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Contraction and Dilatation Operators in a Semilinear Space over Residuated Lattice

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Abstract

The notion of a semilinear space over residuated lattice is introduced. Two problems of solvability of systems of linear-like equations with $\sup -*$ or $\inf \to$ compositions are considered in a finite semilinear space as inverse problems with respect to respective homomorphisms. We prove that a system of equations with $\sup -*$-composition ($\inf \to$) is solvable if and only if its right-hand side is a fixed point of the contraction (dilatation) operator. Moreover, the solution set of the respective system is a class of equivalence of that fixed point which is a homomorphic image of the corresponding right-hand side.

Keywords: Semilinear space, Residuated lattice, Solvability of a system of equations, Contraction operator, Dilatation operator, Fixed point.

1 Introduction

Linear behavior of fuzzy systems has been discovered, e.g., in [3]. Two cases may occur: a behavior of a system is characterized by fuzzy IF-THEN rules (expert knowledge and similar), or it is characterized by a set of input-output pairs of fuzzy sets (monitoring, collecting knowledge, etc.). In the second case, the problem of solving a respective system of fuzzy relation equations arises [4, 7].

In this contribution, we show that a system of fuzzy relation equations can be considered as a system of linear-like equations in a semilinear space over a residuated lattice. We will focus on systems with $\sup -*$ or $\inf \to$ compositions because they are the most popular in practical applications.

The novelty of this contribution consists in considering both systems in parallel. This approach shows interrelation between both systems in what concerns their solvability and their solution sets. To be able to realize this approach we elaborate the notion of a semilinear space and make it useful in proving various properties of systems and their solutions.

We change an angle under which the problem of solvability is usually considered (see, e.g., [2, 4, 6]) and concentrate on characterizations of possible right-hand sides that make the respective systems solvable. We prove that a right-hand side vector must be fixed point of a special operator (contraction or dilatation). Both operators are introduced in this paper. Moreover, we prove that the solution set of the respective system is an equivalence class of that fixed point which is a homomorphic image of the respective right-hand side.

2 Residuated Lattice

The concept of *integral, residuated, commutative l-monoid* also known as *residuated lattice*) has been introduced by U. Höhle. We denote a residuated lattice by $\mathcal{L} = \langle L, \vee, \wedge, *, \to, 0, 1\rangle$ and recall the operation of biresiduation $\leftrightarrow$:

$$x \leftrightarrow y = (x \to y) \wedge (y \leftrightarrow x).$$

The following known algebras are examples of residuated lattice: boolean algebra, Gödel algebra, product (Goguen) algebra, Łukasiewicz algebra. In the sequel, $\mathcal{L}$ will always denote a residuated lattice with a carrier L.

2.1 Semirings and semimodules

In this subsection, we will change the definition of a semiring [5, 3] (we impose weaker requirements), give a definition of a semimodule and consider some examples of both structures. The concept of a semiring was described almost 100 years ago (originating in Dedekind's studies), but a comprehensive study has been published recently [5].

Definition 1
A left semiring $\mathcal{R} = \langle R, +, \cdot, 0, 1\rangle$ is an algebra where

(SR1) $\langle R, +, 0\rangle$ *is a commutative monoid,*

(SR2) $\langle R, \cdot, 1\rangle$ *is a groupoid with the left unit, i.e. for all $a \in R$, $1 \cdot a = a$,*

(SR3) for all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

Remark 1
Let us recall that the notion of a left semiring in [5, 3] contains two stronger requirements: $\langle R, \cdot, 1\rangle$ is supposed to be a monoid and $0 \cdot a = a \cdot 0 = 0$ is supposed to be true for all $a \in R$.

A semiring is *commutative* if $\langle R, \cdot\rangle$ is a commutative groupoid.

A typical example of a commutative semiring is a set $\mathbb{N}$ of non-negative integers with addition and multiplication. We will give two other examples of a semiring, both are reductions of residuated lattice.

Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice. Then its $\vee$-reduct

$$\mathcal{L}_\vee = \langle L, \vee, *, 0, 1\rangle$$

is a commutative semiring, and its $\wedge$-reduct

$$\mathcal{L}_\wedge = \langle L, \wedge, \rightarrow, 1\rangle$$

is a left (non-commutative) semiring where 1 is the unit of the commutative monoid $\langle L, \wedge, 1\rangle$ as well as the left unit of the groupoid $\langle L, \rightarrow, 1\rangle$.

Let us remark that both $\mathcal{L}_\vee$ and $\mathcal{L}_\wedge$ are the idempotent semirings.

In what follows we will adjust the definition of semimodule [5] to the case where it is a lattice ordered structure. We will define $\vee$-semimodule and $\wedge$-semimodule over respective reducts of residuated lattice.

Definition 2
Let $A = \langle A, \vee, \mathbf{0}\rangle$ be a semilattice with the least element $\mathbf{0}$ and $\mathcal{L}_\vee = \langle L, \vee, *, 0, 1\rangle$ a $\vee$-reduct of residuated lattice $\mathcal{L}$. We say that $A_\vee = \langle A, \vee, \bar{*}, \mathbf{0}\rangle$ is a (left) $\vee$-semimodule over $\mathcal{L}_\vee$ if a (left) scalar multiplication $\bar{*}$ by an element from L is defined such that for any $a \in A$ and $p \in L$ there is a uniquely determined element $p\bar{*}a \in A$. The (left) scalar multiplication $\bar{*}$ fulfills the following properties for all $a, b \in A$ and $p, q \in L$:

SM1. $p\bar{*}(a \vee b) = p\bar{*}a \vee p\bar{*}b$,

SM2. $(p \vee q)\bar{*}a = p\bar{*}a \vee q\bar{*}a$,

SM3. $p\bar{*}(q\bar{*}a) = (p * q)\bar{*}a$,

SM4. $1\bar{*}a = a$,

SM5. $p\bar{*}\mathbf{0} = \mathbf{0}$.

If the scalar multiplication is commutative then we will drop the adjective "left" when we refer to it and to the respective semimodule.

Let $\mathcal{L}_\wedge$ be the $\wedge$-reduct of the residuated lattice $\mathcal{L}$. In this case, we will define the notion of $\wedge$-semimodule over $\mathcal{L}_\wedge$ as a dual notion to the above defined $\vee$-semimodule.

Definition 3
Let $A = \langle A, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ be a lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$, $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ a residuated lattice, $A_\vee$ a $\vee$-semimodule over $\mathcal{L}_\vee$. Let $\mathcal{L}_\wedge$ be the left semiring of $\mathcal{L}$. We say that $A_\wedge$ is a dual (with respect to $A_\vee$) left $\wedge$-semimodule over $\mathcal{L}_\wedge$ if a left scalar multiplication $\bar{\rightarrow}$ by an element from L is defined such that for any $a \in A$

and $p \in L$ there is a uniquely determined element $p \bar{\rightarrow} a \in A$. The left scalar multiplication $\bar{\rightarrow}$ fulfills the following properties for all $a, b \in A$ and $p, q \in L$:

dSM1. $p \bar{\rightarrow} (a \wedge b) = (p \bar{\rightarrow} a) \wedge (p \bar{\rightarrow} b)$,

dSM2. $p \bar{\rightarrow} (q \bar{\rightarrow} a) = (p * q) \bar{\rightarrow} a$,

dSM3. $1 \bar{\rightarrow} a = a$,

dSM4. $p \bar{\rightarrow} \mathbf{1} = \mathbf{1}$.

We will give two examples of dual semimodules over respective reducts of a residuated lattice.

Example 1
Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice.

1. *Let $L^n = \langle L^n, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$, $n \geq 1$, be the lattice of n-dimensional vectors over L such that $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$ and*

$$(a_1, \dots, a_n) \vee (b_1, \dots, b_n) = (a_1 \vee b_1, \dots, a_n \vee b_n),$$
$$(a_1, \dots, a_n) \wedge (b_1, \dots, b_n) = (a_1 \wedge b_1, \dots, a_n \wedge b_n).$$

For $\lambda \in L$, we define left scalar multiplications

$$\lambda \bar{*} (a_1, \dots, a_n) = (\lambda * a_1, \dots, \lambda * a_n),$$
$$\lambda \bar{\rightarrow} (a_1, \dots, a_n) = (\lambda \rightarrow a_1, \dots, \lambda \rightarrow a_n).$$

Then $L^n_\vee$ is a semimodule over $\mathcal{L}_\vee$ and $L^n_\wedge$ is a dual left semimodule over $\mathcal{L}_\wedge$.

2. *Let X be a non-empty set and $L^X = \langle L^X, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ the lattice of L-valued functions on X where $\mathbf{0}, \mathbf{1}$ are identically equal to 0 (respectively, 1) functions. Moreover,*

$$(f \vee g)(x) = f(x) \vee g(x), \quad x \in X,$$
$$(f \wedge g)(x) = f(x) \wedge g(x), \quad x \in X.$$

For $\lambda \in L$, we define left scalar multiplications

$$(\lambda \bar{*} f)(x) = \lambda * f(x), \quad x \in X,$$
$$(\lambda \bar{\rightarrow} f)(x) = \lambda \rightarrow f(x), \quad x \in X.$$

Then $L^X_\vee$ is a semimodule over $\mathcal{L}_\vee$ and $L^X_\wedge$ is a dual left semimodule over $\mathcal{L}_\wedge$.

3 Semilinear Spaces

In this section, a *semilinear space* is defined over residuated lattice – a structure which is endowed with two external operations which are connected by the adjunction property. Our definition is different from those ("natural") ones, considered in [5, 3]), where the external operations are fully inverse to each other.

Definition 4
Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice and $A = \langle A, \vee, \mathbf{0}\rangle$ a semilattice and a $\vee$-semimodule over $\mathcal{L}_\vee$ with the scalar multiplication $\bar{*}$.

We say that A is an idempotent semilinear space over residuated lattice (shortly, a semilinear space) if for each $\lambda \in L$ the mapping $h_\lambda : A \longrightarrow A$, defined by $h_\lambda(\mathbf{a}) = \lambda \bar{*} \mathbf{a}$, has a residual, i.e. the isotone mapping $g_\lambda : A \longrightarrow A$ such that

$$(g_\lambda \circ h_\lambda)(\mathbf{a}) \geq \mathbf{a}, \quad (1)$$
$$(h_\lambda \circ g_\lambda)(\mathbf{a}) \leq \mathbf{a}. \quad (2)$$

It follows from the definition that the carrier A of a semilinear space is a partially ordered set where

$$\mathbf{a} \leq \mathbf{b} \quad \text{iff} \quad \mathbf{a} \vee \mathbf{b} = \mathbf{b}.$$

We say that a semilinear space is *lattice-ordered* if its carrier is a lattice with respect to the order given above. The elements of a semilinear space are called *vectors* and denoted by bold characters, and elements of L are called *scalars* and denoted by Greek characters.

Based on the fact (see e.g. [1]) that if a residual mapping exists then it is unique, we define for each $\lambda \in L$ another scalar operation $\bar{\rightarrow}$ on A:

$$\lambda \bar{\rightarrow} \mathbf{a} = g_\lambda(\mathbf{a}). \quad (3)$$

It is true that for any $\lambda \in L$ and any $\mathbf{a} \in A$

$$\lambda \bar{\rightarrow} \mathbf{a} = \max\{\mathbf{b} \in A \mid \lambda \bar{*} \mathbf{b} \leq \mathbf{a}\}$$

if and only if the right-hand side exists. Therefore, if A has the greatest element $\mathbf{1}$ then for any $\lambda \in L$ and any $\mathbf{a} \in A$

$$0 \bar{\rightarrow} \mathbf{a} = \mathbf{1}, \qquad 1 \bar{\rightarrow} \mathbf{a} = \mathbf{a}$$

and

$$\lambda \bar{\rightarrow} \mathbf{1} = \mathbf{1}.$$

Lemma 1
Let $\mathcal{L}$ be a residuated lattice and A a lattice-ordered semilinear space over $\mathcal{L}$. Then for each $\lambda \in L$, the operation g_λ is distributive over $\wedge$, i.e. for all $\mathbf{a}, \mathbf{b} \in A$

$$g_\lambda(\mathbf{a} \wedge \mathbf{b}) = g_\lambda(\mathbf{a}) \wedge g_\lambda(\mathbf{b}).$$

The lemma below proves a powerful property which external operations have - *adjunction.*

Lemma 2
Let $\mathcal{L}$ be a residuated lattice and A a lattice-ordered semilinear space over $\mathcal{L}$. Then for each $\lambda \in L$, the operations h_λ and g_λ constitute an adjoint pair, i.e. for any $\mathbf{a}, \mathbf{b} \in A$ the adjunction property

$$h_\lambda(\mathbf{b}) \leq \mathbf{a} \quad \textit{iff} \quad \mathbf{b} \leq g_\lambda(\mathbf{a}) \tag{4}$$

holds true.

Using the adjunction property, we may establish many other useful properties of semilinear spaces in the same way as it is done in the theory of residuated lattices. The following lemma shows how this can be done. For better reading, we will use the denotation $\lambda \bar{*} \mathbf{a}$ and $\lambda \bar{\rightarrow} \mathbf{a}$ for $h_\lambda(\mathbf{a})$ and $g_\lambda(\mathbf{a})$ respectively.

Lemma 3
Let $\mathcal{L}$ be a residuated lattice and A a lattice-ordered semilinear space over $\mathcal{L}$. Then for each $\lambda, \mu \in L$, $\mathbf{a} \in A$ the following

$$\lambda \bar{\rightarrow} (\mu \bar{\rightarrow} \mathbf{a}) = (\lambda * \mu) \bar{*} \mathbf{a}. \tag{5}$$

holds true.

The following are examples of a semilinear space which we will use in the sequel.

Example 2
1. *Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a residuated lattice on L. The set of n-dimensional vectors L^n, $n \geq 1$, such that*

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \quad \textit{iff} \quad a_1 \leq b_1, \dots, a_n \leq b_n$$

is a lattice ordered semilinear space over $\mathcal{L}$ where for arbitrary $\lambda \in L$

$$\lambda \bar{*} (a_1, \dots, a_n) = (\lambda * a_1, \dots, \lambda * a_n),$$
$$\lambda \bar{\rightarrow} (a_1, \dots, a_n) = (\lambda \rightarrow a_1, \dots, \lambda \rightarrow a_n).$$

The least element in L^n is the vector $\mathbf{0} = (0, \dots, 0)$ and the greatest element in L^n is the vector $\mathbf{1} = (1, \dots, 1)$. The lattice ordered semilinear space L^n will be referred to as a semilinear vector space.

2. *Let $\mathcal{L}$ be a residuated lattice on L, $X \neq \emptyset$ and L^X a set of all L-valued functions on X such that*

$$f \leq g \quad \textit{iff} \quad f(x) \leq g(x), \quad x \in X.$$

Put

$$(\lambda \bar{*} f)(x) = \lambda * f(x), \quad x \in X,$$
$$(\lambda \bar{\rightarrow} f)(x) = \lambda \rightarrow f(x), \quad x \in X.$$

The least element $\mathbf{0}$ in L^X is the function identically equal to zero and the greatest element $\mathbf{1}$ in L^X is the function identically equal to one. The lattice ordered semilinear space L^X will be referred to as a semilinear functional space.

Definition 5
Let $\langle A_1, \vee, \bar{*}, \mathbf{0}\rangle$ and $\langle A_2, \vee, \bar{*}, \mathbf{0}\rangle$ be two semilinear spaces over $\mathcal{L}$. A homomorphism H is a map $H : A_1 \mapsto A_2$ such that for all $\mathbf{a}, \mathbf{b} \in A_1$, $\lambda \in L$

$$H(\mathbf{a} \vee \mathbf{b}) = H(\mathbf{a}) \vee H(\mathbf{b}), \tag{6}$$
$$H(\lambda \bar{*} \mathbf{a}) = \lambda \bar{*} H(\mathbf{a}), \tag{7}$$
$$H(\mathbf{0}) = \mathbf{0},$$

Example 3
Let $A_1 = L^m$, $m \geq 1$, and $A_2 = L^n$, $n \geq 1$, be semilinear vector spaces over $\mathcal{L}$ (see Example 2, case 2). Let R be an $n \times m$ matrix with elements r_{ij} from L. We define a homomorphism $H_R : L^m \longrightarrow L^n$ so that $H_R(\mathbf{a}) = (H_R(\mathbf{a})_1, \dots, H_R(\mathbf{a})_n)$ where

$$H_R(\mathbf{a})_i = \bigvee_{j=1}^{m} (r_{ij} * a_j), \quad i = 1, \dots, n. \tag{8}$$

It is easy to see that $H_R(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{a}, \mathbf{b} \in A_1$, $\lambda, \mu \in L$

$$H_R(\lambda \bar{*} \mathbf{a} \vee \mu \bar{*} \mathbf{b}) = \lambda \bar{*} H_R(\mathbf{a}) \vee \mu \bar{*} H_R(\mathbf{b}).$$

Definition 6
Let $\langle A_1, \vee, \bar{*}, \mathbf{0}\rangle$ *and* $\langle A_2, \vee, \bar{*}, \mathbf{0}\rangle$ *be two semilinear spaces over* $\mathcal{L}$ *and* $H : A_1 \mapsto A_2$ *a homomorphism. A residual (of H) homomorphism* G *is a map* $G : A_2 \mapsto A_1$ *such that for all* $\mathbf{a} \in A_1$, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b} \in A_2$

$$\mathbf{b}_1 \leq \mathbf{b}_2 \Rightarrow G(\mathbf{b}_1) \leq G(\mathbf{b}_2)$$

and

$$(G \circ H)(\mathbf{a}) \geq \mathbf{a}, \tag{9}$$
$$(H \circ G)(\mathbf{b}) \leq \mathbf{b}. \tag{10}$$

Moreover, if A_1 *and* A_2 *have the greatest elements then* $G(\mathbf{1}) = \mathbf{1}$.

Lemma 4
Let $G : A_2 \mapsto A_1$ *be the residual (of H) homomorphism from semilinear space* A_2 *to semilinear space* A_1. *Then for all* $\mathbf{a} \in A_1$, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b} \in A_2$

$$G(\mathbf{b}_1 \wedge \mathbf{b}_2) = G(\mathbf{b}_1) \wedge G(\mathbf{b}_2) \tag{11}$$

and the adjunction property

$$H(\mathbf{a}) \leq \mathbf{b} \quad \textit{iff} \quad \mathbf{a} \leq G(\mathbf{b}).$$

hold true.

It is not difficult to prove that if a homomorphism $H : L^m \mapsto L^n$ is determined by an $n \times m$ matrix R such that $H = H_R$, then its residual homomorphism $G : L^n \mapsto L^m$ is uniquely determined by the transposed matrix R^T and given as follows:

$$G_{R^T}(\mathbf{b}) = \bigwedge_{i=1}^{n} (r_{ij} \to b_i), \quad j = 1, \dots, m. \tag{12}$$

4 Systems of Equations in Semilinear Spaces

In what follows, we fix a residuated lattice with support L and consider L^m and L^n, $m, n \geq 1$, as semilinear spaces over $\mathcal{L}$ (see Example 2, case 2).

Throughout this section, let $A = (a_{ij})$ be a fixed $n \times m$ matrix and $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{d} = (d_1, \dots, d_m)$ vectors, all have components from L. The following two systems of equations

$$\begin{aligned} a_{11} * x_1 \vee \dots \vee a_{1m} * x_m &= b_1, \\ \dots\dots\dots\dots \\ a_{n1} * x_1 \vee \dots \vee a_{nm} * x_m &= b_n, \end{aligned} \tag{13}$$

and

$$\begin{aligned} (a_{11} \to y_1) \wedge \dots \wedge (a_{n1} \to y_n) &= d_1, \\ \dots\dots\dots\dots \\ (a_{1m} \to y_1) \wedge \dots \wedge (a_{nm} \to y_n) &= d_m, \end{aligned} \tag{14}$$

are considered with respect to unknown vectors $\mathbf{x} = (x_1 \dots, x_m)$ and $\mathbf{y} = (y_1 \dots, y_n)$. Matrix A and vectors $\mathbf{b}$ and $\mathbf{d}$ in (13) and (14) will be further referred to as the *matrix of coefficients* and *vectors of right-hand sides* or *right-hand side vectors.*

It is easy to see that both systems represent two inverse problems with respect to two homomorphisms H_A and its residual G_A (determined by the matrix A) respectively, i.e.

$$H_A(\mathbf{x}) = \mathbf{b}, \qquad G_A(\mathbf{y}) = \mathbf{d}.$$

In the literature related to fuzzy sets and systems, the above considered systems are usually denoted by

$$A \circ \mathbf{x} = \mathbf{b}, \qquad A \triangleright \mathbf{y} = \mathbf{d}$$

where $\circ$ is the so called sup $-*$ composition and $\triangleright$ is the inf $\to$ composition.

We say that $\mathbf{x}^0 \in L^m$ ($\mathbf{y}^0 \in L^n$) is a solution of (13) (solution of (14)) if each equality in (13) (in (14)) becomes true after substitution of $\mathbf{x}^0$ (of $\mathbf{y}^0$) for $\mathbf{x}$ (for $\mathbf{y}$).

It has been mentioned in Introduction, that analogues of systems of equations (13) or (14) have been considered in the literature devoted to fuzzy sets and systems (see e.g. [2, 4, 6]). From these sources we took the following results which are put together in the Proposition below.

Proposition 1
Let $A = (a_{ij})$ *be an* $n \times m$ *matrix of coefficients and* $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{d} = (d_1, \dots, d_m)$ *vectors of of right-hand sides in (13) and (14), all have components from* L. *Then*

(i) system (13) is solvable if and only if the vector $\hat{\mathbf{x}} = G_A(\mathbf{b})$ is its solution;

(ii) if (13) is solvable then $\hat{\mathbf{x}} = G_A(\mathbf{b})$ is its greatest solution;

(iii) system (14) is solvable if and only if the vector $\check{\mathbf{y}} = H_A(\mathbf{d})$ is its solution;

(iv) if (14) is solvable then $\check{\mathbf{y}} = H_A(\mathbf{d})$ is its least solution.

Further on, $\hat{\mathbf{x}}$ (respectively, $\check{\mathbf{y}}$) will always denote the vector expressed by $G_A(\mathbf{b})$ (respectively, $H_A(\mathbf{d})$).

If $\mathbf{x}^1 \in L^m$ and $\mathbf{x}^2 \in L^m$ are solutions of (13) then $\mathbf{x}^1 \vee \mathbf{x}^2$ is a solution of (13) too. Therefore, the set of solutions of (13) form a $\vee$-semilattice with "unit" element.

If $\mathbf{y}^1 \in L^n$ and $\mathbf{y}^2 \in L^n$ are solutions of (14) then $\mathbf{y}^1 \wedge \mathbf{y}^2$ is a solution of (14) too. Therefore, the set of solutions of (14) form a $\wedge$-semilattice with "zero" element.

Proposition 1 turns the problem of finding solutions to (13) (with given A and $\mathbf{b}$) or to (14) (with given A and $\mathbf{d}$) to the investigation whether these systems are solvable.

The latter problem will be considered in a new formulation which emphasizes that the solvability of (13) or (14) means the expressibility of the right-hand side vectors by the column-vectors of the matrix A. In the forgoing text we will see that the right-hand side vectors should be fixed points of certain operators in order to guarantee solvability of (13) or (14). Therefore, the new formulation will be called *The Fixed Points Problem* (**FP**):

(FP) Given $n \times m$ matrix A, characterize all vectors $\mathbf{b} \in L^n$ (all vectors $\mathbf{d} \in L^m$) such that (13) (respectively, (14)) is solvable.

Let us remark that the formulation above is similar to the formulation of the problem of solvability of a system of linear equations in linear algebra which puts an emphasis on a matrix of coefficients.

5 Fixed points of the contraction and dilatation operators

In this section, we will introduce two operators of contraction and dilatation connected with the matrix A of coefficients in systems (13) and (14). We will show that the problem (**FP**) is equivalent with the problem of characterization of fixed points of contraction and dilatation operators. Throughout this section, $A = (a_{ij})$ will be a $n \times m$ matrix with components from L, $\mathbf{b} = (b_1, \ldots, b_n) \in L^n$, $\mathbf{d} = (d_1, \ldots, d_m) \in L^m$.

Definition 7
1. *$AA^{\rightarrow} : L^n \mapsto L^n$ is a contraction operator on L^n if it assigns $(AA^{\rightarrow})\mathbf{b} = H_A(G_A(\mathbf{b}))$ to every element $\mathbf{b} \in L^n$ such that*

$$(AA^{\rightarrow})\mathbf{b}_i = \bigvee_{j=1}^{m} (a_{ij} * \bigwedge_{l=1}^{n} (a_{lj} \rightarrow b_l)), i = 1, \ldots, n.$$

2. *$A^{\rightarrow}A : L^m \mapsto L^m$ is a dilatation operator on L^m if it assigns $(A^{\rightarrow}A)\mathbf{d} = G_A(H_A(\mathbf{d}))$ to every element $\mathbf{d} \in L^m$ such that*

$$(A^{\rightarrow}A)\mathbf{d}_j = \bigwedge_{i=1}^{n} (a_{ij} \rightarrow \bigvee_{l=1}^{m} (a_{il} * d_l)), j = 1, \ldots, m.$$

The following proposition easily follows from Proposition 1.

Proposition 2
Let systems of equations (13) and (14) be specified by the $n \times m$ matrix of coefficients $A = (a_{ij})$ and the respective right-hand side vectors $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{d} = (d_1, \ldots, d_m)$, all have components from L. Then

(i) the system $H_A(\mathbf{x}) = \mathbf{b}$ (13) is solvable if and only if

$$(AA^{\rightarrow})\mathbf{b} = \mathbf{b},$$

or if and only if $\mathbf{b} \in L^n$ is a fixed point of the operator $AA^{\rightarrow}$;

(ii) the system $G_A(\mathbf{y}) = \mathbf{d}$ (14) is solvable if and only if

$$(A^{\rightarrow}A)\mathbf{d} = \mathbf{d},$$

or if and only if $\mathbf{d} \in L^m$ is a fixed point of the operator $A^{\rightarrow}A$.

Remark 2
Easy to see that fixed points of $AA^{\rightharpoondown}$ (respectively $A^{\rightharpoondown}A$) are eigenvectors of the respective operators.

Denotation. $\mathcal{F}(AA^{\rightharpoondown})$ ($\mathcal{F}(A^{\rightharpoondown}A)$) is the set of fixed points of $AA^{\rightharpoondown}$ ($A^{\rightharpoondown}A$).

Let us remark that transformations of L^n or L^m given by contraction or dilatation operators, are not completely new. They were investigated in various structures by different names: in lattices and max-plus algebras [1] they are called as compositions of a mapping and its residual and vice versa. In Theorems 1 and 2 below we will combine the known and new facts and reformulate them according to our terminology.

Theorem 1
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Let $AA^{\rightharpoondown} : L^n \mapsto L^n$ be the corresponding contraction operator. Then the following holds true:

a) for all $\mathbf{b} \in L^n$, $(AA^{\rightharpoondown})\mathbf{b} \leq \mathbf{b}$,

b) for all $\mathbf{x} \in L^m$, $H_A(\mathbf{x})$ is a fixed point of $AA^{\rightharpoondown}$,

c) for all $\mathbf{b} \in L^n$, $\mathbf{b}_0 = (AA^{\rightharpoondown})\mathbf{b}$ is a fixed point of $AA^{\rightharpoondown}$,

d) for each $\mathbf{b} \in L^n$ there exists a uniquely determined fixed point $\mathbf{b}_0 \in L^n$ of $AA^{\rightharpoondown}$ such that $G_A(\mathbf{b}_0) = G_A(\mathbf{b})$.

e) for all $\mathbf{b}_1, \mathbf{b}_2 \in L^n$, $\mathbf{b}_1 \leq \mathbf{b}_2$ implies $(AA^{\rightharpoondown})(\mathbf{b}_1) \leq (AA^{\rightharpoondown})(\mathbf{b}_2)$,

f) if $\mathbf{b}_1, \mathbf{b}_2 \in L^n$ are fixed points of $AA^{\rightharpoondown}$ then $\mathbf{b}_1 \vee \mathbf{b}_2$ is a fixed point of $AA^{\rightharpoondown}$ too.

Corollary 1
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Then for all $\mathbf{b} \in L^n$,

$$G_A(H_A(G_A(\mathbf{b}))) = G_A(\mathbf{b}).$$

Theorem 2
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Let $A^{\rightharpoondown}A : L^m \mapsto L^m$ be the corresponding dilatation operator. Then the following holds true:

a) for all $\mathbf{d} \in L^m$, $(A^{\rightharpoondown}A)\mathbf{d} \geq \mathbf{d}$,

b) for all $\mathbf{y} \in L^n$, $G_A(\mathbf{y})$ is a fixed point of $A^{\rightharpoondown}A$,

c) for all $\mathbf{d} \in L^m$, $\mathbf{d}_0 = (A^{\rightharpoondown}A)\mathbf{d}$ is a fixed point of of $A^{\rightharpoondown}A$,

d) for each $\mathbf{d} \in L^m$ there exists a uniquely determined fixed point $\mathbf{d}_0 \in L^m$ of $A^{\rightharpoondown}A$ such that $H_A(\mathbf{d}_0) = H_A(\mathbf{d})$.

e) for all $\mathbf{d}_1, \mathbf{d}_2 \in L^m$, $\mathbf{d}_1 \leq \mathbf{d}_2$ implies $(A^{\rightharpoondown}A)(\mathbf{d}_1) \leq (A^{\rightharpoondown}A)(\mathbf{d}_2)$,

f) if $\mathbf{d}_1, \mathbf{d}_2 \in L^m$ are fixed points of $A^{\rightharpoondown}A$ then $\mathbf{d}_1 \wedge \mathbf{d}_2$ is a fixed point of $A^{\rightharpoondown}A$ too.

Corollary 2
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Then for all $\mathbf{d} \in L^m$,

$$H_A(G_A(H_A(\mathbf{d}))) = H_A(\mathbf{d}).$$

The following theorem shows how fixed points of contraction and dilatation operators are related.

Theorem 3
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L.

(i) If $\mathbf{b}_0 \in L^n$ is a fixed point of $AA^{\rightharpoondown}$ on L^n then there exists a uniquely determined fixed point $\mathbf{d}_0 \in L^m$ of $A^{\rightharpoondown}A$ on L^m such that $H_A(\mathbf{d}_0) = \mathbf{b}_0$ and $\mathbf{d}_0 = G_A(\mathbf{b}_0)$.

(ii) If $\mathbf{d}_0 \in L^m$ is a fixed point of $A^{\rightharpoondown}A$ on L^m then there exists a uniquely determined fixed point $\mathbf{b}_0 \in L^n$ of $AA^{\rightharpoondown}$ on L^n such that $G_A(\mathbf{b}_0) = \mathbf{d}_0$ and $\mathbf{b}_0 = H_A(\mathbf{d}_0)$.

Corollary 3
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Let $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$, $G_A \mid_{\mathcal{F}(AA^{\rightharpoondown})}$ be restrictions of the respective homomorphisms on the respective sets of fixed points. Then $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$ isomorphically maps $\mathcal{F}(A^{\rightharpoondown}A)$ onto $\mathcal{F}(AA^{\rightharpoondown})$. Moreover, $G_A \mid_{\mathcal{F}(AA^{\rightharpoondown})}$ is inverse to $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$ so that for all $\mathbf{d}_0 \in \mathcal{F}(A^{\rightharpoondown}A)$, $\mathbf{b}_0 \in \mathcal{F}(AA^{\rightharpoondown})$

$$H_A(\mathbf{d}_0) = \mathbf{b}_0 \quad \textit{iff} \quad G_A(\mathbf{b}_0) = \mathbf{d}_0.$$

Remark 3
It is worth to be noticed that homomorphic images of fixed points are fixed points too. This is not always true for preimages. This means that if a fixed point $\mathbf{b}_0 \in L^n$ *of* $AA^{\rightharpoonup}$ *is represented by* $H_A(\mathbf{x}) = \mathbf{b}_0$ *(or* $\mathbf{b}_0$ *is an image of* $\mathbf{x}$*) then* $\mathbf{x} \in L^m$ *is not necessarily a fixed point of* $A^{\rightharpoonup}A$*. Similarly for a fixed point* $\mathbf{d}_0 \in L^m$ *of* $A^{\rightharpoonup}A$ *and its representation* $G_A(\mathbf{y}) = \mathbf{d}_0$*.*

5.1 Fixed Points and Solution Sets

Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L.

The following relations

$$\mathbf{b}_1 \equiv_{A^{\rightharpoonup}} \mathbf{b}_2 \quad \text{iff} \quad G_A(\mathbf{b}_1) = G_A(\mathbf{b}_2)$$

and

$$\mathbf{d}_1 \equiv_A \mathbf{d}_2 \quad \text{iff} \quad H_A(\mathbf{d}_1) = H_A(\mathbf{d}_2)$$

are equivalences on L^n and L^m respectively.

Let $\mathbf{b}_0 \in L^n$ ($\mathbf{d}_0 \in L^m$) be a fixed point of $AA^{\rightharpoonup}$ ($A^{\rightharpoonup}A$). The class of equivalence $[\mathbf{b}_0]_{A^{\rightharpoonup}}([\mathbf{d}_0]_A)$ will be denoted $[\mathbf{b}_0]$ ($[\mathbf{d}_0]$).

Lemma 5
Let $\mathbf{b}_0 \in L^n$, $\mathbf{d}_0 \in L^m$ *be fixed points of* $AA^{\rightharpoonup}$, $A^{\rightharpoonup}A$ *respectively, and* $\mathbf{b} \in [\mathbf{b}_0]$, $\mathbf{d} \in [\mathbf{d}_0]$. *Then*

a) $\mathbf{b}_0$ *is the least element in* $[\mathbf{b}_0]$*;*
$\mathbf{d}_0$ *is the great element in* $[\mathbf{d}_0]$.

b) Let moreover, $G_A(\mathbf{b}_0) = \mathbf{d}_0$*. Then*
$A \triangleright \mathbf{b} = \mathbf{d}_0$*;*
$A \circ \mathbf{d} = \mathbf{b}_0$.

The following theorem shows the certain duality between fixed points and solution sets of the respective systems (13) or (14).

Theorem 4
Let all conditions of Lemma 5 be fulfilled. Then

a) the equivalence class $[\mathbf{b}_0]$ *is the solution set of the system (14) specified by*

$$A \triangleright \mathbf{y} = \mathbf{d}_0,$$

b) the equivalence class $[\mathbf{d}_0]$ *is the solution set of the system (13) specified by*

$$A \circ \mathbf{x} = \mathbf{b}_0.$$

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