

UNIVERSITY OF OSTRAVA

Institute for Research and Applications of Fuzzy Modeling

A Plea for the Usefulness of the Deductive Interpretation of Fuzzy Rules in Engineering Applications

Ulrich Bodenhofer, Martina Daňková, Martin Štěpnička and Vilém
Novák

Research report No. 127

2007

Submitted/to appear:

FUZZ-IEEE 2007

Supported by:

MSM6198898701, DAR 1M0572 of the MŠMT ČR and the *Kplus* Competence Center Program

University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478

e-mail: bodenhofer@bioinf.jku.at, martina.dankova@osu.cz, martin.stepnicka@osu.cz, vilem.novak@osu.cz

Abstract

This contribution is intended as a position paper that favors the viewpoint that inference based on deductive rules (i.e., the rules are interpreted using fuzzy implication) can indeed be considered as a valuable inference scheme in real-world applications. For this purpose, we highlight the basic concepts behind the most common fuzzy inference schemes and demonstrate their interpretation by means of illustrative examples. We conclude that, under some reasonable conditions, deductive inference is able to compete with or even outperform the well-known Mamdani-Assilian inference.

1 Introduction

There is no doubt that fuzzy IF-THEN rules (shortly, fuzzy rules) are *the* central concept in fuzzy systems. Fuzzy rules are usually formulated as statements of the form

$$\text{IF } x \text{ is } \mathcal{A}_i \text{ THEN } y \text{ is } \mathcal{B}_i, \quad i = 1, \dots, n \quad (1)$$

where \mathcal{A}_i and \mathcal{B}_i are certain linguistic expressions [1,2]. Such a set of rules (a so-called *fuzzy rule base*) is commonly interpreted as a characterization of dependencies between a given input and the desired output of the fuzzy system (for other interpretations, see, for instance, [3]). Common interpretations model the rule base (1) as a *fuzzy relation*.

Note that, given an *interpretation* of a fuzzy rule base as a fuzzy relation and an actual input that can be either crisp value or a fuzzy set, the process of *inference* can proceed whose result is an output fuzzy set. In this paper, we only consider crisp inputs and so inference is nothing else but computing cuts of the fuzzy relation which models the rule base.

From a naive viewpoint, it seems near at hand to interpret fuzzy rules as some kinds of logical implications. It is well-known, though, that this is widely not the case. A vast majority of existing fuzzy systems makes use of *Mamdani-Assilian interpretation* [4] or the closely related interpolation-based *Takagi-Sugeno model* [5]. Inference schemes based on fuzzy rules interpreted as logical implications—let us call this *deductive interpretation* in the following—are also well-known from literature [6–9], but they have been only rarely applied.

So, what may be the reasons why deductive inference has been neglected in applications? Surely, large parts of the fuzzy control community do not even know about approaches other than Mamdani-Assilian and Takagi-Sugeno inference. Many textbooks do not even mention deductive inference. To our best knowledge, LFLC [10,11] is the only fuzzy systems software that supports deductive inference¹, while the common commercial software products do not. On the other hand, there are certainly fuzzy control researchers and engineers who know about deductive interpretation, but who are convinced that Mamdani-Assilian and Takagi-Sugeno interpretation provide better results. So, this paper serves for two purposes: (1) to provide a basic engineering-oriented introduction to deductive inference and (2) to clarify common misconceptions about its properties. We want to convince the readers that deductive inference is, in any case, a serious and meaningful alternative to Mamdani-Assilian inference.

This paper is organized as follows. Section 2 provides an overview of the most common interpretation of fuzzy rules. In Section 3, we motivate our viewpoint by means of illustrative examples. Finally, in Section 4, we conclude that deductive inference is not only a theoretical option, but also a practically feasible alternative to Mamdani-Assilian inference.

2 Overview of Fuzzy Inference Schemes

2.1 Preliminaries

For simplicity, fuzzy rules (1) can be written with one input only. Extension to more inputs is straightforward by taking a vector input and possible technical or computational complications, which can appear in the multivariate case, play no role in the overview of schemes.

We explicitly distinguish between linguistic expressions and the fuzzy sets modeling them. A linguistic expression is always denoted with a calligraphic letter, while the corresponding fuzzy set is denoted by an italic letter. Let us also remark that, in this paper, we will usually deal with fuzzy sets only disregarding any linguistic interpretation.

We assume that the reader knows the basic concepts of triangular norms and conorms [12]. As usual, $x *_{\mathbf{G}} y = \min(x, y)$ is called *minimum t-norm* or *Gödel t-norm*, $x *_{\mathbf{P}} y = x \cdot y$ is called *product t-norm*, and $x *_{\mathbf{L}} y = \max(x +$

¹Even more, this system implements the so-called *perception-based logical deduction* that mimics human logical inference based on knowledge obtained from genuine linguistic description. Let us note that there exist several successful real applications of this system.

$y - 1, 0)$ is called *Lukasiewicz t-norm*. These three basic t-norms are continuous and induce the following *residual implications*:

$$\begin{aligned} x \rightarrow_{\mathbf{G}} y &= \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases} \\ x \rightarrow_{\mathbf{P}} y &= \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{otherwise,} \end{cases} \\ x \rightarrow_{\mathbf{L}} y &= \min(1 - x + y, 1). \end{aligned}$$

For a detailed investigation and justification of residual implications, see [7, 9, 13, 14].

Recall also the following well-known definitions (A is a fuzzy subset of some universe X):

$$\begin{aligned} \text{height}(A) &= \sup\{A(x) \mid x \in X\}, \\ \text{Core}(A) &= \{x \mid A(x) = 1\}, \\ \text{Supp}(A) &= \{x \mid A(x) > 0\}, \\ \text{Ceil}(A) &= \{x \mid A(x) = \text{height}(A)\}. \end{aligned}$$

In this paper, we will use the two most common defuzzification methods: for a given fuzzy set A on a real-valued universe $X \subseteq \mathbb{R}$, provided that $\text{Ceil}(A)$ is non-empty, we define the *mean of maxima* of A as

$$\text{MOM}(A) = \begin{cases} \frac{\sum_{x \in \text{Ceil}(A)} x}{|\text{Ceil}(A)|} & \text{if } \int_{\text{Ceil}(A)} 1 \, dx = 0, \\ \frac{\int_{\text{Ceil}(A)} x \, dx}{\int_{\text{Ceil}(A)} 1 \, dx} & \text{otherwise.} \end{cases}$$

Given a fuzzy set A on $X \subseteq \mathbb{R}$ such that the membership function of A is integrable and $\int_X A(x) \, dx > 0$, the *center of gravity* of A is defined as

$$\text{COG}(A) = \frac{\int_X x \cdot A(x) \, dx}{\int_X A(x) \, dx}.$$

2.2 Mamdani-Assilian inference

Let us first recall the well-known Mamdani-Assilian inference. As noted above, it has been introduced in the seminal paper [4], but the concept has been foreshadowed already in [15]. In Mamdani-Assilian inference, each rule is interpreted as a Cartesian product of fuzzy sets, then takes the degree to which a rule's condition is fulfilled and computes the output of a rule by truncating the output fuzzy set with this degree. Finally, the output of the rule base is computed by aggregating the output fuzzy sets of the rules using a t-conorm (often the maximum). The last step is usually a defuzzification using the COG method.

More formally, the fuzzy relation associated with the rule base (1) is

$$R(x, y) = \bigvee_{i=1}^n (A_i(x) * B_i(y)). \quad (2)$$

Obviously, $R(x, y)$ is the degree to which y is in the output of the rule base for input x . Moreover, we see that R is nothing else but the disjunction of n fuzzy relations being Cartesian products of input and output fuzzy sets.

The most common choice is to use the minimum (Gödel) t-norm for the Cartesian products (which corresponds to truncating B_i with the degree $A_i(x)$) and to use the maximum for aggregation. This is also the choice used in [4]. Also popular is the combination of maximum for aggregation and product t-norm for building the Cartesian products. Note that it is not so uncommon in applications to use the sum to aggregate the outputs of the rules (which gives rise to *sum-prod inference* and *sum-min inference*). As the sum is not a closed operation on the unit interval, the peculiar situation may occur that the output of the rule is not a fuzzy set anymore. This approach is mainly used in control applications, where a COG defuzzification is used in the sequel anyway. As the COG defuzzification works for all kinds of integrable functions, this is not a practical problem (it remains a conceptual problem though). It is more than obvious that Mamdani-Assilian inference does not make use of any concept of implication.

Instead of interpreting fuzzy rules as logical implications, it builds up an input-output relation from smaller units, and those units are examples of fuzzy input-output pairs. As the conditions of the rules may (or usually do) overlap, we can consider Mamdani-Assilian as an approach that interpolates between known input-output pairs [8, 16].

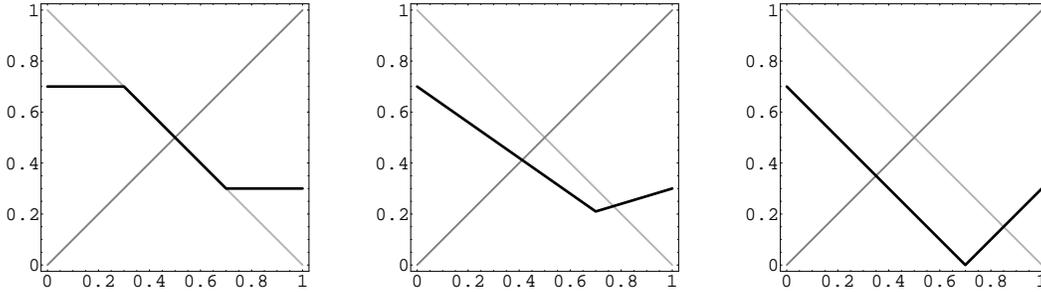


Figure 1: Example 1: output fuzzy sets obtained for $x = 0.3$ with Mamdani-Assilian inference using maximum aggregation, and the three basic t-norms $*_{\mathbf{G}}$ (left), $*_{\mathbf{P}}$ (middle), and $*_{\mathbf{L}}$ (right).

2.3 Deductive inference

We have highlighted above that the terms “IF” and “THEN” may directly support the assumption that implications are the proper concepts to interpret a rule base (1). It is not surprising, therefore, that such inference schemes have been introduced quite early [17, 18] (again, foreshadowed by a seminal work of Zadeh [19]), albeit significantly after Mamdani-Assilian inference.

The interpretation of a single fuzzy rule

$$\text{IF } x \text{ is } \mathcal{A} \text{ THEN } y \text{ is } \mathcal{B} \quad (3)$$

by a fuzzy relation with the membership function $A(x) \rightarrow B(y)$ models a logical constraint in the sense that if the condition \mathcal{A} is true (in some degree) then the conclusion \mathcal{B} is true as well (in a greater or the same degree) but we cannot *deduce* anything about \mathcal{B} if \mathcal{A} is not true. This means that, whenever \mathcal{A} is not true, everything is still possible. So this means that each rule infers constraints on the output from a given input. In the case that a fuzzy rule base consists of n fuzzy rules of the form (3), then all the constraints are valid at the same time and so the interpretation of the whole rule base is modeled by intersection of all individual rule interpretations, i.e., the rules are aggregated by conjunction. Therefore, the terms *deductive interpretation* and *deductive inference* are justified. Similarly as above, the deductive interpretation leads to a fuzzy relation

$$R(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \quad (4)$$

In (4), \rightarrow stands for a fuzzy implication and \wedge is modeled by a t-norm. For specific reasons (for a detailed argumentation, see [7, 9]), it is meaningful to restrict to a residual implication and the minimum/Gödel t-norm for aggregation.

3 Motivating Examples

3.1 Approximation examples

Example 1 As a very simple case, let us first consider a simple rule base that is supposed to simplistically model the identity function on the unit interval. So we consider $x, y \in [0, 1]$ and the following rule base:

$$\begin{aligned} \text{IF } x \text{ is } \mathcal{S} \text{ THEN } y \text{ is } \mathcal{S} \\ \text{IF } x \text{ is } \mathcal{L} \text{ THEN } y \text{ is } \mathcal{L} \end{aligned}$$

Suppose the linguistic labels \mathcal{S} and \mathcal{L} are modeled by the fuzzy sets $S(x) = 1 - x$ and $L(x) = x$, respectively.

Figure 1 shows the resulting output fuzzy sets for $x = 0.3$ if we use Mamdani-Assilian inference with the three basic t-norms and the maximum for aggregation.

Figure 2 shows output fuzzy sets for $x = 0.3$ if we use deductive inference with the residual implications of the three basic t-norms and the minimum for aggregation.

It is obvious—and corresponds to the well-established practice—that the MOM defuzzification is meaningless for the results that are obtained by Mamdani-Assilian inference (regardless which operations we choose). Instead, COG

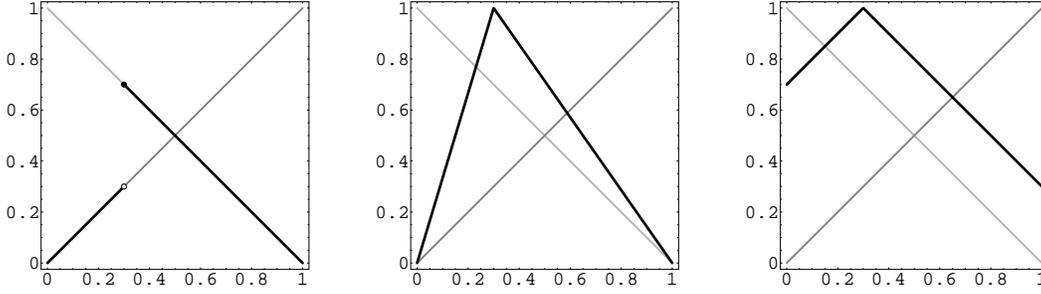


Figure 2: Example 1: output fuzzy sets obtained for $x = 0.3$ with deductive inference using minimum aggregation, and the three fuzzy implications $\rightarrow_{\mathbf{G}}$ (left), $\rightarrow_{\mathbf{P}}$ (middle), and $\rightarrow_{\mathbf{L}}$ (right).

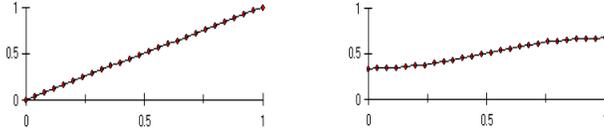


Figure 3: Example 1: input-output function given by deductive inference with MOM and Mamdani-Assilian inference with COG

is the common standard. The opposite is true for the deductive inference. While COG produces inferior results, MOM is better choice.

As can be seen from Figure 3, in Example 1 we obtain the identity function when using deductive inference with MOM (this holds even for any choice of implication), which is not the case for Mamdani-Assilian inference with COG.

Obviously, we do not claim that there is no fuzzy rule base which would provide us with the identity function output based on Mamdani-Assilian inference and COG. But the example was only a motivating visualization demonstrating that Mamdani-Assilian inference does not work as a proper model of fuzzy rules even in very simple and natural rule bases.

Although Example 1 is illustrative, it is too simplistic to draw more profound conclusions. So let us enhance this example a bit.

Example 2 We consider $x, y \in [0, 1]$ again and a simple fuzzy partition consisting of six triangular fuzzy sets that uniformly cover the input domain (i.e. the peaks of the membership functions are 0, 0.2, 0.4, 0.6, 0.8, 1). For the output domain, we also use a fuzzy partition of six triangular fuzzy sets, but the peaks are distributed as 0, 0.04, 0.16, 0.36, 0.64, 1. The rule set is constructed with the intention to approximate the square function.

Figure 4 shows contour plots of the three fuzzy relations that are induced by Mamdani-Assilian inference (cf. (2)) if we use the maximum for aggregation and the three basic t-norms as conjunctions. The graphs also show the final functions that are obtained after COG defuzzification (bold black line). One easily sees the Cartesian products induced by the six rules and how the final output is computed from them as a kind of interpolation. It is also evident, however, that the final functions do not approximate the square function very well.

Figure 5 shows contour plots of the three fuzzy relations induced by deductive inference (cf. (4)) if we use the minimum for aggregation and the residual implications of the three basic t-norms. The graphs moreover show the final function that is obtained after MOM defuzzification (bold black line). We see that the fuzzy relations do not have such an intuitive interpretation as the fuzzy relations induced by Mamdani-Assilian inference (this is particularly true if we look at the case of the Lukasiewicz implication), but the final functions are in all three cases the piecewise linear interpolation of the sampling points induced by the peaks of the membership functions.

3.2 Classification example

Now we consider a simple, yet very illustrative, classification example. Assume we have to classify animals only on the basis of their height h and the number of legs l and assume further that the three crisp non-overlapping classes we consider are dogs, horses and fish, so the output universe is $\{\text{dog, horse, fish}\}$. The first input variable is numerical

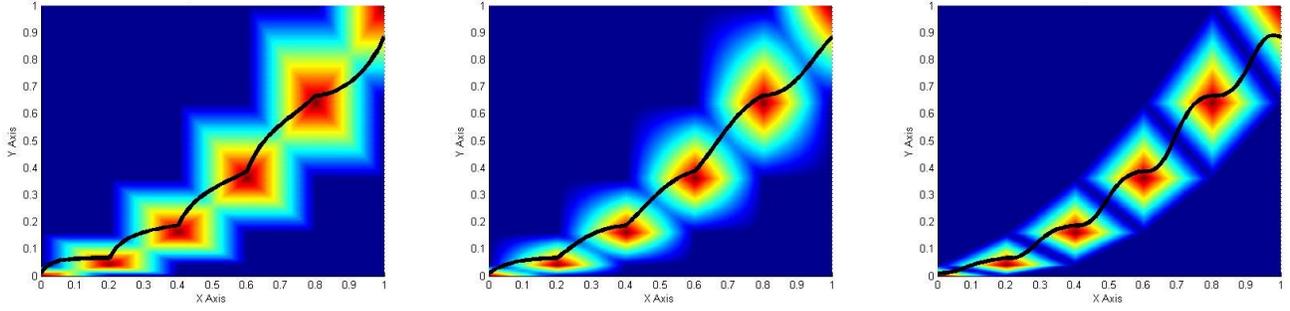


Figure 4: Example 2: fuzzy relations induced by Mamdani-Assilian inference using the maximum for aggregation and the t-norms $*_{\mathbf{G}}$ (left), $*_{\mathbf{P}}$ (middle), and $*_{\mathbf{L}}$ (right).

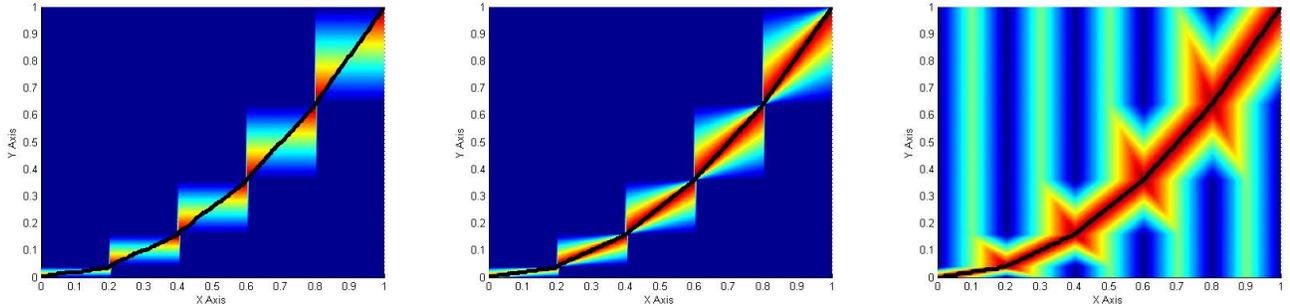


Figure 5: Example 2: fuzzy relations induced by deductive inference using the minimum for aggregation and the fuzzy implications $\rightarrow_{\mathbf{G}}$ (left), $\rightarrow_{\mathbf{P}}$ (middle), and $\rightarrow_{\mathbf{L}}$ (right).

with the range $[0, 300]$ (in cm) and suppose that we have two linguistic expressions \mathcal{S} and \mathcal{L} . The corresponding fuzzy sets are defined as

$$S(x) = \begin{cases} 1 & x < 100 \\ \frac{140-x}{40} & x \in [100, 140] \\ 0 & x > 140 \end{cases}, \quad L(x) = 1 - S(x).$$

Let us consider the following rule base:

$$\text{IF } h \text{ is } \mathcal{S} \text{ AND } l \text{ is } 4 \text{ THEN } c \text{ is } \textit{dog} \tag{R1}$$

$$\text{IF } h \text{ is } \mathcal{L} \text{ AND } l \text{ is } 4 \text{ THEN } c \text{ is } \textit{horse} \tag{R2}$$

$$\text{IF } l \text{ is } 0 \text{ THEN } c \text{ is } \textit{fish} \tag{R3}$$

Now assume that we are given an animal that is 110cm tall and has four legs. If we process these data using max-* inference (since the output classes are crisp and disjoint, the results are independent of the choice of $*$), we obtain that the animal belongs to class *dog* with a degree of 0.75, to the class *horse* with a degree of 0.25 and to class *fish* with a degree of 0. Now let us process the same data using deductive inference. The condition of rule (R1) is fulfilled to a degree of 0.75. Using the Łukasiewicz implication (this time, the choice of the connective does matter!), the output of rule (R1) is the fuzzy set that contains *dog* to a degree of 1 and *horse* and *fish* to a degree of 0.25. The condition of rule (R2) is fulfilled to a degree of 0.25, so the output of rule (R2) is a fuzzy set that contains *horse* to a degree of 1, while it contains *dog* and *fish* to a degree of 0.75. The condition of rule (R3) is not fulfilled at all, so the output is the fuzzy set that contains all three animals to a degree of 1. Finally, if we compute the min-intersection of the three fuzzy sets, we obtain that fuzzy set as output that contains *dog* to a degree of 0.75, *horse* to a degree of 0.25, and *fish* to a degree of **0.25** too.

This example illustrates the fundamental differences of Mamdani-Assilian and deductive inference. Mamdani-Assilian inference gathers evidence in favor of the classes, while deductive inference computes constraints, as noted above. The result that deductive inference yields a degree of 0.25 for class *fish* might appear peculiar, but it is easy to explain: we know that animals with 0 legs are fish, but that does not imply, in turn, that animals having legs are

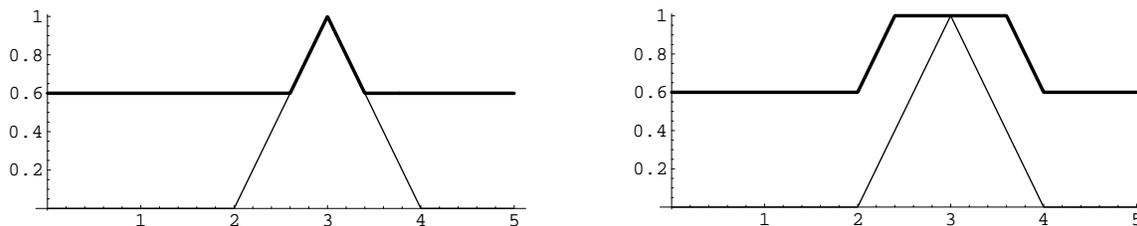


Figure 6: Output of a fuzzy rule whose condition is fulfilled to a degree of 0.4, the output fuzzy set B is triangular (gray in both graphs); the left graph shows the output if deductive inference is employed using $x \rightarrow y = \max(1 - x, y)$ (Kleene-Dienes implication), the right one shows the same for \rightarrow_L .

no fish—not without additional knowledge. So let us add a fourth rule:

$$\text{IF } l \text{ is } 4 \text{ THEN } c \text{ is } \textit{dog} \text{ OR } \textit{horse} \quad (\text{R4})$$

Just like the rules (R1)–(R3), this seems an intuitive piece of knowledge. If we repeat max-min inference with the same data above for the rules (R1)–(R4), we obtain that the considered animal may be *dog* or *horse* to degrees of 1 and *fish* to a degree of 0. So the seemingly intuitive rule has led to significantly less specificity of the final result. In case of deductive inference, this was just the constraint we needed to finally exclude *fish* from the result set, so the result is the fuzzy set that contains *dog* to a degree of 0.75, *horse* to a degree of 0.25, and *fish* to a degree of 0.

Which conclusions can we draw from this example? If a rule base contains rules with overlapping conditions, where some are more and some are less specific, then Mamdani-Assilian inference gives priority to the less specific rules—potentially leading to a loss of information/specificity. If the same situation occurs in deductive inference, the more specific result (constraint) overrides the less specific one.

Furthermore, consider the case that a rule base contains rules that have strongly overlapping conditions but disjoint conclusions. If we use an input from the area where the two conditions overlap to a degree of 1, Mamdani-Assilian inference will yield the union of the two conclusions, while deductive inference will consider this case as a contradictory situation and give a zero output. From that viewpoint, deductive inference implicitly checks the consistency of the rule base.

4 Discussion

Although many authors made considerable efforts to explain that there exist differences in the interpretation and usage of the two different approaches [6–9, 16, 20–22], these efforts seem to have been unheard in the engineering community. Therefore, common misunderstandings sustain and are still distributed.

Misunderstanding 1: t-norms as implications. We have already explained the trivial fact that Mamdani-Assilian inference does not make use of any implication. It should be noted that Mamdani and Assilian [4] did not refer to the notion of implication at all, but later—probably inspired by the naive viewpoint that rules genuinely model implications—wrong and misleading terms like “Mamdani implication”, “minimum implication”, “product implication”, “Larsen implication”, “t-norm implication”, etc. were introduced to denote the t-norms used in Mamdani-Assilian inference (cf. [23–26]). Although this incorrect view has been many times discussed, from the above cited references, we can see that this fact has not been fully adopted yet and therefore it is worth to be stressed.

We should not forget that, as a second side of this misunderstanding, logicians have rejected Mamdani-Assilian inference for quite a long time as an illogical approach, because it does not involve an implication to model rules. In the 1990ies, this misunderstanding was resolved by the works of Hájek and others [7, 9, 22]. They showed that both deductive and Mamdani-Assilian inference have sound logical foundations but from different viewpoints.

Misunderstanding 2: deductive inference gives bad results. The common understanding (exceptions to this oversimplified rule exist and are appreciated) among fuzzy systems engineers is that deductive inference is useless for engineering applications, where intuition is a main argument. Mendel [26], for instance, stresses that the results contradict what he calls “engineering common sense”. He provides an example to illustrate his viewpoint (for a similar example, see Figure 6) and concludes that both fuzzy sets have unbounded support—which does not make sense in his opinion. That may be true if we view the results from the same angle as the results of Mamdani-Assilian inference. The output of deductive rules should better be interpreted as constraints regarding the possibility to which a given value y can be in the output. As noted above already, the COG defuzzification is, therefore, unsuitable for

the output of deductive rules. We have the impression that some opponents of deductive inference have rejected it simply because they considered the shape of the output fuzzy sets meaningless for COG defuzzification.

The examples in Section 3 have clearly confirmed the following facts:

1. Mamdani-Assilian inference is a specific interpolation between fuzzy input-output pairs; deductive inference computes logical constraints.
2. COG defuzzification is suitable for Mamdani-Assilian inference and unsuitable for deductive inference; in turn, MOM defuzzification is suitable for deductive inference and unsuitable for Mamdani-Assilian inference.
3. Deductive inference can handle different levels of specificity of knowledge, whereas Mamdani-Assilian inference cannot (see the classification example).
4. For two contradictory rules, Mamdani-Assilian inference yields the union of the two output fuzzy sets, whereas deductive inference yields a zero output. Hence, deductive inference has a built-in mechanism for detecting inconsistencies in rule bases.
5. If no rule fires (i.e. if all rule conditions are fulfilled only to a degree of 0), Mamdani-Assilian yields a zero output (no evidence for anything), while deductive inference yields a one output (anything possible).
6. Examples 1 and 2 indicate that the local approximation capabilities of deductive inference with MOM can outperform Mamdani-Assilian inference with COG.

From this summary, we dare to conclude that almost none of the arguments that are usually raised against deductive inference are justified. The results in Section 3 have demonstrated that deductive inference has the potential to give results that are as good as those of Mamdani-Assilian inference. In particular, in complex tasks, like in data mining, logic programming, or complex expert systems, the ability of deductive inference to handle different levels of specificity (see classification example), make it a good choice.

Admittedly, one serious problem remains in case that numerical output is needed (e.g. in fuzzy control): MOM does not depend continuously upon its input membership function, therefore, issues regarding the smoothness of the resulting control function may arise. So let us turn our attention to this issue in the following. For all further considerations, *coherence* of the fuzzy relation induced by the rule base will be essential. Coherence means that, for all inputs $x \in X$, there is a $y \in Y$ such that $R(x, y) = 1$ [6, 20]. In other words, coherence requires $\text{Core}(R(x, \cdot))$ to be non-empty.

The coherence property is not a useless issue when dealing with deductive rules. It is a key issue to certify that a given rule base does not contain conflicting rules [20]. Incoherent rule bases with conflicting rules lower the largest membership degrees in the resulting fuzzy relations which is caused by overlapping antecedents and contradictory consequents. A very illustrative example of such a rule base can be found again in [20]:

IF *obstacle* is *left* OR *front* THEN *bypass* is *right*
 IF *obstacle* is *right* OR *front* THEN *bypass* is *left*

In case of having an obstacle in front this rulebase leads to ill behavior no matter which inference is used. Therefore contradiction, which is present in the rule base, not the inference mechanism, is undesirable! Mamdani-Assilian inference does not lower the membership degrees in the resulting fuzzy relations which only hides this feature of a contradiction which is sometimes wrongly understood as an advantage. Vice versa, checking the coherence (possible only for the deductive rule bases) is an effective tool for a detection of possibly conflicting rules.

Proposition 1 *Let R be a fuzzy relation interpreting a fuzzy rule base (1) and let R be coherent. If the functions $\text{IR}, \text{SR} : X \rightarrow Y$ where $\text{IR}(x) = \inf_{y \in Y} \text{Core}(R(x, y))$ and $\text{SR}(x) = \sup_{y \in Y} \text{Core}(R(x, y))$ are continuous, then the output function $\text{MR} : X \rightarrow Y$ of the fuzzy rule based system where $\text{MR}(x) = \text{MOM}(R(x, \cdot))$ is continuous, too.*

Proposition 1 provides us with sufficient (not necessary) conditions yielding the continuity of the output function of a deductive fuzzy rule based system. The question is how to ensure coherence and the continuity of functions IR and SR. Let us leave coherence aside for a moment and concentrate on the second question.

Proposition 2 *Assume we are given fuzzy sets A_i on $X \subseteq \mathbb{R}$ with continuous membership functions and fuzzy sets B_i on $Y \subseteq \mathbb{R}$ whose membership functions are continuous and are strictly monotone to the left and to the right of $\text{Core}(B_i)$ (for all $i = 1, \dots, n$). Furthermore, assume that the fuzzy relation R is given as in (4) and coherent. Then the functions IR and SR are continuous for all $x \notin \bigcup_{i=1}^n (\text{Supp}(A_i) \setminus \text{Supp}(A_i))$.*

Corollary 1 *With the assumptions of Proposition 2 and the additional assumption that $\text{Supp}(A_i) = X$ for all $i = 1, \dots, n$, the output function of the fuzzy rule-based system $\text{MR}(x) = \text{MOM}(R(x, \cdot))$ is a continuous function.*

Corollary 1 is based on removing possible points of discontinuity. It implies that the continuity of the output function can particularly be maintained for a coherent fuzzy relation defined by deductive inference from a fuzzy rule base if Gaussian membership functions or general radial basis functions are used. However, even if $\text{Supp}(A_i) \neq X$, then possible discontinuities of MR can be avoided, as the following proposition demonstrates.

Proposition 3 *With the assumptions of Proposition 2 and the additional assumption that $\text{Supp}(B_i) = Y$ for all $i = 1, \dots, n$, the output function of the fuzzy rule-based system $\text{MR}(x) = \text{MOM}(R(x, \cdot))$ is a continuous function.*

We immediately see from Proposition 3 and Corollary 1 that Gaussian (or GRBF) membership functions either on the antecedent or consequent side ensure that fuzzy rule base (1) interpreted by (4) has a continuous output function. Of course, this only holds with the special technical assumptions of these results—including coherence. How to ensure coherence is the question that we have neglected so far. Note, however, that this has been already studied exhaustively in [20]. Moreover, in [27,28], very encouraging results concerning deductive fuzzy rule bases with Gaussian membership functions can be found. In [27], even an algorithm for automatic learning of coherent deductive inference systems with Gaussian membership functions and other radial basis functions from data is presented.

5 Conclusion

This paper has reviewed the two most important classes of fuzzy inference schemes. We have highlighted arguments that deductive inference is not contradicting engineering common sense. We have seen by examples that good approximation behavior can be obtained and we have successfully addressed the question how continuous output can be maintained for deductive inference with mean of maximum defuzzification. We hope that we can contribute to overcoming common misconceptions, that fuzzy control engineers increasingly pay attention to the alternative that deductive inference, and that fuzzy control researchers increasingly study and advance the topic.

References

- [1] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning I,” *Inform. Sci.*, vol. 8, pp. 199–250, 1975.
- [2] V. Novák, “A comprehensive theory of evaluating linguistic expressions,” Res. Rep. 71, Institute of Research and Applications of Fuzzy Modeling, University of Ostrava, 2005.
- [3] V. Novák and S. Lehmké, “Logical structure of fuzzy IF-THEN rules,” *Fuzzy Sets and Systems*, vol. 157, no. 15, pp. 2003–2029, 2006.
- [4] E. H. Mamdani and S. Assilian, “An experiment in linguistic synthesis with a fuzzy logic controller,” *Int. J. Man-Mach. Stud.*, vol. 7, pp. 1–13, 1975.
- [5] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Trans. Syst. Man Cybern.*, vol. 15, no. 1, pp. 116–132, 1985.
- [6] D. Dubois and H. Prade, “What are fuzzy rules and how to use them,” *Fuzzy Sets and Systems*, vol. 84, pp. 169–185, 1996.
- [7] P. Hájek, *Metamathematics of Fuzzy Logic*, ser. Trends in Logic. Dordrecht: Kluwer Academic Publishers, 1998, vol. 4.
- [8] R. Kruse, J. Gebhardt, and F. Klawonn, *Foundations of Fuzzy Systems*. New York: John Wiley & Sons, 1994.
- [9] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*. Boston: Kluwer Academic Publishers, 1999.
- [10] A. Dvořák, H. Habiballa, V. Novák, , and V. Pavliska, “The concept of LFLC 2000—its specificity, realization and power of applications,” *Comput. Ind.*, vol. 51, pp. 269–280, 2003.

- [11] V. Novák, “Linguistically oriented fuzzy logic controller and its design,” *Internat. J. Approx. Reason.*, vol. 12, no. 3–4, pp. 263–277, 1995.
- [12] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, ser. Trends in Logic. Dordrecht: Kluwer Academic Publishers, 2000, vol. 8.
- [13] J. A. Goguen, “The logic of inexact concepts,” *Synthese*, vol. 19, no. 3–4, pp. 325–373, 1969.
- [14] J. Pavelka, “On fuzzy logic I–III,” *Z. Math. Logik*, vol. 25, pp. 45–52, 119–134, 447–464, 1979.
- [15] L. A. Zadeh, “Outline of a new approach to the analysis of complex systems and decision processes,” *IEEE Trans. Syst. Man Cybern.*, vol. 3, no. 1, pp. 28–44, 1973.
- [16] F. Klawonn and V. Novák, “The relation between inference and interpolation in the framework of fuzzy systems,” *Fuzzy Sets and Systems*, vol. 81, no. 3, pp. 331–354, 1996.
- [17] J. F. Baldwin and N. C. F. Guild, “Modelling controllers using fuzzy relations,” *Kybernetes*, vol. 9, pp. 223–229, 1980.
- [18] M. Mizumoto and H.-J. Zimmermann, “Comparison of fuzzy reasoning methods,” *Fuzzy Sets and Systems*, vol. 8, pp. 253–283, 1982.
- [19] L. A. Zadeh, “Fuzzy sets as a basis for a theory of possibility,” *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 3–28, 1978.
- [20] D. Dubois, H. Prade, and L. Ughetto, “Checking the coherence and redundancy of fuzzy knowledge bases,” *IEEE Trans. Fuzzy Systems*, vol. 5, no. 6, pp. 398–417, August 1997.
- [21] —, “Fuzzy logic, control engineering and artificial intelligence,” in *Fuzzy Algorithms for Control*, ser. International Series in Intelligent Technologies, H. B. Verbruggen, H.-J. Zimmermann, and R. Babuška, Eds. Boston: Kluwer Academic Publishers, 1999, pp. 17–57.
- [22] L. Godo and P. Hájek, “A note on fuzzy inference as deduction,” in *Fuzzy Sets, Logics and Reasoning about Knowledge*, ser. Applied Logic Series, D. Dubois, E. P. Klement, and H. Prade, Eds. Dordrecht: Kluwer Academic Publishers, 1999, vol. 15, pp. 237–241.
- [23] R. Fuller, *Introduction to Neuro-Fuzzy Systems*, ser. Advances in Soft Computing. Heidelberg: Physika-Verlag, 2000, vol. 2.
- [24] W. J. M. Kickert and E. H. Mamdani, “Analysis of a fuzzy logic controller,” *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 29–44, 1978.
- [25] P. M. Larsen, “Industrial applications of fuzzy logic control,” *Int. J. Man-Mach. Stud.*, vol. 12, pp. 3–10, 1980.
- [26] J. M. Mendel, “Fuzzy logic systems for engineering: a tutorial,” *Proc. IEEE*, vol. 83, no. 3, pp. 345–377, 1995.
- [27] D. Coufal, “Radial implicative fuzzy inference systems,” Ph.D. dissertation, University of Pardubice, 2003.
- [28] —, “Coherence of radial implicative fuzzy systems,” in *Proc. 15th IEEE Int. Conf. on Fuzzy Systems*, Vancouver, BC, July 2006, pp. 307–314.