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Institute for Research and Applications of Fuzzy Modeling

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Irina Perfilieva, Hans De Meyer, Bernard De Baets, Dagmar Plšková

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University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-59-7091401 fax: +420-59-6120478
e-mail: irina.perfilieva@osu.cz

Cauchy Problem with Fuzzy Initial Condition and Its Approximate Solution with the Help of Fuzzy Transform .

Irina Perfilieva[a], Hans De Meyer[b], Bernard De Baets[b], Dagmar Plšková[a]

***Abstract*— We investigate the Cauchy problem for ordinary differential equation (ODE) with fuzzy initial condition. We define a solution to this problem and propose a new method of how an approximate solution can be constructed. The proposed method is based on the technique of fuzzy transforms and ends up with a solution of a system of fuzzy relation equations. The approximate solution can be expressed formally and the quality of approximation is estimated.**

I. Introduction

In this paper, we focus on the Cauchy problem for ordinary differential equation (ODE) with fuzzy initial condition. We propose to apply the technique of fuzzy transforms and obtain an approximate solution expressed by the fuzzy relation. The latter gives a solution to a respective system of fuzzy relation equations.

In the theory of fuzzy differential equations (FDE, for short) which goes to [5], [6] and uses the notion of Hukuhara derivative, significant problems arise with any attempt to come up with a formally expressed solution. This is because the solutions of FDE have different properties from the solutions of ordinary differential equations and they lack e.g., stability or periodicity.

There are several approaches to define a solution for a FDE, such as Hukuhara approach [5], [7], [10], differential inclusions [3], quasiflows, differential equations in metric spaces [8], [9] and the so called levelwise approach [19] which probably, is the most "natural". In a certain sense, all these approaches use α-cuts representation of fuzzy sets and thus, the solution they defined is based on this notion too. This is the main reason why there are technical difficulties in expressing the solution formally. Several numerical methods for FDE were proposed to avoid this difficulty too, see e.g. [1], [2], [4].

Apart from this stream, the technique of fuzzy transforms has been proposed in [11] and further investigated in [16], [17]. It turned out, that it is a fairly powerful technique with many nice properties and great potential for various applications, such as special numerical methods, solution of ordinary and partial differential equations [12], [18], applications to signal processing, compression and decompression of images and fusion of images [16], [15]. A numerical method which generalizes the Euler method has been proposed in [12] for an ordinary Cauchy problem. It was challenging to the author to show that the technique of fuzzy transforms copes with that classical problem and has certain advantages in their special cases. In this contribution, we had a purpose to extend the technique of fuzzy transforms to the case of the Cauchy problem for ODE with fuzzy initial condition and moreover, to propose a formal expression for a solution. Due to the complexity of the definition of a precise solution to the above mentioned Cauchy problem, the latter has been realized for an approximate solution only.

The following characterizes the structure and novelty of the proposed contribution. A construction of approximation models on the basis of the F-transform is recalled in Section II. In Sections III, we show how F-transforms can be used in solving ordinary differential equations with ordinary initial conditions. The generalized Euler method has been proposed and justified. The Sections IV focuses on the Cauchy problem with fuzzy initial conditions and its solution. We define a solution of this problem and propose, how an approximate solution can be constructed. The proposed method is based on the technique of fuzzy transforms and uses set of solutions of a system of fuzzy relation equations. Finally, we give graphical illustrations.

II. Fuzzy Partition and Fuzzy Transform

Approximate representation of a continuous function by a system of fuzzy IF-THEN rules proved that a required accuracy can be achieved even if we have a rough information about a behavior of a function. By this we mean that, given average values of a function within intervals which cover the universe, we can reconstruct the function with the respective level of accuracy.

We begin our analysis by recalling basic agreements and notation. Fuzzy sets will be considered on the universe $\mathbb{R}$ of real numbers (or on some interval $[a, b] \subset \mathbb{R}$) and will be identified with their membership functions, i.e. mappings from $\mathbb{R}$ to $[0, 1]$. The set of fuzzy sets on $\mathbb{R}$ (resp. $[a, b]$) will be denoted by $\mathcal{F}(\mathbb{R})$ (resp. $\mathcal{F}([a, b])$). The domain consists of elements of a fuzzy partition of some interval $[a, b]$ of real numbers (universe of discourse).

Definition 1 ([16])
Let $[a, b]$ be a real interval and $x_1 < \cdots < x_n$ be nodes within $[a, b]$ such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1, \ldots, A_n : [a, b] \longrightarrow [0, 1]$, constitute a fuzzy partition *of $[a, b]$ if they fulfil the following conditions for $k = 1, \ldots, n$:*

[a]Address: University of Ostrava, Institute for Research and Applications of Fuzzy Modelling, 30. dubna 22, 70103 Ostrava, Czech Republic (email: {Irina.Perfilieva, Dagmar.Plskova}@osu.cz).

[b]Address: University of Gent, Coupure links 653, B-9000 Gent, Belgium (email: {Bernard.DeBaets, Hans.DeMeyer}@ugent.be)

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1) $A_k : [a,b] \longrightarrow [0,1]$, $A_k(x_k) = 1$;
2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$;
3) $A_k(x)$ is continuous;
4) $A_k(x)$, $k = 2,\dots,n$, monotonically increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1,\dots,n-1$, monotonically decreases on $[x_k, x_{k+1}]$;
5) for all $x \in [a,b]$, the following orthogonality condition holds:

$$\sum_{k=1}^{n} A_k(x) = 1. \tag{1}$$

The membership functions $A_1(x),\dots,A_n(x)$ are called basic functions.

We say that a fuzzy partition $A_1(x),\dots,A_n(x)$, $n > 2$, is *h-uniform* if the nodes $x_1,\dots,x_n$ are h-equidistant, i.e. $x_k = a + h(k-1)$, $k = 1,\dots,n$, where $h = (b-a)/(n-1)$, and two more properties are fulfilled for $k = 2,\dots,n-1$:

6) $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0,h]$,
7) $A_k(x) = A_{k-1}(x-h)$, for all $x \in [x_k, x_{k+1}]$ and $A_{k+1}(x) = A_k(x-h)$, for all $x \in [x_k, x_{k+1}]$.

The reference to h (in the h-uniform partition) will be omitted if h is fixed or known from the context.

Let $C[a,b]$ be the set of continuous functions on interval $[a,b]$. The following definition (see also [16]) introduces the fuzzy transform of a function $f \in C[a,b]$. (For the sake of simplicity, we will consider functions with one variable. However, the fuzzy transform of a function of two or more variables can be easily obtained as a straightforward extension (see [18] for more details) of definitions given below.)

Definition 2
Let $A_1,\dots,A_n$ be basic functions which constitute a fuzzy partition of $[a,b]$ and f be any function from $C[a,b]$. We say that the n-tuple of real numbers $[F_1,\dots,F_n]$ given by

$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \qquad k = 1,\dots,n, \tag{2}$$

is the (integral) F-transform of f with respect to $A_1,\dots,A_n$.

We will denote the F-transform of a function $f \in C[a,b]$ with respect to $A_1,\dots,A_n$ by $\mathbf{F}_n[f]$. Then, according to Definition 2, we can write $\mathbf{F}_n[f] = [F_1,\dots,F_n]$. The elements $F_1,\dots,F_n$ are called *components of the F-transform.*

The F-transform with respect to $A_1,\dots,A_n$ establishes a linear mapping from $C[a,b]$ to $\mathbb{R}^n$ so that

$$\mathbf{F}_n[\alpha f + \beta g] = \alpha\mathbf{F}_n[f] + \beta\mathbf{F}_n[g]$$

for $\alpha, \beta \in \mathbb{R}$ and functions $f, g \in C[a,b]$. This linear mapping is denoted by $\mathbf{F}_n$ where n is dimension of the image space.

At this point we will refer to [16] for some useful properties of the F-transform components. The most important property concerns the following problem: how accurately is the original function f represented by its F-transform? We will show in this contribution that under certain assumptions on the original function, the components of its F-transform are *weighted mean values* of the given function where the weights are given by the basic functions.

Theorem 1 ([16])
Let f be a continuous function on $[a,b]$ and $A_1,\dots,A_n$ be basic functions which constitute a fuzzy partition of $[a,b]$. Then the k-th component of the F-transform minimizes the function

$$\Phi(y) = \int_a^b (f(x) - y)^2 A_k(x)dx \tag{3}$$

defined on $[f(a), f(b)]$.

A. Inverse F-transform

The inverse F-transform (with respect to $A_1,\dots,A_n$) takes an n-dimensional vector of reals $\mathbf{r} = (r_1,\dots,r_n)$ and produces a linear combination of basic functions with coefficients given by $\mathbf{r}$. In the case when $\mathbf{r}$ is the F-transform with respect to $A_1,\dots,A_n$ of some function $f \in C[a,b]$, i.e. $\mathbf{r} = \mathbf{F}_n[f]$, the above mentioned linear combination is an inversion formula.

Definition 3 ([16])
Let $A_1,\dots,A_n$ be basic functions which constitute a fuzzy partition of $[a,b]$ and f be a function from $C[a,b]$. Let $\mathbf{F}_n[f] = [F_1,\dots,F_n]$ be the F-transform of f with respect to $A_1,\dots,A_n$. Then the function

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x) \tag{4}$$

is called the inverse F-transform.

The theorem below shows that the inverse F-transform $f_{F,n}$ can approximate the original continuous function f with an arbitrary precision.

Theorem 2 ([16])
Let f be a continuous function on $[a,b]$. Then for any $\varepsilon > 0$ there exist n_ε and a fuzzy partition $A_1,\dots,A_{n_\varepsilon}$ of $[a,b]$ such that for all $x \in [a,b]$

$$|f(x) - f_{F,n_\varepsilon}(x)| \le \varepsilon \tag{5}$$

where f_{F,n_ε} is the inverse F-transform of f with respect to the fuzzy partition $A_1,\dots,A_{n_\varepsilon}$.

We can reformulate the result of Theorem 2 for the case of uniform fuzzy partitions of $[a,b]$ having in mind the fact that the number of nodes n determines the uniform fuzzy partition up to the shape of membership functions.

Corollary 1
Let f be a continuous function on $[a,b]$ and let $\{(A_1^{(n)},\dots,A_n^{(n)})_n\}$ be a sequence of uniform fuzzy partitions of $[a,b]$, one for each n. Let $\{f_{F,n}(x)\}$ be the sequence of inverse F-transforms, each with respect to the given n-tuple $A_1^{(n)},\dots,A_n^{(n)}$. Then for any $\varepsilon > 0$ there exists n_ε such that for each $n > n_\varepsilon$ and for all $x \in [a,b]$

$$|f(x) - f_{F,n}(x)| \le \varepsilon. \tag{6}$$

III. Approximate Solution to the Ordinary Cauchy Problem

In this section we show how F-transforms can be used in solving ordinary differential equations with ordinary initial conditions. At first, we will investigate the relation between the F-transform components of a function and its derivative.

Lemma 1
Let f be a function having finite derivative f'' everywhere in an open interval (a, b) and assume that f' is continuous on the closed interval $[a, b]$. Let $A_1, \ldots, A_n$, $n > 2$, be basic functions which constitute an h-uniform fuzzy partition of $[a, b]$. Assume that $F_1, \ldots, F_n$ and $F'_1, \ldots, F'n$ are the F-transform components of f and f' (with respect to $A_1, \ldots, A_n$), respectively. Then

$$F'_k = \frac{1}{h}(F_{k+1} - F_k) + O(h), \quad k = 1, \ldots, n-1. \quad (7)$$

Proof: Let all the assumptions formulated above be true. For each $x \in [a, b-h]$ we can replace $f'(x)$ by its approximation $\frac{f(x+h)-f(x)}{h}$ so that

$$f(x+h) = f(x) + hf'(x) + O(h^2). \quad (8)$$

Let us denote $g(x) = f(x+h)$ and consider g as a new function on $[a, b-h]$. Note that the interval $[a, b-h]$ is uniformly partitioned by $A_1, \ldots, A_{n-1}$ where the last function A_{n-1} is restricted to the subinterval $[b-2h, b-h]$. If we denote $G_1, \ldots, G_{n-1}$ the respective F-transform components of g then from (8) and linearity of the F-transform we obtain

$$G_k = F_k + hF'_k + O(h^2), \quad k = 1, \ldots, n-1.$$

It is easy to show that for $k = 1, \ldots, n-1$, $G_k = F_{k+1}$. Indeed, for $k = 2, \ldots, n-2$ we have

$$G_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} f(x+h)A_k(x)dx = \frac{1}{h}\int_{x_k}^{x_{k+2}} f(t)A_{k+1}(t)dt = F_{k+1}.$$

Similar computation can be performed for $k = 1, n-1$. Hence, after the substitution for G_k we obtain

$$F_{k+1} = F_k + hF'_k + O(h^2), \quad k = 1, \ldots, n-1 \quad (9)$$

which easily leads to (7). ■

The Cauchy problem consists in solving the following differential equation on the interval $[x_1, x_n]$:

$$y'(x) = g(x, y) \quad (10)$$

with the initial condition $y(x_1) = y_1$. The solution is any function y defined on the interval $[x_1, x_n]$ and such that it fulfils the initial condition at point x_1 and its derivative fulfils (10). Let us remind that if the function g is defined and continuous on a domain $G \subset [x_1, x_n] \times \mathbb{R}$, containing the point (x_1, y_1), and fulfils the Lipschitz condition with respect to y then the Cauchy problem has only one solution. This proposition is known as the Picard's theorem.

A. The F-Transform Based Method

The following method has been proposed in [12] for obtaining an approximate solution of (10):

- Choose $h > 0$ and make an h-uniform partition of $[x_1, x_n]$ with n basic functions $A_1, \ldots, A_n$.
- Use the following scheme

$$\hat{Y}_1 = y_1, \quad (11)$$

$$\hat{Y}_{k+1} = \hat{Y}_k + h\hat{G}_k, \quad k = 1, \ldots, n-1, \quad (12)$$

$$\hat{G}_k = \frac{\int_a^b g(x, \hat{Y}_k)A_k(x)dx}{\int_a^b A_k(x)dx}, \quad k = 1, \ldots, n-1, \quad (13)$$

  and obtain the approximate F-transform $[\hat{Y}_1, \ldots, \hat{Y}_n]$ of y with respect to $A_1, \ldots, A_n$.
- Obtain the desired approximation for y by the inverse F-transform applied to $[\hat{Y}_1, \ldots, \hat{Y}_n]$.

The proposed method is similar to the well-known Euler method and under similar assumptions, it has the same degree of accuracy.

Theorem 3
Let g be a function having finite derivative g'' everywhere in an open interval (a, b) and assume that g' is continuous on the closed interval $[a, b]$. Let moreover, g fulfil the Lipschitz condition with the constant L_g with respect to y. Then a local error of the scheme (11)-(13) is of the order h^2.

Proof: The following estimations are used in the sequel for $k = 1, \ldots, n-1$:

$$y(x_{k+1}) = y(x_k) + hg(x_k, y_k) + \frac{h^2}{2}g'(\xi_k, y(\xi_k))$$

$$\mid g(x_k, y_k) - g(x_k, \hat{Y}_k) \mid \leq 2L_g h,$$

$$\mid \hat{G}_k - g(x_k, \hat{Y}_k) \mid \leq \frac{1}{12}M_2 h^2$$

where $y_k = y(x_k)$, $M_2 = \max_{x \in (a,b)} \mid g''(x) \mid$.

Let us fix k, $2 \leq k \leq n-1$, and estimate

$$\mid y_{k+1} - \hat{Y}_{k+1} \mid = y(x_k) + hg(x_k, y_k) + \frac{h^2}{2}g'(\xi_k, y(\xi_k)) - (\hat{Y}_k + h\hat{G}_k) + hg(x_k, \hat{Y}_k) - hg(x_k, \hat{Y}_k) \leq \mid y(x_k) - \hat{Y}_k \mid + h \mid g(x_k, y_k) - g(x_k, \hat{Y}_k) \mid + h \mid g(x_k, \hat{Y}_k) - \hat{G}_k \mid + \frac{h^2}{2}M_1 \leq \mid y(x_k) - \hat{Y}_k \mid + 2L_g h^2 + \frac{1}{12}M_2 h^3 + \frac{h^2}{2}M_1 = \mid y(x_k) - \hat{Y}_k \mid + ch^2$$

where $M_1 = \max_{x \in [a,b]} \mid g'(x) \mid$ and c is an appropriate constant which depends on g, but neither on k nor on h. ■

Corollary 2
Let the condition of Theorem 3 be fulfilled. Then there is a constant C such that for any $2 \leq k \leq n$,

$$\mid y_k - \hat{Y}_k \mid \leq Ch.$$

IV. Cauchy Problem with Fuzzy Initial Condition

In this section, we consider the Cauchy problem where the initial condition is not known precisely. This usually occurs when the problem is a result of physical measurements. We will consider the case where the initial condition is a fuzzy set on the real line whose membership function is unimodular. The following is a formulation of what is given: the differential equation (cf. 10)

$$y'(x) = g(x, y)$$

where the function $y = y(x)$ is unknown and considered on the interval $[x_1, x_n]$, and the "value" of y at point x_1

$$y(x_1) = \tilde{Y}_1 \tag{14}$$

where $\tilde{Y}_1$ is a unimodular fuzzy set on $\mathbb{R}$ with the support $[y_{1,l}, y_{1,r}]$ and the kernel $\{y_1\}$. The latter means that $\tilde{Y}_1(y_1) = 1$.

It is not an easy task to define what can be considered as a solution of the differential equation (10) together with the initial condition (14). If we agree that a solution is a function on $[x_1, x_n]$ which fulfils (10) then what does it mean that this function has fuzzy value at point x_1? On the other hand, if a solution is a fuzzy relation which fulfils (14) at point x_1 then what does it mean that this relation fulfils (10)? Needless to say that we can speak about a problem only when we formulate what is given and what is expected to be a solution. On the basis of this discussion, we propose the following generalization of the Cauchy problem.

Definition 4 (Generalized Cauchy problem)
Let the differential equation (10) be considered with respect to the unknown function $y = y(x)$ which is defined on the interval $[x_1, x_n]$ and restricted at point x_1 by (14), i.e. the value $y(x_1)$ is any point u from the set $\mathrm{Supp}\,\tilde{Y}_1$ with the degree of membership $\tilde{Y}_1(u)$.

Let us consider the following three auxiliary Cauchy problems with the same differential equation (10), but different initial conditions:

$$y'(x) = g(x, y), \quad y(x_1) = y_{1,l}, \tag{15}$$

$$y'(x) = g(x, y), \quad y(x_1) = y_1, \tag{16}$$

$$y'(x) = g(x, y), \quad y(x_1) = y_{1,r}. \tag{17}$$

The solution of the Generalized Cauchy problem is a binary fuzzy relation $Y \in \mathcal{F}([x_1, x_n] \times \mathbb{R})$ which fulfils the following conditions:

- *For each $x \in [x_1, x_n]$, $Y(x, u)$ is a unimodal function on $\mathbb{R}$ such that its support is the interval $[v_x, w_x]$ and its kernel is $\{u_x\}$, i.e.*

$$Y(x, u_x) = 1. \tag{18}$$

- *If for each $x \in [x_1, x_n]$ we put $y(x) = u_x$ then the function y fulfils the Cauchy problem (16).*
- *If for each $x \in [x_1, x_n]$ we put $y^l(x) = v_x$ then the function y^l fulfils the Cauchy problem (15).*
- *If for each $x \in [x_1, x_n]$ we put $y^r(x) = w_x$ then the function y^r fulfils the Cauchy problem (17).*
- $Y(x_1, v) = \tilde{Y}_1$.

It is worth to be noticed that the shape of any section (membership function of a respective fuzzy set) of a solution of the Generalized Cauchy problem is not restricted by the formulation of that problem. The restrictions are given of a support of each section as well as on its kernel. Moreover, it is easy to see that if fuzzy set $\tilde{Y}_1$ degenerates to a singleton then the Generalized Cauchy problem coincides with the ordinary Cauchy problem.

The following theorem easily follows from the Picard's theorem.

Theorem 4
Let the function g in (10) be defined and continuous on a domain $G \subseteq [x_1, x_n] \times \mathbb{R}$, containing all points from the set $\{(x_1, y) \mid y_{1,l} \le y \le y_{1,r})\}$. Moreover, let g fulfil the Lipschitz condition with respect to y. Let fuzzy set $\tilde{Y}_1$ in (14) have a triangular shape. Then there exists a solution $Y \in \mathcal{F}([x_1, x_n] \times \mathbb{R})$ of the Generalized Cauchy problem such that for each $x \in [x_1, x_n]$, $Y(x, u)$ is a triangular shaped function.

Remark 1
The assumptions of Theorem 3 are more restrictive that those of Theorem 4. Therefore when we require the former we automatic have the latter.

In the foregoing text, the solution Y characterized in Theorem 4, will referred to as a solution of the Generalized Cauchy problem with triangular sections. The following algorithm computes an approximation to the solution Y with triangular sections:

Algorithm.

- Choose $h > 0$ and make an h-uniform fuzzy partition $A_1, \dots, A_n$ of $[x_1, x_n]$ such that $h(n-1) = (x_n - x_1)$.
- Use the F-transform based method and obtain the approximate F-transforms $\hat{\mathbf{F}}_n[y^l]$, $\hat{\mathbf{F}}_n[y]$ and $\hat{\mathbf{F}}_n[y^r]$ of solutions of (15), (16) and (17) respectively. Let us denote $\hat{\mathbf{F}}_n[y^l] = [\hat{Y}_{l1}, \dots, \hat{Y}_{ln}]$, $\hat{\mathbf{F}}_n[y] = [\hat{Y}_1, \dots, \hat{Y}_n]$ and $\hat{\mathbf{F}}_n[y^r] = [\hat{Y}_{r1}, \dots, \hat{Y}_{rn}]$.
- Create n triangular shaped fuzzy sets $\tilde{Y}_1, \dots, \tilde{Y}_n$, such that for each $k = 1, \dots, n$, the interval $[\hat{Y}_{lk}, \hat{Y}_{rk}]$ is the support of $\tilde{Y}_k$ and $\{\hat{Y}_k\}$ is its kernel.
- Consider the system of fuzzy relation equations

$$\begin{aligned} A_1 \circ R^h &= \tilde{Y}_1, \\ &\dots\dots\dots\dots \\ A_n \circ R^h &= \tilde{Y}_n, \end{aligned} \tag{19}$$

  where $\circ$ is the $\sup -*$ composition and R^h is an unknown fuzzy relation.
- Choose Łukasiewicz t-norm for $*$ and its residuum for $\rightarrow$ and take R^h such that it fulfills (19) with thus

specified $*$ and the following inequality

$$\bigvee_{i=1}^{n}(A_i(x) * \tilde{Y}_i(v)) \le R^h(x,v) \le \bigwedge_{i=1}^{n}(A_i(x) \to \tilde{Y}_i(v)). \tag{20}$$

Remark 2

Due to the orthogonality of $A_1, \ldots, A_n$ (1), the following two relations (lower and upper bounds of R^h in (20))

$$\check{R}^h(x,v) = \bigvee_{i=1}^{n}(A_i(x) * \tilde{Y}_i(v)),$$

$$\hat{R}^h(x,v) = \bigwedge_{i=1}^{n}(A_i(x) \to \tilde{Y}_i(v)),$$

are solutions of (19). This fact has been proved in [13]. Both of $\check{R}^h$ and $\hat{R}^h$ can be chosen for R^h. Moreover, R^h can be chosen in the following form too:

$$\bar{R}^h(x,v) = \sum_{i=1}^{n}(A_i(x) \cdot \tilde{Y}_i(v)). \tag{21}$$

The latter is due to the inequality $\check{R}^h \le \bar{R}^h \le \hat{R}^h$ proved in [14].

The following two statements estimate closeness between (triangular) sections of Y and respective sections of R^h. The first Lemma 2 compares sections of Y and R^h at nodes $x_1, \ldots, x_n$ of the h-uniform (fuzzy) partition of $[x_1, x_n]$. Then the second Lemma 3 compares sections of Y and R^h at other points $x \in [x_1, x_n]$. In both cases, the closeness is expressed by distance in the metric space of continuous functions.

Lemma 2

Let function g fulfil conditions of Theorem 3. Let fuzzy set $\tilde{Y}_1$ in (14) have triangular shape and $Y \in \mathcal{F}([x_1, x_n] \times \mathbb{R})$ be a solution of the Generalized Cauchy problem with triangular sections.

Let R^h be an approximate solution, obtained by the above given Algorithm, based on an h-uniform (fuzzy) partition of $[x_1, x_n]$ with nodes $x_1, \ldots, x_n$. Then at each node x_i, $i = 1, \ldots, n$, the respective section $R^h(x_i, \cdot)$ is a triangular shaped function on $\mathbb{R}$. Moreover, there exists h_0 such that for all $h < h_0$ the distance between Y and R^h at any node x_i is estimated by

$$\max_{v \in \mathbb{R}} \mid R^h(x_i, v) - Y(x_i, v) \mid \le Kh, \qquad i = 2, \ldots, n \tag{22}$$

where the constant K does not depend on i.

Lemma 3

Let function g fulfil conditions of Theorem 3. Let fuzzy set $\tilde{Y}_1$ in (14) have triangular shape and $Y \in \mathcal{F}([x_1, x_n] \times \mathbb{R})$ be a solution of the Generalized Cauchy problem with triangular sections.

Let $\bar{R}^h$ (21) be a solution of (19) with basic functions $A_1, \ldots, A_n$ and triangular shaped fuzzy sets $\tilde{Y}_1, \ldots, \tilde{Y}_n$, obtained by the Algorithm. Then at each point $\tilde{x} \in [x_1, x_n]$, different from any node x_i, $i = 1, \ldots, n$, the section $\bar{R}^h(\tilde{x}, \cdot)$ of the approximate solution given by $\bar{R}$, is close to the solution section $Y(\tilde{x}, \cdot)$ and their difference is estimated by

$$\max_{v \in \mathbb{R}} \mid Y(\tilde{x}, v) - \bar{R}^h(\tilde{x}, v) \mid \le Ph \tag{23}$$

where the constant P does not depend on $\tilde{x}$.

The statement in Lemma 3 is illustrated on Figure 3.

Summarizing both Lemma 2 and Lemma 3, we obtain the following theorem.

Theorem 5

Let function g fulfil conditions of Theorem 3. Let fuzzy set $\tilde{Y}_1$ in (14) have triangular shape and $Y \in \mathcal{F}([x_1, x_n] \times \mathbb{R})$ be a solution of the Generalized Cauchy problem with triangular sections.

Let $\bar{R}^h(x,v) = \sum_{i=1}^{n}(A_i(x) \cdot \tilde{Y}_i(v))$ (cf. 21) be a solution of (19) with basic functions $A_1, \ldots, A_n$ and triangular shaped fuzzy sets $\tilde{Y}_1, \ldots, \tilde{Y}_n$, obtained by the Algorithm. Then there exists h_0 such that for all $h < h_0$

$$\max_{v \in \mathbb{R}} \mid Y(\tilde{x}, v) - \bar{R}^h(\tilde{x}, v) \mid \le Qh, \qquad \tilde{x} \in [x_1, x_n] \tag{24}$$

where the constant Q does not depend on $\tilde{x}$.

The picture on Figure 1 illustrates the approximate solution $\bar{R}^h$ of the below given Cauchy problem with fuzzy initial condition:

$$y'(x) = \sqrt{x} - y,$$
$$y(0) = \tilde{Y}_1,$$

where $\tilde{Y}_1$ is a unimodal fuzzy set on $\mathbb{R}$ such that $\tilde{Y}_1(2) = 1$.

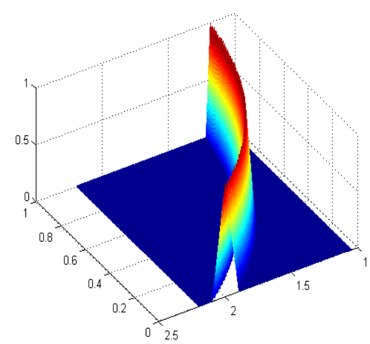

Fig. 1. Approximate solution (fuzzy relation $\bar{R}$) of the Cauchy problem with fuzzy initial condition.

V. Conclusion

The problem of solving an ordinary differential equation (ODE) with fuzzy initial condition has been investigated. We applied the technique of fuzzy transforms and obtained an approximate solution expressed by the fuzzy relation. The latter can be expressed formally with the help of arithmetic operations and operations of Łukasiewicz algebra.

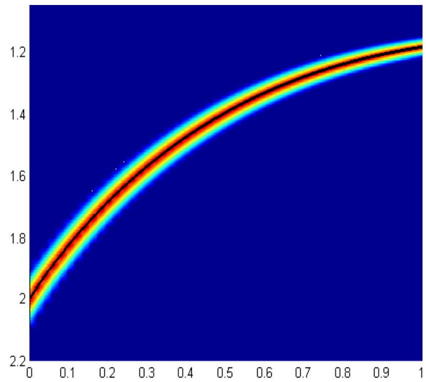

Fig. 2. Approximate solution (fuzzy relation $\bar{R}^h$) of the Cauchy problem with fuzzy initial condition and the precise solution (black thick curve) of the related Cauchy problem (16) with ordinary initial condition $y(x_1) = 2$. View from the above.

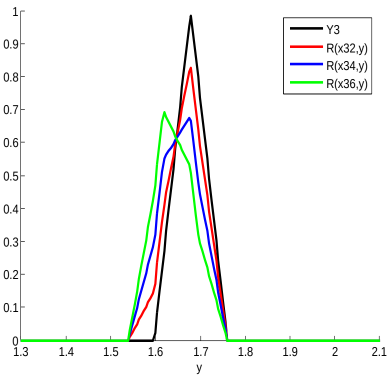

Fig. 3. Various sections of the approximate (fuzzy relation $\bar{R}^h$) and precise (fuzzy relation Y) solutions: $Y(x_3, \cdot)$ (black triangle) and $\bar{R}^h(x_{32}, \cdot)$, $\bar{R}^h(x_{34}, \cdot)$ and $\bar{R}^h(x_{36}, \cdot)$ such that $\mid x_3 - x_{32} \mid < \mid x_3 - x_{34} \mid < \mid x_3 - x_{36} \mid$.

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