^2 f^{oo}(id_{E^1})(a^2)) = \mathbf{1}^2 =$$

$$f^o([id_{E^1} \doteq^1_{oo} id_{E^1}]) = f^o(\bigwedge_{a^1\in E^1} (id_{E^1}(a^1) \sim^1 id_{E^1}(a^1)) = f^o(\mathbf{1}^1).$$

From it follows that f^o is embedding of $\mathcal{E}^1$ in $\mathcal{E}^2$ which together with (15) and Lemma 3 proves that (f^o, f^α) is also embedding of $(\mathcal{E}^1, M^1_\alpha, \stackrel{\circ}{=}^1_\alpha)$ in $(\mathcal{E}^2, M^2_\alpha, \stackrel{\circ}{=}^2_\alpha)$.

□

It follows from Lemma 4 that each homomorphism $\mathfrak{f}$ is necessarily an embedding of the model $\mathcal{M}^1$ in the model $\mathcal{M}^2$.

Example 2
Let us introduce a model $\mathcal{M}^2$ which differs from $\mathcal{M}^1$ in Example 1 by

$$M^2_o = E^2 = \{\mathbf{0}^2 = a^2_0, \dots, a^2_n, \dots, a^2_k = \mathbf{1}^2\}, \qquad k = 2n,$$
$$M^2_\epsilon = \{m^2_0, \dots, m^2_r, \dots, m^2_q\}, \qquad q = 2r.$$

The rest is defined analogously as in Example 2.

Let us now define $f^o : E^1 \longrightarrow E^2$, by $f^o(a^1_i) = a^2_{2i}$, $i = 0, \dots, n-1$. Clearly, $f^o(a^1_n) = a^2_k$. We verify that $f^o(a^1_i \wedge^1 a^1_j) = f^o(a^1_{\min(i,j)}) = a^2_{\min(2i,2j)} = a^2_{2i} \wedge^2 a^2_{2j} = f^o(a^2_i) \wedge^2 f^o(a^2_j)$ for all $i, j \in \{0, \dots, n\}$. Analogously, $f^o(a^1_i \sim^1 a^1_j) = f^o(a^1_{n-|i-j|}) = a^2_{2(n-|i-j|)} = a^2_{2i} \sim^2 a^2_{2j} = f^o(a^1_i) \sim^2 f^o(a^1_j)$. Similarly also for $\otimes^1$ and Δ^1 and we conclude that f^o is an algebra morphism.

Similarly, putting $f^\epsilon(m^1_i) = m^2_{2i}$, $i = 0, \dots, r$ we may show that f^ϵ is homomorphism $f^\epsilon : (M^1_\epsilon, \stackrel{\circ}{=}^1_\epsilon) \longrightarrow (M^2_\epsilon, \stackrel{\circ}{=}^2_\epsilon)$.

Mappings of functions, for example, $f^{o\epsilon}$, are defined as follows. Let $m^1_{o\epsilon} \in (M^1_o)^{M^1_\epsilon}$. Then we choose a function $m^2_{o\epsilon} \in (M^2_o)^{M^2_\epsilon}$ such that

$$m^2_{o\epsilon}(f^\epsilon(m^1_i)) = f^o(m^1_{o\epsilon}(m^1_i)), \qquad i \in \{0, r\} \tag{17}$$

and put $f^{o\epsilon}(m^1_{o\epsilon}) = m^2_{o\epsilon}$.

A special care must be taken of functions of the type $o\alpha$. In this case we choose $m^2_{o\alpha} = f^{o\epsilon}(m^1_{o\epsilon})$ fulfilling (17) and, moreover,

$$m^2_{o\alpha}(m^2_j) \neq \mathbf{1}^2 \quad \text{for all} \quad m^2_j \neq f^\epsilon(m^1_i), \qquad i \in \{0, r\}. \tag{18}$$

The functions (8), (9) interpreting the description operators $\iota_{o(oo)}, \iota_{\epsilon(o\epsilon)}$ must fulfil analogous condition as (17), namely

$$f^{\epsilon(o\epsilon)}(\mathcal{M}^1(\iota_{\epsilon(o\epsilon)})(f^{o\epsilon}(m^1_i)) = f^\epsilon(\mathcal{M}^1(\iota_{\epsilon(o\epsilon)})(m^1_i)) \tag{19}$$

and similarly for the type $o(oo)$. This is assured by (18).

Clearly, $f^o, f^\epsilon, f^{o\epsilon}$ is a commuting triple. Finally, if we continue in analogous way for all $f^{\beta\alpha} : M^1_{\beta\alpha} \longrightarrow M^2_{\beta\alpha}$ to fulfil Definition 4 we conclude that the model $\mathcal{M}^1$ is embedded in $\mathcal{M}^2$.

Lemma 5
Let all functions in (10) be bijections and item (i) of Definition 6 be fulfiled. If $f^{(oo)o}(\mathcal{M}^1(\mathbf{E}_{(oo)o})) = \mathcal{M}^2(\mathbf{E}_{(oo)o})$ then f^o preserves all existing infima.

PROOF: Let $h_{oo}, \bar{h}_{oo} \in (E^1)^{E^1}$. Using (15) and the assumption we obtain

$$f^o([h_{oo} \stackrel{\circ}{=}^1_{oo} \bar{h}]) = [f^{oo}(h) \stackrel{\circ}{=}_{oo} f^{oo}(\bar{h})] = f^o(\bigwedge_{a^1\in E^1} (h(a^1) \sim^1 \bar{h}(a^1))) =$$

$$\bigwedge_{a^1\in E^1} (f^{oo}(h)(f(a^1)) \sim^2 f^{oo}(\bar{h}(a^1))) = \bigwedge_{a^1\in E^1} (f^o(h(a^1)) \sim^2 f^o(\bar{h}(a^1))) =$$

$$\bigwedge_{a^1\in E^1} (f^o(h(a^1) \sim^1 \bar{h}(a^1))).$$

Let us choose $\bar{h}(a) = \mathbf{1}^1$ for all $a \in E^1$. Then from the previous equality and the fact that the EQ-algebra $\mathcal{E}^1$ is good we obtain

$$f^o(\bigwedge_{a^1 \in E^1} (h(a^1))) = \bigwedge_{a^1 \in E^1} (f^o(h(a^1))).$$

Since $h_{oo} \in (E^1)^{E^1}$ is arbitrary, we conclude that f^o preserves existing infima. □

Definition 7
Let $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ be an embedding.

(i) $\mathcal{M}^1$ is a submodel *of $\mathcal{M}^2$, in symbols $\mathcal{M}^1 \subset \mathcal{M}^2$, if f^o and f^ϵ are identities and $\mathcal{E}^1$ is a subalgebra of $\mathcal{E}^2$.*

(ii) Let all functions in (10) be bijections. Moreover, let item (ii) of Definition 6 be modified as follows: $f^o(a) = \mathbf{1}^2$ iff $a = \mathbf{1}^1$. Then $\mathfrak{f}$ is an isomorphism *between $\mathcal{M}^1$ and $\mathcal{M}^2$ and we write*

$$\mathcal{M}^1 \cong \mathcal{M}^2.$$

Remark 1
Note that in the definition of submodel, we may not take the other f^γ for $\gamma \neq \epsilon$ to be identities because the sets M^1_α, M^1_β differ from the corresponding sets M^2_α, M^2_β and so, $M^1_{\beta\alpha}$ cannot be a subset of $M^2_{\beta\alpha}$.

Theorem 3
Let $\mathcal{M}_1 \cong \mathcal{M}_2$ be an isomorphism and A_α a formula of type α. Then

$$f^\alpha(\mathcal{M}^1_p(A_\alpha)) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_\alpha)$$

for any assignment $p \in \mathrm{Asg}(\mathcal{M}^1)$ and the corresponding assignment $p \circ \mathfrak{f} \in \mathrm{Asg}(\mathcal{M}^2)$.

PROOF: If $A_\alpha := c_\alpha$ for a constant c_α then the proof follows from Definition 6(ii).

Let $A_\alpha := x_\alpha$ and $p(x_\alpha) = m \in M^1_\alpha$. Then $(p \circ \mathfrak{f})(x_\alpha) = f^\alpha(m) \in M^2_\alpha$ and we have

$$f^\alpha(\mathcal{M}^1_p(x_\alpha)) = f^\alpha(m) = (p \circ \mathfrak{f})(x_\alpha) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_\alpha).$$

Let the inductive assumption hold and $A_\alpha := C_{\alpha\beta}B_\beta$. Then

$$\mathcal{M}^2_{p\circ\mathfrak{f}}(C_{\alpha\beta}B_\beta) = \mathcal{M}^2_{p\circ\mathfrak{f}}(C_{\alpha\beta})(\mathcal{M}^2_{p\circ\mathfrak{f}}(B_\beta)) = f^{\alpha\beta}(\mathcal{M}^1_p(C_{\alpha\beta}))(f^\beta(\mathcal{M}^1_p(B_\beta))) =$$
$$= f^\alpha(\mathcal{M}^1_p(C_{\alpha\beta})(\mathcal{M}^1_p(B_\beta))) = f^\alpha(\mathcal{M}^1_p(C_{\alpha\beta}B_\beta))$$

because $\langle f^\beta, f^\alpha, f^{\alpha\beta}\rangle$ is a commuting triple.

Let $\alpha = \beta\gamma$ and $A_\alpha := \lambda x_\gamma\, B_\beta$. Then

$$\mathcal{M}^1_p(\lambda x_\gamma\, B_\beta) = F^1 : M^1_\gamma \longrightarrow M^1_\beta$$

where $F^1(m^1_\gamma) = \mathcal{M}^1_{p'}(B_\beta)$ for each assignment $p' = p \setminus x_\gamma$ and $p'(x_\gamma) = m^1_\gamma$. Furthermore, by the inductive assumption, $\mathcal{M}^2_{p'\circ\mathfrak{f}}(B_\beta) = f^\beta(\mathcal{M}^1_{p'}(B_\beta))$ where, because f^γ is a bijection, $(p' \circ \mathfrak{f})(x_\gamma) = m^2_\gamma = f^\gamma(p'(x_\gamma))$ by (12). Hence, we have defined a function $F^2 : M^2_\gamma \longrightarrow M^2_\beta$ such that each element $m^2_\gamma \in M^2_\gamma$ is assigned the element $\mathcal{M}^2_{p\circ\mathfrak{f}}(B_\beta) \in M^2_\beta$. Since $p' = p\setminus x_\gamma$ we have $p' \circ \mathfrak{f} = (p\circ\mathfrak{f})\setminus x_\gamma$ where the assignment $p \circ \mathfrak{f}$ fulfils (12). Because f^γ is a bijection, we conclude that $F^2 = \mathcal{M}^2_{p\circ\mathfrak{f}}(\lambda x_\gamma\, B_\beta)$.

Finally, $\langle f^\gamma, f^\beta, f^{\beta\gamma}\rangle$ is a commuting triple, which means that $f^\beta(F^1(m^1_\gamma)) = F^2(f^\gamma(m^1_\gamma))$. Because $f^{\beta\gamma}$ must be injection, we conclude that $f^{\beta\gamma}(F^1) = F_2$ and so,

$$\mathcal{M}^2_{p\circ\mathfrak{f}}(\lambda x_\gamma\, B_\beta) = f^{\beta\gamma}(\mathcal{M}^1_p(\lambda x_\gamma\, B_\beta)).$$

□

Let us emphasize, that it also follows from the previous proof that

$$\mathcal{M}^2_{p\circ\mathfrak{f}}((\lambda x_\gamma\, B_\beta)C_\gamma) = f^{\beta\gamma}(\mathcal{M}^1_p(\lambda x_\gamma\, B_\beta))(f^\gamma(\mathcal{M}^1_p(C_\gamma))) =$$
$$= f^\beta(\mathcal{M}^1_p((\lambda x_\gamma\, B_\beta)C_\gamma)) = f^\beta(\mathcal{M}^1_p(D_\beta)) = \mathcal{M}^2_{p\circ\mathfrak{f}}(D_\beta)$$

where D_β is obtained from $(\lambda x_\gamma\, B_\beta)C_\gamma$ by λ-conversion.

Theorem 4
Let $\mathcal{M}_1$, $\mathcal{M}_2$ be standard, $\mathcal{E}^1 \cong \mathcal{E}^2$ and let there be an isomorphism $f^\epsilon : (M^1_\epsilon, \mathrel{\dot{=}}^1_\epsilon) \longrightarrow (M^2_\epsilon, \mathrel{\dot{=}}^2_\epsilon)$. Let $\mathfrak{f}$ be a set (10) of bijections induced due to Lemma 1. If $f^\alpha(\mathcal{M}^1(c_\alpha)) = \mathcal{M}^2(c_\alpha)$ for arbitrary constant c_α, $\alpha \in$ Types then $\mathcal{M}_1 \cong \mathcal{M}_2$.

PROOF: It follows from Lemma 1 that all $(f^\alpha, f^\beta, f^{\beta\alpha})$ are commuting triples. Conditions (ii) and (iii) of Definition 6 follow from the assumption. □

To show that each $f^{\beta\alpha}$ in this theorem in fulfils (11), let $f^{\beta\alpha}(\mathcal{M}^1(c_{\beta\alpha})) = \mathcal{M}^2(c_{\beta\alpha})$. Then using Lemma 1 we have:

$$\mathcal{M}^2(c_{\beta\alpha})(m^2_\alpha) = f^{\beta\alpha}(\mathcal{M}^1(c_{\beta\alpha}))(m^2_\alpha) = f^{\beta\alpha}(\mathcal{M}^1(c_{\beta\alpha}))(f^\alpha(m^1_\alpha)) = f^\beta(\mathcal{M}^1(c_{\beta\alpha})((f^\alpha)^{-1}(m^2_\alpha))) = f^\beta(\mathcal{M}^1(c_{\beta\alpha})(m^1_\alpha))$$

i.e.

$$f^\alpha \circ f^{\beta\alpha}(\mathcal{M}^2(c_{\beta\alpha})) = \mathcal{M}^1(c_{\beta\alpha}) \circ f^\beta.$$

Definition 8
(i) Models $\mathcal{M}^1$ and $\mathcal{M}^2$ are elementary equivalent, $\mathcal{M}^1 \equiv \mathcal{M}^2$, if

$$\mathcal{M}^1(A_o) = \mathbf{1}^1 \quad \text{iff} \quad \mathcal{M}^2(A_o) = \mathbf{1}^2 \tag{20}$$

holds for arbitrary sentence $A_o \in Form_o$.

(ii) Models $\mathcal{M}^1$ and $\mathcal{M}^2$ are strongly elementary equivalent, $\mathcal{M}^1 \hat{\equiv} \mathcal{M}^2$, if $\mathcal{E}^1 = \mathcal{E}^2$ and

$$\mathcal{M}^1(A_o) = \mathcal{M}^2(A_o) \tag{21}$$

holds for arbitrary sentence $A_o \in Form_o$.

(iii) An embedding $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ is elementary if

$$f^o(\mathcal{M}^1_p(A_o)) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_o) \tag{22}$$

holds for arbitrary formula $A_o \in Form_o$ and $p \in \mathrm{Asg}(\mathcal{M}^1)$.

(iv) A model $\mathcal{M}^1$ is an elementary submodel of $\mathcal{M}^2$, in symbols $\mathcal{M}^1 \prec \mathcal{M}^2$, if $\mathcal{M}^1 \subset \mathcal{M}^2$ and

$$\mathcal{M}^1_p(A_o) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_o) \tag{23}$$

holds for arbitrary formula $A_o \in Form_o$ and $p \in \mathrm{Asg}(\mathcal{M}^1)$.

Theorem 5
(a) If $\mathcal{M}^1 \hat{\equiv} \mathcal{M}^2$ then $\mathcal{M}^1 \equiv \mathcal{M}^2$. If $\mathcal{M}^1 \cong \mathcal{M}^2$ then $\mathcal{M}^1 \hat{\equiv} \mathcal{M}^2$.

(b) If $\mathcal{M}^1 \prec \mathcal{M}^2$ then $\mathcal{M}^1 \equiv \mathcal{M}^2$.

(c) If $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ and $\mathfrak{g} : \mathcal{M}^2 \longrightarrow \mathcal{M}^3$ are elementary embeddings then $\mathfrak{f} \circ \mathfrak{g} : \mathcal{M}^1 \longrightarrow \mathcal{M}^3$ is an elementary embedding.

(d) If $\mathcal{M}^1 \prec \mathcal{M}^2$ and $\mathcal{M}^2 \prec \mathcal{M}^3$ then $\mathcal{M}^1 \prec \mathcal{M}^3$.

(e) The relations $\cong, \equiv, \hat{\equiv}$ are equivalences.

PROOF: (a) is obvious.

(b) Let $\mathcal{M}^1 \prec \mathcal{M}^2$ and A_o be a sentence. Then the assignment p plays no role. But then $\mathcal{M}^1(A_o) = \mathbf{1}$ iff $\mathcal{M}^2(A_o) = \mathbf{1}$ because $\mathcal{E}^1$ is a subalgebra of $\mathcal{E}^2$.

(c) By Lemma 2, we conclude that all triples $\langle f^\alpha \circ g^\alpha, f^\beta \circ g^\beta, f^{\beta\alpha} \circ g^{\beta\alpha}\rangle$ are commuting. Similarly, composition of homomorphisms of $(f^o \circ g^o, f^\alpha \circ g^\alpha)(M^1_\alpha, \mathrel{\dot{=}}^1_\alpha) \longrightarrow (M^2_\alpha, \mathrel{\dot{=}}^2_\alpha)$ and $(M^2_\alpha, \mathrel{\dot{=}}^2_\alpha) \longrightarrow (M^3_\alpha, \mathrel{\dot{=}}^3_\alpha)$ is a homomorphism. Finally, we verify that $g^\alpha(f^\alpha(\mathcal{M}^1(c_\alpha))) = \mathcal{M}^3(c_\alpha)$ holds for every constant c_α, $\alpha \in$ *Types*.

(d), (e) are obvious. □

The following theorem is analogous to [8, Proposition 11] (Tarski-Vaught test).

Theorem 6

Let $\mathcal{M}^1 \subset \mathcal{M}^2$. Then the following is equivalent:

(a) $\mathcal{M}^1 \prec \mathcal{M}^2$.

(b) For all types α, β, formulas $D_{\beta\alpha}, G_{\beta\alpha}$ and the assignment $p \in Asg(\mathcal{M}^1)$ the following equality holds true:

$$\bigwedge\{\mathcal{M}^1_p(G_{\beta\alpha})(p'(x_\alpha)) \stackrel{\circ}{=}^1_\beta \mathcal{M}^1_p(D_{\beta\alpha})(p'(x_\alpha)) \mid p, p' \in \mathrm{Asg}(\mathcal{M}^1), p' = p \setminus x_\alpha\} =$$
$$\bigwedge\{\mathcal{M}^2_{p\circ\mathfrak{f}}(G_{\beta\alpha})(p''(x_\alpha)) \stackrel{\circ}{=}^2_\beta \mathcal{M}^2_{p\circ\mathfrak{f}}(D_{\beta\alpha})(p''(x_\alpha)) \mid$$
$$p \in \mathrm{Asg}(\mathcal{M}^1), p'' \in \mathrm{Asg}(\mathcal{M}^2), p'' = (p \circ \mathfrak{f}) \setminus x_\alpha\}. \quad (24)$$

PROOF: (a)⇒(b): It follows from (a) that

$$\mathcal{M}^1_p(A_o) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_o) \quad (25)$$

for all formulas A_o. This includes also a special case $A_o := (G_{\beta\alpha} \equiv D_{\beta\alpha})$ where $D_{\beta\alpha}, G_{\beta\alpha}$ are formulas of the complex type $\beta\alpha$, $\alpha, \beta \in$ *Types*. But then from $\mathcal{M}^1_p(G_{\beta\alpha} \equiv D_{\beta\alpha}) = \mathcal{M}^2_{p\circ\mathfrak{f}}(G_{\beta\alpha} \equiv D_{\beta\alpha})$ we obtain (24) by (5).

(b)⇒(a): By induction on the complexity of the formula.

Let $A_o := c_o$ for some constant c_o. Then $\mathcal{M}^1_p(A_o) = \mathcal{M}^2_{p\circ\mathfrak{f}}(A_o)$ because $E^1 \subseteq E^2$ due to definition of submodel. Similarly for $A_o := x_o$.

Let $A_o := B_{o\alpha} G_\alpha$. Then (25) holds because $\langle f^o, f^\alpha, f^{o\alpha} \rangle$ is a commuting triple (cf. Definition 4).

It remains to check the special case $A_o := (\mathbf{E}_{(o(\beta\alpha))(\beta\alpha)} G_{\beta\alpha}) D_{\beta\alpha}$, which in a more readable way is the formula $A_o := (G_{\beta\alpha} \equiv D_{\beta\alpha})$. Then, with respect to the definition (5) we conclude from the assumption (b) that (25) holds in this case as well.

□

3.5. Cardinality of models

Given a model $\mathcal{M}$, let us discuss its cardinality. This entirely depends on the cardinality of the basic sets M_o and M_ϵ. Moreover, in general model, cardinalities of the sets $M_{\beta\alpha}$ are generally smaller than cardinalities of $M_\beta^{M_\alpha}$.

Let us assume that cardinalities of M_o, M_ϵ are regular, i.e. that $\mathrm{Card}(cf(M)) = \mathrm{Card}(M)$ where $cf(M)$ is a cofinality of M. We will also assume AC (Axiom of Choice). Then the following classical formula holds true ([3]):

$$\aleph_\eta^{\aleph_\theta} = \begin{cases} \aleph_\eta, & \text{if } \aleph_\theta < \aleph_\eta, \\ \aleph_{\theta+1} & \text{if } \eta \leq \theta \end{cases} \quad (26)$$

where η, θ are ordinal numbers.

The following special cases follow from (26) for a standard model $\mathcal{M}$:

(i) If M_o, M_ϵ are finite then, clearly, all M_α for $\alpha \in$ *Types* are finite. In this case, we say that the whole *model is finite.*

(ii) Let $\mathrm{Card}(M_\epsilon) < \aleph_0$ and $\mathrm{Card}(M_o) \in \{\aleph_0, \aleph_1\}$. Then $\mathrm{Card}(M_{\epsilon o}) \in \{\aleph_1, \aleph_2\}$ while $\mathrm{Card}(M_{o\epsilon}) < \aleph_0$. If α does not contain the type o then $\mathrm{Card}(M_\alpha) < \aleph_0$. Thus, if $\mathrm{Card}(M_\alpha) = \aleph_\eta$ and $\mathrm{Card}(M_\beta) < \aleph_0$ then $\mathrm{Card}(M_{\beta\alpha}) = \aleph_{\eta+1}$ and $\mathrm{Card}(M_{\alpha\beta}) = \aleph_\eta$.

It is clear that analogous properties are obtained for $\mathrm{Card}(M_o) < \aleph_0$ and $\mathrm{Card}(M_\epsilon) \in \{\aleph_0, \aleph_1\}$.

(iii) If $\mathrm{Card}(M_o), \mathrm{Card}\, M_\epsilon \in \{\aleph_0, \aleph_1\}$, or one (or both) of the former are finite then $\mathrm{Card}(M_\alpha) < \aleph_\omega$ for all $\alpha \in$ *Types.*

4. Conclusion

In this paper, we introduced few basic concepts of model theory in fuzzy type theory and some of their properties. We focused on the concepts related to homomorphism, isomorphism, submodel, elementary embedding, etc. The paper is introductory to the topic demonstrating specificities of model theory for fuzzy type theory. In the future, we will focus on questions concerning construction of models, their cardinality, and various interrelations among them.

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