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TOWARDS FUZZY INTERPOLATION WITH “AT LEAST–AT MOST” FUZZY RULE BASES

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Fuzzy interpolation property is among the most important properties of fuzzy inference systems. It has been showed that the normality plus Ruspini condition applying to the antecedent fuzzy sets is a sufficient condition with a high practical impact.

Another important property is the monotone behavior of the resulting control function (after a defuzzification) derived from a monotone fuzzy rule base. Unfortunately, this goal may be often reached only when applying *at least* and *at most* modifiers which is in collision with the Ruspini condition.

This paper tries to answer the question whether this collision is an unavoidable obstacle for the interpolation property or whether the “lost” Ruspini condition does not cause losing the interpolation.

Keywords: Fuzzy interpolation; Fuzzy relation equation; At least and at most modifiers; Fuzzy inference systems.

1. Introduction

Fuzzy inference mechanism as one of keynote issues in fuzzy modeling. It employs fuzzy rules

$$\mathbf{IF } x \mathbf{ is } A_i \mathbf{ THEN } y \mathbf{ is } B_i, \quad x \in X, y \in Y, \quad i = 1, \dots, n \quad (1)$$

and an inference mechanism, which is generally an *image*^a of a fuzzy sets under a fuzzy relation stemming from a *fuzzy relational composition* denoted by @. In Ref. 1, it has been showed, that unlike the other ones,

^aBy the word “image” we do not mean a result of a mapping but the mapping itself, as it is used often in literature. Such an image is defined as a composition of a fuzzy set and fuzzy relation so, it is not correct to talk about composition of fuzzy relations.

the *Computational Rule of Inference* (CRI) denoted by \circ and the *Bandler-Kohout subproduct* denoted by \triangleleft are the only (equally) appropriate images/compositions.

Let us fix a residuated lattice

$$\mathcal{L} = \langle [0, 1], \wedge, \vee, *, \rightarrow, 0, 1 \rangle$$

where $*$ is a left-continuous t-norm and \rightarrow its adjoint residuum, as the basic algebraic structure for the whole paper.

Fuzzy rule base (1) is then modelled by fuzzy relation $\hat{R} \subseteq X \times Y$:

$$\hat{R}(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \quad (2)$$

which keeps the conditional nature of the rules. An alternative to the approach given above consists in modelling the rules by fuzzy relation $\check{R} \subseteq X \times Y$:

$$\check{R}(x, y) = \bigvee_{i=1}^n (A_i(x) * B_i(y)). \quad (3)$$

Approach given by (2) is sometimes called implicative² while the one given by (3) is called conjunctive or Mamdani-Assilian approach. The conjunctive approach obviously does not implement any kind of implication (no conditional nature) however, both approaches have sound logical backgrounds.³

2. Fuzzy inference systems - properties

There are some desirable properties which should be kept by any fuzzy inference system.

2.1. Fuzzy interpolation

One of the fundamental properties is the fuzzy interpolation which leads to systems of fuzzy relation equations

$$A_i @ R = B_i, \quad i = 1, \dots, n, \quad (4)$$

solved with respect to the unknown $R \in \mathcal{F}(X \times Y)$.

Remark 2.1. Let us stress, that we abstract from the problems of higher input dimension due to the fact that they have no impact on our investigation where $x \in X$ may be a vector variable.

Let us recall some main results.

Theorem 2.1. *System (4) with $@ \equiv \circ$ is solvable if and only if \hat{R} is a solution of the system and moreover, \hat{R} is the greatest solution of the system.*

Although Theorem 2.1 states the necessary and sufficient condition of the solvability of the system with CRI we still do not know, when \hat{R} is the solution, i.e., how to guarantee the solvability.

Theorem 2.2. *Let A_i for $i = 1, \dots, n$ be normal. Then \check{R} is a solution of (4) with $@ \equiv \circ$ if and only if the following condition*

$$\bigvee_{x \in X} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in Y} (B_i(y) \leftrightarrow B_j(y)), \quad (5)$$

holds for arbitrary $i, j \in \{1, \dots, n\}$.

Theorem 2.2 specifies a sufficient condition under which the system is solvable. Moreover, it ensures that not only \hat{R} but also \check{R} is a solution of the given system with \circ composition.

Analogous results are valid even for the Bandler-Kohout subproduct.

Theorem 2.3. *System (4) with $@ \equiv \triangleleft$ is solvable if and only if \check{R} is a solution of the system and moreover, \check{R} is the least solution of the system.*

Theorem 2.4. *Let A_i for $i = 1, \dots, n$ be normal. Then \hat{R} is a solution of (4) with $@ \equiv \triangleleft$ if and only if condition (5) holds for arbitrary $i, j \in \{1, \dots, n\}$.*

Condition (5) to which Theorems 2.2,2.4 refer to, is not very convenient from a practical point of view. Sufficient conditions for solvability of both systems with a high practical importance is published in Refs. 4,5.

Theorem 2.5. *Let A_i for $i = 1, \dots, n$ be normal and fulfill the Ruspini condition*

$$\sum_{i=1}^n A_i(x) = 1, \quad x \in X. \quad (6)$$

Then the system (4) with $@ \in \{\circ, \triangleleft\}$ is solvable.

The advantage of Theorem 2.5 lies in the fact that it imposes properties only on antecedent fuzzy sets. Therefore, they may be easily fulfilled independently on the consequents already when building the fuzzy rule base.

2.2. Monotonicity - motivation for the study

Some processes lead to a monotone “*the bigger antecedent, the bigger consequent*” knowledge expressed by a fuzzy rule base.

If we introduce an appropriate partial ordering of fuzzy sets \leq_f , e.g., based on the ordering of α -cuts, monotone fuzzy rule bases may be defined. Let us recall that Remark 2.1 is relevant even for this part of the text.

Definition 2.1. Fuzzy rule base (1) is called *monotone* if for any two rules:

IF x **is** A_i **THEN** y **is** B_i

IF x **is** A_j **THEN** y **is** B_j

such that $A_i \leq_f A_j$, the inequality $B_i \leq_f B_j$ holds.

In Ref. 6, it has been observed, that the monotonicity of the fuzzy rule base does not guarantee the monotonicity of the resulting “control function”^b that connects crisp inputs with crisp outputs upon defuzzification. This serious problem has been already investigated.^{7,8}

In case of the implicative interpretation, the monotonicity can be hardly guaranteed. On the other hand, if we employ the idea of “at least - at most” fuzzy rule bases, the conditions may be found easily.⁸

At least (\mathcal{L}) and *at most* (\mathcal{M}) modifiers have been proposed in Ref. 9. For $C \subseteq X \subseteq \mathbb{R}$, fuzzy sets $\mathcal{L}(C), \mathcal{M}(C) \subseteq X$ may be defined as follows

$$\mathcal{L}(C)(u) = \sup\{C(t) \mid t \in U, t \leq u\},$$

$$\mathcal{M}(C)(u) = \sup\{C(t) \mid t \in U, t \geq u\}.$$

In Ref.,⁸ every single rule

IF x **is** A_i **THEN** y **is** B_i

is understood as *at least* rule or/and as *at most* rule

IF x **is** $\mathcal{L}(A_i)$ **THEN** y **is** $\mathcal{L}(B_i)$,

IF x **is** $\mathcal{M}(A_i)$ **THEN** y **is** $\mathcal{M}(B_i)$,

respectively. So, we consider the so called *at least* fuzzy rule base, *at most* fuzzy rule base which are modelled by fuzzy relations given by (2,3) with the only difference that the antecedent and consequent fuzzy sets are obtained with help of the *at least* and *at most* modifiers.

^bWe use the term “control function”, although it is clear that all concepts can be applied in other domains beside control equally well.

Conditions for the monotonicity of the resulting (control) function of such adjoint fuzzy rule bases connected to appropriate defuzzification methods have been found.⁸

Unfortunately, the antecedent fuzzy sets $\mathcal{L}(A_i), \mathcal{M}(A_i)$ of the above mentioned fuzzy rule bases do not meet the Ruspini condition. Therefore, we cannot use Theorem 2.5 to guarantee the solvability of the adjoint systems of fuzzy relation equations:

$$\mathcal{L}(A_i) @ R = \mathcal{L}(B_i), \quad i = 1, \dots, n, \quad (7)$$

$$\mathcal{M}(A_i) @ R = \mathcal{M}(B_i), \quad i = 1, \dots, n, \quad (8)$$

with respect to unknown $R \in \mathcal{F}(X \times Y)$. So, if we want to ensure the monotonicity of the resulting function after an appropriate defuzzification, by applying the modifiers, we cannot be sure about the interpolation property. The questions, whether these two elementary properties may be ensured in advance, is the main motivation for the investigation presented in the next Section.

3. Main results

3.1. Single rule problem

First, let us concentrate on the result valid just for $n = 1$, i.e., we study the adjoint fuzzy relation equation related to a single fuzzy rule problem.

Proposition 3.1. *Let A be normal. Then $\mathcal{L}(\hat{R})$ is a solution to*

$$\mathcal{L}(A) \circ R = \mathcal{L}(B).$$

Proposition 3.2. *Let A be normal. Then $\mathcal{L}(\check{R})$ is a solution to*

$$\mathcal{L}(A) \triangleleft R = \mathcal{L}(B).$$

Analogous results may be obtained for the adjoint fuzzy relation equations employing the *at most* modifiers as well.

Proposition 3.3. *Let A be normal. Then $\mathcal{M}(\hat{R})$ is a solution to*

$$\mathcal{M}(A) \circ R = \mathcal{M}(B).$$

Proposition 3.4. *Let A be normal. Then $\mathcal{M}(\check{R})$ is a solution to*

$$\mathcal{M}(A) \triangleleft R = \mathcal{M}(B).$$

We have showed, that the normality of the original fuzzy sets is a sufficient condition for the solvability of the adjoint fuzzy relation equations employing *at least* or *at most* modifiers. The results are valid for both, the CRI as well as the Bandler-Kohout subproduct. Moreover, the Propositions above, also determine the solutions. These solutions are the standard interpretations of fuzzy rules to which the modifiers are applied.

Although without deeper practical impact and just preliminary, this is an encouraging result promising that the adjoint systems with n fuzzy relation equations could also be solvable by standard interpretations with the modifiers.

3.2. Multiple rules

Let us recall, that we follow the idea of Ref. 8 and we consider the partial ordering of convex fuzzy sets \leq_f based on the α -cuts:

$$A_i \leq_f A_j \quad \text{if} \quad \inf[A_i]_\alpha \leq \inf[A_j]_\alpha \quad \text{and} \quad \sup[A_i]_\alpha \leq \sup[A_j]_\alpha \quad \forall \alpha \in (0, 1],$$

where $[\cdot]_\alpha$ denotes the α -cut.

Although the antecedent fuzzy sets $\mathcal{L}(A_i), \mathcal{M}(A_i)$ of the adjoint fuzzy rule bases cannot fulfill the Ruspini condition, the original fuzzy rule base antecedent fuzzy sets A_i were assumed to fulfill this requirement to meet the assumptions of Theorem 2.5, throughout the monotonicity problem investigation.⁸ Therefore, these antecedent fuzzy sets may be linearly ordered and consequently denoted by indexes in such a way that $i \leq j$ implies $A_i \leq_f A_j$.

Theorem 3.1. *Let fuzzy rule base (1) be monotone. Let A_i be normal, fulfill the Ruspini condition and be ordered in such a way that $A_i \leq_f A_j$ iff $i \leq j$. Then $\mathcal{L}(\hat{R})$ is a solution to system (7) with $@ \equiv \circ$.*

Theorem 3.2. *Let fuzzy rule base (1) be monotone. Let A_i be normal, fulfill the Ruspini condition and be ordered in such a way that $A_i \leq_f A_j$ iff $i \leq j$. Then $\mathcal{M}(\check{R})$ is a solution to system (8) with $@ \equiv \triangleleft$.*

Let us again stress, we have known nothing about solvability of (7) with $@ \equiv \circ$ and (8) with $@ \equiv \triangleleft$ before. The Ruspini condition guaranteed the solvability only for (4) with either \circ or \triangleleft .

4. Conclusion

Fuzzy interpolation property is among the most important properties of fuzzy inference systems. It leads to the study of solvability of systems of

fuzzy relation equations. We recall the condition published, e.g., in Refs. 4, 5. The main importance of this condition lies in very simple assumptions (normality plus Ruspini) only for antecedent fuzzy sets which may be easily fulfilled in advance to ensure the solvability independently on consequents.

Monotonicity of the resulting control function is another important property. Some particular results in this topic show, that especially for the implicational interpretation the monotonicity is hardly directly feasible. The solution comes from applying the *at least* and *at most* modifiers. Nevertheless, applying these modifiers to antecedent fuzzy sets does not meet the Ruspini condition anymore. It is questionable, if the Ruspini condition of the original antecedent fuzzy sets ensuring the solvability of original system of fuzzy relation equations is sufficient even for the solvability of the adjoint systems of fuzzy relation equations.

Partially positive answers to this question are given by Propositions 3.1-3.4 and especially by Theorems 3.1-3.2.

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