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RESEARCH ARTICLE

# Fuzzy Logic: A Powerful Tool for Modeling of Vagueness

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In this paper, we briefly discuss the indeterminacy phenomenon as a phenomenon having two facets: uncertainty and vagueness. Then, we describe some basic principles of modern mathematical fuzzy logic and show that it constitutes a group of well established formal systems. We argue that until now, fuzzy logic provides the most successful solutions when trying to grasp vagueness. Its model of vagueness is based on introducing degrees of truth with well established and substantiated structure that non-trivially generalizes classical logic. We also outline formal theories that model some of the essential manifestations of vagueness headed by an analysis of sorites.

**Keywords:** fuzzy logic; vagueness; truth values.

## 1. Introduction

The goal of this paper is twofold: *first*, to present fuzzy logic (more precisely, mathematical fuzzy logic) as a group of well established formal systems and *second*, to give arguments for its use as a tool for modeling the vagueness phenomenon. Fuzzy logic has for a long time been a subject of controversy among philosophers, logicians, and fuzzy logicians. While the first raise many arguments against the ability of fuzzy logic to capture the vagueness phenomenon, the latter argue the opposite. One surprising fact is that philosophers quite often cite papers that are 30–40 years old (cf. Keefe (2000), Read (1995)) in their arguments, thus ignoring that fuzzy logic has made significant progress. However, the situation is improving, and the discussion is gradually becoming well substantiated and fruitful for both sides. One of the goals of this paper is to help fill in this gap and to make an attempt at clarification of some misinterpretations occurring in the philosophical literature.

One of the objections against fuzzy logic is the allegation that it lacks the rule of modus ponens. This would indeed be a serious deficiency. However, the first precisely introduced formal system of fuzzy logic published by J. Pavelka in (Pavelka 1979), of course, includes a sound modus ponens rule. The first author of this paper has extended Pavelka's logic to first-order in (Novák 1990) and proved a generalization of the famous Gödel completeness theorem. Since then, a lot of other works elaborating on formal fuzzy logic in detail have been published. As a consequence,

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many arguments against fuzzy logic turn out to be barren; they are either wrong or non-substantiated.

The fact is that vagueness is a fairly mysterious phenomenon that plays a significant (if not crucial) role in human thinking. Thus, one may agree that a working mathematical theory of vagueness phenomenon is necessary. We argue that until now, this goal is most successfully accomplished by fuzzy logic, which attempts to grasp vagueness by introducing a well established and substantiated structure of truth degrees and by using the latter for modeling the way vagueness manifests itself in various situations.

Let us stress that we do not have the pretension to claim that fuzzy logic explains vagueness. We argue that it is a reasonable mathematical model having enough power to deal with vagueness – nothing more, nothing less.

It seems that the use of degrees is very natural for the human mind. For example, we often say "almost white dress", "very strong engine", "too unpleasant a situation", etc. We also judge practically everything in our lives using various kinds of degrees that can be either specific (temperature, money) or abstract. To characterize vagueness, certain degrees of intensity of the property of interest are introduced. The degrees are taken from a scale that must have some necessary properties to serve as a good means; except for being an ordered set, it must capture the continuity feature of vagueness and so, it must be uncountable. Furthermore, to be able to represent various operations with the properties possessed by objects only partly, we must endow it with additional operations. The result is a specific algebra — the *algebra of truth values*. A lot of important mathematical results have been obtained in the theory of these algebras. For example, see the books (Hájek 1998, Cignoli *et al.* 2000, Novák *et al.* 1999, Novák and Perfilieva 2000, Turunen 1999) and many papers scattered throughout journals.

We also argue that the rise of vagueness is closely related to some *indiscernibility relation*. For example, a movie is a sequence of pictures. When projected at sufficient speed, we are unable to distinguish them one from another and the result is a vague phenomenon that we regard as a continuous movement. Similarly, the shape of a heap of stones is also vague, and when adding or removing one stone, its shape changes indiscernibly. This is the core of the *sorites paradox*. Another example is the so-called *ostensive definition*, that is, learning by example: e.g., a mother shows her child a given plant and says "this is a tree". After several repeats with different trees, the child captures this idea and can correctly point out various trees that may be significantly different from those originally shown. This means that he/she captured a certain indiscernibility relation enabling him/her to classify trees[1].

The following hypothesis has been extensively discussed in (Novák 1992):

Hypothesis 1.1 Vagueness of natural phenomena, i.e. phenomena recordable by the human mind, usually using natural language, is a consequence of the indiscernibility phenomenon.

There are at least two relevant ways to construct a mathematical model of the indiscernibility relation. One way is developed inside the alternative set theory (Vopěnka 1979) and the second one is based on introducing degrees. This leads to the concept of fuzzy equality (fuzzy equivalence), an idea contained in the works of many authors (e.g., Höhle (2007), Klawonn and Kruse (1993), Parikh (1983)). It can be demonstrated that membership degrees in fuzzy sets can be quite naturally

[1] One may argue that, in fact, some relevant properties are abstracted which are the same for all trees. However, this is not convincing because extension of such properties — a grouping of objects being trees — still does not have clear boundaries and remains vague. This argument can be better a core of explanation where does the considered indiscernibility come from.

derived from fuzzy equality (cf. Novák (1992), Pultr (1984)).

This paper is structured as follows: In Section 2, we characterize the vagueness phenomenon as one of two facets (together with uncertainty) of a more general phenomenon of indeterminacy. Hence, while the mathematical model of uncertainty is probability theory, fuzzy logic is a mathematical model of vagueness. In Section 3, the general principles of fuzzy logic are presented. Section 4 provides an overview of fuzzy logic in a narrow sense, and Section 5 deals with a fuzzy-logic model of the semantics of the, so called, evaluative linguistic expressions that are essential bearers of vagueness in natural language. Finally, Section 6 focuses on other issues of vagueness analyzed from the point of view of fuzzy logic, starting with the sorites paradox and also including a model of higher-order vagueness. For brevity, we had to relax strict precision in some places to maintain clarity of our explanation.

## 2. Indeterminacy

We argue that vagueness is one facet of a more general phenomenon that should be called *indeterminacy*. Among these facets we can distinguish *uncertainty* and *vagueness*. Both of them characterize situations in which the amount and extent of our knowledge is crucial. There is, however, an essential distinction between both facets and it must be respected by any theory that claims to deal with indeterminacy (cf., e.g., Dvořák and Novák (2005), Hájek and Paris (1997), Novák *et al.* (1999), Novák (2006b)).

Note that in reality, we often meet indeterminacy *with both its facets* present, i.e. *vague* phenomena are at the same time *uncertain*. Let us briefly summarize the main features of both facets.

### 2.1 *Uncertainty*

The *uncertainty* phenomenon emerges when there is a *lack of knowledge* about the *occurrence* of some *event*. The event is encountered when an experiment (process, test, etc.) is to proceed, the result of which is not known to us. It may also refer to a variety of potential outcomes, ways of solution, choices, etc. *Randomness* is a specific form of uncertainty arising in connection with time. There is no randomness (uncertainty) once the experiment is realized and the result is known to us. From this point of view, uncertainty is epistemological concept. Note that it is connected with the question of whether a given event may be observed within some time period, or not. This becomes apparent in the typical example of tossing a dice. The phenomenon to occur is *the number of dots on the dice* and it occurs after the experiment (i.e. tossing the dice once) has been realized. Thus, we refer here to the future. However, the variety of potential events may raise an even more abstract uncertainty that is less dependent on time. We may, for example, analyze uncertainty in potentiality (that is, lack of knowledge) without a necessary reference to time, or with reference to the past (such as a posterior Bayesian probability). Note that pure uncertainty in an abstract way refrains from the character of events of concern, i.e., they can be either crisply or vaguely delineated.

The mathematical model (i.e. quantified characterization) of uncertainty is provided especially by *probability theory*. In everyday terminology, probability can be thought of as a numerical measure of the likelihood that a particular event will occur. There are also other mathematical theories addressing the mentioned abstract uncertainty, for example possibility theory, belief measures, and others.

### 2.2 *Vagueness*

The *vagueness phenomenon* arises when trying to *group* together objects that have a certain property $\varphi$. The result is an *actualized* grouping $X$ of objects which is not necessarily a set because the property $\varphi$ may be *vague*. That is, it may be impossible to characterize all the elements of the given grouping precisely and unambiguously — there can exist *borderline* objects for which it is unclear whether they have the property $\varphi$ (and thus, whether they belong to $X$), or not. On the other hand, it is always possible to characterize at least some *typical objects* (prototypes), i.e. objects having typically the property in concern. For example, everybody can point to a "blue jacket" or a "long bridge," but it is impossible to show all blue jackets and all long bridges.

Vagueness is opposite to exactness and we argue that it cannot be avoided in the human way of regarding the world. Any attempt to explain a description with extensive amount of details necessarily leads us to the use of vague concepts. The reason is that precise description contains an abundant number of details (cf. the *incompatibility principle* formulated by L. A. Zadeh in (Zadeh 1973)). To understand such a description, we must group these details together — and this can hardly be done precisely. Natural language plays a non-substitutable role here . However, the problem lies deeper in the way people actually regard the phenomena around them. Unlike uncertainty where we always have to consider whether some phenomenon *occurs or not*, vagueness concerns the way the *phenomenon itself* is delineated, no matter whether it will occur or not.

A typical feature of vagueness is its continuity: a small difference between objects cannot lead to abrupt change in the decision of whether either of them has, or does not have a vague property (cf. Black's "museum of applied logic" in (Black 1937)). The transition from having a (vague) property to not having it is smooth.

There is also discussion about the role of fuzzy logic with respect to probability and possibility theory. This can be, in our opinion, well explained on the basis of the concepts of actuality and potentiality. This idea was presented in (Novák 2006b).

## 3. General principles of fuzzy logic

### 3.1 *Global characterization of fuzzy logic*

Informally, we may say that fuzzy logic (FL) is a special many-valued logic addressing the vagueness phenomenon and developing tools for its modeling via truth degrees taken from an ordered scale. It is expected to preserve as many properties of classical logic as possible. Furthermore, the following is required from FL:

(a) It must be a well established sound formal system.
(b) It must be a system open to new connectives, generalized quantifiers, non-commutativity of conjunction and disjunction, and possibly to other specific phenomena.
(c) It should provide tools suitable for providing solutions to specific problems arising when dealing with vagueness, and for development of special techniques and concepts. Among such tools we can rank evaluative linguistic expressions, linguistic variables, fuzzy IF-THEN rules, fuzzy quantification, defuzzification, fuzzy equality, etc.
(d) Special inference schemata including sophisticated inference schemes of human reasoning (e.g., compositional rule of inference, reasoning based on natural language expressions, non-monotonic reasoning, abduction, etc.) should be

expressible in it.

Let us stress that many of these requirements are already fulfilled by the available formal systems of FL.

The fundamental classification of FL is *fuzzy logic in a narrow sense* (FLn) and that *in a broader sense* (FLb). The latter is as an extension of FLn whose aim is to develop a formal theory of *the human way of reasoning that would include a mathematical model of the meaning of some expressions of natural language (evaluative linguistic expressions), the theory of generalized quantifiers and their use in human reasoning.* One of the foreseen goals of FLb is to develop a formal logic that could be applied in human-like behaving robots. The history of FL has been nicely summarized by P. Hájek in (Hájek 2006). The following is a (uncomplete) list of relevant books on fuzzy logic: Gottwald (2001), Hájek (1998), Klement *et al.* (2000), Cignoli *et al.* (2000), Nguyen and Walker (2000), Novák *et al.* (1999), Novák and Perfilieva (2000).

### 3.2 *Truth degrees*

A lot of controversy arose about degrees of truth. According to our analysis above, they naturally emerge when we try to characterize an actualized grouping. If the latter is not delineated sharply then two truth degrees become insufficient and we must to resort to a richer scale of them. Thus, truth degrees provide a reasonable technical means for dealing with vagueness.

Let us stress that they have little use alone. A frequent example[1] is "I love you in the degree 0.954867283", or even worse — "...in the degree $\sqrt{2}/2$". Of course, nobody will ever say such a strange sentence. However, it is quite natural to say "I love you *very much*". This sentence contains hidden degrees which, however, are not specified precisely but using the *evaluative linguistic expression* — see below.

We conclude that *specific truth values are assigned only in the model* and in practice we never deal with them directly without a wider context. It is important, however, to *compare them.* Consequently, what is important is the course of truth values w.r.t. various kinds of models and not their specific values. In other words, a fuzzy set being a mathematical model of the above discussed grouping of objects is a function $A : U \longrightarrow L$ ($U$ is a universe and $L$ is a set of truth degrees) whose most important characteristic is its *shape*, i.e. the way of $A$'s course.

## 4. Fuzzy logic in narrow sense

### 4.1 *Structure of truth values*

The reasoning behind the structure of truth values in fuzzy logic has been initiated by J. A. Goguen in his famous paper (Goguen 1969). Extensive analysis on what the structure of truth values should look like can also be found in the book (Novák *et al.* 1999), Section 4.2. Goguen gave a lot of arguments in favor of the assumption that there must be two conjunctions in fuzzy logic. The most essential argument is that the ordinary conjunction which joins formulas is not the same as the conjunction joining premises in modus ponens. This conjunction is seemingly unique when considering two truth values only. When introducing more of them, these conjunctions split into two different (in general) operations.

[1] P. Hájek, personal discussion.

From the algebraic point of view we arrive at the assumption that truth degrees should form a *residuated lattice*:

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

which has the following properties:

(a) $\mathcal{L} = \langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with the smallest element $\mathbf{0}$ and greatest element $\mathbf{1}$ (i.e., $a \leq b$ iff $a \wedge b = a$, as usual).
(b) The operation $\otimes$ is usually called *product* and it is associative ($(a \otimes b) \otimes c = a \otimes (b \otimes c)$), commutative ($a \otimes b = b \otimes a$) and $a \otimes \mathbf{1} = a$, $\quad a, b, c \in L$.
(c) *Adjunction:* $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, $\quad a, b, c \in L$ where $\rightarrow$ is called *residuation* (sometimes also *(fuzzy) implication*).

When using this structure in logical analysis, it turned out that the following property is also indispensable:

(d) *Prelinearity:* $(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$, $a, b \in L$.

A prelinear residuated lattice is called an *MTL-algebra*.

A more specific and quite often used structure is that of BL-algebra. This is MTL algebra fulfilling, moreover, also the *divisibility* $a \otimes (a \rightarrow b) = a \wedge b$. Even more specific is MV-algebra, which is the closest nontrivial generalization of boolean algebra.

On the basis of this definition, we can introduce the following additional operations: *Negation* $\neg a = a \rightarrow \mathbf{0}$, *power* $a^n = a \otimes a \otimes \cdots \otimes a$ and *biresiduation* $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ and also *sum* $a \oplus b = \neg(\neg a \otimes \neg b)$. The algebra is linearly ordered if for all $a, b \in L$, $a \leq b$ or $b \leq a$.

It can be shown that a residuated lattice is a generalization of a boolean algebra where in general, however,

$$a \vee \neg a \neq \mathbf{1}$$

as well as

$$a \wedge \neg a \neq \mathbf{0}.$$

On the other hand, $a \otimes \neg a = \mathbf{0}$ and $a \oplus \neg a = \mathbf{1}$ for all $a \in L$. Moreover, the adjunction (property (c)) is algebraic formulation of modus ponens (for explanation, see (Novák *et al.* 1999), Section 4.2). If we put $L = \{\mathbf{0}, \mathbf{1}\}$ then the residuated lattice reduces just to the algebra for classical logic where $\wedge = \otimes$ is conjunction, $\vee = \oplus$ is disjunction, $\rightarrow$ is implication and $\neg$ is negation.

An essential property which holds in all residuated lattices is the following:

$$a \otimes (a \rightarrow b) \leq b, \qquad a, b \in L. \tag{2}$$

This property is an easy consequence of the adjunction (property (c)) and it assures us that modus ponens is a sound inference rule.

We may add various conditions to residuated lattices to obtain stronger structures. These lead to various formal systems of fuzzy logic. Let us name, for example, *basic fuzzy logic* (BL), MTL-logic (MTL), IMTL-logic (IMTL), Łukasiewicz logic (Ł), Gödel logic (G), product logic Π, and others. There are also fairly complicated ŁΠ-algebras which have two (different) products and two implications and which determine ŁΠ fuzzy logic (ŁΠ). For the details see (Cintula 2001, Esteva and Godo 2001, Gottwald 2001, Hájek 1998, Novák *et al.* 1999) and elsewhere.

*4.1.1 Fundamental examples.*

There are three fundamental examples of the structure of truth values, namely *standard Gödel, standard product, and standard Łukasiewicz MV-algebra*, which play essential roles in the theory of fuzzy logic. All of them have the support $L = [0, 1]$ and so, the operations $\vee, \wedge$ are minimum and maximum, respectively. Furthermore,

| *Gödel algebra* | *product algebra* |
|---|---|
| $a \otimes b = a \wedge b$ | $a \otimes b = a \cdot b$ |
| $a \to b = \begin{cases} 1 & \text{if } a \le b \\ b & \text{otherwise} \end{cases}$ | $a \to b = \begin{cases} 1 & \text{if } a \le b \\ \frac{b}{a} & \text{otherwise} \end{cases}$ |
| $\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$ | $\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$ |

In *standard Łukasiewicz MV-algebra*, we set

$$a \otimes b = 0 \vee (a + b - 1),$$
$$a \to b = 1 \wedge (1 - a + b),$$
$$\neg a = 1 - a.$$

For many properties of all these algebras see (Cignoli *et al.* 2000, Hájek 1998, Novák *et al.* 1999).

*4.1.2 Subdirect representation.*

One of the objections to fuzzy logic is the linearity of the ordering of the set of truth values. Though this is indeed true in the examples above, there is an infinite number of various examples of algebras of truth values that are either linearly or only partially ordered. Any of them can be used when modeling the vagueness phenomenon. The following general algebraic result is of a special importance.

**Theorem 4.1:** *Every MTL-algebra $\mathcal{L}$ is isomorphic to a subdirect product of* linearly ordered *MTL-algebras.*

The essential informal consequence of this theorem is that fuzzy logic may be confined only to linearly ordered structures of truth values! This result significantly reduces objections towards the usually assumed linearity of ordering of truth values since the required generality is not significantly harmed. If we prove something for an arbitrary linearly ordered algebra then this result holds also for any partially ordered algebra.

On the other hand, one cannot apply Theorem 4.1 arbitrarily. We must never forget that truth values are mainly technical means for modeling more complicated vague phenomena. When considering, e.g., the evaluative predications "John is tall" and "Mary is rich," then given concrete *John* and *Mary*, these predications can be true in various degrees. There is no reason to reject their comparison, i.e., John can be tall in a degree, say 0.9, while Mary can be rich in a degree, say 0.6. These *truth values* can be compared but *not the meaning* of these clearly incomparable statements. We should not mix the meaning of such statements. That would require a more sophisticated model (one possible is discussed below) than simple truth value assignment in a model that bears only partial information.

### 4.2 *Fuzzy and classical logic*

As all formal logical systems, fuzzy logic strictly distinguishes syntax from semantics. There are two essential approaches: fuzzy logic with *traditional syntax* and that with *evaluated syntax* ($\mathrm{Ev_L}$).

#### *4.2.1 Fuzzy logic with traditional syntax*

This approach is promoted by many mathematicians[1]. In fact, this is a class of various logical systems which are determined by the assumed structure of truth values. They generalize syntax of classical logic by adding a new connective of *strong conjunction* (**&**) (and, possibly, some other ones) and modifying special axioms. Each of these systems has a finite list of schemes of logical axioms and inference rules of *modus ponens*. The fundamental concept of provability is classical, i.e., a formula $A$ is *provable* in a theory $T$, $T \vdash A$, if there exists its formal proof.

As mentioned, all these formal systems of fuzzy logic differ from classical logic by considering two conjunction connectives, namely *minimum conjunction* $A \boldsymbol{\wedge} B$ which is interpreted by minimum ($\wedge$), and *strong conjunction* $A \mathbin{\&} B$ which is interpreted by the product $\otimes$. These conjunctions can also be taken as models of two kinds of coordination — close and free. Their use depends on a local character of the conjuncts, i.e., the resulting truth degree should be computed from truth degrees of the respective conjuncts with respect to their meaning, too. For example *slow and safe car* would better lead to strong conjunction while *big and beautiful house* to minimum conjunction. In general, we should use strong conjunction whenever we join formulas without a priori knowledge about their content since it is safer.

Other connectives are the following: *implication* $A \Rightarrow B$ interpreted by $\rightarrow$, *negation* $\neg A := A \Rightarrow \bot$ interpreted by $\neg a$, *equivalence* $A \Leftrightarrow B$ interpreted by $\leftrightarrow$, *disjunction* $A \boldsymbol{\vee} B$ interpreted by ($\vee$) and *strong disjunction* $A \boldsymbol{\nabla} B := \neg(\neg A \mathbin{\&} \neg B)$.

As mentioned, the inference rules are *modus ponens* (from $A$ and $A \Rightarrow B$ infer $B$) and also *generalization* (from $A$ infer $(\forall x)A$).

Given a language $J$ of predicate fuzzy logic we may naturally introduce the concept of a structure for $J$ as follows:

$$\mathcal{M} = \langle \langle M, \mathcal{L} \rangle, \{F_f \mid f \in \mathrm{Func}_J\}, \{r_P \mid P \in \mathrm{Pred}_J\}, \{m_{\mathbf{u}} \mid \mathbf{u} \in \mathrm{OConst}_J\} \rangle, \qquad (3)$$

in Tarski's sense where $M$ is a set, $F_f : M^n \longrightarrow M$ is an $n$-ary function assigned to each $n$-ary functional symbol $f$, $r_P : M^n \longrightarrow L$ is an $n$-ary fuzzy relation assigned to each $n$-ary predicate symbol and $m_{\mathbf{u}} \in M$ is an element representing object constant. Given an assignment $p$ of elements from $M$ to variables, we define a truth value $\mathcal{M}_p(A) \in L$ of each formula $A$ of the given language.

The following completeness theorem holds for many kinds of fuzzy logics (e.g., MTL, IMTL, BL, G and other ones). We will often use the term "fuzzy logic" without closer specification of the concrete system if unnecessary. The details including full proofs can be found in the books (Hájek 1998, Cignoli *et al.* 2000, Novák and Perfilieva 2000, Novák *et al.* 1999) and in the papers (Cintula 2001, Esteva and Godo 2001).

**Theorem 4.2 :** *Let $T$ be a theory of fuzzy logic. Then $T \vdash A$ iff $\mathcal{M}(A) = 1$ for each model $\mathcal{M} \models T$.*

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[1]P. Hájek, F. Esteva, S. Gottwald, L. Godo, F. Montagna, D. Mundici, and others.

We can also introduce the concept of a *crisp formula* that is a formula $A$ for which $\vdash A \vee \neg A$. Since the boolean lattice $\{\mathbf{0}, \mathbf{1}\}$ is also an MTL-algebra we may take *classical logic as a sort of limit "fuzzy" logic*, i.e. it is a fuzzy logic in which *all formulas are crisp.*

### *4.2.2 Fuzzy logic with evaluated syntax*

This approach was initiated by Pavelka in (Pavelka 1979) and extended to first-order in (Novák 1990). It is a more radical departure from classical logic since it allows evaluation of formulas also in syntax simply by assuming that axioms may not be fully convincing, i.e., not fully true. Consequently, axioms may form a fuzzy set. But this means a departure from the traditional conception of syntax.

Let $F_J$ be the set of all well-formed formulas of language $J$. The fundamental concept is that of an *evaluated formula* $a/A$ where $A \in F_J$ is a formula and $a \in L$ is its syntactic evaluation. This means that on a syntactical level, we a priori assume that a certain formula may be true at least in the degree $a$, which can be generally different from $\mathbf{1}$. Hence, the designated truth values are replaced by the *maximality principle* saying that if the same formula is assigned more truth values then its final truth assignment, then it is equal to the maximum (supremum) of all of them. Consequently, all truth values are equally important.

*Syntactical derivation* in $\mathrm{Ev}_{\text{Ł}}$ may end up with a formula that is true in *arbitrary degree* during the derivation and to progress the degree further (unlike traditional syntax where the derivation ends up only with a formula true in the degree $\mathbf{1}$). It can be demonstrated that these principles lead to transparent generalization of classical logic both in semantics as well as in syntax.

The truth values of this logic must form the standard Łukasiewicz MV-algebra. It can be proved that only this algebra (and its isomorphs) can ensure completeness — see below.

Since the syntax deals with evaluated formulas, we must introduce evaluated inference rules. For example, *modus ponens* takes the form

$$\frac{a/A, c/A \Rightarrow B}{a \otimes c/B}$$

and similarly also generalization. This rule is *sound* in the sense that it is not possible to derive a conclusion $B$ with the truth value smaller than the conjunction of truth values of the respective premises $A$ and $A \Rightarrow B$ in any interpretation. This is assured by the property (2) of residuated lattices.

A *fuzzy theory* $T$ is a fuzzy set of formulas. More specifically, it is determined by a triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle$$

where LAx is a fuzzy set of logical axioms, SAx is a fuzzy set of special axioms and $R$ a set of evaluated inference rules. This definition is motivated by the idea that axioms need not be fully true (convincing). A typical example are the assumptions leading to sorites paradox (the axiom $(\forall n)(H(n) \Rightarrow H(n+1))$ saying that "if $n$ stones do not form a heap then $n+1$ do not form it as well" can hardly be fully convincing).

We also introduce the concept of *evaluated proof* which means that each proof $w_A$ of a formula $A$ has its value $\mathrm{Val}(w_A) \in L$. In correspondence with the maximality principle we obtain the *provability degree* of a formula

$$T \vdash_a A \quad \text{iff} \quad a = \bigvee\{\mathrm{Val}(w_A) \mid w_A \text{ is a proof of } A \text{ in } T\}$$

where $\mathrm{Val}(w_A)$ is a *value* of evaluated proof, see (Novák *et al.* 1999), Definition 4.4, page 99. This means that we can have various proofs of the same formula which are "valuable" in various degrees. The final provability degree is the upper limit value of all possible proofs. Note that this is a straightforward generalization of classical provability where each formula in the proof can be taken as evaluated by the degree **1** (and so, in classical case it is sufficient to find one proof only).

The truth degree $T \models_a A$ is the infimum of all truth values $\mathcal{M}_p(A)$ of $A$ in all models $\mathcal{M} \models T$ where $\mathcal{M}$ is a structure (3) such that

$$\mathcal{M}_p(A) \geq \mathrm{SAx}(A)$$

for all special axioms of $T$. This means that the truth of a formula is the lower limit of all possibilities — we are most pessimistic.

The following is a generalization of the Gödel completeness theorem ($J(T)$ is the language of a fuzzy theory $T$).

**Theorem 4.3 :** *For every fuzzy theory $T$ and every formula $A \in F_{J(T)}$*

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A.$$

For more details including all proofs see (Novák *et al.* 1999).

### 4.3 *Fuzzy type theory*

It turns out that for full-fledged treatment of vagueness we cannot make do with predicate fuzzy logic, especially if we want to model the meaning of natural language expressions. Thus, the need for higher-order fuzzy logic arises. Such logic is fuzzy type theory (FTT) which is a generalization of classical type theory which was initiated by B. Russel, A. Church and L. Henkin. For full treatment of FTT we refer particularly to (Novák 2005a). Here, we provide only a sketchy idea. Let us also mention that there exists another version of higher-order fuzzy logic (Běhounek and Cintula 2005) which focuses especially on the construction of special classes, and thus aims to become the general formal frame for "fuzzy mathematics".

The structure of truth values of FTT is supposed to form one of the following: a complete IMTL$_\Delta$-algebra (see Esteva and Godo (2001)), the standard Łukasiewicz$_\Delta$ MV-algebra, or a BL$_\Delta$-algebra.

The most important for applications in linguistics is the standard Łukasiewicz$_\Delta$ MV-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \oplus, \Delta, \rightarrow, 0, 1 \rangle$$

where

$$\wedge = \text{minimum}, \qquad \vee = \text{maximum},$$
$$a \otimes b = 0 \vee (a + b - 1), \qquad a \to b = 1 \wedge (1 - a + b),$$
$$\neg a = a \to 0 = 1 - a, \qquad a \oplus b = 1 \wedge |a - b|,$$
$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding FTT is denoted by Ł-FTT.

*4.3.1 Fuzzy equality.*

The crucial concept in FTT is that of a *fuzzy equality*. This is a fuzzy relation $\doteq: M \times M \longrightarrow L$ which has the following properties:

(i) reflexivity $[m \doteq m] = \mathbf{1}$,
(ii) symmetry $[m \doteq m'] = [m' \doteq m]$,
(iii) $\otimes$-transitivity $[m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m'']$

for all $m, m', m'' \in M$ where $[m \doteq m']$ denotes a *truth value* of $m \doteq m'$.

A special case of fuzzy equality on the algebra of truth values is *biresiduation* (see definition on page 6). An example of a fuzzy equality on $M = \mathbb{R}$ with respect to standard Łukasiewicz$_\Delta$ MV-algebra is

$$[m \doteq n] = 1 - (1 \wedge |m - n|), \qquad m, n \in \mathbb{R}.$$

The reason for considering fuzzy equality in FTT is motivated by the fact that the basic connective in classical type theory is the equality. This turned out to be in nice correspondence with Hypothesis 1.1, as will be noted below[1].

*4.3.2 Syntax of FTT*

It is a generalization of the lambda-calculus and is constructed in a classical way. The main difference from classical type theory is in the definition of additional special connectives. Note that all essential syntactical elements of FTT are formulas (alternatively, they can be called lambda-terms which is quite often in classical type theory).

As usual, each formula $A$ has a certain type. The basic types are $o$ (truth values) and $\epsilon$ (elements). These can be then iterated to more complex types.

Formulas of type $o$ (truth value) can be joined by the following connectives: $\equiv$ (equivalence), $\boldsymbol{\vee}$ (disjunction), $\boldsymbol{\wedge}$ (conjunction), **&** (strong conjunction), $\boldsymbol{\nabla}$ (strong disjunction), $\Rightarrow$ (implication). General ($\forall$) and existential ($\exists$) quantifiers are defined as special formulas. For the details about their definition and semantics — see (Novák 2005a).

In general, if $A_{\beta\alpha}$ is a formula then it represents a function assigning elements of type $\beta$ to elements of type $\alpha$. Hence, $A_{o\alpha}$ represents a fuzzy set of elements. It can also be understood as a first-order property of elements of the type $\alpha$. Similarly, $A_{(o\alpha)\alpha}$ represents a fuzzy relation (between elements of type $\alpha$).

There are 17 *logical axioms* in IMTL$_\Delta$-FTT (its structure of truth values is IMTL$_\Delta$-algebra). This number is influenced mainly by the structure of truth values

[1]This fact also led to search of a special algebra of truth values where fuzzy equality is the basic operation. The first exposition of the new fuzzy type theory based on this algebra can be found in (Novák 2008).

and it can vary for other FTT's. Because of lack of space, we will not present these axioms in detail.

Let us only mention *axioms of descriptions* $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha$, where $\alpha = o, \epsilon$ and $\iota_{\alpha(o\alpha)}$ is a description operator which assigns to a fuzzy set an element which belongs in the fuzzy set to degree 1 (i.e., interpretation of this operator is a *defuzzification operation*). $\mathbf{E}_{(o\alpha)\alpha}$ is a special constant which represents a fuzzy equality.

FTT has two inference rules and the classical concept of provability. The rules of modus ponens and generalization are derived rules.

A *theory* $T$ is a set of formulas of type $o$. We may also consider a specific formula $\dagger$ of type $o$ which represents the most indefinite truth value for which $\vdash \neg\dagger \equiv \dagger$ holds. Note that such a truth value, in general, need not be present in the given algebra of truth values. In the standard Łukasiewicz MV-algebra is $\dagger$ interpreted by the truth value 0.5.

A formula $\boldsymbol{\Delta} A_o$ is *crisp* (in the sense introduced above), i.e., its interpretation is either 0 or 1. There are formulas which are not crisp.

### *4.3.3 Semantics of FTT*

is defined using a generalization of the concept of *frame* which is a system

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{L} \rangle$$

where $=_\alpha$ is a special fuzzy equality in each set $M_\alpha$ and $\mathcal{L}$ is an algebra of truth values. Moreover, for any types $\alpha, \beta \in \mathit{Types}$, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$.

To define an interpretation of a given formula we must start with an assignment $p$ of elements from $M_\alpha$ to variables $x_\alpha$, $\alpha \in \mathit{Types}$. Then each formula $A_{\beta\alpha}$ is interpreted in $\mathcal{M}$ with respect to the assignment $p$ by a function $\mathcal{M}_p(A_{\beta\alpha}) : M_\alpha \longrightarrow M_\beta$.

A frame $\mathcal{M}$ is a *general model* of a theory $T$ if interpretation of all its formulas is defined for any assignment $p$ and all axioms of $T$ are true in the degree $\mathbf{1}$ in $\mathcal{M}$. A formula $A_o$ is true in $T$, $T \models A_o$ if it is true in the degree $\mathbf{1}$ in all models of $T$.

The following theorem has been proved in (Novák 2005a).

**Theorem 4.4:** *$T \vdash A_o$ iff $T \models A_o$ holds for every theory $T$ and a formula $A_o$.*

We claim that most (if not all) essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model.

The following special crisp formulas are quite useful:

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(\neg z_o),$$

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(z_o \vee \neg z_o).$$

The $\Upsilon_{oo}$ expresses a property of being a non-zero truth value and $\hat{\Upsilon}_{oo}$ a general one (i.e. between $\mathbf{0}$ and $\mathbf{1}$).

**Remark 1:** The $\boldsymbol{\Delta}$ corresponds to the $D$-operator of supervaluation theory (e.g., $A \vdash C$ implies $\vdash \boldsymbol{\Delta} A \Rightarrow C$ as well as $\neg C \vdash \neg\boldsymbol{\Delta} A$) and $\hat{\Upsilon}$ corresponds to $I$-operator *indefinitely*.

We will need the following property:

**Lemma 4.5:** *$\vdash \hat{\Upsilon}(\hat{\Upsilon} x_o \wedge y_o) \equiv (\hat{\Upsilon} x_o \wedge \hat{\Upsilon} y_o)$.*

**Proof:** Let $\mathcal{M}$ be a model and $p$ an assignment. If $p(x_o) \in (0, 1)$ then $\mathcal{M}_p(\hat{\Upsilon}(\hat{\Upsilon} x_o \wedge y_o)) = \mathcal{M}_p(\hat{\Upsilon} y_o) = \mathcal{M}_p(\hat{\Upsilon} x_o) \wedge \mathcal{M}_p(\hat{\Upsilon} y_o)$. If $p(x_o) \in \{0, 1\}$ then $\mathcal{M}_p(\hat{\Upsilon} x_o) = 0$

and so, $\mathcal{M}_p(\hat{\Upsilon}(\hat{\Upsilon}x_o \wedge y_o)) = \mathcal{M}_p(\hat{\Upsilon}x_o) \wedge \mathcal{M}_p(\hat{\Upsilon}y_o) = 0$. The lemma holds by the completeness theorem. □

### 4.4 *Scheme of the most important fuzzy logics*

We have already mentioned that there exist many formal systems of fuzzy logic. We argue that not all these systems indeed deserve to be called "fuzzy logic". According to us, the main criterion should be the ability of such a system to serve as a powerful formal grounds for characterization of the vagueness phenomenon, fulfilling the agenda of fuzzy logic and potential for the development of fuzzy logic in a broader sense as a logic of natural human reasoning. The fuzzy logic systems that, according to their properties and strength, seem to fit best this aim are depicted in Figure 1. One can find there also axioms that must be added to obtain

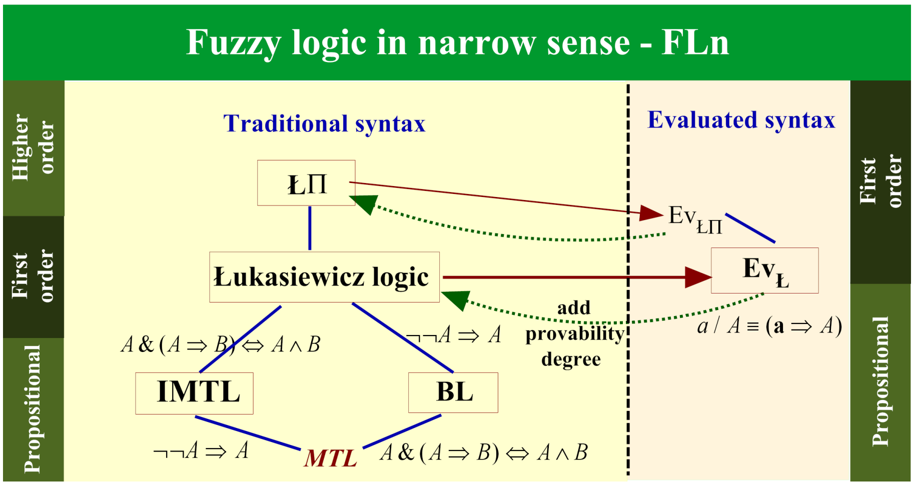


Figure 1. Scheme of the most important formal fuzzy logic systems that are relevant for modeling of the vagueness phenomenon.

another logical system. The scheme outlines also the shift from Ł to $\mathrm{Ev}_{\text{Ł}}$, and in a sense, also a similar shift from ŁΠ. The dotted arrows back express representability in the corresponding logic expanded by logical (truth) constants without evaluated syntax, provided that the concept of provability is added as a special definition. Existence of higher order fuzzy logic (fuzzy type theory) for each of the emphasized logics with traditional syntax (including, of course, classical one) is also indicated; such a logic is ready for extension to FLb.

## 5. Trichotomous evaluative linguistic expressions

Trichotomous evaluative linguistic expressions (or, simply, evaluative expressions) are expressions of natural language, for example, *small, medium, big, about twenty five, roughly one hundred, very short, more or less deep, not very tall, roughly warm or medium hot, quite roughly strong, roughly medium size*, and many others. They form a small but very important part of natural language and are present in its everyday use at practically any time. The reason is that people very often need to evaluate phenomena around them and make important decisions on the basis of them. Besides others, evaluative expressions are used when learning how to control (e.g., driving a car), and in many other activities. Motivation for their study in fuzzy logic has been provided by L. A. Zadeh in (Zadeh 1975).

We argue that the meaning of evaluative linguistic expressions is the fundamental bearer of the vagueness phenomenon and, namely, that vagueness of their meaning is a consequence of the indiscernibility between objects. According to our discussion after Hypothesis 1.1, indiscernibility is in fuzzy logic modeled by fuzzy equality.

All the details about the formal theory of evaluative linguistic expressions can be found in (Novák 2006a, 2008a). As usual, we distinguish *intension* (a property), and *extension* in a given *context* of use (i.e., a *possible world*; see Fitting (2006))[1].

The mathematical representation of intension is a function defined on a set of contexts which assigns to each context a fuzzy set of elements. Intension leads to different truth values in various contexts but is invariant with respect to them. *Extension* is a class of elements (i.e., a fuzzy set) determined by the intension when setting a *specific context*. It depends on the particular context of use and changes whenever the context is changed. For example, the expression "tall" is a name of an intension being a property of some feature of objects, i.e. of their height. Its meaning can be, e.g., 30 cm when a beetle needs to climb a straw, 30 m for an electrical pylon, 200 m or more for a skyscraper, etc.

Though the structure of evaluative expressions is more complicated, we will consider in this paper only the *pure evaluative expressions* that are expressions of the form ⟨linguistic hedge⟩⟨atomic evaluative expression⟩, where linguistic hedges are e.g. *very*, *more or less* and atomic evaluative expressions are *small*, *medium* and *big*. The essential characteristics of their meaning is the following:

(i) Extensions are classes of elements taken from nonempty, linearly ordered and bounded scale representing *context*, in which three distinguished limit points can be determined: *left bound*, *right bound*, and a *central point*.
(ii) Each of the above limit points is a starting point of some horizon running towards the next limit point in the sense of the ordering and vanishing beyond. Each horizon is determined by a reasoning analogous to that leading to the sorites paradox. Then extension of any evaluative expression is delineated by a specific horizon resulting from a shift of the above horizon. The modification corresponds to a linguistic hedge and is "small for big truth values" and "big for small ones".
(iii) Each scale is vaguely partitioned by the fundamental evaluative trichotomy consisting of a pair of antonyms, and a middle member (typically, "small, medium, big").

A formal logical theory of the meaning of evaluative linguistic expressions $T^{Ev}$ in FTT is formed on the basis of the above characteristics. It is determined by 11 special axioms and its language has means by which we can express formally the notions of context, horizon, and many others. The leading principle is based on Hypothesis 1.1, i.e. on a construction of a special fuzzy equality (the model of indiscernibility relation) characterizing the meaning of evaluative expressions.

The $T^{Ev}$ also has means to distinguish the meaning of evaluative expressions from the meaning of *evaluative predications* that are special linguistic expressions of the form

$$X \text{ is } \langle\text{linguistic hedge}\rangle\langle\text{TE-adjective}\rangle \tag{4}$$

where $X$ is a variable for some specific *feature* of objects (e.g., temperature, pressure, height, depth, etc.) and TE-adjective is an adjective of the kind "small,

[1]We follow the possible world semantics. In the theory of evaluative linguistic expressions, however, it is more convenient to replace the general term "possible world" by a more apt term "context".

medium, big", for example "long, shallow, high", etc. The former are *canonical* TE-adjectives.

Intensions of evaluative expressions and predications are represented by specific formulas which are denoted by $(Sm\,\boldsymbol{\nu})$ (corresponds to "small"), $(Me\,\boldsymbol{\nu})$ (corresponds to "medium"), $(Bi\,\boldsymbol{\nu})$ (corresponds to "big") and where $\boldsymbol{\nu}$ is a formula assigned to a linguistic hedge. For the sake of brevity, we do not demonstrate the structure of them and so, we refer the reader to (Novák 2008a).

Possible extensions of evaluative expressions in a model are depicted in Fig. 2. In the picture, the context is determined by a triple $\langle v_L, v_S, v_R\rangle$ where $v_L$ is a

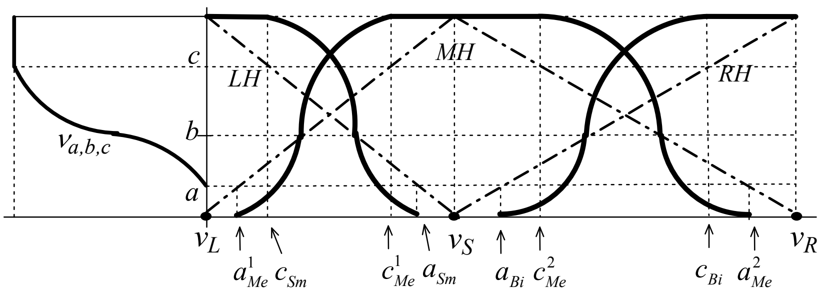


Figure 2. Scheme of the construction of extensions of evaluative expressions

left bound, $v_S$ central point and $v_R$ the right bound. The $LH$, $MH$ and $RH$ are fuzzy sets interpreting the left, medium and right horizon, respectively. These fuzzy sets are determined by a special fuzzy equality $\approx_w$ constructed for each context $w$ from one universal fuzzy equality (cf. Hypothesis 1.1). The $\nu_{a,b,c}$ is a function specific for each linguistic hedge $\boldsymbol{\nu}$ which represents the corresponding horizon shift. The composition of each horizon and its shift provides extension of the evaluative predication (4) being a fuzzy set of elements.

It is possible to construct a model of the theory $T^{Ev}$ of evaluative expressions. Hence, using the completeness theorem, we can prove the following:

**Theorem 5.1 :** *The theory of evaluative linguistic expressions is consistent.*

## 6. Some problems of vagueness from the point of view of fuzzy logic

### 6.1 *Sorites paradox*

The *sorites paradox* is usually considered as the crucial display of the vagueness phenomenon. It has been discussed in many papers and books both from a philosophical as well as from a mathematical point of view — see, e.g., (Beall 2003, Black 1937, Goguen 1969, Hájek and Novák 2003, Keefe 2000, Novák *et al.* 1999, Parikh 1971, Williamson 1996) and elsewhere.

Sorites paradox has several equivalent formulations and is essentially the same as the other well known paradox, the so called falakros (bald men) paradox. Formally it can be presented as a sequence of deductions

$$\mathbb{FN}(0), \mathbb{FN}(0) \Rightarrow \mathbb{FN}(1), \mathbb{FN}(1), \ldots, \mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1), \mathbb{FN}(n+1), \ldots$$

where the predicate $\mathbb{FN}(n)$ means, e.g., "*n does not form a heap*", "*n is small*", "*n is feasible*", etc. Each $\mathbb{FN}(n+1)$ is obtained from $\mathbb{FN}(n)$ and $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$

using modus ponens. The paradox arises when assuming the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ to be true.

We argue that the correct analysis of sorites should be based on the following assumptions:

(i) "0 does not form a heap" is valid.
(ii) "If $n$ does not form a heap then $n+1$ does not form it" is *practically valid* but not completely valid.
(iii) "There is an $n$ forming a heap" is valid.

We will show that in fuzzy logic these assumptions do not lead to contradiction.

The following theorem can be proved in $\mathrm{Ev}_{\mathrm{Ł}}$(see Hájek and Novák (2003), Novák *et al.* (1999)).

**Theorem 6.1:** *T be a consistent fuzzy theory in which all Peano axioms are accepted in the degree 1. Let* $1 \geq \varepsilon > 0$ *and* $\mathbb{FN} \notin J(T)$ *be a new predicate. Then*

$$T^{+} \quad = \quad T \ \cup \ \left\{{}^{1}/\mathbb{FN}(0), {}^{1-\varepsilon}/(\forall n)(\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)), {}^{1}/(\exists n)\neg\mathbb{FN}(n)\right\}$$

*is a conservative extension of* $T$.

In this theorem ${}^{1}/\mathbb{FN}(0)$ is an evaluated formula expressing that $\mathbb{FN}(0)$ is true — this is a formalization of the assumption (i). Similarly for (iii). The assumption (ii) requires the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ to be practically valid. Setting values of $\varepsilon$ provides various possible models of sorites.

Note that a specific heap, in fact, depends on a context. For example, a heap of stones of size 1 cm may have much smaller number of stones than a heap of stones of size 1 mm. But this is a matter of concrete model (!), the logic itself is not influenced. The context is in our solution accomplished by specific value of $\varepsilon$.

We can also provide a solution of sorites in a logic with traditional syntax, namely BL (cf. Hájek and Novák (2003)). This is achieved using a special predicate $At$ interpreted as "almost true" which acts as a special unary connective. This solution takes a more specific form in connection with the concept of horizon as formalized in FTT discussed below.

### 6.2 *Horizon*

The concept of horizon plays a leading role in formation of the *alternative set theory* (AST) by P. Vopěnka (Vopěnka 1979, 1989). It can be informally characterized as follows.

(i) It is a threshold terminating *our view* of the world.
(ii) The world is not terminated by the horizon and continues beyond it.
(iii) The part of the world before horizon is delineated non-sharply, our ability to discern phenomena diminishes when getting closer to the horizon.
(iv) It is not fixed and can be moved along in the world.

Note that the concept of horizon is used in AST to develop the theory of *natural infinity* — an infinity that emerges in large sets that are finite but not located as a whole before the horizon.

We argue that the concept of horizon is analogous to sorites. Within the theory of evaluative expressions $T^{Ev}$ outlined above, it is formalized using a special fuzzy equality $\approx_w$ introduced w.r.t. a context $w$. The sorites then can be obtained as follows.

Assume the theory of Peano arithmetics and let the context be $w_N = \langle 0, p, q\rangle$. Here, $p$ represents a central point of the context $w_N$ (see discussion on p. 14). Furthermore, we put $\mathbb{FN}(n) = [0 \approx_{w_N} n]$ — the truth values of the statement "$n$ is approximately equal to 0 in the context $w$" which can be informally interpreted as "$n$ stones do not form a heap".

For proofs of the following theorems see (Novák 2008a).

**Theorem 6.2:**

*(a)* $\vdash \boldsymbol{\Delta}\mathbb{FN}(0)$,
*(b)* $\vdash (\forall n)(\boldsymbol{\Delta}(p \leq n) \Rightarrow \neg\mathbb{FN}(n))$,
*(c)* $\vdash (\exists m)(0 < m \,\&\, \hat{\Upsilon}(\mathbb{FN}(m)))$,
*(d)* $\vdash \neg(\exists n)(\boldsymbol{\Delta}\mathbb{FN}(n) \,\&\, \boldsymbol{\Delta}\neg\mathbb{FN}(n+1))$,
*(e)* $\vdash (\forall n)(\mathbb{FN}(n) \Rightarrow \cdot(n \approx_{w_N} n+1) \Rightarrow \mathbb{FN}(n+1))$.

The property (a) states classically that 0 surely does not form a heap; (b) states that each $n$ behind central point $p$ already forms a heap. This corresponds to our practical experience — in the given context, one can always show a heap. The property (c) states that there is a borderline number $m$ which is neither surely not a heap nor surely the opposite. (d) states that there is no number $n$ such that is surely not a heap and $n+1$ is surely a heap. Finally, (e) states that the implication $\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)$ is *almost true* in the sense introduced by P. Hájek in (Hájek and Novák 2003) where the degree of being "almost true" is measured by the degree in which the fuzzy equality $(n \approx_{w_N} n+1)$ is true. This is again in correspondence with the experience since adding a stone to what is not yet a heap imperceptibly changes the shape of the former towards the heap.

*Sorites in evaluative expressions.*

The above theorem is a basis for analogous theorem concerning the meaning of evaluative expressions of the form "⟨linguistic hedge⟩*small*" (for example, *small, very small, more or less small*, etc.).

**Theorem 6.3:** *Let* $\boldsymbol{\nu}$ *be a linguistic hedge. Then*

*(a)* $\vdash \boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(0)$,
*(b)* $\vdash (\exists p)(\boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(p))$,
*(c)* $\vdash \neg(\exists n)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(n) \,\&\, \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(n+1))$,
*(d)* $\vdash (\forall n)((Sm\,\boldsymbol{\nu})(n) \Rightarrow At((Sm\,\boldsymbol{\nu})(n+1))$
*where At is measured by a special fuzzy equality* $\boldsymbol{\nu}(n \approx n+1)$.

This means that 0 is surely small, there is a number that surely is not small, there is no surely small number $n$ such that $n+1$ surely is not small, and if $n$ is small, then it is almost true that $n+1$ is also small. An analogous theorem can be proved also about *big* and *medium.*

**Theorem 6.4:**

$$T^{Ev} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Sm\,\boldsymbol{\nu})(x) \,\&(x < y \Rightarrow \boldsymbol{\Delta}\neg(Sm\,\boldsymbol{\nu})(y))),$$

$$T^{Ev} \vdash \neg(\exists x)(\forall y)(\boldsymbol{\Delta}(Bi\,\boldsymbol{\nu})(x) \,\&(y < x \Rightarrow \boldsymbol{\Delta}\neg(Bi\,\boldsymbol{\nu})(y))).$$

According to this theorem, there is no "surely ⟨*linguistic hedge*⟩*small* $x$" followed by a "surely not ⟨*linguistic hedge*⟩*small* $y$" which we can paraphrase as there is no last "surely ⟨*linguistic hedge*⟩*small* $x$". Similarly, there is no first "surely ⟨*linguistic hedge*⟩*big* $x$" ($x, y$ need not represent natural numbers). It must be noted that both theorems above hold *in arbitrary context* (we have omitted the details for simplicity).

### 6.3 *Tall and Taller*

In (Keefe 2000), one can encounter a somewhat unclear discussion about "tall and taller" aiming at the critiques of fuzzy logic. Let us propose the following.

We will interpret *Tall* by evaluative expression "big" (in the formal theory $T^{Ev}$ outlined above) and *Taller* as a binary crisp relation

$$\vdash y\ \mathit{Taller}\ x \equiv \mathbf{\Delta}(x < y)$$

(it is not a problem to interpret it also as a fuzzy relation).

Using the properties of $T^{Ev}$, we can easily prove the following.

**Theorem 6.5:**

$$\vdash \mathit{Tall}\ (x)\ \&\ (y\ \mathit{Taller}\ x) \Rightarrow \mathit{Tall}\ (y).$$

The interpretation is straightforward: if $x$ is tall and $y$ is taller than $x$ then $y$ is also tall — the truth degree of $y$ being tall is greater than or equal to the truth degree of $x$ being tall.

From the theorem, we can derive, for example,

$$\nvdash \mathit{Short}(x)\ \&\ (y\ \mathit{Taller}\ x) \Rightarrow \mathit{Tall}\ (y),$$
$$\vdash \mathit{Tall}\ (x)\ \&\ \mathit{not}\ \mathit{Tall}\ (x) \equiv \bot.$$

Hence, it is *not provable* that, if $x$ is short and $y$ is taller than $x$ then $y$ is tall. At the same time, the strong conjunction of $x$ being tall and also not tall is false. One can see that there is no controversy in this analysis.

### 6.4 *Higher-order vagueness*

Higher-order vagueness seems to be, in a sense, an artificial problem that arises when deliberating about consequences of the definition of a vague property that should have definitely positive, negative, and borderline cases. The main idea consists of thinking about the fact that vagueness has no "end", i.e. that there are no sharp boundaries in any respect. According to (Keefe 2000), there is neither a sharp boundary between positive and negative cases, nor between borderline cases and other sharp boundaries. It means, besides others, that if a vague property $A(x)$ of elements is given, then there are values of $x$ for which "$A(x)$ is borderline" is itself borderline. In other words, the property "to be borderline" is also vague. Then we can iterate, i.e. there are borderline borderline cases as well as borderline cases of "definitely $A(x)$", "definitely $\neg A(x)$", etc. We can also say that higher-order vagueness stems from vagueness of the property "to be borderline", i.e., that there are elements "more or less borderline". Thus, elements having a given property $A$ that are "typically borderline" should be distinguishable from elements that are borderline only "a little", i.e. close to prototypical examples of $A$ or not $A$. We propose a fuzzy logic model of higher-order vagueness in Ł-FTT.

How can we recognize a borderline element in the fuzzy logic model? The answer follows from the answer to the question, what does it actually mean that a given property is vague? In fuzzy logic, this should be done in accordance with the assigned truth degrees.

For simplicity, we will consider only first-order properties of type $A_{o\alpha}$[1]. Let us

[1] Of course, we can consider also possible worlds (contexts) in our theory.

now define the property "to be vague" formally as follows:

$$Vag := \lambda u_{o\alpha}\,(\forall t_o)(\exists y_\alpha)\boldsymbol{\Delta}(u_{o\alpha}y_\alpha \equiv t_o).$$

The formula $Vag\,A_{o\alpha}$ simply says that a formula $A_{(o\alpha_n)}$ is vague if its interpretation is a surjective function to the set of all truth values. This definition can be justified by the very principle of fuzzy logic — to model vagueness via assignment of truth values from a scale (basically infinite) to elements so that $\mathcal{M}_p(A_{o\alpha}(x_\alpha)) = a \in L$ expresses a truth value of the statement "an element $m_\alpha = p(x_\alpha)$ has the property $A$". The truth values $a \neq \mathbf{0}, \mathbf{1}$ characterize borderline cases. Since there should be no truth value gaps, a vague property must principally attain any truth value from the scale (algebra) $L$. Note that, e.g. in Ł-FTT, this assumption enables us to model the Black's "museum of applied logic" (cf. Black (1937)) — cutting arbitrary small piece of a chair corresponds to lowering the truth value $\mathcal{M}_p(\text{Chair}(x))$ by arbitrary small $\varepsilon$.

Because in Ł-FTT, the equivalence $y_o \equiv z_o$ is interpreted by the continuous operation of biresiduation $\leftrightarrow$, $\vdash Vag(y_o \equiv)^2$ (twice application of the strong conjunction **&**) holds true.

If the degree of $\mathcal{M}_p(A_{o\alpha}(x_\alpha))$ is close to 0 or 1 then the element $p(x_\alpha)$ is more definite, i.e., less borderline. The closer it is to the neutral value †[1] the more it is borderline. Therefore, we will introduce below a formula $Brd_{o\alpha(o\alpha)}$ expressing that an element $x_\alpha$ of type $\alpha$ is a *borderline* case of a property $A_{o\alpha}$. We will do it, moreover, iteratively for arbitrary order.

First, we define recursively the following special formulas:

$$C_0 := \lambda z_o\, z_o, \tag{5}$$

$$C_1 := \lambda z_o\,(C_0\, z_o \equiv \dagger)^2, \tag{6}$$

$$\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_n := \lambda z_o\,(C_{n-1} z_o \equiv \dagger)^2. \tag{7}$$

**Lemma 6.6:** *If* $\vdash Vag\,u_{o\alpha}$ *then* $\vdash Vag(C_n u_{o\alpha})$ *for all* $n$.

**Proof:** The statement is trivial for $n = 0$. Let the property hold for some $n$, $\mathcal{M}$ be a model and $p$ an assignment to variables such that $p(t_o) = a \in L$. Then $\mathcal{M}_p((C_n z_o \equiv \dagger)^2 \equiv t_o) = a$ if we find some $b \in L$ such that $p(z_o) = b$. Such $b$, however, exists by the assumption. Therefore, $\mathcal{M}_p((\exists z_o)\boldsymbol{\Delta}((C_n z_o \equiv \dagger)^2 \equiv t_o)) = 1$ for arbitrary $p$ which leads to $\mathcal{M}_p((\forall t_o)(\exists z_o)\boldsymbol{\Delta}((C_n z_o \equiv \dagger)^2 \equiv t_o)) = 1$. We conclude that $\vdash Vag(C_{n+1}u_{o\alpha})$. □

**Lemma 6.7:**

*(a)* $\vdash \hat{\Upsilon}(z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.
*(b)* $\vdash \hat{\Upsilon} z_o \wedge \hat{\Upsilon}(z_o \equiv \dagger)^2 \equiv \hat{\Upsilon}(z_o \equiv \dagger)^2$.
*(c)* $\vdash (z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.
*(d)* $\vdash \hat{\Upsilon} z_o \wedge (z_o \equiv \dagger)^2 \equiv (z_o \equiv \dagger)^2$.

**Proof:** We will use the completeness theorem. Let $\mathcal{M}$ be a model and $p$ an assignment of elements of $\mathcal{M}$ to variables.

(a) If $\mathcal{M}_p(\hat{\Upsilon}(z_o \equiv \dagger)^2) = 0$ then the formula in (a) is true in the degree 1 in $\mathcal{M}_p$. Let $\mathcal{M}_p(\hat{\Upsilon}(z_o \equiv \dagger)^2) = 1$. Then $\mathcal{M}_p(z_o) \notin \{0, 0.5, 1\}$ and so $\mathcal{M}_p(\hat{\Upsilon} z_o) = 1$. Consequently, (a) is provable.

[1]Recall that in the standard semantics, interpretation of † is the truth value 0.5.

(b) is a consequence of (a).

(c) If $\mathcal{M}_p(\hat{\Upsilon} z_o) = 1$ then the truth value of (c) in $\mathcal{M}_p$ is also 1. Let $\mathcal{M}_p(\hat{\Upsilon} z_o) = 0$. Then $\mathcal{M}_p(z_o) \in \{0, 1\}$ and so, $\mathcal{M}_p((z_o \equiv \dagger)^2) = 0$ and so, the truth value of (c) in $\mathcal{M}_p$ is 1. Consequently, (c) is provable.

(d) is a consequence of (c). □

Now, we define the predicate "to be borderline" on the level $n$ as follows:

$$Brd^{(1)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2 \tag{8}$$

$$Brd^{(n)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(Brd^{(n-1)}\, u_{o\alpha} x_\alpha) \wedge (Brd^{(n-1)}\, u_{o\alpha} x_\alpha \equiv \dagger)^2 \tag{9}$$

where $(Ax \equiv \dagger)^2$ means $(Ax \equiv \dagger)\,\&(Ax \equiv \dagger)$. This definition is based on the following idea: all elements $x$ for which $Ax$ is true in a degree different from 0, 1 are borderline. The closer this degree is to 0.5 the more indefinite this element is; if the truth degree is 0.5 then it is "typically borderline", i.e. neither close to definitely $Ax$, nor close to definitely not $Ax$. Similar reasoning can be done on any level $n$.

**Lemma 6.8:** *The following holds true:*

$$\vdash Brd^{(n)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_{n-1}(u_{o\alpha} x_\alpha)) \wedge C_n(u_{o\alpha} x_\alpha) \tag{10}$$

*for all $n \geq 1$.*

**Proof:** After rewriting, we obtain for $n = 1$

$$\vdash Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2,$$

i.e. $\vdash Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge C_1(u_{o\alpha} x_\alpha)$. Furthermore, for $n = 2$ we have

$$\vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(\hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2) \wedge \\ ((\hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_1(u_{o\alpha} x_\alpha) \equiv \dagger)^2) \equiv \dagger)^2. \tag{11}$$

Then, using Lemma 6.7 and the properties of FTT we obtain

$$\vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_1(u_{o\alpha} x_\alpha)) \wedge (C_1(u_{o\alpha} x_\alpha) \equiv \dagger)^2,$$

i.e. $\vdash Brd^{(2)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_1(u_{o\alpha} x_\alpha)) \wedge C_2(u_{o\alpha} x_\alpha)$. Analogously we proceed up to arbitrary $n$. □

**Theorem 6.9:**

*(a) If $\vdash Vag\, u_{o\alpha}$ then $\vdash Vag(Brd^{(n)}\, u_{o\alpha})$ for all $n \geq 1$.*

*(b) To every formula $A_{o\alpha}$ (vague property) and every $n \geq 1$ the following is provable:*

*(i) $\vdash (\exists x_\alpha)\, Brd^{(n)}\, A_{o\alpha} x_\alpha$,*

*(ii) $\vdash (\exists x_\alpha) \hat{\Upsilon}(Brd^{(n)}\, A_{o\alpha} x_\alpha)$.*

**Proof:** (a) is a consequence of Lemmas 6.6 and 6.8.

(b) Both statements follow from (a).

□

This theorem states that the property "to be borderline" is vague. Moreover, the case (b) states that there is an element that is "borderline borderline

... borderline", and also an element that is "definitely borderline ... borderline".

Another possibility to model higher-order vagueness in fuzzy logic is based on the use of the sophisticated theory of fuzzy IF-THEN rules in FLb (see Novák (2005b), Novák and Lehmke (2006)), i.e. the rules of the form

$$\mathsf{IF}\ X \text{ is } \mathcal{A}\ \mathsf{THEN}\ Y \text{ is } \mathcal{B}$$

where $\mathcal{A}, \mathcal{B}$ are trichotomous evaluative linguistic expressions of natural language, or they can be arbitrary vague properties. Using the rules we can characterize the vagueness of a given predicate of arbitrary order. Then using the *perception-based logical deduction* (Novák 2005b) we can show a given element $p(x_\alpha)$ to be a borderline one in arbitrary order.

## 7. Conclusions

Mathematical fuzzy logic is a group of well established and powerful formal logical tools. We argue that it provides a reasonable and well working mathematical model of the vagueness phenomenon. The main idea consists in introducing a specific algebra of truth values by which the truth value of the statement "an element $x$ has the property $A$" is expressed. Note that all available analyzes of vagueness exhibit a clear classification of elements to *typical* ones, i.e. those having a property $A$, those not having it, and *borderline* ones. This immediately gives rise to three truth values. More truth values just provide a more subtle classification. We are convinced that any attempt to avoid degrees is hopeless and can lead only to impoverishing the theory. Though the degree-theoretical approach need not be omnipotent, it certainly can be very helpful.

The degrees are hidden also in the *supervaluation* theory (see, for example, Varzi (2000)). Its main idea is that vague phenomena can be made precise in a variety of different ways. The truth that an element has a vague property is its supervaluation, which is a function of the tentative ordinary (classical) truth valuations of this proposition. For each way of making it precise, we get a new tentative classical valuation indicating whether the proposition, as thus made precise, is true or false. If every way of making the proposition precise makes it (classically) true, all of its tentative valuations will be true. If every precise version of the proposition is false, all of the tentative classical valuations are false. Otherwise, we get a mixture of tentative valuations. The result of supervaluation of the vague proposition is true if all the tentative valuations are true, and it is false if they are all false; otherwise it is undefined. We argue that this situation can be embedded into fuzzy logic and thus, in fuzzy set theory. One attempt has already been done in (Fermüller and Kosik 2006), where the authors suggest the introduction of some measure on a class $K$ of models. More detailed research still needs to be done.

We have also outlined one possible way to model higher-order vagueness in fuzzy logic. Though the theory is not yet finished, the results seem to be promising.

The great power of fuzzy logic lies also in its ability to capture the meaning of vague expressions of natural language. Until now, the theory was confined to a small though important class of evaluative linguistic expressions, conditional clauses, and partly also to the class of linguistic quantifiers (Novák 2008b). We argue that the results are in accordance with the intuitive meaning and have a wide potential for various kinds of applications.

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