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The Bandler-Kohout Subproduct as a Suitable Inference Mechanism

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Abstract— *The compositional rule of inference (CRI) introduced by Zadeh is widely used in approximate reasoning schemes using fuzzy sets. In this work we show that the Bandler-Kohout subproduct does possess all the important properties such as equivalent and reasonable conditions for their solvability, their interpolative properties and the preservation of the indistinguishability that may be inherent in the input fuzzy sets, as cited in favor of using CRI.*

Keywords— Bandler-Kohout subproduct, compositional rule of inference, correctness and continuity of inference, fuzzy relation equations.

1   Introduction

Fuzzy systems are one of the best known applications of fuzzy logic. These are usually based on a set of fuzzy rules. Systems using fuzzy rules have been applied in a wide variety of applications, viz., automatic control, decision making, risk analysis, etc.

If X is a classical set, we denote the set of all fuzzy sets on X by $\mathcal{F}(X)$. Given two non-empty classical sets X, Y, a fuzzy rule is usually given in the following form:

$$\textbf{IF } x \text{ is } \mathcal{A} \textbf{ THEN } y \text{ is } \mathcal{B} , \qquad (1)$$

where $\mathcal{A}, \mathcal{B}$ are membership predicates represented by fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, respectively. Given a fuzzy observation $A' \in \mathcal{F}(X)$ a corresponding output fuzzy set $B' \in \mathcal{F}(Y)$ is deduced using an inference mechanism.

1.1   Fuzzy Relational Inference

Fuzzy inference systems can be broadly classified as those inference mechanisms that are fuzzy relation based, i.e., those that interpret a given fuzzy rule base as a fuzzy relation, and those that are not. Similarity Based Reasoning (SBR) (see, for instance, TÜRKSEN [1]) and Inverse Truth-functional Modification of BALDWIN [2] are two of the representative examples of the latter type of inference mechanisms that do not use fuzzy relations and which are well established in the literature.

In this work, we focus only on those inference mechanisms that are fuzzy relation based. However, it should be mentioned that, under certain conditions, an equivalent fuzzy relation based description of some of these inference mechanisms can be given (see [3, 4]).

1.2   Compositional Rule of Inference

The earliest inference scheme, proposed by Zadeh himself and well established in the literature, is the Compositional Rule of Inference (CRI) [5]. In CRI, a fuzzy rule is interpreted by a fuzzy relation $R \in \mathcal{F}(X \times Y)$. Given a fuzzy observation $A' \in \mathcal{F}(X)$ an output fuzzy set $B' \in \mathcal{F}(Y)$ is deduced by the CRI based on the fuzzy relation R. From the fuzzy relational point of view the CRI is defined as follows:

$$B'(y) = \bigvee_{x \in X} (A'(x) * R(x, y)), \quad y \in Y , \qquad (2)$$

where $*$ is a fuzzy conjunction, typically a t-norm (see KLEMENT et al. [6] for more details). We use the following notation to indicate the CRI scheme:

$$B' = A' \circ R . \qquad (3)$$

1.3   Popularity of CRI

The CRI mechanism is most often and widely used for the following reasons:

(i) An important issue in the applicability of CRI scheme is to determine an appropriate fuzzy relation R based on the given fuzzy rules so as to obtain meaningful conclusions. This pertains to the solvability of the fuzzy relation equations. A necessary and sufficient condition for the solvability of fuzzy relation equations has been well established for long, see DI NOLA et al [7]. Moreover, recently, PERFILIEVA and LEHMKE [8] showed its solvability under conditions that are more favourable and prevalent in many applicational contexts.

(ii) In PERFILIEVA et al. [8, 9] the authors have dealt with the continuity of a fuzzy function adjoint to the CRI mechanism and a fuzzy relation interpreting fuzzy rules (8). The authors, defining continuity suitably, have shown that it is equivalent to the correctness of the model under consideration.

(iii) KLAWONN and CASTRO [10] have proven two important and interesting results about the CRI scheme using fuzzy rules and the indistinguishability inherent to the fuzzy sets considered. They showed the robustness of the CRI mechanism in scenarios where there can be slight discrepancies between the intended input and the actual input. They have also shown that the indistinguishability induced by the fuzzy set representing the linguistic expression in the premise of the rule cannot be overcome.

However, it should be noted that there are still some drawbacks that are usually cited about CRI. Recently, JAYARAM

[11] has proposed a modified form of CRI called the Hierarchical CRI that addresses many of these drawbacks. The above properties will be dealt with in more detail in Sec. 3.

1.4 Motivation for this work: Bandler-Kohout Subproduct

Generally, the CRI - represented by ∘ - may be replaced by any appropriate image of a fuzzy set under the fuzzy relation R denoted by @, which fulfills some required properties. Besides the most often used CRI inference scheme, another scheme of inference is the Bandler-Kohout subproduct ◁ (BK-Subproduct, for short) proposed by BANDLER and KOHOUT [12]. Considering the fuzzy rule as given in (1), for a given input $A' \in \mathcal{F}(X)$ the inference obtained using the BK-Subproduct is given as follows:

$$B'(y) = \bigwedge_{x \in X} (A'(x) \to R(x, y)), \quad y \in Y, \qquad (4)$$

where $\to$ is a fuzzy implication (see Sec. 2 for more details), $B' \in \mathcal{F}(Y)$ and R is the fuzzy relation that interprets the fuzzy rule (1).

Later on, PEDRYCZ [13] has shown certain solvability conditions for the BK-Subproduct inference as given in (4). Buoyed by this important result on the BK-Subproduct, this paper is an attempt at studying the suitability of the BK-Subproduct as an inference mechanism, by showing that the BK-Subproduct does have all the above desirable properties possessed by CRI.

1.5 Organisation of the work

In Section 2 we recall the preliminaries required for the rest of the paper. Also the structure and inference in the Compositional Rule of Inference (CRI) mechanism is discussed. Following this, in Section 3 we discuss some of the important properties cited in favor of using CRI,e.g., equivalent and reasonable conditions for their solvability, their interpolative properties and the preservation of the indistinguishability that may be inherent in the input fuzzy sets. Section 4, which contains our main work, shows that all the above mentioned properties are valid even in the case of the BK-Subproduct.

2 Preliminaries

2.1 Fuzzy Inference Mechanisms

An inference mechanism acting upon fuzzy rules is usually based on fuzzy logic connectives. In this work, we restrict ourselves to inference schemes based on fuzzy logic connectives. Let us fix a complete residuated lattice $\mathcal{L} = ([0, 1], \wedge, \vee, *, \to, 0, 1)$ (see, for example, NOVÁK et al. [14]) as the basic algebraic structure for the whole paper.

Since the set $[0, 1]$ is totally ordered, $\mathcal{L}$ becomes an MTL-algebra and hence, in our context, $*$ becomes a left-continuous t-norm and $\to$ is the residual implication *uniquely* given as follows:

$$a \to b = \bigvee \{t \in \mathcal{L} | a * t \leq b\}, \qquad (5)$$

which is also a fuzzy implication, for more details we refer the readers to [6, 14, 15].

From the two operations $\wedge, \to$ of $\mathcal{L}$ we can derive yet another operation known as the *biresiduum* which is defined as follows:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a), \qquad a, b \in \mathcal{L}. \qquad (6)$$

Finally, by an extension of an algebraic operation on $\mathcal{L}$ to an operation between fuzzy sets we mean the following:

$$(C \star D)(u, v) = C(u) \star D(v), \qquad u \in U, v \in V, \qquad (7)$$

where $\star \in \{\wedge, \vee, *, \to, \leftrightarrow\}$ and where C, D are arbitrary fuzzy sets on arbitrary universes U, V, respectively.

2.2 Structure and Inference in CRI

The CRI inference mechanism given in (2) is for a fuzzy relation $R \in \mathcal{F}(X \times Y)$ that interprets a single fuzzy rule. However, usually multiple fuzzy rules are a necessity. Let us denote the multiple fuzzy rules as follows:

$$\textbf{IF } x \text{ is } \mathcal{A}_i \qquad \textbf{THEN} \qquad y \text{ is } \mathcal{B}_i, \qquad (8)$$

where $\mathcal{A}_i, \mathcal{B}_i$ are membership predicates represented by fuzzy sets $A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(Y)$, respectively, for $i = 1, \ldots, n$.

Then a fuzzy relation interpreting the whole *fuzzy rule base* (altern. *linguistic description*), composed of the n fuzzy rules (8), has to be constructed. basically, there are two main approaches to the construction of such a fuzzy relation.

The following fuzzy relation $\check{R} \in \mathcal{F}(X \times Y)$

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)), \qquad (9)$$

is used in most of the real world applications. This is mainly due to the successful applications of this, say Cartesian product approach, published by Mamdani and Assilian in [16], which was followed by a huge number of researchers and practitioners, see e.g. [17, 18].

Alternatively, to keep the conditional IF-THEN form of the fuzzy rules (8) fuzzy relation $\hat{R} \in \mathcal{F}(X \times Y)$ given as follows

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)), \qquad (10)$$

can be chosen to interpret the fuzzy rule base. It deals with a mathematically correct extension of a classical implication.

To stress the difference between both the approaches, let us recall the work of DUBOIS et al. [19], where the authors state: "*In the view given by* (10)*, each piece of information (fuzzy rule) is viewed as a constraint. This view naturally leads to a conjunctive way of merging the individual pieces of information since the more information, the more constraints and the less possible values to satisfy them.*" While the same authors describe the second approach proposed by Mamdani and Assilian as follows: "*It seems that fuzzy rules modelled by* $\check{R}$ *are not viewed as constraints but are considered as pieces of data. Then the maximum in* (9) *expresses accumulation of data*".

In other words, instead of interpreting fuzzy rules as logical implications, the approach given using $\check{R}$ builds up an input-output relation from smaller units, and those units are examples of fuzzy input-output pairs. And then Mamdani-Assilian approach is a method that interpolates between known input-output pairs [20]. For an extensive study of different fuzzy rules we refer to [21, 22].

It should be stressed that both approaches have sound logical foundations but from different viewpoints, see e.g. [23, 24, 14]. However, only the approach using $\check{R}$ was widely used in applications although the implicational approach using $\hat{R}$ is probably as useful as the Mamdani-Assilian one, see [25]. Nevertheless, as we show in Section 4, the implicational approach using $\hat{R}$ does have an important role to play in the case of BK-Subproducts (see Theorem 4.7).

3 Some desirable properties of CRI

Fuzzy rules may be viewed as a partial mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ assigning $B_i \in \mathcal{F}(Y)$ to $A_i \in \mathcal{F}(X)$ for every $i = 1, \ldots, n$. The inference process then may be viewed as an extension of this partial mapping to a total one [26]. For better understanding, let us adopt the notation from [8] and consider the following structure

$$S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @) ,$$

where $@ : \mathcal{F}(X) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$ is an image of a fuzzy set under a fuzzy relation. For instance, $@$ could be one of $\circ$ or $\triangleleft$. Now, by the choice of the fuzzy relation R interpreting the fuzzy rule base and by the choice of $@$, we define a fuzzy function $f_R(A) : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $f_R(A) = A @ R$, for arbitrary $A \in \mathcal{F}(X)$.

3.1 Solvability of systems of fuzzy relation equations with the $\circ$ image

One of the fundamental properties of such a mapping is its interpolativity, i.e., $f_R(A_i) = B_i$. In this case, we say that R is a *correct model* of given fuzzy rules in the given structure S [8].

This leads us to deal with a system of fuzzy relation equations [27] where generally the system of equations

$$A_i @ R = B_i, \quad i = 1, \ldots, n , \tag{11}$$

is solved with respect to the known $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$ and unknown $R \in \mathcal{F}(X \times Y)$. If R is a solution to (11) then the adjoint fuzzy function fulfills $f_R(A_i) = B_i$.

In the case of CRI, the above system of equations, viz., (11), reduces to the following:

$$A_i \circ R = B_i, \quad i = 1, \ldots, n . \tag{12}$$

Let us recall some main results which may be found, e.g., in [7, 27, 28, 29].

Theorem 3.1 *System (12) is solvable if and only if $\hat{R}$ is a solution of the system and moreover, $\hat{R}$ is the greatest solution of (12).*

On the one hand, Theorem 3.1 states the necessary and sufficient condition of the solvability of system (12) and it determines the solution. Moreover, it ensures that the given solution is the greatest one. On the other hand, we still do not know, when $\check{R}$ is the solution, i.e., how to ensure the solvability.

Theorem 3.2 [29] *Let A_i for $i = 1, \ldots, n$ be normal. Then $\check{R}$ is a solution of (12) if and only if the following condition*

$$\bigvee_{x \in X} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in Y} (B_i(y) \leftrightarrow B_j(y)) , \tag{13}$$

holds for arbitrary $i, j \in \{1, \ldots, n\}$.

Theorem 3.2 specifies a sufficient condition under which the system is solvable and moreover, it ensures that not only $\hat{R}$ but also even $\check{R}$ is a solution of system (12).

It is worth mentioning that condition (13) appearing in Theorem 3.2 is from the practical point of view not very convenient. However, if the antecedent fuzzy sets form the so called $*$-semi-partition it forces fulfilling the discussed condition in advance, see DE BAETS and MESIAR [30] for more details.

Another sufficient condition for solvability of the systems with a high practical importance was published in [31].

Theorem 3.3 (Theorem 7 in [31]) *Let A_i for $i = 1, \ldots, n$ be normal and fulfill the Ruspini condition*

$$\sum_{i=1}^n A_i(x) = 1, \quad x \in X. \tag{14}$$

Then the system (12) is solvable.

3.2 Continuity of a model of fuzzy rules

In [8, 9] the authors have dealt with the continuity of a fuzzy function adjoint to the CRI mechanism and a fuzzy relation interpreting fuzzy rules (8). They have defined continuity suitably and have shown that it is equivalent to the correctness of the model under consideration.

Although the original definition in [8] of a continuous model was given for the particular inference mechanism CRI, i.e., for $@ \equiv \circ$, the particular image plays absolutely no role in the proof of Theorem 3.6 explaining the nature of the definiton and hence can be generalized for an arbitrary image of a fuzzy set under a fuzzy relation.

Definition 3.4 A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is said to be a *continuous model of fuzzy rules* (8) *in a structure* $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$ if for each $i \in I$ and for each $A \in \mathcal{F}(X)$ the following inequality holds:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A @ R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) . \tag{15}$$

Inequality (15) can be rewritten in terms of the adjoint fuzzy function f_R as follows:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (f_R(A))(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \tag{16}$$

Remark 3.5 Let us explain why formula (15) expresses the continuity. The closeness between fuzzy sets is measured by the biresiduation operation $\leftrightarrow$, i.e. it is a dual information to the metric one. Let us consider a continuous Archimedean t-norm $*$ with an additive generator $g : [0, 1] \to [0, +\infty]$. Then the biresiduum may be written in the form

$$a \leftrightarrow b = g^{-1}(|g(a) - g(b)|) \tag{17}$$

where $g^{-1} : [0, \infty] \to [0, 1]$ is the inverse function and where in the case of $g(0) = \infty$ we define $g(0) - g(0) = 0$. Now, for an arbitrary non-empty universe X it is possible to define a metric D_g on $\mathcal{F}(X)$ generated by g as follows:

$$D_g(A, B) = \bigvee_{x \in X} |g(A(x)) - g(B(x))|. \tag{18}$$

The following theorem justifies the use of the notion of continuity in Definition 3.4. For more details, see PERFILIEVA and LEHMKE [8].

Theorem 3.6 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$ be a structure for fuzzy rules (8) such that $\mathcal{L}$ be a residuated lattice on $[0, 1]$ with a continuous Archimedean t-norm $*$ having a continuous additive generator g. A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a continuous model of the fuzzy rules in the given structure S if and only if*

$$D_g(B_i, (A@R)) \leq D_g(A_i, A), \quad i = 1, \ldots, n \qquad (19)$$

for each fuzzy set $A \in \mathcal{F}(X)$.

The following lemma was crucial for further results published in [8].

Lemma 3.7 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ be a structure for fuzzy rules (8) and let $R \in \mathcal{F}(X \times Y)$. Then for any $A \in \mathcal{F}(X)$ and all $i = 1, \ldots, n$ and $y \in Y$ it is true that*

$$B_i(y) \leftrightarrow (A \circ R)(y) \geq \delta_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)), \quad (20)$$

where $\delta_{R,i}(y) = B_i(y) \leftrightarrow (A_i \circ R)(y)$.

Finally, we recall here the main result in PERFILIEVA et al. [8, 9] concerning the relationship of the above mentioned continuity and the interpolativity for the CRI.

Theorem 3.8 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ be a structure for fuzzy rules (8). A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of fuzzy rules (8) in the given structure S if and only if it is a continuous model of these rules in S.*

3.3 CRI and the Indistinguishability of Premises

Let X be a classical set and let $\sim$ be an equivalence relation defined on X, i.e., $\sim$ is reflexive, symmetric and transitive. Immediately, $\sim$ partitions X into equivalence classes. It is well-known then that an $M \subseteq X$ belongs to this partition if, and only if, whenever $x \in M$ and $x \sim y$ for some $y \in X$ then $y \in M$. In a sense, the elements of M are indistinguishable and can be represented mathematically as follows:

$$x \in M \text{ and } x \sim y \text{ implies } y \in M.$$

A similar relation between fuzzy equivalence relations and fuzzy sets on X was introduced by KLAWONN and CASTRO [10]. Once again the operation $*$ comes from the residuated lattice $\mathcal{L}$.

Definition 3.9 A fuzzy subset E of the Cartesian product X^2 is called a *fuzzy equivalence* relation on X if the following properties are satisfied for all $x, y, z \in X$:

(Reflexivity)	$E(x, x) = 1,$	(ER)
(Symmetry)	$E(x, y) = E(y, x),$	(ES)
(Transitivity)	$E(x, z) \geq E(x, y) * E(y, z).$	(ET)

Definition 3.10 A fuzzy set $\mu \in \mathcal{F}(X)$ is called *extensional* with respect to a fuzzy equivalence relation E on X if

$$\mu(x) * E(x, y) \leq \mu(y), \qquad x, y \in X. \qquad (21)$$

If a fuzzy set μ is not extensional with respect to the considered fuzzy equivalence relation E, instead one considers the smallest fuzzy set that is extensional with respect to E and contains μ.

Definition 3.11 Let $\mu \in \mathcal{F}(X)$ and let E be a fuzzy equivalence relation on X. The fuzzy set

$$\widetilde{\mu}(x) = \bigwedge \{\nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional with respect to } E\},$$
$$(22)$$

is called the *extensional hull* of μ. Note that by $\mu \leq \nu$ we mean that for all $x \in X$, $\mu(x) \leq \nu(x)$, i.e., we mean ordering in the sense of inclusion, not in the sense of ordering fuzzy quantities.

Proposition 3.12 [10] *Let $\mu \in \mathcal{F}(X)$ and let E be a fuzzy equivalence relation on X.*

(i) $\widetilde{\mu}(x) = \bigvee \{\mu(y) * E(x, y) \mid y \in X\},$

(ii) $\widetilde{\mu}$ *is extensional with respect to E,*

(iii) $\widetilde{\widetilde{\mu}} = \widetilde{\mu}.$

KLAWONN and CASTRO [10] have proven the following two important and interesting results about the CRI scheme using fuzzy rules and the indistinguishability inherent to the fuzzy sets considered.

Theorem 3.13 *Let $S = (X, Y, \{A, B\}, \mathcal{L}, \circ)$ be a structure for fuzzy rule (1). Let E be a fuzzy equivalence relation on X with respect to which A is extensional. Let $A' \in \mathcal{F}(X)$ be any fuzzy set, then*

$$A' \circ \hat{R} = \widetilde{A'} \circ \hat{R},$$
$$A' \circ \check{R} = \widetilde{A'} \circ \check{R}.$$

The following interpretation of the above result is given in [10]: *The output obtained from CRI for a given fuzzy rule and an input fuzzy set A' does not change if we substitute A' by its extensional hull $\widetilde{A'}$. The indistinguishability inherent in the fuzzy set A cannot be avoided even if the input fuzzy set A' stands for a crisp value. Further, a fuzzified input does not change the outcome of a rule as long as the fuzzy set obtained by the fuzzification is contained in the extensional hull of the original crisp input value.* They finally conclude that *it does not make sense to measure more exactly than the indistinguishability admits.*

In other words, this shows the robustness of the inference in scenarios where there can be slight discrepancies between the intended input and the actual input.

It is immediate now, as already observed in [10], that *the indistinguishability induced by the fuzzy set representing the linguistic expression in the premise of the rule cannot be overcome.*

Let us stress that the proof of Theorem 3.13 may be easily modified in order to capture the situation with n fuzzy rules. It can be observed that only the Mamdani-Assilian approach $\check{R}$ generally works in the combination with the CRI.

Theorem 3.14 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ be a structure for fuzzy rules (8). Let E be a fuzzy equivalence relation on X with respect to which A_i is extensional for arbitrary $i = 1, \ldots, n$. Let $A' \in \mathcal{F}(X)$ be any fuzzy set, then*

$$A' \circ \check{R} = \widetilde{A'} \circ \check{R}.$$

4 BK-Subproduct and some desirable properties

4.1 Solvability of systems of fuzzy relation equations with the $\lhd$ image

In the case of BK-Subproduct, the system of equations (11), reduces to the following:

$$A_i \lhd R = B_i, \quad i = 1, \ldots, n . \tag{23}$$

Concerning system (23), let us recall the following two basic theorems .

Theorem 4.1 [13] *System (23) is solvable if and only if $\check{R}$ is a solution of the system and moreover, $\check{R}$ is the least solution of system (23).*

Theorem 4.2 [32] *Let A_i for $i = 1, \ldots, n$ be normal. Then $\hat{R}$ is a solution of (23) if and only if the condition (13) holds for arbitrary $i, j \in \{1, \ldots, n\}$.*

Again, condition (13) to which Theorem 4.2 refers to, is not very convenient from a practical point of view and may be a priori fulfilled by using antecedent fuzzy sets which form the $*$-semi-partition.

Fortunately, that the sufficient condition for solvability of the systems with a high practical importance stated in Theorem 3.3 is valid even for system (23).

Theorem 4.3 (Theorem 7 in [31]) *Let A_i for $i = 1, \ldots, n$ be normal and fulfill the Ruspini condition (14). Then the system (23) is solvable.*

4.2 Continuity of BK-Subproduct

The BK-Subproduct, unlike the CRI, was not motivated by approximate reasoning, see BANDLER and KOHOUT [12]. However, as mentioned in GOTTWALD [33], it is not necessary to insist on purely logical foundations for an inference mechanism, and it can simply be a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ fulfilling certain properties. Let us also recall that it was PEDRYCZ [13] who firstly proposed the BK-Subproduct as an inference scheme.

We may again state an analogous lemma to Lemma 3.7 also for the case of the BK-Subproduct.

Lemma 4.4 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \lhd)$ be a structure for fuzzy rules (8) and let $R \in \mathcal{F}(X \times Y)$. Then for any $A \in \mathcal{F}(X)$ and all $i = 1, \ldots, n$ and $y \in Y$ it is true that*

$$B_i(y) \leftrightarrow (A \lhd R)(y) \geq \delta_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) , \tag{24}$$

where $\delta_{R,i}(y) = B_i(y) \leftrightarrow (A_i \lhd R)(y)$.

Due to Lemma 4.4, we may state the following analogous theorem to Theorem 3.8 which again shows that the BK subproduct as an inference mechanism carries the same property as the CRI.

Theorem 4.5 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \lhd)$ be a structure for fuzzy rules (8). A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of fuzzy rules (8) in the given structure S if and only if it is a continuous model of these rules in S.*

4.3 BK-Subproduct and the Indistinguishability of Premises

In this subsection, we show the robustness of the BK-Subproduct inference mechanism along similar lines as KLAWONN and CASTRO [10]. Once again the employed operations come from the residuated lattice $\mathcal{L}$. Firstly note that if a fuzzy set $\mu \in \mathcal{F}(X)$ is extensional with respect to a fuzzy equivalence relation E on X then

$$E(x, y) \to \mu(y) \geq \mu(x), \ x, y \in X . \tag{25}$$

Proposition 4.6 *Let $\mu \in \mathcal{F}(X)$ and E a fuzzy equivalence relation on X. Then*

$$\widetilde{\mu}(x) = \bigwedge \{E(x, y) \to \mu(y) \mid y \in X\} . \tag{26}$$

Now, we present the main result analogous to Theorem 3.13.

Theorem 4.7 *Let $S = (X, Y, \{A, B\}, \mathcal{L}, \lhd)$ be a structure for fuzzy rule (1). Let E be a fuzzy equivalence relation on X with respect to which A is extensional. Let $A' \in \mathcal{F}(X)$ be any fuzzy set, then*

$$A' \lhd \hat{R} = \widetilde{A'} \lhd \hat{R},$$
$$A' \lhd \check{R} = \widetilde{A'} \lhd \check{R}.$$

The above result, as already noted in the case of CRI, shows the robustness of the BK-Subproduct inference in scenarios where there can be slight discrepancies between the intended input and the actual input and reinforces the fact that even in the case of BK-Subproduct the indistinguishability induced by the fuzzy set representing the linguistic expression in the premise of the rule cannot be overcome.

And again as in case of the CRI, we may generalize the result concerning the indistinguishability of the premises for an arbitrary finite number of rules. Note that in the case of the BK-Subproduct the $\hat{R}$ is plays the main role.

Theorem 4.8 *Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \lhd)$ be a structure for fuzzy rules (8). Let E be a fuzzy equivalence relation on X with respect to which each A_i is extensional, for arbitrary $i = 1, \ldots, n$. Let $A' \in \mathcal{F}(X)$ be any fuzzy set, then*

$$A' \lhd \hat{R} = \widetilde{A'} \lhd \hat{R}.$$

5 Conclusions

In this work, after recalling some of the properties that are usually cited in favor of using the Compositional Rule of Inference (CRI) introduced by ZADEH [5], viz., equivalent and reasonable conditions for their solvability, their interpolative properties and the preservation of the indistinguishability that may be inherent in the input fuzzy sets, we have shown that the Bandler-Kohout subproduct introduced in [12] does possess all the above properties and hence is equally suitable for consideration when reasoning with a system of fuzzy rules. Towards this end some new but equivalent results on indistinguishability operations has also been presented.

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