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Reasoning about Mathematical Fuzzy Logic and its Future

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Abstract

This paper is devoted to reasoning about fuzzy logic which is based on various personal observations of the author. Our goal is to think of the state of the art in mathematical fuzzy logic (MFL) and to outline some of the tasks on which, in the author's opinion, MFL should focus in the future. In our discussion, we will mention not only the basic theory, but also its extension called *fuzzy logic in broader sense* (FLb). The paradigm of the latter is to be the logic of natural human reasoning, whose most essential characteristic is the use of natural language. Besides brief description of FLb, we will also mention some of its applications. On the basis of that, we will ponder on other possible directions for research, namely the possibility of using FLn as a metatheory of fuzzy mathematics, as a proper tool for modeling of the main manifestations of the phenomenon of vagueness, and as a reasonable tool for developing models of linguistic semantics.

Keywords: Fuzzy logic in narrow sense, fuzzy logic in broader sense, evaluative linguistic expressions, intermediate quantifiers, vagueness, fuzzy/linguistic IF-THEN rules, perception-based logical deduction, approximate reasoning, fuzzy mathematics, linguistic semantics

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1. Introduction

This paper is devoted to reasoning about fuzzy logic which is based on various personal observations of the author who stood at the roots of fuzzy logic, contributed to the development of some of its constituents and also realized several kinds of its applications. We will concentrate on the following goals: (a) to provide a brief survey of the present situation in MFL and to demonstrate that this is already a well-established formal theory; (b) to speculate about ways how it could be further developed. We think that it should be directed more towards development of mathematical models of human thinking, characterized by the essential use of natural language; (c) to outline some potential applications, both in the development of fuzzy mathematics itself (including the models due to (b)) as well as specific applications in practical human activities such as business or industrial production.

The structure of the paper is as follows: in Section 2, we will briefly overview the basic constituent of MFL, which is the *fuzzy logic in narrow sense* (FLn). In Section 3, we will focus on an extension of FLn called *fuzzy logic in broader sense* (FLb)[2]. Since FLb is much less well known, we will give slightly more details. In Section 4, we will present several theorems showing that the ancient *sorites* and *falakros* paradoxes are not paradoxical in MFL. Section 5 is devoted to an interesting program of *fuzzy mathematics* — an attempt at the systematic development of this branch of mathematics on the basis of FLn as a metatheory. Finally, Section 6 is a succinct presentation of the author's view of the problems on which MFL should be focused in the future.

The paper is in many places quite brief. Let us emphasize that its goal is not to provide an exhaustive presentation of fuzzy logic in narrow sense because there are several good existing papers on this topic, e.g., [14, 33, 34, 38] among others. We will pinpoint some particular points which are, in our opinion, interesting and suggest ways for the further development of the field.

2. Fuzzy Logic in Narrow Sense

2.1. Initial remarks

Mathematical fuzzy logic (MFL) has been established as a sound formal system whose objective is to provide tools for the development of a working mathematical model of vagueness phenomenon and to offer well justified applications of it. The mathematization of vagueness is based on the introduction of degrees of truth, which form a special algebra.

The history began by the seminal paper on fuzzy logic written by J. A. Goguen in [31], which appeared quite shortly after the seminal paper on fuzzy set theory written by L. A. Zadeh [96]. Goguen proposed to split the operation representing conjunction into two operations (minimum and multiplication) and to consider the residuated lattice as a convenient structure of truth values, and thus as a convenient basis for the semantics of fuzzy logics. The paper contains a lot of solid material accompanied by detailed philosophical discussion, but it is not sufficiently formal. The first highly mathematically sophisticated

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[2]Let us emphasize that FLb is a specific paradigm that should not be confused with L. A Zadeh's concept of "fuzzy logic in the wide sense"!

paper on the subject was written by J. Pavelka [83]. Since then, significant developments were achieved in FLn. The Pavelka's work has been continued by V. Novák [61]. The most essential moment for the subsequent development of MFL was publication of the monograph written by P. Hájek [35]. A crucial influence on the development of logic had also results in algebra especially thanks to U. Höhle [45] and D. Mundici (cf. [10, 58]). Other people who have significantly contributed to the development of MFL are F. Esteva, L. Godo, S. Gottwald, A. DiNola, F. Montagna, R. Cignoli, G. Gerla, E. Turunen, P. Cintula, C. Noguera, and others.

We can distinguish two directions in FLn:

(i) Fuzzy logic with traditional syntax, and

(ii) fuzzy logic with evaluated syntax.

As a matter of fact, we are not speaking about two logics but about *two classes* of logics, which are distinguished, first by the way in which their syntax is established, and second, by the properties determined by the chosen structure of the truth values. While the first class is wide, the second class is quite narrow being limited to structures of truth values with continuous connectives (namely classically continuous if the support is $[0, 1]$). This immediately raises the question: *Which of these logics is the "real fuzzy logic"?*[3] In other words, *which of these logics is the most proper tool for the construction of an acceptable model of vagueness ?* More deeply, *which algebraic structure is the best structure of truth values for this purpose?*

There are several partial answers to these questions. In the first place, we must answer the question: *Can we obtain a suitable model of the phenomenon of vagueness using fuzzy logic?* Many arguments in favor of the *positive* answer have been given, e.g., in [36, 37, 64, 68]. It must be emphasized, however, that fuzzy logic is not the logic of vagueness but the *logic of ordered structures*! Hence, the core of the answer lies in accepting (or rejecting) the idea of using degrees when describing vagueness. Consequently, we cannot say that vagueness follows the laws of some of our logics, but the other way around — using our logic(s) we can develop models that behave as if they were affected by vague phenomena (at least in the important aspects). This makes our situation easier.

Possible answers to the other questions posed above can be found in [5, 38, 41, 69]. One idea is to delineate fuzzy logics mathematically. A reasonable criterion is that a logic is fuzzy if it is complete w.r.t. *linearly ordered* matrices. However, the class of the remaining logics is still too wide, though very interesting. A pragmatic but less explicit point of view requires a fuzzy logic to be an open system enabling the development of sophisticated inference schemes of human reasoning. This suggests basic requirements that the logic in concern should satisfy. For example, it turns out that such a logic should satisfy the law of contraposition, (and, consequently, the law of double negation). In general, however, we cannot expect just one logical system to be the "real fuzzy logic" but rather a sufficiently narrow class whose members could be used in various situations for various purposes.

Below, we will provide a brief overview of FLn. This is a quite difficult task because the interrelation among all the considered logics is very intricate, and its full description would require several tens of pages. Therefore, we will pinpoint only a few aspects and refer the reader to the many existing expositions, e.g., [12, 13, 14, 15, 25, 35, 39, 80].

2.2. Algebraic structures of truth values

As mentioned, the fundamental algebra of truth values for fuzzy logic is a residuated lattice (more precisely, a (commutative) integral, bounded, lattice-ordered residuated monoid)

$$\mathscr{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle , \tag{1}$$

where $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with bottom element $\mathbf{0}$ and top element $\mathbf{1}$, and $\langle L, \otimes, \mathbf{1} \rangle$ is a monoid, which is usually considered to be commutative (but not necessarily). The operation $\rightarrow$ is the *residuation*

[3] This question was raised during the discussions that took place on the conference "The Challenge of Semantics", Vienna, July 2004 and on XXVI$^{\text{th}}$ Linz seminar, Linz, February 2005. Partial answers were gathered in the special issue of Fuzzy Sets and Systems **157**(2005), ed. by V. Novák.

operation satisfying the adjointness property: for all $a, b, c \in L$,

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c\,. \tag{2}$$

In the case that $\otimes$ is not commutative, this condition enforces two different implications (residuations). It is then questionable how the second implication should be interpreted. A deep algebraic analysis of general residuated lattices is provided in [29].

The following are special derived operations:

$$\neg a = a \rightarrow \mathbf{0}, \qquad \text{(negation)}$$
$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \qquad \text{(biresiduation)}$$

Additional properties that may be considered include the following:

$$(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}, \qquad \text{(prelinearity)}$$
$$\neg\neg a = a, \qquad \text{(double negation)}$$
$$a \otimes (a \rightarrow b) = a \wedge b, \qquad \text{(divisibility)}$$
$$(a \rightarrow b) \rightarrow b = a \vee b, \qquad \text{(MV)}$$
$$\neg\neg c \leq (a \otimes c \rightarrow b \otimes c) \rightarrow (a \rightarrow b), \qquad (\Pi_1)$$
$$a \wedge \neg a = \mathbf{0}, \qquad (\Pi_2)$$
$$a \otimes a = a, \qquad \text{(idempotence)}$$
$$(a \otimes b \rightarrow \mathbf{0})) \vee ((a \otimes b) \rightarrow (a \wedge b)) = \mathbf{1}. \qquad \text{(WNM)}$$

These properties give rise to special classes of algebras. Essential for fuzzy logic are *MTL-algebras*[4], which satisfy *prelinearity*. Further types of algebras are *strict MTL-algebras* (prelinearity, (Π_2)), *cancellative MTL-algebras* (prelinearity, $(\Pi_1), (\Pi_2)$), *BL-algebras*[5] (prelinearity, divisibility), *IMTL-algebras* (prelinearity, double negation), *MV-algebras* ((MV)), *Gödel algebras* (prelinearity, divisibility, idempotence), *product algebras* (prelinearity, divisibility, $(\Pi_1), (\Pi_2)$), WNM-algebras[6] (prelinearity, (WNM)), NM-algebras[7] (prelinearity, double negation, WNM), and several other kinds of algebras.

If $L = [0, 1]$, then the corresponding algebra is called *standard*. In the standard MTL-algebra, the multiplication $\otimes$ is a left-continuous t-norm, in the standard BL-algebra it is a continuous t-norm, and in the standard product algebra it is the ordinary product of real numbers. An exceptional role is played by the standard Łukasiewicz MV-algebra

$$\mathscr{L}_{\text{L}} = \langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle$$

where

$$a \otimes b = \max(0, a + b - 1), \qquad \text{(Łukasiewicz conjunction)}$$
$$a \rightarrow b = \min(1, 1 - a + b). \qquad \text{(Łukasiewicz implication)}$$

This algebra is determined by the fact that it is the only standard residuated lattice (up to isomorphism) whose residuation operation is continuous. This has important consequences both for the mathematics and for its applications.

There is also a finite MV-chain (finite Łukasiewicz algebra)

$$\langle \{a_0, \ldots, a_m\}, \vee, \wedge, \otimes, \rightarrow, a_0, a_m \rangle$$

with the operations $a_i \otimes a_k = a_{\max(0, i+k-m)}$ and $a_i \rightarrow a_k = a_{\min(m, m-i+k)}$.

[4]MTL means Monoidal T-norm-based Logic.

[5]BL-means Basic (fuzzy) Logic.

[6]WNM means Weak Nilpotent-Minimum.

[7]NM means Nilpotent Minimum.

A particularly important operation is the Δ operation, which is defined in $[0,1]$ by

$$\Delta(a) = \begin{cases} 1, & \text{if } a = 1, \\ 0 & \text{otherwise} \end{cases}$$

(in general, Δ is characterized by 6 axioms; see [35]). This operation extracts, in a sense, a boolean structure from the corresponding algebra.

The most complicated residuated-lattice-based algebra is ŁΠ ([11, 26]):

$$\mathscr{L} = \langle L, \vee, \wedge, \rightarrow_{Ł}, \rightarrow_{\Pi}, \otimes_{Ł}, \otimes_{\Pi}, \mathbf{0}, \mathbf{1}\rangle$$

where $\mathscr{L} = \langle L, \vee, \wedge, \rightarrow_{Ł}, \otimes_{\Pi}, \Delta, \mathbf{0}, \mathbf{1}\rangle$ is an MV_{Δ}-algebra, $\langle L, \otimes_{\Pi}, \mathbf{1}\rangle$ is a commutative monoid, and the following properties hold:

$$\begin{aligned}
a \otimes_{\Pi} (b \otimes_{Ł} \neg_{Ł} c) &= (a \otimes_{\Pi} b) \otimes_{Ł} \neg_{Ł}(a \otimes_{\Pi} c), \\
\Delta(a \leftrightarrow_{Ł} b) \wedge \Delta(c \leftrightarrow_{Ł} d) &\leq ((a \otimes c) \leftrightarrow_{Ł} (b \otimes d)), \quad \otimes \in \{\rightarrow_{\Pi}, \otimes_{\Pi}\}, \\
a \wedge (a \rightarrow_{\Pi} \mathbf{0}) &= \mathbf{0}, \\
\Delta(a \rightarrow_{Ł} b) &\leq (a \rightarrow_{Ł} b), \\
\Delta(a \rightarrow_{Ł} b) &\leq (a \otimes_{\Pi} (a \rightarrow_{\Pi} b) \leftrightarrow_{Ł} b).
\end{aligned}$$

All these structures have one important common property, which follows from the prelinearity property.

Theorem 1
Each MTL-algebra (and thus all the algebras considered above) is isomorphic to a subdirect product of linearly ordered MTL algebras.

This theorem has substantial impact on fuzzy logics, as discussed below.

Quite recently, a new special algebra, called a (non-commutative) EQ-algebra, has been introduced [22, 23, 75]. The idea behind it is to introduce an algebra of truth values in which implication, as one of the basic operations, is replaced by *fuzzy equality* (equivalence). Hence, it is an algebra of type (2, 2, 2, 0)

$$\mathscr{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle\,, \tag{3}$$

where for all $a, b, c, d \in E$,

(E1) $\langle E, \wedge, \mathbf{1}\rangle$ is a $\wedge$-semilattice with the top element $\mathbf{1}$,

(E2) $\langle E, \otimes, \mathbf{1}\rangle$ is a monoid and $\otimes$ is isotone w.r.t. $\leq$,

(E3) $a \sim a = \mathbf{1}$,

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$,

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$,

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,

(E7) $a \otimes b \leq a \sim b$ [8].

The implication operation is derived using the formula[9]

$$a \rightarrow b = (a \wedge b) \sim a.$$

[8] In [75], also the axiom (E8) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$ was introduced. It was shown in [23], however, that it is provable from the other ones.

[9] It is interesting to note that this formula was, in principle, already proposed by G. W. Leibniz.

Unlike the non-commutative residuated lattice, the EQ-algebra has only one implication independently of the (non-)commutativity of $\otimes$.

For logic, the most interesting are the so called *good EQ-algebras*, which have the property $a \sim \mathbf{1} = a$. If $\mathscr{E}$ contains the bottom element $\mathbf{0}$, then it is possible to introduce negation by $\neg a = a \sim \mathbf{0}$. An IEQ-algebra is an EQ-algebra satisfying the law of double negation $\neg\neg a = a$. A theorem on the subdirect representation of special good EQ-algebras has also been proved (see [22]).

Let us remark that EQ-algebras share many properties with residuated lattices but are not equivalent to them. Namely, each residuated lattice gives rise to a good EQ-algebra, but there are EQ-algebras (including good ones) that are not residuated (cf. [75]) and thus do not give rise to residuated lattices.

All the above mentioned algebras, considered as structures of truth values, have amazing properties with very subtle and complicated interrelations. Note that all of them generalize boolean algebra for classical logic in more or less natural ways.

2.3. Fuzzy logics with traditional syntax

This class of logics has been elaborated in a deep and seminal way by P. Hájek in the monograph [35]. Formulas in these logics are dealt with classically, i.e., the language of fuzzy logic differs from that of classical logic in particular by having more connectives. On the other hand, the concepts of inference rule, theory, proof and many other ones remain classical. The language $\mathscr{J}$ of fuzzy logics contains *disjunction* $\vee$, *conjunction* $\wedge$, *implication* $\Rightarrow$, a specific new connective of *strong conjunction* $\mathbin{\&}$, and a logical (truth) constant $\bot$ for falsity. Of course, depending on the chosen logic, some of the connectives can be defined in terms of the others. The connectives of *negation* $\neg$ and *equivalence* $\Leftrightarrow$ are derived.

The semantics is many valued based on a corresponding algebra, as discussed above. For convenience, we will speak about a $\mathscr{J}$-algebra, where each connective from $\mathscr{J}$ is interpreted by the corresponding algebraic operation (namely, $\vee$ is interpreted by $\vee$, $\wedge$ by $\wedge$, $\mathbin{\&}$ by $\otimes$, $\Rightarrow$ by $\rightarrow$, and accordingly for all other possible connectives). An extensive survey of various kinds of algebraic semantics is presented in [13].

The most prominent fuzzy logic is MTL, whose algebraic semantics is formed by MTL-algebras. Its axioms are as follows:

(A1) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

(A2) $(A \wedge B) \Rightarrow A$

(A3) $(A \wedge B) \Rightarrow (B \wedge A)$

(A4) $(A \mathbin{\&} (A \Rightarrow B)) \Rightarrow (A \wedge B)$

(A5a) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \mathbin{\&} B) \Rightarrow C)$

(A5b) $((A \mathbin{\&} B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

(A6) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

(A7) $\bot \Rightarrow A$

(for the details, see [24][10]). This logic is the basis for the so called *core fuzzy logics* introduced in [15, 39]. The latter are expansions of MTL-logic satisfying the following additional properties:

(i) For any formulas A, B, C in the language of MTL, $A \Leftrightarrow B \vdash C(A) \Leftrightarrow C(B)$.

(ii) For every theory T and formulas A, B, the following holds (*Local Deduction Theorem*):

$$T \cup \{A\} \vdash B \quad \text{iff} \quad T \vdash A^n \Rightarrow B$$

for some n, where A^n is a short for $A \mathbin{\&} \cdots \mathbin{\&} A$ (n-times).

[10] The list originally also contained the axioms $(A \mathbin{\&} B) \Rightarrow A$ and $(A \mathbin{\&} B) \Rightarrow (B \mathbin{\&} A)$. It was proved in [9] that these axioms are provable from the others; the latter are already independent.

Prominent examples of core fuzzy logics include IMTL, BL, product, and Łukasiewicz, among others.

All the core fuzzy logics enjoy the following form of completeness.

Theorem 2
For every formula A and theory T, the following are equivalent:

(a) $T \vdash A$,

(b) A is true in degree **1** *in every linearly ordered $\mathscr{J}$-algebra that is a model of T.*

(c) A is true in degree **1** *in every $\mathscr{J}$-algebra that is a model of T.*

This theorem gave rise to the idea that a given logic deserves to be called "fuzzy" if it is complete w.r.t. linearly ordered $\mathscr{J}$-algebras. Even more, it raised the question whether it is possible to confine our attention only to the standard algebras.

The answer is partly positive. We call a logic *standard* complete if it is complete w.r.t. standard chains. It is *hyperreal-chain* complete if it is complete w.r.t. a class of chains whose lattice reduct is an ultrapower of $[0,1]$. At the same time, if the theory T in Theorem 2 is arbitrary, then we speak about *strong* completeness. If it is only allowed to be finite, then we speak about *finite strong* completeness. If $T = \emptyset$, then it is complete.

Clearly, strong standard completeness implies finite strong standard as well as standard completeness. Moreover, it also implies strong hyperreal-chain as well as finite strong hyperreal-chain and hyperreal-chain completeness.

The following hold for some of the core fuzzy logics (cf. [13]).

Logic	standard complete	finite strong st. complete	strong st. complete
MTL, IMTL, SMTL, Gödel, WNM, NM	Yes	Yes	Yes
BL, SBL, Łukasiewicz, product, ΠMTL	Yes	Yes	No

At present, there are many core fuzzy logics, both in propositional and in predicate versions. In the paper [15], 57 different fuzzy logics are studied.

FLn has been studied in detail, especially from the algebraic and metamathematical points of view. One can hardly estimate how far this work may reach. It seems that FLn offers a deeply justified technique enabling us to model various manifestations of the phenomenon of vagueness and to develop other applications. However, there are not yet many results on the applied side. We know a lot about fuzzy logics "from the outside" but there are only a few relevant results inside them.

2.4. Fuzzy logic with evaluated syntax ($Ev_Ł$)

This logic is specific by introducing formulas that are also evaluated on the syntactic level. The basic concept is that of an *evaluated formula* a/A, where A is a formula and $a \in L$ is its *syntactic* evaluation. This has a nice interpretation since it allows us to consider axioms that need not be fully satisfactory, so their initial truth values can be lower than **1**. Moreover, it enables us to derive statements about intermediate truth values.

A propositional version of this logic was established by J. Pavelka in [83] and extended to a predicate version by V. Novák in [61] and in the book [80]. As usual, $Ev_Ł$ is a mathematical logic with clearly distinguished syntax and semantics. The syntax, however, is generalized, and in addition to evaluated formulas, it also has precise definitions of evaluated proof, fuzzy theory, evaluated provability, model, etc. Its semantics is determined by a finite Łukasiewicz algebra, or, in the case of $L = [0,1]$, the standard Łukasiewicz algebra only (up to isomorphism). The reason for the latter follows from [83, Part III, Theorem 1.7] (cf. also [80, Theorem 4.2]) where it has been proved that completeness of the syntax w.r.t.

semantics requires the corresponding algebra to satisfy the following four equations:

$$\bigvee_{i\in I}(a \to b_i) = a \to \bigvee_{i\in I} b_i, \qquad \bigwedge_{i\in I}(a \to b_i) = a \to \bigwedge_{i\in I}(b_i) \tag{4}$$

$$\bigvee_{i\in I}(a_i \to b) = \bigwedge_{i\in I} a_i \to b, \qquad \bigwedge_{i\in I}(a_i \to b) = \bigvee_{i\in I} a_i \to b \tag{5}$$

These equations are equivalent in $[0, 1]$ to the continuity of $\to$, which is satisfied only by the Łukasiewicz implication and its isomorphs.

The resulting logic is quite strong and has a lot of interesting properties. As mentioned, its syntax is evaluated, which means that evaluated formulas are manipulated using special inference rules. A formal *fuzzy theory* T is determined by a triple $T = \langle \mathrm{LAx}, \mathrm{SAx}, R\rangle$ where $\mathrm{LAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of special axioms and R is a set of inference rules.

The *evaluated formal proof* of A in T is a sequence of evaluated formulas where each (evaluated) formula is an axiom (in a certain degree) or has been derived using an inference rule. The evaluation of the last formula in the proof is a *value* of the proof. This gives rise to the notion of *provability degree*[11]:

$$a = \bigvee\{\mathrm{Val}(w) \mid w \text{ is a proof of } A \text{ in } T\}. \tag{6}$$

Then we say that a formula A is a *theorem in degree a* in T and write $T \vdash_a A$. Note that (6) naturally generalizes the classical definition of provability where finding one proof is sufficient, while here, the provability is obtained as a supremum of values of still "better and better" proofs.

The completeness of $\mathrm{Ev_L}$ takes the form of a generalization of the Gödel-Henkin completeness theorem:

$$T \vdash_a A \quad \text{iff} \quad T \models_a A, \qquad a \in L, \tag{7}$$

for all all formulas A and all *fuzzy theories* T, where a in $T \models_a A$ is the infimum of the truth values of A in all models of T.

It is a special property of this logic that its language, in addition to the connectives introduced above, also contains *logical (truth) constants* as names for the truth values. In its original version, all the values from L were considered, which in the case of $L = [0, 1]$ makes the language uncountable. However, a simplification to only countably many of them has been presented in [35][12], in [80] as a possible restriction, and again in [70] in a concise way from the point of view of $\mathrm{Ev_L}$.

The system of $\mathrm{Ev_L}$ is open to extension by new connectives, which makes it a fairly rich logical system. However, in the case of $L = [0, 1]$, all the connectives must be continuous. On the other hand, the fact that it is the only logic of this kind (up to isomorphism) leaves many researchers unsatisfied with it. This has led to attempts at partial approaches to $\mathrm{Ev_L}$ via the introduction of logical constants $\mathbf{a}$ for rational truth values in the language of the corresponding logic (cf. e.g., [27, 91]), which enables us to take into account also evaluated formulas; these are identified with formulas $\mathbf{a} \Rightarrow A$ which is possible because in any model $\mathscr{M}$, $\mathscr{M}(\mathbf{a} \Rightarrow A) = \mathbf{1}$ is equivalent with $\mathscr{M}(A) \geq a$. Completeness (in the classical sense) for a wide class of such logics has been proved. However, the generalized completeness (7) does not hold.

2.5. Fuzzy type theory

It turns out that a fully-fledged treatment of vagueness, especially if we want to model the semantics of natural language expressions, is not adequate when only first-order fuzzy logic is considered (cf. [53, 54]). Thus, the need for higher-order fuzzy logic arises. Such logic is a *fuzzy type theory* (FTT), which is a generalization of the classical type theory initiated by B. Russell, A. Church and L. Henkin (cf. [1, 8, 43, 90]). Note that the above mentioned program of fuzzy mathematics uses the so called *fuzzy class theory*, which is another kind of higher-order fuzzy logic (see [3]).

[11] Let us emphasize that formula (6) is not definition of the provability degree but conclusion of a theorem resulting from a more general definition of the graded logical consequence; the details can be found in [80].

[12] This is a different approach, presented under the name "Rational Pavelka Logic". In fact, this is Łukasiewicz logic with logical constants for all rational truth values, and the provability degree is introduced as an additional concept that is not generic constituent of this logic.

FTT was initiated by V. Novák in [65], where the first formal system of IMTL-FTT based on an IMTL_Δ-algebra was introduced. Other systems of FTT are BL-FTT (based on BL_Δ algebra), Ł-FTT(based on the standard Łukasiewicz$_\Delta$ algebra), and also the new EQ-FTT, which is based on a good EQ-algebra of truth values (see [74]).

The syntax of FTT is a generalization of the lambda-calculus constructed in a classical way. The main difference from classical type theory is thus in the definition of additional special connectives and the corresponding axioms[13]. Note that all the essential syntactical elements of FTT are formulas[14].

As usual, each formula A has a certain type. The basic types are o (truth values) and ϵ (elements), which can be then combined iteratively to form more complex types. Formulas of type o (truth value) can be joined by the following connectives: $\equiv$ (equivalence), $\vee$ (disjunction), $\wedge$ (conjunction), **&** (strong conjunction), ∇ (strong disjunction), $\Rightarrow$ (implication), $\neg$ (negation), and $\boldsymbol{\Delta}$ (the unary delta-connective). The quantifiers $\forall$ and $\exists$ are defined as special formulas.

If $A \in Form_{o\alpha}$, then A is interpreted as a fuzzy set of elements. It can also be understood as a first-order property of elements of the type α. Similarly, $A_{(o\alpha)\alpha}$ is interpreted as a binary fuzzy relation (between elements of type α). The crucial connective in FTT is a fuzzy equality $\equiv$ so that

$$A_\alpha \equiv B_\alpha$$

is a formula of type o representing fuzzy equality between objects of type α. This corresponds to Henkin's approach to classical type theory in which equality is the sole connective (cf. [44]). The *description operator* $\iota_{\alpha(o\alpha)}$ can also be introduced in FTT. Its interpretation is the *defuzzification operation* assigning an element from the kernel of the fuzzy set of elements of type α.

A theory T is a set of formulas of type o (truth value). It is easy to characterize crisp formulas (their interpretation is either **0** or **1**) as well as general ones and, moreover, to characterize formulas that can attain only general truth values distinct from **0** and **1**. Hence, we have the means to can realize, for example, the D-operator (definitely) as well as I-operator *indefinitely* considered in the supervaluation theory (cf. [48, 93]).

The semantics of FTT is defined using a generalization of the classical concept of frame which is a system $\langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathscr{L}\rangle$ where M_α is a set and $=_\alpha$ is a special fuzzy equality defined on each set M_α. The construction of the frame is iterative. Namely we start with some arbitrary set $(M_\epsilon, =_\epsilon)$ and the set of truth values $(L, \leftrightarrow)$. Furthermore, if $\alpha = \gamma\beta$ then $M_\alpha \subseteq M_\gamma^{M_\beta}$. Thus, the interpretation of a formula $A_{\gamma\beta}$, in general, is a function assigning to every $m \in M_\beta$ an element from M_γ.

Theorem 3 (completeness)
(a) A theory T of FTT is consistent iff T has a general model.

(b) $T \vdash A_o$ iff $T \models A_o$ holds for every theory T and a formula A_o.

We claim that all essential properties of vague predicates are formally expressible in FTT, and thus, they have a many-valued model.

3. Fuzzy Logic in Broader Sense

3.1. The paradigm of FLb

As a reaction to the concept of fuzzy logic in wide sense which is too wide and, in fact, includes also theories which are far from logic, in 1995 the paradigm of fuzzy logic in the broader sense (FLb) was proposed by V. Novák in [62] as a program to extend FLn as follows: to develop *a formal theory of natural human reasoning, which is characterized by the use of natural language. Thus, the theory should encompass mathematical models of special expressions of natural language, take into account their vagueness and develop specific reasoning schemes.* This paradigm overlaps with two other paradigms proposed in the literature, namely *common sense reasoning* and *precisiated natural language* (PNL).

[13]Unlike 9 axioms of classical TT, there are up to 19 axioms of FTT.
[14]Alternatively, they can also be called lambda-terms, as is usual in computer-science-oriented type theory.

The idea of common sense reasoning has been proposed by J. McCarthy in [55] as a part of the program of logic-based artificial intelligence. Its paradigm is to develop formal common sense theories and systems using mathematical logics that exhibit common sense behavior. The reason is that common sense reasoning is a central part of human behavior, and no real intelligence is possible without it. The main drawback of the up-to-date formalizations of common sense reasoning, in our opinion, is that it neglects the vagueness present in the meaning of natural language expressions (cf. [16] and the citations therein).

The concept of a *precisiated natural language* has been proposed by L. A. Zadeh in [100]. Its idea is to develop a "reasonable working formalization of the semantics of natural language without pretensions to capture it in detail and fineness." The goal is to provide an acceptable and applicable technical solution. It should also be noted that the term *perception* is not considered here as a psychological term but rather as a result of intrinsically imprecise human measurement. The concept of PNL is based on two main premises:

(a) much of the world's knowledge is perception based,

(b) perception based information is intrinsically fuzzy.

The PNL methodology requires presence of *World Knowledge Database* and *Multiagent, Modular Deduction Database*. The former contains all the necessary information, including perception based propositions describing the knowledge acquired by direct human experience, which can be used in the deduction process. The latter contains various rules of deduction. Until recently, however, no exact formalization of PNL had been developed, so it should be considered mainly as a reasonable methodology.

The concept of FLb is a sort of glue between the two paradigms that takes the best of each. It has been slowly developed over the years, and at present, FLb consists of the following theories[15]:

(a) Formal theory of evaluative linguistic expressions, which is explained in detail in [72] (see also [71]).

(b) Formal theory of fuzzy IF-THEN rules and approximate reasoning (derivation of a conclusion), presented in [18, 20, 66, 77, 78].

(c) Formal theory of intermediate and generalized quantifiers, presented in [19, 46, 67, 73].

Let us remark that the first attempt to develop these theories was based on Ev_{L}. The essential constituent of FLb, however, is a model of natural language semantics. Many logicians and linguists (cf. [53, 54, 92]) have argued that the first order logic is not sufficient for this task. Although the first-order Ev_{L} has greater explicative power than classical first-order logic, the formalization of items (a) and (b) above using Ev_{L}, as presented, e.g., in [80, Chapter 6], is not fully satisfactory. Therefore, a more suitable formal system has been chosen as the basis for further development of FLb, namely the *fuzzy type theory.*

3.2. Theory of Evaluative Linguistic Expressions

This is the main constituent of FLb, which, at the same time, has itself many kinds of applications. Recall that evaluative linguistic expressions are special expressions of natural language with which people evaluate phenomena around them. They include them the following classes of linguistic expressions:

(i) *Basic trichotomous evaluative expressions* (small, medium, big; weak, medium strong, strong; silly, normal, intelligent, etc.)[16].

(ii) *Fuzzy numbers* (twenty five, roughly one hundred, etc.).

(iii) *Simple evaluative expressions* (very short, more or less strong, more or less medium, roughly big, about twenty five, etc.).

(iv) *Compound evaluative expressions* (roughly small or medium, small but not very (small), etc.).

[15] There are some other papers whose topics relate to the topic of FLb (cf. [50, 95]). None of them, however, can be considered as a contribution to the consistent development of FLb as a logical theory.

[16] Note that these expressions also include the gradable adjectives

(v) *Negative evaluative expressions* (not small, not very big, etc.).

An indispensable role in the formation of these expressions is played by *hedges*, which are formed on a surface level by a special subclass of adverbs. We distinguish *narrowing hedges* (very, extremely, significantly, etc.), *widening hedges* (more or less, roughly, very roughly, etc.) and *specifying hedges* (approximately, about, rather, precisely, etc.).

When joining evaluative expressions with nouns, we obtain *evaluative linguistic predications* (temperature is low, very intelligent man, more or less weak force, medium tension, extremely long bridge, short distance and pleasant walk, roughly small or medium speed, etc.). In FLb, we usually simplify evaluative predications to

$$X \text{ is } \mathscr{A},$$

where X is a variable whose values are the values of some measurable *feature of the noun*, while the noun itself is omitted from consideration. The $\mathscr{A}$ is an evaluative expression.

Since evaluative expressions are omnipresent in natural language, they occur in the description of any process, decision situation, procedure, characterization of objects, etc. Therefore, the mathematical model of their meaning initiated in the seminal works by L. A. Zadeh (see [97, 98]) can be ranked among the most important contributions of fuzzy logic.

The formalization of the semantics of evaluative expressions is based on the standard assumptions of the theory of semantics developed in linguistics and logic (cf. [53, 54, 42]). Namely, the fundamental concepts are context (=possible world), intension, and extension. In FLb, a formal theory T^{Ev} is developed as a special theory of Łukasiewicz fuzzy type theory (Ł-FTT) formalizing certain general characteristics of the semantics of evaluative expressions.

The *intension* of an evaluative expression (or predication) $\mathscr{A}$ is obtained as interpretation of a formula $\lambda w\, \lambda x\, (Aw)x$ of Ł-FTT in a special model $\mathscr{M}$: Let U be a linearly ordered set U (we usually put $U = \mathbb{R}$).

- Recall that by a *context* (a *possible world*) we mean, in general, a state of the world at a given point in time and space. It is very difficult to formalize such a definition. However, in [72], it is argued that extensions of evaluative expressions are classes of elements taken from scale representing

  The *context* is a nonempty, linearly ordered and bounded, in which three distinguished limit points can be determined: a *left bound* v_L, a *right bound* v_R, and a *central point* v_S. Hence, each context is identified with an ordered triple

  $$w = \langle v_L, v_S, v_R \rangle$$

  where $v_L, v_S, v_R \in U$. A straightforward example is the predication "$\mathscr{A}$ town", for example "small town", "very big town", etc. Then, the corresponding context for the Czech Republic can be $\langle 3\,000, 50\,000, 1\,000\,000 \rangle$, while for the USA it can be $\langle 30\,000, 200\,000, 10\,000\,000 \rangle$.

  We introduce a set W of contexts. Each element $w \in W$ gives rise to an interval $w = [v_L, v_R] \subset U$.

- The *intension* $\mathrm{Int}(\mathscr{A})$ of an evaluative expression $\mathscr{A}$ is a property that attains various truth values in various contexts but is invariant with respect to them. Therefore, it is modeled as a function $\mathrm{Int}(\mathscr{A}) : W \longrightarrow \mathscr{F}(U)$ where $\mathscr{F}(U)$ is a set of all fuzzy sets over U.

- The *extension* $\mathrm{Ext}_w(\mathscr{A})$ of an evaluative expression $\mathscr{A}$ in the context $w \in W$ is a fuzzy set of elements

  $$\mathrm{Ext}_w(\mathscr{A}) = \mathrm{Int}(w)(\mathscr{A}) \underset{\sim}{\subset} w.$$

In our example, the truth value of a "small town having 30 000 inhabitants" could be, for example, 0.7 in the Czech Republic and 1 in the USA.

In the theory T^{Ev}, the *extension* of an evaluative expression is obtained as a shifted horizon where the shift corresponds to a linguistic hedge, which is thus modeled by a function $L \longrightarrow L$. A graphical scheme of such an interpretation in a specific context can be seen in Figure 1. The full formal theory, and also its informal justification, can be found in [72].

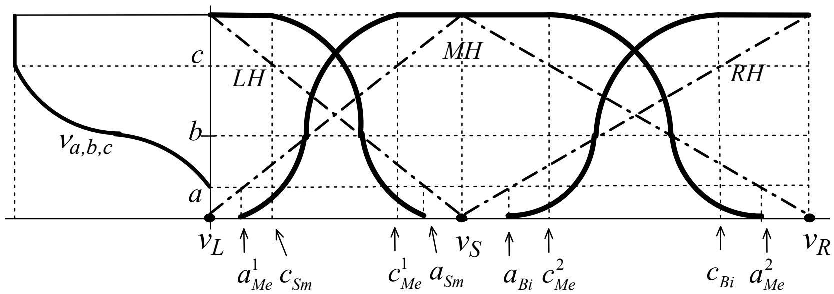

Figure 1: Graphical scheme of a construction of extensions of evaluative expressions in a given context. Each extension is obtained as a composition of a function representing a respective horizon LH, MH, RH (in the figure, it is linear because of the use of Ł-FTT) and the deformation function $\nu_{a,b,c}$ whose graph is for convenience depicted turned 90° degrees anticlockwise.

3.3. Fuzzy/linguistic IF-THEN rules and approximate reasoning

The theory of fuzzy IF-THEN rules is the most widely discussed and most powerful area of fuzzy logic, which has a wide variety of applications. However, there is still no general agreement about what we are, in fact, discussing. First, let us recall the general form of a fuzzy IF-THEN rule:

$$\mathsf{IF}\ X \text{ is } \mathscr{A}\ \mathsf{THEN}\ Y \text{ is } \mathscr{B}, \tag{8}$$

where X is $\mathscr{A}, Y$ is $\mathscr{B}$ are evaluative predications[17]. A typical example is

IF the temperature is small, THEN the amount of gas is very big.

Observe that such a rule has a surface form of a conditional linguistic clause. A (finite) set of rules (8) is called a *linguistic description*.

The rules (8) (and the linguistic descriptions) apparently characterize some kind of relation between values of X and Y. There are two basic approaches to how the rules can be construed: relational and logical/linguistic.

Relational approach. The relational approach assumes some chosen formal system of predicate fuzzy logic. Then, certain first-order formulas $A(x), B(x)$ are assigned to the evaluative predications 'X is $\mathscr{A}$', 'Y is $\mathscr{B}$', respectively, and are interpreted in a suitable formal model. Although the surface form of the rules (8) is linguistic, they are not treated in this way. The whole linguistic description is construed as a fuzzy relation resulting from the interpretation of one of two normal forms: disjunctive and conjunctive (see, e.g., the book [32], and many other publications). This approach has been well-elaborated inside BL-fuzzy logic in [35, Chapter 7] and [17, 84, 85, 86], and also inside $\mathrm{Ev}_{\text{Ł}}$ in [80, Chapters 5,6]. This method of interpretation of fuzzy IF-THEN rules is very convenient when we need a nice tool for the approximation of functions but it is less convenient as a model of human reasoning. Therefore, it does not fit the paradigm of FLb.

Logical/linguistic approach. The logical/linguistic approach follows the paradigm of FLb, namely, the rules (8) are taken as genuine conditional clauses of natural language and the linguistic description is taken as a text characterizing some situation, strategy of behavior, control of some process, etc. The goal of the constructed FLb model is to mimic the way how people understand natural language. Then, a formal theory of FTT is considered (it must include a formal theory T^{Ev} of evaluative expressions) so that the intension of each rule (8) can be constructed:

$$\mathrm{Int}(\mathscr{R}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot Ev^A wx \Rightarrow Ev^C w'y \tag{9}$$

[17] For simplicity, we consider only one variable in the antecedent.

where w, w' are contexts of the antecedent and consequent of (8), respectively, Ev^A is the intension of the predication in the antecedent and Ev^C the intension of the predication in the consequent. The linguistic description is interpreted as a set of intensions (9) (see [71, 77] for the details). When considering a suitable model, we obtain a formal interpretation of (9) as a function that assigns to each pair of contexts $w, w' \in W$ a fuzzy relation among objects[18]. It is important to realize that in this case, we introduce a consistent model of the context and provide a general rule for the construction of the extension in every context.

A further widely discussed problem concerns *approximate reasoning* (i.e., the derivation of a conclusion): given a linguistic description consisting of n rules of the form (8), let us learn that 'X is $\mathscr{A}_0$' where $\mathscr{A}_0$ differs slightly from all the $\mathscr{A}_1, \dots, \mathscr{A}_n$. How should we construct a conclusion Y is $\mathscr{B}_0$, i.e. ,what should be $\mathscr{B}_0$?

A detailed logical analysis of the relational-based approximate reasoning inside the predicate fuzzy logic with traditional syntax has been provided in [35] (see also [87]) and inside the predicate Ev_{L} in [80]. When taking a specific model of such logic, we derive a conclusion as a certain composition of fuzzy relations. This interpretation, however, does not lie in the realm of FLb because natural language is neglected. The main outcome of this approach is the well founded and nicely working approximation of functions that are specified imprecisely (we speak of the *fuzzy approximation* of functions).

The most elaborated approximate reasoning method in logical/linguistic interpretation is the so called *perception-based logical deduction* (see [66, 78]). The main idea is to consider the linguistic description as a specific text, which has a *topic* (what we are speaking about) and *focus* (what is the new information — for the detailed linguistic analysis of these concepts, see, e.g., [42]). Each rule is understood as local but vague information about the relation between X and Y. The given predication 'X is $\mathscr{A}_0$' is taken as a *perception* of some specific value of X. On this basis, the most proper rule from the linguistic description is applied (fires), and the best value of Y with respect to this rule is taken as a result. Hence, despite the vagueness of the rules forming the linguistic description, the procedure can distinguish among them[19].

3.4. Generalized (fuzzy) quantifiers

Generalized quantifiers occur quite often in natural language. Recall that these are words such as *most, a lot of, many, a few, a great deal of, a large part of, etc.*. A general theory of generalized quantifiers was initiated by A. Mostowski in [57] and further elaborated by P. Lindström, D. Westerståhl, E. L. Keenan, J. Barwise, R. Cooper ([49, 52, 94, 88]). Generalized quantifiers were introduced into fuzzy logic by L. A. Zadeh [99] and further elaborated by P. Hájek [35, Chapter 8], I. Glöckner [30], M. Holčapek and A. Dvořák [19, 46] and by a few other people.

A class of particular interest for FLb consists of the so called *intermediate quantifiers*, for example, "many, a lot of, most, almost all", etc. A detailed analysis of these quantifiers can be found in [89]. Clearly, they are quantifiers which lay between the classical quantifiers "for all' and "exists". Hence, the main idea of how their semantics can be captured is the following: Intermediate quantifiers refer to elements taken from a class that is "smaller" than the original universe in a specific way. Namely, they are classical quantifiers "for all" or "exists" taken over a class of elements that is determined using an appropriate evaluative expression. Classical logic has no substantiation for why and how the range of quantification should be made smaller. In fuzzy logic, we can apply the theory of evaluative linguistic expressions as follows (for the details, see [73]). Let $Ev \in \mathit{Form}_{(o\alpha)(\alpha o)}$ be an evaluative predication.

[18]Note that for specific elements assigned to x, y, the intension (9) provides a truth value.

[19]The rule of perception-based logical deduction can be formally expressed as

$$r_{PbLD} : \frac{LPerc^{LD}(x_0, w) = \mathrm{Int}(X \text{ is } \mathscr{A}_{i_0}), \quad LD}{Eval(\hat{y}_{i_0}, w', \mathscr{B}_{i_0})},$$

where LD is a linguistic description, $\mathrm{Int}(X \text{ is } \mathscr{A}_{i_0}) \in Topic^{LD}$, $\mathrm{Int}(Y \text{ is } \mathscr{B}_{i_0}) \in Focus^{LD}$ and $\hat{y}_{i_0}$ is the resulting best value of Y, provided that the perception $\mathscr{A}_{i_0}$ is given and the dependence between X and Y is locally characterized by LD.

Then we define

$$(Q^{\forall}_{Ev}\, x_\alpha)(B_{o\alpha}, A_{o\alpha}) := (\exists z_{o\alpha})((\boldsymbol{\Delta}(z_{o\alpha} \subseteq B_{o\alpha})\, \&(\forall x_\alpha)(z_{o\alpha}\, x_\alpha \Rightarrow A_{o\alpha}x_\alpha)) \wedge Ev((\mu B_{o\alpha})z_{o\alpha})), \tag{10}$$

$$(Q^{\exists}_{Ev}\, x_\alpha)(B_{o\alpha}, A_{o\alpha}) := (\exists z_{o\alpha})((\boldsymbol{\Delta}(z_{o\alpha} \subseteq B_{o\alpha})\, \&(\exists x_\alpha)(z_{o\alpha}\, x_\alpha \wedge A_{o\alpha}x_\alpha)) \wedge Ev((\mu B_{o\alpha})z_{o\alpha})). \tag{11}$$

The interpretation of formula (10) is as follows: there is a fuzzy set $z_{o\alpha}$ of objects having the property $B_{o\alpha}$, the size of which (determined by a measure μ) is characterized by the evaluative expression *Ev*, and all these objects also have the property $A_{o\alpha}$. The interpretation of (11) is similar. The property is here, for simplicity, represented by a fuzzy set. However, it is also possible to introduce possible worlds.

Note that the theory also includes classical quantifiers. Special natural language quantifiers can be specified, e.g., by the following formulas:

$$\begin{aligned} &Most := Q^{\forall}_{Very\ big} && Many := Q^{\forall}_{Big} \\ &Several := Q^{\forall}_{Small} && Some := Q^{\exists}_{Small} \end{aligned}$$

In [89], altogether total of 105 *generalized syllogisms* were informally introduced (including the basic Aristotelian syllogisms). All of them are also valid in this theory, for example:

ATK:

All M are Y
Most X are M
———
Many X are Y

AKK:

All M are Y
Many X are M
———
Many X are Y

3.5. A model of common sense human reasoning

The principal goal of FLb specified in the beginning of this section is to develop a model of natural human (common sense) reasoning. In this section, we will briefly demonstrate a possible way to approach this goal.

A typical example of human reasoning is the reasoning of detectives. Therefore, we have chosen a simple story based on one episode from the famous TV series about Lt. Columbo. Let us emphasize that the method outlined below can be taken as a more general methodology that has a variety of specific applications (cf. [21]).

The story:

Mr. John Smith has been shot dead in his house. He was found by his friend, Mr. Robert Brown. Lt. Columbo suspects Mr. Brown to be the murderer.

Mr. Brown's testimony:

I have started from my home at about 6:30, arrived at John's house at about 7, found John dead and went immediately to the phone booth to call police. They came immediately.

Evidence of Lt. Columbo:

Mr. Smith had a high quality suit and a broken wristwatch stopped at 5:45. There was no evidence of a hard blow to his body. Lt. Columbo touched the engine of Mr. Brown's car and found it to be more or less cold.

Lt. Columbo concluded that Mr. Brown was lying because of the following:

(i) Mr. Brown's car engine is *more or less cold*, so he must have been waiting *long* (more than about 30 minutes). Therefore, he could not have arrived and called the police (who came *immediately*).

(ii) A *high quality wristwatch* does not break after *not too hard blow*. A man having *high quality dress* and a *luxurious house* is supposed to also have a *high quality wristwatch*. The wristwatch of John Smith is of *low quality*, so it does not belong to him. It does not display the time of Mr. Smith's death.

The reasoning of Lt. Columbo inside FLb can be modeled as a combination of logical rules, world knowledge and evidence with the help of *non-monotonic reasoning*. This procedure is realized syntactically in Ł-FTT.

The world knowledge includes common sense knowledge of the context and further knowledge, which can be characterized using linguistic descriptions:

- *Context*:
  - (a) Drive duration to heat the engine (minutes): $w_D = \langle 0, 5, 30\rangle$.
  - (b) Temperature of engine (degrees Celsius): $w_T = \langle 0, 45, 100\rangle$.
  - (c) Abstract degrees: quality, state, strike strength: $\langle 0, 0.5, 1\rangle$
- *Logical rules.* These are logical theorems of Ł-FTT and theorems given by some considered theory, for example

$$\textsf{IF } X \text{ is } Sm_\nu \textsf{ THEN } X \text{ is} \neg Bi,$$
$$\textsf{IF } X \text{ is } Bi_\nu \textsf{ THEN } X \text{ is} \neg Sm\,.$$

- *Common sense knowledge from physics:*

$$\textsf{IF } \textit{drive duration} \text{ is } Bi \textsf{ THEN } \textit{engine temperature} \text{ is } Bi,$$
$$\textsf{IF } \textit{drive duration} \text{ is } Sm \textsf{ THEN } \textit{engine temperature} \text{ is } ML\,Sm,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

- *Common sense knowledge of customs of people:*

$$\textsf{IF } \text{quality of } x\text{'s suit is } Bi \textsf{ AND } \text{quality of } x\text{'s house is } Ve\,Bi$$
$$\textsf{THEN } \text{wealth of } x \text{ is } Bi,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

- Some other kinds of common sense knowledge, for example, properties of products, etc.

Finally, we construct a specific model $\mathscr{M}$ determined by the evidence, which includes the *context* (for example, the wealth or quality of Mr. Smith's house) and *perceptions*, for example:

- Touching Mr. Brown's car engine does not burn the hand, i.e., its temperature is more or less low.
- The quality of Mr. Smith's house is very high.

On the basis of a formal analysis which includes the use of perception-based logical deduction, Lt. Columbo concludes that the two special constructed theories are contradictory. Since his perceptions and the evidence cannot be doubted, Mr. Brown is lying, so he had an opportunity to kill Mr. Smith. It is important to emphasize that the contradictory theories have been constructed as nodes of a graph representing the structure of *non-monotonic* reasoning.

3.6. Other applications of FLb

There are several other successful applications of FLb, which we will very briefly mention in this section to demonstrate the power of FLb. Let us emphasize that we confine only to specific applications in which the *formal theory of FLb* (and, consequently, of FLn) was applied.

Applications of the theory of evaluative expressions. Because evaluative linguistic expressions are very often used in human reasoning, they have a great potential for practical applications. One of such applications described in [63] was a geological problem of identifying rock sequences on the basis of expert geologists' knowledge. The problem is normally solved at the table by a geologist on the basis of his/her expert knowledge, which contains evaluative expressions such as "too thin", "too thick", "sufficiently thick", etc. The algorithm developed using FLb works with at least 81% success in comparison with the geologist's results.

Linguistic control of technological processes. This is the most successful area of applications. The main tool is the perception-based logical deduction that employs linguistic descriptions of the control strategy. The ability to imitate human way of reasoning makes this method of control attractive. The main idea is to make the computer act like a "human partner" who understands and follows the linguistic description of the control strategy. Unlike classical fuzzy control (cf., e.g., [56]) where the control strategy described in natural language serves only as a rough guide to how a relational interpretation of linguistic description can be constructed, in linguistic control the control strategy is formed directly using the conditional linguistic clauses (8).

As a simple example, consider the old classical problem of how to avoid an obstacle on the basis of the following strategy: if the obstacle is *very near* then avoid it to the left (turn the steering wheel to the left), and if it is *near* then avoid it to the right. Otherwise do nothing. The corresponding linguistic description is the following:

	distance	⇒ turn of steering wheel
1.	VeSm	−Bi
2.	Sm	+Bi
3.	Me	Ze
4.	Bi	Ze

(Sm, Bi, Me means "small, big, medium", respectively, Ze means "zero" and Ve means "very"). If

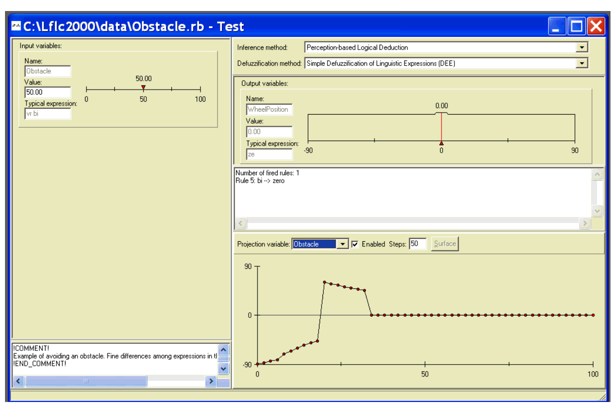

Figure 2: The position of the steering wheel when avoiding the obstacle (lower part). The x-axis is the distance in the context $\langle 0, 50, 100\rangle$ and y-axis is position of the steering wheel.

we construct a fuzzy relation using triangular fuzzy sets, then the approximate reasoning using the composition of fuzzy relations leads to colliding with the obstacle. The FLb solution (using PbLD rule) successfully avoids it.

We have developed a general methodology for successful linguistic control and tested it on many practical examples. We can also learn linguistic descriptions on the basis of monitoring the run of successful control (see [7]). Moreover, when specifying a suitable context, the linguistic description provides control of various kinds of processes if their control strategies are similar. A nice practical example is described in [59], where the control of 5 aluminum-melting furnaces is developed using the same linguistic description for each furnace.

Decision making. The decision situation can be effectively described by a set of linguistic Descriptions, which consist of rules, such as

IF price is rather small AND size of garden is very big, THEN the house is very good.

This method has the following advantages:

- The decision situation is well understandable to people.
- It is easy to include non-quantifiable information.
- The degree of importance is naturally included in the linguistic characterization of the decision situation, so there is no need to assign weights.

More details can be found in [79], along with several other working applications of FLb, for example, in time series analysis and forecasting where the proper linguistic description is learned from the previous development and then using the PbLD method, the future development of the time series is forecasted (to find more about this technique, see [81, 82]).

4. Sorites and Falakros Paradoxes in MFL

In this section, we will briefly recall how fuzzy logic is able to model some manifestations of the vagueness phenomenon. First, there is the *sorites paradox*:

> *One grain does not make a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.*

The falakros paradox[20] is of a similar character.

Let $\mathbb{FN}(n)$ denote the property "n stones do not form a heap" (or, "a man having n hairs is bald"). The paradox is in classical logic obtained when assuming the following axioms:

(i) $\mathbb{FN}(0)$, i.e. "0 stones does not form a heap".

(ii) $(\forall n)(\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1))$, i.e. ,"if n stones do not form a heap then $n+1$ stones also do not form a heap" for arbitrary n.

The analysis of this paradox reveals that the problem lies in the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$. If we relax Axiom (ii) in the sense that it is only "practically valid" then the paradox disappears. The problem is that there is no way to express "practically valid" in classical logic. Fuzzy logic, however, has suitable means for such characterization.

We will consider a predicate fuzzy logic and a formal theory of natural numbers expanded by a new unary predicate $\mathbb{FN}$.

Predicate logic BL$\forall$. Let us consider the language of Peano arithmetic expanded by a new unary predicate $\mathbb{FN}(n)$ and a new unary connective At that means "the given property is almost true", which is interpreted by a certain non-decreasing function on L (for the details see [40]). Furthermore, let us introduce a theory T_{at} as follows:

(i) The equality $=$ is crisp, i.e. $(\forall n)(\forall m)(n = m \vee n \neq m)$.

(ii) T_{at} contains all axioms of Peano arithmetic.

(iii) $(\forall n)(\forall m)(n < m \Rightarrow (\mathbb{FN}(m) \Rightarrow \mathbb{FN}(n)))$,

(iv) $(\forall n)(\mathbb{FN}(n) \Rightarrow (At(\mathbb{FN}(n+1)) \wedge At(\mathbb{FN}(n+n)) \wedge At(\mathbb{FN}(n \cdot n)))$.

Then we can prove the following theorem.

[20] A man with no hair is bald. A man with one hair more than a bald man is still bald. Consequently, all men are bald.

Theorem 4 (Hájek)
Let a theory T_{FN} be obtained from T_{at} by

$$T_{FN} = T_{at} \cup \{\mathbb{FN}(0), (\forall n)(\mathbb{FN}(n) \Rightarrow At(\mathbb{FN}(n+1))), (\exists n)\neg\mathbb{FN}(n)\}\,.$$

Then T_{FN} is a consistent theory.

Clearly, the connective At provides a model of "being practically valid,"so we have a consistent theory T_{FN} in which there "exist heaps".

Predicate $Ev_Ł$. The property of "being practically valid" is expressed in $\mathrm{Ev_Ł}$ by assuming Axiom (ii) in a degree smaller than 1. Then, we can prove the following theorem.

Theorem 5 ([80])
Let T be a consistent fuzzy theory in which all Peano axioms are crisp and accepted in degree 1. Let $1 \geq \varepsilon > 0$ and $\mathbb{FN} \notin J(T)$ be a new predicate. Then

$$T^{+} = T \cup \left\{{}^{1}\!/\mathbb{FN}(0), {}^{1-\varepsilon}\!/(\forall n)(\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)), {}^{1}\!/(\exists n)\neg\mathbb{FN}(n)\right\}$$

is a conservative extension of T.

Note that we do not exclude the possibility that $\varepsilon = 1$. In this case, we obtain a crisp theory in which only 0 does not form a heap, but all the other natural numbers do.

The theory of evaluative linguistic expressions. Recall the formal theory T^{Ev} introduced in Łukasiewicz FTT. The concept of context plays an essential role in this theory. First, we again consider Peano arithmetic and a context $w = \langle 0, p, q\rangle$ where p, q are some numerals. The meaning of evaluative expressions is constructed using a specific fuzzy equality $\approx$ so that we can define $\mathbb{FN}(n) = [0 \approx_{w_N} n]$. Finally, let $\boldsymbol{\nu}$ be a linguistic hedge (e.g., *extremely, very, more or less, roughly*, etc.).

Theorem 6 ([72])
The following are provable in T^{Ev} for an arbitrary context w:

(a) $T^{\mathrm{Ev}} \vdash (Sm\,\boldsymbol{\nu})w\,0,$

(b) $T^{\mathrm{Ev}} \vdash (\exists m)(\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})w\,m,$

(c) $T^{\mathrm{Ev}} \vdash \neg(\exists n)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})w\,n \,\&\, \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})w\,(n+1)),$

(d) $T^{\mathrm{Ev}} \vdash (\forall n)((Sm\,\boldsymbol{\nu})w\,n \Rightarrow At((Sm\,\boldsymbol{\nu})w\,(n+1))$

where $At(C)$ for some property C means "it is almost true that C" and is defined by $At(C) := ((Sm\,\boldsymbol{\nu})w\,n \Rightarrow (Sm\,\boldsymbol{\nu})w\,(n+1)) \Rightarrow C$.

Recall that $\boldsymbol{\nu}$ interprets the meaning of some linguistic hedge. Theorem 6 says the following: we may prove in T^{Ev} for an arbitrary context w that (a) "0 is $\boldsymbol{\nu}$*small*" (for example, "very small", "extremely small", "roughly small", etc.); (b) there is a number m which surely is not $\boldsymbol{\nu}$*small*; (c) there is no surely $\boldsymbol{\nu}$*small* number n such that $n+1$ would surely be not $\boldsymbol{\nu}$*small*; (d) if n is $\boldsymbol{\nu}$*small* then it is "almost true" that $n+1$ is also $\boldsymbol{\nu}$*small*. Let us emphasize that similar theorems can also be proved for $\boldsymbol{\nu}$ *big* and $\boldsymbol{\nu}$ *medium*.

We can also construct a model of T^{Ev} in which the set of elements is a subset of the real numbers. This means that we can also characterize "small" or "big" real numbers (of course, in a specific context).

Finally, we will present the following theorem, which is strongly related to the phenomenon of vagueness. Note that the proved properties relate to the context w, which is a bounded (possibly finite) interval.

Theorem 7
The following are provable in the theory T^{Ev} in each context w:

(a) $T^{\mathrm{Ev}} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(\boldsymbol{\nu}\, Sm)w\,x\,\&(x <_w y \Rightarrow \boldsymbol{\Delta}\neg(\boldsymbol{\nu}\, Sm)w\,y))$,

(b) $T^{\mathrm{Ev}} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(\boldsymbol{\nu}\, Bi)wx\,\&(y <_w x \Rightarrow \boldsymbol{\Delta}\neg(\boldsymbol{\nu}\, Bi)w\,y)$.

This theorem formally states that (a) "there is no last surely small x" and (b) "there is no first surely big x". In [76], a fuzzy logic model of higher order vagueness has also been proposed.

5. Fuzzy Class Theory

Besides FLb as an attempt at developing an extended logic, a nice idea to use FLn as a metatheory for all of fuzzy set theory (*fuzzy mathematics*) has been proposed by P. Cintula and L. Běhounek in [3, 4][21].

The starting point is axiomatization of the notion of fuzzy set. This was first based on the many-sorted predicate second-order ŁΠ fuzzy logic. Later (e.g., in [2]) it was simplified to MTL_Δ (second-order) fuzzy logic. This enables us, in fact, to consider any kind of stronger logic if necessary.

The theory in which the fuzzy mathematics is formulated is called *fuzzy class theory* (FCT). The primitive predicates are *identity* $=$ which is treated classically and *membership* $\in$ between successive sorts. For example, $x \in X$ is a formula expressing membership of an individual x of the first order in an individual X of the second order.

The basic axioms are the following:

($\in$1) $y \in \{x \mid A(x)\} \Leftrightarrow A(y)$,

($\in$2) $(\forall x)\boldsymbol{\Delta}(x \in A \Leftrightarrow x \in B) \Rightarrow A = B$.

The essential feature is the fact that theorems are proved using MTL_Δ. This has several consequences. First, formulas occurring in theorems may be true in a degree possibly different from $\mathbf{1}$. Thus, if the theorem takes the role of implication in the form

$$\vdash A \Rightarrow B$$

then it in fact says that

"the *truth* of B is *at least* as high as the *truth* of A".

Second, it is quite often necessary to consider the formula

$$A^n := \underbrace{A\,\&\cdots\&\,A}_{n-\text{times}}$$

in the given theorem. This means that A must be used n times to maintain the validity of the former.

The models are systems of fuzzy sets over a crisp universe U where the membership functions of fuzzy subsets take values in some MTL_Δ-chain. The intended models are those that contain all fuzzy subsets and fuzzy relations. Models in which the MTL_Δ-chain is standard, i.e., given by a left-continuous t-norm on $[0,1]$, correspond to Zadeh's original notion of fuzzy set (cf. [96]).

The definitions in FCT are very natural, for example:

$$\begin{aligned}
\emptyset &:= \{x \mid \mathbf{0}\},\\
V &:= \{x \mid \mathbf{1}\},\\
Ker(A) &:= \{x \mid \boldsymbol{\Delta}(x \in A)\},\\
A \cap B &:= \{x \mid x \in A \wedge x \in B\},\\
A \cup B &:= \{x \mid x \in A \vee x \in B\},\\
\mathrm{Refl}(R) &:= (\forall x)R(x,x) &(12)\\
f(A,B) &:= \{z \mid (\exists x,y)(z = f(x,y) \wedge x \in A \wedge x \in B)\}. &(13)
\end{aligned}$$

[21] Let us remark that the idea of developing fuzzy set theory (its concepts and operations) as well as its applications as a consequence of the fundamental formal logical theory was already anticipated in [60]. The logic considered there was $\mathrm{Ev}_Ł$, which, however, does not seem to be appropriate as the metatheory of fuzzy mathematics.

Similarly, we can introduce many other known concepts. Note that (13) is Zadeh's extension principle. An example of a theorem in FCT is the following:

$$R \subseteq S \,\&\, \mathrm{Refl}(R) \Rightarrow \mathrm{Refl}(S). \tag{14}$$

Its interpretation says that "the more it is true that the fuzzy relation R is a fuzzy subset of S and R is reflexive, the more it is true that S is also reflexive". This has a good sense. For example, let there be only one element x such that the truth of $R(x, x)$ is close to but different from $\mathbf{1}$. Then we may take such R to be practically reflexive. According to (14), we know that the truth degree of reflexivity of S cannot be smaller than that of $R(x, x)$. If we take (14) classically, then only the truth value $\mathbf{1}$ is considered.

The program to develop fuzzy mathematics in this way seems to be very promising because it encompasses and quite often generalizes most (if not all) of the results in fuzzy set theory obtained so far (see, e.g., [2, 6, 47, 51]).

Let us remark that fuzzy class theory can be extended without difficulties to higher-order fuzzy logic and, consequently, the whole program of fuzzy mathematics can also be formulated in FTT, where there are natural means for this task. For example, the special comprehension term $\{x \mid A(x)\}$ is just the formula $\lambda x_\alpha \, A_o$ (for arbitrary type α). The comprehension axiom and extensionality are logical axioms of FTT, and all the definitions can be naturally formulated for all orders.

6. Possible Future Directions in MFL

Let us now summarize how the current situation in mathematical fuzzy logic can be regarded. There is a deeply developed fuzzy logic in narrow sense, in which metamathematics of many formal (residuated lattice-based) logical systems has been studied in detail, including their interrelations. Therefore, the structure of FLn is already quite well known. It is still not fully clear which logic is the most convenient to solve problems related to models of vagueness and their applications. Therefore, we think that it is already time to turn our attention to the proclaimed goals of fuzzy logic so that its power can be fully recognized. As can be seen from the presentation above, some steps in this direction have already been accomplished.

We think that there are good reasons to continue the development of models of vagueness using the tools of FLn. We also propose to continue the development of FLb so that the intricate principles of human (i.e., common sense) reasoning can be more properly captured. The resulting logic would have many important applications (some of them were outlined above) in artificial intelligence, robotics, and in many other areas of human activities. We see also great potential in the development of fuzzy mathematics in the frame of MFL.

As discussed in detail in [68], vagueness is one facet of a wider phenomenon of *indeterminacy*, another facet of which is uncertainty. The latter is usually mathematically modeled using probability theory (but also using possibility or belief theory). Thus, a promising direction is also to consider uncertainty in the sense that has been nicely established by T. Flaminio and F. Montagna in [28] and to develop the corresponding models.

Below is a short list of some of the possible future tasks in the study of MFL and its applications[22]:

(a) Develop or improve mathematical models of the phenomenon of vagueness with the goal of capturing the ways in which the latter manifests itself in various situations (without pretensions to capture its substance in all details).

(b) Extend the repertoire of evaluative linguistic expressions for which a reasonable working mathematical model can be elaborated.

(c) Extend the theory of generalized quantifiers using the formalism of FLn. (some steps have already been accomplished, cf. [19, 30, 46]). Introduce models of a wider class of specific natural language quantifiers.

[22]The author admits that the list is influenced by his interests but believes that the tasks presented here should receive wider attention.

(d) Find a reasonable class of "intended models" for the theory of intermediate and generalized quantifiers.

(e) Study various forms of common sense human reasoning and search for a reasonable formalization of them. As a subgoal, extend the list of intermediate quantifiers and the list of generalized syllogisms (cf. [73, 89]).

(f) Develop a reasonable formalization of FLb on the basis of which the above mentioned constituents of the PNL methodology (namely the World Knowledge Database and Multiagent, Modular Deduction Database) could be formed. Furthermore, extend the results obtained in the AI *theory of common sense reasoning*.

(g) Extend the technique by joining fuzzy logic with probability theory so as to be able to include also uncertainty inside FLb and thus to develop a concise theory of the phenomenon of indeterminacy.

(h) Study the ways in which other formal systems of FLn (besides the fuzzy type theory) could be used for the solution of problems raised in FLb and develop them accordingly.

(i) Develop other aspects of fuzzy mathematics. Try to find problems specific to fuzzy mathematics, which are either unsolvable or meaningless in classical mathematics.

In general, we propose to shift the interest in MFL from its outer structure (metamathematics of fuzzy logics) to the inner structure of some reasonably chosen systems. One of the problems we face quite often is also the great complexity of formal means when they are applied. The question is raised as to whether or not it is possible to simplify the formalization without losing the expressive power. We thus face a wide variety of interesting problems, both inside mathematical fuzzy logic and in its applications, that are interesting to solve in the future.

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