University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

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Michal Holčapek

Research report No. 131

2007

Submitted/to appear:

Fuzzy Sets and Systems

Supported by:

Project MSM6198898701 of MŠMT ČR and the Grant of GA ČR 201/04/1033.

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478
e-mail: michal.holcapek@osu.cz

An axiomatic approach to cardinalities of finite L-fuzzy sets ⋆

Michal Holčapek

Institute for Research and Applications of Fuzzy Modeling
University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic
michal.holcapek@seznam.cz

Abstract

The aim of this paper is to propose two axiomatic systems for the cardinalities of finite **L**-fuzzy sets which are, in some sense, dual to each other. The first axiomatic system is a generalization of the system introduced by Casasnovas and Torrens. The other is defined as a dual to the first one and enables to define further cardinalities that also include some examples of scalar cardinalities. Further, cardinalities of both axiomatic systems are represented by two adequate homomorphisms. Finally, selected properties of cardinalities of finite **L**-fuzzy sets are investigated.

Key words: cardinality of fuzzy sets, convex fuzzy sets, generalized natural numbers, equipotence of fuzzy sets

1 Introduction

In fuzzy mathematics, the cardinality of a fuzzy set is a measure of the "number of elements belonging to the fuzzy set". Analogously to the cardinal theory of sets, there are two approaches to cardinality of fuzzy sets - one which is based on the relation between fuzzy sets of being equipotent (equipollent, bijective), and another which uses ordinary cardinal or ordinal numbers, real numbers or some fuzzy generalization of cardinal or ordinal numbers. The first approach to cardinality of fuzzy sets was investigated in e.g. [10, 11, 15]. Since the results based on the comparison of two fuzzy sets are rather theoretical than practical, the main attention in fuzzy cardinal theory is focused on the second approach. According to suitable objects describing the size of fuzzy

⋆ The paper has been partially supported by the Institutional Research Plan MSM 6198898701 and the Grant of GA ČR 201/04/1033.

sets, we can distinguish two directions - one includes *scalar* and another *fuzzy* approaches.

In the scalar approaches the cardinalities of fuzzy sets are defined as a mapping that to each (mainly finite) fuzzy set assigns a single ordinary cardinal number or a non-negative real number. Note that a finite fuzzy set is understood as the fuzzy set with a finite support. A basic definition of the scalar cardinality was proposed by A. De Luca and S. Termini in [5]. In this simple case, the scalar cardinality of a finite fuzzy set $A : X \to [0,1]$ is defined as the sum of membership degrees of finite fuzzy set A, i.e. $|A| = \sum_{x\in X} A(x)$. Other definitions of scalar cardinalities as well as their properties could be found in e.g. [8,9,11,14,21,24,32]. An axiomatic approach to the scalar cardinalities of finite fuzzy sets was proposed by M. Wygralak in [28] and some relationships between fuzzy mappings and scalar cardinalities of fuzzy sets were stated in [3,13,22].

In the fuzzy approaches the cardinalities of fuzzy sets are defined as a mapping that assigns to each fuzzy set a suitable (often convex) fuzzy set over the set of natural numbers or universes containing a broader class of cardinal or ordinal numbers. The convex fuzzy sets over the set of natural numbers or over a class of cardinals are usually referred to *generalized natural numbers* or *generalized cardinals* (see e.g. [8,24–27]), respectively. Although the study of cardinalities of infinite fuzzy sets seems to be very interesting, the cardinalities of finite fuzzy sets play the central role in the research. The first definition of cardinality of finite fuzzy sets, by means of mappings from the set of natural numbers to the interval $[0,1]$, was proposed by L.A. Zadeh in [33]. In order to model the truth value of statements with natural language quantifiers as e.g. "for nearly all students visited the lecture", "few single women stayed at home", "about half questions were not answered" L.A. Zadeh in [32] introduced three types of cardinality of finite fuzzy sets, namely *FGCount*, *FLCount* and *FECount*. If we keep Zadeh's notation, then $\mathit{FGCount}(A)(k) = \bigvee\{a \mid |A_a| \geq k\}$ expresses a degree to which A contains *at least* k elements. The dual variant $\mathit{FLCount}(A)(k) = 1 - \mathit{FGCount}(A)(k+1)$ determines a degree to which A has *at most* k elements. A degree to which A has *exactly* k elements is then expressed by $\mathit{FECount}(A)(k) = \mathit{FGCount}(A)(k) \wedge \mathit{FLCount}(A)(k)$. Note that cardinalities *FGCount*, *FLCount* and *FECount* are defined using general natural numbers contrary to Zadeh's first definition of cardinality, where the non-convex fuzzy sets are also supposed. Other approaches to the definition of fuzzy cardinality for finite fuzzy sets could be found in e.g. [7,8,21,23,29,30]. An axiomatic approach to cardinalities of finite fuzzy sets was proposed by J. Casasnovas and J. Torrens in [2]. The proposed system of axioms enables us to define an infinite class of cardinalities of finite fuzzy sets that contains a lot of the above referred cardinalities. An extension of *FGCount* for fuzzy sets with the membership degrees in a totally ordered lattice was proposed by P. Lubczonok in [18]. An approach to the non-convex cardinality of fuzzy sets could be found in [6].

As we have mentioned in the previous part, the cardinalities of fuzzy sets are closely connected with generalized quantifiers in fuzzy logic, where the residuated lattices and its special cases as e.g. BL-algebras, MV-algebras, IMTL-algebras etc. form a general framework for the interpretation of truth values in fuzzy logic (see e.g. [1, 19]). Hence, it seems to be very useful to introduce the cardinality theory for finite fuzzy sets which membership degrees are interpreted in residuated lattices. Nevertheless, only residuated lattices are not sufficient to establish our axiomatic systems for fuzzy sets in a reasonable way. In particular, we need other operations which are, in some sense, dual to the operations of the residuated lattices. Therefore, we extend the residuated lattices by two other operations, precisely by addition and difference, and define the so-called residuated-dually residuated lattices denoted by $\mathbf{L}$. The main goal of this paper is to introduce two systems of reasonable axioms which determine a broad family of cardinalities of finite $\mathbf{L}$-fuzzy sets covering the most of well-known examples of fuzzy cardinalities and some examples of scalar cardinalities. The first system generalizes the axiomatic system proposed by Casasnovas and Torrens in [2] and enables us to define, for instance, some cardinalities based on the triangular norms suggested by Wygralak in [29]. The other system is introduced as a dual to the first one. A motivation to introduce such dual system is to propose, among others, a new view on scalar cardinalities (cf. [28]). Analogously to the representations of scalar cardinalities and fuzzy cardinalities (see [2, 28]) by adequate mappings we show that each cardinality of finite $\mathbf{L}$-fuzzy sets satisfying some axiomatic system can be represented by two suitable homomorphisms between reducts of residuated-dually residuated lattice $\mathbf{L}$. Finally, we prove some selected properties of cardinalities of $\mathbf{L}$-fuzzy sets which are usually investigated with regard to the cardinalities of fuzzy sets.

2 Preliminaries

2.1 Algebraic structures of membership degrees of fuzzy sets

In this paper, we will interpret the membership degrees of fuzzy set in a complete residuated lattice which is, moreover, extended by an adjoint couple of operations. We say that an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, \bot, \top \rangle$ with six binary operations and two constants is a *residuated-dually residuated lattice* (*rdr*-lattice for short), if

(i) $\langle L, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, respectively,

(ii) $\langle L, \otimes, \top \rangle$ and $\langle L, \oplus, \bot \rangle$ are commutative monoids,

(iii) it satisfies the adjointness property and dual adjointness property, i.e.

$$a \le b \to c \quad \text{if and only if} \quad a \otimes b \le c, \tag{1}$$
$$a \le b \oplus c \quad \text{if and only if} \quad a \ominus b \le c \tag{2}$$

hold for each $a, b, c \in L$ ($\le$ denotes the corresponding lattice ordering).

The operations $\otimes$, $\to$, $\oplus$ and $\ominus$ are called *multiplication*, *residuum*, *addition* and *difference*, respectively. We say that an *rdr*-lattice is *complete* (*linearly ordered*), if $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete (linearly ordered) lattice, respectively. Further, an *rdr*-lattice is *divisible*, if $a \otimes (a \to b) = a \wedge b$ holds for all $a, b, c \in L$, and *dually divisible*, if $(a \ominus b) \oplus b = a \vee b$ holds for all $a, b, c \in L$.

Remark 1 *Obviously, if the dually adjoint couple $\langle \oplus, \ominus \rangle$ is forgotten in an* rdr*-lattice* $\mathbf{L}$*, then we obtain a reduct of* $\mathbf{L}$ *called the* residuated lattice *that will be further denoted by* $\mathbf{L}^{\mathrm{r}}$*. For more information about the residuated lattices we refer to e.g. [1]. If the adjoint couple $\langle \otimes, \to \rangle$ is forgotten, then we obtain a reduct of* $\mathbf{L}$ *called the* dually residuated lattice *that will be denoted* $\mathbf{L}^{\mathrm{rd}}$*. Hence, the* rdr*-lattices are a structure of residuated and dually residuated lattices with the same bounded lattice, where no identities combining operations of residuated and dually residuated lattices are supposed in general.*

It is known that the operations $\wedge$ and $\otimes$ (or $\vee$ and $\oplus$) have a lot of common properties which can be used for various alternative constructions. Therefore, we denote them by the common symbol $\odot$ (or dually $\overline{\odot}$) and we often deal with both operations in general. If we have $a_1 \otimes \cdots \otimes a_n$, then we will also write $\bigotimes_{i=1}^n a_i$ or $\bigotimes_{i \in I} a_i$, where $I = \{1, \ldots, n\}$. Analogously, we will write $\bigoplus_{i=1}^n a_i$, $\bigodot_{i=1}^n a_i$ and $\overline{\bigodot}_{i=1}^n a_i$ or $\bigoplus_{i \in I} a_i$, $\bigodot_{i \in I} a_i$ and $\overline{\bigodot}_{i \in I} a_i$. Moreover, we put $\bigodot_{i \in \emptyset} a_i = \top$ and $\overline{\bigodot}_{i \in \emptyset} a_i = \bot$. Now, let us show several examples of complete *rdr*-lattices.

Example 2 *Let $\langle L, \wedge, \vee, \bot, \top \rangle$ be a complete lattice. Let us put $\otimes = \wedge$, $\oplus = \vee$ and define*

$$a \to b = \bigvee \{x \in L \mid a \wedge x \le b\}$$
$$a \ominus b = \bigwedge \{x \in L \mid b \vee x \ge a\}.$$

Then $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, \oplus, \ominus, \bot, \top \rangle$ is a complete rdr-lattice, where $\mathbf{L}^{\mathrm{r}}$ is a complete Heyting algebra and $\mathbf{L}^{\mathrm{rd}}$ is a complete dual Heyting algebra.

Example 3 *Let T and S be a left continuous t-norm and a right continuous t-conorm, respectively, and define*

$$a \to_T b = \bigvee \{c \in [0,1] \mid T(a,c) \le b\}$$
$$a \ominus_S b = \bigwedge \{c \in [0,1] \mid S(b,c) \ge a\}.$$

Then $\mathbf{L} = \langle [0,1], \min, \max, T, \to_T, S, \ominus_S, 0, 1\rangle$ *is a complete rdr-lattice, where* $\mathbf{L}^{\mathrm{r}}$ *is a complete residuated lattice determined by a left continuous t-norm* T *(see e.g. [1, 12, 16]) and* $\mathbf{L}^{\mathrm{rd}}$ *is a complete dually residuated lattice determined by a right continuous t-conorm* S *(the proof of this could be done by analogy to that of* $\mathbf{L}^{\mathrm{r}}$*).*

Example 4 *Let* $\mathbf{L}^{\mathrm{r}} = \langle L, \wedge, \vee, \otimes, \to, \bot, \top\rangle$ *be an MV-algebra, i.e. a residuated lattice satisfying the prelinearity axiom (i.e.* $(a \to b) \vee (b \to a) = \top$ *holds for all* $a, b \in L$*) and the law of double negation (i.e.* $(a \to \bot) \to \bot = a$ *holds for all* $a \in L$*). Let us put* $\neg a = a \to \bot$ *and define* $a \oplus b = \neg(\neg a \otimes \neg b)$ *and* $a \ominus b = a \otimes \neg b$ *(see e.g. [4, 20]). Then* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, \oplus, \ominus, \bot, \top\rangle$ *is an rdr-lattice. Obviously, if* $\mathbf{L}^{\mathrm{r}}$ *is complete MV-algebra, then also* $\mathbf{L}$ *is a complete rdr-lattice.*

Example 5 *Let* $[0, \infty]$ *be the set of non-negative real numbers extended by the infinity and* $\frac{1}{a}$ *denote the inverse value of* a*, where* $\frac{1}{\infty} = 0$ *and* $\frac{1}{0} = \infty$*. An algebra* $\mathbf{L} = \langle [0,\infty], \wedge, \vee, \otimes, \to, \oplus, \ominus, 0, \infty\rangle$*, where for each* $a, b \in [0, \infty]$ *we have (+ and − are the common addition and difference of real numbers, respectively)*

$$a \oplus b = \begin{cases} a + b, & a, b \in [0, \infty), \\ \infty, & \text{otherwise} \end{cases} \quad \text{and} \quad a \ominus b = \begin{cases} 0 \vee (a - b), & a, b \in [0, \infty), \\ 0, & b = \infty, \\ \infty, & \text{otherwise}, \end{cases}$$

$$a \otimes b = \frac{1}{\frac{1}{a} \oplus \frac{1}{b}} \quad \text{and} \quad a \to b = \frac{1}{\frac{1}{b} \ominus \frac{1}{a}},$$

is a complete rdr-lattice of the non-negative real numbers.

Proposition 6 *Let* $\mathbf{L}$ *be a complete rdr-lattice. Then for arbitrary* $a \in L$ *and* $\{b_i \in L \mid i \in I\}$*, where* I *is an index set, we have*

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i) \quad \text{and} \quad a \oplus \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \oplus b_i). \tag{3}$$

Moreover, if $\mathbf{L}$ *is divisible or dually divisible, then we have*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i) \quad \text{or} \quad a \vee \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \vee b_i), \tag{4}$$

respectively.

PROOF. The proof of (3) for $\otimes$ and (4) for $\wedge$ could be found in [1]. Let $a, b, c \in L$ and $\{b_i \mid i \in I\}$ be an index set. Obviously, $a \oplus b = a \oplus b$ implies $(a \oplus b) \ominus a \le b$ (according to the dual adjointness property). Now, if $b \le c$ then $(a \oplus b) \ominus a \le b \le c$ implies $a \oplus b \le a \oplus c$ (isotonicity of $\oplus$ in both arguments).

Hence, the inequality $a \oplus \bigwedge_{i\in I} b_i \leq \bigwedge_{i\in I}(a \oplus b_i)$ is a straightforward consequence of isotonicity of $\oplus$. On the other hand, obviously $\bigwedge_{i\in I}(a \oplus b_i) \leq a \oplus b_i$ holds for every $i \in I$. By adjointness we have $\bigwedge_{i\in I}(a \oplus b_i) \ominus a \leq b_i$ for every $i \in I$ and thus $\bigwedge_{i\in I}(a \oplus b_i) \ominus a \leq \bigwedge_{i\in I} b_i$. Hence, we have $\bigwedge_{i\in I}(a \oplus b_i) \leq a \oplus \bigwedge_{i\in I} b_i$ and (3) for $\oplus$ is proved. Let us suppose that $\mathbf{L}$ is divisible, i.e. $(a \ominus b) \oplus b = a \vee b$. Obviously, $a \vee \bigwedge_{i\in I} b_i \leq \bigwedge_{i\in I}(a \vee b_i)$. On the other hand, obviously $a \ominus b = a \ominus b$ implies $a \leq (a \ominus b) \oplus b$ (according to the dual adjointness property). Now, if $b \leq c$ then from isotonicity of $\oplus$ we have $a \leq (a \ominus b) \oplus b \leq (a \ominus b) \oplus c$. By adjointness we obtain $a \ominus c \leq a \ominus b$ (antitonicity of $\ominus$ in the second argument). Hence, we can write $\bigwedge_{i\in I}(a \vee b_i) = \bigvee_{i\in I}((a \ominus b_i) \oplus b_i) \leq \bigwedge_{i\in I}((a \ominus \bigwedge_{i\in I} b_i) \oplus b_i = (a \ominus \bigwedge_{i\in I} b_i) \oplus \bigwedge_{i\in I} b_i = a \vee \bigwedge_{i\in I} b_i$ and the proof of (4) for $\vee$ is finished. □

2.2 $\mathbf{L}$*-fuzzy sets*

Let $\mathbf{L}$ be a complete *rdr*-lattice and $\mathbb{X}$ be a non-empty set. An $\mathbf{L}$-*fuzzy set* in $\mathbb{X}$ is a mapping $A : \mathbb{X} \to L$, where L is the support of $\mathbf{L}$. The set $\mathbb{X}$ is called the *universe of discovering* (or *universe* for short) and the set of all $\mathbf{L}$-fuzzy sets will be denoted by $\mathcal{F}_{\mathbf{L}}(\mathbb{X})$. Let A be an $\mathbf{L}$-fuzzy set in $\mathbb{X}$, then the set $A_a = \{x \in \mathbb{X} \mid A(x) \geq a\}$ is called the *a-cut* of A, $A^{\mathrm{d}}_a = \{x \in \mathbb{X} \mid A(x) \leq a\}$ is called the *dual a-cut* of A and $\mathrm{Supp}(A) = \{x \in \mathbb{X} \mid A(x) > 0\}$ is called the *support* of A. An $\mathbf{L}$-fuzzy set A is called *empty* (denoted by $\emptyset$) or *crisp*, if $A(x) = \bot$ or $A(x) \in \{\bot, \top\}$ for each $x \in \mathbb{X}$, respectively. An $\mathbf{L}$-fuzzy set A is called *singleton*, if $A(y) = \bot$ for all $y \in \mathbb{X}$ that are different from some $x \in \mathbb{X}$. If A is a singleton with $A(x) = a$ and $A(y) = \bot$ for all $y \neq x$, then we will write $A = \{a/x\}$. Note that $\emptyset$ is a special singleton and thus $\emptyset = \{\bot/x\}$ for any $x \in \mathbb{X}$. Further, each $\mathbf{L}$-fuzzy set is uniquely determined by its singletons. We say that an $\mathbf{L}$-fuzzy set A is *finite*, if $\mathrm{Supp}(A)$ is a finite set. Note that finite $\mathbf{L}$-fuzzy sets can be also defined over infinite universes. The set of all finite $\mathbf{L}$-fuzzy sets will be denoted by $\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Further, let us define the intersection of $\mathbf{L}$-fuzzy sets by $(A \cap B)(x) = A(x) \wedge B(x)$ and the union of $\mathbf{L}$-fuzzy sets by $(A \cup B)(x) = A(x) \vee B(x)$. Finally, we say that A is *less than or equal* to B and denote by $A \leq B$, if $A(x) \leq B(x)$ holds for each $x \in \mathbb{X}$. It is easy to see that this relation is a partial ordering on the set of all $\mathbf{L}$-fuzzy sets.

2.3 Convexity of $\mathbf{L}$*-fuzzy sets and generalized extension principle*

In this section, we will extend the notion of convex fuzzy sets. Let $\mathbf{L}$ be a complete *rdr*-lattice and $(\mathbb{X}, \leq)$ be an ordered set. We say that an $\mathbf{L}$-fuzzy set A in $\mathbb{X}$ is $\odot$-*convex* (or $\overline{\odot}$-*convex*), if $A(y) \geq A(x) \odot A(z)$ (or $A(y) \leq A(x) \,\overline{\odot}\, A(z)$) holds for arbitrary $x, y, z \in \mathbb{X}$ such that $x \leq y \leq z$. The set of all $\odot$-convex (or $\overline{\odot}$-convex) $\mathbf{L}$-fuzzy sets over $\mathbb{X}$ will be denoted by $\mathcal{CV}_{\mathbf{L}}^{\odot}(\mathbb{X})$ (or $\mathcal{CV}_{\mathbf{L}}^{\overline{\odot}}(\mathbb{X})$).

Example 7 *Let* $\mathbf{L}$ *be the rdr-lattice determined by the Łukasiewicz t-norm* T *and t-conorm* S*. Recall that* $T(a,b) = \max(a+b-1,0)$ *and* $S(a,b) = \min(a+b,1)$ *for all* $a,b \in [0,1]$*. Let* $(\mathbb{N}_{\omega}^{-0}, \leq)$ *be the set of natural numbers without zero extended by the first infinite cardinal* ω *and* $\leq$ *is the common ordering of extended natural numbers (*$n \leq \omega$ *for any* $n \in \mathbb{N}_{\omega}^{-0}$*). Put* $A(n) = \frac{1}{n}$ *for any* $n \neq \omega$ *and* $A(\omega) = 0$*. Then obviously* A *is* $\odot$*-convex and* $\overline{\odot}$*-convex* $\mathbf{L}$*-fuzzy set for all operations. Further, let*

$$A(n) = \begin{cases} 0.7, & n \text{ is an even number,} \\ 0.5, & n \text{ is an odd number or } n = \omega, \end{cases}$$

Then A *is* $\otimes$*-convex and* $\oplus$*-convex* $\mathbf{L}$*-fuzzy set which is not* $\wedge$*-convex and* $\vee$*-convex (e.g.* $0.7 = A(1) \wedge A(3) \not\leq A(2) = 0.5$*). Obviously,* $\otimes$*-convex* $\mathbf{L}$*-fuzzy sets do not keep the original idea based on the convex a-cuts (cf. [17]).*

Let $f : \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \to \mathbb{Y}$ be an arbitrary crisp mapping. If we want to extend f onto an $\mathbf{L}$-mapping $\widehat{f}^{\odot} : \mathcal{F}_{\mathbf{L}}(\mathbb{X}_1) \times \cdots \times \mathcal{F}_{\mathbf{L}}(\mathbb{X}_n) \to \mathcal{F}_{\mathbf{L}}(\mathbb{Y})$, we can apply the Zadeh extension principle (cf. [31]) which was and still is a very powerful tool of the fuzzy set theory with many practical applications. Its more general form is the following one

$$\widehat{f}^{\odot}(A_1, \ldots, A_n)(y) = \bigvee_{\substack{(x_1,\ldots,x_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(x_1,\ldots,x_n)=y}} A_1(x_1) \odot \cdots \odot A_n(x_n), \tag{5}$$

where $(A_1, \ldots, A_n) \in \mathcal{F}_{\mathbf{L}}(\mathbb{X}_1) \times \cdots \times \mathcal{F}_{\mathbf{L}}(\mathbb{X}_n)$. Analogously, we can define a dual generalized Zadeh extension principle as follows

$$\widehat{f}^{\overline{\odot}}(A_1, \ldots, A_n)(y) = \bigwedge_{\substack{(x_1,\ldots,x_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(x_1,\ldots,x_n)=y}} A_1(x_1) \,\overline{\odot} \cdots \overline{\odot}\, A_n(x_n), \tag{6}$$

where $(A_1, \ldots, A_n) \in \mathcal{F}_{\mathbf{L}}(\mathbb{X}_1) \times \cdots \times \mathcal{F}_{\mathbf{L}}(\mathbb{X}_n)$. Now, we may ask the question. What properties the mapping f has to have in order to the $\mathbf{L}$-mapping $\widehat{f}^{\odot}$ (or $\widehat{f}^{\overline{\odot}}$) preserves the $\odot$-convexity (or $\overline{\odot}$-convexity) of $\mathbf{L}$-fuzzy sets, it means that $\widehat{f}^{\odot}(A_1, \ldots, A_n) \in \mathcal{CV}_{\mathbf{L}}^{\odot}(\mathbb{Y})$ holds for any $(A_1, \ldots, A_n) \in \mathcal{CV}_{\mathbf{L}}^{\odot}(\mathbb{X}_1) \times \cdots \times \mathcal{CV}_{\mathbf{L}}^{\odot}(\mathbb{X}_n)$ (and similarly for $\widehat{f}^{\overline{\odot}}$). The following theorem gives an answer.

Theorem 8 *Let* $(\mathbb{X}_1, \leq_1), \ldots, (\mathbb{X}_n, \leq_n)$ *be linearly ordered sets and* $(\mathbb{Y}, \leq)$ *be an ordered set. Let* $f : \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \to \mathbb{Y}$ *be a surjective mapping such that for each* $(x_1, \ldots, x_n), (z_1, \ldots, z_n) \in \prod_{i=1}^n \mathbb{X}_i$ *and for each* $y \in \mathbb{Y}$ *with* $f(x_1, \ldots, x_n) \leq y \leq f(z_1, \ldots, z_n)$ *there exists* $(y_1, \ldots, y_n) \in \prod_{i=1}^n \mathbb{X}_i$ *satisfying the following conditions*

(i) $f(y_1, \dots, y_n) = y$,
(ii) $x_i \le_i y_i \le_i z_i$ *or* $z_i \le_i y_i \le_i x_i$ *hold for every* $i = 1, \dots, n$.

Then $\widehat{f}^{\odot}$ *preserves the* $\odot$*-convexity of* $\mathbf{L}$*-fuzzy sets, where the divisibility of* $\mathbf{L}$ *is supposed for* $\odot = \wedge$*, and* $\widehat{f}^{\overline{\odot}}$ *preserves the* $\overline{\odot}$*-convexity of* $\mathbf{L}$*-fuzzy sets, where the dual divisibility of* $\mathbf{L}$ *is supposed for* $\overline{\odot} = \vee$*.*

PROOF. Here, we will prove only the preservation of $\otimes$-convexity of $\mathbf{L}$-fuzzy sets. The rest could be done by analogy, where Lemma 6 is used for the operations $\wedge$ and $\vee$. Let $f : \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \to \mathbb{Y}$ be a mapping satisfying the presumptions of theorem. Further, let $A_1, \dots, A_n$ be arbitrary $\otimes$-convex $\mathbf{L}$-fuzzy sets in $\mathbb{X}_1, \dots, \mathbb{X}_n$ and $x \le y \le z$ be arbitrary elements from $\mathbb{Y}$. Since f is the surjective mapping, then there exist n-tuples $(x_1, \dots, x_n)$ and $(z_1, \dots, z_n)$ from $\prod_{i=1}^n \mathbb{X}_i$ such that $f(x_1, \dots, x_n) = x$ and $f(z_1, \dots, z_n) = z$. Moreover, from the linearity of $\le_i$ we have $x_i \le z_i$ or $z_i \le x_i$ for every $i = 1, \dots, n$. According to the presumptions there exists $(y_1, \dots, y_n) \in \prod_{i=1}^n \mathbb{X}_i$ that satisfies the conditions (i) and (ii). Since $A_1, \dots, A_n$ are the $\otimes$-convex $\mathbf{L}$-fuzzy sets, then we have $A_i(y_i) \ge A_i(x_i) \otimes A_i(z_i)$ for every $i = 1, \dots, n$. Hence, we have

$$\widehat{f}^{\otimes}(A_1, \dots, A_n)(y) = \bigvee_{\substack{(y'_1,\dots,y'_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(y'_1,\dots,y'_n) = y}} \bigotimes_{i=1}^n A_i(y'_i) \ge$$

$$\bigvee_{\substack{(x_1,\dots,x_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(x_1,\dots,x_n) = x}} \bigvee_{\substack{(z_1,\dots,z_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(z_1,\dots,z_n) = z}} \Big(\bigotimes_{i=1}^n (A_i(x_i) \otimes A_i(z_i)) \Big) =$$

$$\bigvee_{\substack{(x_1,\dots,x_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(x_1,\dots,x_n) = x}} \bigotimes_{i=1}^n A_i(x_i) \otimes \bigvee_{\substack{(z_1,\dots,z_n) \in \prod_{i=1}^n \mathbb{X}_i \\ f(z_1,\dots,z_n) = z}} \bigotimes_{i=1}^n A_i(z_i) =$$

$$\widehat{f}^{\otimes}(A_1, \dots, A_n)(x) \otimes \widehat{f}^{\otimes}(A_1, \dots, A_n)(z)$$

and thus $\hat{f}^{\otimes}$ preserves the $\otimes$-convexity of $\mathbf{L}$-fuzzy sets. □

3 Structures for cardinalities of L-fuzzy sets

As we have mentioned in the introduction, the generalized natural numbers (in our denotation it means the $\wedge$-convex $\mathbf{L}$-fuzzy sets in the set of natural numbers $\mathbb{N}$) form a suitable fuzzy counterpart of natural numbers for the expression of finite fuzzy cardinals. In this section we will extend the notion of generalized natural numbers and establish two mathematical structures of generalized natural numbers for cardinalities of $\mathbf{L}$-fuzzy sets.

Let us denote $\mathbb{N}_n = \{0, 1, \ldots, n\}$, where $n > 0$ and, moreover, $n = \omega$ is also possible. Further, let us define the addition $\boxplus$ on $\mathbb{N}_n$ as $a \boxplus b = \min(a + b, n)$, where $+$ is the common addition of natural numbers and, moreover, $a + \omega = \omega + a = \omega$ for $n = \omega$. Let $\mathbf{L}$ be a complete *rdr*-lattice. Then we denote $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ the set of all $\odot$-convex $\mathbf{L}$-fuzzy sets in $\mathbb{N}_n$ and $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ the set of all $\overline{\odot}$-convex $\mathbf{L}$-fuzzy sets in $\mathbb{N}_n$. These sets seem to be a natural generalization of the generalized natural numbers. In order to defined cardinalities of finite $\mathbf{L}$-fuzzy sets we have to introduce the operation of addition on $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ and $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$. For this purpose, it is natural to use the generalized Zadeh extension principle and its dual form.

First, let us extend the addition $\boxplus$ of $\mathbb{N}_n$ to the operation of addition $+^{\odot}$ on $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ using the generalized Zadeh extension principle and put $E(k) = \top$, if $k = 0$, and $E(k) = \bot$ otherwise. It is easy to see that E is the $\odot$-convex $\mathbf{L}$-fuzzy set in $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$. The following theorem shows that $+^{\odot}$ is defined correctly and has some natural properties of addition.

Theorem 9 *Let $\mathbf{L}$ be a complete rdr-lattice. Then $(\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}, +^{\odot}, E)$ is the commutative monoid with the neutral element E, where the divisibility of $\mathbf{L}$ is supposed for $\odot = \wedge$.*

PROOF. In order to prove that $+^{\odot}$ is the operation on $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ it is sufficient (according to Theorem 8) to show that for arbitrary $i \leq j \leq k$ from $\mathbb{N}_n$ and for arbitrary $i_1, i_2, k_1, k_2 \in \mathbb{N}_n$ such that $i_1 \boxplus i_2 = i$ and $k_1 \boxplus k_2 = k$ there exist $j_1, j_2 \in N_n$ with $j_1 \boxplus j_2 = j$ and $i_t \leq j_t \leq k_t$ or $k_t \leq j_t \leq i_t$ hold for $t = 1, 2$. Obviously, if $a \geq b$ (suppose $a \neq \omega$ in $\mathbb{N}_\omega$), then we can put $a \boxminus b = c$ if and only if $a = b \boxplus c$. First, let us suppose that $i_1 \leq k_1$, $i_2 \leq k_2$ and $k_1 \neq \omega \neq k_2$. If $j \leq i_1 \boxplus k_2$, then it is sufficient to put $j_1 = i_1$ and $j_2 = j \boxminus i_1$. Obviously, $i_2 \leq j_2 \leq k_2$, because $i_1 \boxplus i_2 \leq j$. If $j > i_1 \boxplus k_2$, then it is sufficient to put $j_2 = k_2$ and $j_1 = j \boxminus k_2$. Again, we obtain $i_1 \leq j_1 \leq k_1$, because $j \leq k_1 \boxplus k_2$. Further, let us suppose that $n = \omega$ and $i_1 \leq k_1 = \omega$ and $i_2 \leq k_2$. If $j = \omega$, then we put $j_1 = \omega$ and $j_2 = i_2$. If $j < \omega$ (and necessary $i < \omega$), then it is sufficient to put $j_1 = i_1$ and $j_2 = j \boxminus i_1$. The analogical results could be obtained for $k_1 < \omega$ and $k_2 = \omega$ or $k_1 < \omega$ and $k_2 < \omega$. Finally, the cases $k_1 \leq i_1$, $i_2 \leq k_2$ and $i_1 \leq k_1$, $k_2 \leq i_2$ could be proved by analogy and thus $(\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}, +^{\odot})$ is the grupoid, where the divisibility of $\mathbf{L}$ has to be supposed in the case $\odot = \wedge$. Since the operation $\odot$ are commutative, then $+^{\odot}$ is also commutative and $(\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}, +^{\odot})$ is the commutative grupoid. Let A be an $\odot$-convex $\mathbf{L}$-fuzzy set. Then

$$(A +^{\odot} E)(i) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 \boxplus i_2 = i}} A(i_1) \odot E(i_2) = A(i) \odot E(0) = A(i) \odot \top = A(i).$$

holds for any $i \in \mathbb{N}_n$ and $A +^{\odot} E = E +^{\odot} A = A$ follows from the commutativity of $+^{\odot}$. Hence, E is a neutral element in the commutative groupoid $(\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}, +^{\odot})$. Finally, for arbitrary $A, B, C \in \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ and $i \in \mathbb{N}_n$ we have

$$((A +^{\odot} B) +^{\odot} C)(i) = \bigvee_{\substack{j,k\in\mathbb{N}_n \\ j\boxplus k=i}} \Big(\bigvee_{\substack{j_1,j_2\in\mathbb{N}_n \\ j_1\boxplus j_2=j}} A(j_1) \odot B(j_2)\Big) \odot C(k) =$$

$$\bigvee_{\substack{j,k\in\mathbb{N}_n \\ j+k=i}} \bigvee_{\substack{j_1,j_2\in\mathbb{N} \\ j_1\boxplus j_2=j}} A(j_1) \odot B(j_2) \odot C(k) = \bigvee_{\substack{j_1,j_2,k\in\mathbb{N}_n \\ j_1\boxplus j_2\boxplus k=i}} A(j_1) \odot B(j_2) \odot C(k) =$$

$$\bigvee_{\substack{j_1,j\in\mathbb{N}_n \\ j_1\boxplus j=i}} A(j_1) \odot \Big(\bigvee_{\substack{j_2,k\in\mathbb{N}_n \\ j_2\boxplus k=j}} B(j_2) \odot C(k)\Big) = (A +^{\odot} (B +^{\odot} C))(i)$$

and thus the operation $+^{\odot}$ is associative. Hence, $(\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}, +^{\odot})$ is the commutative monoid, where the divisibility of $\mathbf{L}$ is supposed for $\odot = \wedge$. □

Further, let us extend the addition $\boxplus$ of $\mathbb{N}$ to the operation of addition $+^{\overline{\odot}}$ on $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ using the dual generalized Zadeh extension principle and put $E(k) = \bot$, if $k = 0$, and $E(k) = \top$ otherwise. Again, E is the $\overline{\odot}$-convex $\mathbf{L}$-fuzzy set in $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ and the following theorem shows the correctness of $+^{\overline{\odot}}$ and its properties.

Theorem 10 *Let* $\mathbf{L}$ *be a complete rdr-lattice. Then* $(\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}, +^{\overline{\odot}}, E)$ *is a commutative monoid with the neutral element, where the dual divisibility of* $\mathbf{L}$ *is supposed for* $\overline{\odot} = \vee$.

PROOF. It could be done by analogy to the proof of Theorem 9. □

Obviously, both defined structures have many common properties with the classical structures $(\mathbb{N}_n, \boxplus, 1)$ of (restricted) natural numbers. So we will call these structures, in general, as the *structures of generalized natural numbers restricted to* n and the cardinalities of finite $\mathbf{L}$-fuzzy sets will be described by the generalized natural numbers from these structures. For simplicity, in the following parts, if we suppose the general structures $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ and $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$, then the complete *rdr*-lattice $\mathbf{L}$ will be always divisible for $\odot = \wedge$ and dually divisible for $\overline{\odot} = \vee$.

4 Axiomatic approach to $\odot$-cardinalities of finite L-fuzzy sets

The cardinalities of finite fuzzy sets are usually defined as a mapping or more precisely a measure $\mathbb{C}$ from the set of all finite fuzzy sets in $\mathbb{X}$ to the set of generalized natural numbers which satisfies the additivity property, i.e. $\mathbb{C}(A\cup$

$B) = \mathbb{C}(A)+\mathbb{C}(B)$ holds for each A, B such that $A \cap B = \emptyset$, where $+$ is defined by Zadeh extension principle. J. Casasnovas and J. Torrens in [2] proposed a system containing four reasonable axioms which characterize some important properties of the well-known cardinalities of finite fuzzy sets. Further, they proved that each cardinality defined inside their axiomatic system can be uniquely identified with two special mappings (cf. [2, 28]). Nevertheless, there are some reasonable definition of cardinality, e.g. the cardinalities based on the triangular norms (cf. [29]), that can not be defined inside Casasnovas-Torrens axiomatic system. The reason is that the operation of minimum has the primary role in this axiomatic system. In this section, a generalization of Casasnovas-Torrens axiomatic system will be introduced. This generalization is not trivial and the original Casasnovas-Torrens axiomatic system could be obtained from it, if we restrict ourselves to the complete *rdr*-lattices defined on the unit interval and put $\odot = \wedge$.

4.1 Definition and examples

Let $i \in \mathbb{N}$ be an arbitrary natural number. Then we define $i \boxdot 1$ by induction as follows $0 \boxdot 1 = 0$ and $i \boxdot 1 = ((i-1) \boxdot 1) \boxplus 1$ for each $i = 1, 2, \dots,$ where $\boxplus$ is the addition in $\mathbb{N}_n$. Recall that $\mathbf{L}$ is divisible, if $\mathcal{CV}^{\wedge}_{\mathbf{L},\mathrm{n}}$ is supposed.

Definition 11 *Let* $\mathbf{L}$ *be a complete rdr-lattice. A mapping* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ *is an* $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets*, if it satisfies the following axioms for arbitrary* $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$

(C1) if $A \cap B = \emptyset$*, then* $\mathbb{C}(A \cup B) = \mathbb{C}(A) +^{\odot} \mathbb{C}(B)$*,*
(C2) if $i, j \in \mathbb{N}_n$ *and* $i > |\mathrm{Supp}(A)|$*,* $j > |\mathrm{Supp}(B)|$*, then* $\mathbb{C}(A)(i) = \mathbb{C}(B)(j)$*,*
(C3) if A *is a crisp set, then* $\mathbb{C}(A)$ *is a crisp set and* $\mathbb{C}(A)(|A| \boxdot 1) = \top$*,*
(C4) if $a \in L$*,* $x, y \in \mathbb{X}$ *and* $i \in \mathbb{N}_n$*, then* $\mathbb{C}(\{a/x\})(i) = \mathbb{C}(\{a/y\})(i)$*,*
(C5) if $a, b \in L$ *and* $x \in \mathbb{X}$*, then*

$$\mathbb{C}(\{a \,\overline{\odot}\, b/x\})(0) = \mathbb{C}(\{a/x\})(0) \odot \mathbb{C}(\{b/x\})(0), \tag{7}$$

$$\mathbb{C}(\{a \odot b/x\})(1) = \mathbb{C}(\{a/x\})(1) \odot \mathbb{C}(\{b/x\})(1). \tag{8}$$

The axioms C1-C5 are called the *additivity, variability, consistency, singleton independency, preservation of non-existence* and *existence*, respectively. The terms of the first three axioms are used from [2]. Let us point out some motivation as well as the meaning for each one of these axioms. The additivity of $\odot$-cardinality is the property of cardinality of sets and it seems to be natural. The idea of variability is that the $\odot$-cardinality of $\mathbf{L}$-fuzzy sets is only influenced by the elements of universe that belong to supports of $\mathbf{L}$-fuzzy sets. The axiom of consistency requires the $\odot$-cardinality of finite crisp set to be again a crisp set in $\mathbb{N}_n$, because each element from a given universe "absolutely" belongs or not to the crisp set. Moreover, it seems to be reasonable to put

$\mathbb{C}(A)(i) = \top$, if $|A| = i \leq n$. The axiom is a generalization of the previous consideration for arbitrary i (i.e. also $i > n$). The singleton independency abstracts away from particular elements of a given universe. Finally, if A is a finite **L**-fuzzy set and $x \in \mathbb{X}$, then the value $\mathbb{C}(\{A(x)/x\})(0)$ could express a measure of how much the element x does not exist in A and analogously, the value $\mathbb{C}(\{A(x)/x\})(1)$ could express a measure of how much the element x does exist in A. In other words, $\mathbb{C}(\{A(x)/x\})(0)$ could be understood as the degree of that $\{A(x)/x\}$ is the empty set and analogously, $\mathbb{C}(\{A(x)/x\})(1)$ as the degree of that $\{A(x)/x\}$ is the one element set $\{x\}$. So the axioms of non-existence and existence preservations define the relations between the non-existence and the existence of the element x in A. The following three propositions show several examples of $\odot$-cardinalities of finite **L**-fuzzy sets (cf. [2, 29, 30]).

Proposition 12 *Let* **L** *be a complete divisible rdr-lattice. Then a mapping* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\wedge}_{\mathbf{L},\mathrm{n}}$*, defined for each* $i \in \mathbb{N}_n$ *by*

$$\mathbb{C}_0(A)(i) = \bigvee\{a \mid a \in L \text{ and } |A_a| \geq i\}, \tag{9}$$

is a $\wedge$*-cardinality of finite* **L***-fuzzy sets (determined by a-cuts).*

PROOF. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $i, j \in \mathbb{N}_n$ such that $i \leq j$. Obviously, if $|A_a| \geq j$ then also $|A_a| \geq i$. Hence, we have $\mathbb{C}_0(A)(i) \geq \mathbb{C}_0(A)(j)$. A simple consequence of this is that $\mathbb{C}_0(A)$ is a $\wedge$-convex **L**-fuzzy set for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Now, let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $A \cap B = \emptyset$. Then $|(A \cup B)_a| = |A_a| + |B_a| \geq \min(|A_a| + |B_a|, n) = |A_a| \boxplus |B_a|$. Further, $|(A \cup B)_a| \geq i$ if and only if $|A_a| \boxplus |B_a| \geq i$ holds for each $i \in \mathbb{N}_n$ and $|A_a| \leq |A_b|$, if $b \leq a$. Let $k, l, i \in \mathbb{N}_n$ and $k \boxplus l = i$. According to the distributivity $\wedge$ w.r.t. $\vee$ (**L** is divisible) and the fact that $|A_{a\wedge b}| \boxplus |B_{a \wedge b}| \geq |A_a| \boxplus |B_b| \geq i$, we have

$$\mathbb{C}_0(A)(k) \wedge \mathbb{C}_0(B)(l) = \Big(\bigvee_{\substack{a \in L \\ |A_a| \geq k}} a\Big) \wedge \Big(\bigvee_{\substack{b \in L \\ |B_b| \geq l}} b\Big) = \bigvee_{\substack{a \in L \\ |A_a| \geq k}} \bigvee_{\substack{b \in L \\ |B_b| \geq l}} (a \wedge b) \leq \bigvee_{\substack{a,b \in L \\ |A_a| \boxplus |B_b| \geq i}} (a \wedge b) = \bigvee_{\substack{a,b \in L \\ |A_{a\wedge b}| \boxplus |B_{a\wedge b}| \geq i}} (a \wedge b) = \bigvee_{\substack{c \in L \\ |A_c| \boxplus |B_c| \geq i}} c = \mathbb{C}_0(A \cup B)(i).$$

Hence, for each $i \in \mathbb{N}_n$ we obtain

$$(\mathbb{C}_0(A) +^{\wedge} \mathbb{C}_0(B))(i) = \bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} (\mathbb{C}_0(A)(k) \wedge \mathbb{C}_0(B)(l)) \leq \mathbb{C}_0(A \cup B)(i).$$

Let $c \in L$ such that $|(A \cup B)_c| \geq i$. Then there exist $k_c, l_c \in N_n$ such that $|A_c| \geq k_c$, $|B_c| \geq l_c$ and $k_c \boxplus l_c = i$ (e.g. if $|A_c| \leq i$, then put $k_c = |A_c|$ and

$l_c = i - k_c$). Moreover, we have

$$(\bigvee_{\substack{a\in L\\ |A_a|\geq k_c}} a) \wedge (\bigvee_{\substack{b\in L\\ |B_b|\geq l_c}} b) \geq c \wedge c = c.$$

Hence, we obtain for each $i \in \mathbb{N}_n$

$$\mathbb{C}_0(A\cup B)(i) = \bigvee_{\substack{c\in L\\ |(A\cup B)_c|\geq i}} c \leq \bigvee_{\substack{k,l\in N_n\\ k\boxplus l=i}} (\bigvee_{\substack{a\in L\\ |A_a|\geq k}} a) \wedge (\bigvee_{\substack{b\in L\\ |B_b|\geq l}} b) = (\mathbb{C}_0(A) +^{\wedge} \mathbb{C}_0(B))(i)$$

and $\mathbb{C}_0$ satisfies the axiom of additivity. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $|\mathrm{Supp}(A)| < i$. Then clearly $|A_a| < i$ for each $a \in L$ and $a \neq \bot$. Hence, we have $\mathbb{C}_0(A)(i) = \bigvee \emptyset = \bot$, if $|\mathbb{X}| < i$, or $\mathbb{C}_0(A)(i) = \bigvee \bot = \bot$, if $|\mathbb{X}| \geq i$. Thus we have proved the axiom of variability. Let $A \subseteq \mathbb{X}$ be a finite crisp set. If $|A| \geq i$, then also $|A_\top| \geq i$ and thus $\mathbb{C}_0(A)(i) = \top$. In particular, we have $|A| \geq |A| \boxdot 1$ and thus $\mathbb{C}_0(A)(|A| \boxdot 1) = \top$. Further, if $|A| < i$, then also $|\mathrm{Supp}(A)| < i$ and $\mathbb{C}_0(A)(i) = \bot$ follows from the axiom of variability. Hence, $\mathbb{C}_0$ satisfies the axiom of consistency. The singleton independency follows from the fact that $|\{c/x\}_a| = |\{c/y\}_a|$ holds for each a-cut of singletons $\{a/x\}$ and $\{c/y\}$. Finally, we have $\mathbb{C}_0(\{c/x\})(0) = \bigvee\{a \in L \mid |\{c/x\}_a| \geq 0\} = \top$ for each $c \in L$, which implies the validity of the axiom of non-existence preservation, and $\mathbb{C}_0(\{c/x\})(1) = \bigvee\{a \in L \mid |\{c/x\}_a| \geq 1\} = c$ for each $c \in L$, which implies the validity of the axiom of existence preservation. □

Remark 13 *It is easy to show that $\mathbb{C}_0$ is not a $\otimes$-cardinality of finite* **L***-fuzzy sets in general.*

Proposition 14 *Let* **L** *be a complete rdr-lattice and $\mathbb{C}_0$ is the $\wedge$-cardinality of finite* **L***-fuzzy sets determined by a-cuts defined in Proposition 12. Then*

$$\mathbb{C}_1^a(A) = \mathbb{C}_0(A_a), \tag{10}$$

where $a \in L\setminus\{\bot\}$, defines a $\wedge$-cardinality of finite **L***-fuzzy sets. If* **L** *is linearly ordered, then*

$$\mathbb{C}_2(A) = \mathbb{C}_0(\mathrm{Supp}(A)) \tag{11}$$

defines a $\wedge$-cardinality of finite **L***-fuzzy sets.*

PROOF. First, we will show that $\mathbb{C}_1^a$ is a $\wedge$-cardinality for any $a \in L \setminus \{\bot\}$. Let $a \in L \setminus \{\bot\}$ be arbitrary. Obviously, $\mathbb{C}_1^a(A)$ is a $\wedge$-convex **L**-fuzzy set for any $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Further, according to the axiom C3 we have $\mathbb{C}_1^a(A)(i) \in \{\bot, \top\}$ for all $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $i \in \mathbb{N}_n$. Hence, we needn't suppose the divisibility of **L**, because the distributivity $\wedge$ w.r.t. $\vee$ is satisfied, if we deal only with $\bot$ and $\top$. Obviously, if $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ such that $A \cap B = \emptyset$, then

$A_a \cap B_a =$ for all $a \in L \setminus \{\bot\}$. Moreover, $(A \cup B)_a = A_a \cup B_a$. Hence, for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ such that $A \cap B = \emptyset$ we have $\mathbb{C}_1^a(A \cup B) = \mathbb{C}_0((A \cup B)_a) = \mathbb{C}_0(A_a \cup B_a) = \mathbb{C}_0(A_a) +^\wedge \mathbb{C}_0(B_a) = \mathbb{C}_1^a(A) +^\wedge \mathbb{C}_1^a(B)$. Thus $\mathbb{C}_1$ satisfies C1. Since $A_a \subseteq \mathrm{Supp}(A)$ for all $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$, we have $\mathbb{C}_1^a(A)(i) = \mathbb{C}_0(A_a)(i) = \mathbb{C}_0(B_a)(j) = \mathbb{C}_1^a(B)(j)$ for arbitrary $i > |\mathrm{Supp}(A)|$ and $j > |\mathrm{Supp}(B)|$ and thus $\mathbb{C}_1$ satisfies C2. The satisfaction of C3 follows from the fact that $A = A_a$ holds for each crisp set $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. If $b \in L$ and $x, y \in \mathbb{X}$, then for each $i \in \mathbb{N}_n$ we have $\mathbb{C}_1^a(\{b/x\})(i) = \mathbb{C}_0(\emptyset)(i) = \mathbb{C}_1^a(\{b/y\})(i)$, if $b \not\geq a$, and $\mathbb{C}_1^a(\{b/x\})(i) = \mathbb{C}_0(\{x\})(i) = \mathbb{C}_0(\{y\})(i) = \mathbb{C}_1^a(\{b/y\})(i)$ otherwise. Hence, $\mathbb{C}_1$ satisfies C4. Finally, let $x \in \mathbb{X}$, $a \in L \setminus \{\bot\}$ and $b, c \in L$. Since $\mathbb{C}_1(\{d/x\})(0) = \mathbb{C}_0(\{\{d/x\}_a) = \top$ for each $d \in L$, then $\mathbb{C}_1$ trivially satisfies the preservation of non-existence. Further, if $b \wedge c \geq a$, then $b \geq a$ and $c \geq a$. Hence, we have $\mathbb{C}_1(\{b \wedge c/x\})(1) = \mathbb{C}_0(\{x\})(1) = \top = \mathbb{C}_0(\{x\})(1) \wedge \mathbb{C}_0(\{x\})(1) = \mathbb{C}_0(\{b/x\}_a)(1) \wedge \mathbb{C}_0(\{c/x\}_a)(1) = \mathbb{C}_1(\{b/x\})(1) \wedge \mathbb{C}_1(\{c/x\})(1)$. If $b \wedge c \not\geq a$, then $b \not\geq a$ or $c \not\geq a$. In the case $b \not\geq a$ we obtain $\mathbb{C}_1(\{b \wedge c/x\})(1) = \mathbb{C}_0(\emptyset)(1) = \bot = \mathbb{C}_0(\emptyset)(1) \wedge \mathbb{C}_0(\{c/x\})(1) = \mathbb{C}_1(\{b/x\})(1) \wedge \mathbb{C}_1(\{c/x\})(1)$ and the same result we obtain in the case $c \not\geq a$. Hence, $\mathbb{C}_1^a$ satisfies the preservation of existence and $\mathbb{C}_1^a$ is a $\wedge$-cardinality of finite $\mathbf{L}$-fuzzy sets. Further, let us suppose that $\mathbf{L}$ is linearly ordered. The proof of the satisfaction of axioms C1-C4 and the preservation of non-existence by $\mathbb{C}_2$ could be done by analogy to the previous proof of $\mathbb{C}_1^a$. Here, we will only prove the last condition, i.e. the preservation of existence. Let $a, b \in L$ and $x \in \mathbb{X}$. If $a \wedge b > \bot$, then $a > \bot$, $b > \bot$ and hence we have $\mathbb{C}_2(\{a \wedge b/x\})(1) = \mathbb{C}_0(\{x\})(1) = \top = \mathbb{C}_0(\{x\})(1) \wedge \mathbb{C}_0(\{x\})(1) = \mathbb{C}_2(\{a/x\})(1) \wedge \mathbb{C}_2(\{b/x\})(1)$. If $a \wedge b = \bot$, then $a = \bot$ or $b = \bot$ (according to the linearity of $\mathbf{L}$) and hence $\mathbb{C}_2(\{a \wedge b/x\})(1) = \mathbb{C}_0(\emptyset)(1) = \bot = \mathbb{C}_2(\{a/x\})(1) \wedge \mathbb{C}_2(\{b/x\})(1)$, where $\mathbb{C}_2(\{a/x\})(1) = \mathbb{C}_0(\emptyset) = \bot$ or $\mathbb{C}_2(\{b/x\})(1) = \mathbb{C}_0(\emptyset) = \bot$. Thus $\mathbb{C}_2$ satisfies the preservation of existence and the proof of the proposition is finished. □

Proposition 15 *Let* $\mathbf{L}$ *be a complete linearly ordered rdr-lattice and* $\mathbb{C}_0$ *is the* $\wedge$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets determined by a-cuts defined in Proposition 12. Then*

$$\mathbb{C}_3(A) = \mathbb{C}_0(A_\top) \tag{12}$$

$$\mathbb{C}_4(A)(i) = \begin{cases} \top, & i = 0, \\ \mathbb{C}_4(A)(i-1) \otimes \mathbb{C}_0(A)(i), & \textit{otherwise}, \end{cases} \tag{13}$$

for each $i \in \mathbb{N}_\omega$*, define* $\otimes$*-cardinalities of finite* $\mathbf{L}$*-fuzzy sets.*

PROOF. First, we will show that $\mathbb{C}_3$ is a $\otimes$-cardinality of finite $\mathbf{L}$-fuzzy sets. Obviously, $\mathbb{C}_3 = \mathbb{C}_1^\top$ and thus $\mathbb{C}_3(A)(i) \in \{\bot, \top\}$. Further, it is easy to see that the operations $\wedge$ and $\otimes$ coincide on the restricted set of membership degrees $\{\bot,\top\}$. From the coincidence of $\wedge$ and $\otimes$ on $\{\bot,\top\}$ we can write

$$(\mathbb{C}(A) +^{\wedge} \mathbb{C}(B))(i) = \bigwedge_{\substack{j,k\in\mathbb{N}\\ j\boxplus k=i}} \mathbb{C}_3(A)(j) \wedge \mathbb{C}_3(B)(k) =$$
$$\bigwedge_{\substack{j,k\in\mathbb{N}\\ j\boxplus k=i}} \mathbb{C}_3(A)(j) \otimes \mathbb{C}_3(B)(k) = (\mathbb{C}_3(A) +^{\otimes} \mathbb{C}_3(B))(i)$$

for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $i \in \mathbb{N}_n$. Since $\mathbb{C}_1^{\top}$ is a $\wedge$-cardinality, then clearly $\mathbb{C}_3$ satisfies the axioms C1-C4. Let $a, b \in L$ and $x \in \mathbb{X}$ be arbitrary. From the definition of $\mathbb{C}_0$ we have $\mathbb{C}_3(\{a \oplus b/x\})(0) = \top = \top \otimes \top = \mathbb{C}_3(\{a/x\})(0) \otimes \mathbb{C}_3(\{b/x\})(0)$ and thus $\mathbb{C}_3$ satisfies the preservation of non-existence. Now, if $a \otimes b = \top$, then $a = b = \top$ and $\mathbb{C}_3(\{a \otimes b/x\})(1) = \mathbb{C}_0^{\top}(\{x\})(1) = \top = \mathbb{C}_0^{\top}(\{x\})(1) \otimes \mathbb{C}_0^{\top}(\{x\})(1) = \mathbb{C}_3(\{a/x\})(1) \otimes \mathbb{C}_3(\{b/x\})(1)$. Finally, if $a \otimes b < \top$, then $a < \top$ or $b < \top$ and thus $\mathbb{C}_3(\{a \otimes b/x\})(1) = \mathbb{C}_0^{\top}(\emptyset)(1) = \bot = \mathbb{C}_3(\{a/x\})(1) \otimes \mathbb{C}_3(\{b/x\})(1)$, where $\mathbb{C}_3(\{a/x\})(1) = \mathbb{C}_0^{\top}(\emptyset) = \bot$ or $\mathbb{C}_3(\{b/x\})(1) = \mathbb{C}_0^{\top}(\emptyset) = \bot$. Thus $\mathbb{C}_3$ satisfies the preservation of existence and $\mathbb{C}_3$ is a $\otimes$-cardinality of finite **L**-fuzzy sets. Further, we will prove that (13) also defines a $\otimes$-cardinality of finite **L**-fuzzy sets. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary and $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$. Obviously, due to the linearity of **L**, we can order the membership degrees of A, i.e. $A(x_{k_1}) \geq A(x_{k_2}) \geq \cdots \geq A(x_{k_r})$, and (13) can be rewritten as follows

$$\mathbb{C}_4(A)(i) = \begin{cases} \top, & i = 0, \\ A(x_{k_1}) \otimes \cdots \otimes A(x_{k_i}), & 0 < i \leq r \\ \bot, & i > r. \end{cases} \tag{14}$$

Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $A \cap B = \emptyset$. Let $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \dots, y_s\}$. From (14) we have for $i = 0$

$$\mathbb{C}_4(A \cup B)(0) = \top = \mathbb{C}_4(A)(0) \otimes \mathbb{C}_4(B)(0) = (\mathbb{C}_4(A) +^{\otimes} \mathbb{C}_4(B))(i)$$

and for $i > r + s$

$$\mathbb{C}_4(A \cup B)(i) = \bot = \bigvee_{\substack{p,q\in\mathbb{N}_\omega\\ p\boxplus q=p+q=i}} \mathbb{C}_4(A)(p) \otimes \mathbb{C}_4(B)(q) = (\mathbb{C}_4(A) +^{\otimes} \mathbb{C}_4(B))(i),$$

because of $p > r$ or $q > s$ are held for arbitrary $p, q \in \mathbb{N}_\omega$ such that $p + q = i$ and $p > r$ implies $\mathbb{C}_4(A)(p) = \bot$ and $q > s$ implies $\mathbb{C}_4(B)(q) = \bot$. Finally, let $0 < i \leq r + s$. Then

$$\mathbb{C}_4(A \cup B)(i) = A(x_{k_1}) \otimes \cdots \otimes A(x_{k_p}) \otimes B(y_{k_1}) \otimes \cdots \otimes B(y_{k_q}), \tag{15}$$

where $p + q = i$ and, moreover, $\min(A(x_{k_1}), \dots, B(y_{k_q})) \geq A(x_k)$ for each $k = k_{p+1}, \dots, k_r$ and $\min(A(x_{k_1}), \dots, B(y_{k_q})) \geq B(y_k)$ for each $k = k_{q+1}, \dots, k_s$.

Thus the value $\mathbb{C}(A\cup B)(i)$ is the product of i greatest membership degrees from $A\cup B$. i.e. from $A(x_1),\dots,A(x_r),B(y_1),\dots,B(y_s)$. It is easy to show that $\mathbb{C}_4(A)(p)\otimes\mathbb{C}_4(B)(q)=A(x_{k_1})\otimes\cdots\otimes A(x_{k_p})\otimes B(y_{k_1})\otimes\cdots\otimes B(y_{k_q})\geq \mathbb{C}_4(A)(p')\otimes\mathbb{C}_4(B)(q')$ for arbitrary $p',q'\in\mathbb{N}_\omega$ such that $p'+q'=i$. Hence, we have

$$\mathbb{C}_4(A\cup B)(i)=\mathbb{C}_4(A)(p)\otimes\mathbb{C}_4(B)(q)= \bigvee_{\substack{p',q'\in\mathbb{N}_\omega\\ p'+q'=i}}\mathbb{C}_4(A)(p')\otimes\mathbb{C}_4(B)(q')=(\mathbb{C}_4(A)+^{\otimes}\mathbb{C}_4(B))(i)$$

and thus $\mathbb{C}_4$ satisfies the axiom of additivity. The axioms C2-C4 and the preservation of non-existence follow immediately from the definition (14). Finally, let $a,b\in L$ and $x\in\mathbb{X}$ be arbitrary. Then $\mathbb{C}_4(\{a\otimes b/x\})(1)=a\otimes b=\mathbb{C}_4(\{a/x\})(1)\otimes\mathbb{C}_4(\{b/x\})(1)$ and $\mathbb{C}_4$ satisfies the preservation of existence. Hence, $\mathbb{C}_4$ is a $\otimes$-cardinality of finite $\mathbf{L}$-fuzzy sets and the proof is finished. □

Recall that E denotes the neutral element in $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$. Further, let us put $\mathbf{1}(i)=\top$ for each $i\in\mathbb{N}_n$. Obviously, $\mathbf{1}$ is the greatest element of $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ w.r.t. the ordering $\leq$ of $\mathbf{L}$-fuzzy sets in $\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$.

Proposition 16 *Let $\mathbb{C}:\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})\to\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ be an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets. Then $\mathbb{C}(\emptyset)=E$ or $\mathbb{C}(\emptyset)=\mathbf{1}$. If $\mathbb{C}(\emptyset)=\mathbf{1}$, then $\mathbb{C}(A)(i)\leq\mathbb{C}(A)(j)$ holds for each $i,j\in\mathbb{N}_n$ such that $i\leq j$.*

PROOF. According to the consistency, we have $\mathbb{C}(\emptyset)(i)\in\{\bot,\top\}$ for each $i\in\mathbb{N}_n$ and $\mathbb{C}(\emptyset)(0)=\top$. A consequence of the variability is that $\mathbb{C}(\emptyset)(i)=\mathbb{C}(\emptyset)(j)$ for every $i,j>0$. Hence, we have either $\mathbb{C}(\emptyset)(i)=\bot$ for each $i>0$ or $\mathbb{C}(\emptyset)(i)=\top$ for each $i>0$ and thus $\mathbb{C}(\emptyset)=E$ or $\mathbb{C}(\emptyset)=\mathbf{1}$. Let $\mathbb{C}(\emptyset)=\mathbf{1}$. Then we have

$$\mathbb{C}(A)(i)=\mathbb{C}(A\cup\emptyset)(i)=(\mathbb{C}(A)+^{\odot}\mathbb{C}(\emptyset))(i)= \bigvee_{\substack{k,l\in\mathbb{N}_n\\ k\boxplus l=i}}(\mathbb{C}(A)(k)\odot\mathbb{C}(\emptyset)(l))=\bigvee_{\substack{k,l\in\mathbb{N}_n\\ k\boxplus l=i}}(\mathbb{C}(A)(k)\odot\top)=\bigvee_{\substack{k\in\mathbb{N}_n\\ k\leq i}}\mathbb{C}(A)(k)$$

for each $i\in\mathbb{N}_n$. Hence, we obtain $\mathbb{C}(A)(i)\leq\mathbb{C}(A)(j)$ for each $i,j\in\mathbb{N}_n$ such that $i\leq j$. □

Proposition 17 *Let $\mathbb{C}:\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})\to\mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ be an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets and $A\in\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ such that $\mathrm{Supp}(A)\subseteq\{x_1,\dots,x_m\}$. Then we have*

$$\mathbb{C}(A)(i)=\bigvee_{\substack{i_1,\dots,i_m\in\mathbb{N}_n\\ i_1\boxplus\cdots\boxplus i_m=i}}\bigodot_{k=1}^{m}\mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{16}$$

for each $i \in \mathbb{N}_n$.

PROOF. It is a straightforward consequence of the additivity which is applied on the singletons. $\square$

Theorem 18 *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ *be an* $\odot$*-cardinality of finite* **L***-fuzzy sets and* $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ *such that* $\mathrm{Supp}(A) \subseteq \{x_1, \ldots, x_m\} \subseteq \mathbb{X}$*. Then we have*

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\ldots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{17}$$

for each $i \in \mathbb{N}_n$*, where* $i \leq m$*. Moreover, if* $m < n$*, then* $\mathbb{C}(A)(i) = \bot$ *or* $\mathbb{C}(A)(i) = \top$ *holds for each* $m < i \leq n$*, respectively.*

PROOF. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$, where $\mathrm{Supp}(A) \subseteq \{x_1, \ldots, x_m\} \subseteq \mathbb{X}$. First, let us suppose that $m < n$. Obviously, (17) is true for $i \leq 1$. Let $1 < i \leq m$ and $M = \{i_1, \ldots, i_m\}$, where $i_1, \ldots, i_m \in \mathbb{N}_n$, such that $i_1 \boxplus \cdots \boxplus i_m = i$. Put $I_M = \{i_k \mid i_k \in M \text{ and } i_k \notin \{0, 1\}\}$. Obviously, if to each M there exists $M' = \{i'_1, \ldots, i'_m\}$ such that $I_{M'} = \emptyset$ and

$$\bigodot_{i_k \in M} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{i'_k \in M'} \mathbb{C}(\{A(x_k)/x_k\})(i'_k), \tag{18}$$

then (17) is true. In the following part, we will show that there exists $M' = \{i'_1, \ldots, i'_m\}$ such that $I_{M'} \subset I_M$ and (18) is satisfied. Let $i_{k_0} \in I_M$ be an arbitrary element. Then there exist at least $i_{k_0} - 1$ elements of M which are equal to 0. In fact, suppose $s < i_{k_0} - 1$ is the maximal number of elements of M that are equal to 0. Then $(m-1) - s$ elements of M are greater than 0 and different from i_{k_0}. Hence, we can write $i_1 \boxplus \cdots \boxplus i_m = i_1 + \cdots + i_m \geq i_k + ((m-1) - s) > i_{k_0} + ((m-1) - (i_{k_0} - 1)) = m \geq i$, a contradiction. Thus, we can choose the elements $i_{k_1} = \cdots = i_{k_r} = 0$ of M, where $r = i_{k_0} - 1$. Due to the axioms of variability and consistency, we have $\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) = \mathbb{C}(\{\bot/x_{k_0}\})(2) = \mathbb{C}(\{\bot/x_{k_0}\})(2) = \mathbb{C}(\{A(x_l)/x_l\})(2) \in \{\bot, \top\}$, where $l \in \{k_1, \ldots, k_r\}$. From the $\odot$-convexity of $\mathbb{C}$, the axiom of existence preservation and $\mathbb{C}(\{\bot/x_{k_0}\})(0) = \top$ we obtain

$$\begin{aligned} &\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_l)/x_l\})(0) = \\ &\quad \mathbb{C}(\{A(x_l)/x_l\})(2) \odot \mathbb{C}(\{A(x_l)/x_l\})(0) \leq \mathbb{C}(\{A(x_l)/x_l\})(1) \end{aligned}$$

for each $l \in \{k_1, \dots, k_r\}$ and

$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) = \mathbb{C}(\{\bot/x_{k_0}\})(2) \odot \top =$$
$$\mathbb{C}(\{\bot/x_{k_0}\})(2) \odot \mathbb{C}(\{\bot/x_{k_0}\})(0) \leq$$
$$\mathbb{C}(\{\bot/x_{k_0}\})(1) = \mathbb{C}(\{\bot \odot A(x_{k_0})/x_{k_0}\})(1) =$$
$$\mathbb{C}(\{\bot/x_{k_0}\})(1) \odot \mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(1) \leq \mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(1).$$

Since $\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \in \{\bot, \top\}$ and $\bot$, $\top$ are the idempotent elements of $\mathbf{L}$ w.r.t $\odot$, then we can write according to the previous inequalities

$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0) \odot \cdots \odot \mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(0) =$$
$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0) \odot \cdots \odot$$
$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(0) \leq$$
$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(1) \odot \mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1) \odot \cdots \odot \mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(1).$$

If we define $M' = \{i'_1, \dots, i'_m\}$ such that $i'_{k_0} = i'_{k_1} = \cdots = i'_{k_r} = 1$ and $i'_l = i_l$ for all $l \in M \setminus \{k_0, k_1, \dots, k_r\}$, then we obtain $i'_1 \boxplus \cdots \boxplus i'_m = i$, where $I_{M'} = \{i'_k \mid i'_k \in M' \text{ and } i'_k \notin \{0,1\}\} = I_M \setminus \{i_k\} \subset I_M$, and

$$\bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i'_k).$$

Obviously, the mentioned procedure may be repeated as long as all elements from I_M are not removed. Hence, to each M there exists M' such that $I_{M'} = \emptyset$ and (18) is satisfied. Thus (17) is true for all $1 < i \leq m$. Further, let us suppose that $n \leq m$. If $i < n$, then we can apply the same procedure as in the previous case. Let $i = n$ and $i_1 \boxplus \cdots \boxplus i_m = n$. If $i_{k_1}, \dots, i_{k_r}$ are all elements from $M = \{i_1, \dots, i_m\}$ that are equal to 0 and $i_{k_0} > 1$, then we have (see above)

$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(i_{k_0}) \odot \mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0) \odot \cdots \odot \mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(0) \leq$$
$$\mathbb{C}(\{A(x_{k_0})/x_{k_0}\})(1) \odot \mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1) \odot \cdots \odot \mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(1).$$

Moreover, the inequality $\mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \mathbb{C}(\{A(x_k)/x_k\})(1)$ holds for all $i_k \in M$, where $i_k \geq 1$. Hence, we have

$$\bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(1)$$

and thus the first part of the theorem is proved. Let $m < n$. Then, according to the axiom of variability and Proposition 16, we have $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \bot$ or $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \top$ for each $|\mathrm{Supp}(A)| \leq m < i \leq n$ and the second part of the theorem is also proved. $\square$

Remark 19 *A simple consequence of the previous theorem is that* $\mathbb{C}(A)(n) = \top$ *holds for each crisp set* A*, where* $\mathbb{N}_n$ *is supposed and* $|A| \geq n$*.*

In [2] there is shown a representation of cardinalities of finite fuzzy sets using two monotonic mappings $f, g : [0,1] \to [0,1]$. Precisely, f is a non-decreasing mapping, g is a non-increasing mapping and further $f(0), g(1) \in \{0,1\}$, $f(1) = 1$ and $g(0) = 0$. In order to introduce some analogical representation for the $\odot$-cardinalities of finite **L**-fuzzy sets, we have to establish several notions. Let $\mathbf{L}_i = \langle L_i, \wedge_i, \vee_i, \otimes_i, \rightarrow_i, \oplus_i, \ominus_i, \bot_i, \top_i \rangle$, where $i = 1, 2$, be rdr-lattices and $h : L_1 \to L_2$ be a mapping. We say that h is an $\odot$-*homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2$, if h is an homomorphism from the reduct $(L_1, \odot_1, \top_1)$ of the rdr-lattice $\mathbf{L}_1$ to the reduct $(L_2, \odot_2, \top_2)$ of the rdr-lattice $\mathbf{L}_2$, i.e. $h(a \odot_1 b) = h(a) \odot_2 h(b)$ and $h(\top_1) = \top_2$. Obviously, each homomorphism between rdr-lattices (or residuated lattices which are the reducts of original rdr-lattices) is also an $\odot$-homomorphism. Further, we say that h is an $\overline{\odot}_{\mathrm{d}}$-*homomorphism*, if h is a homomorphism from the reduct $(L_1, \overline{\odot}_1, \bot_1)$ of the rdr-lattice $\mathbf{L}_1$ to the reduct $(L_2, \odot_2, \top_2)$ of the rdr-lattice $\mathbf{L}_2$, i.e. $h(a \,\overline{\odot}_1\, b) = h(a) \odot h(b)$ and $h(\bot_1) = \top_2$. Again, each homomorphism between rdr-lattices (or homomorphism from a dually residuated lattice to a residuated lattice which are the reducts of original rdr-lattices) is also an $\odot_{\mathrm{d}}$-homomorphism.

Lemma 20 *Let* $\mathbf{L}$ *be a complete* rdr*-lattice and* $f, g : L \to L$ *be* $\odot$*- and* $\overline{\odot}_{\mathrm{d}}$*-homomorphisms from* $\mathbf{L}$ *to* $\mathbf{L}$ *such that* $f(\bot) \in \{\bot, \top\}$ *and* $g(\top) \in \{\bot, \top\}$. *Let* $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ *be a mapping defined by the induction as follows*

$$\mathbb{C}_{f,g}(\{a/x\})(0) = g(a),\ \mathbb{C}_{f,g}(\{a/x\})(1) = f(a) \text{ and}$$
$$\mathbb{C}_{f,g}(\{a/x\})(k) = f(\bot),\ k > 1$$

hold for each singleton $\{a/x\} \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ *and*

$$\mathbb{C}_{f,g}(A) = \mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\odot} \cdots +^{\odot} \mathbb{C}_{f,g}(\{A(x_m)/x_m\})$$

holds for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$, *where* $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$. *Then the mapping* $\mathbb{C}_{f,g}$ *is an* $\odot$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets (generated by the* $\odot$*- and* $\overline{\odot}_{\mathrm{d}}$*-homomorphisms* f *and* g*), respectively.*

PROOF. First, we will prove that $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ is defined correctly, i.e. $\mathbb{C}_{f,g}(A)$ is an $\odot$-convex $\mathbf{L}$-fuzzy set for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Let $\{a/x\} \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be a singleton. If $n = 1$, then $\mathbb{C}_{f,g}(\{a/x\})$ is clearly an $\odot$-convex $\mathbf{L}$-fuzzy set. Let $n > 1$. Since $f(\bot) = f(\bot \odot a) = f(\bot) \odot f(a) \le f(a)$ holds for any $a \in L$, then we have

$$\mathbb{C}_{f,g}(\{a/x\})(0) \odot \mathbb{C}_{f,g}(\{a/x\})(2) = g(a) \odot f(\bot) \le f(a) = \mathbb{C}_{f,g}(\{a/x\})(1).$$

Moreover, obviously $\mathbb{C}_{f,g}(\{a/x\})(i) \odot \mathbb{C}_{f,g}(\{a/x\})(k) = \mathbb{C}_{f,g}(\{a/x\})(j)$ is trivially fulfilled for each triplet $0 < i \le j \le k$ from $\mathbb{N}_n$ and thus $\mathbb{C}_{f,g}$ assigns an

$\odot$-convex $\mathbf{L}$-fuzzy set to each singleton from $\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. According to Theorem 9, the sum of $\odot$-convex singletons is an $\odot$-convex $\mathbf{L}$-fuzzy set and thus $\mathbb{C}_{f,g}(A)$ is an $\odot$-convex $\mathbf{L}$-fuzzy set for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Hence, the $\mathbb{C}_{f,g}$ is defined correctly. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary disjoint $\mathbf{L}$-fuzzy sets, where $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \dots, y_s\}$. Due to the associativity of the operation $+^{\odot}$ and the definition of $\mathbb{C}_{f,g}$, we have

$$\begin{aligned}
&\mathbb{C}_{f,g}(A \cup B) = \\
&\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\odot} \cdots +^{\odot} \mathbb{C}_{f,g}(\{A(x_r)/x_r\}) +^{\odot} \mathbb{C}_{f,g}(\{B(y_1)/y_1\}) +^{\odot} \cdots \\
&\quad +^{\odot} \mathbb{C}_{f,g}(\{B(y_s)/y_s\}) = \Big(\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\odot} \cdots +^{\odot} \mathbb{C}_{f,g}(\{A(x_r)/x_r\})\Big) +^{\odot} \\
&\qquad \Big(\mathbb{C}_{f,g}(\{B(y_1/y_1\}) +^{\odot} \cdots +^{\odot} \mathbb{C}_{f,g}(\{B(y_s)/y_s\})\Big) = \mathbb{C}_{f,g}(A) +^{\odot} \mathbb{C}_{f,g}(B).
\end{aligned}$$

Hence, the mapping $\mathbb{C}_{f,g}$ satisfies the axiom of additivity. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be an $\mathbf{L}$-fuzzy set with $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$. From the additivity of $\mathbb{C}_{f,g}$ applied on singletons we have

$$\mathbb{C}_{f,g}(A)(i) = \bigvee_{\substack{i_1,\dots,i_r \in \mathbb{N}_n \\ i_1+\cdots+i_m = i}} \mathbb{C}(\{A(x_1)/x_1\})(i_1) \odot \cdots \odot \mathbb{C}(\{A(x_m)/x_m\})(i_m) \quad (19)$$

for each $i \in \mathbb{N}_n$. Let $i \in \mathbb{N}_n$ be an arbitrary natural number. Then denote the set of all m-dimensional vector $(i_1, \dots, i_m)$ over $\mathbb{N}_n$ having $i_1 \boxplus \cdots \boxplus i_m = i$ by $\mathcal{V}_i^m$ and put $K_{\mathbf{i}} = \{k \mid i_k = 0\}$, $L_{\mathbf{i}} = \{l \mid i_l = 1\}$ and $R_{\mathbf{i}} = \{r \mid i_r > 1\}$ for each $\mathbf{i} \in \mathcal{I}$. Obviously, the sets $K_{\mathbf{i}}, L_{\mathbf{i}}, R_{\mathbf{i}}$ are mutually disjoint and cover the set $\{1, \dots, m\}$. Finally, let us denote $a_{K_{\mathbf{i}}} = \bigodot_{k \in K_{\mathbf{i}}} g(A(x_k))$, $a_{L_{\mathbf{i}}} = \bigodot_{l \in L_{\mathbf{i}}} f(A(x_l))$ and $a_{R_{\mathbf{i}}} = \bigodot_{r \in R_{\mathbf{i}}} f(\bot) = f(\bot) \in \{\bot, \top\}$. Recall that $\bigodot_{a \in \emptyset} a = \top$. Obviously, the formula (19) can be rewritten as follows

$$\mathbb{C}_{f,g}(A)(i) = \bigvee_{\mathbf{i} \in \mathcal{V}_i^m} a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}}. \quad (20)$$

Let us suppose that $i > m$. Then necessarily $R_{\mathbf{i}} \neq \emptyset$ for each $\mathbf{i} \in \mathcal{V}_i^m$. Since $a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}} \leq a_{R_{\mathbf{i}}} = f(\bot)$ holds for each $\mathbf{i} \in \mathcal{V}_i^m$, then $\mathbb{C}_{f,g}(A)(i) = \bigvee_{\mathbf{i} \in \mathcal{V}_i^m} a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}} \leq \bigvee_{\mathbf{i} \in \mathcal{V}_i^m} f(\bot) = f(\bot)$. On the other hand, there exists $\mathbf{i} \in \mathcal{V}_i^m$ such that $K_{\mathbf{i}} = \emptyset$ and thus

$$a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}} = \top \odot \bigodot_{l \in L_{\mathbf{i}}} f(A(x_l)) \odot f(\bot) = f(\bigodot_{l \in L_{\mathbf{i}}} A(x_l) \odot \bot) = f(\bot)$$

for some $\mathbf{i} \in \mathcal{V}_i^m$, since f is an $\odot$-homomorphism from $\mathbf{L}$ to $\mathbf{L}$ and $\bot$ is an annihilator in $\mathbf{L}$. Hence, we obtain $\mathbb{C}_{f,g}(A)(i) = f(\bot) \in \{\bot, \top\}$ and thus the axiom of variability is satisfied. The axiom of consistency is a simple consequence of the previous consideration. In fact, suppose that $A \subseteq \mathbb{X}$ is a crisp set (i.e. $|A| = m$). If $i > |A|$, then $\mathbb{C}_{f,g}(A)(i) = f(\bot) \in \{\bot, \top\}$. If $i \leq |A|$, then clearly $a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}} \in \{\bot, \top\}$ for each $\mathbf{i} \in \mathcal{V}_i^m$, because $a_{K_{\mathbf{i}}}, a_{L_{\mathbf{i}}}, a_{R_{\mathbf{i}}} \in \{\bot, \top\}$, where $a_{K_{\mathbf{i}}} = g(\top \,\overline{\odot} \cdots \overline{\odot}\, \top) = g(\top) \in \{\bot, \top\}$, if $K_{\mathbf{i}} \neq \emptyset$, and $a_{K_{\mathbf{i}}} = \top$, if

$K_{\mathbf{i}} = \emptyset$. Hence, we obtain $\mathbb{C}(A)(i) = \bigvee_{\mathbf{i}\in\mathcal{I}} a_{K_{\mathbf{i}}} \odot a_{L_{\mathbf{i}}} \odot a_{R_{\mathbf{i}}} \in \{\bot, \top\}$. Moreover, if $|A| \leq n$, then there exists $\mathbf{i} \in \mathcal{V}_i^m$ such that $K_{\mathbf{i}} = R_{\mathbf{i}} = \emptyset$. Hence, we have $\mathbb{C}(A)(|A|) = \mathbb{C}(A)(|A| \boxdot 1) \geq a_{L_{\mathbf{i}}} = f(\top \odot \cdots \odot \top) = \top$. Further, if $|A| > n$, then $n = |A| \boxdot 1$ and again there exists $\mathbf{i} \in \mathcal{V}_i^m$ such that $K_{\mathbf{i}} = M_{\mathbf{i}} = \emptyset$. Analogously, we have $\mathbb{C}(|A| \boxdot 1) = a_{L_{\mathbf{i}}} = \top$. The axioms of singleton independency and non-existence and existence preservations follow immediately from the definition of $\odot$- and $\overline{\odot}_{\mathrm{d}}$-homomorphism f and g, respectively. So the mapping $\mathbb{C}_{f,g}$ satisfies all axioms C1–C5 and thus $\mathbb{C}_{f,g}$ is an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets. $\Box$

Theorem 21 (Representation of $\odot$-cardinality) *Let $\mathbf{L}$ be a complete rdr-lattice and $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ be a mapping which satisfies the axiom of additivity. Then the following statements are equivalent:*

(i) $\mathbb{C}$ is an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets,

(ii) there exist an $\odot$-homomorphism $f : \mathbf{L} \to \mathbf{L}$ and an $\overline{\odot}_{\mathrm{d}}$-homomorphism $g : \mathbf{L} \to \mathbf{L}$, such that $f(\bot) \in \{\bot, \top\}$, $g(\top) \in \{\bot, \top\}$ and

$$\mathbb{C}(\{a/x\})(0) = g(a),\ \mathbb{C}(\{a/x\})(1) = f(a),\ \mathbb{C}(\{a/x\})(k) = f(\bot)$$

hold for arbitrary $a \in L$, $x \in \mathbb{X}$ and $k \in \mathbb{N}_n$, where $k > 1$.

PROOF. First, we will show that (i) implies (ii). Let us suppose that $\mathbb{C}$ is an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets. Define two mappings $f, g : L \to L$ as follows

$$f(a) = \mathbb{C}(\{a/x\})(1), \tag{21}$$
$$g(a) = \mathbb{C}(\{a/x\})(0). \tag{22}$$

According to the axiom C5, we have

$$f(a \odot b) = \mathbb{C}(\{a \odot b/x\})(1) = \mathbb{C}(\{a/x\})(1) \odot \mathbb{C}(\{b/x\})(1) = f(a) \odot f(b),$$
$$g(a \,\overline{\odot}\, b) = \mathbb{C}(\{a \,\overline{\odot}\, b/x\})(0) = \mathbb{C}(\{a/x\})(0) \odot \mathbb{C}(\{b/x\})(0) = g(a) \odot g(b)$$

and, moreover, $f(\top) = \mathbb{C}(\{\top/x\})(1) = \top$, $g(\bot) = \mathbb{C}(\{\bot/x\}(0) = \top$ hold due to the axiom C3. Hence, f is an $\odot$-homomorphism and g is an $\overline{\odot}_{\mathrm{d}}$-homomorphism of the relevant structures. The rest conditions of f and g are a simple consequence of the axiom C3 and Proposition 16. Further, we will show that (ii) implies (i). Let $\mathbb{C}_{f,g}$ be the $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets generated by the $\odot$-homomorphism f and $\overline{\odot}_{\mathrm{d}}$-homomorphisms g defined in Lemma 20. Since $\mathbb{C}_{f,g}(\{a/x\}) = \mathbb{C}(\{a/x\})$ for any singleton from $\mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $\mathbb{C}$ satisfies the axiom of additivity, then also $\mathbb{C}_{f,g}(A) = \mathbb{C}(A)$ for any $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Hence, we obtain that $\mathbb{C}$ is an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets. $\Box$

Remark 22 *According to the previous theorem, each $\odot$-cardinality $\mathbb{C}$ of finite $\mathbf{L}$-fuzzy sets is generated by an $\odot$-homomorphism f and $\overline{\odot}_{\rm d}$-homomorphism g satisfying the conditions of (ii). This representation enables us to investigate the $\odot$-cardinalities from the perspective of various types of $\odot$- and $\overline{\odot}_{\rm d}$-homomorphisms.*

4.3 Selected properties

It is well known that cardinality of sets preserves the ordering of the set determined by the inclusion relation to the ordering of cardinals, i.e. $A \subseteq B$ implies $|A| \leq |B|$. In the first part of this section, we will study some analogical question for $\odot$-cardinalities of finite $\mathbf{L}$-fuzzy sets. Before we give a claim, let us establish two special types of homomorphisms. First, we say that an $\odot$-homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$ is an $\odot$-o-*homomorphism*, if $h(a) \leq h(b)$ holds for arbitrary $a \leq b$. Analogously, an $\overline{\odot}_{\rm d}$-homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$ is an $\overline{\odot}_{\rm d}$-o-*homomorphism*, if $h(a) \leq h(b)$ holds for arbitrary $a \geq b$.

Remark 23 *Obviously, an arbitrary $\wedge$- or $\vee$-homomorphism is also $\wedge$-o- or $\vee$-o-homomorphism, respectively. For example, if h is a $\wedge$-homomorphism and $a \leq_1 b$, then $h(a) = h(a \wedge_1 b) = h(a) \wedge_2 h(b)$ and thus $h(a) \leq_2 h(b)$.*

Example 24 *Let $\mathbf{L}$ be a (complete) rdr-lattice. First, let $f : L \to L$ be a mapping defined by $f(a) = \top$ for all $a \in L$. Obviously, $f(a \odot b) = \top = \top \odot \top = f(a) \odot f(b)$ for each $a, b \in L$ and $f(a) = \top \leq \top = f(b)$ for each $a, b \in L$, where $a \leq b$. Hence, f is an $\odot$-o-homomorphism called the* trivial $\odot$-o-homomorphism. *Second, let $f : L \to L$ be a mapping defined by $f(a) = a \otimes \cdots \otimes a = a^n$ for some $n \in \mathbb{N}$, where $n \geq 1$. Then $f(\top) = \top^n = \top$ and $f(a \otimes b) = (a \otimes b)^n = a^n \otimes b^n = f(a) \otimes f(b)$. Moreover, if $a \leq b$, then $f(a) = a^n \leq b^n = f(b)$ follows from the monotony of $\otimes$. Hence, f is an $\otimes$-o-homomorphism.*

Example 25 *Let $\mathbf{L}$ be a (complete) rdr-lattice from Example 4. Let $g : L \to L$ be defined by $g(a) = \top$ for all $a \in L$. Then clearly f is an $\overline{\odot}_{\rm d}$-o-homomorphism which will be called the* trivial $\overline{\odot}_{\rm d}$-o-homomorphism. *Further, let $g : L \to L$ be defined by $g(a) = (\neg a)^n$. Then g is an $\overline{\odot}_{\rm d}$-o-homomorphism. In fact, we have $g(a \oplus b) = \neg(a \oplus b) = \neg\neg(\neg a \otimes \neg b) = \neg a \otimes \neg b = g(a) \otimes g(b)$ and, moreover, if $a \leq b$, then $\neg b \leq \neg a$ and thus $g(b) \leq g(a)$.*

An $\odot$-cardinality $\mathbb{C}$ of finite $\mathbf{L}$-fuzzy sets generated by just an $\odot$-homomorphism f (i.e. g is the trivial $\overline{\odot}_{\rm d}$-o-homomorphism and thus it has no effect to the $\odot$-cardinality) will be denoted by $\mathbb{C}_f$. Analogously, an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets generated by just an $\overline{\odot}_{\rm d}$-homomorphism g (i.g. f is the trivial $\odot$-o-homomorphism and thus it has no effect to the $\odot$-cardinality) will be denoted by $\mathbb{C}_g$.

We say that an $\odot$-cardinality $\mathbb{C}$ *preserves* or *reverses the ordering* (of **L**-fuzzy sets), if $\mathbb{C}(A) \leq \mathbb{C}(B)$ holds for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ such that $A \leq B$ or $B \leq A$, respectively.

Theorem 26 *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ be an $\odot$-cardinality of **L**-fuzzy sets generated by $\odot$-o- and $\overline{\odot}$-o-homomorphisms f and g, respectively. Then*

(i) $\mathbb{C}_{f,g}$ preserves the ordering if and only if g is trivial, i.e. $\mathbb{C}_{f,g} = \mathbb{C}_g$.
(ii) $\mathbb{C}_{f,g}$ reverses the ordering if and only if f is trivial, i.e. $\mathbb{C}_{f,g} = \mathbb{C}_f$.

PROOF. Here, we will prove just the first statement, the second one could be done by analogy. First, let $\mathbb{C}_{f,g}$ preserves the ordering of **L**-fuzzy sets. If g is a non-trivial $\overline{\odot}_{\mathrm{d}}$-o-homomorphism, then there exists $a \in L$, where $\perp < a$, such that $g(a) < \top$. Then obviously $\{\perp/x\} \leq \{a/x\}$ for some $x \in \mathbb{X}$ and simultaneously $\mathbb{C}_{f,g}(\{\perp/x\})(0) = g(\perp) > g(a) = \mathbb{C}_{f,g}(\{a/x\})(0)$, a contradiction. Hence, g has to be the trivial $\overline{\odot}_{\mathrm{d}}$-o-homomorphism and (i) implies (ii). Further, let f be an $\odot$-o-homomorphism and g be the trivial $\overline{\odot}_{\mathrm{d}}$-o-homomorphism. If $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ such that $A \leq B$, then $\mathbb{C}_f(\{A(x)/x\})(0) = g(A(x)) = \top = g(B(x)) = \mathbb{C}_f(\{B(x)/x\})(0)$ and $\mathbb{C}_f(\{A(x)/x\})(1) = f(A(x)) \leq f(B(x)) = \mathbb{C}_f(\{B(x)/x\})(1)$ hold for any $x \in \mathbb{X}$. The inequality $\mathbb{C}_{f,g}(A) \leq \mathbb{C}_{f,g}(B)$ is a simple consequence of Theorem 18 and the monotony of $\odot$. Hence, (ii) implies (i) and the proof is completed. □

In the cardinal theory there is a very important property of cardinality called the valuation property, i.e. $|A \cap B| + |A \cup B| = |A| + |B|$ holds for arbitrary sets A and B. Unfortunately, this property is not satisfied, in general, for all $\odot$-cardinalities. For example, if we suppose a complete *rdr*-lattice determined by an Archimedean continuous t-norm T and t-conorm S (see Example 3) and we introduce the T-intersection of **L**-fuzzy sets by $(A \cap_T B)(x) = T(A(x), B(x))$ and the S-union of **L**-fuzzy sets by $(A \cup_S B)(x) = S(A(x), B(x))$, then it easy to show that the valuation property is not satisfied for the T-cardinality (cf. Theorem 4.15. in [29]). In our case, the situation is even more complicated because we suppose more general structures than complete *rdr*-lattices determined by Archimedean continuous t-norms and t-conorms.

Example 27 *Let $L = \{\perp, a, b, \top\}$ be a set and $<$ be the ordering relation on L such that $\perp < a, b < \top$ and $a||b$ (a and b are two indistinguishable elements w.r.t. $<$). Let us consider the rdr-lattice $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \ominus, \perp, \top\rangle$, where $\wedge$ and $\vee$ are determined by $<$ and $\rightarrow$ and $\ominus$ are defined analogously as in Example 2. Now, let us define mappings $f, g : L \to L$ as follows*

$$f(x) = x, \text{ for each } x \in L$$

$$g(x) = \begin{cases} x, & \textit{if } x \in \{a, b\}, \\ \bot, & \textit{if } x = \top, \\ \top, & \textit{if } x = \bot. \end{cases}$$

Obviously, f is a $\wedge$-o-homomorphism and g a $\vee_{\mathrm{d}}$-o-homomorphism (e.g. we can write $g(a \vee b) = g(\top) = \bot = g(a) \wedge g(b)$). Let $\mathbb{C}_{f,g}$ be a $\wedge$-cardinality of $\mathbf{L}$-fuzzy sets generated by f and g and $A = \{a/x\}$ and $B = \{b/x\}$ be two $\mathbf{L}$-fuzzy sets. Then we have

$$((\mathbb{C}_{f,g}(A \cap B) +^{\wedge} \mathbb{C}_{f,g}(A \cup B))(1) = (\mathbb{C}_{f,g}(\{a \wedge b/x\}) +^{\wedge} \mathbb{C}_{f,g}(\{a \vee b/x\}))(1) =$$
$$(\mathbb{C}_{f,g}(\{\bot/x\})(0) \wedge \mathbb{C}_{f,g}(\{\top/x\})(1)) \vee (\mathbb{C}_{f,g}(\{\bot/x\})(1) \wedge \mathbb{C}_{f,g}(\{\top/x\})(0)) =$$
$$(g(\bot) \wedge f(\top)) \vee (g(\top) \wedge f(\bot)) = (\top \wedge \top) \vee (\bot \wedge \bot) = \top.$$

On the other hand, we have

$$(\mathbb{C}_{f,g}(A) +^{\wedge} \mathbb{C}_{f,g}(B))(1) = (\mathbb{C}_{f,g}(\{a/x\}) +^{\wedge} \mathbb{C}_{f,g}(\{b/x\}))(1) =$$
$$(\mathbb{C}_{f,g}(\{a/x\})(0) \wedge \mathbb{C}_{f,g}(\{b/x\})(1)) \vee (\mathbb{C}_{f,g}(\{a/x\})(1) \wedge \mathbb{C}_{f,g}(\{b/x\})(0)) =$$
$$(g(a) \wedge f(b)) \vee (f(a) \wedge g(b)) = (a \wedge b) \vee (a \wedge b) = \bot \vee \bot = \bot.$$

Hence, we can see that the valuation property needn't be satisfied, if we consider a residuated lattice with indistinguishable elements.

Lemma 28 *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\wedge}_{\mathbf{L},\mathrm{n}}$ be a $\wedge$-cardinality of finite $\mathbf{L}$-fuzzy sets. Then*

(i) $\mathbb{C}(\{a \wedge b/x\})(0) \geq \mathbb{C}(\{a/x\})(0) \vee \mathbb{C}(\{b/x\})(0)$,
(ii) $\mathbb{C}(\{a \vee b/x\})(1) \geq \mathbb{C}(\{a/x\})(1) \vee \mathbb{C}(\{b/x\})(1)$,
(iii) $\mathbb{C}(\{a \wedge b/x\})(t) \wedge \mathbb{C}(\{a \vee b/x\})(s) \geq \mathbb{C}(\{a/x\})(t) \wedge \mathbb{C}(\{b/x\})(s)$

hold for arbitrary $a, b \in L$, $x \in \mathbb{X}$ and $t, s \in \{0, 1\}$ such that $t \leq s$.

PROOF. Let $a, b \in L$. Then

$$\mathbb{C}(\{a/x\})(0) = \mathbb{C}(\{a \vee (a \wedge b)/x\})(0) = \mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{a \wedge b/x\})(0).$$

Hence, we have $\mathbb{C}(\{a \wedge b/x\})(0) \geq \mathbb{C}(\{a/x\})(0)$. Analogously, we obtain $\mathbb{C}(\{a \wedge b/x\})(0) \geq \mathbb{C}(\{b/x\})(0)$ and thus (i) is true. Further, we have

$$\mathbb{C}(\{a/x\})(1) = \mathbb{C}(\{a \wedge (a \vee b)/x\})(1) = \mathbb{C}(\{a/x\})(1) \wedge \mathbb{C}(\{a \vee b/x\})(1)$$

and thus $\mathbb{C}(\{a \vee b/x\})(1) \geq \mathbb{C}(\{a/x\})(1)$. Analogously, we obtain $\mathbb{C}(\{a \vee b/x\})(1) \geq \mathbb{C}(\{b/x\})(1)$ and thus (ii) is true. Let $t = s = 0$, then we can write (according to (i) and the preservation of non-existence)

$$\mathbb{C}(\{a\wedge b/x\})(0) \wedge \mathbb{C}(\{a \vee b/x\})(0) \geq (\mathbb{C}(\{a/x\})(0) \vee \mathbb{C}(\{b/x\})(0)) \wedge \\ (\mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{b/x\})(0)) = \mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{b/x\})(0).$$

Further, let $t = 0$ and $s = 1$. Then we have (according to (i) and (ii))

$$\mathbb{C}(\{a\wedge b/x\})(0) \wedge \mathbb{C}(\{a \vee b/x\})(1) \geq (\mathbb{C}(\{a/x\})(0) \vee \mathbb{C}(\{b/x\})(0)) \wedge \\ (\mathbb{C}(\{a/x\})(1) \vee \mathbb{C}(\{b/x\})(1)) \geq \mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{b/x\})(1).$$

Finally, let $t = s = 1$. Then we have (according to (ii) and the preservation of existence)

$$\mathbb{C}(\{a\wedge b/x\})(1) \wedge \mathbb{C}(\{a \vee b/x\})(1) \geq (\mathbb{C}(\{a/x\})(1) \wedge \mathbb{C}(\{b/x\})(1)) \wedge \\ (\mathbb{C}(\{a/x\})(1) \vee \mathbb{C}(\{b/x\})(1)) \geq \mathbb{C}(\{a/x\})(1) \wedge \mathbb{C}(\{b/x\})(1)$$

and the proof of (iii) is completed. □

Theorem 29 *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\wedge}_{\mathbf{L},\mathrm{n}}$ be a $\wedge$-cardinality of finite $\mathbf{L}$-fuzzy sets. Then*

$$\mathbb{C}(A \cap B) +^{\wedge} \mathbb{C}(A \cup B) \geq \mathbb{C}(A) +^{\wedge} \mathbb{C}(B) \tag{23}$$

holds for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. If $\mathbf{L}$ is linearly ordered, then $\mathbb{C}$ fulfils the valuation property.

PROOF. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary finite $\mathbf{L}$-fuzzy sets and $\mathrm{Supp}(A)$, $\mathrm{Supp}(B) \subseteq \{x_1, \dots, x_m\}$, where we suppose (without lost of generality) $m \geq n$. Obviously, we have $\mathrm{Supp}(A \cap B), \mathrm{Supp}(A \cup B) \subseteq \{x_1, \dots, x_m\}$. Let us notice that according to the axiom C5 and (iii) from Lemma 28

$$\mathbb{C}(\{a \wedge b/x\})(1) \wedge \mathbb{C}(\{a \vee b/x\})(0) = (\mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{b/x\})(0)) \wedge \\ (\mathbb{C}(\{a/x\})(1) \wedge \mathbb{C}(\{b/x\})(1)) \leq \mathbb{C}(\{a \wedge b/x\})(0) \wedge \mathbb{C}(\{a \vee b/x\})(1) \tag{24}$$

holds for arbitrary $a, b \in L$. Further, we assert the following claim: Let $k, l, i \in \mathbb{N}_n$, where $k \boxplus l = i$, and $k_1, \dots, k_m, l_1, \dots, l_m \in \{0, 1\}$, where $k_1 \boxplus \cdots \boxplus k_m = k$ and $l_1 \boxplus \cdots \boxplus l_m = l$. Then there exist $k', l' \in \mathbb{N}_n$, where $k' \boxplus l' = i$ and $k' \leq l'$, and $k'_1, \dots, k'_m, l'_1, \dots, l'_m \in \{0, 1\}$, where $k'_1 \boxplus \cdots \boxplus k'_m = k'$ and $l'_1 \boxplus \cdots \boxplus l'_m = l$, such that

$$\mathbb{C}(\{A(x_r) \wedge B(x_r)/x_r\})(k'_r) \wedge \mathbb{C}(\{A(x_r) \vee B(x_r)/x_r\})(l'_r) \geq \\ \mathbb{C}(\{A(x_r) \wedge B(x_r)/x_r\})(k_r) \wedge \mathbb{C}(\{A(x_r) \vee B(x_r)/x_r\})(l_r) \tag{25}$$

holds for each $r = 1, \dots, m$. In fact, let us put $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{l} = (l_1, \dots, l_m)$ and denote $K_{\mathbf{k}} = \{r \mid k_r = 0\}$, $L_{\mathbf{k}} = \{r \mid k_r = 1\}$, $K_{\mathbf{l}} = \{r \mid l_r = 0\}$ and

$L_{\mathbf{l}} = \{r \mid l_r = 1\}$ (see also the proof of Lemma 20). Then we can define $\mathbf{k}' = (k'_1, \dots, k'_m)$, $\mathbf{l}' = (l'_1, \dots, l'_m) \in \{0,1\}^m$ such that

$$K_{\mathbf{k}'} = K_{\mathbf{k}} \cup L_{\mathbf{k}} \setminus L_{\mathbf{l}}, \qquad L_{\mathbf{k}'} = L_{\mathbf{k}} \cap L_{\mathbf{l}}, \tag{26}$$
$$K_{\mathbf{l}'} = K_{\mathbf{k}} \cap K_{\mathbf{l}}, \qquad L_{\mathbf{l}'} = L_{\mathbf{l}} \cup K_{\mathbf{l}} \setminus K_{\mathbf{k}}. \tag{27}$$

Obviously, $|K_{\mathbf{k}'} \cup L_{\mathbf{k}'}| = |K_{\mathbf{l}'} \cup L_{\mathbf{l}'}| = m$, $|L_{\mathbf{k}'} \cup L_{\mathbf{l}'}| = |L_{\mathbf{k}} \cup L_{\mathbf{l}}| = i$ and $K_{\mathbf{l}'} \subseteq K_{\mathbf{k}'}$ and $L_{\mathbf{k}'} \subseteq L_{\mathbf{l}'}$. Since $k'_r \leq l'_r$ for all $r = 1, \dots, m$, then $k' = k'_1 \boxplus \cdots \boxplus k'_m \leq l'_1 \boxplus \cdots \boxplus l'_m = l'$. Let us notice that if $k_r \leq l_r$ then $k_r = k'_r$ and $l_r = l'_r$. In fact, if $r \in K_{\mathbf{k}} \cap K_{\mathbf{l}}$, then also $r \in K_{\mathbf{k}'} \cap K_{\mathbf{l}'}$. If $r \in K_{\mathbf{k}} \cap L_{\mathbf{l}}$, then also $r \in K_{\mathbf{k}'} \cap L_{\mathbf{l}'}$, and if $r \in L_{\mathbf{k}} \cap L_{\mathbf{l}}$, then also $r \in L_{\mathbf{k}'} \cap L_{\mathbf{l}'}$. Hence, (25) is true for all $k_r \leq l_r$. Finally, let $k_r = 1$ and $l_r = 0$. Then $k'_r = 0$ and $l'_r = 1$ and (25) is a straightforward consequence of (24). So the claim is proved. Let us denote $\mathcal{R}_i = \{(\mathbf{k}, \mathbf{l}) \in \{0,1\}^m \times \{0,1\}^m \mid k_1 \boxplus \cdots \boxplus k_m \boxplus l_1 \boxplus \cdots \boxplus l_m = i\}$ and define the mapping $\pi^i : \mathcal{R}_i \to \mathcal{R}_i$ such that $\pi^i(\mathbf{k}, \mathbf{l}) = (\mathbf{k}', \mathbf{l}')$, where $\mathbf{k}'$ and $\mathbf{l}'$ are defined by (26) and (27), respectively. Denote $\pi^i_{1r}(\mathbf{k}, \mathbf{l}) = k'_r$ and $\pi^i_{2r}(\mathbf{k}, \mathbf{l}) = l'_r$ and $A(x_r) = a_r$ and $B(x_r) = b_r$ for arbitrary $\mathbf{L}$-fuzzy sets $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $r = 1, \dots, m$. Now, we can write according to Theorem 18, the above claim and (iii) of Lemma 28 (recall that $\pi^i_{1r}(\mathbf{k}, \mathbf{l}) \leq \pi^i_{2r}(\mathbf{k}, \mathbf{l})$ for any $r = 1, \dots, m$))

$$
\begin{aligned}
(\mathbb{C}(A \cap B) +^{\wedge} \mathbb{C}(A \cup B))(i) &= \bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} \Big(\bigvee_{\substack{\mathbf{k} \in \{0,1\}^m \\ k_1 \boxplus \cdots \boxplus k_m = k}} \bigwedge_{r=1}^{m} \mathbb{C}(\{a_r \wedge b_r / x_r\})(k_r) \Big) \wedge \\
&\qquad \Big(\bigvee_{\substack{\mathbf{l} \in \{0,1\}^m \\ l_1 \boxplus \cdots \boxplus l_m = l}} \bigwedge_{s=1}^{m} \mathbb{C}(\{a_s \vee b_s / x_s\})(l_s) \Big) \geq \\
&\bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} \bigvee_{\substack{\mathbf{k} \in \{0,1\}^m \\ k_1 \boxplus \cdots \boxplus k_m = k}} \bigvee_{\substack{\mathbf{l} \in \{0,1\}^m \\ l_1 \boxplus \cdots \boxplus l_m = l}} \bigwedge_{r=1}^{m} \Big(\mathbb{C}(\{(a_r \wedge b_r / x_r\})(k_r) \wedge \mathbb{C}(\{a_r \vee b_r / x_r\})(l_r) \Big) = \\
&\bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} \bigvee_{\substack{\mathbf{k} \in \{0,1\}^m \\ k_1 \boxplus \cdots \boxplus k_m = k}} \bigvee_{\substack{\mathbf{l} \in \{0,1\}^m \\ l_1 \boxplus \cdots \boxplus l_m = l}} \bigwedge_{r=1}^{m} \Big(\mathbb{C}(\{(a_r \wedge b_r / x_r\})(\pi^i_{1,r}(\mathbf{k}, \mathbf{l})) \\
&\qquad \wedge \mathbb{C}(\{a_r \vee b_r / x_r\})(\pi^i_{2r}(\mathbf{k}, \mathbf{l})) \Big) \geq \\
&\bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} \bigvee_{\substack{\mathbf{k} \in \{0,1\}^m \\ k_1 \boxplus \cdots \boxplus k_m = k}} \bigvee_{\substack{\mathbf{l} \in \{0,1\}^m \\ l_1 \boxplus \cdots \boxplus l_m = l}} \bigwedge_{r=1}^{m} \Big((\mathbb{C}(\{a_r / x_r\})(k'_r) \wedge \mathbb{C}(\{b_r / x_r\})(l'_r)) \\
&\qquad \vee (\mathbb{C}(\{b_r / x_r\})(k'_r) \wedge \mathbb{C}(\{a_r / x_r\})(l'_r)) \Big) \geq \\
&\bigvee_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} \bigvee_{\substack{\mathbf{k} \in \{0,1\}^m \\ k_1 \boxplus \cdots \boxplus k_m = k}} \bigvee_{\substack{\mathbf{l} \in \{0,1\}^m \\ l_1 \boxplus \cdots \boxplus l_m = l}} \bigwedge_{r=1}^{m} (\mathbb{C}(\{a_r / x_r\})(k_r) \wedge \mathbb{C}(\{b_r / x_r\})(l_r)) =
\end{aligned}
$$

$$\bigvee_{\substack{k,l\in\mathbb{N}_n\\ k\boxplus l=i}} \Big(\bigvee_{\mathbf{k}\in\{0,1\}^m k_1\boxplus\cdots\boxplus k_m=k} \bigwedge_{r=1}^{m} \mathbb{C}(\{a_r/x_r\})(k_r)$$

$$\wedge \bigvee_{\substack{\mathbf{l}\in\{0,1\}^m\\ l_1\boxplus\cdots\boxplus l_m=l}} \bigwedge_{r=1}^{m} \mathbb{C}(\{b_r/x_r\})(l_r)\Big) = (\mathbb{C}(A) +^{\wedge} \mathbb{C}(B))(i),$$

where

$$(\mathbb{C}(\{a_r/x_r\})(k'_r) \wedge \mathbb{C}(\{b_r/x_r\})(l'_r)) \vee (\mathbb{C}(\{a_r/x_r\})(l'_r) \wedge \mathbb{C}(\{b_r/x_r\})(k'_r)) \geq \mathbb{C}(\{a_r/x_r\})(k_r) \wedge \mathbb{C}(\{b_r/x_r\})(l_r)$$

follows from the fact that $k_r = k'_r$ and $l_r = l'_r$ for $k_r \leq l_r$ and $k_r = l'_r$ and $l_r = k'_r$ for $k_r > l_r$. Hence, (25) is proved. Let us suppose that $\mathbf{L}$ is a linearly ordered residuated lattice and $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Denote $X_A = \{x \in \mathrm{Supp}(A \cup B) \mid A(x) < B(x)\}$ and $X_B = \{x \in \mathrm{Supp}(A \cup B) \mid A(x) \geq B(x)\}$. According to the axiom of additivity, we have

$$\mathbb{C}(A) +^{\wedge} \mathbb{C}(B) = \sum_{x\in X_A}^{\wedge} \mathbb{C}(\{A(x)/x\}) +^{\wedge} \sum_{x\in X_B}^{\wedge} \mathbb{C}(\{A(x)/x\}) +^{\wedge}$$
$$\sum_{x\in X_A}^{\wedge} \mathbb{C}(\{B(x)/x\}) +^{\wedge} \sum_{x\in X_B}^{\wedge} \mathbb{C}(\{B(x)/x\}) = \sum_{x\in X_A}^{\wedge} \mathbb{C}(\{A(x)/x\}) +^{\wedge}$$
$$\sum_{x\in X_B}^{\wedge} \mathbb{C}(\{B(x)/x\}) +^{\wedge} \sum_{x\in X_A}^{\wedge} \mathbb{C}(\{B(x)/x\}) +^{\wedge} \sum_{x\in X_B}^{\wedge} \mathbb{C}(\{A(x)/x\}) =$$
$$\sum_{x\in \mathrm{Supp}(A\cup B)}^{\wedge} \mathbb{C}(\{A(x) \wedge B(x)/x\}) +^{\wedge} \sum_{x\in \mathrm{Supp}(A\cup B)}^{\wedge} \mathbb{C}(\{A(x) \vee B(x)/x\}) =$$
$$\mathbb{C}(A \cap B) +^{\wedge} \mathbb{C}(A \cup B),$$

where e.g. $\sum^{\wedge}_{x\in X_A} \mathbb{C}(\{A(x)/x\}) = \mathbb{C}(\{A(x_1)/x_1\}) +^{\wedge} \cdots +^{\wedge} \mathbb{C}(\{A(x_k)/x_k\})$ for $X_A = \{x_1, \dots, x_k\}$. Hence, the valuation property is satisfied for all finite $\mathbf{L}$-fuzzy sets, whenever $\mathbf{L}$ is linearly ordered, and thus the proof is completed. □

In the cardinal theory there is a fundamental relation between the cardinality of sets and bijective mappings. More precisely, two sets has the same cardinality if and only if there exists a bijective mapping between them. Note that we usually say that two sets are equipotent or also equipollent, if there exists a bijective mapping between them. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary $\mathbf{L}$-fuzzy sets. We say that A and B be equipotent, if there exists a bijective mapping $f : \mathbb{X} \to \mathbb{X}$ such that $A(x) = B(f(x)$ for each $x \in \mathbb{X}$. The fact that A and B are equipotent will be denote by $A \equiv B$.

Theorem 30 *Let $\mathbb{C}$ be an $\odot$-cardinality of finite $\mathbf{L}$-fuzzy sets and $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. If $A \equiv B$, then $\mathbb{C}(A) = \mathbb{C}(B)$.*

PROOF. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary and $A \equiv B$. Then there exists a bijective mapping $f : \mathbb{X} \to \mathbb{X}$ such that $A(x) = B(f(x))$ for each $\mathbb{X}$. Hence, obviously $f(\mathrm{Supp}(A)) = \mathrm{Supp}(B)$ and for each $i \in \mathbb{N}_n$ we have (according to Theorem 18)

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) =$$
$$\bigvee_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigodot_{k=1}^{m} \mathbb{C}(\{B(f(x_k))/f(x_k)\})(i_k) = \mathbb{C}(B)(i),$$

where $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$. Thus $\mathbb{C}(A) = \mathbb{C}(B)$ and the statement is proved. □

Remark 31 *Obviously, the opposite implication in the previous theorem is not true in general (cf. [29]).*

5 Axiomatic approach to $\overline{\odot}$-cardinalities of finite L- sets

In this section, we try to define another axiomatic system for cardinalities of **L**-fuzzy sets which is, in a certain sense, dual to the previous one. These cardinalities are then called $\overline{\odot}$-cardinalities. They were proposed as a way to generalize the scalar cardinalities of fuzzy sets. The idea was based on the fact that the operation addition which is used in the computation of scalar cardinalities can be described as an addition in *rdr*-lattices. Nevertheless, the axiomatic system presented below contains a broader family of $\overline{\odot}$-cardinalities of finite **L**-fuzzy sets and the scalar cardinalities could be understood as special elements of this family. The following subsections keep the same outline as the previous ones and the statements have the "dual" forms. Therefore, some of the comments and proofs are omitted.

5.1 Definition and examples

Recall that the dual divisibility (and thus the distributivity of $\vee$ over $\wedge$ is held) of **L** in the case $\mathcal{CV}^{\vee}_{\mathbf{L}^{\mathrm{rd}}}(\mathbb{N}_n)$ is always assumed and it will not be mentioned in the following text.

Definition 32 *Let* **L** *be a complete rdr-lattice. A mapping* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *is an* $\overline{\odot}$*-cardinality of* **L***-fuzzy sets, if it satisfies the following axioms for arbitrary* $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$

($\overline{C}$1) if $A \cap B = \emptyset$*, then* $\mathbb{C}(A \cup B) = \mathbb{C}(A) +^{\overline{\odot}} \mathbb{C}(B)$.
($\overline{C}$2) if $i, j \in \mathbb{N}_n$ *and* $i > |\mathrm{Supp}(A)|$, $j > |\mathrm{Supp}(B)|$*, then* $\mathbb{C}(A)(i) = \mathbb{C}(B)(j)$,

($\overline{C}$3) *if A is a crisp set, then $\mathbb{C}(A)$ is a crisp set and $\mathbb{C}(A)(|A| \boxdot 1) = \bot$,*
($\overline{C}$4) *if $a \in L$, $x, y \in \mathbb{X}$ and $i \in \mathbb{N}_n$, then $\mathbb{C}(\{a/x\})(i) = \mathbb{C}(\{a/y\})(i)$,*
($\overline{C}$5) *if $a, b \in L$, then*

$$\mathbb{C}(\{a \odot b/x\})(1) = \mathbb{C}(\{a/x\})(1) \,\overline{\odot}\, \mathbb{C}(\{b/x\})(1), \tag{28}$$
$$\mathbb{C}(\{a \,\overline{\odot}\, b/x\})(0) = \mathbb{C}(\{a/x\})(0) \,\overline{\odot}\, \mathbb{C}(\{b/x\})(0). \tag{29}$$

The mentioned axioms are again called the *additivity*, *variability*, *consistency*, *singleton independency*, *preservation of non-existence* and *existence*, respectively. The first two axioms have the same meaning as the axioms of additivity and variability for the $\odot$-cardinalities. The axiom of consistency also states that the values of $\overline{\odot}$-cardinalities must belong to $\{\bot, \top\}$ for the crisp sets. However, if $A \subseteq \mathbb{X}$ is a crisp set then we have $\mathbb{C}(A)(|A| \boxdot 1) = \bot$, contrary to $\mathbb{C}(A)(|A| \boxdot 1) = \top$ for the $\odot$-cardinalities. This value could be interpreted as a truth value of the statement that the crisp set A has not the cardinality which is equal to $|A|$. For instance, the cardinality of the empty set is equal to 0 and therefore $\mathbb{C}(\emptyset)(0) = \bot$ holds for any $\overline{\odot}$-cardinality. A consequence of the mentioned consideration is the fact that the membership degree of the $\overline{\odot}$-cardinality for a finite **L**-fuzzy set A in the value i expresses an extent of "truth" of the statement that the **L**-fuzzy set A has not i elements. In other words, the value $\mathbb{C}(A)(i)$ describes the degree of "truth" of the statement that A has either more or less (but not equal) elements than i. Further, the axiom of singleton independency has the same meaning as in the case of the $\odot$-cardinalities. The last two axioms are proposed to be dual to the corresponding axioms of preservation of non-existence and existence for the $\odot$-cardinalities. Moreover, these axioms support our interpretation of the $\overline{\odot}$-cardinalities values. In fact, let us suppose that $a \leq b$ are elements of a linearly ordered *rdr*-lattice and $\overline{\odot} = \vee$. From $a \leq b$ we can say that the **L**-fuzzy set $B = \{b/x\}$ has more elements than the **L**-fuzzy set $A = \{a/x\}$ [1]. Hence, it is natural to expect that the degree of truth of the statement that the **L**-fuzzy set has no element must be greater for B than A and thus $\mathbb{C}(A)(0) \leq \mathbb{C}(B)(0)$. It is easy to see that this inequality is, however, a simple consequence of the axiom of existence preservation. Analogously, the degree of truth of the statement that the **L**-fuzzy set has not just one element must be greater for A than B and thus $\mathbb{C}(A)(1) \geq \mathbb{C}(B)(1)$. Again, this inequality is a simple consequence of the axiom of non-existence preservation.

The following propositions are examples of $\overline{\odot}$-cardinalities of finite **L**-fuzzy sets.

Proposition 33 *Let* **L** *be a complete dually divisible rdr-lattice. Then a map-*

[1] It is a consequence of the fact that the element x belongs to B with the greater membership degree than to A.

ping $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *defined for each* $i \in \mathbb{N}_n$ *by*

$$\mathbb{C}(A)(i) = \begin{cases} \bigwedge\{a \mid a \in L \text{ and } |\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq i\}, & i \neq n \\ \bot, & i = n, \end{cases} \tag{30}$$

where A^{d}_a *denotes the dual* a*-cut, is a* $\vee$*-cardinality of finite* **L***-fuzzy sets.*

PROOF. Note that the definition (30) could be simplified for $n = \omega$ and we can write only

$$\mathbb{C}(A)(i) = \bigwedge\{a \mid a \in L \text{ and } |\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq i\}, \tag{31}$$

because $\mathbb{C}(A)(\omega) = \bigwedge\{a \mid a \in L \text{ and } |\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq \omega\} = \bot$ holds for any $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Let us show that $\mathbb{C}$ is a $\vee$-cardinality of finite **L**-fuzzy sets. Let $i, j \in \mathbb{N}_m$ be arbitrary such that $i \leq j$. If $|\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq i$ for some $a \in L$ then also $|\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq j$. Hence, we obtain

$$\mathbb{C}(A)(j) = \bigwedge_{\substack{a \in L \\ |\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq j}} \leq \bigwedge_{\substack{a \in L \\ |\mathbb{X} \setminus A^{\mathrm{d}}_a| \leq i}} = \mathbb{C}(A)(i)$$

whenever $i \leq j$, and thus $\mathbb{C}(A)$ is the $\vee$-convex **L**-fuzzy set for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary pairwise disjoint finite **L**-fuzzy sets. If $i = n$, then we have

$$(\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(n) = \bigwedge_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = n}} (\mathbb{C}(A)(k) \vee \mathbb{C}(B)(l)) = \mathbb{C}(A)(n) \vee \mathbb{C}(B)(n) = \bot.$$

Hence, we obtain $(\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(n) = \mathbb{C}(A \cup B)(n)$ for all disjoint **L**-fuzzy sets $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$. Further, let us suppose that $i < n$. Since $A \cap B = \emptyset$ and thus $A(x) = \bot$ or $B(x) = \bot$ holds for any $x \in \mathbb{X}$, then we have $\mathbb{X} \setminus (A \cup B)^{\mathrm{d}}_c = \{x \in \mathbb{X} \mid A(x) \vee B(x) \not\leq c\} = \{x \in \mathbb{X} \mid A(x) \not\leq c \text{ or } B(x) \not\leq c\} = \{x \in \mathbb{X} \mid A(x) \not\leq c\} \cup \{x \in \mathbb{X} \mid B(x) \not\leq c\} = \mathbb{X} \setminus A^{\mathrm{d}}_c \cup \mathbb{X} \setminus B^{\mathrm{d}}_c$. Moreover, obviously $\mathbb{X} \setminus A^{\mathrm{d}}_c \subseteq \mathbb{X} \setminus A^{\mathrm{d}}_\bot = \mathrm{Supp}(A)$ and $\mathbb{X} \setminus B^{\mathrm{d}}_c \subseteq \mathbb{X} \setminus B^{\mathrm{d}}_\bot = \mathrm{Supp}(B)$ and thus $\mathbb{X} \setminus A^{\mathrm{d}}_c \cap \mathbb{X} \setminus B^{\mathrm{d}}_c = \emptyset$. Hence, we obtain $|\mathbb{X} \setminus (A \cup B)^{\mathrm{d}}_c| = |\mathbb{X} \setminus A^{\mathrm{d}}_c| + |\mathbb{X} \setminus B^{\mathrm{d}}_c|$ for arbitrary disjoint **L**-fuzzy sets $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ and $c \in L$. Moreover, $\mathbb{X} \setminus A^{\mathrm{d}}_a \subseteq \mathbb{X} \setminus A^{\mathrm{d}}_b$, whenever $a \geq b$. Further, we have $i = k \boxplus l = k + l$ for all $i < n$. Hence, we can write for each $i < n$ (according to the dual divisibility of $\mathbf{L}$)

$$(\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(i) = \bigwedge_{\substack{k,l\in\mathbb{N}_n \\ k+l=i}} (\mathbb{C}(A)(k) \vee \mathbb{C}(B)(l)) =$$

$$\bigwedge_{\substack{k,l\in\mathbb{N}_n \\ k+l=i}} (\bigwedge_{\substack{a\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_a|\le k}} a \vee \bigwedge_{\substack{b\in L \\ |\mathbb{X}\setminus B^{\mathrm{d}}_b|\le l}} b) = \bigwedge_{\substack{k,l\in\mathbb{N}_n \\ k+l=i}} \bigwedge_{\substack{a\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_a|\le k}} \bigwedge_{\substack{b\in L \\ |\mathbb{X}\setminus B^{\mathrm{d}}_b|\le l}} (a \vee b) \ge$$

$$\bigwedge_{\substack{a,b\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_{a\vee b}|+|\mathbb{X}\setminus B^{\mathrm{d}}_{a\vee b}|\le i}} (a \vee b) = \bigwedge_{\substack{c\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_c|+|\mathbb{X}\setminus B^{\mathrm{d}}_c|\le i}} c = \bigwedge_{\substack{c\in L \\ |\mathbb{X}\setminus (A\cup B)^{\mathrm{d}}_c|\le i}} c = \mathbb{C}(A\cup B)(i).$$

Conversely, let $|\mathbb{X} \setminus (A \cup B)^{\mathrm{d}}_c| \le i$ for some $c \in L$. Then obviously there exist $k_c, l_c \in \mathbb{N}_n$ such that $|\mathbb{X} \setminus A^{\mathrm{d}}_c| \le k_c$, $|\mathbb{X} \setminus B^{\mathrm{d}}_c| \le l_c$ and $k_c + l_c = i$. Hence, we obtain the following inequality

$$\bigwedge_{\substack{a\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_a|\le k_c}} a \vee \bigwedge_{\substack{b\in L \\ |\mathbb{X}\setminus B^{\mathrm{d}}_b|\le l_c}} b \le c \vee c = c.$$

Since there exist $k_c, l_c \in \mathbb{N}_n$ with the considered properties to each $c \in L$ with $|X \setminus (A \cup B)^{\mathrm{d}}_c| \le i$, then we can write

$$\mathbb{C}(A\cup B)(i) = \bigwedge_{\substack{c\in L \\ |X\setminus (A\cup B)^{\mathrm{d}}_c|\le i}} c \ge$$

$$\bigwedge_{\substack{k,l\in\mathbb{N}_n \\ k\boxplus l=i}} (\bigwedge_{\substack{a\in L \\ |\mathbb{X}\setminus A^{\mathrm{d}}_a|\le k}} a \vee \bigwedge_{\substack{b\in L \\ |\mathbb{X}\setminus B^{\mathrm{d}}_b|\le l}} b) = (\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(i)$$

and thus $\mathbb{C}$ satisfies the axiom of additivity. Further, if $i > |\mathrm{Supp}(A)|$, then $|\mathbb{X}\setminus A^{\mathrm{d}}_{\bot}| = |\mathrm{Supp}(A)| < i$ and thus $\mathbb{C}(A)(i) = \bot$. Hence, the axiom of variability is fulfilled. Let $A \subseteq \mathbb{X}$ be a crisp set. If $i < |A|$, then $\mathbb{C}(A)(i) = \bigwedge\{a \in L \mid |\mathbb{X} \setminus A^{\mathrm{d}}_a| \le i\} = \top \in \{\bot, \top\}$. Moreover, if $|A| = i < n$, then $|\mathbb{X} \setminus A^{\mathrm{d}}_{\bot}| = i \le i$ and thus $\mathbb{C}(A)(|A| \boxdot 1) = \mathbb{C}(A)(i) = \bot$. If $|A| = i \ge n$, then $\mathbb{C}(A)(i \boxdot 1) = \mathbb{C}(A)(n) = \bot$. Hence, the axiom of consistency is also satisfied. The singleton independency is clearly fulfilled. Finally, we have $\mathbb{C}(\{a/x\})(0) = a$, since $|\mathbb{X} \setminus \{a/x\}^{\mathrm{d}}_a| = 0$ and a is the least element with the desired property, and $\mathbb{C}(\{a/x\})(1) = \bot$, since $|\mathbb{X} \setminus \{\bot/x\}^{\mathrm{d}}_{\bot}| = 0$ and $\mathbb{X} \setminus \{a/x\}^{\mathrm{d}}_{\bot}| = 1$ for any $a > \bot$. Hence, we have $\mathbb{C}(\{a \wedge b/x\}(1) = \bot = \mathbb{C}(\{a/x\})(1) \vee \mathbb{C}(\{b/x\})(1)$ and $\mathbb{C}(\{a \vee b/x\}(0) = a \vee b = \mathbb{C}(\{a/x\})(0) \vee \mathbb{C}(\{b/x\})(0)$. Thus the preservation of non-existence and the preservation of existence are also fulfilled and the proof is finished. □

Remark 34 *The above defined* $\vee$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets is a dual cardinality to the* $\wedge$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets* $\mathbb{C}_0$ *introduced in Proposition 12.*

Let $\{a_i \mid i \in I\}$ be an index set, where I is a finite set (possibly empty). Recall that $\bigoplus_{i\in I} a_i = \bot$, if $I = \emptyset$, and $\bigoplus_{i\in I} a_i = a_{i_1} \oplus \cdots \oplus a_{i_n}$, if $I = \{i_1, \ldots, i_n\}$.

Proposition 35 *Let* $\mathbf{L}$ *be a complete rdr-lattices. Then a mapping* $\mathbb{C}_{\mathrm{sc}} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}_{\mathbf{L},1}$ *defined as follows*

$$\mathbb{C}_{\mathrm{sc}}(A)(i) = \begin{cases} \bigoplus_{x\in \mathrm{Supp}(A)} A(x), & i = 0, \\ \bot, & i = 1, \end{cases} \tag{32}$$

is an $\oplus$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets.*

PROOF. Let $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary. Then obviously $\mathbb{C}_{\mathrm{sc}}(A)$ is trivially the $\oplus$-convex $\mathbf{L}$-fuzzy set. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ be arbitrary disjoint $\mathbf{L}$-fuzzy sets. Since $\mathrm{Supp}(A \cup B) = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$, then we have

$$(\mathbb{C}_{\mathrm{sc}}(A) +^{\oplus} \mathbb{C}_{\mathrm{sc}}(B))(0) = \bigwedge_{\substack{k,l\in\{0,1\}\\ k\boxplus l=0}} \mathbb{C}_{\mathrm{sc}}(A)(k) \oplus \mathbb{C}_{\mathrm{sc}}(B)(l) =$$
$$\mathbb{C}_{\mathrm{sc}}(A)(0) \oplus \mathbb{C}_{\mathrm{sc}}(B)(0) = \bigoplus_{x\in\mathrm{Supp}(A)} A(x) \oplus \bigoplus_{y\in\mathrm{Supp}(B)} B(y) =$$
$$\bigoplus_{x\in\mathrm{Supp}(A\cup B)} (A \cup B)(x) = \mathbb{C}_{\mathrm{sc}}(A \cup B)(0)$$

and

$$(\mathbb{C}_{\mathrm{sc}}(A) +^{\oplus} \mathbb{C}_{\mathrm{sc}}(B))(1) = \bigwedge_{\substack{k,l\in\{0,1\}\\ k\boxplus l=1}} \mathbb{C}_{\mathrm{sc}}(A)(k)\oplus\mathbb{C}_{\mathrm{sc}}(B)(l) =$$
$$\mathbb{C}_{\mathrm{sc}}(A)(1) \oplus \mathbb{C}_{\mathrm{sc}}(B)(1) = \bot \oplus \bot = \bot = \mathbb{C}_{\mathrm{sc}}(A \cup B)(1)$$

and hence the axiom of additivity is satisfied. Obviously, if $i > |\mathrm{Supp}(A)|$ and simultaneously $j > |\mathrm{Supp}(B)|$ hold for some $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$, then $i = j = 1$ and $A = B = \emptyset$. Hence, we have $\mathbb{C}(\emptyset)(1) = \mathbb{C}(\emptyset)(1) = \bot$ and thus the axiom of variability is satisfied. If $A \subseteq \mathbb{X}$ is a crisp set, then $\mathbb{C}(A)(0)$ and $\mathbb{C}(A)(1)$ clearly belong to $\{\bot, \top\}$. Moreover, $\mathbb{C}_{\mathrm{sc}}(\emptyset)(|\emptyset| \boxdot 1) = \mathbb{C}_{\mathrm{sc}}(\emptyset)(0) = \bot$ and if $|A| > 0$, then $\mathbb{C}_{\mathrm{sc}}(A)(|A| \boxdot 1) = \mathbb{C}_{\mathrm{sc}}(A)(1) = \bot$. Hence, the axiom of consistency is satisfied. Finally, we have $\mathbb{C}_{\mathrm{sc}}(\{a/x\})(0) = a$ and $\mathbb{C}_{\mathrm{sc}}(\{a/x\})(1) = \bot$ for any $x \in \mathbb{X}$ and $a \in L$. Hence, the singleton independency is fulfilled. Furthermore, we have $\mathbb{C}_{\mathrm{sc}}(\{a \oplus b/x\})(0) = a \oplus b = \mathbb{C}_{\mathrm{sc}}(\{a/x\})(0) \oplus \mathbb{C}_{\mathrm{sc}}(\{b/x\})(0)$ and $\mathbb{C}_{\mathrm{sc}}(\{a \otimes b/x\})(1) = \bot = \mathbb{C}_{\mathrm{sc}}(\{a/x\})(1) \oplus \mathbb{C}_{\mathrm{sc}}(\{b/x\})(1)$. Hence, the axiom of preservation of existence and non-existence is also satisfied and $\mathbb{C}_{\mathrm{sc}}$ is an $\oplus$-cardinality of finite $\mathbf{L}$-fuzzy sets. □

Remark 36 *Obviously, the above defined* $\oplus$*-cardinality is independent on the operation of multiplication, because the axiom of non-existence preservation is trivially satisfied.*

Remark 37 *It is easy to see that if we consider the rdr-lattice from Example 5 then the scalar cardinality of a fuzzy set* $A : \mathbb{X} \to [0, 1]$ *introduced by*

De Luca and Termini in [5] can be defined as the membership value of $\mathbb{C}_{\mathrm{sc}}(A)$ *for the element* 0, *i.e.* $|A| = \mathbb{C}_{\mathrm{sc}}(A)(0)$. *Analogously, we can define further examples of scalar cardinalities, where different operations of rdr-lattices are supposed. Note that not all scalar cardinalities introduced by Wygralak's axiomatic system can be established using* $\overline{\odot}$*-cardinalities. The reason is that the* $\overline{\odot}$*-cardinalities have to satisfy the axioms of non-existence and existence preservations which are much stronger than the Wygralak's axiom of singleton monotonicity (cf. [28, 30]). On the other hand, using the presented approach a new forms of the scalar cardinalities could be introduced.*

Recall that E denotes the neutral element in $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$. Further, let us define $\mathbf{0}(i) = 0$ for each $i \in \mathbb{N}_n$. Obviously, $\mathbf{0}$ is the least element of $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ w.r.t. the ordering $\leq$ of $\mathbf{L}$-fuzzy sets in $\mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$.

Proposition 38 *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *be an* $\odot$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets. Then* $\mathbb{C}(\emptyset) = E$ *or* $\mathbb{C}(\emptyset) = \mathbf{0}$. *If* $\mathbb{C}(\emptyset) = \mathbf{0}$, *then* $\mathbb{C}(A)(i) \leq \mathbb{C}(A)(j)$ *holds for each* $i, j \in \mathbb{N}_n$ *such that* $i \geq j$.

PROOF. According to the consistency, we have $\mathbb{C}(\emptyset)(i) \in \{\bot, \top\}$ for each $i \in \mathbb{N}_n$ and $\mathbb{C}(\emptyset)(0) = \bot$. A consequence of the variability is that $\mathbb{C}(\emptyset)(i) = \mathbb{C}(\emptyset)(j)$ for every $i, j > 0$. Hence, we have either $\mathbb{C}(\emptyset)(i) = \bot$ for each $i > 0$ or $\mathbb{C}(\emptyset)(i) = \top$ for each $i > 0$ and thus $\mathbb{C}(\emptyset) = E$ or $\mathbb{C}(\emptyset) = \mathbf{0}$. Let $\mathbb{C}(\emptyset) = \mathbf{0}$. Then we have

$$\mathbb{C}(A)(i) = \mathbb{C}(A \cup \emptyset)(i) = (\mathbb{C}(A) +^{\overline{\odot}} \mathbb{C}(\emptyset))(i) = \bigwedge_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} (\mathbb{C}(A)(k) \,\overline{\odot}\, \mathbb{C}(\emptyset)(l)) = \bigwedge_{\substack{k,l \in \mathbb{N}_n \\ k \boxplus l = i}} (\mathbb{C}(A)(k) \,\overline{\odot}\, \bot) = \bigwedge_{\substack{k \in \mathbb{N}_n \\ k \leq i}} \mathbb{C}(A)(k)$$

for each $i \in \mathbb{N}_n$. Hence, we obtain $\mathbb{C}(A)(i) \leq \mathbb{C}(A)(j)$ for each $i, j \in \mathbb{N}_n$ such that $i \geq j$. $\square$

Theorem 39 *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\odot}_{\mathbf{L},\mathrm{n}}$ *be an* $\overline{\odot}$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets and* $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ *such that* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} \subseteq \mathbb{X}$. *Then we have*

$$\mathbb{C}(A)(i) = \bigwedge_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \overline{\bigodot_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{33}$$

for each $i \in \mathbb{N}_n$, *where* $i \leq m$. *Moreover, if* $m < n$, *then* $\mathbb{C}(A)(i) = \bot$ *or* $\mathbb{C}(A)(i) = \top$ *holds for each* $m < i \leq n$, *respectively.*

PROOF. It could be done by analogy to the proof of Theorem 18.

Before we introduce an analogical representation of the $\overline{\odot}$-cardinalities of finite $\mathbf{L}$-fuzzy sets, we introduce two further types of homomorphisms between reducts of rdr-lattices. Let $\mathbf{L}_i = \langle L_i, \wedge_i, \vee_i, \otimes_i, \rightarrow_i, \oplus_i, \ominus_i, \bot_i, \top_i \rangle$, where $i = 1, 2$, be arbitrary rdr-lattices and $h : L_1 \rightarrow L_2$ be a mapping. We say that h is an $\overline{\odot}$-*homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2$, if h is an homomorphism from the reduct $(L_1, \overline{\odot}_1, \bot_1)$ of the rdr-lattice $\mathbf{L}_1$ to the reduct $(L_2, \overline{\odot}_2, \bot_2)$ of the rdr-lattice $\mathbf{L}_2$, i.e. $h(a\,\overline{\odot}_1\,b) = h(a)\,\overline{\odot}_2\,h(b)$ and $h(\bot_1) = \bot_2$. Obviously, each homomorphism between rdr-lattices (or dually residuated lattices which are the reducts of original rdr-lattices) is also an $\overline{\odot}$-homomorphism. Further, we say that h is an $\odot_{\mathrm{d}}$-*homomorphism*, if h is a homomorphism from the reduct $(L_1, \odot_1, \top_1)$ of the rdr-lattice $\mathbf{L}_1$ to the reduct $(L_2, \overline{\odot}_2, \bot_2)$ of the rdr-lattice $\mathbf{L}_2$, i.e. $h(a \odot_1 b) = h(a)\,\overline{\odot}\,h(b)$ and $h(\top_1) = \bot_2$. Again, each homomorphism between rdr-lattices (or homomorphism from a residuated lattice to a dually residuated lattice which are the reducts of original rdr-lattices) is also an $\odot_{\mathrm{d}}$-homomorphism.

Lemma 40 *Let* $\mathbf{L}$ *be a complete* rdr*-lattice and* $f, g : L \rightarrow L$ *be* $\overline{\odot}$*- and* $\odot_{\mathrm{d}}$*-homomorphisms from* $\mathbf{L}$ *to* $\mathbf{L}$ *such that* $f(\top) \in \{\bot, \top\}$ *and* $g(\bot) \in \{\bot, \top\}$. *Let* $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \rightarrow \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *be a mapping defined by the induction as follows*

$$\mathbb{C}_{f,g}(\{a/x\})(0) = f(a),\ \mathbb{C}_{f,g}(\{a/x\})(1) = g(a) \text{ and}$$
$$\mathbb{C}_{f,g}(\{a/x\})(k) = g(\bot),\ k > 1$$

hold for each singleton $\{a/x\} \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$ *and*

$$\mathbb{C}_{f,g}(A) = \mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\overline{\odot}} \cdots +^{\overline{\odot}} \mathbb{C}_{f,g}(\{A(x_m)/x_m\})$$

holds for each $A \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$, *where* $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$. *Then the mapping* $\mathbb{C}_{f,g}$ *is an* $\overline{\odot}$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets (generated by the* $\overline{\odot}$*- and* $\odot_{\mathrm{d}}$*-homomorphisms* f *and* g*), respectively.*

PROOF. It could be done by analogy to the proof of Lemma 20.

Theorem 41 (Representation of $\overline{\odot}$-cardinality) *Let* $\mathbf{L}$ *be a complete* rdr*-lattice and* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \rightarrow \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *be a mapping which satisfies the axiom of additivity. Then the following statements are equivalent:*

(i) $\mathbb{C}$ *is an* $\overline{\odot}$*-cardinality of finite* $\mathbf{L}$*-fuzzy sets,*
(ii) there exist an $\overline{\odot}$*-homomorphism* $f : \mathbf{L} \rightarrow \mathbf{L}$ *and an* $\odot_{\mathrm{d}}$*-homomorphism* $g : \mathbf{L} \rightarrow \mathbf{L}$, *such that* $f(\top) \in \{\bot, \top\}$, $g(\bot) \in \{\bot, \top\}$ *and*

$$\mathbb{C}(\{a/x\})(0) = f(a),\ \mathbb{C}(\{a/x\})(1) = g(a),\ \mathbb{C}(\{a/x\})(k) = g(\bot)$$

hold for arbitrary $a \in L$, $x \in \mathbb{X}$ and $k \in \mathbb{N}_n$, where $k > 1$.

PROOF. It could be done by analogy to the proof of Theorem 21.

5.3 Selected properties

In this section we present some properties of $\overline{\odot}$-cardinalities which are dual, in some sense, to the properties of $\odot$-cardinalities of finite **L**-fuzzy sets. First, we show the preservation or reversation of the ordering relation of **L**-fuzzy sets by $\overline{\odot}$-cardinalities.

We say that an $\overline{\odot}$-homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$ is an $\overline{\odot}$-o-*homomorphism*, if $h(a) \leq h(b)$ holds for arbitrary $a, b \in L_1$ with $a \leq b$. Analogously, we say that an $\odot_{\mathrm{d}}$-homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$ is an $\odot_{\mathrm{d}}$-o-*homomorphism*, if $h(a) \leq h(b)$ holds for arbitrary $a, b \in L_1$ with $a \geq b$. If h is an $\overline{\odot}$-homomorphism such that $h(a) = \bot_2$ for any $a \in L_1$, then obviously h is an example of $\overline{\odot}$-o-homomorphism which will be called the *trivial* $\overline{\odot}$-homomorphism. Analogously, if h is an $\odot_{\mathrm{d}}$-homomorphism such that $h(a) = \bot$ for any $a \in L_1$, then h is an $\odot_{\mathrm{d}}$-o-homomorphism which will be called the *trivial* $\odot_{\mathrm{d}}$-homomorphism.

An $\overline{\odot}$-cardinality $\mathbb{C}_{f,g}$ that is generated by the trivial $\overline{\odot}$-homomorphism f (g is an $\odot_{\mathrm{d}}$-homomorphism) will be denoted by $\mathbb{C}_g$ and $\mathbb{C}_{f,g}$ that is generated by the trivial $\odot_{\mathrm{d}}$-homomorphism g (f is an $\overline{\odot}$-homomorphism) will be denoted by $\mathbb{C}_f$.

Theorem 42 *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ be an $\overline{\odot}$-cardinality of finite* **L***-fuzzy sets generated by $\overline{\odot}$-o- and $\odot_{\mathrm{d}}$-o-homomorphism f and g, respectively. Then we have*

(i) $\mathbb{C}_{f,g}$ preserves the ordering if and only if g is trivial, i.e. $\mathbb{C}_{f,g} = \mathbb{C}_f$,
(ii) $\mathbb{C}_{f,g}$ reverses the ordering if and only if f is trivial, i.e. $\mathbb{C}_{f,g} = \mathbb{C}_g$.

PROOF. It could be done by analogy to the proof of Theorem 26.

Lemma 43 *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ be a $\vee$-cardinality of finite* **L***-fuzzy sets. Then*

(i) $\mathbb{C}(\{a \vee b/x\})(1) \leq \mathbb{C}(\{a/x\})(1) \wedge \mathbb{C}(\{b/x\})(1)$,
(ii) $\mathbb{C}(\{a \wedge b/x\})(0) \leq \mathbb{C}(\{a/x\})(0) \wedge \mathbb{C}(\{b/x\})(0)$,
(iii) $\mathbb{C}(\{a \vee b/x\})(t) \vee \mathbb{C}(\{a \wedge b/x\})(s) \leq \mathbb{C}(\{a/x\})(t) \vee \mathbb{C}(\{b/x\})(s)$

hold for arbitrary $a, b \in L$, $x \in \mathbb{X}$ and $t, s \in \{0, 1\}$ such that $t \geq s$.

PROOF. It could be done by analogy to the proof of Lemma 28.

The following theorem shows the satisfaction of the valuation property for $\vee$-cardinalities of finite **L**-fuzzy sets.

Theorem 44 *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(\mathbb{X}) \to \mathcal{CV}^{\overline{\odot}}_{\mathbf{L},\mathrm{n}}$ *be a* $\vee$*-cardinality of finite* **L***-fuzzy sets. Then*

$$\mathbb{C}(A \cap B) +^{\vee} \mathbb{C}(A \cup B) \leq \mathbb{C}(A) +^{\vee} \mathbb{C}(B) \qquad (34)$$

holds for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$*. If* **L** *is linearly ordered, then* $\mathbb{C}$ *fulfils the valuation property.*

PROOF. It could be done by analogy to the proof of Theorem 29.

Recall that **L**-fuzzy sets A and B are equipotent (denoted by $A \equiv B$), if there exists a bijective mapping $f : \mathbb{X} \to \mathbb{X}$ such that $A(x) = B(f(x))$ for any $x \in \mathbb{X}$.

Theorem 45 *Let* $\mathbb{C}$ *be an* $\overline{\odot}$*-cardinality of finite* **L***-fuzzy sets and* $A, B \in \mathcal{FIN}_{\mathbf{L}}(\mathbb{X})$*. If* $A \equiv B$*, then* $\mathbb{C}(A) = \mathbb{C}(B)$*.*

PROOF. It could be done by analogy to the proof of Theorem 30.

6 Conclusion

In this paper two axiomatic systems for cardinalities of finite **L**-fuzzy sets were introduced. In particular, the first one generalized the axiomatic systems proposed by J. Casasnovas and J. Torrens in [2]. The second axiomatic system was then defined as a “dual” system to the first one which could give a possibility to describe also some family of scalar cardinalities. We proved that cardinalities of both axiomatic systems can be represented by two adequate homomorphisms between reducts of residuated-dually residuated lattices. Selected properties as the preservation and reversal of the ordering relation of **L**-fuzzy sets by cardinalities, the valuation property and the equality of cardinality for equipotent **L**-fuzzy sets are presented. Construction of cardinality theory based on the proposed axiomatic systems seems to be an interesting topic for further research.

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