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EQ-algebra-based Fuzzy Type Theory and Its Extensions

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Abstract

In this paper, we introduce a new algebra called 'EQ-algebra', which is an alternative algebra of truth values for formal fuzzy logics. It is specified by replacing implication as the main operation with a fuzzy equality. Namely, EQ-algebra is a semilattice endowed with a binary operation of fuzzy equality and a binary operation of multiplication. Implication is derived from the fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ-algebras overlap with residuated lattices but are not identical with them. We choose one class of suitable EQ-algebras (good EQ-algebras) and develop a formal theory of higher-order fuzzy logic called 'basic fuzzy type theory' (FTT). We develop in detail its syntax and semantics, and we prove some basic properties, including the completeness theorem with respect to generalized models. The paper also provides an overview of the present state of the art of FTT.

Keywords: mathematical fuzzy logic; fuzzy type theory; EQ-algebra; residuated lattice

1 Introduction

The development of mathematical fuzzy logic started in 1969 with the paper of J. A. Goguen [14]. His ideas were first followed in the papers of J. Pavelka [33] and V. Novák [22], who established the branch of fuzzy logic, characterised by the introduction of evaluated syntax. This means that axioms can be only partially true; there are fuzzy theories in which formulas can be proved to some truth degree generally less than 1. This work reached its crown in the monograph of V. Novák et al. [32].

Another branch initiated by the monograph of P. Hájek [15] preserves traditional syntax. The semantics is established on the basis of various structures of truth values being special kinds of residuated lattice. Many special formal systems of propositional and predicate first-order fuzzy logics are proposed (e.g., in [7], 57 systems are studied) that are, in principle, extensions of the MTL-fuzzy logic, a logic based on the MTL-algebra (a prelinear residuated lattice) of truth values [10]. These logics are examples of *core* fuzzy logics (cf. [17]).

The development of fuzzy logic has been naturally continued to higher order one, having been initiated by the author of this paper in [23, 24] and continued in detail in the paper [26]. The resulting logic is called a *fuzzy type theory* (FTT) and is a generalization of the classical simple type theory introduced by B. Russell in [35] and formally elaborated as a higher-order mathematical logic especially by A. Church, L. Henkin and P. Andrews (see [1, 6, 18, 19]). A good overview of the main principles and virtues of simple type theory has been recently explained by W. Farmer in [12].

There are several reasons for introducing fuzziness into higher order logic:

(i) There is a long term endeavour to capture the semantics of concepts and natural language expressions (cf., e.g., [13, 21]) in general logic. Because (classical) type theory is the main mathematical tool there, replacing it by FTT may enable the inclusion of vagueness in the developed models.

(ii) We believe that FTT can become a powerful tool for the modelling of commonsense reasoning, which is inseparably tied to (the semantics of) natural language and in which vagueness plays a

significant role. Thus FTT may bring the formal theory of commonsense reasoning closer to the human way of thinking.

(iii) FTT provides a model of various deep manifestations of the vagueness phenomenon, including ones such as is higher order vagueness.

(iv) A secondary motivation comes from the effort to establish foundations of the "fuzzy" mathematics as a whole. For this purpose, a special formal system called *fuzzy class theory* has been developed by L. Běhounek and P. Cintula in [4]. All their results can be equivalently expressed using FTT.

At first sight, it seems natural that FTT should be based on the same structures of truth values as the above mentioned core fuzzy logics, i.e. that this structure should be an an extension of the MTL-algebra. Indeed, this has been done and so far, the following distinguished formal systems of FTT are described: IMTL-FTT, which is based on IMTL$_\Delta$-algebra (see [26]); Ł-FTT, which is based on Łukasiewicz$_\Delta$ algebra; and BL-FTT, which is based on BL$_\Delta$-algebra (see [25])*). All of these logics enjoy the generalized completeness property, i.e. completeness w.r.t. generalized models. This direction can be called *implication-based*.

In should be emphasized, however, that L. Henkin developed the type theory [19] using identity as a sole primitive constant (a detailed description of this theory can be also found in the monograph [1]). This choice is interesting because the endeavour to develop logic on the basis of identity (equality) as the principal connective can be traced to G. W. Leibnitz, L. Wittgenstein, and F. P. Ramsey (cf. [34]). For example, Leibnitz proclaimed that "a fully satisfactory logical calculus must be an equational one" (cf. [5]). Note that there are also works developing classical propositional logic on the basis of equivalence as the basic connective (a recent book presenting logic in this way is [36]), but the equivalence relation cannot be the sole connective.

This second direction, which may be called *equality-based*, treats equality between elements on the basis of the same principles independent of their character. Thus, logical equivalence is understood as an equality between truth values. Therefore, we can use one common term — *equality* — both for logical equivalence (i.e. equality between truth values) as well as for equality between arbitrary elements.

An interesting question is raised whether fuzzy (many-valued) logic can also be developed on the basis of generalized equality called *fuzzy equality* as the main connective. How should fuzzy equality be interpreted in the structure of truth values? In [26], we used the biresiduation operation $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. One can see that this operation is derived from implication, and therefore we face a fundamental methodological discrepancy: the *basic connective* is interpreted by a *derived* algebraic operation. Hence, it is natural to ask: is there a typical algebraic structure of truth values in which the fuzzy equality (i.e., fuzzy equivalence) is basic while the implication is derived?

A possible answer to this question has been given by the author of this paper in [27, 28]. The resulting structure called *EQ-algebra* has been elaborated in detail in [30]. Its axioms are quite specific, and we explain their motivation in the next section. The essential feature of EQ-algebras is that implication and multiplication are no longer closely tied by the adjunction. This enables us to refrain from commutativity of multiplication, which gives rise to the development of a fuzzy logic with one noncommutative conjunction connective. Unlike noncommutative implication-based fuzzy logics, we need neither two implications (left and right), nor a special inference rule for them (cf. [16]).

It should be noted that the class of EQ-algebras is interesting in itself because they give us information about the behaviour of the nontrivial concept of equality and shed new light on the residuated lattices. It also notable that each EQ-algebra is also a BCK-algebra and so, the interrelations among all considered algebras are quite intricate. A detailed study has not yet been pursued.

This paper is an extended version of some parts of [29]. We briefly review EQ-algebras, give reasons for choosing one specific class of EQ-algebras suitable as algebras for truth values for a formal system of fuzzy type theory, and develop in detail syntax and semantics of the latter (occasionally, we will denote it by EQ-FTT).

There are several outcomes of this paper:

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*)The MTL-FTT has not been studied but it is certainly possible.

(i) We develop a *basic fuzzy type theory* that lies even deeper than the MTL-algebra-based one. Thus, the former can be naturally extended to more specific fuzzy type theories, namely IEQ-FTT (involutive EQ-algebra based FTT) and also to all the above-mentioned residuated lattice-based FTTs.

(ii) Our results raise interesting philosophical questions. For example, (i) suggests that (fuzzy) equality is a more fundamental (and simpler) concept than (fuzzy) implication. This result may shed light on the long-existing question regarding the essence of implication. Another question: why are the "goodness axiom" ($a \sim \mathbf{1} = a$; cf. also (FT-tval3)) and, at least, "separateness" necessary in logic, even though the general character of equality does not enforce them?

(iii) From the technical point of view, this paper also provides an overview of the present state of the art of FTT. Let us remark that in various places, we follow closely the methodology developed previously in [26] (and also in [1] and elsewhere) but, at the same time, we bring various improvements to it. As we are dealing with a new algebra that is not a simple modification of MTL, we think that it is necessary to write the proofs carefully in detail to preserve the readability of this paper.

The paper is structured as follows: the next section contains a theory of EQ-algebras focusing on their role as structures of truth values for FTT. Sections 3–5 are devoted to a formal system of basic FTT: its main properties, construction of the canonical model, and proof of the completeness theorem. In Section 6, we will briefly outline further some extensions of the basic FTT.

2   EQ-algebras

In this section, we present the theory of EQ-algebras focusing on the properties needed for the development of EQ-algebra-based fuzzy type theory. We will follow the results presented in the papers [8, 9, 30].

2.1   Basic definitions

What properties should an algebra have in which fuzzy equality is the basic operation? At first glance, it seems clear that these properties should be reflexivity, symmetry, and transitivity. We are convinced, however, that fuzzy equality should be characterized by a deeper property from which symmetry and transitivity would follow.

Recall some of the ideas of Leibnitz (cf. [5]). According to his conception of equality, "if A equals B then A can be replaced by B wherever A occurs"[†]. We are convinced that this is the strongest principle that should be reflected in fuzzy equality. We will call its algebraic formulation given below the *substitution principle*. Furthermore, let $A \equiv B$ be an equality of concepts and AB their intersection. Then Leibnitz interprets "if A then B" as $AB \equiv A$. Note that if we understand AB as a conjunction then we obtain a formula giving the implication $A \Rightarrow B$; cf. e.g., [19] and elsewhere. An analogous formula ($a \wedge b = a$) is used in lattice theory to define ordering (note that this complies with the ordering via residuation in residuated lattices); or if we construe A, B as sets then this formula represents $A \subseteq B$. Thus, $AB \equiv A$ can be also construed as "A is contained in B".

Because the basic idea behind the fuzzy approach is the introduction of comparable degrees, we will use the meet operation to introduce *ordering*. Hence, the skeleton of our algebra is a meet-semilattice. Furthermore, we must also express full truth. Thus, the algebra should contain the greatest element $\mathbf{1}$. Next, we introduce the fuzzy equality (denoted by $\sim$ in the sequel). Finally, we need a combination principle that need not necessarily be identical with the meet operation because the principal role of the latter is to provide ordering. Therefore, we will introduce a third operation of multiplication "$\otimes$" which will not participate on the ordering but only to enable us to "fuse things together". Consequently, its basic properties are limited. On the basis of this discussion, we come to the following definition.

Definition 1
A non-commutative EQ-algebra is an algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle \tag{1}$$

[†] This is often called "Leibnitz's Law of Indiscernability of Identicals".

of type (2, 2, 2, 0) fulfilling the following axioms for all $a, b, c \in E$:

(E1) $\langle E, \wedge, \mathbf{1} \rangle$ *is a commutative idempotent monoid ($\wedge$-semilattice with the top element $\mathbf{1}$). We put $a \leq b$ iff $a \wedge b = a$, as usual.*

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ *is a monoid such that $\otimes$ is isotone w.r.t. $\leq$.*

(E3) $a \sim a = \mathbf{1}$,   *(reflexivity)*

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$,   *(substitution)*

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$,   *(congruence)*

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,   *(isotonicity of implication)*

(E7) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$,   *(antitonicity of implication)*

(E8) $a \otimes b \leq a \sim b$.   *(boundedness)*

EQ-algebra is commutative *is $\otimes$ is a commutative operation.*

Axiom (E4) is the main axiom representing the substitution principle. Axioms (E6) and (E7) characterize more complex combinations and can be explained in terms of Leibnitz ideas. Namely, if A is contained in the intersection of B, C then A is contained in B only. And also, if A is contained in B then the intersection of A, C is also contained in B.

We will put

$$a \to b = (a \wedge b) \sim a, \tag{2}$$

and

$$\tilde{a} = a \sim \mathbf{1} \tag{3}$$

where $a, b \in E$. The derived operation (2) is called *implication*. Using it, (E6),(E7) can be rewritten into

$$a \to (b \wedge c) \leq a \to b, \tag{E6'}$$
$$a \to b \leq (a \wedge c) \to b, \tag{E7'}$$

respectively. It is not difficult to show that (E6) implies isotonicity of $\to$ in the second variable and (E7) its antitonicity in the first one. Moreover, since the implication is obtained from the fuzzy equality without using the multiplication, the adjunction condition is not preserved. Consequently, EQ-algebras are not, in general, residuated.

If $\mathcal{E}$ contains also the bottom element $\mathbf{0}$ then we put

$$\neg a = a \sim \mathbf{0}, \qquad a \in E \tag{4}$$

and call this operation a *negation*.

The following special classes of EQ-algebras can be introduced.

Definition 2
An EQ-algebra $\mathcal{E}$ is:

(i) semiseparated *if for all $a \in E$,*

    (E9) $a \sim \mathbf{1} = \mathbf{1}$ *implies* $a = \mathbf{1}$.

(ii) separated *if for all $a, b \in E$,*

    (E10) $a \sim b = \mathbf{1}$ *implies* $a = b$.

(iii) spanned *if*

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(E11) $\tilde{\mathbf{0}} = \mathbf{0}$.

(iv) good *if for all* $a \in E$,

(E12) $a \sim \mathbf{1} = a$, *i.e. if* $\tilde{a} = a$.

(v) residuated *if for all* $a, b, c \in E$,

(E13) $(a \otimes b) \wedge c = a \otimes b \quad$ iff $\quad a \wedge ((b \wedge c) \sim b) = a$.

(vi) involutive *(IEQ-algebra) if for all* $a \in E$,

(E14) $\neg\neg a = a$.

An EQ-algebra $\mathcal{E}$ is *complete* if $\inf(K)$ exists for every $K \subseteq E$. This immediately implies that complete EQ-algebras are complete lattices. A lattice ordered EQ-algebra has also the binary operation $\vee$ so that $\langle E, \wedge, \vee \rangle$ is a lattice. An EQ-algebra $\mathcal{E}$ is a *lattice EQ-algebra* (ℓEQ-algebra) if it is lattice ordered and, moreover, the following additional substitution axiom holds for all $a, b, c, d \in E$:

(E15) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$.

It is easy to see that a complete EQ-algebra is a complete lattice ordered EQ-algebra (cf. [3]). Hence, every finite EQ-algebra is latice ordered.

2.2 Main properties

We will demonstrate that the $\rightarrow$ operation defined in (2) has properties that justify it to become a reasonable candidate for interpretation of the implication connective in FTT. However, this is not true in all EQ-algebras. In general, $a \leq b$ implies $a \rightarrow b = \mathbf{1}$ but not vice-versa. Therefore, there are algebras with comparable elements a, b such that $b > a$ and, at the same time, it holds that $a \sim b = \mathbf{1}$ and $b \rightarrow a = \mathbf{1}$. An EQ-algebra is *regular* if no such couples of elements exist in it. It is easy to see that each separated EQ-algebra is regular. Namely, if an EQ-algebra $\mathcal{E}$ is separated then $a \leq b$ iff $a \rightarrow b = \mathbf{1}$ holds for all $a \in E$.

The following has been proved in [30].

Lemma 1
Let $\mathcal{E}$ be an EQ-algebra and $a, b, c \in E$. Then:

(a) $a \sim b = b \sim a$,   *(symmetry)*

(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,   *(transitivity)*

(c) $(a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c$,   *(transitivity of implication)*

(d) $a \otimes (a \rightarrow b) \leq \tilde{b}$,

(e) *If* $a \leq b \rightarrow c$, *then* $a \otimes b \leq \tilde{c}$,

(f) $(a \rightarrow b) \otimes (b \rightarrow a) \leq a \sim b \leq (a \rightarrow b) \wedge (b \rightarrow a)$. *As a special case, if $\mathcal{E}$ is linearly ordered then*

$(a \rightarrow b) \otimes (b \rightarrow a) = a \sim b = (a \rightarrow b) \wedge (b \rightarrow a)$.

(g) $b \leq a \rightarrow b$,

(h) $a \rightarrow (b \rightarrow a) = \mathbf{1}$,

(i) $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$,

(j) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$.

(k) $(c \rightarrow a) \otimes (c \rightarrow b) \leq c \rightarrow a \wedge b$.

Note that in good EQ-algebras, item (e) is one implication of the adjunction condition. The opposite implication, however, does not hold in general.

Example 1
Let $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ be a residuated lattice. We may introduce two biresiduation operations:

$$a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a), \tag{5}$$

$$a \Leftrightarrow b = (a \Rightarrow b) \otimes (b \Rightarrow a). \tag{6}$$

Both operations are natural interpretations of fuzzy equivalence since they are reflexive, symmetric, and transitive in the following sense:

$$(a \square b) \otimes (b \square c) \leq a \square c$$

for all $a, b, c \in L$ where $\square \in \{\Leftrightarrow, \Leftrightarrow\}$.

 (i) $\mathcal{E}_\mathcal{L} = \langle L, \wedge, \otimes, \Leftrightarrow, \mathbf{1} \rangle$ is a separated EQ-algebra.

 (ii) If $\mathcal{L}$ is linearly ordered then also $\hat{\mathcal{E}}_\mathcal{L} = \langle L, \wedge, \otimes, \Leftrightarrow, \mathbf{1} \rangle$ is a separated EQ-algebra (since both $\Leftrightarrow$ and $\Leftrightarrow$ concide).

 (iii) Let $*$ be a monoidal operation on L such that $* \leq \otimes$. Then the algebra $\mathcal{E} = \langle L, \wedge, *, \Leftrightarrow, \mathbf{1} \rangle$ is a separated EQ-algebra. If $* < \otimes$ then $\mathcal{E}$ is not residuated.

Example 2
Let $E_5 = \{0 < a < b < c < 1\}$ be a chain and put

$\otimes$	$\mathbf{0}$	a	b	c	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a
b	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a	b
c	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a	c
$\mathbf{1}$	$\mathbf{0}$	a	b	c	$\mathbf{1}$

$\sim$	$\mathbf{0}$	a	b	c	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	c	b	a	$\mathbf{0}$
a	c	$\mathbf{1}$	c	b	a
b	b	c	$\mathbf{1}$	c	b
c	a	b	c	$\mathbf{1}$	c
$\mathbf{1}$	$\mathbf{0}$	a	b	c	$\mathbf{1}$

$\rightarrow$	$\mathbf{0}$	a	b	c	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	b	c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
c	a	b	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a	b	c	$\mathbf{1}$

It can be verified that $\langle E_5, \wedge, \otimes, \sim, \mathbf{1} \rangle$ is a linearly ordered IEQ-algebra which is non-residuated. Indeed, since the first column of $\sim$ gives negation, we can see that $\neg\neg x = x$. At the same time, e.g., $c \otimes c = a \leq a$, but $c \not\leq c \rightarrow a = b$.

Example 3
Example of a finite non-trivial non-residuated IEQ-algebra is the following: its (semi)lattice structure is in Figure 1. Product and fuzzy equality are defined as follows:

$\otimes$	$\mathbf{0}$	a	b	c	d	e	f	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a
b	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	b
c	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	c
d	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	d	d	d
e	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	e	d	e
f	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	d	d	f
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	d	f	$\mathbf{1}$

Figure 1: Eight elements IEQ-algebra.

$\sim$	**0**	a	b	c	d	e	f	**1**
0	**1**	e	f	d	c	a	b	**0**
a	e	**1**	d	f	c	a	c	a
b	f	d	**1**	e	c	c	b	b
c	d	f	e	**1**	c	c	c	c
d	c	c	c	c	**1**	f	e	d
e	a	a	c	c	f	**1**	d	e
f	b	c	b	c	e	d	**1**	f
1	**0**	a	b	c	d	e	f	**1**

$\to$	**0**	a	b	c	d	e	f	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
a	e	**1**	e	**1**	**1**	**1**	**1**	**1**
b	f	f	**1**	**1**	**1**	**1**	**1**	**1**
c	d	f	e	**1**	**1**	**1**	**1**	**1**
d	c	c	c	c	**1**	**1**	**1**	**1**
e	a	a	c	c	f	**1**	f	**1**
f	b	c	b	c	e	e	**1**	**1**
1	**0**	a	b	c	d	e	f	**1**

This algebra is non-residuated since, e.g., $\mathbf{0} = a \otimes f \le b$, but $a \not\le f \to b = b$.

Example 4

Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$ be an EQ-algebra and X a set. Let E^X be a set of all functions on X and define the operations $\wedge^f, \otimes^f, \sim^f$ pointwise, i.e. $(f \Box^f g)(x) = f(x) \Box g(x)$ and put $\mathbf{1}^f(x) = \mathbf{1}$ for all $x \in X$ where $\Box \in \{\wedge, \otimes, \sim\}$. Then

$$\mathcal{E}^f = \langle E^X, \wedge^f, \otimes^f, \sim^f, \mathbf{1}^f \rangle$$

is an EQ-algebra.

One can see from these examples that, in general, neither non-linear nor linear EQ-algebras coincide with residuated lattices.

Lemma 2

Let $\mathcal{E}$ be a good EQ-algebra. Then the following holds for all $a, b \in E$:

(a) $\mathcal{E}$ is separated and axiom (E8) is provable from the other EQ-axioms.

(b) $a \le (a \sim b) \sim b$,

(c) $a \le (a \to b) \to b$,

(d) $a \otimes (a \to b) \le b$,

(e) $a \to (b \to c) = b \to (a \to c)$.

(f) If $a \to b_i = \mathbf{1}$ for all $i \in I$ and $\bigwedge_{i \in I} b_i$ exists then $a \to \bigwedge_{i \in I} b_i = \mathbf{1}$.

(g) $a \le b \to c$ iff $b \le a \to c$.

(h) $\bigwedge_{i \in I}(a_i \to b) = \bigvee_{i \in I} a_i \to b$,
provided that both infimum as well as supremum exist.

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PROOF: (a), (b) were proved in [30].

(c)[†] By (b) and (E6) we obtain

$$a \leq (a \sim (a \wedge b)) \sim (a \wedge b) \leq (a \rightarrow b) \rightarrow (a \wedge b) \leq (a \rightarrow b) \rightarrow b.$$

(d) implies immediately from Lemma 1(d).

(e) By (c), Lemma 1(j) and the properties of implication we have

$$b \rightarrow (a \rightarrow c) \leq ((a \rightarrow c) \rightarrow c) \rightarrow (b \rightarrow c) \leq a \rightarrow (b \rightarrow c).$$

(f) It follows from the condition that $a \leq b_i$, $i \in I$ and so, $a \leq \bigwedge_{i \in I} b_i$.

(g) was proved in [9].

(h) From (g), (e), and the properties of supremum we obtain the inequality $\leq$ in (h). The opposite inequality follows from the properties of implication. □

It is easy to show that a corresponding reduct of a good EQ-algebra is a BCK-semilattice (BCK-algebra). In [30], it has been demonstrated that there are two different good EQ-algebras with the same implication operation. Therefore, good EQ-algebras are not simply BCK-semilattics (BCK-algebras) extended by a semicopula operation.

Lemma 3
Let $\mathcal{E}$ be an EQ-algebra. Then for all $a, b, c \in E$:

(a) $\mathcal{E}$ is residuated if and only if it is separated and $(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$.

(b) If $\mathcal{E}$ is ℓEQ-algebra then $(a \rightarrow c) \otimes (b \rightarrow c) \leq (a \vee b) \rightarrow c$.

(c) $\mathcal{E}$ is good iff $a \otimes (a \rightarrow b) \leq b$.

PROOF: (a) was proved in [30, Lemma 14(b)].

(b) Since $a \rightarrow c = (a \vee c) \sim c$, we have

$$((a \vee c) \sim c) \otimes ((b \vee c) \sim c) \leq (a \vee b) \vee c \sim c = (a \vee b) \rightarrow c$$

using Axiom (E15).

(c) If $\mathcal{E}$ is good then the inequality follows from Lemma 1(d). The converse follows from the properties of $\rightarrow$ by [30], Lemma 12(c). □

Note that the inequality (c) in the above lemma is the algebraic characterization of many-valued modus ponens (cf. [15, 32]).

An EQ-algebra $\mathcal{E}$ is *prelinear* if for all $a, b \in E$, $\mathbf{1}$ is the unique upper bound of $\{a \rightarrow b, b \rightarrow a\}$.

Lemma 4 ([8])
The following is equivalent:

(a) $\mathcal{E}$ is prelinear.

(b) $((a \rightarrow b) \rightarrow c) \leq ((b \rightarrow a) \rightarrow c) \rightarrow c$

If $\mathcal{E}$ is prelinear EQ-algebra then it is *lattice ordered* where the join is defined by

$$a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a). \tag{7}$$

Algebras of special interest for FTT, and especially for its applications, are IEQ-algebras, i.e. EQ-algebras in which negation is involutive.

Lemma 5
Let $\mathcal{E}$ be an IEQ-algebra. Then the following holds for all $a, b \in E$:

[†] The proofs of this item and items (e) and (h) were adopted from [9].

(a) Put $a \vee b = \neg(\neg a \wedge \neg b)$. Then $\vee$ is a supremum of a, b and so, $\mathcal{E}$ is a lattice.

(b) $a \sim b = \neg a \sim \neg b$,

(c) $a \leq b$ iff $\neg b \leq \neg a$,

(d) $\neg b \rightarrow \neg a = a \rightarrow b$.

PROOF: The proof can be found in [30].

$\square$

Let us remark that the contraposition property (d) is not equivalent with the double negation in EQ-algebras.

The proof of the following theorem can be again found in [30].

Theorem 1
Every IEQ-algebra $\mathcal{E}$ is good, spanned and separated ℓEQ-algebra.

Definition 3
Let $\mathcal{E}$ be a spanned EQ-algebra. A delta operation in $\mathcal{E}$ is an operation $\Delta : E \longrightarrow E$ fulfilling the following axioms:

(i) $\Delta \mathbf{1} = \mathbf{1}$,

(ii) $\Delta a \leq a$,

(iii) $\Delta a \leq \Delta\Delta a$,

(iv) $\Delta(a \sim b) \leq \Delta\tilde{a} \sim \Delta\tilde{b}$,

(v) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.

If $\mathcal{E}$ is also lattice ordered then Δ must also fulfil the following:

(vi) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,

(vii) $\Delta a \vee \neg\Delta a = \mathbf{1}$.

Lemma 6
(a) If $a \leq b$ then $\Delta a \leq \Delta b$.

(b) $\Delta(a \rightarrow b) \leq \Delta\tilde{a} \rightarrow \Delta\tilde{b}$.

(c) If $\mathcal{E}$ is good then $\Delta(a \rightarrow b) \leq \Delta a \rightarrow \Delta b$.

PROOF: (a) follows immediately from the property (v) of Δ.

(b) Note that from $\widetilde{a \wedge b} \leq \tilde{a} \wedge \tilde{b} \leq \tilde{a}$ we obtain by [30], Proposition 2 and property (iv) the following inequality:

$$\Delta((a \wedge b) \sim a) \leq \Delta(\widetilde{a \wedge b}) \sim \Delta\tilde{a} \leq (\Delta\tilde{a} \wedge \Delta\tilde{b}) \sim \Delta\tilde{a} = \Delta\tilde{a} \rightarrow \Delta\tilde{b}.$$

(c) is obvious.

$\square$

If the algebra is linearly ordered then we can define Δ-operation by $\Delta(\mathbf{1}) = \mathbf{1}$ and $\Delta(x) = \mathbf{0}$ otherwise. Surprisingly, it is not possible to define the Δ operation on the IEQ-algebra from Example 3. From it follows that not every EQ-algebra can be extended by Δ. On the other hand, Example 4 gives clue how EQ$_\Delta$-algebras can be constructed, for example, from linearly ordered ones.

At the end of this subsection, we will present some results from the filter theory of EQ-algebras. More details can be found in [8, 9, 30].

Definition 4
A set $F \subseteq E$ is a prefilter of $\mathcal{E}$ if

  (i) $\mathbf{1} \in F$,

  (ii) *if $a, a \rightarrow b \in F$ then $b \in F$,*

  (iii) *if $a, b \in F$ the $a \otimes b \in F$*

for all $a, b \in E$. A prefilter F is a filter if $a \rightarrow b \in F$ implies $(a \otimes c \rightarrow b \otimes c) \in F$ as well as $(c \otimes a \rightarrow c \otimes b) \in F$ for all $a, b, c \in E$. A prefilter F is prime if $a \rightarrow b \in F$ or $b \rightarrow a \in F$ for every $a, b \in E$.

    Given a prefilter F, an equivalence $a \approx_F b$ is defined in a standard way. An equivalence class of $a \in E$ is denoted by $[a]_F$; the operations on $E|F$ are defined pointwise as usual.

Theorem 2
Let $\mathcal{E}$ be an EQ-algebra and F a prefilter of it.

  (a) *If F is a filter then $\approx_F$ is a congruence and $\mathcal{E}|F = \langle E|F, \wedge_F, \otimes_F, \sim_F, [\mathbf{1}]_F \rangle$ is a separated EQ-algebra such that $f : a \in [a]_F$ is a homomorphism of $\mathcal{E}$ onto $\mathcal{E}|F$.*

  (b) *Let $\mathcal{E}$ be good and prelinear. Then $\langle E|F, \wedge_F, \vee_F, [\mathbf{1}]_F \rangle$ is a distributive lattice with the top element $[\mathbf{1}]_F$ where the join is induced by the equality (7).*

  (c) *If $\mathcal{E}$ is good then the lattice of filters of $\mathcal{E}$ is isomorphic to its lattice of congruences.*

Let us remark, however, that there are EQ-algebras with no filter. The reason is the possible existence of pathological couples. Thus, we cannot conclude from (a) that every EQ-algebra can be homomorphically embedded into a separated one. On the other hand, if $\mathcal{E}$ is separated then $\{\mathbf{1}\} \subset E$ is always a filter.

    The following theorem has been proved in [8].

Theorem 3
Let $\mathcal{E}$ be a good EQ-algebra. The following is equivalent:

  (a) *$\mathcal{E}$ is subdirectly embeddable into a product of linearly ordered good EQ-algebras, i.e. $\mathcal{E}$ is representable.*

  (b) *$\mathcal{E}$ is prelinear and every minimal prime prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.*

  (c) *$\mathcal{E}$ satisfies the inequality*

$$(d \rightarrow (d \otimes (c \rightarrow ((b \rightarrow a) \otimes c)))) \rightarrow u \le ((a \rightarrow b) \rightarrow u) \rightarrow u \qquad (8)$$

2.3 Fuzzy equality between arbitrary objects and weakly extensional functions

The fuzzy equality $\sim$ considered in the EQ-algebra is the many-valued equality between truth values, i.e. it is a fuzzy equivalence. However, in FTT we must introduce also a *fuzzy equality* on an arbitrary set M.

Definition 5
Let $\mathcal{E}$ be an EQ-algebra with the support E and M be a set. A fuzzy equality $\overset{\circ}{=}$ on M is a binary fuzzy relation on M, i.e. a function

$$\overset{\circ}{=}: M \times M \longrightarrow E$$

such that the following holds for all $m, m', m'' \in M$:

  (i) $\overset{\circ}{=}(m, m) = \mathbf{1}$,   *(reflexivity)*

  (ii) $\overset{\circ}{=}(m, m') = \overset{\circ}{=}(m', m)$,   *(symmetry)*

  (iii) $\overset{\circ}{=}(m, m') \otimes \overset{\circ}{=}(m', m'') \le \overset{\circ}{=}(m, m'')$.   *($\otimes$-transitivity)*

If $m, m' \in M$ then we will usually write $[m \stackrel{\circ}{=} m']$ instead of $\stackrel{\circ}{=}(m, m')$. We say that $\stackrel{\circ}{=}$ is *separated* if

$$[m \stackrel{\circ}{=} m'] = \mathbf{1} \quad \text{iff} \quad m = m'$$

holds for all $m, m' \in M$.

Definition 5 enables us to introduce a fuzzy equality as an arbitrary function fulfilling the given axioms. However, in EQ-algebra we already have at disposal the fuzzy equality $\sim$ between truth values. Therefore, it seems natural to put all the fuzzy equalities on arbitrary set on a unique basis. This can be done using a special function $f : M \longrightarrow E$. The precise statement is in the following lemma.

Lemma 7
Let M be a set, $\mathcal{E}$ an EQ-algebra and $f : M \longrightarrow E$ a function. Then the equation

$$[m \stackrel{\circ}{=} m'] = f(m) \sim f(m'), \qquad m, m' \in M \tag{9}$$

induces a fuzzy equality on M. If f is an injection and $\sim$ is separated then $\stackrel{\circ}{=}$ is separated, too.

PROOF: It follows from (E3) and Lemma 1(a), (b) that $\sim$ is a fuzzy equality on E in the sense of Definition 5. Verifying that (9) then induces a fuzzy equality on M as well as its separateness for an injection is straightforward. □

We will call the function f in the above lemma a *generating function* for a fuzzy equality (on M). As a special case, objects of the set M can be functions. Then the fuzzy equality can be induced in the sense of the following lemma.

Lemma 8 ([26])
Let M_α, M_β be sets and $\stackrel{\circ}{=}_\beta$ be a fuzzy equality on a set M_β. Then the function $\stackrel{\circ}{=}_{\beta\alpha} : M_\beta^{M_\alpha} \times M_\beta^{M_\alpha} \longrightarrow E$ defined for every $h, h' \in M_\beta^{M_\alpha}$ by

$$[h \stackrel{\circ}{=}_{\beta\alpha} h'] = \bigwedge_{m \in M_\alpha} [h(m) \stackrel{\circ}{=}_\beta h'(m)] \tag{10}$$

is a fuzzy equality. If $\stackrel{\circ}{=}_\beta$ is separated then $\stackrel{\circ}{=}_{\beta\alpha}$ is also separated.

Finally, we need the following concept. Let $F : M \longrightarrow N$ be a function and $\stackrel{\circ}{=}_1, \stackrel{\circ}{=}_2$ be fuzzy equalities defined in M, N, respectively. We say that F is *weakly extensional* if for all $m, m' \in M$,

$$[m \stackrel{\circ}{=}_1 m'] = \mathbf{1} \quad \text{implies that} \quad [F(m) \stackrel{\circ}{=}_2 F(m')] = \mathbf{1}. \tag{11}$$

3 Basic fuzzy type theory

By Theorem 2(a), each EQ-algebra having a filter can be homomorphically embedded into a separated EQ-algebra. Moreover, when trying to develop the syntax of a logic based on EQ-algebras (see below; cf. also [31]), it turns out that the following syntactic derivation is necessary: given a provable formula A, derive the equivalence $A \equiv \top$ (i.e., equivalence of A with the representation of truth), and vice-versa. From it follows that good EQ-algebras (which are necessarily separated) are reasonable candidates to become algebras of truth values for FTT. Furthermore, we need them to be lattices and also, we need the Δ operation to take the values $\mathbf{1}, \mathbf{0}$ only since otherwise several unpleasant difficulties occur (e.g., the important rule of two cases introduced below does not hold). Therefore, we will suppose that the structure of truth values is formed by a linearly ordered good EQ_Δ-algebra. On the basis of it, we will introduce a basic FTT. The presented technique is analogous to IMTL-FTT (introduced in detail in [26]) and follows also the methods of [1, 19]). Since basic principles of the proofs in IMTL-FTT are valid also in the basic FTT, we will often refer to the former. We will quite often use the (meta-)symbol ":=" which means "is defined by".

3.1 Syntax

Let ϵ, o be distinct objects. The set of types is the smallest set *Types* satisfying:

(i) $\epsilon, o \in$ *Types*,

(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

The type ϵ represents elements and o truth values. Higher-order types represent functions (of one variable[†].).

A *language* J of FTT consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ where $\alpha \in$ *Types*, auxiliary symbol λ, and brackets. A set of formulas[‡] over the language J is the smallest set such that for each $\alpha, \beta \in$ *Types* the following is specified:

(i) If $x_\alpha \in J$ is a variable, $\alpha \in$ *Types*, then x_α is a formula of type α.

(ii) If $c_\alpha \in J$ is a constant, $\alpha \in$ *Types*, then c_α is a formula of type α.

(iii) If $B_{\beta\alpha}$ is a formula of type $\beta\alpha$ and A_α a formula of type α then $(B_{\beta\alpha}A_\alpha)$ is a formula of type β.

(iv) If A_β is a formula of type β and $x_\alpha \in J$ a variable of type α then $\lambda x_\alpha A_\beta$ is a formula of type $\beta\alpha$.

The set of formulas of type α, $\alpha \in$ *Types*, is denoted by *Form*$_\alpha$. If $A \in$ *Form*$_\alpha$ is a formula of type $\alpha \in$ *Types* then we will often write A_α. This means that if $\alpha \neq \beta$ then A_α and A_β are *different* formulas. A set of all formulas of the language J is *Form* $= \bigcup_{\alpha \in Types}$ *Form*$_\alpha$.

Specific constants always present in the language of FTT are the following.

(i) $\mathbf{E}_{(oo)o}, \mathbf{E}_{(o(\beta\alpha))(\beta\alpha)}, \qquad \alpha, \beta \in$ *Types*, $\hfill$ (fuzzy equality)

(ii) $\mathbf{G}_{o\epsilon}$, $\hfill$ (generating function)

(iii) $\mathbf{C}_{(oo)o}$, $\hfill$ (conjunction)

(iv) $\mathbf{S}_{(oo)o}$, $\hfill$ (strong conjunction)

(v) $\mathbf{D}_{oo}$, $\hfill$ (delta)

(vi) $\iota_{\epsilon(o\epsilon)}, \iota_{o(oo)}$. $\hfill$ (description operators)

The fundamental connective in FTT is the *fuzzy equality* defined as follows:

(i) $\equiv_{(oo)o} \; := \; \lambda x_o \, \lambda y_o \, (\mathbf{E}_{(oo)o} \, y_o) x_o$,

(ii) $\equiv_{(o\epsilon)\epsilon} \; := \; \lambda x_\epsilon \, \lambda y_\epsilon \, (\mathbf{E}_{(oo)o} \, (\mathbf{G}_{o\epsilon} y_\epsilon))(\mathbf{G}_{o\epsilon} x_\epsilon)$,

(iii) $\equiv_{(o(\beta\alpha))(\beta\alpha)} \; := \; \lambda f_{\beta\alpha} \, \lambda g_{\beta\alpha} \, (\mathbf{E}_{(o(\beta\alpha))(\beta\alpha)} \, g_{\beta\alpha}) f_{\beta\alpha}$.

We will write fuzzy equality in the form $(A_\alpha \equiv B_\alpha)$. Note this is a formula of type o. We usually do not write a type at "$\equiv$" since this is clear from the types of A_α, B_α. Recall that, if $\alpha = o$ then "$\equiv$" is due to (i) the logical (fuzzy) equivalence. As already stated, we will always speak about fuzzy equality only.

Further basic formulas in FTT are the following:

(a) Representation of truth and falsity:

$$\top := (\lambda x_o \, x_o \equiv \lambda x_o \, x_o), \qquad \bot := (\lambda x_o \, x_o \equiv \lambda x_o \, \top).$$

[†] This is sufficient due to the trick of M. Schönfinkel, Über die Bausteine der mathematischen Logik. Mathematische Annalen **92**(1924), 150–182.

[‡] In the present literature on classical type theory, especially that focused on computer science, our formulas are often called *lambda-terms*. We will keep the original logical term introduced by A. Church and L. Henkin to emphasize that we are developing the *logic*.

(b) Conjunction:
$$\wedge := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o} \, y_o) x_o.$$

(c) Strong conjunction (fusion):
$$\& := \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o} \, y_o) x_o.$$

(d) Delta:
$$\boldsymbol{\Delta} := \lambda x_o \, \mathbf{D}_{oo} \, x_o.$$

(e) Negation:
$$\neg := \lambda x_o (\bot \equiv x_o).$$

(f) Implication:
$$\Rightarrow := \lambda x_o \lambda y_o ((x_o \wedge y_o) \equiv x_o).$$

(g) Quantifiers: Let $A_o \in Form_o$ and x_α be a variable of type α and y_o does not occur in A_o. Then we put:

$$(\forall x_\alpha) A_o := (\lambda x_\alpha \, A_o \equiv \lambda x_\alpha \top), \qquad\qquad (general\ quantifier)$$
$$(\exists x_\alpha) A_o := (\forall y_o)((\forall x_\alpha)\boldsymbol{\Delta}(A_o \Rightarrow y_o) \Rightarrow y_o). \qquad (existential\ quantifier)$$

(h) Disjunction:
$$\vee := \lambda x_o \lambda y_o ((x_o \Rightarrow y_o) \Rightarrow y_o) \wedge ((y_o \Rightarrow x_o) \Rightarrow x_o).$$

The n-fold strong conjunction is $A_o^n := A_o \& \cdots \& A_o$ (n-times).

3.2 Semantics

3.2.1 General frame

Semantics of FTT is, analogously as in classical type theory, defined using the concept of a general frame. In FTT, however, we must consider with each set of a given type also a fuzzy equality defined on it.

Definition 6
A general frame for the language J is a tuple

$$\mathcal{M} = \langle (M_\alpha, \doteq_\alpha)_{\alpha \in Types}, \mathcal{E}_\Delta, G \rangle$$

where $\mathcal{E}_\Delta$ is a linearly ordered good EQ_Δ-algebra of truth values

$$\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \mathbf{0}, \mathbf{1}, \Delta \rangle \tag{12}$$

and $G : M_\epsilon \longrightarrow E$ is a generating function of a fuzzy equality on M_ϵ.
Each set M_α is associated with the corresponding type as follows:

 (i) *$M_o = E$.*

 (ii) *M_ϵ is some arbitrary (non-empty) set.*

 (iii) *If $\beta\alpha$ is a non-elementary type then the corresponding set $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$.*

 (iv) *Each set $M_{oo} \cup M_{(oo)o}$ is closed w.r.t. all the operations from $\mathcal{E}_\Delta$.*

 (v) *For each $\alpha \in Types$, $\doteq_\alpha$ is a fuzzy equality on M_α which is either of the following:*

  (a) *If $\alpha = o$ then $\doteq_\alpha := \sim$.*

  (b) *If $\alpha = \epsilon$ then $\doteq_\alpha$ is the fuzzy equality (9) defined using the generating function G.*

  (c) *If $\alpha = \gamma\beta$ then $\doteq_\alpha$ is the fuzzy equality (10).*

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3.2.2 Interpretation of formulas

To define interpretation of formulas in a frame $\mathcal{M}$, we must consider an assignment p of elements from $\mathcal{M}$ to variables. Namely, p is a function from the set of all variables to elements from $\mathcal{M}$ in keeping with the corresponding types. If p and p' are two assignments differing only in the variable x_α then we will write $p = p'\backslash x_\alpha$. The set of all assignments over $\mathcal{M}$ is denoted by $\mathrm{Asg}(\mathcal{M})$.

Interpretation of formulas in a general frame $\mathcal{M}$ is a function that assigns to every formula A_α, and to every assignment p a corresponding element from the set M_α. Let A_α, $\alpha \in$ *Types*, be a formula, $\mathcal{M}$ a general frame and $p \in \mathrm{Asg}(\mathcal{M})$ an assignment of elements from M_α to variables of J. Then the interpretation of A_α is denoted by $\mathcal{M}_p(A_\alpha)^{\dagger)}$

Interpretation of constants

(i) $\mathcal{M}_p(\mathbf{E}_{(oo)o}) := \sim,$

(ii) $\mathcal{M}_p(\mathbf{E}_{(o(\beta\alpha))(\beta\alpha)}) := \overset{\circ}{=}_{\beta\alpha},$

(iii) $\mathcal{M}_p(\mathbf{G}_{o\epsilon}) := G,$

(iv) $\mathcal{M}_p(\mathbf{C}_{(oo)o}) := \wedge,$

(v) $\mathcal{M}_p(\mathbf{S}_{(oo)o}) := \otimes,$

(vi) $\mathcal{M}_p(\mathbf{D}_{oo}) := \Delta.$

(vii) Interpretation of the description operator $\mathcal{M}_p(\iota_{\epsilon(o\epsilon)})$ (or $\mathcal{M}_p(\iota_{o(oo)})$) is a function assigning to each non-empty fuzzy set in M_ϵ (or in M_o) an element from its kernel$^{\ddagger)}$ provided that the latter is nonempty; otherwise it is not determined.

Note that the description operator represents, in fact, a defuzzification operation (cf. [32], Chapter 5).

Interpretation of complex formulas

(i) Interpretation of a formula $B_{\beta\alpha}A_\alpha$, which is of type β, is

$$\mathcal{M}_p(B_{\beta\alpha}A_\alpha) = \mathcal{M}_p(B_{\beta\alpha})(\mathcal{M}_p(A_\alpha)).$$

(ii) Interpretation of a formula $\lambda x_\alpha\, A_\beta$, which is of type $\beta\alpha$, is a function

$$\mathcal{M}_p(\lambda x_\alpha\, A_\beta) = F : M_\alpha \longrightarrow M_\beta$$

which assigns to each $m_\alpha \in M_\alpha$ the element $F(m_\alpha) = \mathcal{M}_{p'}(A_\beta)$ determined by an assignment p' such that $p' = p\backslash x_\alpha$ and $p'(x_\alpha) = m_\alpha$. It follows from this definition that F is weakly extensional w.r.t "$=_\alpha$" and "$=_\beta$".

Lemma 9
The following holds true for every assignment $p \in \mathrm{Asg}(\mathcal{M})$:

(a) $\mathcal{M}_p(A_o \equiv B_o) = \mathcal{M}_p(A_o) \sim \mathcal{M}_p(B_o),$

(b) $\mathcal{M}_p(A_\epsilon \equiv B_\epsilon) = G(\mathcal{M}_p(A_\epsilon)) \sim G(\mathcal{M}_p(B_\epsilon)),$

(c)
$$\mathcal{M}_p(A_{\beta\alpha} \equiv B_{\beta\alpha}) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p\backslash x_\alpha}} (\mathcal{M}_{p'}(A_{\beta\alpha})(m_\alpha) \overset{\circ}{=}_\beta \mathcal{M}_{p'}(A_{\beta\alpha})(m_\alpha)).$$

$^{\dagger)}$ This function is in [26] written as $\mathcal{I}_p^{\mathcal{M}}(A_\alpha)$. We can also use the notation $\|A_\alpha\|_p^{\mathcal{M}}$ adopted from [15]. Our notation minimizes overburden of symbols by subscripts.

$^{\ddagger)}$ If $A \in E^M$ is a fuzzy set then its kernel is a set $\mathrm{Ker}(A) = \{m \mid A(m) = \mathbf{1}\}$.

PROOF: This follows immediately from the previous definitions. □

Lemma 10
Let $A_o, B_o \in Form_o$. Then the following holds true for every assignment p of elements from $\mathcal{M}$ to variables:

(a) $\mathcal{M}_p(\top) = \mathbf{1}$, $\mathcal{M}_p(\bot) = \mathbf{0}$,

(b) $\mathcal{M}_p(\neg A_o) = \mathcal{M}_p(A_o) \sim \mathbf{0}$,

(c) $\mathcal{M}_p(A_o \Rightarrow B_o) = \mathcal{M}_p(A_o) \rightarrow \mathcal{M}_p(B_o)$,

(d) $\mathcal{M}_p(A_o \,\&\, B_o) = \mathcal{M}_p(A_o) \otimes \mathcal{M}_p(B_o)$,

(e)
$$\mathcal{M}_p((\forall x_\alpha)A_o) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \backslash x_\alpha}} \mathcal{M}_{p'}(A_o),$$

(f)
$$\mathcal{M}_p((\exists x_\alpha)A_o) = \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \backslash x_\alpha}} \mathcal{M}_{p'}(A_o),$$

(g) $\mathcal{M}_p(A_o \vee B_o) = \mathcal{M}_p(A_o) \vee \mathcal{M}_p(B_o)$.

PROOF: The proof of (a)–(e), (g) follows from the assumed properties of $\mathcal{E}$ (cf. [26]).
(f)

$$\mathcal{M}_p((\exists x_\alpha)A_o) = \bigwedge_{\substack{a = p'(y_o) \in E \\ p' = p \backslash x_\alpha, y_o}} \Big(\bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \backslash x_\alpha, y_o}} \mathcal{M}_{p'}(\mathbf{\Delta}(A_o \Rightarrow y_o) \Rightarrow y_o) =$$

$$= \bigwedge \{ a \mid \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \backslash x_\alpha}} \mathcal{M}_{p'}(A_o) \leq a, a \in E \} = \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \backslash x_\alpha}} \mathcal{M}_{p'}(A_o)$$

because $\mathcal{M}_{p'}(\mathbf{\Delta}(A_o \Rightarrow y_o) \Rightarrow y_o)$ is equal to $a = p'(y_o)$, if $\mathcal{M}_{p'}(A_o) \leq a$ and it is equal to $\mathbf{1}$ otherwise. □

3.3   Axioms and inference rules

We will now specify formulas of type o which take the role of logical axioms of FTT. Note that their number is greater than in the classical type theory because we must characterize the structure of truth values. The types $\alpha, \beta \in Types$ are arbitrary.

Fundamental axioms

(FT-fund1) $\mathbf{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$,

(FT-fund2) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$,

(FT-fund3) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$,

(FT-fund4) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

Axioms of truth values. As usual in fuzzy logic, we have two kinds of conjunction, namely the "ordinary" conjunction $\wedge$ and the *strong conjunction* $\&$. Let $\bigcirc \in \{\wedge, \&\}$.

(FT-tval1) $(x_o \wedge y_o) \equiv (y_o \wedge x_o)$,

(FT-tval2) $(x_o \bigcirc y_o) \bigcirc z_o \equiv x_o \bigcirc (y_o \bigcirc z_o)$,

(FT-tval3) $(x_o \equiv \top) \equiv x_o$,

(FT-tval4a) $(x_o \bigcirc \top) \equiv x_o$,

(FT-tval4b) $(\top \& x_o) \equiv x_o$,

(FT-tval5) $(x_o \wedge x_o) \equiv x_o$,

(FT-tval6) $((x_o \wedge y_o) \equiv z_o) \& (t_o \equiv x_o) \Rightarrow (z_o \equiv (t_o \wedge y_o))$,

(FT-tval7) $(x_o \equiv y_o) \& (z_o \equiv t_o) \Rightarrow (x_o \equiv z_o) \equiv (y_o \equiv t_o)$

(FT-tval8) $(x_o \Rightarrow (y_o \wedge z_o)) \Rightarrow (x_o \Rightarrow y_o)$

(FT-tval9) $(x_o \Rightarrow y_o) \Rightarrow ((x_o \wedge z_o) \Rightarrow y_o)$

(FT-tval10a) $\boldsymbol{\Delta}(x_o \Rightarrow y_o) \Rightarrow (x_o \& z_o \Rightarrow y_o \& z_o)$

(FT-tval10b) $\boldsymbol{\Delta}(x_o \Rightarrow y_o) \Rightarrow (z_o \& x_o \Rightarrow z_o \& x_o)$

(FT-tval11) $((x_o \Rightarrow y_o) \Rightarrow z_o) \Rightarrow ((y_o \Rightarrow x_o) \Rightarrow z_o) \Rightarrow z_o$

Note that axiom (FT-tval3) expresses that the underlying EQ-algebra is good. Axioms (FT-tval7)–(FT-tval9) correspond with the axioms of EQ-algebra for the fuzzy equality. Axioms (FT-tval10a) and (FT-tval10b) expresses isotonicity of $\&$ and axiom (FT-tval11) is the prelinearity axiom. Note also that $\&$ is non-commutative.

Axioms of delta

(FT-delta1) $(g_{oo}(\boldsymbol{\Delta} x_o) \wedge g_{oo}(\neg \boldsymbol{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\boldsymbol{\Delta} y_o)$

(FT-delta2) $\boldsymbol{\Delta}(x_o \wedge y_o) \equiv \boldsymbol{\Delta} x_o \wedge \boldsymbol{\Delta} y_o$

(FT-delta3) $\boldsymbol{\Delta}(x_o \vee y_o) \Rightarrow \boldsymbol{\Delta} x_o \vee \boldsymbol{\Delta} y_o$

(FT-delta4) $\boldsymbol{\Delta} x_o \vee \neg \boldsymbol{\Delta} x_o$

Axioms of quantifiers

(FT-quant1) $\boldsymbol{\Delta}(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$, $\quad x_\alpha$ is not free in A_o

(FT-quant2) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha)A_o \Rightarrow B_o)$, $\quad x_\alpha$ is not free in B_o

(FT-quant3) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha)A_o \vee B_o)$, $\quad x_\alpha$ is not free in B_o

Axioms of descriptions

(FT-descr1) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha} y_\alpha) \equiv y_\alpha$, $\qquad \alpha = o, \epsilon$

Lemma 11
If A_o is an axiom from the above list then for every general frame $\mathcal{M}$ and an assignment p, $\mathcal{M}_p(A_o) = \mathbf{1}$ holds true.

PROOF: The equality for the fundamental axioms, axioms of descriptions, and (FT-delta1) have been proved in [23, 26]. The other axioms of delta and truth values follow from the EQ-algebra axioms and the assumed properties. Axioms (FT-quant1) and (FT-quant2) follow from Lemma 2(f) and (h), respectively. The proof for (FT-quant3) is the same as in [15], Lemma 5.1.9. □

Inference rules

(R)   Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then, infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.

(N)   Let $A_o \in Form_o$. Then, from A_o infer $\mathbf{\Delta} A_o$.

A theory T is a set of formulas of type o (determined by a subset of special axioms, as usual). Provability is defined as usual. If T is a theory and A_o a formula then $T \vdash A_o$ means that A_o is provable in T.

Lemma 12

The inference rules (R) and (N) are sound, i.e. the following holds for every general frame $\mathcal{M}$ and an assignment $p \in Asg(\mathcal{M})$:

Rule (R):   if $\mathcal{M}_p(A_\alpha \equiv A'_\alpha) = \mathbf{1}$ then $\mathcal{M}_p(B_o) = \mathcal{M}_p(B'_o)$.
Rule (N):   if $\mathcal{M}_p(A_o) = \mathbf{1}$ then also $\mathcal{M}_p(\mathbf{\Delta} A_o) = \mathbf{1}$.

PROOF: Soundness of the rules (R) and (N) has been proved in [26]. Note that the full equality in case of the rule (R) follows from the fact that the considered EQ-algebra is good and, hence, separated.   □

3.3.1   Models of FTT

Definition 7

(a) *A safe general model is a general frame $\mathcal{M}$ such that for every assignment $p \in Asg(\mathcal{M})$, $\mathcal{M}_p(A_\alpha) \in M_\alpha$ holds true for all $\alpha \in Types$. This means that each set M_α from the frame $\mathcal{M}$ has enough elements so that the interpretation $\mathcal{M}_p(A_\alpha)$ is always defined. As a special case (analogously as in the corresponding concept of a safe general model in [15]), if a formula A_α contains quantifiers then all the necessary suprema and infima in $\mathcal{E}$ exist.*

(b) *If T is a theory then a model of T (in symbols $T \models \mathcal{M}$) is a safe general model, in which all special axioms of T are true in the degree $\mathbf{1}$.*

Let T be a theory. A formula A_o is *true* in the degree $a \in E$ in T, if

$$a = \bigwedge \{ \mathcal{M}_p(A_o) \mid \mathcal{M} \models T, p \in Asg(\mathcal{M}) \}, \tag{13}$$

(provided that the infimum exists). Occasionally, we may write (13) as $T \models_a A_o$ (we omit the subscript if $a = \mathbf{1}$).

As a consequence of Lemmas 11 and 12 we obtain the following.

Theorem 4 (Soundness)

The basic FTT is sound in the following sense: If $\vdash A_o$ then $\mathcal{M}_p(A_o) = \mathbf{1}$ holds for every assignment p and every safe general model $\mathcal{M}$.

4   Main properties of the basic FTT

In this section, we will present some of the properties of the basic FTT.

The proofs of the following lemmas are technical and similar to analogous proofs in [26]. Therefore, we omit them.

Lemma 13

(a) $\vdash (A_\epsilon \equiv B_\epsilon) \equiv (\mathbf{G}_{o\epsilon} A_\epsilon \equiv \mathbf{G}_{o\epsilon} B_\epsilon)$,

(b) $\vdash \top$,

(c) $\vdash A_\alpha \equiv A_\alpha, \qquad \alpha \in Types$.

(d) $A_{o,x_\alpha}[\top], A_{o,x_\alpha}[\bot] \vdash A_{o,x_\alpha}[\boldsymbol{\Delta} y_o]$. *(Rule of Two Cases)*

(e) $A_o, A_o \Rightarrow B_o \vdash B_o$. *(Modus Ponens)*

(f) $A_o, B_o \vdash A_o \,\&\, B_o$.

(g) $\vdash (A_o \equiv B_o) \equiv (B_o \equiv A_o)$,

(h) $\vdash (A_o \equiv B_o) \,\&\, (B_o \equiv C_o) \Rightarrow (A_o \equiv C_o)$,

(i) $\vdash (A_o \Rightarrow B_o) \,\&\, (B_o \Rightarrow A_o) \Rightarrow (A_o \equiv B_o)$

(j) $\vdash (\top \Rightarrow A_o) \equiv A_o$.

(k) $A_o \Rightarrow B_o, B_o \Rightarrow C_o \vdash A_o \Rightarrow C_o$.

(l) $\vdash A \Rightarrow (B \Rightarrow C) \vdash (A \,\&\, B) \Rightarrow C)$.

(m) $A_o \Rightarrow B_o \vdash (A_o \bigcirc C_o) \Rightarrow (B_o \bigcirc C_o)$ where $\bigcirc \in \{\land, \&\}$
   (and also $A_o \Rightarrow B_o \vdash (C_o \,\&\, A_o) \Rightarrow (C_o \,\&\, B_o)$).

(n) $A_o \Rightarrow B_o, B_o \Rightarrow A_o \vdash A_o \equiv B_o$.

Note that in general, the converse of (l) does not hold (cf. examples).

Lemma 14
(a) $\vdash (A_o \Rightarrow B_o) \,\&\, (B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o)$,

(b) $\vdash (A_o \land B_o) \Rightarrow A_o$

(c) $\vdash (C_o \Rightarrow A_o) \,\&\, (C_o \Rightarrow B_o) \Rightarrow (C_o \Rightarrow (A_o \land B_o))$,

(d) $\vdash (A_o \equiv B_o) \Rightarrow (A_o \Rightarrow B_o)$,

(e) $\vdash A_o \Rightarrow (B_o \Rightarrow A_o)$,

(f) $\vdash A_o \Rightarrow ((A_o \equiv B_o) \equiv B_o)$

(g) $\vdash A_o \Rightarrow ((A_o \Rightarrow B_o) \Rightarrow B_o)$

Theorem 5 (Equality theorem)
(a) $\vdash \boldsymbol{\Delta}(x_\beta \equiv y_\beta) \Rightarrow (\boldsymbol{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta))$.

(b) $\vdash (\boldsymbol{\Delta}(x_\beta \equiv y_\beta) \,\&\, \boldsymbol{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta})) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta)$.

Lemma 15
(a) $A_o \vdash (\forall x_\alpha) A_o$. *(Rule of Generalization)*

(b) $\vdash (\forall x_\alpha) B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$, *(∀-substitution)*

(c) $\vdash \bot \Rightarrow A_o$,

(d) Let $T \vdash B_o$ and $A_{\alpha_1}, \ldots, A_{\alpha_n}$ be substitutable for all free occurrences of $x_{\alpha_1}, \ldots, x_{\alpha_n}$ in B_o. Then
   $T \vdash B_{o,x_{\alpha_1},\ldots,x_{\alpha_n}}[A_{\alpha_1}, \ldots, A_{\alpha_n}]$.

Lemma 16
(a) $\vdash A_o \Rightarrow (A_o \lor B_o)$,

(b) $A_o \Rightarrow C_o, B_o \Rightarrow C_o \vdash (A_o \lor B_o) \Rightarrow C_o$.

Lemma 17
(a) $\vdash \boldsymbol{\Delta}\top \equiv \top$,

(b) $\vdash \boldsymbol{\Delta} x_o \Rightarrow x_o,$

(c) $\vdash \boldsymbol{\Delta} x_o \Rightarrow \boldsymbol{\Delta}\boldsymbol{\Delta} x_o,$

(d) $\vdash \boldsymbol{\Delta}(x_o \equiv y_o) \Rightarrow (\boldsymbol{\Delta} x_o \equiv \boldsymbol{\Delta} y_o),$

(e) $\vdash \boldsymbol{\Delta}(x_o \Rightarrow y_o) \Rightarrow (\boldsymbol{\Delta} x_o \Rightarrow \boldsymbol{\Delta} y_o),$

(f) $\vdash (\boldsymbol{\Delta} A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow ((\boldsymbol{\Delta} A_o \Rightarrow B_o) \Rightarrow (\boldsymbol{\Delta} A_o \Rightarrow C_o)).$

Lemma 18
(a) $\vdash B_{o\alpha} A_\alpha \Rightarrow (\exists x_\alpha) B_{o\alpha} x_\alpha.$ $\qquad\qquad\qquad\qquad\qquad\qquad$ ($\exists$-substitution)

(b) $(\forall x_\alpha)(A_o \Rightarrow B_o) \vdash A_o \Rightarrow (\forall x_\alpha) B_o$ and $A_o \Rightarrow (\forall x_\alpha) B_o \vdash (\forall x_\alpha)(A_o \Rightarrow B_o)$, where x_α is not free in A_o.

(c) $\vdash (\forall x_\alpha)(A_o \equiv B_o) \vdash (\forall x_\alpha) A_o \equiv (\forall x_\alpha) B_o.$

Theorem 6

$$\vdash (\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \equiv (f_{\beta\alpha} \equiv g_{\beta\alpha}).$$

Theorem 7
Let $\alpha \in$ Types. Then

(a) $\vdash (x_\alpha \equiv y_\alpha) \equiv (y_\alpha \equiv x_\alpha),$

(b) $\vdash (x_\alpha \equiv y_\alpha) \,\&\, (y_\alpha \equiv z_\alpha) \Rightarrow (x_\alpha \equiv x_\alpha).$

PROOF: The proof proceeds by induction on the complexity of type.
　　(a) For $\alpha = o$, the provability follows from Lemma 13(g).
　　Let $\alpha = \epsilon$.

(L.1) $\vdash (x_\epsilon \equiv y_\epsilon) \equiv (\mathbf{G}_{o\epsilon} x_\epsilon \equiv \mathbf{G}_{o\epsilon} y_\epsilon)$ $\qquad\qquad\qquad$ (definition of equality, (FT-fund4))

(L.2) $\vdash (y_\epsilon \equiv x_\epsilon) \equiv (\mathbf{G}_{o\epsilon} y_\epsilon \equiv \mathbf{G}_{o\epsilon} x_\epsilon)$ $\qquad\qquad\qquad$ (definition of equality, (FT-fund4))

(L.3) $\vdash (\mathbf{G}_{o\epsilon} x_\epsilon \equiv \mathbf{G}_{o\epsilon} y_\epsilon) \equiv (\mathbf{G}_{o\epsilon} y_\epsilon \equiv \mathbf{G}_{o\epsilon} x_\epsilon)$ $\qquad\qquad\qquad$ (provable for $\alpha = o$)

(L.4) $\vdash (x_\epsilon \equiv y_\epsilon) \equiv (y_\epsilon \equiv x_\epsilon)$ $\qquad\qquad\qquad\qquad\qquad$ (L.1, L.2, L.3, rule (R))

　　Let $\alpha = \gamma\beta$. Furthermore, note that $f_{\gamma\beta}\, x_\beta$ is a formula of type γ. Therefore we may start with the inductive assumption
$$\vdash (f_{\gamma\beta}\, x_\beta \equiv g_{\gamma\beta}\, x_\beta) \equiv (g_{\gamma\beta}\, x_\beta \equiv f_{\gamma\beta}\, x_\beta).$$
Then use generalization, Lemma 18(c), Theorem 6 and rule (R).
　　(b) If $\alpha = o$ then the property follows from Lemma 14(h). For $\alpha = \epsilon$ we prove the property analogously as above with the use of its provability for $\alpha = o$.
　　By the inductive assumption, we have

$$\vdash (f_{\gamma\beta}\, x_\beta \equiv g_{\gamma\beta}\, x_\beta) \,\&\, (g_{\gamma\beta}\, x_\beta \equiv h_{\gamma\beta}\, x_\beta) \Rightarrow (f_{\gamma\beta}\, x_\beta \equiv h_{\gamma\beta}\, x_\beta).$$

Then using Lemma 14(d), Lemma 18(c) and (b) we obtain

$$\vdash (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv g_{\gamma\beta}\, x_\beta) \,\&\, (\forall x_\beta)(g_{\gamma\beta}\, x_\beta \equiv h_{\gamma\beta}\, x_\beta) \Rightarrow (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv h_{\gamma\beta}\, x_\beta).$$

Finally use Theorem 6 and rule (R). $\hfill\square$

　　The following theorem characterizes basic properties of the description operator.

Theorem 8
(a) $\vdash \iota_{\gamma(o\gamma)}(\mathbf{E}_{(o\gamma)\gamma}\, y_\gamma) \equiv y_\gamma$ *holds for every type $\gamma \in$ Types.*

(b) $\vdash \boldsymbol{\Delta}(p_{o\alpha} \equiv q_{o\alpha}) \Rightarrow (\iota_{\alpha(o\alpha)}p_{o\alpha} \equiv \iota_{\alpha(o\alpha)}q_{o\alpha})$,

(c) $\vdash (\exists y_\alpha)(\exists x_\alpha)(\boldsymbol{\Delta}(p_{o\alpha}x_\alpha \equiv (E_{(o\alpha)\alpha}y_\alpha)x_\alpha)) \Rightarrow p_{o\alpha}(\iota_{\alpha(o\alpha)}p_{o\alpha})$,

(d) $\vdash (\exists x_\alpha)\boldsymbol{\Delta}A_{o\alpha} \Rightarrow A_{o\alpha}(\iota_{\alpha(o\alpha)}A_{o\alpha})$.

PROOF: (a)–(c) have been proved in [23]. (d) is a consequence of (c). □

The property (d) of this theorem explicitly states that if the kernel of the fuzzy set (represented by) $A_{o\alpha}$ is non-empty then the element $\iota_{\alpha(o\alpha)}A_{o\alpha}$ belongs to it, as well. In other words, this element always belongs to a kernel of a normal fuzzy set.

Theorem 9 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \text{iff} \quad T \vdash \boldsymbol{\Delta}A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

PROOF: The proof is analogous to that in [26], under the reserve that Lemmas 13 and 14 should be used in some steps. □

Definition 8
Let T be a theory. We say that:

(i) T is contradictory if $T \vdash \bot$. Otherwise it is consistent.

(ii) T is maximal consistent if each its extension T', $T' \supset T$ is inconsistent.

(iii) T is linear[†] if for every two formulas A_o, B_o

$$T \vdash A_o \Rightarrow B_o \quad \text{or} \quad T \vdash B_o \Rightarrow A_o.$$

(iv) T is extensionally complete if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \nvdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ it follows that there is a closed formula C_α such that $T \nvdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha$.

The following theorem immediately follows from Lemma 15(c).

Theorem 10
A theory T is contradictory iff each formula $A_o \in Form_o$ is provable in it.

The following two important properties which hold in residuated fuzzy logics (cf. [17]) hold also in basic FTT.

Lemma 19
Let T be a theory.

(a) If $T \cup \{A_o\} \vdash C_o$ and $T \cup \{B_o\} \vdash C_o$ then $T \cup \{A_o \lor B_o\} \vdash C_o$.

(Proof by cases)

(b) $T \cup \{A_o \Rightarrow B_o\} \vdash C_o$ and $T \cup \{B_o \Rightarrow A_o\} \vdash C_o$ then $T \vdash C_o$

(Prelinearity property)

PROOF: (a) By the assumption and the deduction theorem, we obtain $T \vdash \boldsymbol{\Delta}A_o \Rightarrow C_o$ and $T \vdash \boldsymbol{\Delta}B_o \Rightarrow C_o$. Then use Lemmas 13(f), (k) and 16(b), axiom (FT-delta3), and Modus Ponens.
(b) follows from (a) using the deduction theorem, axiom (FT-tval10a), rules (N) and Modus Ponens.
□

[†]Such theory is in [15] and also in [26] called complete.

Theorem 11
Every consistent theory T can be extended to a maximally consistent complete theory.

PROOF: The proof proceeds in the same way as the proof of item 3 of Lemma 2 from [17] because the prelinearity property is provable in FTT. □

Theorem 12
Every consistent theory T can be extended to an extensionally complete consistent theory $\overline{T}$.

PROOF: We will adapt the proofs of Theorem 20 from [26] and item 1 of Lemma 2 from [17].

Let us denote $\pi = \mathrm{Card}(J(T))$. Let K_α be a well ordered set of new constants of type $\alpha \in Types$, $\mathrm{Card}(K_\alpha) \leq \pi$ and put $K = \bigcup_{\alpha \in Types} K_\alpha$ and $J^+(T) = J(T) \cup K$. We will also enumerate all closed formulas of the language $J(T)$ by ordinal numbers.

In the proof, we construct a sequence of theories T_μ, $\mu \leq \pi$ and a sequence of special sets[†] of formulas Ψ_μ such that:

(a) $T_0 = T$ and $T_\nu \subset T_\mu$ for $\nu < \mu$,

(b) $\Psi_\nu \subset \Psi_\mu$ and $T_\mu \nvdash \Psi_\mu$.

The construction proceeds by transfinite recursion. We suppose that $T_{<\mu} = \bigcup_{\nu \in \mu} T_\nu$ and $\Psi_{<\mu} = \bigcup_{\nu \in \mu} \Psi_\nu$ are already constructed.

Let $D \in \Psi_{<\mu}$. Furthermore, let $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be the first not yet processed formula of this form, and let $c_\alpha \in K$ be the first not yet used constant (both in the given well ordering). We distinguish two cases:

(i) Let $T_{<\mu} \vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$. Then we put $\Psi_\mu = \Psi_{<\mu}$ and $T_\mu = T_{<\mu} \cup \{A_{\beta\alpha} \equiv B_{\beta\alpha}\}$.

(ii) Let $T_{<\mu} \nvdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$. Then we put $\Psi_\mu = \Psi_{<\mu} \cup \{D \vee (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha) \mid D \in \Psi_{<\mu}\}$ and $T_\mu = T_{<\mu}$.

Now we must show that in both cases $T_\mu \nvdash \Psi_\mu$. This is done analogously as in [17]. In case (ii), we must show that the assumption $T_\mu \vdash D \vee (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$ leads to contradiction; indeed, using (FT-quant3) and Lemma 13(m) we obtain $T_\mu \vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$.

To show, finally, that T_π is extensionally complete, let $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be a formula processed in step μ and $T_\pi \nvdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Then it had to be processed as the case (ii), i.e. $T_\mu \nvdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$, otherwise we get a contradiction. But the previous reasoning, together with Lemma 16(a), leads to $T_\pi \nvdash (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$ for some constant c_α. □

5 Canonical model of FTT

This subsection is devoted to construction of the canonical (general) model of a consistent theory of FTT. The construction is analogous to the classical way, i.e. we use syntactical material for the construction. For each type $\alpha \in Types$, we construct the corresponding set M_α. For elementary types, this construction is straightforward but for complex types $\beta\alpha$ we have to construct sets $M_{\beta\alpha}$ as sets of weakly extensional functions.

5.1 Construction of the good EQ-algebra of truth values

We start with the construction of the set M_o of truth values and its appropriate algebraic structure. Let T be a theory.

Let us define an equivalence on the set of closed formulas from $Form_o$ by

$$A_o \approx B_o \quad \text{iff} \quad T \vdash A_o \equiv B_o. \tag{14}$$

[†] The sets must be directed — the details are in [17].

Using Lemmas 13(c), (g) and 14(h) and rule (R) we can verify that $\sim$ is indeed the equivalence. The equivalence class of a formula A_o is denoted by $|A_o|$ and we put $M_o = Form_o|\approx$.

Now we define the following operations on the set M_o:

$$|A_o| \wedge_T |B_o| = |A_o \wedge B_o|, \tag{15}$$
$$|A_o| \otimes_T |B_o| = |A_o \,\&\, B_o|, \tag{16}$$
$$|A_o| \sim_T |B_o| = |A_o \equiv B_o|, \tag{17}$$
$$\Delta_T(|A_o|) = |\boldsymbol{\Delta} A_o|, \tag{18}$$
$$|A_o| \vee_T |B_o| = |A_o \vee B_o|, \tag{19}$$
$$\mathbf{1}_T = |\top|, \qquad \mathbf{0}_T = |\bot|. \tag{20}$$

Note that we immediately obtain that $|A_o| \rightarrow_T |B_o| = |(A_o \wedge B_o) \equiv A_o|$.

Theorem 13
Let T be a linear extensionally complete theory. Then the algebra

$$\mathcal{E}_T = \langle M_o, \wedge_T, \otimes_T, \sim_T, \Delta_T, \mathbf{1}_T, \mathbf{0}_T \rangle \tag{21}$$

is a linearly ordered good EQ_Δ-algebra which is lattice ordered with the join given by (19).

PROOF: Let us verify the axioms of EQ-algebra:

(E1) follows from (FT-tval1), (FT-tval2), (FT-tval5). Note that we have

$$|A_o| \leq |B_o| \quad \text{iff} \quad |A_o| \wedge_T |B_o| = |A_o| \quad \text{iff} \quad T \vdash (A_o \wedge B_o) \equiv A_o$$
$$\text{iff} \quad T \vdash A_o \Rightarrow B_o \quad \text{iff} \quad T \vdash (A_o \Rightarrow B_o) \equiv \top \quad \text{iff} \quad |A_o| \rightarrow_T |B_o| = |\top|. \tag{22}$$

(E2) follows from (FT-tval1), (FT-tval2),(FT-tval10a), (FT-tval4a) and (FT-tval4b).

(E3) follows from Lemma 13(c). (E4) follows from (FT-tval6), (E5) from (FT-tval7), (E6) from (FT-tval8), (E7) from (FT-tval9). Axiom (FT-tval3) implies that $\mathcal{E}_T$ is good. Therefore, by Lemma 2(a), axiom (E8) is provable.

From (22) and the linearity of T we conclude that $\mathcal{E}_T$ is linearly ordered, thus prelinear and so, (19) is a join in M_o.

As for axioms of Δ_T from Definition 3, we see that (i)–(iv) follow from Lemma 17(a)–(d), respectively and (v)–(vii) follow from (FT-delta2)–(FT-delta4), respectively. However, because $\mathcal{E}_T$ is linearly ordered, we can replace definition (18) by

$$\Delta_T(|A_o|) = \begin{cases} |\top| & \text{if } T \vdash A_o, \\ |\bot| & \text{otherwise.} \end{cases} \tag{23}$$

We must now prove that

$$|(\forall x_\alpha)A_o| = \bigwedge \{|A_{o,x_\alpha}[c_\alpha]| \,\big|\, \text{all constants } c_\alpha \in Form_\alpha\}. \tag{24}$$

The proof is analogous to that of Lemma 5.2.6 from [15]. Namely, the inequality $\leq$ follows from Lemma 15(a). Conversely, let $|D| \leq |A_{o,x_\alpha}[c_\alpha]|$ for all $c_\alpha \in Form_\alpha$. If $|D| \not\leq |(\forall x_\alpha)A_o|$ then $T \not\vdash D \Rightarrow (\forall x_\alpha)A_o$ and so, $T \not\vdash (\forall x_\alpha)(D \Rightarrow A_o)$ by Lemma 18(b). By the definition of $\forall$, this means that

$$T \not\vdash \lambda x_\alpha (D \Rightarrow A_o)x_\alpha \equiv \lambda x_\alpha \top.$$

Since T is extensionally complete, this means that

$$T \not\vdash D \Rightarrow A_{o,x_\alpha}[c_\alpha] \equiv \top$$

for some constant $c_\alpha \in Form_\alpha$ and we conclude that $|D| \not\leq |A_{o,x_\alpha}[c_\alpha]|$ — a contradiction.

From (24), analogously as in Lemma 10(f), we also obtain that

$$|(\exists x_\alpha)A_o| = \bigvee \{|A_{o,x_\alpha}[c_\alpha]| \,\big|\, \text{all constants } c_\alpha \in Form_\alpha\}.$$

Thus, we conclude that $\mathcal{E}_T$ is a good linearly ordered EQ_Δ-algebra. $\qquad\square$

5.2 Construction of the canonical frame

We now need to define all the domains of the canonical frame which, in general, consist of weakly extensional functions. Therefore, we define a special function $\mathcal{V}$, whose domain and range are formulas or their equivalence classes. Then we define the sets of the canonical frame by

$$M_\alpha = \{\mathcal{V}(A_\alpha) \mid A_\alpha \in Form_\alpha\}, \qquad \alpha \in Types. \tag{25}$$

The construction proceeds inductively:

(i) If $\alpha = o$ then $\mathcal{V}(A_o) = |A_o|$, i.e. $M_o = Form_o|\approx$. Furthermore, we put $\stackrel{\circ}{=}_o := \sim_T$ where $\sim_T$ is defined in (17).

(ii) If $\alpha = \epsilon$ then $\mathcal{V}(A_\epsilon) = A_\epsilon$, i.e. $M_\epsilon = Form_\epsilon$.

(iii) If $\alpha = \gamma\beta$ then we put $\mathcal{V}(A_{\gamma\beta}) \subseteq M_\beta \times M_\gamma$ which is a relation consisting of couples

$$\langle \mathcal{V}(B_\beta), \mathcal{V}(A_{\gamma\beta}B_\beta) \rangle$$

for all closed $B_\beta \in Form_\beta$ and $A_{\gamma\beta} \in Form_{\gamma\beta}$.

(iv) The generating function G_T is defined by $G_T : \mathcal{V}(A_\epsilon) \mapsto \mathcal{V}(\mathbf{G}_{o\epsilon}A_\epsilon)$ for all $A_\epsilon \in Form_\epsilon$.

The fuzzy equality in each set M_α, $\alpha \neq o$, is defined by

$$\stackrel{\circ}{=}_\alpha (\mathcal{V}(A_\alpha), \mathcal{V}(B_\alpha)) := |A_\alpha \equiv B_\alpha|. \tag{26}$$

We will write $[\mathcal{V}(A_\alpha) \stackrel{\circ}{=}_\alpha \mathcal{V}(B_\alpha)]$ instead of $\stackrel{\circ}{=}_\alpha (\mathcal{V}(A_\alpha), \mathcal{V}(B_\alpha))$ in the sequel.

It follows from (15)–(20) and this description that the operations from $\mathcal{E}_T$ are included in $M_{oo} \cup M_{o(oo)}$.

Lemma 20
The relation (26) is a fuzzy equality on M_α, $\alpha \in Types$.

PROOF: To prove that it is a fuzzy equality, note that definition (17) of a fuzzy equality on M_o has the same form as (26). Therefore, the reflexivity of (26) follows from Lemma 13(c). Symmetry and transitivity follows from Theorem 7(a) and (b), respectively. □

The proof of the following lemma is closely analogous to the proofs of [26, Lemmas 15 and 16].

Lemma 21
(a) $[\mathcal{V}(A_\epsilon) \stackrel{\circ}{=}_\epsilon \mathcal{V}(B_\epsilon)] = |\mathbf{G}_{o\epsilon}A_\epsilon \equiv \mathbf{G}_{o\epsilon}B_\epsilon| = G_T(\mathcal{V}(A_\epsilon)) \sim_T G_T(\mathcal{V}(B_\epsilon))$.

(b) If T is an extensionally complete theory then for all types $\alpha = \gamma\beta$

$$[\mathcal{V}(A_{\gamma\beta}) \stackrel{\circ}{=}_{\gamma\beta} \mathcal{V}(B_{\gamma\beta})] = \bigwedge_{C_\beta \in Form_\beta} [\mathcal{V}(A_{\gamma\beta}C_\beta) \stackrel{\circ}{=}_\gamma \mathcal{V}(B_{\gamma\beta}C_\beta)] \tag{27}$$

where the formulas C_β in (27) are closed.

(c) For all $\alpha \in Types$

$$[\mathcal{V}(A_\alpha) \stackrel{\circ}{=}_\alpha \mathcal{V}(B_\alpha)] = \mathbf{1} \quad iff \quad T \vdash A_\alpha \equiv B_\alpha. \tag{28}$$

PROOF: (a) follows from (26), Lemma 13(a) and the definition of G_T.

(b) By axiom (FT-fund3) we have

$$T \vdash (A_{\gamma\beta} \equiv B_{\gamma\beta}) \Rightarrow (A_{\gamma\beta}C_\beta \equiv B_{\gamma\beta}C_\beta)$$

for every closed C_β. From it follows that

$$[\mathcal{V}(A_{\gamma\beta}) \stackrel{\circ}{=}_{\gamma\beta} \mathcal{V}(B_{\gamma\beta})] \leq \bigwedge_{C_\beta \in Form_\beta} [\mathcal{V}(A_{\gamma\beta}C_\beta) \stackrel{\circ}{=}_\gamma \mathcal{V}(B_{\gamma\beta}C_\beta)]. \tag{29}$$

The opposite inequality can be proved analogously as in the proof of Theorem 13. Let $T \vdash D_o \Rightarrow (A_{\gamma\beta}\, x_\beta \equiv B_{\gamma\beta}\, x_\beta)$ for all x_β and let us suppose that $\mathcal{V}(D_o) \not\leq \mathcal{V}(A_{\gamma\beta} \equiv B_{\gamma\beta})$. Then

$$T \not\vdash D_o \Rightarrow (A_{\gamma\beta} \equiv B_{\gamma\beta}).$$

Because T is extensionally complete, there is a closed C_β such that $T \not\vdash D_o \Rightarrow (A_{\gamma\beta}\, C_\beta \equiv B_{\gamma\beta}\, C_\beta)$ — a contradiction. This means that the right-hand side of (27) is infimum.

(c) Recall that by (20), $\mathbf{1}_T = |\top|$. Hence, by (26), Lemma 20, and (14) we obtain

$$[\mathcal{V}(A_\alpha) \overset{\circ}{=}_\alpha \mathcal{V}(B_\alpha)] = |A_\alpha \equiv B_\alpha| = |\top| \quad \text{iff} \quad T \vdash A_\alpha \equiv B_\alpha.$$

$\square$

Lemma 22
Let T be an extensionally complete theory. Then $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function.

PROOF: The extensionality means by (11) that $[\mathcal{V}(B_\alpha) \overset{\circ}{=}_\alpha \mathcal{V}(B'_\alpha)] = \mathbf{1}_T$ implies $[\mathcal{V}(A_{\beta\alpha}B_\alpha) \overset{\circ}{=}_\beta \mathcal{V}(A_{\beta\alpha}B'_\alpha)] = \mathbf{1}_T$.

We know that $[\mathcal{V}(B_\alpha) \overset{\circ}{=}_\alpha \mathcal{V}(B'_\alpha)] = \mathbf{1}_T$ iff $T \vdash B_\alpha \equiv B'_\alpha$. Since $T \vdash A_{\beta\alpha} \equiv A_{\beta\alpha}$ we conclude using Theorem 5 that $T \vdash A_{\beta\alpha}B_\alpha \equiv A_{\beta\alpha}B'_\alpha$, i.e. $[\mathcal{V}(A_{\beta\alpha}B_\alpha) = \mathcal{V}(A_{\beta\alpha}B'_\alpha)] = \mathbf{1}_T$. This means that $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function.

$\square$

It follows from the above construction that

$$\mathcal{M}^T = \langle (M_\alpha, \overset{\circ}{=}_\alpha)_{\alpha \in Types}, \mathcal{E}_T, G_T \rangle \tag{30}$$

is a general frame which will be called a *canonical frame* of FTT.

5.3   Canonical model of FTT

Now, we can define the interpretation of formulas in the canonical frame. More specifically, let p be an assignment of elements to variables, i.e. $p(x_\alpha) = \mathcal{V}(A_\alpha) \in M_\alpha$ for all $\alpha \in Types$. Then we put:

(i) If x_α is a variable then $\mathcal{M}_p^T(x_\alpha) = p(x_\alpha)$.

(ii) If c_α, $\alpha \neq o$ is a constant then $\mathcal{M}_p^T(c_\alpha) = \mathcal{V}(c_\alpha) \in M_\alpha$. Furthermore,

(a) $\mathcal{M}_p^T(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality (26), $\alpha \in Types$.

(b) Interpretation of the conjunction $\mathbf{C}_{(oo)o}$ and strong conjunction $\mathbf{S}_{(oo)o}$ are the operations (15) and (16), respectively. Interpretation of the delta operation $\mathbf{D}_{(oo)o}$ is given by (23).

(c) Interpretation of $\mathbf{G}_{o\epsilon}$ is the generating function G_T.

(iii) Interpretation of the formula $\lambda x_\alpha\, A_\beta$ of type $\beta\alpha$ is the function

$$\mathcal{M}_p^T(\lambda x_\alpha\, A_\beta) : \mathcal{V}(B_\alpha) \mapsto \mathcal{V}((\lambda x_\alpha\, A_\beta)B_\alpha) \tag{31}$$

for each assignment $p' = p \backslash x_\alpha$, where $p'(x_\alpha) = \mathcal{V}(B_\alpha)$.

(iv) Interpretation of the description operator is the function

$$\mathcal{M}_p^T(\iota_{\alpha(o\alpha)}) : \mathcal{V}(A_{o\alpha}) \mapsto \mathcal{V}(\iota_{\alpha(o\alpha)}A_{o\alpha}), \qquad \alpha = o, \epsilon. \tag{32}$$

Let us show weak extensionality of (31) and (32). Let

$$\mathcal{M}_p^T(B_\alpha \equiv B'_\alpha) = \mathbf{1}_T.$$

Then $T \vdash B_\alpha \equiv B'_\alpha$ and so, $T \vdash A_{\beta,x_\alpha}[B_\alpha] \equiv A_{\beta,x_\alpha}[B'_\alpha]$ by rule (R), i.e. $[\mathcal{V}(A_{\beta,x_\alpha}[B_\alpha]) \overset{\circ}{=}_\beta \mathcal{V}(A_{\beta,x_\alpha}[B'_\alpha])] = \mathbf{1}_T$. Analogously, we obtain the weak extensionality of (32) from Theorem 8(b).

It also follows from Theorem 8(d) and (32) that, if interpretation $\mathcal{M}_p^T(A_{o\alpha})$ of the formula $A_{o\alpha}$ is a normal fuzzy set then the membership degree

$$\mathcal{M}_p^T(A_{o\alpha}(\iota_{\alpha(o\alpha)}A_{o\alpha})) = \mathbf{1}_T.$$

Otherwise the membership degree of $\mathcal{M}_p^T(\iota_{\alpha(o\alpha)}A_{o\alpha})$ in $\mathcal{M}_p^T(A_{o\alpha})$ is not determined in advance.

From the above construction we conclude that (30) is a general safe *canonical model* of FTT.

5.4 Completeness of FTT

With respect to the above results, we are able to prove the following version of completeness of FTT.

The proofs of the following theorems are close to the proofs of [26, Theorems 23 and 24].

Theorem 14
A theory T is consistent iff it has a safe general model $\mathcal{M}$.

PROOF: If T is inconsistent then $T \vdash \bot$. Thus, if $\mathcal{M} \models T$ then $\mathcal{M}_p(\bot) = \mathbf{1}$, which is impossible.

Conversely: first, we extend T to a linear extensionally complete theory T' and construct its canonical model. It follows from the construction of the canonical model in Subsection ?? that $\mathcal{M}_p^T(A_\alpha) \in M_\alpha$ holds for every formula A_α and for every assignment $p \in \mathrm{Asg}(\mathcal{M}^T)$. Let A_o be an axiom of T. Then $T \vdash A_o$ and so, $T' \vdash A_o$, too. By (FT-tval3) and rule (R) we have $T' \vdash A_o \equiv \top$ (as well as $T \vdash A_o \equiv \top$) and so, by (14) we obtain

$$\mathcal{M}_p^T(A_o) = |\top| = \mathbf{1}_T.$$

This means that $\mathcal{M}^T$ is a model of T.   □

Theorem 15
For every theory T and a formula A_o

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

PROOF: In the same way as in classical logic, the implication left-to-right is the soundness theorem.

Conversely (cf. [15]), we will show that $T \nvdash A_o$ implies that there is a general model $\mathcal{M}$ of T and an assignment p such that $\mathcal{M}_p(A_o) \neq \mathbf{1}$.

Let us consider the canonical model $\mathcal{M}^T$ of T and let $\mathcal{M}_p^T(A_o) = \mathbf{1} = |\top|$ for some assignment p. This means that $T \vdash A_o \equiv \top$, i.e. $T \vdash A_o$. Hence, $T \nvdash A_o$ means that $\mathcal{M}_p^T(A_o) \neq \mathbf{1}_T$.   □

Of course, this theorem is only existential, i.e. it says nothing about the complexity of proving in FTT. It is almost certain that the complexity is, in general, very high. However, the system described in [2] encourages us to suppose that proving could be partially automated even in FTT.

6 Extensions of the basic FTT

6.1 Involutive EQ-algebra-based FTT

A simplification of the basic FTT is obtained when considering its algebra of truth values to be an involutive EQ-algebra. The resulting theory is called IEQ-FTT. Recall that by Theorem 1, each IEQ-algebra is good ℓEQ-algebra. This means, besides others, that axiom (FT-tval3) can be omitted.

The following modified definitions should be introduced:

(i) Disjunction: $\vee := \lambda x_o \lambda y_o \neg(\neg x_o \wedge \neg y_o)$.

(ii) Existential quantifier: $(\exists x_\alpha)A_o := \neg(\forall x_\alpha)\neg A_o$.

The axioms of IEQ-FTT are the following: (FT-fund1)–(FT-fund4), (FT-tval1), (FT-tval2), (FT-tval4a)–(FT-tval11), (FT-delta1)–(FT-delta4), (FT-quant3), and (FT-descr1).

Axiom (FT-tval3) is replaced by

(FT-tval3) $\neg\neg x_o \equiv x_o$

and axiom (FT-quant1) is modified as follows:

(FT-quant1) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$, $\quad x_\alpha$ is not free in A_o.

Some properties of IEQ-FTT are stronger because the contraposition and double negation are provable.

Theorem 16
A theory T of IEQ-FTT is consistent iff it has a safe general model $\mathcal{M}$.

6.2 Residuated lattice-based FTT

Residuated lattice-based FTTs have already been developed for several kinds of residuated lattices. However, they can be obtained also by extending the basic FTT. The basic example is IMTL-FTT studied in detail in [26] which is obtained from IEQ-FTT by adding the following axiom:

(RFT1) $((A_o \,\&\, B_o) \Rightarrow C_o) \equiv (A_o \Rightarrow (B_o \Rightarrow C_o))$

The definitions of basic formulas are the same. Of course, we can also add (RFT1) to the basic FTT. This would lead to MTL-FTT which, though, has yet not been studied.

Two more extensions can be considered. The first, most important is Łukasiewicz FTT (Ł-FTT). It differs from IMTL-FTT by the following definitions:

$$\vee := \lambda x_o(\lambda y_o(x_o \Rightarrow y_o) \Rightarrow y_o), \hspace{3cm} \text{(disjunction)}$$
$$\& := \lambda x_o(\lambda y_o(\neg(x_o \Rightarrow \neg y_o))). \hspace{2.5cm} \text{(strong conjunction)}$$

Its axioms are those of IEQ-FTT, (RFT1), and also

(LFT1) $(A_o \vee B_o) \equiv (B_o \vee A_o)$.

There is also a simpler alternative which uses Rose-Rosser implication axioms for characterization of the structure of truth values.

Theorem 17
A theory T of Ł-FTT is consistent iff it has a general model $\mathcal{M}$.

Finally, we can also consider extension to BL-FTT. Recall that BL stands for *basic fuzzy logic* developed by P. Hájek in [15]. Axioms of BL-FTT are those of the basic FTT, (RFT1) and also the following:

(BL-FT1) $(A_o \wedge B_o) \equiv A \,\&\, (A_o \Rightarrow B_o)$

Theorem 18
A theory T of BL-FTT is consistent iff it has a safe general model $\mathcal{M}$.

7 Conclusion

In this paper, we considered a new kind of algebra called EQ-algebra that has been proposed as a structure of truth values for fuzzy type theory. The outcome of a new algebra for FTT is twofold: first, the basic connective of a fuzzy equality (equivalence) in FTT is interpreted by a primary algebraic operation. Second, implication is not as closely tied with multiplication as in residuated lattices. This fact places both concepts in a different light and enables the development of the non-commutative FTT with only one implication. We are convinced that this lone implication is more natural than two implications that are necessary if a non-commutative residuated lattice is considered (see [16]). Note that non-commutativity of the conjunction in natural language[†] is common and thus, the formal logic must be able to model it (for example, if it looks to be a model of commonsense reasoning).

[†] This phenomenon is in linguistics called "coordination".

We have developed the formal calculus of the basic FTT. Many of its formal properties were described and the completeness theorem was proved. Our reason for calling our FTT "basic" is that it can be extended to stronger theories. First, if we introduce the double negation axiom, then we obtain an IEQ-FTT that is more compact than the basic FTT. Furthermore, introducing axiom (RFT1), results in residuated-lattice-based fuzzy type theories, namely MTL-FTT and IMTL-FTT, whose underlying structure of truth values is isomorphic with an MTL or IMTL algebra, respectively. By introducing further axioms, we obtain Ł-FTT (Łukasiewicz) or BL-FTT (based on BL-algebra of truth values). While the latter is an interesting generalization of BL-fuzzy logic of P. Hájek, the former is a fundamental higher order logic which is sufficiently powerful to be applied in modelling of various phenomena in natural language semantics, in modelling the vagueness phenomenon, or in modelling of the commonsense reasoning. The role of IEQ-FTT in this respect has not yet been investigated. One of the conclusions following from our theory is that all the core fuzzy logics in the sense of [17] can be extended to a corresponding fuzzy type theory.

Further investigation can be anticipated in two directions. First, to investigate our basic FTT as the basic formal calculus for all stronger fuzzy type theories. The second direction, initiated in classical type theory by W. Farmer (see [11]), is to modify the syntax so that the general model can also contain partial functions. This direction of research is motivated especially by the desire to model the meaning of concepts (see [20, 21]). One of the additional tasks is also the study of the complexity of sets of tautologies of all FTT's. Finally, let us remark that our theory suggests philosophical questions concerning the role of (fuzzy) equality/equivalence in (fuzzy) logic with respect to implication.

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