Soft Computing: Overview and Recent Developments in Fuzzy Optimization

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Abstract

Soft Computing (SC) represents a significant paradigm shift in the aims of computing, which reflects the fact that the human mind, unlike present day computers, possesses a remarkable ability to store and process information which is pervasively imprecise, uncertain and lacking in categoricity. At this juncture, the principal constituents of Soft Computing (SC) are: Fuzzy Systems (FS), including Fuzzy Logic (FL); Evolutionary Computation (EC), including Genetic Algorithms (GA); Neural Networks (NN), including Neural Computing (NC); Machine Learning (ML); and Probabilistic Reasoning (PR). In this work, we focus on fuzzy methodologies and fuzzy systems, as they bring basic ideas to other SC methodologies. The other constituents of SC are also briefly surveyed here but for details we refer to the existing vast literature. In Part 1 we present an overview of developments in the individual parts of SC. For each constituent of SC we overview its background, main problems, methodologies and recent developments. We focus mainly on Fuzzy Systems, for which the main literature, main professional journals and other relevant information is also supplied. The other constituencies of SC are reviewed shortly. In Part 2 we investigate some fuzzy optimization systems. First, we investigate Fuzzy Sets - we define fuzzy sets within the classical set theory by nested families of sets, and discuss how this concept is related to the usual definition by membership functions. Further, we will bring some important applications of the theory based on generalizations of concave functions. We study a decision problem, i.e. the problem to find a "best" decision in the set of feasible alternatives with respect to several (i.e. more than one) criteria functions. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. We also study the compromise decisions maximizing some aggregation of the criteria. The criteria considered here will be functions defined on the set of feasible alternatives with the values in the unit interval. In Fuzzy mathematical programming problems (FMP) the values of the objective function describe effects from choices of the alternatives. Among others we show that the class of all MP problems with (crisp) parameters can be naturally embedded into the class of FMP problems with fuzzy parameters. Finally, we deal with a class of fuzzy linear programming problems. We show that the class of crisp (classical) LP problems can be embedded into the class of FLP ones. Moreover, for FLP problems we define the concept of duality and prove the weak and strong duality theorems. Further, we investigate special classes of FLP - interval LP problems, flexible LP problems, LP problems with interactive coefficients and LP problems with centered coefficients. We present here an original mathematically oriented and unified approach.

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Part I

Soft Computing - Overview

Chapter 1

Introduction

1.1 Guiding Principle of Soft Computing

Soft computing is tolerant of imprecision, uncertainty, partial truth, and approximation. In effect, the role model for soft computing is the *human mind*. The guiding principle of soft computing is: Exploit the tolerance for imprecision, uncertainty, partial truth, and approximation to achieve tractability, robustness and low solution cost and solve the fundamental problem associated with the current technological development: the lack of the required intelligence of the recent information technology that enables *human-centered functionality*. The basic ideas underlying soft computing in its current incarnation have links to many earlier influences, among them Zadeh's 1965 paper on fuzzy sets; the 1975 paper on the analysis of complex systems and decision processes; and the 1979 report (1981 paper) on possibility theory and soft data analysis. The inclusion of neural computing and genetic computing in soft computing came at a later point.

At this juncture, the principal constituents of Soft Computing (SC) are:

- Fuzzy Systems (FS), including Fuzzy Logic (FL);
- Evolutionary Computation (EC), including Genetic Algorithms (GA);
- Neural Networks (NN), including Neural Computing (NC);
- Machine Learning (ML);
- Probabilistic Reasoning (PR).

Fuzzy theory plays a leading role in soft computing and this stems from the fact that human reasoning is not crisp and admits degrees. What is important to note is that soft computing is not a melange. Rather, it is a partnership in which each of the partners contributes a distinct methodology for addressing problems in its domain. In this perspective, the principal constituent methodologies in

SC are complementary rather than competitive. Furthermore, soft computing may be viewed as a foundation component for the emerging field of conceptual intelligence.

1.2 Importance of Soft Computing

The complementarity of FS, NN, EC, ML and PR has an important consequence: in many cases a problem can be solved most effectively by using FS, NN, EC, ML and PR in combination rather than exclusively. A striking example of a particularly effective combination is what has come to be known as "neurofuzzy systems." Such systems are becoming increasingly visible as consumer products ranging from air conditioners and washing machines to photocopiers and camcorders. Less visible but perhaps even more important are neurofuzzy systems in industrial applications. What is particularly significant is that in both consumer products and industrial systems, the employment of soft computing techniques leads to systems which have high MIQ (Machine Intelligence Quotient). In large measure, it is the high MIQ of SC-based systems that accounts for the rapid growth in the number and variety of applications of soft computing.

The conceptual structure of soft computing suggests that future university students should be trained not just in fuzzy logic, neurocomputing, genetic algorithms, or probabilistic reasoning but in all of the associated methodologies, though not necessarily to the same degree.

For example, at present, the BISC Group (Berkeley Initiative on Soft Computing) comprises close to 1000 students, professors, employees of private and non-private organizations and, more generally, individuals who have interest or are active in soft computing or related areas. Currently, BISC has over 50 Institutional Affiliates, with their ranks continuing to grow in number. At Berkeley, U.S.A., BISC provides a supportive environment for visitors, postdocs and students who are interested in soft computing and its applications. In the main, support for BISC comes from member companies. More details on the web page:

http://www-bisc.cs.berkley.edu

1.3 The Contents of the Study

The successful applications of soft computing suggest that the impact of soft computing will be felt increasingly in coming years of the new millennium. Soft computing is likely to play an especially important role in science and engineering, but eventually its influence may extend much farther. Building human-centered systems is an imperative task for scientists and engineers in the new millennium.

In many ways, soft computing represents a significant paradigm shift in the aims of computing - a shift which reflects the fact that the human mind, unlike present day computers, possesses a remarkable ability to store and process information which is pervasively imprecise, uncertain and lacking in categoricity.

In this work, we focus primarily on fuzzy methodologies and fuzzy systems, as they bring basic ideas to other SC methodologies. The other constituents of SC are also surveyed here but for details we refer to the existing vast literature.

In Part 1 we present an overview of developments in the individual parts of SC. For each constituent of SC we briefly overview its background, main problems, methodologies and recent developments. We deal mainly with Fuzzy Systems in which area we have been researching for about 20 years. Here, the main literature, main professional journals and other relevant information is also supplied. The other constituencies of SC are reviewed only shortly.

In Part 2 we investigate some fuzzy optimization systems. In a way, this work is a continuation of the former research report: J. Ramik and M. Vlach, Generalized concavity as a basis for optimization and decision analysis. Research report IS-RR-2001-003, JAIST Hokuriku 2001, 116 p.

In Chapter 8 we deal with fuzzy sets. Already in the early stages of the development of fuzzy set theory, it has been recognized that fuzzy sets can be defined and represented in several different ways. Here we define fuzzy sets within the classical set theory by nested families of sets, and then we discuss how this concept is related to the usual definition by membership functions. Binary and valued relations are extended to fuzzy relations and their properties are extensively investigated. Moreover, fuzzy extensions of real functions are studied, particularly the problem of the existence of sufficient conditions under which the membership function of the function value is quasiconcave. Sufficient conditions for commuting the diagram "mapping - α-cutting" is presented in the form of classical Nguyen's result.

In the second part - Applications we will bring some important applications of the theory presented in the first part of the book, based on generalizations of concave functions.

In Chapter 9, we consider a decision problem, i.e. the problem to find a "best" decision in the set of feasible alternatives with respect to several (i.e. more than one) criteria functions. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. We study also the compromise decisions maximizing some aggregation of the criteria. The criteria considered here will be functions defined on the set of feasible alternatives with the values in the unit interval. The results by Ramik and Vlach (2001) are extended and presented in the framework of multi-criteria decision making.

Fuzzy mathematical programming problems (FMP) investigated in Chapter 10 form a subclass of decision - making problems where preferences between alternatives are described by means of objective function(s) defined on the set of alternatives in such a way that greater values of the function(s) correspond to more preferable alternatives (if "higher value" is "better"). The values of the objective function describe effects from choices of the alternatives. In this chapter we begin with the formulation a FMP problem associated with the classical MP problem. After that we define a feasible solution of FMP problem and optimal solution of FMP problem as special fuzzy sets. From practical point of view, α-cuts of these fuzzy sets are important, particularly the α-cuts with the maximal α. Among others we show that the class of all MP problems with (crisp) parameters can be naturally embedded into the class of FMP problems with fuzzy parameters.

In Chapter 11 we deal with a class of fuzzy linear programming problems and again introduce the concepts of feasible and optimal solutions - the necessary tools for dealing with such problems. In this way we show that the class of crisp (classical) LP problems can be embedded into the class of FLP ones. Moreover, for FLP problems we define the concept of duality and prove the weak and strong duality theorems. Further, we investigate special classes of FLP - interval LP problems, flexible LP problems, LP problems with interactive coefficients and LP problems with centered coefficients.

In the both last chapters we take advantage of an original unified approach by which a number of new and yet unpublished results are acquired.

Our approach to SC presented in this work is mathematically oriented as the author is a mathematician. There exist, however, other approaches to SC, e.g. human science and computer approach, putting more stress on other aspects of he subject.

Chapter 2

Fuzzy Systems

2.1 Introduction

Fuzzy systems are based on fuzzy logic, a generalization of conventional (Boolean) logic that has been extended to handle the concept of partial truth – truth values between "completely true" and "completely false". It was introduced by L. A. Zadeh of University of California, Berkeley, U.S.A., in the 1960's, as a means to model the uncertainty of natural language. Zadeh himself says that rather than regarding fuzzy theory as a single theory, we should regard the process of "fuzzification" as a methodology to generalize any specific theory from a crisp (discrete) to a continuous (fuzzy) form.

2.2 Fuzzy Sets

The theory of fuzzy sets now encompasses a well organized corpus of basic notions including (and not restricted to) aggregation operations, a generalized theory of relations, specific measures of information content, a calculus of fuzzy numbers. Fuzzy sets are also the cornerstone of a non-additive uncertainty theory, namely possibility theory, and of a versatile tool for both linguistic and numerical modeling: fuzzy rule-based systems. Numerous works now combine fuzzy concepts with other scientific disciplines as well as modern technologies.

In mathematics fuzzy sets have triggered new research topics in connection with category theory, topology, algebra, analysis. Fuzzy sets are also part of a recent trend in the study of generalized measures and integrals, and are combined with statistical methods. Furthermore, fuzzy sets have strong logical underpinnings in the tradition of many-valued logics.

Fuzzy set-based techniques are also an important ingredient in the development of information technologies. In the field of information processing fuzzy sets are important in clustering, data analysis and data fusion, pattern recognition and computer vision. Fuzzy rule-based modeling has been combined with other techniques such as neural nets and evolutionary computing and applied to systems and control engineering, with applications to robotics, complex process control and supervision. In the field of information systems, fuzzy sets play a role in the development of intelligent and flexible man-machine interfaces and the storage of imprecise linguistic information. In Artificial Intelligence various forms of knowledge representation and automated reasoning frameworks benefit from fuzzy set-based techniques, for instance in interpolative reasoning, non-monotonic reasoning, diagnosis, logic programming, constraint-directed reasoning, etc. Fuzzy expert systems have been devised for fault diagnosis, and also in medical science. In decision and organization sciences, fuzzy sets has had a great impact in preference modeling and multicriteria evaluation, and has helped bringing optimization techniques closer to the users needs. Applications can be found in many areas such as management, production research, and finance. Moreover concepts and methods of fuzzy set theory have attracted scientists in many other disciplines pertaining to human-oriented studies such as cognitive psychology and some aspects of social sciences.

In classical set theory, a *subset* U of a set S can be considered as a mapping from the elements of S to the elements of the set $\{0, 1\}$, consisting of the two elements 0 and 1, i.e.:

$$U : S \to \{0, 1\}.$$

This mapping may be represented as a set of ordered pairs, with exactly one ordered pair present for each element of S. The first element of the ordered pair is an element of the set S, and the second element is an element of the set $\{0, 1\}$. The value zero is used to represent non-membership, and the value one is used to represent membership. The truth or falsity of the statement:

$$x \text{ is in } U$$

is determined by finding the ordered pair whose first element is x. The statement is true if the second element of the ordered pair is 1, and the statement is false if it is 0.

Similarly, a *fuzzy subset* (or *fuzzy set*) F of a set S can be defined as a set of ordered pairs, each with the first element from S, and the second element from the interval $[0, 1]$, with exactly one ordered pair present for each element of S. This defines a mapping between elements of the set S and values in the interval $[0, 1]$. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in-between are used to represent intermediate *degrees of membership*. The set S is referred to as the *universe of discourse* for the fuzzy subset F. Frequently, the mapping is described as a function, the *membership function* of F. The ordinary sets are considered as special cases of fuzzy sets with the membership functions equal to the characteristic functions. They are called *crisp sets*.

The above definition of fuzzy set brings the equivalence between a fuzzy set as such, intuitively a set-based concept, and its membership function, a mapping from the universe of discourse to the unit interval $[0, 1]$, or, more generally, to some lattice L. Here, the operations with fuzzy sets are defined by the operations with functions.

In Chapter 8, specially devoted to fuzzy sets, our approach is reversed: we define a fuzzy set as a family of (crisp) sets, where each member of the family corresponds to a specific grade of membership from the unit interval $[0, 1]$. Doing this we define easily the corresponding membership function. This approach is intuitively well understandable and practically easily tractable, as it is natural to work with (crisp) sets with the membership grade grater or equal to some level. Moreover, this approach seems to be more elegant to some mathematicians who are rather reluctant to speak about "sets" having in mind "functions".

The fuzzy set based on the concept of family of nested sets, enjoys, among others, the following advantages:

- it makes possible to create consistent (mathematical) theory,
- no "artificial" identification of a fuzzy set with its membership function is necessary,
- nonfuzzy sets can be naturally embedded into the fuzzy sets,
- nonfuzzy concepts may be extended to represent fuzzy ones,
- any fuzzy problem can be viewed as a family of nonfuzzy ones,
- practical tractability is achieved.

2.3 Fuzzy Logic

The degree to which the statement

$$x \text{ is in } F$$

is true is determined by finding the ordered pair whose first element is x. The *degree of truth* of the statement is the second element of the ordered pair. In practice, the terms "membership function" and fuzzy subset get used interchangeably. That is a lot of mathematical baggage, so here is an example. Let us talk about people and "tallness" expressed as their $HEIGHT$. In this case the set S (the universe of discourse) is the set of people. Let us define a fuzzy subset *tall*, which will answer the question "to what degree is person x tall?" Zadeh describes $HEIGHT$ as a *linguistic variable*, which represents our cognitive category of "tallness". The values of this linguistic variable are fuzzy subsets as *tall*, *very_tall*, or *short*. To each person in the universe of discourse, we assign a degree of membership in the fuzzy subset *tall*. The easiest way to do this is with a membership function based on the real function h ("height of a person in cm") which is defined for each person $x \in S$:

$$tall(x) = \{ \begin{array}{ll} 0 & \text{if } h(x) < 150, \\ \frac{h(x)-150}{50} & \text{if } 150 < h(x) \leq 200, \\ 1 & \text{if } h(x) > 200. \end{array}$$

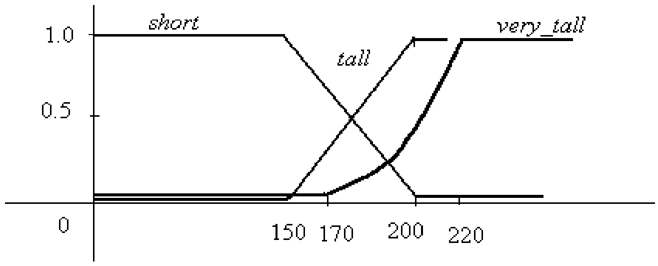

Figure 2.1:

The fuzzy subset *very_tall* may be defined by a nonlinear function of $h(x)$:

$$very\_tall(x) = \{ \begin{array}{ll} 0 & \text{if } h(x) < 170, \\ \left(\frac{h(x)-170}{50}\right)^2 & \text{if } 170 < h(x) \leq 220, \\ 1 & \text{if } h(x) > 220. \end{array}$$

On the other hand, the fuzzy subset *short* is defined as follows:

$$short(x) = \{ \begin{array}{ll} 1 & \text{if } h(x) < 150, \\ \frac{200-h(x)}{50} & \text{if } 150 < h(x) \leq 200, \\ 0 & \text{if } h(x) > 200. \end{array}$$

A graphs of these membership functions look like in Figure 2.1.
Given this definition, here are some example values:

Person x	$h(x)$	$tall(x)$
Mikio	135	0.00
Hideki	173	0.46
Atsuko	155	0.10
Masato	195	0.90

Expressions like "x is A" can be interpreted as degrees of truth, e.g., "Hideki is *tall*" $= 0.46$.

Remark 1 *Membership functions used in most applications almost never have as simple a shape as tall(x) in the above stated example. At minimum, they* *tend to be triangles pointing up, and they can be much more complex than that. Also, the discussion characterizes membership functions as if they always are based on a single criterion, but this is not always the case, although it is quite common. One could, for example, want to have the membership function for tall depend on both a person's height and their age, e.g. "somebody is tall for his age". This is perfectly legitimate, and occasionally used in practice. It is referred to as a two-dimensional membership function, or a "fuzzy relation". It is also possible to have even more criteria, or to have the membership function depend on elements from two completely different universes of discourse.*

Now that we know what a statement like "x is *LOW*" means in fuzzy logic, how do we interpret a statement like:

$$(x \text{ is } \mathit{low}) \text{ AND } (y \text{ is } \mathit{high}) \text{ OR } (\text{NOT } z \text{ is } \mathit{medium}).$$

The standard definitions in fuzzy logic are:

$$\begin{aligned} \mathit{truth}(\text{NOT } x) &= 1.0 - \mathit{truth}(x), \\ \mathit{truth}(x \text{ AND } y) &= \min\{\mathit{truth}(x), \mathit{truth}(y)\}, \\ \mathit{truth}(x \text{ OR } y) &= \max\{\mathit{truth}(x), \mathit{truth}(y)\}. \end{aligned} \tag{2.1}$$

Note that if you plug just the values zero and one into these definitions, you get the same truth tables as you would obtain from conventional predicate logic, particularly, from fuzzy logic operations (2.1) we obtain predicate logic operations on condition all fuzzy membership grades are restricted to the traditional set $\{0, 1\}$. This effectively establishes fuzzy subsets and logic as a true generalization of classical set theory and logic. In fact, by this reasoning all crisp (traditional) subsets are fuzzy subsets of this very special type; and there is no conflict between fuzzy and crisp methods.

Example 2 *Assume the same definition of tall as above, and in addition, assume that we have a fuzzy subset old defined by the membership function:*

$$\mathit{old}(x) = \{ \begin{array}{cl} 0 & \mathit{if}\ a(x) < 18, \\ \frac{a(x)-18}{42} & \mathit{if}\ 18 < a(x) \leq 60, \\ 1 & \mathit{if}\ a(x) > 60. \end{array}$$

where a is a function defined on the set of all people S ("age of a person in years"). Moreover, let

$$\begin{aligned} a &= (x \text{ is } \mathit{tall}) \text{ AND } (x \text{ is } \mathit{old}), \\ b &= (x \text{ is } \mathit{tall}) \text{ OR } (x \text{ is } \mathit{old}), \\ c &= \text{NOT}(x \text{ is } \mathit{tall}). \end{aligned}$$

Then we can compute the following values:

Person x	$h(x)$	$a(x)$	x is *tall*	x is *old*	a	b	c
Mikio	135	10	0.00	0.00	0.00	0.00	1.00
Hideki	173	85	0.46	1.00	0.46	1.00	0.54
Atsuko	155	40	0.10	0.52	0.10	0.52	0.90
Masato	192	22	0.90	0.10	0.10	0.90	0.10

After this simple introductory examples we try to explain shortly, what is fuzzy logic about.

Generally speaking, logic, as a mathematical theory, studies the notions of consequence. It deals with propositions (sentences), sets of propositions and the relation of consequence among them, see Hájek (1998). The task of formal logic is to present all this by means of well-defined logical calculi admitting exact investigation. Various calculi differ in their definitions of sentences and concepts of consequences, e.g. propositional/predicate logics, modal propositional/predicate logics, many-valued propositional/predicate logics etc. Often a logical calculus has two notions of consequence: syntactical (based on a notion of proof) and semantical (based on a notion of truth). The natural questions of soundness (does probability implies truth?) pose themselves.

Fuzziness is imprecision or vagueness; a fuzzy proposition may be true to some degree. Standard examples of fuzzy propositions use a linguistic variable as, $HEIGHT$ or AGE with possible values *short*, *very tall* and *young*, *old* etc.

Fuzzy logic is viewed as a formal mathematical theory for the representation of uncertainty. Uncertainty is crucial for the management of real systems: if you had to park your car *precisely* in one place, it would not be possible. Instead, you work within, say, 10 cm tolerances. The presence of uncertainty is the price you pay for handling a complex system. Nevertheless, fuzzy logic is a mathematical formalism, and a membership grade is a precise number. What's crucial to realize is that fuzzy logic is a logic *of* fuzziness, not a logic which is *itself* fuzzy. But that is natural: just as the laws of probability are not random, so the laws of fuzziness are not vague.

Fuzzy logic is used directly in a number many applications, e.g. the Sony PalmTop apparently uses a fuzzy logic decision tree algorithm to perform handwritten (computer light-pen) Kanji character recognition. Most applications of fuzzy logic use it as the underlying logic system for fuzzy expert systems, e.g. cameras, video-cameras, washing machines, blood-pressure measuring devices, rice-cookers, air-conditioners etc.

Relevant literature about fuzzy sets and fuzzy logic:

Kantrowitz, M., Horstkotte, E. and Joslyn, C., "Answers to Frequently Asked Questions about Fuzzy Logic and Fuzzy Expert Systems", comp.ai.fuzzy, <month, <year, ftp.cs.cmu.edu: /user/ai/pubs/faqs/fuzzy/ fuzzy.faq

Bezdek, J. C., "Fuzzy Models — What Are They, and Why?", IEEE Transactions on Fuzzy Systems, 1993, 1:1, 1-6 .

Bandler, W. and Kohout, L. J., "Fuzzy Power Sets and Fuzzy Implication Operators", Fuzzy Sets and Systems 4,1980, 13-30.

Dubois, D. and Prade, H., "A Class of Fuzzy Measures Based on Triangle Inequalities", Internat. J. General Systems, 8, 36-48.

Gottwald, S., Fuzzy Sets and Fuzzy Logic, Vieweg, Wiesbaden, 1993.

P. Hájek, *Metamathematics of fuzzy logic.* Kluwer Acad. Publ., Series Trends in Logic, Dordrecht /Boston /London, 1998.

Hőle, U. and Klement, P., Eds., Non-Classical Logics and their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht /Boston /London 1995.

Novák, V., Perfilieva I. and Mockor, J., Mathematical Principles of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht /Boston /London, 1999.

Ramik, J. and Vlach, M., Generalized concavity as a basis for optimization and decision analysis. Research report IS-RR-2001-003, JAIST Hokuriku 2001.

Zadeh, L.A. "Fuzzy sets". *Inform. Control* (8) 1965, 338-353.

Zadeh, L. A. "The Calculus of Fuzzy Restrictions", In: Fuzzy Sets and Applications to Cognitive and Decision Making Processes, Eds. Zadeh, L.A. et. al., Academic Press, New York, 1975, 1-39.

Zadeh, L.A. "The concept of a linguistic variable and its application to approximate reasoning". *Information Sciences*, Part I: 8, 1975, 199-249; Part II: 8, 301-357; Part III: 43-80.

2.4 Fuzzy Numbers and Fuzzy Arithmetic

Fuzzy numbers are fuzzy subsets of the real line. They have a peak or plateau with membership grade 1, over which the members of the universe are completely in the set. The membership function is increasing towards the peak and decreasing away from it. Fuzzy numbers are used very widely in fuzzy control applications. A typical case is the triangular fuzzy number which is one form of the fuzzy number. Slope and trapezoidal functions are also used, as well as exponential curves similar to Gaussian probability densities. For more information, see Chapter 8 and also the extensive literature:

Relevant literature about fuzzy numbers:

Dubois, D. and Prade, H., "Fuzzy Numbers: An Overview", in Analysis of Fuzzy Information 1:3-39, CRC Press, Boca Raton, 1987.

Dubois, D. and Prade, H., "Mean Value of a Fuzzy Number", Fuzzy Sets and Systems 24(3):279-300, 1987.

Kaufmann, A., and Gupta, M., "Introduction to Fuzzy Arithmetic", Reinhold, New York, 1985.

2.5 Determination of Membership Functions

Determination methods for constructing membership functions of fuzzy sets break down broadly into the following categories:

2.5.1 Subjective evaluation and elicitation

As fuzzy sets are usually intended to model people's cognitive states, they can be determined from either simple or sophisticated elicitation procedures. At they very least, subjects simply draw or otherwise specify different membership curves appropriate to a given problem. These subjects are typically experts in the problem area, or they are given a more constrained set of possible curves from which they choose. Under more complex methods, users can be tested using psychological methods.

2.5.2 Ad-hoc forms and methods

While there is a vast (infinite) array of possible membership function forms, most actual fuzzy control operations draw from a very small set of different curves, for example simple forms of fuzzy numbers. This simplifies the problem, for example to choosing just the central value and the slope on either side, see Chapter 8.

2.5.3 Converted frequencies or probabilities

Sometimes information taken in the form of frequency histograms or other probability curves are used as the basis to construct a membership function. There are a variety of possible conversion methods, each with its own mathematical and methodological strengths and weaknesses. However, it should always be remembered that membership functions are not (necessarily) probabilities.

2.5.4 Physical measurement

Many applications of fuzzy logic use physical measurement, but almost none measure the membership grade directly. Instead, a membership function is provided by another method, and then the individual membership grades of data are calculated from it (see Turksen, below).

Relevant literature about membership functions:

Z.Q. Liu and S. Miyamoto, Eds.: Soft Computing and Human - Centered Machines, Springer, Tokyo-Berlin-Heidelberg-New York, 2000.

Turksen, I.B., "Measurement of Fuzziness: Interpretation of the Axioms of Measure", in Proceeding of the Conference on Fuzzy Information and Knowledge Representation for Decision Analysis. IFAC, Oxford, 1984,97-102.

2.6 Membership Degrees Versus Probabilities

This problem can be solved in two ways:

- how does fuzzy theory differ from probability theory mathematically,

- how does it differ in interpretation and application.

At the mathematical level, fuzzy values are commonly misunderstood to be probabilities, or fuzzy logic is interpreted as some new way of handling probabilities. But this is not the case. A minimum requirement of probabilities is *additivity*, that is that they must add together to one, or the integral of their density functions must be one. However, this does not hold in general with membership grades. And while membership grades can be determined with probability densities in mind, there are other methods as well which have nothing to do with frequencies or probabilities. Because of this, fuzzy researchers have gone to great pains to distance themselves from probability. But in so doing, many of them have lost track of another point, which is that the converse in some sense does hold: probability distributions can be converted to fuzzy sets. As fuzzy sets and fuzzy logic generalize Boolean sets and logic, they also generalize probability. In fact, from a mathematical perspective, fuzzy sets and probability exist as parts of a greater Generalized Information Theory which includes many formalisms for representing uncertainty (including random sets, Dempster-Shafer evidence theory, probability intervals, possibility theory, general fuzzy measures, interval analysis, etc.). Furthermore, one can also talk about random fuzzy events and fuzzy random events. This whole issue is beyond the scope of this survey. We refer to the books and papers cited below.

Semantically, the distinction between fuzzy logic and probability theory has to do with the difference between the notions of probability and a degree of membership. Probability statements are about the likelihoods of outcomes: an event either occurs or does not, and you can bet on it. With fuzziness, one cannot say unequivocally whether an event occurred or not, and instead you are trying to model the *extent* to which an event occurs.

Relevant literature about fuzzy versus probability:

Bezdek, J. C., "Fuzzy Models — What Are They, and Why?", IEEE Transactions on Fuzzy Systems, 1:1,1-6.

Delgado, M., and Moral, S., "On the Concept of Possibility-Probability Consistency", Fuzzy Sets and Systems 21,1987,311-318.

Dempster, A.P., "Upper and Lower Probabilities Induced by a Multivalued Mapping", Annals of Math. Stat. 38,1967,325-339.

Henkind, S. J. and Harrison, M. C., "Analysis of Four Uncertainty Calculi", IEEE Trans. Man Sys. Cyb. 18(5),1988,700-714.

Kampe, D. and Feriet, J., "Interpretation of Membership Functions of Fuzzy Sets in Terms of Plausibility and Belief", in Fuzzy Information and Decision Process, M.M. Gupta and E. Sanchez, Eds., North-Holland, Amsterdam, 1982, 93-98.

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Klir, G. and Parviz, B., "Probability-Possibility Transformations: A Comparison". Int. J. Gen. Sys. 21(1),1992,291-310.

Kosko, B., "Fuzziness vs. Probability", Int. J. Gen. Sys. 17(2-3), 1990,211-240.

Puri, M.L. and Ralescu, D.A., "Fuzzy Random Variables", J. Math. Analysis and Applications, 114,1986,409-422.

Shafer, G., "A Mathematical Theory of Evidence", Princeton University, Princeton, 1976.

Shanahan, J.G., Soft Computing for Knowledge Discovery, Kluwer Acad. Publ., Boston /Dordrecht /London, 2000.

2.7 Possibility Theory

Possibility theory is another recent form of information theory which is related to but independent of both fuzzy sets and probability theory. Technically, a possibility distribution is a normal fuzzy set (at least one membership grade equals 1). For example, all fuzzy numbers are possibility distributions. However, possibility theory can also be derived without reference to fuzzy sets. The rules of possibility theory are similar to probability theory, but the possibility calculus differs to the calculus of probability theory. Also, possibilistic *nonspecificity* is available as a measure of information similar to the stochastic *entropy*.

Possibility theory has a methodological advantage over probability theory as a representation of nondeterminism in systems, because the "plus/times" calculus does not validly generalize nondeterministic processes, while "max/min" do.

Relevant literature about fuzzy versus probability:

Dubois, D. and Prade, H., "Possibility Theory", Plenum Press, New York, 1988.

Joslyn, C., "Possibilistic Measurement and Set Statistics", In: Proc. of the 1992 NAFIPS Conference, 2, NASA, 1992, 458-467.

Joslyn, C., "Possibilistic Semantics and Measurement Methods in Complex Systems", In: Proc. of the 2nd International Symposium on Uncertainty Modeling and Analysis, B. Ayyub, Ed., IEEE Computer Society 1993.

Wang, Z. and Klir, G., "Fuzzy Measure Theory", Plenum Press, New York, 1991.

Zadeh, L., "Fuzzy Sets as the Basis for a Theory of Possibility", Fuzzy Sets and Systems 1:1978, 3-28.

2.8 Fuzzy Expert Systems

A fuzzy expert system is an expert system that uses a collection of fuzzy membership functions and rules, instead of Boolean logic, to reason about data. The rules in a fuzzy expert system are usually of a form similar to the following:

$$\text{IF } (x \text{ is } LOW) \text{ AND } (y \text{ is } HIGH) \text{ THEN } (z \text{ is } MEDIUM),$$

where x and y are input variables (names for know data values), z is an output variable (a name for a data value to be computed), LOW is a membership function (fuzzy subset) defined on the set of x, $HIGH$ is a membership function defined on the set of y, and $MEDIUM$ is a membership function defined on the set of z. The *antecedent* (the rule's *premise*, between IF and THEN) describes to what degree the rule applies, while the *consequent* (the rule's *conclusion*, following THEN) assigns a membership function to each of one or more output variables. Most tools for working with fuzzy expert systems allow more than one conclusion per rule (a compound consequent). The set of rules in a fuzzy expert system is known as the *rule base* or *knowledge base*. The general inference process proceeds in 4 steps:

1. Under *fuzzification*, the membership functions defined on the input variables are applied to their actual values, to determine the degree of truth for each rule premise.

2. Under *inference*, the truth value for the premise of each rule is computed, and applied to the conclusion part of each rule. This results in one fuzzy subset to be assigned to each output variable for each rule. Usually only min or $\cdot$ ("product") are used as inference rules. In min inferencing, the output membership function is clipped off at a height corresponding to the rule premise's computed degree of truth (fuzzy logic AND). In "product" inferencing, the output membership function is scaled by the rule premise's computed degree of truth.

3. Under *composition*, all of the fuzzy subsets assigned to each output variable are combined together to form a single fuzzy subset for each output variable. Again, usually max or $\sum$ "sum" are used. In max composition, the combined output fuzzy subset is constructed by taking the pointwise maximum over all of the fuzzy subsets assigned to variable by the inference rule (fuzzy logic OR). In "sum" composition, the combined output fuzzy subset is constructed by taking the pointwise sum over all of the fuzzy subsets assigned to the output variable by the inference rule.

4. Finally is the (optional) *defuzzification*, which is used when it is useful to convert the fuzzy output set to a crisp number. There are many defuzzification methods (at least 30). Two of the more common techniques are the *centroid* and *maximum* methods. In the centroid method, the crisp value of the output variable is computed by finding the variable value of the center of gravity of the membership function for the fuzzy value. In

the maximum method, one of the variable values at which the fuzzy subset has its maximum truth value is chosen as the crisp value for the output variable.

Let us demonstrate the process on a simple example. Assume that the variables x, y, and z all take on values in the interval $[0, 10]$, and that the following membership functions and rules are defined as follows:

$$low(t) = 1 - \frac{t}{10}, \tag{2.2}$$

$$high(t) = \frac{t}{10}. \tag{2.3}$$

Rule 1: IF x is LOW AND y is LOW THEN z is $HIGH$
Rule 2: IF x is LOW AND y is $HIGH$ THEN z is LOW
Rule 3: IF x is $HIGH$ AND y is LOW THEN z is LOW
Rule 4: IF x is $HIGH$ AND y is $HIGH$ THEN z is $HIGH$

Notice that instead of assigning a single value to the output variable z, each rule assigns an entire fuzzy subset (LOW or $HIGH$).

Notice that we have:

$$low(t) + high(t) = 1.0$$

for all t. This is not obligatorily required, but it is common.

The value of t at which $low(t)$ is maximal is the same as the value of t at which $high(t)$ is minimal, and vice-versa. This is also not obligatorily required, but fairly common.

The same membership functions are used for all variables. This is not required, but *not* common.

In the fuzzification subprocess, the membership functions defined on the input variables are applied to their actual values, to determine the degree of truth for each rule premise. The degree of truth for a rule's premise is sometimes referred to as its level *alpha*, a number from $[0, 1]$. If a rule's premise has a nonzero degree of truth, then the rule is said to *fire*. For example, let $x = 3.2$, $y = 3.3$. By (2.2), (2.3) we compute:

$$low(x) = 0.68, high(x) = 0.32, low(y) = 0.67, high(y) = 0.33.$$

Each rule has its own alpha, namely:

$$alpha1 = 0.67, alpha2 = 0.33, alpha3 = 0, 32, alpha4 = 0.32.$$

In the inference subprocess, the truth value for the premise of each rule is computed, and applied to the conclusion part of each rule. This results in one fuzzy subset to be assigned to each output variable for each rule. As it has been already mentioned min and $\cdot$ are two *inference methods* or *inference rules*. In min inferencing, the output membership function is clipped off at a height corresponding to the rule premise's computed degree of truth. This corresponds to the traditional interpretation of the fuzzy logic AND operation. In $\cdot$ (product) inferencing, the output membership function is scaled by the rule premise's computed degree of truth.

More generally, minimum and product are two particular cases of triangular norms, or t-norms. For an inference method any triangular norm can be applied, see Klement et al. (2000).

Let's look, for example, at Rule 1 for $x = 0.0$, $y = 3.2$. As can be easily computed, the premise degree of truth is 0.68. For this rule, min inferencing will assign z the fuzzy subset defined by the membership function:

$$Rule1(z) = \{ \begin{array}{ll} \frac{z}{10} & \text{if } z \leq 6.8, \\ 0.68 & \text{if } z > 6.8. \end{array}$$

For the same conditions, $\cdot$ (product) inferencing will assign z the fuzzy subset defined by the membership function:

$$Rule1(z) = 0.68 \cdot \frac{z}{10}$$

The terminology used here is slightly nonstandard. In most texts, the term "inference method" is used to mean the combination of the things referred to separately here as "inference" and "composition". Thus, you can see such terms as "max-min inference" and "sum-product inference" in the literature. They are the combination of max composition and min inference, or $\sum$ composition and $\cdot$ inference, respectively. You'll also see the reverse terms "min-max" and "product-sum" – these mean the same things in the reverse order. It seems clearer to describe the two processes separately.

In the composition subprocess, all of the fuzzy subsets assigned to each output variable are combined together to form a single fuzzy subset for each output variable. Max composition and sum composition are two *composition rules*. In max composition, the combined output fuzzy subset is constructed by taking the pointwise maximum over all of the fuzzy subsets assigned to the output variable by the inference rule. In sum composition, the combined output fuzzy subset is constructed by taking the pointwise sum over all of the fuzzy subsets assigned to the output variable by the inference rule.

Note that this can result in truth values greater than one! For this reason, sum composition is only used when it will be followed by a defuzzification method, such as the c*entroid* method, that doesn't have a problem with this odd case. Otherwise sum composition can be combined with *normalization* and is therefore a general purpose method again. For example, assume $x = 0.0$ and

$y = 3.2$. Min inferencing would assign the following four fuzzy subsets to z:

$$
\begin{aligned}
Rule1(z) &= \{ \begin{array}{ll} \frac{z}{10} & \text{if } z \leq 6.8, \\ 0.68 & \text{if } z > 6.8, \end{array} \\
Rule1(z) &= \{ \begin{array}{ll} 0.32 & \text{if } z \leq 6.8, \\ 1 - \frac{z}{10} & \text{if } z > 6.8, \end{array} \\
Rule3(z) &= 0.0, \\
Rule4(z) &= 0.0.
\end{aligned}
$$

Max composition would result in the fuzzy subset:

$$
Rules(z) = \{ \begin{array}{ll} 0.32 & \text{if } z \leq 3.2, \\ 1 - \frac{z}{10} & \text{if } 3.2 < z \leq 6.8, \\ 0.68 & \text{if } z > 6.8. \end{array}
$$

Product inferencing would assign the following four fuzzy subsets to z:

$$
\begin{aligned}
Rule1(z) &= 0.068 \cdot z, \\
Rule1(z) &= 0.32 - 0.032 \cdot z, \\
Rule3(z) &= 0.0, \\
Rule4(z) &= 0.0.
\end{aligned}
$$

Sum composition would result in the fuzzy subset:

$$
Rules(z) = 0.32 + 0.036 \cdot z.
$$

More generally, maximum and sum are two particular cases of triangular conorms, or t-conorms. For an composition rules any triangular conorm can be applied, see Ramik and Vlach (2001). Hence, "inference" and "composition" in an expert system may be created by a couple (T, S), where T is a t-norm and S is a t-conorm. In the "best" couple, T and S are mutually dual. More about this subject can be found in Klement et al. (2000).

Sometimes it is useful to just examine the fuzzy subsets that are the result of the composition process, but more often, this *fuzzy value* needs to be converted to a single number – a *crisp value*. This is what the defuzzification subprocess does. There exist many defuzzification methods, for example, a couple of years ago, Mizumoto (1989) published a paper that compared ten defuzzification methods. Two of the more common techniques are the *centroid* and *maximum* methods.

In the centroid method, the crisp value of the output variable is computed by finding the variable value of the center of gravity of the membership function for the fuzzy value.

In the maximum method, one of the variable values at which the fuzzy subset has its maximum truth value is chosen as the crisp value for the output variable.

There are several variations of the maximum method that differ only in what they do when there is more than one variable value at which this maximum truth value occurs. One of these, the *average of maxima* method, returns the average of the variable values at which the maximum truth value occurs.

To compute the centroid of the function $f(x)$, you divide the *moment* of the function by the *area* of the function. To compute the moment of $f(x)$, you compute the integral $\int xf(x)dx$, and to compute the area of $f(x)$, you compute the integral $\int f(x)dx$.

Sometimes the composition and defuzzification processes are combined, taking advantage of mathematical relationships that simplify the process of computing the final output variable values.

To date, fuzzy expert systems are the most common use of fuzzy logic. They are used in several wide-ranging fields, including:

- Linear and Nonlinear Control,
- Pattern Recognition,
- Financial Systems,
- Operation Research,
- Data Analysis.

2.9 Fuzzy Control

The purpose of control is to influence the behavior of a system by changing an input or inputs to that system according to a rule or set of rules that model how the system operates. The system being controlled may be mechanical, electrical, chemical or any combination of these. Classic control theory uses a mathematical model to define a relationship that transforms the desired state (requested) and observed state (measured) of the system into an input or inputs that will alter the future state of that system, see the following figure:

reference→0→(SYSTEM)→+→ output
↑ ↓
+←(MODEL)←+ feedback

The most common example of a control model is the PID (proportional-integral- derivative) controller. This takes the output of the system and compares it with the desired state of the system. It adjusts the input value based on the difference between the two values according to the following equation.

$$output = A.e + B.INT(e)dt + C.de/dt$$

where, A, B and C are constants, e is the error term, $INT(e)dt$ is the integral of the error over time and de/dt is the change in the error term. The major drawback of this system is that it usually assumes that the system being modelled in linear or at least behaves in some fashion that is a monotonic function. As the complexity of the system increases it becomes more difficult to formulate that mathematical model. Fuzzy control replaces, in the picture above, the role of the mathematical model and replaces it with another that is build from a number of smaller rules that in general only describe a small section of the whole system. The process of inference binding them together to produce the desired outputs. That is, a fuzzy model has replaced the mathematical one. The inputs and outputs of the system have remained unchanged. The Sendai subway is the prototypical example application of fuzzy control.

Relevant literature about fuzzy expert systems and fuzzy control:

Driankov, D., Hellendoorn, H. and Reinfrank, M., "An Introduction to Fuzzy Control", Springer-Verlag, New York, 1993.

Chen, G., Ying, M. and Cai, K.-Y., Fuzzy Logic and Soft Computing, Kluwer Acad. Publ., Boston/Dordrecht/London, 1999.

Harris, C.J., Moore, C.G. and Brown, M., "Intelligent Control, Aspects of Fuzzy Logic and Neural Nets", World Scientific, 1997.

Mizumoto, M., "Improvement Methods of Fuzzy Controls", In: Proceedings of the 3rd IFSA Congress, Seattle, 1989, 60-62.

Terano, T., Asai, K. and Sugeno, M., "Fuzzy Systems Theory and Applications", Academic Press, 1992.

Yager, R.R., and Zadeh, L. A., "An Introduction to Fuzzy Logic Applications in Intelligent Systems", Kluwer Academic Publishers, Boston / Dordrecht / London, 1991.

Zimmermann, H.J., "Fuzzy set Theory", Kluwer Acad. Publ., Boston / Dordrecht / London, 1991.

2.10 Fuzzy Clustering

Clustering (also cluster analysis) is referred to a group of classification methods for data analysis. Since 1970's, fuzzy set theory has been applied to clustering and a number of clustering methods and techniques have been developed, see the relevant literature below. One of the most frequent application area is the pattern recognition. Clustering methods can be, as usual, divided into two categories:

- nonhierarchical fuzzy clustering,
- hierarchical fuzzy clustering.

The most popular method of nonhierarchical clustering is the fuzzy c-means method which is a direct extension of nonfuzzy (crisp) nonhierarchical clustering.

On the other hand, fuzzy hierarchical clustering is not a direct generalization of crisp methods, it provides rather direction for agglomerative clustering.

In general, clustering requires that objects to be classified should be put in one of a number of classes depending on some characterization exemplified by the concept of distance. The object being close in the sense of the distance are given in the same class whereas those being far to each other are to be put in different classes. More precisely, the quality of classification is measured by an objective function which is usually minimized so as to obtain the optimal solution, i.e. the resulting classification of the objects in question. Centers of the clusters are allocated and the distance of objects is measured with respect to those centers. The underlying idea of fuzzy clustering is that an object belongs to more than one cluster but with possibly different degrees of membership.

Standard method of fuzzy c-means take advantage of the rectangular matrix with elements from the unit interval $[0, 1]$ serving as weighting parameters in the objective function and at the same time as membership degrees of the classified objects of individual clusters. numerous techniques have been proposed in the literature to improve the results of classification for numerous types of problem data. For more information about classification techniques and algorithms as well as applications of clustering, see the recommended literature.

Relevant literature about fuzzy clustering:

http:/www.fuzzy-clustering.de - extensive bibliographical database, includes papers, books and software, pdf, ps files to download, etc.

Anderberg, M.R., "Cluster Analysis for Applications", Academic Press, New York, 1973.

Bezdek, J.C., Patern Recognition with Fuzzy Objective Function Algorithms, Plenum Press, New York, 1981.

Everitt, B.S., "Cluster Analysis", Arnold Publ House, London, 1993.

Hopfer, F., Klawon, F., Kruse, R. and Runkler, T., "Fuzzy Cluster Analysis", J. Wiley, Amazon, 1999.

Miyamoto, S., "Fuzzy Sets in Information Retrieval and Cluster Analysis", Kluwer Acad. Publ., Dordrecht, 1990.

Miyamoto, S. and Umayahara, K., "Methods in Hard and Fuzzy Clustering", in: Soft Computing and Human-Centered Machines, Liu, Z. and Miyamoto, S., Eds., Springer, Tokyo, Berlin, 2000.

Nakamori, Y., Ryoke, M. and Umayahara, K., "Multivariate analysis for Fuzzy Modeling", Proc. of the 7th World IFSA Congress, Prague, Academia, Praha, 1997, 93-98.

2.11 Decision Making in Fuzzy Environment

When dealing with practical decision problems, we often have to take into consideration uncertainty in the problem data. It may arise from errors in measuring physical quantities, from errors caused by representing some data in a computer, from the fact that some data are approximate solutions of other problems or estimations by human experts, etc. In some of these situations, the fuzzy set approach may be applicable. In the context of multicriteria decision making, functions mapping the set of feasible alternatives into the unit interval $[0, 1]$ of real numbers representing normalized utility functions can be interpreted as membership functions of fuzzy subsets of the underlying set of alternatives. However, functions with the range $[0, 1]$ arise in more contexts.

A *decision problem in* X, i.e. the problem to find a "best" decision in the set of feasible alternatives X with respect to several (i.e. more than one) criteria functions is considered. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions - commonly called Pareto-optimal decisions.

In Chapter 9 we study also the compromise decisions maximizing some aggregation of the criteria. This problem was introduced originally by Bellman and Zadeh for the minimum aggregation function. The criteria considered here will be functions defined on the set X of feasible alternatives with the values in the unit interval $[0, 1]$. Such functions can be interpreted as membership functions of fuzzy subsets of X and will be called here *fuzzy criteria*.

The set X of feasible alternatives is a convex subset, or a generalized convex subset of n-dimensional Euclidean space $\mathbf{R}^n$, frequently we consider $X = \mathbf{R}^n$. The main subject of our interest in Chapter 9 is to derive some important relations between Pareto-optimal decisions and compromise decisions. The relevant literature to the subject can be found also in Chapter 9.

2.12 Fuzzy Mathematical Programming

Mathematical programming problems (MP) form a subclass of decision-making problems where preferences between alternatives are described by means of objective function(s) defined on the set of alternatives in such a way that greater values of the function(s) correspond to more preferable alternatives (if "higher" is "better"). The values of the objective function describe effects from choices of the alternatives. In economic problems, for example, these values may reflect profits obtained when using various means of production. The set of feasible alternatives in MP problems is described implicitly by means of constraints - equations or inequalities, or both - representing relevant relationships between alternatives. In any case the results of the analysis using given formulation of the MP problem depend largely upon how adequately various factors of the real system are reflected in the description of the objective function(s) and of the constraint(s).

Descriptions of the objective function and of the constraints in a MP problem usually include some parameters. For example, in problems of resources allocation such parameters may represent economic parameters like costs of various types of production, labor costs requirements, shipment costs, etc. The nature of those parameters depends, of course, on the detailization accepted for the model representation, and their values are considered as data that should be exogenously used for the analysis.

Clearly, the values of such parameters depend on multiple factors not included into the formulation of the problem. Trying to make the model more representative, we often include the corresponding complex relations into it, causing that the model becomes more cumbersome and analytically unsolvable. Moreover, it can happen that such attempts to increase "the precision" of the model will be of no practical value due to the impossibility of measuring the parameters accurately. On the other hand, the model with some fixed values of its parameters may be too crude, since these values are often chosen in a quite arbitrary way.

An intermediate approach is based on the introduction into the model the means of a more adequate representation of experts´understanding of the nature of the parameters in the form of fuzzy sets of their possible values. The resultant model, although not taking into account many details of the real system in question could be a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. On this way we obtain a new type of MP problems containing fuzzy parameters. Treating such problems requires the application of fuzzy-set-theoretic tools in a logically consistent manner. Such treatment forms an essence of *fuzzy mathematical programming* (FMP) investigated in this chapter.

FMP and related problems have been extensively analyzed and many papers have been published displaying a variety of formulations and approaches. Most approaches to FMP problems are based on the straightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function. This approach has been mentioned by Bellman and Zadeh already in their paper published in the early seventies. Later on many papers have been devoted to the problem of mathematical programming with fuzzy parameters, known under different names, mostly as fuzzy mathematical programming, but sometimes as possibilistic programming, flexible programming, vague programming, inexact programming etc. For an extensive bibliography see the overview paper Rommelfanger and Slowinski (1998).

In Chapter 10 we present a general approach based on a systematic extension of the traditional formulation of the MP problem. This approach is based on the numerous former works of the author of this work, and also on the works of many other authors, see the literature to Chapter 10.

The *fuzzy mathematical programming problem* (FMP problem) is denoted by:

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{f}(x;\tilde{c}) \\ \text{subject to} & \\ & \tilde{g}_i(x;\tilde{a}_i) \ \ \tilde{R}_i \ \tilde{b}_i, \ i \in \mathcal{M} = \{1,2,...,m\}, \end{array} \tag{2.4}$$

where $\tilde{R}_i$, $i \in \mathcal{M}$, are fuzzy relations on $\mathcal{F}(\mathbf{R})$, the set of all fuzzy subsets of

R. Formulation (2.4) is not an optimization problem in a classical sense, as it is not yet defined, how the objective function $\tilde{f}(x;\tilde{c})$ can be "maximized", and how the constraints $\tilde{g}_i(x;\tilde{a}_i)\ \ \tilde{R}_i\ \tilde{b}_i$ can be treated. In fact, we need a concept of feasible solution and also that of optimal solution.

Most important mathematical programming problems (10.4) are those where the functions f and g_i are linear. The *fuzzy linear programming problem* (FLP problem) is denoted as

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \\ & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n\ \tilde{R}_i\ \tilde{b}_i,\ i \in \mathcal{M}, \\ & \qquad\qquad x_j \geq 0,\ j \in \mathcal{N} = \{1, 2, ..., n\}. \end{array} \tag{2.5}$$

All these problems are thoroughly investigated in Chapter 10 and 11.

2.13 Mailing Lists

In this section we inform about *Internet* based mailing lists dealing with fuzzy systems. This list is far from being complete as the development in this area is very fast.

NAFIPS Fuzzy Logic Mailing List:

This is a mailing list for the discussion of fuzzy logic, NAFIPS and related topics, located at the Georgia State University. Recently, there are about 500 subscribers, located primarily in North America. Postings to the mailing list are automatically archived. The mailing list server itself is like most of those in use on the Internet. If you are already familiar with Internet mailing lists, the only thing you will need to know is that the name of the server is

listproc@listproc.gsu.edu

and the name of the mailing list itself is

nafips-l@listproc.gsu.edu

If you are not familiar with this type of mailing list server, the easiest way to get started is to send the following message to listproc@listproc.gsu.edu: help

You will receive a brief set of instructions by e-mail within a short time. Once you have subscribed, you will begin receiving a copy of each message that is sent by anyone to

nafips-l@listproc.gsu.edu

and any message that you send to that address will be sent to all of the other subscribers.

Fuzzy-Mail Mailing List:

This is a mailing list for the discussion of fuzzy logic and related topics, located at the Technical University of Vienna in Austria. Recently, there are more than 1000 subscribers, located primarily in Europe. After more than 5 years of experience, the author would recommend this mailing list to any person interested in fuzzy systems and fuzzy logic. Frequently, the discussion about the hot topics of theory and practice of FS is interesting, deep and stimulating. Information about new conferences and seminars, journals and books are also of practical use. The list is slightly moderated (only irrelevant mails are rejected) and is two-way gatewayed to the aforementioned NAFIPS-L list and to the comp.ai.fuzzy internet newsgroup. Messages should therefore be sent only to one of the three media, although some mechanism for mail-loop avoidance and duplicate-message avoidance is activated. In addition to the mailing list itself, the list server gives access to some files, including archives and the "Who is Who in Fuzzy Logic" database. The name of the server is

listproc@dbai.tuwien.ac.at

and the name of the mailing list is

fuzzy-mail@dbai.tuwien.ac.at

If you are not familiar with this type of mailing list server, the easiest way to get started is to send the following message to

listproc@dbai.tuwien.ac.at: get fuzzy-mail info

You will receive a brief set of instructions by e-mail within a short time. Once you have subscribed, you will begin receiving a copy of each message that is sent by anyone to

fuzzy-mail@dbai.tuwien.ac.at

and any message that you send to that address will be sent to all of the other subscribers.

Mailing lists for fuzzy systems in Japan:

We mention two mailing lists for fuzzy systems in Japan. Both forward many articles from the international mailing lists, but the other direction is not automatic.

Asian Fuzzy Mailing System (AFMS):

afuzzy@ea5.yz.yamagata-u.ac.jp

To subscribe, send a message to

aserver@ea5.yz.yamagata-u.ac.jp

with your name and email address. Membership is restricted to within Asia as a general rule. The list is maintained by Prof. Mikio Nakatsuyama, Department of Electronic Engineering, Yamagata University, 4-3-16 Jonan, Yonezawa 992 Japan, E-mail: nakatsu@ea5.yz.yamagata-u.ac.jp.

All messages to the list have the Subject line replaced with "AFMS". The language of the list is English.

Fuzzy Mailing List - Japan:

fuzzy-jp@sys.es.osaka-u.ac.jp

This is an unmoderated list, with mostly original contributions in Japanese (JIS-code). To subscribe, send subscriptions to the listserver

fuzzy-jp-request@sys.es.osaka-u.ac.jp

If you need to speak to a human being, send mail to the list owners Itsuo Hatono and Motohide Umano of Osaka University to

fuzzy-admin@tamlab.sys.es.osaka-u.ac.jp.

2.14 Main International Journals

FUZZY SETS AND SYSTEMS (FSS)

International Journal of Soft Computing and Intelligence. The official publication of the International Fuzzy Systems Association (IFSA). Subscription is free to members of IFSA.

Published annually, 24 times in 2001

Publisher: Elsevier Science

ISSN: 0165-0114

Since its launching in 1978, the journal Fuzzy Sets and Systems has been devoted to the international advancement of the theory and application of fuzzy sets and systems.

The scope of the journal Fuzzy Sets and Systems has expanded so as to account for all facets of the field while emphasizing its specificity as bridging the gap between the flexibility of human representations and the precision and clarity of mathematical or computerized representations, be they numerical or symbolic.

The journal welcomes original and significant contributions in the area of Fuzzy Sets whether on empirical or mathematical foundations, or their applications to any domain of information technology, and more generally to any field of investigation where fuzzy sets are relevant. Applied papers demonstrating the usefulness of fuzzy methodology in practical problems are particularly welcome.

Fuzzy Sets and Systems publishes high-quality research articles, surveys as well as case studies. Separate sections are Recent Literature, and the Bulletin, which offers research reports, book reviews, conference announcements and various news items. Invited review articles on topics of general interest are included and special issues are published regularly.

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING (IJAR)

Fuzzy Logic in Recognition and Search.

Published 8 times annually.

Publisher: Elsevier Science

ISSN 0888-613X.

International Journal of Approximate Reasoning is dedicated to the dissemination of research results from the field of approximate reasoning and its applications, with emphasis on the design and implementation of intelligent systems for scientific and engineering applications. Approximate reasoning is computational modeling of any part of the process used by humans to reason about natural phenomena.

The journal welcomes archival research papers, surveys, short notes and communications, and book reviews. Current areas of interest include, but are not limited to, applications and/or theories pertaining to computer vision, engineering and expert systems, fuzzy logic and control, information retrieval and database design, machine learning, neurocomputing, pattern recognition and robotics.

The journal is affiliated with the North American Fuzzy Information Processing Society (NAFIPS).

IEEE TRANSACTIONS ON FUZZY SYSTEMS (TFS)
A publication of the IEEE Neural Network Council
Published 4 times annually.
ISSN 1063-6706

Transactions on Fuzzy Systems is published quarterly. TFS will consider papers that deal with the theory, design or an application of fuzzy systems ranging from hardware to software. Authors are encouraged to submit articles which disclose significant technical achievements, exploratory developments, or performance studies of fielded systems based on fuzzy models. Emphasis is given to engineering applications.

TFS publishes three types of articles: papers, letters, and correspondence. All contributions are handled in the same fashion. Review management is under the direction of an associate editor, who will solicit four reviews for each submission. The associate editor ordinarily waits for at least three reports before a decision is reached. Often, reviews take six to nine months to obtain, and the publication process after acceptance can take an additional six months.

INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS
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IJUFKS is forum for research on various methodologies for management of imprecise, vague, unceertain or incomplete information. The aim of the journal is to promote theoretical, methodological or practical works dealing with all kinds of methods to represent and manipulate imperfectly described pieces of knowledge. It is published by print and also electronically on Internet.

2.15 Web Pages

Recently, a number of Web pages relevant to the area of Fuzzy Systems or Soft Computing approaches infinity, below we mention only a few. Interested reader may visit these pages and surfing on Internet he/she could find any kind of information.

- http://www-bisc.cs.berkley.edu - official web page of Berkley Initiative for Soft Computing, UCLA, with BISC Mailing List, special interest groups and lot of other information and activities.

- http://ic-www.arc.nasa.gov/ne.html - Neuro-Engeneering and Soft Computing web page by NASA research

- http://www.flll.uni-linz.ac.at - web page of Fuzzy Logic Laboratory Linz, Johannes Kepler University, Austria

- http://www.engineering.missouri.edu/academic/cecs/fuzzy - the University of Missouri - Columbia web page of one of the largest US research group in fuzzy solutions field

- http://www.mitgmbh.de/mit - web page of Management Intelligent Technologies GmbH in Aachen,Germany

- http://www.osu.cz/irafm - web page of Institute for Research and Application of Fuzzy Modeling, University of Ostrava, The Czech Republic
- http://lisp.vse.cz - web page of Laboratory for Intelligent Systems, University of Economics, Prague, The Czech Republic
- http://www.jaist.ac.jp/ks/index-e.html - web page of Graduate School of Knowledge Science, Japan Institute of Science and Technology.

2.16 Fuzzy Researchers

A list of "Who's Who in Fuzzy Logic" (researchers and research organizations in the field of fuzzy logic and fuzzy expert systems) may be obtained by sending a message to

listproc@vexpert.dbai.tuwien.ac.at

with

GET LISTPROC WHOISWHOINFUZZY

in the message body.

Chapter 3

Evolutionary Computation

The universe of Evolutionary Computation (EC); that in turn is only a small footpath to a voluminous scientific universe, that, incorporating Fuzzy Systems, and Artificial Neural Networks, is sometimes referred to as Computational Intelligence (CI) or Computational Science; that in turn is only part of an even more advanced scientific universe of enormous complexity, that incorporating Artificial Life, Fractal Geometry, and other Complex Systems Sciences might someday be referred to as Natural Computation (NC). Over the course of the past years, *global optimization* algorithms imitating certain principles of nature have proved their usefulness in various domains of applications. Especially worth copying are those principles where nature has found "stable islands" in a "turbulent ocean" of solution possibilities. Such phenomena can be found in annealing processes, central nervous systems and biological evolution, which in turn have lead to the following *optimization* methods: Simulated Annealing (SA), Artificial Neural Networks (ANNs) and the broad field of *Evolutionary Computing* (EC). EC may currently be characterized by the following pathways:

Genetic Algorithms (GA),
Evolutionary Programming (EP),
Evolution Strategies (ES),
Classifier Systems (CFS),
Genetic Programming (GP),

and several other problem solving strategies, that are based upon biological observations, that date back to Charles Darwin's discoveries in the 19th century: the means of *natural selection* and the *survival of the fittest*, i.e. the theory of evolution. The inspired algorithms are thus termed *Evolutionary Algorithms* (EA).

Evolutionary algorithm is an umbrella term used to describe computer-based problem solving systems which use computational models of evolutionary processes as key elements in their design and implementation. A variety of evolutionary algorithms have been proposed. The major ones are:

Genetic Algorithms (GA),
Evolutionary Programming (EP),

Evolutionary Strategies (ES),
Classifiers Systems (CS), and
Genetic Programming (GP).

They all share a common conceptual base of simulating the *evolution* of individual structures via processes of *selection*, *mutation*, and *reproduction*. The processes depend on the perceived performance of the individual structures as defined by an environment. More precisely, EAs maintain a *population* of structures, that evolve according to rules of selection and other operators, that are referred to as "search operators", (or genetic operators), such as *recombination* and *mutation*. Each *individual* in the population receives a measure of it's *fitness* in the *environment*. Reproduction focuses attention on high fitness individuals, thus exploiting the available fitness information. Recombination and mutation perturb those individuals, providing general heuristics for *exploration*. Although simplistic from a biologist's viewpoint, these algorithms are sufficiently complex to provide robust and powerful adaptive search mechanisms.

3.1 Genetic Algorithm (GA)

The *Genetic Algorithm* is a model of machine learning which derives its behavior from a metaphor of the processes of evolution in nature. This is done by the creation within a machine of a population of individuals represented by *chromosomes*, in essence a set of character strings that are analogous to the base-4 chromosomes that we see in our own DNA. The individuals in the population then go through a process of evolution. We should note that evolution (in nature or anywhere else) is not a purposive or directed process. That is, there is no evidence to support the assertion that the goal of evolution is to produce Mankind. Indeed, the processes of nature seem to boil down to different individuals competing for resources in the environment. Some are better than others. Those that are better are more likely to survive and propagate their genetic material.

In nature, we see that the encoding for our genetic information (genome) is done in a way that admits asexual reproduction (such as by budding) typically results in offspring that are genetically identical to the parent. Sexual reproduction allows the creation of genetically radically different offspring that are still of the same general flavor (species).

At the molecular level what occurs (wild oversimplification alert!) is that a pair of chromosomes bump into one another, exchange chunks of genetic information and drift apart. This is the recombination operation, which GA/GPers generally refer to as crossover because of the way that genetic material crosses over from one chromosome to another. The crossover operation happens in an environment where the selection of who gets to mate is a function of the fitness of the individual, i.e. how good the individual is at competing in its environment. Some GAs use a simple function of the fitness measure to select individuals (probabilistically) to undergo genetic operations such as crossover or asexual reproduction (the propagation of genetic material unaltered). This is fitness-proportionate selection.

Other implementations use a model in which certain randomly selected individuals in a subgroup compete and the fittest is selected. This is called *tournament selection* and is the form of selection we see in nature when stags rut to vie for the privilege of mating with a herd of hinds. The two processes that most contribute to evolution are crossover and fitness based selection/reproduction. As it turns out, there are mathematical proofs that indicate that the process of fitness proportionate reproduction is, in fact, near optimal in some senses. Mutation also plays a role in this process, although how important its role is continues to be a matter of debate (some refer to it as a background operator, while others view it as playing the dominant role in the evolutionary process).

It cannot be stressed too strongly that the GA (as a simulation of a genetic process) is not a random search for a solution to a problem. The genetic algorithm uses stochastic processes, but the result is distinctly non-random. GAs are used for a number of different application areas. An example of this would be multidimensional optimization problems in which the character string of the chromosome can be used to encode the values for the different parameters being optimized. In practice, therefore, we can implement this genetic model of computation by having arrays of bits or characters to represent the chromosomes. Simple bit manipulation operations allow the implementation of crossover, mutation and other operations.

Although a substantial amount of research has been performed on variable-length strings and other structures, the majority of work with GAs is focussed on fixed-length character strings. We should focus on both this aspect of fixed-lengthness and the need to encode the representation of the solution being sought as a character string, since these are crucial aspects that distinguish GP, which does not have a fixed length representation and there is typically no encoding of the problem.

When the GA is implemented it is usually done in a manner that involves the following cycle: Evaluate the fitness of all of the individuals in the population. Create a new population by performing operations such as crossover, fitness-proportionate reproduction and mutation on the individuals whose fitness has just been measured. Discard the old population and iterate using the new population. One iteration of this loop is referred to as a generation. There is no theoretical reason for this as an implementation model. Indeed, we do not see this punctuated behavior in populations in nature as a whole, but it is a convenient implementation model. The first generation (generation 0) of this process operates on a population of randomly generated individuals. From there on, the genetic operations, in concert with the fitness measure, operate to improve the population.

Pseudo Code of Genetic Algorithm:

Algorithm GA **is**
start with an initial time
$t := 0;$

initialize a usually random population of individuals
    initpopulation $P(t)$;
evaluate fitness of all initial individuals in population
    evaluate $P(t)$;
test for termination criterion (time, fitness, etc.)
    while not done **do**
        increase the time counter $t := t + 1$;
        select sub-population for offspring production
            $P\prime :=$ select parents $P(t)$;
        ·recombine the "genes" of selected parents
            recombine $P\prime(t)$;
        ·perturb the mated population stochastically
            mutate $P\prime(t)$;
        ·evaluate it's new fitness
            evaluate $P\prime(t)$;
        ·select the survivors from actual fitness
            $P :=$ survive $P, P\prime(t)$;
    od
end GA.

3.2 Evolutionary Programming (EP)

Evolutionary programming, originally conceived by L. J. Fogel in 1960, is a stochastic optimization strategy similar to GA, but instead places emphasis on the behavioral linkage between parents and their offsprings, rather than seeking to emulate specific genetic operators as observed in nature. EP is similar to evolution strategies, although the two approaches developed independently (see below). Like both ES and GAs, EP is a useful method of optimization when other techniques such as gradient descent or direct, analytical discovery are not possible. Combinatoric and real-valued function optimization in which the optimization surface or fitness landscape is "rugged", possessing many locally optimal solutions, are well suited for EP.

The 1966 book, "Artificial Intelligence Through Simulated Evolution" by Fogel, Owens and Walsh is the landmark publication for EP applications, although many other papers appear earlier in the literature. In the book, finite state automata were evolved to predict symbol strings generated from Markov processes and non- stationary time series. Such evolutionary prediction was motivated by a recognition that prediction is a keystone to intelligent behavior (defined in terms of adaptive behavior, in that the intelligent organism must anticipate events in order to adapt behavior in light of a goal). Since then EP attracted a diverse group of academic, commercial and military researchers engaged in both developing the theory of the EP technique and in applying EP to a wide range of optimization problems, both in engineering and biology.

For EP, like GAs, there is an underlying assumption that a fitness landscape can be characterized in terms of variables, and that there is an optimum solution (or multiple such optima) in terms of those variables. For example, if one were trying to find the shortest path in a Traveling Salesman Problem, each solution would be a path. The length of the path could be expressed as a number, which would serve as the solution's fitness. The fitness landscape for this problem could be characterized as a hypersurface proportional to the path lengths in a space of possible paths. The goal would be to find the globally shortest path in that space, or more practically, to find very short tours very quickly. The basic EP method involves 3 steps (Repeat until a threshold for iteration is exceeded or an adequate solution is obtained):

Step 1. Choose an initial population of trial solutions at random. The

number of solutions in a population is highly relevant to the speed of optimization, but no definite answers are available as to how many solutions are appropriate (other than 1) and how many solutions are just wasteful.

Step 2. Each solution is replicated into a new population. Each of these offspring solutions are mutated according to a distribution of mutation types, ranging from minor to extreme with a continuum of mutation types between. The severity of mutation is judged on the basis of the functional change imposed on the parents.

Step 3. Each offspring solution is assessed by computing it's fitness.

Typically, a stochastic tournament is held to determine n solutions to be retained for the population of solutions, although this is occasionally performed deterministically. There is no requirement that the population size be held constant, however, nor that only a single offspring be generated from each parent. It should be pointed out that EP typically does not use any crossover as a genetic operator. EP and GAs

There are two important ways in which EP differs from GA. First, there is no constraint on the representation. The typical GA approach involves encoding the problem solutions as a string of representative tokens, the genome. In EP, the representation follows from the problem. A neural network can be represented in the same manner as it is implemented, for example, because the mutation operation does not demand a linear encoding. (In this case, for a fixed topology, real- valued weights could be coded directly as their real values and mutation operates by perturbing a weight vector with a zero mean multivariate Gaussian perturbation. For variable topologies, the architecture is also perturbed, often using Poisson distributed additions and deletions.)

Second, the mutation operation simply changes aspects of the solution according to a statistical distribution which weights minor variations in the behavior of the offsprings as highly probable and substantial variations as increasingly unlikely. Further, the severity of mutations is often reduced as the global optimum is approached. There is a certain tautology here: if the global optimum is not already known, how can the spread of the mutation operation be damped as the solutions approach it? Several techniques have been proposed and implemented which address this difficulty, the most widely studied being the "Meta-Evolutionary" technique in which the variance of the mutation distribution is subject to mutation by a fixed variance mutation operator and evolves along with the solution.

Despite their independent development over 30 years, EP and ES share many similarities. When implemented to solve real-valued function optimization problems, both typically operate on the real values themselves (rather than any coding of the real values as is often done in GAs). Multivariate zero mean Gaussian mutations are applied to each parent in a population and a selection mechanism is applied to determine which solutions to remove from the population. The similarities extend to the use of self-adaptive methods for determining the appropriate mutations to use – methods in which each parent carries not only a potential solution to the problem at hand, but also information on how it will distribute new trials (offspring). Most of the theoretical results on convergence (both asymptotic and velocity) developed for ES or EP also apply directly to the other. The main differences between ES and EP are:

1. SELECTION: EP typically uses stochastic selection via a

tournament. Each trial solution in the population faces competition against a preselected number of opponents and receives a "win" if it is at least as good as its opponent in each encounter. Selection then eliminates those solutions with the least wins. In contrast, ES typically uses deterministic selection in which the worst solutions are purged from the population based directly on their function evaluation.

2. RECOMBINATION: EP is an abstraction of evolution at the level of

reproductive population (i.e., species) and thus no recombination mechanisms are typically used because recombination does not occur between species (by definition: see Mayr's biological species concept). In contrast, ES is an abstraction of evolution at the level of individual behavior. When self-adaptive information is incorporated this is purely genetic information and thus some forms of recombination are reasonable and many forms of recombination have been implemented within ES. Again, the effectiveness of such operators depends on the problem at hand.

Pseudo Code of Evaluation Programming:

Algorithm EP **is**
start with an initial time
  $t := 0$;
initialize a usually random population of individuals
  initpopulation $P(t)$;
evaluate fitness of all initial individuals in population
  evaluate $P(t)$;
test for termination criterion (time, fitness, etc.)
  while not done **do**
    ·perturb the whole population stochastically
      mutate $P\prime(t)$;
    ·evaluate it's new fitness
      evaluate $P\prime(t)$;
    ·stochastically select the survivors from actual fitness
      $P :=$ survive $P, P\prime(t)$;

·increase the time counter $t := t + 1$;

od

end EP.

3.3 Evolution Strategies (ES)

Evolution strategies were invented to solve technical OPTIMIZATION problems like e.g. constructing an optimal flashing nozzle, and ES were primarily only known in civil engineering as an alternative to standard methods. Usually no closed form analytical objective function is available for optimization problems and hence, a few applicable optimization method exists. The first attempts to imitate principles of organic evolution on a computer still resembled the iterative optimization methods. In a two-membered ES, one parent generates one offspring per generation by applying normally distributed mutations, i.e. smaller steps occur more likely than big ones, until a child performs better than its ancestor and takes its place. Because of this simple structure, theoretical results for stepsize control and speed of convergence could be derived. The ratio between successful and all mutations should come to 1/5: the so- called *1/5 success rule* was discovered. This first algorithm, using mutation only, has then been enhanced to an $(\mu + \lambda)$ strategy which incorporated recombination due to several, i.e. μ parents being available. The mutation scheme and the exogenous stepsize control were taken across unchanged from $(1+1)$ ESs. Later, this rule was generalized to the multimembered ES now denoted by $(\mu + \lambda)$ and (μ, λ) which imitates the following basic principles of organic evolution: a population, leading to the possibility of recombination with random mating, mutation and selection. These strategies are termed *plus strategy* and *comma strategy*, respectively. In the plus case, the parental generation is taken into account during selection, while in the comma case only the offspring undergoes selection, and the parents die off. By μ the population size is denoted, and λ denotes the number of offspring generated per generation.

ESs are capable of solving high dimensional, multimodal, nonlinear problems subject to linear and/or nonlinear constraints. The objective function can also, e.g. be the result of a simulation, it does not have to be given in a closed form. This also holds for the constraints which may represent the outcome of, e.g. a finite elements method (FEM). ESs have been adapted to vector optimization problems, and they can also serve as a heuristic for NP-complete combinatorial problems like the travelling salesman problem or problems with a noisy or changing response surface.

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Schwefel, H.P. (1977) "Numerische Optimierung von Computermodellen mittels der Evolutionsstrategie", Birkhaeuser, Basel.

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3.4 Classifier Systems (CS)

No other paradigm of EC has undergone more changes to it's name space than this one. Initially, Holland, see the references below, called his cognitive models "Classifier Systems" (CS). Whence Riolo came into play in 1986 and Holland added a reinforcement component to the overall design of a CS, that emphasized its ability to learn. So, the word "learning" was pretended to the name, to make: "Learning Classifier Systems" (LCS). LCSs are sometimes subsumed under a "new" machine learning paradigm called "Evolutionary Reinforcement Learning" (ERL), see also the chapter Machine Learning below. Classifier systems are systems which take a set of inputs, and produce a set of outputs which indicate some classification on the inputs. It is often regarded as *artificial life* rather than EC. CS can be seen as one of the early applications of GAs, for CSs use this evolutionary algorithm to adapt their behavior toward a changing environment. A cognitive system is capable of classifying the goings on in its environment, and then reacting to these goings on appropriately. We need

(1) an environment;

(2) receptors that tell our system about the goings on;

(3) effectors, that let our system manipulate its environment; and

(4) the system itself, conveniently a "black box" in this first approach, that has (2) and (3) attached to it, and "lives" in (1).

The most primitive "black box" we can think of is a computer. It has inputs (2), and outputs (3), and a message passing system in-between, that converts (i.e., computes), certain input messages into output messages, according to a set of rules, usually called the "program" of that computer. From the theory of computer science, we now borrow the simplest of all program structures, that is something called "production system" (PS). Although it merely consists of a set of if-then rules, it still resembles a full- fledged computer. We now term a single "if-then" rule a "classifier", and choose a representation that makes it easy to manipulate these, for example by encoding them into binary strings. We then term the set of classifiers, a "classifier population", and immediately know how to breed new rules for our system: just use a GA to generate new rules/classifiers from the current population. All that is left are the messages floating through the black box. They should also be simple strings of zeroes and ones, and are to be kept in a data structure, we call "the message list". With all this given, we can imagine the goings on inside the black box as follows: The input interface (2) generates messages, i.e., 0/1 strings, that are written on the message list. Then these messages are matched against the condition-part of all classifiers, to find out which actions are to be triggered. The message list is then emptied, and the encoded actions, themselves just messages, are posted to the message list. Then, the output interface (3) checks the message list for messages concerning the effectors. And the cycle restarts. Note, that it is possible in this set-up to have "internal messages", because the message list is not emptied after (3) has checked; thus, the input interface messages are added to the initially empty list. The general idea of the CS is to start from scratch, i.e., from tabula rasa (without any knowledge) using a randomly generated classifier population, and let the system learn its program by induction, this reduces the input stream to recurrent input patterns, that must be repeated over and over again, to enable the animat to classify its current situation/context and react on the goings on appropriately.

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Rich, E. (1983) "Artificial Intelligence".McGraw-Hill, N. York.

3.5 Genetic Programming (GP)

Genetic programming is the extension of the genetic model of learning into the space of programs. That is, the objects that constitute the population are not fixed-length character strings that encode possible solutions to the problem at hand, they are programs that, when executed, "are" the candidate solutions to the problem. These programs are expressed in genetic programming as parse trees, rather than as lines of code. Because this is a very simple thing to do in the programming language Lisp, many GP people tend to use Lisp. However, this is simply an implementation detail. There are straightforward methods to implement GP using a non-Lisp programming environment. The programs in the population are composed of elements from the *function set* and the *terminal set*, which are typically fixed sets of symbols selected to be appropriate to the solution of problems in the domain of interest. In GP the crossover operation is implemented by taking randomly selected subtrees in the individuals (selected according to fitness) and exchanging them. It should be pointed out that GP usually does not use any mutation as a genetic operator.

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Koza, J.R., "Genetic Programming II", A Bradford Book, MIT Press, Cambridge, 1994.

Shanahan, J.G., Soft Computing for Knowledge Discovery, Kluwer Acad. Publ., Boston /Dordrecht /London, 2000.

Chapter 4

Neural Networks

4.1 Introduction

There is no universally accepted definition of *neural networks* (NN), a common characterization says that an NN is a network of many simple processors ("units"), each possibly having a small amount of local memory. The units are connected by communication channels ("connections") which usually carry numeric (as opposed to symbolic) data, encoded by any of various means. The units operate only on their local data and on the inputs they receive via the connections. The restriction to local operations is often relaxed during training.

Some NNs are models of biological neural networks and some are not, but historically, much of the inspiration for the field of NNs came from the desire to produce artificial systems capable of sophisticated, perhaps "intelligent", computations similar to those that the human brain routinely performs, and thereby possibly to enhance our understanding of the human brain.

Most NNs have some sort of "training" rule whereby the weights of connections are adjusted on the basis of data. In other words, NNs "learn" from examples (as children learn to recognize dogs from examples of dogs) and exhibit some capability for generalization beyond the training data.

NNs normally have great potential for parallelism, since the computations of the components are largely independent of each other. Some people regard massive parallelism and high connectivity to be defining characteristics of NNs, but such requirements rule out various simple models, such as simple linear regression (a minimal feedforward net with only two units plus bias), which are usefully regarded as special cases of NNs.

Here are some of definitions from the books:

According to Haykin, S. (1994) "Neural Networks: A Comprehensive Foundation". Macmillan, New York, p. 2:

"A neural network is a massively parallel distributed processor that has a natural propensity for storing experiential knowledge and making it available for use. It resembles the brain in two respects:

1. Knowledge is acquired by the network through a learning process.

2. Interneuron connection strengths known as synaptic weights are used to store the knowledge".

According to Nigrin, A. (1993) "Neural Networks for Pattern Recognition". The MIT Press, Cambridge, MA, p. 11:

"A neural network is a circuit composed of a very large number of simple processing elements that are neurally based. Each element operates only on local information. Furthermore each element operates asynchronously; thus there is no overall system clock".

According to Zurada, J.M. (1992) "Introduction to Artificial Neural Systems". PWS Publishing Company, Boston, p. 15:

"Artificial neural systems, or neural networks, are physical cellular systems which can acquire, store, and utilize experiential knowledge".

Below we list some exposition texts about NN on Internet:

http://www.dontveter.com/bpr/bpr.html
http://gannoo.uce.ac.uk/bpr/bpr.html
http://www.shef.ac.uk/psychology/gurney/notes/index.html
http://www.statsoft.com/textbook/stathome.html
ftp://ftp.sas.com/pub/neural/FAQ.html

4.2 Principles of Neural Networks

In principle, NNs can compute any computable function, i.e., they can do everything a normal digital computer can do, or perhaps even more.

In practice, NNs are especially useful for classification and function approximation/mapping problems which are tolerant of some imprecision, which have lots of training data available, but to which hard and fast rules (such as those that might be used in an expert system) cannot easily be applied. Almost any finite-dimensional vector function on a compact set can be approximated to arbitrary precision by feedforward NNs (which are the type most often used in practical applications) if you have enough data and enough computing resources.

To be more precise, *feedforward networks* with a single hidden layer and trained by least-squares are statistically consistent estimators of arbitrary square-integrable regression functions under certain practically-satisfiable assumptions regarding sampling, target noise, number of hidden units, size of weights, and form of hidden-unit activation function. Such networks can also be trained as statistically consistent estimators of derivatives of regression functions and quantiles of the conditional noise distribution. Feedforward networks with a single hidden layer using threshold or activation functions are universally consistent estimators of binary classifications under similar assumptions. Note that these results are stronger than the universal approximation theorems that merely show the existence of weights for arbitrarily accurate approximations, without demonstrating that such weights can be obtained by learning.

Unfortunately, the above consistency results depend on one impractical assumption: that the networks are trained by an error minimization technique that comes arbitrarily close to the global minimum. Such minimization is computationally intractable except in small or simple problems. In practice, however, you can usually get good results without doing a full-blown global optimization; e.g., using multiple (say, 10 to 1000) random weight initializations is usually sufficient.

One example of a function that a typical neural net cannot learn is $f(x) = 1/x$ on the open interval $(0, 1)$. An open interval is not a compact set. With any bounded output activation function, the error will get arbitrarily large as the input approaches zero. Of course, you could make the output activation function a reciprocal function and easily get a perfect fit, but NNs are most often used in situations where you do not have enough prior knowledge to set the activation function in such a clever way. There are also many other important problems that are so difficult that a neural network will be unable to learn them without memorizing the entire training set, such as:

· Predicting random or pseudo-random numbers.

· Factoring large integers.

· Determining whether a large integer is prime or composite.

· Decrypting anything encrypted by a good algorithm.

It is important to understand that there are no methods for training NNs that can magically create information that is not contained in the training data.

Feedforward NNs are restricted to finite-dimensional input and output spaces. Recurrent NNs can in theory process arbitrarily long strings of numbers or symbols. But training recurrent NNs has posed much more serious practical difficulties than training feedforward networks. NNs are, at least today, difficult to apply successfully to problems that concern manipulation of symbols and rules, but much research is being done.

As for simulating human consciousness and emotion, that's still in the realm of science fiction. Consciousness is still one of the world's great mysteries. Artificial NNs may be useful for modeling some aspects of or prerequisites for consciousness, such as perception and cognition, but NNs provide no insight so far into what is called the "hard problem":

Many books and articles on consciousness have appeared in the past few years, and one might think we are making progress. But on a closer look, most of this work leaves the hardest problems about consciousness untouched. Often, such work addresses what might be called the "easy problems" of consciousness: How does the brain process environmental stimulation? How does it integrate information? How do we produce reports on internal states? These are important questions, but to answer them is not to solve the hard problem: Why is all this processing accompanied by an experienced inner life?

Neural Networks are interesting for quite a lot of very different people for various reasons:

· Computer scientists want to find out about the properties of non-symbolic information processing with neural nets and about learning systems in general.

· Statisticians use neural nets as flexible, nonlinear regression and classification models.

· Engineers of many kinds exploit the capabilities of neural networks in many areas, such as signal processing and automatic control.

· Cognitive scientists view neural networks as a possible apparatus to describe models of thinking and consciousness (High-level brain function).

· Neuro-physiologists use neural networks to describe and explore medium-level brain function (e.g. memory, sensory system, motorics).

· Physicists use neural networks to model phenomena in statistical mechanics and for a lot of other tasks.

· Biologists use Neural Networks to interpret nucleotide sequences.

· Philosophers.

For world-wide lists of groups doing research on NNs, see the Foundation for Neural Networks's (SNN) web page:

http://www.mbfys.kun.nl/snn/pointers/groups.html

and Neural Networks Research on the IEEE Neural Network Council's homepage:

http://www.ieee.org/nnc.

There are many kinds of NNs by now, new ones (or at least variations of old ones) are invented continuously. Below is a collection of some of the most well known methods, not claiming to be complete.

4.3 Learning Methods in NNs

The two main kinds of learning algorithms are supervised and unsupervised.

· In *supervised learning*, the correct results (target values, desired outputs) are known and are given to the NN during training so that the NN can adjust its weights to try match its outputs to the target values. After training, the NN is tested by giving it only input values, not target values, and seeing how close it comes to outputting the correct target values.

· In *unsupervised learning*, the NN is not provided with the correct results during training. Unsupervised NNs usually perform some kind of data compression, such as dimensionality reduction or clustering. See "What does unsupervised learning learn?"

The distinction between supervised and unsupervised methods is not always clear-cut. An unsupervised method can learn a summary of a probability distribution, then that summarized distribution can be used to make predictions. Furthermore, supervised methods come in two subvarieties: auto-associative and hetero-associative. In auto-associative learning, the target values are the same as the inputs, whereas in hetero-associative learning, the targets are generally different from the inputs. Many unsupervised methods are equivalent to auto-associative supervised methods. For more details, see "What does unsupervised learning learn?"

Two major kinds of network topology are feedforward and feedback.

· In a *feedforward* NN, the connections between units do not form cycles. Feedforward NNs usually produce a response to an input quickly. Most feedforward NNs can be trained using a wide variety of efficient conventional numerical methods in addition to algorithms invented by NN reserachers.

· In a *feedback* or *recurrent* NN, there are cycles in the connections. In some feedback NNs, each time an input is presented, the NN must iterate for a potentially long time before it produces a response. Feedback NNs are usually more difficult to train than feedforward NNs.

Some kinds of NNs (such as those with winner-take-all units) can be implemented as either feedforward or feedback networks.

NNs also differ in the kinds of data they accept. Two major kinds of data are categorical and quantitative.

· Categorical variables take only a finite (technically, countable) number of possible values, and there are usually several or more cases falling into each category. Categorical variables may have symbolic values (e.g., "male" and "female", or "red", "green" and "blue") that must be encoded into numbers before being given to the network. Both supervised learning with categorical target values and unsupervised learning with categorical outputs are called "classification."

· Quantitative variables are numerical measurements of some attribute, such as length in meters. The measurements must be made in such a way that at least some arithmetic relations among the measurements reflect analogous relations among the attributes of the objects that are measured.

Some variables can be treated as either categorical or quantitative, such as number of children or any binary variable. Most regression algorithms can also be used for supervised classification by encoding categorical target values as 0/1 binary variables and using those binary variables as target values for the regression algorithm. The outputs of the network are posterior probabilities when any of the most common training methods are used.

4.4 Well-Known Kinds of NNs

Below we classify the well known kinds of NNs.

1. Supervised
   - Feedforward
     - Linear
       - Hebbian - Hebb (1949), Fausett (1994)
       - Perceptron - Rosenblatt (1958), Minsky and Papert (1969/1988)
       - Adaline - Widrow and Hoff (1960), Fausett (1994)
     - MLP: Multilayer perceptron - Bishop (1995), Reed and Marks (1999)
       - Backprop - Rumelhart, Hinton, and Williams (1986)
       - Cascade Correlation - Fahlman and Lebiere (1990), Fausett (1994)

        - Quickprop - Fahlman (1989)
        - RPROP - Riedmiller and Braun (1993)
    - RBF networks - Bishop (1995), Moody and Darken (1989), Orr (1996)
        - OLS: Orthogonal least squares - Chen, Cowan and Grant (1991)
    - CMAC: Cerebellar Model Articulation Controller - Albus (1975), Brown and Harris (1994)
    - Classification only
        - LVQ: Learning Vector Quantization - Kohonen (1988), Fausett (1994)
        - PNN: Prob abilistic Neural Network - Specht (1990), Masters (1993), Hand (1982), Fausett (1994)
    - Regression only
        - GNN: General Regression Neural Network - Specht (1991), Nadaraya (1964), Watson (1964)
- Feedback - Hertz, Krogh, and Palmer (1991), Medsker and Jain (2000)
    - BAM: Bidirectional associative memory - Kosko (1992), Fausett (1994)
    - Boltzman machine - Ackley et al. (1985), Fausett (1994)
    - Recurrent time series
        - Backpropagtion through time - Werbos (1990)
        - Elman - Elman (1990)
        - FIR: Finite impulse response - Wan (1990)
        - Jordan - Jordan (1986)
        - Real-time recurrent network - Williams and Zipser (1989)
        - Recurrent backpropagation - Pineda (1989), Fausett (1994)
        - TDNN: Time delay NN - Lang, Waibel and Hinton (1990)
- Competitive
    - ARTMAP - Carpenter, Grossberg and Reynolds (1991)
    - Fuzzy ARTMAP - Carpenter, Grossberg, Markuzon, Reynolds and Rosen (1992), Kasuba (1993)
    - Gaussian ARTMAP - Williamson (1995)
    - Counterpropagation - Hecht-Nielsen (1987; 1988; 1990), Fausett (1994)
    - Neocognitron - Fukushima, Miyake, and Ito (1983), Fukushima, (1988), Fausett (1994)

2. Unsupervised - Hertz, Krogh, and Palmer (1991)

- Competitive
  - Vector Quantization
    - Grossberg - Grossberg (1976)
    - Kohonen - Kohonen (1984)
    - Conscience - Desieno (1988)
  - Self-Organizing Map
    - Kohonen - Kohonen (1995), Fausett (1994)
    - GTM: - Bishop, Svens\'en and Williams (1997)
    - Local Linear - Mulier and Cherkassky (1995)
  - Adaptive resonance theory
    - ART 1 - Carpenter and Grossberg (1987a), Moore (1988), Fausett (1994)
    - ART 2 - Carpenter and Grossberg (1987b), Fausett (1994)
    - ART 2-A - Carpenter, Grossberg and Rosen (1991a)
    - ART 3 - Carpenter and Grossberg (1990)
    - Fuzzy ART - Carpenter, Grossberg and Rosen (1991b)
  - DCL: Differential competitive learning - Kosko (1992)
- Dimension Reduction - Diamantaras and Kung (1996)
  - Hebbian - Hebb (1949), Fausett (1994)
  - Oja - Oja (1989)
  - Sanger - Sanger (1989)
  - Differential Hebbian - Kosko (1992)
- Autoassociation
  - Linear autoassociator - Anderson et al. (1977), Fausett (1994)
  - BSB: Brain State in a Box - Anderson et al. (1977), Fausett (1994)
  - Hopfield - Hopfield (1982), Fausett (1994)

3. Nonlearning
   - Hopfield - Hertz, Krogh, and Palmer (1991)
   - Various networks for optimization - Cichocki and Unbehauen (1993), etc.

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The amount of available references is enormous, below we cite only those associated with the general problems of NNs and kinds of NNs. For more detailed references, see

ftp://ftp.sas.com/pub/neural/FAQ.html, or http://www.soft-computing.de.

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Chapter 5

Machine Learning

5.1 Introduction

The main goal of *machine learning* is to built computer systems that can adapt and learn from the experience. Different learning techniques have been developed for different performance tasks. The primary tasks that have been investigated are supervised learning for discrete decision making process, supervised learning for continuous prediction, reinforcement learning for sequential decision making and unsupervised learning. Here we consider a more general setting not necessarily based on NNs, see the previous chapter.

The best understood task is one-shot decision making, see Willson, (1999): the computer is given a description of an object (event, situation, etc.) and it must output a classification of that object. For example, an optical recognizer must input a digitized image of a character and output the name of that character ("A" through "Z"). A machine leaning approach to constructing such a system would begin by collecting training examples, each consisting of a digitized image of a character and the correct name of the character. This would be analyzed by learning algorithm to produce an optical character recognizer for classifying new images.

Machine learning algorithms search a space of candidate classifiers for one that performs well on the training examples and is expected to work well to new classes. Learning methods for classification problems include decision trees, NNs, rule learning algorithms, clustering methods and Bayesian networks, see the next chapter.

There exist for basic questions to answer when developing a new machine learning system:

1. How is the classifier represented?
2. How are examples represented?
3. What objective function should be employed to evaluate candidate classifiers?

4. What search algorithm should be used?

Let us illustrate these four questions using two of the most popular learning algorithms: C4.5 (by Quinlan, 1993) and backpropagation.

The C4.5 algorithm represents a classifier as a decision tree. Each example is represented by a vector of features, e.g. one feature describing a printed character might be whether it has a long vertical line segment (such as the letters B, D, E, etc.). Each node in the decision tree tests the value of one of the features and branches to one of its children, depending on the results of the test. A new example is classified by starting at the root of the tree and applying the test at that node. If the test is true, it branches to the left child; otherwise it branches to the right child. The test at the child node is then applied recursively, until one of the leaf nodes of the tree is reached. The leaf node gives the predicted classification of the example.

C4.5 searches the space of decision trees through a constructive search. It first considers all trees consisting of only a single root node and chooses one of those. Then it considers all trees having that root node and various left children, and chooses one of those, and so on. This process constructs the tree incrementally with the goal of finding the decision tree that miniimzes so called pessimistic error, which is an estimate of classification error of the tree on new training examples. It is based on taking the upper endpoint of a confidence interval for the error of the tree computed separately for each leaf. Although C4.5 constructs its classifier, other learning algorithms begin with a complete classifier and modify it. The backpropagation algorithm for learning NNs begins with an initial NN and computes the classification error of that network on the training data. It then makes small adjustments in the weights of the network to reduce this error. This process is repeated until the error is minimized.

5.2 Three Basic Theories

There are two fundamentally different theories of machine learning. The *classical theory* takes the view that, before analyzing the training examples, the learning algorithms makes a "guess" about an appropriate space of classifiers to consider, e.g., it guesses that decision trees will be better than neural networks. The algorithm then searches the chosen space of classifiers hoping to find a good fit to the data. The *Bayesian theory* take the view that the designer of a learning algorithm encodes all of his/her prior knowledge in the form of a prior probability distribution over the space of candidate classifiers. The learning algorithm then analyzes the training examples and computes the posterior probability distribution over the space of classifiers. In this view the training data serve to reduce our remaining uncertainty about the unknown classifier.

Based on the development of fuzzy set theory, reently fuzzy theory has gained a popularity. Similar to Bayesian approach, learning algorithms encode the prior knowledge in the form of prior possibility distributions. The resulting algorithms output the possibility distributions and in this way the training data again reduce our uncertainty about the unknown classifier.

These three theories lead to different practical approaches. The first theory encourages the development of large, flexible hypothesis space (such as decision trees and NNs) that can represent many different classifiers. The second and third theory implies the development of representational systems that can readily express prior knowledge, such as Bayesian and fuzzy networks and other stochastic /fuzzy models.

5.3 Supervised machine Learning

Here the discussion has focused on discrete classification, but the same issues arise for the second learning task: supervised learning for continuous prediction, or regression. In this task, the computer is given a description of an object and it must output a real number. For example, given a description of prospective student (by the high-school grade-point average (GPA), etc.), the system must predict the student's college GPA. The machine learning approach is the same: a collection of training examples describing students and their GPAs is provided to the learning algorithm, which outputs a predictor to predict college GPA. Learning methods for continuous prediction include neural networks, regression trees, linear and additive models, etc. Classification and prediction are often called supervised learning tasks, because the training data include not only the input objects but also the corresponding output values.

5.4 Reinforcement Machine Learning

As far as *reinforcement learning* tasks are concerned, in these tasks, each decision made by the computer affects subsequent decisions. Consider, for example, a computer controlled robot attempting to navigate from a hospital kitchen to a patient's room. At each point in time, the computer must decide whether to move the robot forward, left, right or backward. Each decision changes the location of the robot, so that the next decision will depend on previous decision. After each decision, the environment provides a real valued reward. For example, robot may receive a positive reward for delivering a meal to the correct patient and a negative reward for bumping into walls. the goal of the robot is to choose sequences of actions to maximize its long-term reward. This is very different from the standard supervised learning task, where each classification decision is completely independent of other decisions.

5.5 Unsupervised machine Learning

The final learning task we discuss here is *unsupervised learning*, where the computer is given a collection of objects and is asked to construct a model to explain the observed properties of these objects. No teacher provides desired output or rewards. For example, given a collection of astronomical objects, the learning system should group the objects into stars, planets, and galaxies and describe each group of its electromagnetic spectrum, distance from earth, and so on.

Unsupervised learning can be understood in a much wider range of tasks in cluster analysis, see the chapter devoted to fuzzy systems. One useful formulation of unsupervised learning is density estimation. Define a probability distribution $P(X)$ to be the probability that object X will be observed. Then the goal of unsupervised learning is to find this probability distribution on the samples of objects. This may be typically accomplished by defining a family of possible stochastic models and choosing the model that best fits for the data.

Considering again the above example with astronomical objects, the probability distribution $P(X)$ describing the whole collection of astronomical objects could then be modeled as a mixture of normal distributions - one distribution for each of objects. The learning process determines the number of groups and the mean and covariance matrix of each multivariate distribution. The AUTOCLASS program discovered a new class of astronomical objects in just this way, see Cheeseman at al. (1988). Similar method has been applied in HMM speech recognition model, see Rabiner (1989).

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Chapter 6

Probabilistic Reasoning

6.1 Introduction

Probabilistic reasoning (PR) reefers to the formation of probability judgements and subjective beliefs about the likelihoods of outcomes and the frequencies of events. The judgements that people make are often about things that are only indirectly observable and only partly predictable. For example, the weather, a game of sports, a project at work, or whatever it could be, our willingness to engage in an endeavor and the actions that we take depend on our estimated likelihood of the relevant outcomes. How likely id our team to win? How frequently have projects like this failed before? Like other areas of reasoning and decision making, we distinguish normative, descriptive and prescriptive approaches.

The normative approach to PR is constrained by the same mathematical rules that govern the classical, set-theoretic concept of probability. In particular, probability judgements are said to be coherent, if they satisfy Kolmogorov's axioms:

1. No probabilities are negative.
2. The probability of tautology is 1.
3. The probability of a disjunction of two logically exclusive statements equals to the sum of their respective probabilities.
4. The probability of a conjunction of two statements equals the product of probability of the first and the probability of the second, assuming that the second statement is true.

Whereas the first three axioms involve unconditional probabilities, the fourth introduces conditional probabilities. When applied to hypotheses and data in inferential contexts, simple arithmetic manipulation of rule 4. leads to the result that the (posterior) probability of a hypothesis conditional on the data is equal to the probability of the data conditional on the hypothesis multiplied by the (prior) probability of the hypothesis, all divided by the probability of the data. Although mathematically trivial, this is of central importance in the context of Bayesian inference, which underlies theories of belief updating and is considered by many to be a normative requirement of probabilistic reasoning. This is the main difference to reasoning that is based on logic, either classical or fuzzy logic (or multi-valued logic), see the chapter dealing with fuzzy systems.

6.2 Markov and Bayesian Networks

Using the above axiomatic basis, structural properties of probabilistic models can be identified and captured by graphical representations, particularly Markov networks and Bayesian networks. A Markov network is an undirected graph whose links represent symmetrical probabilistic dependences, while a Bayesian network is a directed acyclic graph whose arrows represent causal influences or class-property relationships. A formal semantics of both network types has been established and knowledge representation schemes in inference systems is explored with its power and limitations.

The impact of each new piece of evidence is viewed as a perturbation that propagates through the network via message-passing between neighboring variables, with minimal external supervision. Belief parameters, communication messages and updating rules to guarantee some equilibrium can be reached in time proportional to the longest path in the network.

In belief updating the impact of each new piece of evidence is viewed as a perturbation that propagates through the network, at equilibrium, each variable should be bound to a fixed value that together with all other value assignments is the best interpretation of the evidence. This approach is called the *distributed computation.*

6.3 Decision Analysis based on PR

In the decision analysis and making rational decisions PR provides coherent prescriptions for choosing actions and meaningfully guarantees of the quality of these choices. Whereas judgements about the likelihoods of the events is qualified by probabilities, judgements about the desirability of action consequences are qualified by utilities. Bayesian metodologies regard the expected utilities as a gauge of the merit of actions and therefore treat them as prescriptions for choosing among alternatives. These attitudes are captured by what is known as the axioms of utility theory introduced by von Neumann and Morgestern in their famous book from 1947.

While the arguments for the expected-value criterion are normally based on long-run accumulation of payoffs from a long series of repetitive decisions, e.g. gambling, the expected utility criterion is also justified in single-decision situation, as the summation operator originates not with additive accumulation of payoffs but with the additive axiom of probability theory.

6.4 Learning Structure from Data

Taking Bayesian belief networks as the basic scheme of knowledge representation, the learning task separates into two additional subtasks:

(1) *Parameters learning* - learning the numerical parameters (i.e. the conditional probabilities) for a given network topology.

(2) *Structure learning* - identifying the network topology.

The subtasks are not mutually independent because the set of parameters needed depends largely on topology assumed, and conversely. The chief role is played by the structure learning. In PR a number of sophisticated methods and techniques for discovering structures in empirical data have been developed, see the literature, among the m tree structuring method of Chow and Liu (1968) are of significant importance. Dechter (1987) used the similar method to decompose the general n-ary relations into trees of binary relations. The polytree recovery algorithm was developed by Rebane and Pearl (1987). A general probabilistic account of causation was developed under the name of minimal causal model by Pearl and Verma (1991).

6.5 Dampster-Shaffer's Theory

Belief functions were introduced by Dempster (1967) as a generalization of Bayesian inference wherein probabilities are assigned to sets rather than to individual points. In this interpretation, however, it is hard to justify the exclusion of models that are consistent with the information available but that take into account the possibility that the observation could have turned out differently. Shaffer (1976) has reinterpreted Dempster's theory as a model of evidential reasoning including two interactive frames:

- probabilistic frame representing the evidence,
- possibilistic frame where categorical compatibility relations are defined.

In this sense the Dampster-Shaffer's theory serves as a bridge between probabilistic and possibilistic (fuzzy) reasoning, see also chapter Fuzzy Systems.

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Chapter 7

Conclusion

The impact of soft computing has been felt increasingly strong in the recent years. Soft computing is likely to play an especially important role in science and engineering, but eventually its influence may extend much farther. Building human-centered systems is an imperative task for scientists and engineers in the new millennium.

In many ways, soft computing represents a significant paradigm shift in the aims of computing - a shift which reflects the fact that the human mind, unlike present day computers, possesses a remarkable ability to store and process information which is pervasively imprecise, uncertain and lacking in categoricity.

In this overview, we have focused primarily on fuzzy methodologies and fuzzy systems, as they bring basic ideas to other SC methodologies. The other constituents of SC have been also surveyed here but for details we refer to the existing vast literature.

Part II

Fuzzy Optimization

Chapter 8

Fuzzy Sets

8.1 Introduction

A well known fact in the theory of sets is that properties of subsets of a given set X and their mutual relations can be studied by means of their characteristic functions, see e.g. [9] - [23] and [38]. While this may be advantageous in some contexts, we should notice that the notion of a characteristic function is more complex than the notion of a subset. Indeed, the characteristic function χ_A of a subset A of X is defined by

$$\chi_A(x) = \{ \begin{array}{l} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{array}$$

Since χ_A is a function we need not only the underlying set X and its subset A but also one additional set, in this case the set $\{0, 1\}$ or any other two-element set. Moreover, we also need the notion of Cartesian product because functions are specially structured binary relations, in this case special subsets of $X \times \{0, 1\}$.

If we define fuzzy sets by means of their membership functions, that is, by replacing the range $\{0, 1\}$ of characteristic functions with a lattice, for example, the naturally ordered unit interval $[0, 1]$ of real numbers, then we should be aware of the following fact. Such functions may be related to certain objects (build from subsets of the underlying set) in an analogous way how the characteristic functions are related to subsets. This may explain why the fuzzy community (rightly?) hesitates to accept the view that a fuzzy subset of a given set is nothing else than its membership function. Then, a natural question arises. Namely, what are those objects? Obviously, it can be expected that they are more complex than just subsets because the class of functions mapping X into a lattice can be much richer than the class of characteristic functions. In the next section, we show that it is advantageous to define these objects as nested families of subsets satisfying certain mild conditions.

Even if it is not the purpose of this chapter to deal with interpretations of the concepts involved, it should be noted that fuzzy sets and membership functions are closely related to the inherent imprecision of linguistic expressions in natural languages. Probability theory does not provide a way out, as it usually deals with crisp events and the uncertainty is whether this event will occur or not. However, in fuzzy logic, this is a matter of degree of truth rather than a simple "yes or no" decision.

Such generalized characteristic functions have found numerous connotations in different areas of mathematics, variety of philosophical interpretations and lot of real applications, see e.g. books [9], [21], [23], [38] and [52].

In the context of multicriteria decision making, functions mapping the underlying space into the unit interval $[0,1]$ and representing normalized utility functions can be also interpreted as membership functions of fuzzy sets of the underlying space, see [23].

The main purpose of this chapter is investigation of some properties of fuzzy sets, primarily with respect to generalized concave membership functions and with the prospect of applications in optimization and decision analysis.

8.2 Definition and Basic Properties

In order to define the concept of a fuzzy subset of a given set X within the framework of standard set theory we are motivated by the concept of upper level set of a function introduced in [79], see also [59] and [35]. Throughout this chapter, X is a nonempty set.

Definition 3 *Let X be a set, a* fuzzy subset A of X *is the family of subsets $A_\alpha \subset X$, where $\alpha \in [0,1]$, satisfying the following properties:*

$$\begin{array}{ll} (i) & A_0 = X, \\ (ii) & A_\beta \subset A_\alpha \text{ whenever } 0 \leq \alpha < \beta \leq 1, \\ (iii) & A_\beta = \bigcap_{0 \leq \alpha < \beta} A_\alpha. \end{array} \tag{8.1}$$

A fuzzy subset A of X will be also called a fuzzy set.
A membership function $\mu_A : X \to [0,1]$ of the fuzzy set A *is defined as follows:*

$$\mu_A(x) = \sup\{\alpha | \alpha \in [0,1], x \in A_\alpha\}, \tag{8.2}$$

for each $x \in X$.
The core of A, *$Core(A)$, is given by*

$$Core(A) = \{x \in X \mid \mu_A(x) = 1\}. \tag{8.3}$$

A fuzzy subset A of X is called normalized *if its core is nonempty.*
The support of A, *$Supp(A)$, is given by*

$$Supp(A) = Cl(\{x \in X \mid \mu_A(x) > 0\}). \tag{8.4}$$

The height of A, *$Hgt(A)$, is given by*

$$Hgt(A) = \sup\{\mu_A(x) | x \in X\}. \tag{8.5}$$

For each $\alpha \in [0,1]$ *the* α-cut of A, $[A]_\alpha$, *is given by*

$$[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}. \tag{8.6}$$

For $x \in X$ *the function value* $\mu_A(x)$ *is called the* membership degree of x in the fuzzy set A. *The class of all fuzzy subsets of* X *is denoted by* $\mathcal{F}(X)$.

If A is normalized, i.e. $Core(A) \neq \emptyset$, then $Hgt(A) = 1$, but not vice versa.

In the following two propositions, we show that the family generated by the upper level sets of a function $\mu : X \to [0,1]$ satisfies conditions (8.1), thus, it generates a fuzzy subset of X and the membership function μ_A defined by (8.2) coincides with μ. Moreover, for a given fuzzy set $A = \{A_\alpha\}_{\alpha\in[0,1]}$, every α-cut $[A]_\alpha$ given by (8.6) coincides with the corresponding A_α.

Proposition 4 *Let* $\mu : X \to [0,1]$ *be a function and let* $A = \{A_\alpha\}_{\alpha\in[0,1]}$ *be a family of its upper-level sets, i.e.* $A_\alpha = U(\mu,\alpha)$ *for all* $\alpha \in [0,1]$. *Then* A *is a fuzzy subset of* X *and* μ *is the membership function of* A.

Proof. First, we prove that $A = \{A_\alpha\}_{\alpha\in[0,1]}$, where $A_\alpha = U(\mu,\alpha)$ for all $\alpha \in [0,1]$ satisfies conditions (8.1). Indeed, conditions (i) and (ii) hold easily. For condition (iii), we observe that by (ii) it follows that $A_\beta \subset \bigcap\limits_{0\leq\alpha<\beta} A_\alpha$. To prove the opposite inclusion, let

$$x \in \bigcap_{0\leq\alpha<\beta} A_\alpha. \tag{8.7}$$

Assume the contrary, that is let $x \notin A_\beta$. Then $\mu(x) < \beta$ and there exists α', such that $\mu(x) < \alpha' < \beta$. By (8.7) we have $x \in A_{\alpha'}$, thus $\mu(x) \geq \alpha'$, a contradiction.

It remains to prove that $\mu = \mu_A$, where μ_A is a membership function of A. For this purpose let $x \in X$ and let us show that $\mu(x) = \mu_A(x)$.

By definition (8.2) we have

$$\begin{aligned}\mu_A(x) = \sup\{\alpha|\alpha \in [0,1], x \in A_\alpha\} &= \sup\{\alpha|\alpha \in [0,1], x \in U(\mu,\alpha)\}\\ &= \sup\{\alpha|\alpha \in [0,1], \mu(x) \geq \alpha\},\end{aligned}$$

therefore, $\mu(x) = \mu_A(x)$. ■

Proposition 5 *Let* $A = \{A_\alpha\}_{\alpha\in[0,1]}$ *be a fuzzy subset of* X *and let* $\mu_A : X \to [0,1]$ *be the membership function of* A. *Then for all* $\alpha \in [0,1]$ *the* α*-cuts* $[A]_\alpha$ *coincide with* A_α, *i.e.* $[A]_\alpha = A_\alpha$.

Proof. Let $\beta \in [0,1]$. By definition (8.2), observe that $A_\beta \subset [A]_\beta$. It suffices to prove the opposite inclusion.

Let $x \in [A]_\beta$. Then $\mu_A(x) \geq \beta$, or, equivalently, $\sup\{\alpha|\alpha \in [0,1], x \in A_\alpha\} \geq \beta$. It follows that for every β' with $\beta' < \beta$ there exists α' with $\beta' \leq \alpha' \leq \beta$, such that $x \in A_{\alpha'}$. By monotonicity condition (ii) in (8.1) we have $x \in A_\alpha$ for all $0 \leq \alpha \leq \alpha'$. Hence, $x \in \bigcap_{0\leq\alpha<\beta} A_\alpha$, however, applying (iii) in (8.1) we get $x \in A_\beta$. Consequently, $[A]_\beta \subset A_\beta$. ■

These results allow for introducing a natural one-to-one correspondence between fuzzy subsets of X and real functions mapping X to $[0,1]$. Any fuzzy subset A of X is given by its membership function μ_A and vice-versa, any function $\mu : X \to [0,1]$ uniquely determines a fuzzy subset A of X, with the property that the membership function μ_A of A is μ.

The notions of inclusion and equality extend to fuzzy subsets as follows. Let $A = \{A_\alpha\}_{\alpha\in[0,1]}$, $B = \{B_\alpha\}_{\alpha\in[0,1]}$ be fuzzy subsets of X. Then

$$A \subset B \text{ if } A_\alpha \subset B_\alpha \text{ for each } \alpha \in [0,1], \tag{8.8}$$
$$A = B \text{ if } A_\alpha = B_\alpha \text{ for each } \alpha \in [0,1]. \tag{8.9}$$

Proposition 6 *Let A and B be fuzzy subsets of X, $A = \{A_\alpha\}_{\alpha\in[0,1]}$, $B = \{B_\alpha\}_{\alpha\in[0,1]}$. Then the following holds:*

$$A \subset B \textit{ if and only if } \mu_A(x) \leq \mu_B(x) \textit{ for all } x \in X, \tag{8.10}$$
$$A = B \textit{ if and only if } \mu_A(x) = \mu_B(x) \textit{ for all } x \in X. \tag{8.11}$$

Proof. We prove only (8.10), the proof of statement (8.11) is analogical. Let $x \in X$, $A \subset B$. Then by definition (8.8), $A_\alpha \subset B_\alpha$ for each $\alpha \in [0,1]$. Using (8.2) we obtain

$$\begin{aligned} \mu_A(x) &= \sup\{\alpha|\alpha \in [0,1], x \in A_\alpha\} \\ &\leq \sup\{\alpha|\alpha \in [0,1], x \in B_\alpha\} = \mu_B(x). \end{aligned}$$

Suppose that $\mu_A(x) \leq \mu_B(x)$ holds for all $x \in X$ and let $\alpha \in [0,1]$. We have to show that $A_\alpha \subset B_\alpha$. Indeed, for an arbitrary $u \in A_\alpha$, we have

$$\sup\{\beta|\beta \in [0,1], u \in A_\beta\} \leq \sup\{\beta|\beta \in [0,1], u \in B_\beta\}.$$

From here, $\sup\{\beta|\beta \in [0,1], u \in B_\beta\} \geq \alpha$, therefore, for each $\beta < \alpha$ it follows that $u \in B_\beta$. Hence, by (iii) in Definition 3, we obtain $u \in B_\alpha$. ■

Classical sets can be considered as special fuzzy sets where the families contain the same elements. We obtain the following definition.

Definition 7 *Let A be a subset of X. The fuzzy subset $\{A_\alpha\}_{\alpha\in[0,1]}$ of X defined by $A_\alpha = A$ for all $\alpha \in (0,1]$ is called a* crisp fuzzy subset of X generated by A. *A fuzzy subset of X generated by some $A \subset X$ is called a* crisp fuzzy subset of X *or briefly a crisp subset of X.*

Proposition 8 *Let $A = \{A\}_{\alpha\in[0,1]}$ be a crisp subset of X generated by A. Then the membership function of A is equal to the characteristic function of A.*

Proof. Let μ be the membership function of $A = \{A\}_{\alpha\in[0,1]}$. We wish to prove that

$$\mu(x) = \{ \begin{array}{ll} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{array}$$

Let $x \in X, x \notin A$. Then by definition (8.2) and Definition 7, $\mu(x) = \sup\{\alpha|\alpha \in [0,1], x \in A_\alpha\} = 0$.

Let $x \in X, x \in A$. Then again by definition (8.2) and Definition 7,

$$\begin{aligned} \mu(x) &= \sup\{\alpha|\alpha \in [0,1], x \in A_\alpha\} \\ &= \sup\{\alpha|\alpha \in [0,1], x \in A\} = 1. \end{aligned}$$

■

By Definition 7, the set $\mathcal{P}(X)$ of all subsets of X can naturally be embedded into the set of all fuzzy subsets of X and we can write $A = \{A\}_{\alpha\in[0,1]}$ if $\{A\}_{\alpha\in[0,1]}$ is generated by $A \subset X$. According to Proposition 8, we have in this case $\mu_A = \chi_A$. In particular, if A contains one element a of X, that is $A = \{a\}$, then we write $a \in \mathcal{F}(X)$ instead of $\{a\} \in \mathcal{F}(X)$ and χ_a instead of $\chi_{\{a\}}$.

Example 9 *Let $\mu : \mathbf{R} \to [0,1]$ be such that*

$$\mu(x) = e^{-x^2} \text{ if } x \in \mathbf{R}.$$

Let $A' = \{A'_\alpha\}_{\alpha\in[0,1]}$, $A'' = \{A''_\alpha\}_{\alpha\in[0,1]}$ be two families of subsets in $\mathbf{R}$ defined as follows:

$$\begin{aligned} A'_\alpha &= \{x|x \in \mathbf{R}, \mu(x) > \alpha\}, \\ A''_\alpha &= \{x|x \in \mathbf{R}, \mu(x) \geq \alpha\}. \end{aligned}$$

Clearly, A'' is a fuzzy subset of $\mathbf{R}$ and $A' \neq A''$. Observe that (i) and (ii) are satisfied for A' and A'', however $A'_1 = \emptyset$, $\bigcap_{0\leq\alpha<1} A'_\alpha = \{0\}$, thus (iii) in (8.1) is not satisfied. Hence A' is not a fuzzy subset of $\mathbf{R}$.

8.3 Operations with Fuzzy Sets

In order to generalize the set operations of intersection, union and complement to fuzzy set operations, it is natural to use triangular norms, triangular conorms and fuzzy negations, introduced in [79], respectively.

Given a De Morgan triple (T, S, N), i.e. a t-norm T, a t-conorm S and a fuzzy negation N, we can define the operations *intersection* $\cap_T$, *union* $\cup_S$ and *complement* $\mathcal{C}_N$ on $\mathcal{F}(X)$, where for $A, B \in \mathcal{F}(X)$ given by the membership functions μ_A, μ_B, the membership functions of the fuzzy subsets $A \cap_T B$, $A \cup_S B$ and $\mathcal{C}_N A$ of X are defined for each $x \in X$ as follows, see [38]:

$$\begin{aligned} &\mu_{A\cap_T B}(x) = T(\mu_A(x), \mu_B(x)), \\ &\mu_{A\cup_S B}(x) = S(\mu_A(x), \mu_B(x)), \\ &\mu_{\mathcal{C}_N A}(x) = N(\mu_A(x)). \end{aligned} \tag{8.12}$$

The operations introduced by L. Zadeh in [38] have been originally based on $T = T_M = \min$, $S = S_M = \max$ and standard negation N. The properties of the operations intersection $\cap_T$, union $\cup_S$ and complement $\mathcal{C}_N$ can be derived directly from the corresponding properties of the t-norm T, t-conorm S and fuzzy negation N. For brevity, in case of $T = \min$ and $S = \max$, we write only $\cap$ and $\cup$, instead of $\cap_T$ and $\cup_S$.

Notice that for all $A \in \mathcal{F}(X)$ we do not necessarily obtain properties which hold for crisp sets, namely

$$A \cap_T \mathcal{C}_N A = \emptyset, \tag{8.13}$$

$$A \cup_S \mathcal{C}_N A = X. \tag{8.14}$$

If the t-norm T in the De Morgan triple (T, S, N) does not have zero divisors, e.g. $T = \min$, then these properties never hold unless A is a crisp set. On the other hand, for the De Morgan triple (T_L, S_L, N) based on Lukaszewicz t-norm $T = T_L$, properties (8.13) and (8.13) are satisfied.

Given a t-norm T and fuzzy subsets A and B of X and Y, respectively, the *Cartesian product* $A \times_T B$ is the fuzzy subset of $X \times Y$ with the following membership function:

$$\mu_{A \times_T B}(x, y) = T(\mu_A(x), \mu_B(y)) \text{ for } (x, y) \in X \times Y. \tag{8.15}$$

An interesting and natural question arises, whether the α-cuts of the intersection $A \cap_T B$, union $A \cup_S B$ and Cartesian product $A \times_T B$ of $A, B \in \mathcal{F}(X)$, coincide with the intersection, union and Cartesian product, respectively, of the corresponding α-cuts $[A]_\alpha$ and $[B]_\alpha$. We have the following result, see [38].

Proposition 10 *Let T be a t-norm, S be a t-conorm, $\alpha \in [0, 1]$. Then the equalities*

$$\begin{array}{l} [A \cap_T B]_\alpha = [A]_\alpha \cap [B]_\alpha, \\ [A \cup_S B]_\alpha = [A]_\alpha \cup [B]_\alpha, \\ [A \times_T B]_\alpha = [A]_\alpha \times [B]_\alpha, \end{array} \tag{8.16}$$

hold for all fuzzy sets $A, B \in \mathcal{F}(X)$ if and only if α is an idempotent element of both T and S.

In particular, this result means that identities (8.16) hold for all $\alpha \in [0, 1]$ and for all fuzzy sets $A, B \in \mathcal{F}(X)$ if and only if $T = T_M$ and $S = S_M$.

8.4 Extension Principle

The purpose of the extension principle proposed by L. Zadeh in [98], [99], is to extend functions or operations having crisp arguments to functions or operations with fuzzy set arguments. Zadeh's methodology can be cast in the more general setting of carrying a membership function via a mapping, see e.g. [21]. There exist other generalizations for set-to-set mappings, see e.g. [21], [61]. From now on, X and Y are nonempty sets.

Definition 11 (Extension Principle)
Let X, Y be sets, $f : X \to Y$ be a mapping. The mapping $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined for all $A \in \mathcal{F}(X)$ with $\mu_A : X \to [0,1]$ and all $y \in Y$ by

$$\mu_{\tilde{f}(A)}(y) = \{ \begin{array}{ll} \sup\{\mu_A(x) | x \in X, f(x) = y\} & \textit{if } f^{-1}(y) \neq \emptyset, \\ 0 & \textit{otherwise,} \end{array} \tag{8.17}$$

is called a fuzzy extension of f.

By formula (8.17) we define the membership function of the image of the fuzzy set A by fuzzy extension $\tilde{f}$. A justification of this concept is given in the following theorem stating that the mapping $\tilde{f}$ is a true extension of the mapping f when considering the natural embedding of $\mathcal{P}(X)$ into $\mathcal{F}(X)$ and $\mathcal{P}(Y)$ into $\mathcal{F}(Y)$.

Proposition 12 *Let X, Y be sets, $f : X \to Y$ be a mapping, $x_0 \in X, y_0 = f(x_0)$. If $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is defined by (8.17), then*

$$\tilde{f}(x_0) = y_0,$$

and the membership function $\mu_{\tilde{f}(x_0)}$ of the fuzzy set $\tilde{f}(x_0)$ is a characteristic function of y_0, i.e.

$$\mu_{\tilde{f}(x_0)} = \chi_{y_0}. \tag{8.18}$$

Proof. To prove the theorem, it is sufficient to prove (8.18). Remember that we identify subsets and points of X and Y with the corresponding crisp fuzzy subsets.

Let $y \in Y$, we will show that

$$\mu_{\tilde{f}(x_0)}(y) = \chi_{y_0}(y). \tag{8.19}$$

Let $y = y_0$. Since $y_0 = f(x_0)$ we obtain by (8.17) that $\mu_{\tilde{f}(x_0)}(y) = \chi_{x_0}(x_0) = 1$. Moreover, by the definition of characteristic function we have $\chi_{y_0}(y_0) = 1$, thus (8.19) is satisfied.

On the other hand, let $y \neq y_0$. Again, by the definition of characteristic function we have $\chi_{y_0}(y) = 0$. As $y \neq f(x_0)$ we obtain for all $x \in X$ with $y = f(x)$ that $x \neq x_0$. Clearly, $\chi_{x_0}(x) = 0$ and by (8.17) it follows that $\mu_{\tilde{f}(x_0)}(y) = 0$, which was required. ■

A more general form of Proposition 12 says that the image of a crisp set by a fuzzy extension of a function is again crisp.

Theorem 13 *Let X, Y be sets, $f : X \to Y$ be a mapping, $A \subset X$. Then*

$$\tilde{f}(A) = f(A)$$

and the membership function $\mu_{\tilde{f}(A)}$ of $\tilde{f}(A)$ is a characteristic function of the set $f(A)$, i.e.

$$\mu_{\tilde{f}(A)} = \chi_{f(A)}. \tag{8.20}$$

Proof. We prove only (8.20). Let $y \in Y$. Since A is crisp, $\mu_A = \chi_A$. By (8.17) we obtain

$$\mu_{\tilde{f}(A)} = \max\{0, \sup\{\chi_A(t)|t \in X, f(t) = y\} = \{ \begin{array}{l} 1 \text{ if } y \in f(A), \\ 0 \text{ otherwise.} \end{array} .$$

Consequently, $\mu_{\tilde{f}(A)}(y) = \chi_{f(A)}(y)$. ■

In the following sections the extension principle will be used in different settings for various sets X and Y, and also for different classes of mappings.

The mathematics of fuzzy sets is, in a narrow sense, a mathematics of the space of membership functions. In this chapter we shall deal with some properties of this space, primarily with respect to a subset of it - the (generalized) quasiconcave functions.

8.5 Binary and Valued Relations

In a classical set theory, a *binary relation* R between the elements of sets X and Y is defined as a subset of the Cartesian product $X \times Y$, that is, $R \subset X \times Y$. A valued relation on $X \times Y$ will be a fuzzy subset of $X \times Y$.

Definition 14 *A* valued relation *R on $X \times Y$ is a fuzzy subset of $X \times Y$.*

The valued relations are sometimes called fuzzy relations, however, we reserve this name for valued relations defined on $\mathcal{F}(X) \times \mathcal{F}(Y)$, which will be defined later.

Any binary relation R, where $R \subset X \times Y$, is embedded into the class of valued relations by its characteristic function χ_R being understood as its membership function μ_R. In this sense, any binary relation is valued.

Particularly, any function $f : X \to Y$ is considered as a binary relation, that is, as a subset R_f of $X \times Y$, where

$$R_f = \{(x, y) \in X \times Y | y = f(x)\}. \tag{8.21}$$

Here, R_f may be identified with the valued relation by its characteristic function

$$\mu_{R_f}(x, y) = \chi_{R_f}(x, y) \tag{8.22}$$

for all $(x, y) \in X \times Y$, where

$$\chi_{R_f}(x, y) = \chi_{f(x)}(y). \tag{8.23}$$

In particular, if $Y = X$, then any valued relation R on $X \times X$ is a fuzzy subset of $X \times X$.

Definition 15 *A* valued relation *R on X is a valued relation on $X \times X$. A valued relation R on X is* *(i)* reflexive *if for each* $x \in X$

$$\mu_R(x,x) = 1;$$

(ii) symmetric *if for each* $x, y \in X$

$$\mu_R(x,y) = \mu_R(y,x);$$

(iii) T-transitive *if for a t-norm* T *and each* $x, y, z \in X$

$$T(\mu_R(x,y), \mu_R(y,z)) \leq \mu_R(x,z);$$

(iv) separable *if*

$$\mu_R(x,y) = 1 \text{ if and only if } x = y;$$

(v) T-equivalence *if* R *is reflexive, symmetric and* T*-transitive, where* T *is a t-norm;*

(vi) T-equality *if* R *is reflexive, symmetric,* T*-transitive and separable, where* T *is a t-norm.*

Definition 16 *Let a be a valued relation* R *on* X*.*

(i) A valued relation R^{-1} *on* X *is* inverse to R *if for each* $x, y \in X$

$$\mu_{R^{-1}}(x,y) = \mu_R(y,x);$$

(ii) Let N *be a negation,* $N : [0,1] \to [0,1]$*. A valued relation* $\mathcal{C}_N R$ *on* X *is the* complement relation to R *if for each* $x, y \in X$

$$\mu_{\mathcal{C}_N R}(x,y) = N(\mu_R(x,y)).$$

(iii) A valued relation R^s *on* X *is the* strict relation to R *if for each* $x, y \in X$

$$\mu_{R^s}(x,y) = \{ \begin{array}{ll} \mu_R(x,y) & \text{if } \mu_R(y,x) = 0, \\ 0 & \text{otherwise.} \end{array}$$

A valued relation R *on* X *is* strict *if* $R = R^s$*.*

(iv) A valued relation R *on* X *is* closed *if* μ_R *is USC on* $X \times X$*.*

For more information about valued relations see also [23].

Example 17 *Let* $\varphi : \mathbf{R} \to [0,1]$ *be a function. Then* R *defined by the membership function* μ_R *for all* $x, y \in \mathbf{R}$ *by*

$$\mu_R(x,y) = \varphi(x-y) \tag{8.24}$$

is a valued relation on $\mathbf{R}$. *If*

$$\varphi(t) = \begin{cases} 1 \text{ if } & t \leq 0 \\ 0 \text{ otherwise} & \end{cases},$$

then R *defined by (8.24) is the usual binary relation* $\leq$ *on* $\mathbf{R}$. *If*

$$\varphi(t) = \begin{cases} 1 \text{ if } & t \geq 0 \\ 0 \text{ otherwise} & \end{cases},$$

then R *defined by (8.24) is the usual binary relation* $\geq$ *on* $\mathbf{R}$. *If*

$$\varphi(t) = \begin{cases} 1 \text{ if } & t = 0 \\ 0 \text{ otherwise} & \end{cases},$$

then R *defined by (8.24) is the usual binary relation* $=$ *on* $\mathbf{R}$.

8.6 Fuzzy Relations

Consider a valued relation R on $X \times Y$ given by the membership function $\mu_R : X \times Y \to [0,1]$. In order to extend this function with crisp arguments to function with fuzzy arguments, we apply the extension principle (8.63) in Definition 11. Then we obtain a mapping $\tilde{\mu}_R : \mathcal{F}(X \times Y) \to \mathcal{F}([0,1])$, that is, values of $\tilde{\mu}_R$ are fuzzy subsets of $[0,1]$.

Since $\mathcal{F}([0,1])$ can be considered as a lattice, we can consider $\tilde{\mu}_R$ as the membership function of an L-fuzzy set.

However, we do not folow this way, instead, we follow a more practical way and define fuzzy relations as valued relations on $\mathcal{F}(X) \times \mathcal{F}(Y)$.

Definition 18 *Let* X, Y *be nonempty sets. A fuzzy subset of* $\mathcal{F}(X) \times \mathcal{F}(Y)$ *is called a* fuzzy relation *on* $\mathcal{F}(X) \times \mathcal{F}(Y)$.

Definition 19 *Let* X, Y *be sets. Let* R *be a valued relation on* $X \times Y$. *A fuzzy relation* $\tilde{R}$ *on* $\mathcal{F}(X) \times \mathcal{F}(Y)$ *given by the membership function* $\mu_{\tilde{R}} : \mathcal{F}(X) \times \mathcal{F}(Y) \to [0,1]$ *is called a* fuzzy extension of relation R, *if for each* $x \in X$, $y \in Y$, *it holds*

$$\mu_{\tilde{R}}(x,y) = \mu_R(x,y). \tag{8.25}$$

Let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ be fuzzy sets. When appropriate, we shall use also the notation $A \tilde{R} B$, instead of $\mu_{\tilde{R}}(A,B) = 1$.

Definition 20 *Let* X *be a nonempty set. Let* R *be a valued relation on* X. *Let* $\widetilde{{}^cR}$ *be a fuzzy extension of relation* ${}^cR$. *A fuzzy relation* ${}^*\tilde{R}$ *on* $\mathcal{F}(X) \times \mathcal{F}(X)$ *defined for all* $A, B \in \mathcal{F}(X)$ *by*

$$\mu_{{}^*\tilde{R}}(A,B) = 1 - \mu_{\widetilde{{}^cR}}(B,A) \tag{8.26}$$

is called dual to fuzzy extension $\widetilde{{}^cR}$ of relation ${}^cR$.

Now, we define an important special fuzzy extension of a valued relation R.

Definition 21 *Let X, Y be nonempty sets, T be a t-norm. Let R be a valued relation on $X \times Y$. A fuzzy relation $\tilde{R}^T$ on $\mathcal{F}(X) \times \mathcal{F}(Y)$ defined for all fuzzy sets A, B with the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively, by*

$$\mu_{\tilde{R}^T}(A,B) = \sup\{T(\mu_R(x,y), T(\mu_A(x), \mu_B(y)))|x \in X, y \in Y\}, \tag{8.27}$$

*is called a T-*fuzzy extension *of relation R.*

It is easy to show that any T-fuzzy extension $\tilde{R}^T$ of relation R is a fuzzy extension $\tilde{R}$ of relation R on $\mathcal{F}(X) \times \mathcal{F}(Y)$ in the sense of Definition 19.

Proposition 22 *Let X, Y be nonempty sets, T be a t-norm. Let R be a valued relation on $X \times Y$. If $\tilde{R}^T$ is a T-fuzzy extension of relation R, then $\tilde{R}^T$ is a fuzzy extension of relation R. Moreover, if $A', A'' \in \mathcal{F}(X)$, $B', B'' \in \mathcal{F}(Y)$ and*

$$A' \subset A'', B' \subset B'',$$

then

$$\mu_{\tilde{R}^T}(A', B') \leq \mu_{\tilde{R}^T}(A'', B''). \tag{8.28}$$

Proof. Let $x \in X, y \in Y$. By (8.27) we obtain

$$\begin{aligned} \mu_{\tilde{R}^T}(x,y) &= \sup\{T(\mu_R(u,v), T(\chi_x(u), \chi_y(\nu))|u \in X, v \in Y\} \\ &= T(\mu_R(x,y), T(1,1)) = \mu_R(x,y). \end{aligned} \tag{8.29}$$

Observe that for all $u \in X, v \in Y$, $\mu_{A'}(u) \leq \mu_{A''}(u)$ and $\mu_{B'}(v) \leq \mu_{B''}(v)$. Clearly, (8.28) follows by monotonicity of the t-norm T. ■

For any t-norm T, we obtain the following properties of a T-fuzzy extension of the valued relation.

Proposition 23 *Let X, Y be nonempty sets, T be a t-norm. Let R be a valued relation on $X \times Y$. Let $\tilde{R}^T$ be a T-fuzzy extension of relation R. If A and B are crisp sets, $A \subset X$, $B \subset Y$, then*

$$\sup\{\mu_R(x,y)|x \in A, y \in B\} \leq \mu_{\tilde{R}^T}(A,B). \tag{8.30}$$

Furthermore, if R is a binary relation on $X \times Y$ and there exists $a \in A$ and $b \in B$ with $\mu_R(a,b) = 1$, then

$$\mu_{\tilde{R}^T}(A,B) = 1. \tag{8.31}$$

Proof. Let $x \in A, y \in B$. By Definition 21 and (8.28) it follows that $\mu_R(x,y) = \mu_{\tilde{R}^T}(x,y) \leq \mu_{\tilde{R}^T}(A,B)$. Then

$$\sup\{\mu_R(x,y)|x \in A, y \in B\} \leq \mu_{\tilde{R}^T}(A,B). \tag{8.32}$$

If R is a binary relation on $X \times Y$, then $\mu_R(u,v) \in \{0,1\}$ for all $u \in X, v \in Y$. Following (8.32) and taking into account that $\mu_R(a,b) = 1$, we obtain $\mu_{\tilde{R}^T}(A,B) = 1$. ■

It is clear that any dual ${}^*\tilde{R}^T$ to a T-fuzzy extension $\widetilde{{}^cR}^T$ of a valued relation ${}^cR$ is a fuzzy extension of a valued relation R' defined as follows

$$\mu_{R'}(x,y) = 1 - \mu_{{}^cR}(y,x)$$

for all $x \in X, y \in Y$. Moreover, ${}^*\tilde{R}^T$ is monotone in the sense of the following proposition.

Proposition 24 *Let X, Y be nonempty sets, T be a t-norm. Let R be a valued relation on $X \times Y$.*
If ${}^*\tilde{R}^T$ is the dual fuzzy relation to $\widetilde{{}^cR}^T$ of ${}^cR$ with $A', A'' \in \mathcal{F}(X)$, $B', B'' \in \mathcal{F}(Y)$ and

$$A' \subset A'', B' \subset B'',$$

then

$$\mu_{{}^*\tilde{R}^T}(A',B') \geq \mu_{{}^*\tilde{R}^T}(A'',B''). \tag{8.33}$$

Proof. Let $x \in X, y \in Y$. By (8.27) and (8.26) we obtain

$$\mu_{{}^*\tilde{R}^T}(A',B') = 1 - \mu_{\widetilde{{}^cR}^T}(B',A')$$
$$= 1 - \sup\{T(\mu_{{}^cR}(v,u), T(\mu_{B'}(u), \mu_{A'}(\nu))|u \in X, v \in Y\}.$$

Observe that for all $u \in X, v \in Y$, $\mu_{A'}(u) \leq \mu_{A''}(u)$ and $\mu_{B'}(v) \leq \mu_{B''}(v)$. Clearly, (8.33) follows by monotonicity of the t-norm T. ■

Example 25 *Let X, Y be nonempty sets, $f : X \to Y$ be a function, let the corresponding relation R_f be defined by (8.21) and (8.22). Let T be a t-norm, let A and B be fuzzy subsets of X and Y given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let $y \in Y$ and let B be defined for all $z \in Y$ as follows: $\mu_B(z) = \chi_y(z)$. Then by (8.27) we get the T-fuzzy extension $\tilde{R}_f^T$ of relation R_f as*

$$\mu_{\tilde{R}_f^T}(A,B) = \max\left\{0, \sup\{T(\mu_{R_f}(x,z), T\left(\mu_A(x), \chi_y(z)\right) | x \in X, z \in Y\}\right\}. \tag{8.34}$$

The value $\mu_{\tilde{R}_f^T}(A,B)$ expresses the degree in which $y \in Y$ is considered as the image of $A \in \mathcal{F}(X)$ through the function f.

The following proposition says that extension principle (8.17) is a special t-norm independent fuzzy extension of relation (8.21).

Proposition 26 *Let X, Y be nonempty sets, $f : X \to Y$ be a function, let the corresponding relation R_f be defined by (8.21) and (8.22). Let T be a t-norm and A, B be fuzzy subsets with the corresponding membership functions*

$\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let $y \in Y$ and let μ_B be defined for all $z \in Y$ by $\mu_B(z) = \chi_y(z)$. Then for the membership function of the T-fuzzy extension $\tilde{R}_f^T$ of relation R_f, it holds

$$\mu_{\tilde{R}_f^T}(A, B) = \mu_{\tilde{f}(A)}(y),$$

where $\mu_{\tilde{f}(A)}(y)$ is defined by (8.17).

Proof. Let $x \in f^{-1}(y)$. For $z = y$, we get

$$T\left(\mu_A(x), \chi_y(z)\right) = T\left(\mu_A(x), 1\right) = \mu_A(x)$$

and by (8.22), (8.23), $\mu_{R_f}(x, y) = \chi_{f(x)}(y) = 1$. It follows from (8.34) that

$$\begin{aligned}\mu_{\tilde{R}_f^T}(A, B) &= \sup\{T(1, \mu_A(x)) | x \in X, f(x) = y\} \\ &= \sup\{\mu_A(x) | x \in X, f(x) = y\} = \mu_{\tilde{f}(A)}(y).\end{aligned}$$

Next, if $f^{-1}(y) = \emptyset$, then $\mu_{R_f}(x, z) = 0$ for all $x \in X, z \in Y$. By (8.34) we obtain

$$\begin{aligned}\mu_{\tilde{R}_f^T}(A, B) &= \sup\{T(\mu_{R_f}(x, z), T\left(\mu_A(x), \chi_y(z)\right)) | x \in X, z \in Y\} \\ &= \sup\{T(0, T\left(\mu_A(x), \chi_y(z)\right)) | x \in X, z \in Y\} = 0.\end{aligned}$$

However, by (8.17), $\mu_{\tilde{f}(A)}(y) = 0$. ■

A natural question may arise, whether there exists some extension of a valued relation, which is not a T-fuzzy extension. In the following section we shall introduce another fuzzy extensions of valued relations.

8.7 Fuzzy Extensions of Valued Relations

In the preceding section, Definition 21, we have introduced a T-fuzzy extension $\tilde{R}^T$ of a valued relation R, where T has been a t-norm. For arbitrary fuzzy sets A, B given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively, the T-fuzzy extension $\tilde{R}^T$ of a valued relation R has been defined by

$$\mu_{\tilde{R}^T}(A, B) = \sup\{T(T(\mu_A(x), \mu_B(y)), \mu_R(x, y)) | x \in X, y \in Y\}. \qquad (8.35)$$

The T-fuzzy extension of a valued relation is the most common in applications in the area of decision making. However, in possibility theory the other fuzzy relations based on t-norms, t-conorms, possibility and necessity measures are well known, see e.g. [29].

In the following definition we introduce six fuzzy extensions of the valued relation R, including the previously defined T-fuzzy extension $\tilde{R}^T$. Later on, these relations will be used for comparing left and right sides of the constraints in mathematical programming problems.

Definition 27 *Let X, Y be nonempty sets, T be a t-norm, S be a t-conorm. Let R be a valued relation on $X \times Y$.*

(i) *A fuzzy relation $\tilde{R}^T$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}^T}(A, B) = \sup\{T(T(\mu_A(x), \mu_B(y)), \mu_R(x, y)) | x \in X, y \in Y\}. \tag{8.36}$$

(ii) *A fuzzy relation $\tilde{R}_S$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}_S}(A, B) = \inf\{S(S(1 - \mu_A(x), 1 - \mu_B(y)), \mu_R(x, y)) | x \in X, y \in Y\}. \tag{8.37}$$

(iii) *A fuzzy relation $\tilde{R}^{T,S}$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}^{T,S}}(A, B) = \sup\{\inf\{T(\mu_A(x), S(1 - \mu_B(y), \mu_R(x, y))) | y \in Y\} | x \in X\}. \tag{8.38}$$

(iv) *A fuzzy relation $\tilde{R}_{T,S}$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}_{T,S}}(A, B) = \inf\{\sup\{S(T(\mu_A(x), \mu_R(x, y)), 1 - \mu_B(y)) | x \in X\} | y \in Y\}. \tag{8.39}$$

(v) *A fuzzy relation $\tilde{R}^{S,T}$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}^{S,T}}(A, B) = \sup\{\inf\{T(S(1 - \mu_A(x), \mu_R(x, y)), \mu_B(y)) | x \in X\} | y \in Y\} \tag{8.40}$$

(vi) *A fuzzy relation $\tilde{R}_{S,T}$ of $\mathcal{F}(X) \times \mathcal{F}(Y)$ is defined for all fuzzy sets $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ by*

$$\mu_{\tilde{R}_{S,T}}(A, B) = \inf\{\sup\{S(1 - \mu_A(x), T(\mu_B(y), \mu_R(x, y))) | y \in Y\} | x \in X\}. \tag{8.41}$$

Now, we shall study the above defined fuzzy extensions. First, we prove that all fuzzy extensions of a valued relation defined in Definition 27 are fuzzy extensions in the sense of Definition 19, see the first part of Proposition 22. Then we present some monotonicity properties similar to that of the second part of Proposition 22.

Proposition 28 *Let X, Y be sets, T be a t-norm, S be a t-conorm. Let R be a valued relation on $X \times Y$ given by the membership function $\mu_R : X \times Y \to [0, 1]$. If*

$$\tilde{R} \in \{\tilde{R}^T, \tilde{R}_S, \tilde{R}^{T,S}, \tilde{R}_{T,S}, \tilde{R}^{S,T}, \tilde{R}_{S,T}\}, \tag{8.42}$$

then $\hat{R}$ is a fuzzy extension, that is, for each $x \in X$, $y \in Y$

$$\mu_{\tilde{R}}(x, y) = \mu_R(x, y). \tag{8.43}$$

Proof. Let $x \in X, y \in Y$. By (8.36) we obtain

$$\mu_{\tilde{R}^T}(x,y) = \sup\{T(T\left(\chi_x(u), \chi_y(v)\right), \mu_R(u,v)) | u \in X, v \in Y\}$$
$$= T(T(1,1), \mu_R(x,y)) = \mu_R(x,y).$$

By (8.37) we obtain

$$\mu_{\tilde{R}_S}(x,y) = \inf\left\{S\left(S(1-\chi_x(u), 1-\chi_y(v)), \mu_R(u,v)\right) | u \in X, v \in Y\right\}$$
$$= S\left(S(1-\chi_x(x), 1-\chi_y(y)), \mu_R(x,y)\right) = S(S(0,0), \mu_R(x,y)) = \mu_R(x,y).$$

By (8.38) we obtain

$$\mu_{\tilde{R}^{T,S}}(x,y) = \sup\left\{\inf\{T(\chi_x(u), S(1-\chi_y(v), \mu_R(u,v))) | v \in Y\} | u \in X\right\}$$
$$= T(\chi_x(x), S(1-\chi_y(y), \mu_R(x,y))) = T(1, \mu_R(x,y)) = \mu_R(x,y).$$

By (8.39) we obtain

$$\mu_{\tilde{R}_{T,S}}(x,y) = \inf\left\{\sup\{S(T(\chi_x(u), \mu_R(u,v)), 1-\chi_y(v)) | u \in X\} | v \in Y\right\}$$
$$= S(T(\chi_x(x), \mu_R(x,y)), 1-\chi_y(y)) = S(0, \mu_R(x,y)) = \mu_R(x,y).$$

By (8.40) we obtain

$$\mu_{\tilde{R}^{S,T}}(x,y) = \sup\left\{\inf\{T(\chi_y(v), S(1-\chi_x(u), \mu_R(u,v))) | u \in X\} | v \in Y\right\}$$
$$= T(\chi_y(y), S(1-\chi_x(x), \mu_R(x,y))) = T(1, \mu_R(x,y)) = \mu_R(x,y).$$

By (8.41) we obtain

$$\mu_{\hat{R}_{S,T}}(x,y) = \inf\left\{\sup\{S(1-\chi_x(u), T(\chi_y(v), \mu_R(u,v))) | v \in Y\} | u \in X\right\}$$
$$= S(1-\chi_x(x), T(\chi_y(y), \mu_R(x,y))) = S(0, \mu_R(x,y)) = \mu_R(x,y).$$

■

Proposition 29 *Let X, Y be sets, T be a t-norm, S be a t-conorm. Let R be a valued relation on $X \times Y$ given by the membership function $\mu_R : X \times Y \to [0,1]$. Let $A', A'' \in \mathcal{F}(X)$, $B', B'' \in \mathcal{F}(Y)$.*
(i) *If*

$$A' \subset A'', B' \subset B'', \tag{8.44}$$

then

$$\mu_{\tilde{R}^T}(A', B') \leq \mu_{\tilde{R}^T}(A'', B'') \tag{8.45}$$

and

$$\mu_{\tilde{R}_S}(A', B') \geq \mu_{\tilde{R}_S}(A'', B''). \tag{8.46}$$

(ii) *If $\tilde{R} \in \{\tilde{R}^{T,S}, \tilde{R}_{T,S}\}$ and*

$$A' \subset A'', B' \supset B'', \tag{8.47}$$

then
$$\mu_{\tilde{R}}(A', B') \le \mu_{\tilde{R}}(A'', B''). \tag{8.48}$$

(iii) *If* $\tilde{R} \in \{\tilde{R}^{S,T}, \tilde{R}_{S,T}\}$ *and*
$$A' \supset A'', B' \subset B'', \tag{8.49}$$

then
$$\mu_{\tilde{R}}(A', B') \le \mu_{\tilde{R}}(A'', B''). \tag{8.50}$$

Proof. (i) Observe that by (8.44) for all $u \in X, v \in Y$, $\mu_{A'}(u) \le \mu_{A''}(u)$ and $\mu_{B'}(v) \le \mu_{B''}(v)$. Clearly, (8.45) follows from (8.36) by monotonicity of the t-norm T. Similarly, for all $u \in X, v \in Y$,

$$1 - \mu_{A'}(u) \ge 1 - \mu_{A''}(u) \text{ and } 1 - \mu_{B'}(v) \ge 1 - \mu_{B''}(v).$$

Then (8.46) follows from (8.37) by monotonicity of the t-conorm S.

(ii) Let $\tilde{R} = \tilde{R}^{T,S}$, $u \in X, v \in Y$. Then by (8.47) we have

$$\mu_{A'}(u) \le \mu_{A''}(u) \text{ and } 1 - \mu_{B'}(v) \le 1 - \mu_{B''}(v).$$

Inequality (8.48) follows from (8.38) by monotonicity of the t-norm T and t-conorm S.

Analogically we can prove the case $\tilde{R} = \tilde{R}_{T,S}$.

(iii) Let $\tilde{R} = \tilde{R}^{S,T}$, $u \in X, v \in Y$. Then by (8.49) we have

$$1 - \mu_{A'}(u) \le 1 - \mu_{A''}(u) \text{ and } \mu_{B'}(v) \le \mu_{B''}(v).$$

Inequality (8.50) follows from (8.40) by monotonicity of the t-norm T and t-conorm S.

Analogically, we can prove the case $\tilde{R} = \tilde{R}_{S,T}$, we left it to the reader. ■

Further on, we shall deal with properties of fuzzy extensions of binary relations on $X \times Y$.

Proposition 30 *Let R be a binary relation on $X \times Y$, A and B be nonempty crisp subsets of X and Y, respectively. Let T be a t-norm, S be a t-conorm. Then for the membership functions of fuzzy extensions of R it holds*

(i)
$$\mu_{\tilde{R}^T}(A, B) = 1$$

if and only if there exist $a \in A$ and $b \in B$ such that $\mu_R(a, b) = 1$;

(ii)
$$\mu_{\tilde{R}_S}(A, B) = 1$$

if and only if for every $a \in A$ and every $b \in B$ it holds $\mu_R(a, b) = 1$;

(iii)
$$\mu_{\tilde{R}^{T,S}}(A, B) = 1$$

if and only if there exists $a \in A$ *such that for every* $b \in B$ *it holds* $\mu_R(a,b) = 1$;

(iv)

$$\mu_{\tilde{R}_{T,S}}(A,B) = 1$$

if and only if for every $b \in B$ *there exists* $a \in A$ *such that* $\mu_R(a,b) = 1$;

(v)

$$\mu_{\tilde{R}^{S,T}}(A,B) = 1$$

if and only if there exists $b \in B$ *such that for every* $a \in A$ *it holds* $\mu_R(a,b) = 1$;

(vi)

$$\mu_{\tilde{R}_{S,T}}(A,B) = 1$$

if and only if for every $a \in A$ *there exists* $b \in B$ *such that* $\mu_R(a,b) = 1$.

Proof. (i) By (8.36), we obtain

$$\begin{aligned}\mu_{\tilde{R}^T}(A,B) &= \sup\{T(T(\mu_A(x), \mu_B(y)), \mu_R(x,y)) | x \in X, y \in Y\} \\ &= \sup\{T(T(\chi_A(x), \chi_B(y)), \mu_R(x,y)) | x \in X, y \in Y\} \\ &= \begin{cases} 1 & \text{if } \exists a \in A, \exists b \in B : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(ii) By (8.37), we obtain

$$\begin{aligned}\mu_{\hat{R}_S}(A,B) &= \inf\{S(S(1-\mu_A(x), 1-\mu_B(y)), \mu_R(x,y)) | x \in X, y \in Y\} \\ &= \inf\{S(S(1-\chi_A(x), 1-\chi_B(y)), \mu_R(x,y)) | x \in X, y \in Y\} \\ &= \begin{cases} 1 & \text{if } \forall a \in A, \forall b \in B : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(iii) By (8.38), we obtain

$$\begin{aligned}\mu_{\tilde{R}^{T,S}}(A,B) &= \sup\{\inf\{T(\mu_A(x), S(1-\mu_B(y), \mu_R(x,y))) | y \in Y\} | x \in X\} \\ &= \sup\{\inf\{T(\chi_A(x), S(1-\chi_B(y), \mu_R(x,y))) | y \in Y\} | x \in X\} \\ &= \begin{cases} 1 & \text{if } \exists a \in A, \forall b \in B : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(iv) By (8.39), we obtain

$$\begin{aligned}\mu_{\tilde{R}_{T,S}}(A,B) &= \inf\{\sup\{S(T(\chi_A(x), \mu_R(x,y)), 1-\chi_B(y)) | x \in X\} | y \in Y\} \\ &= \begin{cases} 1 & \text{if } \forall b \in B, \exists a \in A : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(v) By (8.40), we obtain

$$\begin{aligned}\mu_{\tilde{R}^{S,T}}(A,B) &= \sup\{\inf\{T(\chi_B(y), S(1-\chi_A(x), \mu_R(x,y))) | x \in X\} | y \in Y\} \\ &= \begin{cases} 1 & \text{if } \exists b \in B, \forall a \in A : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(vi) By (8.41), we obtain

$$\mu_{\tilde{R}_{S,T}}(A,B) = \inf\{\sup\{S(1-\chi_A(x), T(\chi_B(y), \mu_R(x,y)))|y \in Y\}|x \in X\}$$
$$= \begin{cases} 1 & \text{if } \forall a \in A, \exists b \in B : \mu_R(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

■

In the following proposition some relationships between fuzzy extensions and the dual fuzzy extensions are mentioned. When $T = \min$ and $S = \max$, the same results can be also found in [29].

Proposition 31 *Let X, Y be sets, T be a t-norm, S be a t-conorm dual to T. Let R be a valued relation on $X \times Y$. Then for the duals of fuzzy extensions of R it holds*
(i) $\mu_{*\tilde{R}^T}(A,B) = \mu_{\tilde{R}_S}(B,A)$,
(ii) $\mu_{*\tilde{R}_S}(A,B) = \mu_{\tilde{R}^T}(B,A)$,
(iii) $\mu_{*\tilde{R}^{T,S}}(A,B) = \mu_{\tilde{R}_{S,T}}(B,A)$,
(iv) $\mu_{*\tilde{R}_{T,S}}(A,B) = \mu_{\tilde{R}^{S,T}}(B,A)$,
(v) $\mu_{*\tilde{R}^{S,T}}(A,B) = \mu_{\tilde{R}_{T,S}}(B,A)$,
(vi) $\mu_{*\tilde{R}_{S,T}}(A,B) = \mu_{\tilde{R}^{T,S}}(B,A)$,
whenever $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$.

A number of other properties in case of $T = \min$ and $S = \max$ can be found in [29].

Some more properties of the fuzzy extensions of valued relations for the case $X = Y = \mathbf{R}^m$ shall be derived in the last section of this chapter.

8.8 Fuzzy Quantities and Fuzzy Numbers

We start our investigation with the simplest case of fuzzy subsets of the real line: one-dimensional space of real numbers $\mathbf{R}$, therefore we have $X = \mathbf{R}$ and $\mathcal{F}(X) = \mathcal{F}(\mathbf{R})$.

Definition 32 *(i) A fuzzy subset $A = \{A_\alpha\}_{\alpha\in[0,1]}$ of $\mathbf{R}$ is called a* fuzzy quantity*. The set of all fuzzy quantities will be denoted by $\mathcal{F}(\mathbf{R})$.*
(ii) A fuzzy quantity $A = \{A_\alpha\}_{\alpha\in[0,1]}$ is called a fuzzy interval *if A_α is non-empty and convex subset of $\mathbf{R}$ for all $\alpha \in [0,1]$. The set of all fuzzy intervals will be denoted by $\mathcal{F}_I(\mathbf{R})$.*
(iii) A fuzzy interval A is called a fuzzy number *if its core $Core(A)$ is a singleton. The set of all fuzzy numbers will be denoted by $\mathcal{F}_N(\mathbf{R})$.*

Notice that the membership function $\mu_A : \mathbf{R} \to [0,1]$ of the fuzzy interval A is quasiconcave on $\mathbf{R}$, and for all $x, y \in \mathbf{R}$, $x \neq y, \lambda \in (0,1)$, the following inequality holds:

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}. \qquad (8.51)$$

By Definition 32, any fuzzy interval is normalized, since $Core(A) = [A]_1$ is nonempty, that is, there exists an element $x_0 \in \mathbf{R}$ with $\mu_A(x_0) = 1$. Then $Hgt(A) = 1$. Moreover, the restriction of the membership function μ_A to $(-\infty, x_0]$ is non-decreasing and the restriction of μ_A to $[x_0, +\infty)$ is a non-increasing function.

From the point of view of applications, there are some subclasses of the class of fuzzy intervals $\mathcal{F}_I(\mathbf{R})$, we shall investigate them in the sequel.

A *closed fuzzy interval* A has an upper semicontinuous membership function μ_A or, equivalently, for all $\alpha \in (0, 1]$ the α-cut $[A]_\alpha$ is a closed subinterval in $\mathbf{R}$. Such a membership function μ_A, and correspondingly the fuzzy interval A, can be fully described by a quadruple (l, r, F, G), where $l, r, \in \mathbf{R}$ with $l \leq r$, and F, G are non-increasing left continuous functions mapping $(0, +\infty)$ into $[0, 1]$, i.e. $F, G : (0, +\infty) \to [0, 1]$. Moreover, for each $x \in \mathbf{R}$ let

$$\mu_A(x) = \{ \begin{array}{ll} F(l-x) & \text{if } x \in (-\infty, l), \\ 1 & \text{if } x \in [l, r], \\ G(x-r) & \text{if } x \in (r, +\infty). \end{array} \tag{8.52}$$

We shall briefly write $A = (l, r, F, G)$ and the set of all closed fuzzy intervals will be denoted by $\mathcal{F}_{CI}(\mathbf{R})$. As the ranges of F and G are included in $[0, 1)$, we have $Core(A) = [l, r]$. We can see that the functions F, G describe the left and right "shape" of μ_A, respectively. Observe also that a crisp number $x_0 \in \mathbf{R}$ and crisp interval $[a, b] \subset \mathbf{R}$ belongs to $\mathcal{F}_{CI}(\mathbf{R})$, as they may be equivalently expressed by the characteristic functions $\chi_{\{x_0\}}$and $\chi_{[a,b]}$, respectively. These characteristic functions can be also described in the form (8.52) with $F(x) = G(x) = 0$ for all $x \in (0, +\infty)$.

Example 33 Gaussian fuzzy number
Let $A = (a, a, G, G)$, *where* $a \in \mathbf{R}$, $\gamma \in (0, +\infty)$ *and for all* $x \in (0, +\infty)$

$$G(x) = e^{-\frac{x^2}{\gamma}}.$$

The membership function μ_A *of* A *is for all* $x \in \mathbf{R}$ *as follows*

$$\mu_A(x) = e^{-\frac{(x-a)^2}{\gamma}},$$

see Figure 8.1, where $\gamma = 2, a = 3$.

A class of more specific fuzzy intervals of $\mathcal{F}_{CI}(\mathbf{R})$ is obtained, if the α-cuts are considered to be bounded intervals. Let $l, r, \in \mathbf{R}$ with $l \leq r$, let $\gamma, \delta \in [0, +\infty)$ and let L, R be non-increasing non-constant functions mapping interval $(0, 1]$ into $[0, +\infty)$, i.e. $L, R : (0, 1] \to [0, +\infty)$. Moreover, assume that $L(1) = R(1) = 0$, define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$, and for each $x \in \mathbf{R}$

$$\mu_A(x) = \begin{cases} L^{(-1)}\left(\frac{l-x}{\gamma}\right) & \text{if } x \in (l-\gamma, l),\ \gamma > 0, \\ 1 & \text{if } x \in [l, r], \\ R^{(-1)}\left(\frac{x-r}{\delta}\right) & \text{if } x \in (r, r+\delta),\ \delta > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{8.53}$$

Figure 8.1:

where $L^{(-1)}, R^{(-1)}$ are pseudo-inverse functions of L, R, respectively. We shall write $A = (l, r, \gamma, \delta)_{LR}$, the fuzzy interval A is called an *(L,R)-fuzzy interval* and the set of all (L, R)-fuzzy intervals will be denoted by $\mathcal{F}_{LR}(\mathbf{R})$, see also [38]. The values of γ, δ are called the *left* and the *right spread of* A, respectively. Observe that $Supp(A) = [l - \gamma, r + \delta]$, $Core(A) = [l, r]$ and $[A]_\alpha$ is a compact interval for every $\alpha \in (0, 1]$. It is obvious that the class of fuzzy intervals extends the class of crisp closed intervals $[a, b] \subset \mathbf{R}$ including the case $a = b$, i.e. crisp numbers.

Particularly important fuzzy intervals are so called *piecewise linear fuzzy intervals* where $L(x) = R(x) = 1 - x$ for all $x \in [0, 1]$. In this case, the subscript LR will be omitted in the notation, we simply write $A = (l, r, \gamma, \delta)$. If $l = r$, then $A = (r, r, \gamma, \delta)$ is called a *triangular fuzzy number* and the notation is simplified as follows: $A = (r, \gamma, \delta)$.

Interesting classes of fuzzy quantities are based on the concept of basis of generators, see [42], [41].

Definition 34 *A fuzzy quantity A of* $\mathbf{R}$ *given by the membership function* $\mu_A : \mathbf{R} \to [0, 1]$ *is called a* generator in $\mathbf{R}$ *if*

$$\begin{array}{ll} (i) & 0 \in Core(A), \\ (ii) & \mu_A \text{ is quasiconcave on } \mathbf{R}. \end{array} \tag{8.54}$$

Notice that the generator A is a special fuzzy interval that satisfies (i).

Definition 35 *A set* $\mathcal{B} = \{g | g$ *is a generators in* $\mathbf{R}\}$ *is called a* basis of generators in $\mathbf{R}$ *if*

$$\begin{array}{ll} (i) & \chi_{\{0\}} \in \mathcal{B}\ ,\ \chi_{\mathbf{R}} \in \mathcal{B}\ , \\ (ii) & \max\{f, g\} \in \mathcal{B} \text{ and } \min\{f, g\} \in \mathcal{B} \text{ whenever } f, g \in \mathcal{B}\ . \end{array} \tag{8.55}$$

Remember that by χ_A we denote the characteristic function of a set A.

Definition 36 *Let $\mathcal{B}$ be a basis of generators. A fuzzy quantity A of $\mathbf{R}$ given by the membership function $\mu_A : \mathbf{R} \to [0,1]$ is called a $\mathcal{B}$ -*fuzzy interval *if there exists $a_A \in \mathbf{R}$ and $g_A \in \mathcal{B}$ such that for each $x \in \mathbf{R}$*

$$\mu_A(x) = g_A(x - a_A). \tag{8.56}$$

The set of all $\mathcal{B}$ -fuzzy intervals will be denoted by $\mathcal{F}_\mathcal{B}(\mathbf{R})$. Any $A \in \mathcal{F}_\mathcal{B}(\mathbf{R})$ is represented by a couple (a_A, g_A), we write $A = (a_A, g_A)$.
An ordering relation $\leq_\mathcal{B}$ is defined on $\mathcal{F}_\mathcal{B}(\mathbf{R})$ as follows. For $A, B \in \mathcal{F}_\mathcal{B}(\mathbf{R})$, $A = (a_A, g_A)$, $B = (a_B, g_B)$

$$A \leq_\mathcal{B} B$$

if

$$(a_A < a_B) \text{ or } (a_A = a_B \text{ and } g_A \leq g_B). \tag{8.57}$$

Notice that $\leq_\mathcal{B}$ is a partial ordering on $\mathcal{F}_\mathcal{B}(\mathbf{R})$. The proof of the following proposition follows directly from (8.55).

Proposition 37 *A couple $(\mathcal{B}, \leq)$, where $\mathcal{B}$ is a basis of generators and $\leq$ is a pointwise ordering of functions, is a lattice with the maximal element $\chi_\mathbf{R}$ and minimal element $\chi_{\{0\}}$.*

Example 38 *The following sets of functions form a basis of generators in $\mathbf{R}$:*
(i) $\mathcal{B}_D = \{\chi_{\{0\}}, \chi_\mathbf{R}\}$- discrete basis,
(ii) $\mathcal{B}_I = \{\chi_{[a,b]} | -\infty \leq a \leq b \leq +\infty\}$- interval basis,
(iii) $\mathcal{B}_G = \{\mu | \mu(x) = g^{(-1)}(\frac{|x|}{d})$ for each $x \in \mathbf{R}, d > 0\} \cup \{\chi_{\{0\}}, \chi_\mathbf{R}\}$, where $g : (0,1] \to [0, +\infty)$ is non-increasing non-constant function, $g(1) = 0$, $g(0) = \lim\limits_{x \to 0} g(x)$. Evidently, the pointwise relation $\leq$ between function values is a complete ordering on $\mathcal{B}_G$.

Example 39 *$\mathcal{F}_{\mathcal{B}_G}(\mathbf{R}) = \{\mu |$ there exists $a \in \mathbf{R}$ and $g \in \mathcal{B}_G$, such that $\mu(x) = g(x - a)$ for each $x \in \mathbf{R}\}$. Evidently, the relation $\leq_\mathcal{B}$ is a complete ordering on $\mathcal{F}_{\mathcal{B}_G}(\mathbf{R})$.*

8.9 Fuzzy Extensions of Real Functions

Now, we shall deal with the problem of fuzzy extension of a real function f, where $f : \mathbf{R}^m \to \mathbf{R}$, $m \geq 1$, to a function $\tilde{f} : \mathcal{F}(\mathbf{R}) \times \cdots \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$, applying the extension principle from Definition 11. Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1, 2, ..., m$. Let T be a t-norm and let a fuzzy set $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$, for all $x = (x_1, ..., x_m) \in \mathbf{R}^m$ as follows:

$$\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_m}(x_m)). \tag{8.58}$$

The fuzzy set $A \in \mathcal{F}(\mathbf{R}^m)$ given by the membership function (8.58) is called the *fuzzy vector of non-interactive fuzzy quantities*, see [35]. Applying (8.17), we obtain for all $y \in \mathbf{R}$:

$$\mu_{\tilde{f}(A)}(y) = \begin{cases} \sup\{T(\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)) | & x = (x_1, ..., x_m) \in \mathbf{R}^m, f(x) = y\} \\ & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{8.59}$$

Let $D = (d_1, d_2, \ldots, d_m)$ be a nonsingular $m \times m$ matrix, where all $d_i \in \mathbf{R}^m$ are column vectors, $i = 1, 2, ..., m$. Let a fuzzy set $B \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_B : \mathbf{R}^m \to [0, 1]$, for all $x = (x_1, ..., x_m) \in \mathbf{R}^m$ as follows:

$$\mu_B(x) = T(\mu_{A_1}(\langle d_1, x\rangle), \mu_{A_2}(\langle d_2, x\rangle), ..., \mu_{A_m}(\langle d_m, x\rangle)). \tag{8.60}$$

The fuzzy set $B \in \mathcal{F}(\mathbf{R}^m)$ given by the membership function (8.60) is called the *fuzzy vector of interactive fuzzy quantities*, or the *oblique fuzzy vector*, and the matrix D is called the *obliquity matrix*, see [35].

Notice that if D is equal to the identity matrix $E = (e_1, e_2, \ldots, e_m)$, $e_i = (0, ..., 0, 1, 0, ..., 0)$, where 1 is only at the i-th position, then the corresponding vector of interactive fuzzy quantities is a noninteractive one. Interactive fuzzy numbers have been extensively studied e.g. in [32], [35], [69] and [70]. In this study, we shall deal with them again in Chapter 11.

Now, we shall continue our investigation of the non-interactive fuzzy quantities.

Example 40 *Let $m = 2$, $f : \mathbf{R}^2 \to \mathbf{R}$, be defined for all $(x_1, x_2) \in \mathbf{R}^2$ as follows: $f(x_1, x_2) = x_1 * x_2$, where $*$ is a binary operation on $\mathbf{R}$, e.g. one of the four arithmetic operations $(+, -, \cdot, /)$. Let $A_1, A_2 \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by membership functions $\mu_{A_i} : \mathbf{R} \to [0, 1]$, $i = 1, 2$. Then, for a given t-norm T, the fuzzy extension $\tilde{f} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$ defined by (8.59) as*

$$\mu_{A_1 \circledast_T A_2}(y) = \max\{0, \sup\{T(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) | \ x_1 * x_2 = y\}\} \tag{8.61}$$

corresponds to the operation $\circledast_T$ on $\mathcal{F}(\mathbf{R})$. It is obvious that $\circledast_T$ is an extension of $*$, since for any two crisp subsets $A_1, A_2 \in \mathcal{P}(\mathbf{R})$ we obtain

$$A_1 \circledast_T A_2 = A_1 * A_2, \tag{8.62}$$

and, as a special case thereof, for any two crisp numbers $a, b \in \mathbf{R}$,

$$a \circledast_T b = a * b.$$

If $A_1, A_2 \in \mathcal{F}(\mathbf{R})$ are fuzzy quantities, we obtain (8.62) in terms of α-cuts as follows

$$[A_1 \circledast_T A_2]_\alpha = [A_1]_\alpha * [A_2]_\alpha, \tag{8.63}$$

where $\alpha \in (0, 1]$, or in terms of mapping f, (8.63) can be written as

$$[\tilde{f}(A_1, A_2)]_\alpha = f([A_1]_\alpha, [A_2]_\alpha). \tag{8.64}$$

Further on, we shall investigate equality (8.64) in a more general setting as a commutation of a diagram of two operations: mapping by f or $\tilde{f}$ and α-cutting of A or $\tilde{f}(A)$. Considering (8.58) and (8.59), we are interested in the following equality

$$[\tilde{f}(A)]_\alpha = f([A_1]_\alpha, ..., [A_m]_\alpha). \tag{8.65}$$

The process of forming of the left side and parallely the right side of (8.65) may be visualized by the diagram depicted in Fig. 8.2.

Figure 8.2:

Observe that by (8.58) and by definition (8.15) we obtain

$$A = A_1 \times_T A_2 \times_T \cdots \times_T A_m. \tag{8.66}$$

If the equality at the top of this diagram is satisfied, we say, that the *diagram commutes*. For the beginning, we derive several results concerning some convexity properties of the individual elements in the diagram. The first result is a simple generalization of the similar result from [79] for more than two membership functions. Notice that the membership functions in question are not assumed to be normalized.

Proposition 41 *Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1, 2, ..., m$. Let T be a t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ defined by (8.58). If μ_{A_i} are T-quasiconcave on $\mathbf{R}$ for all $i = 1, 2, ..., m$, then μ_A is T-quasiconcave on $\mathbf{R}^m$.*

If we assume that $A_i \in \mathcal{F}(\mathbf{R})$ are normalized, then T-quasiconcavity on $\mathbf{R}$ is equivalent to quasiconcavity of μ_A on $\mathbf{R}$. We have also the following proposition.

Proposition 42 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be fuzzy intervals given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1,2,...,m$. Let $G = \{G_k\}_{k=1}^{\infty}$ be an aggregation operator and let $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ for all $x \in \mathbf{R}^m$ by*

$$\mu_A(x) = G_m(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_m}(x_m)). \tag{8.67}$$

Then μ_A is upper-starshaped on $\mathbf{R}^m$.

Proof. Let us define for $i = 1,2,...,m$

$$\mu_i(x_1,...,x_m) = \mu_{A_i}(x_i). \tag{8.68}$$

Then μ_i is normalized and quasiconcave on $\mathbf{R}^m$. First, we have to show that $Core(\mu_1) \cap \cdots \cap Core(\mu_m) \neq \emptyset$. We prove even more, particularly

$$Core(\mu_1) \cap \cdots \cap Core(\mu_m) = Core(\mu_{A_1}) \times \cdots \times Core(\mu_{A_m}) \neq \emptyset. \tag{8.69}$$

Indeed, if $x = (x_1,...,x_m) \in Core(\mu_1) \cap \cdots \cap Core(\mu_m)$, then for all $i = 1,2,...,m$, $\mu_i(x) = 1$ and by (8.68) we obtain $\mu_{A_i}(x_i) = 1$. Consequently, for all $i = 1,2,...,m$, $x_i \in Core(\mu_{A_i})$, therefore, $x = (x_1,...,x_m) \in Core(\mu_{A_1}) \times \cdots \times Core(\mu_{A_m})$.

Conversely, let $x = (x_1,...,x_m) \in Core(\mu_{A_1}) \times \cdots \times Core(\mu_{A_m})$, then for each $x_i \in Core(\mu_{A_i})$ and all $i = 1,2,...,m$, and by (8.68) it follows that. $\mu_i(x) = 1$ for all i, thus we obtain

$$x = (x_1,...,x_m) \in Core(\mu_1) \cap \cdots \cap Core(\mu_m).$$

Finally, since by the assumption we have $A_i \in \mathcal{F}_I(\mathbf{R})$, and $Core(\mu_{A_i})$ is nonempty for all $i = 1,2,...,m$, then

$$Core(\mu_{A_1}) \times \cdots \times Core(\mu_{A_m}) \neq \emptyset.$$

The rest of the proposition can be proven analogously to the Proposition from [79] with G being an aggregation operator. Thus, $\mu_A = G_m(\mu_{A_1},...,\mu_{A_m})$ is upper-starshaped on $\mathbf{R}^m$. ■

The next example shows that Proposition 42 cannot be strenghten e.g. in such a way that μ_A is quasiconcave on $\mathbf{R}^m$.

Example 43 *Let $X = \mathbf{R}^2$ and let $\mu_{A_i} : \mathbf{R} \to [0,1]$, $i = 1,2$, be defined as follows:*

$$\mu_{A_1}(x_1) = \max\{0, 1 - \sqrt{|x_1|}\},\ \mu_{A_2}(x_2) = \max\{0, 1 - \sqrt{|x_2|}\}.$$

Let $T = T_P$ be a product t-norm. Following (8.58) define for all $(x_1,x_2) \in X$:

$$\mu_A(x_1,x_2) = \max\{0, 1 - \sqrt{|x_1|}\} \cdot \max\{0, 1 - \sqrt{|x_2|}\}. \tag{8.70}$$

It is evident that μ_{A_i} is normalized quasiconcave functions on $\mathbf{R}$ for $i = 1,2$. By Proposition (42), μ_A defined by (8.70) is upper starshaped. In Fig. 8.3,

Figure 8.3:

the contours of some α-cuts of the fuzzy set A given by (8.70), are depicted. This picture demonstrates that μ_A is not quasiconcave on X, as some of its α-cuts are not convex. This fact can be verified by looking closely at the curves $\mu_A(x_1, x_2) = \alpha$ for $\alpha \in (0, 1]$. All α-cuts are, however, starshaped sets.

The next two results concern the α-cuts of the fuzzy quantities.

Proposition 44 *Let $A_i \in \mathcal{F}(\mathbf{R})$ be fuzzy quantities given by the membership functions $\mu_{A_i} : \mathbf{R} \to [0, 1]$, $i = 1, 2, ..., m$. Let T be a t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0, 1]$ in (8.58).*

(i) *If $\alpha \in (0, 1]$, then*

$$[A]_\alpha \subset [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha. \tag{8.71}$$

(ii) *The equation*

$$[A]_\alpha = [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha, \tag{8.72}$$

holds for all $\alpha \in (0, 1]$, if and only if $T = T_M$.

Proof. (i) Let $x = (x_1, ..., x_m) \in [A]_\alpha$, i.e. $\mu_A(x) = T(\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)) \geq \alpha$. Since $\min\{\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)\} \geq T(\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m))$, we obtain $\mu_{A_i}(x_i) \geq \alpha$ for all $i = 1, 2, ..., m$. Consequently, for all $i = 1, 2, ..., m$ we have $x_i \in [A_i]_\alpha$ and also $x = (x_1, ..., x_m) \in [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$.

(ii) Let $T \neq \min$. Then there exists $x = (x_1, ..., x_m) \in \mathbf{R}^m$ such that $\min\{\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)\} > T(\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m))$.

Putting $\beta = \min\{\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)\}$, we have $\beta > 0$ and $x_i \in [A_i]_\beta$ for all $i = 1, 2, ..., m$, i.e. $x = (x_1, ..., x_m) \in [A_1]_\beta \times [A_2]_\beta \times \cdots \times [A_m]_\beta$. However, $\mu_A(x) = T(\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)) < \beta$ and therefore $x = (x_1, ..., x_m) \notin [A]_\beta$, a contradiction with (8.72). Thus, $T = \min$.

On the other hand, if $T = \min$, then

$$\mu_A = \min\{\mu_{A_1}, ..., \mu_{A_m}\}. \tag{8.73}$$

Let $\alpha \in (0, 1]$ be arbitrary and $x = (x_1, ..., x_m) \in \mathbf{R}^m$ be also arbitrary with $x \in [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$. Then $\mu_{A_i}(x_i) \geq \alpha$ for all $i = 1, 2, ..., m$ and it follows that $\min\{\mu_{A_1}(x_1), ..., \mu_{A_m}(x_m)\} \geq \alpha$. Hence, by (8.73) we have $x \in [A]_\alpha$. We have just proven inclusion $[A]_\alpha \supset [A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$, the opposite inclusion (8.71) is true by (i). Consequently, we have the required result (8.72). ■

Now, we shall deal with $\tilde{f}$ being a fuzzy extension of the mapping f by using the extension principle (8.59). Some sufficient conditions under which $\tilde{f}(A)$ is quasiconcave on $\mathbf{R}$ will be given in the next section as the consequence of a more general result. The problem of commuting of the diagram in Fig. 8.2 will be also resolved there.

8.10 Higher Dimensional Fuzzy Quantities

In the previous section we assumed that the fuzzy subset $A \in \mathcal{F}(\mathbf{R}^m)$ was given in the special form (8.58), or eventually (8.60). In this section, we shall investigate a fuzzy subsets of the m-dimensional real vector space $\mathbf{R}^m$, where m is a positive integer. The set of all fuzzy subsets of $\mathbf{R}^m$, denoted by $\mathcal{F}(\mathbf{R}^m)$, is called the set of all *m-dimensional fuzzy quantities.* Sometimes the expression m-dimensional is omitted. We shall investigate the problem of extension a real function $f : \mathbf{R}^m \to \mathbf{R}$, $m \geq 1$, to a function $\tilde{f} : \mathcal{F}(\mathbf{R}^m) \to \mathcal{F}(\mathbf{R})$. The process of commuting of the operations of mapping and α-cutting is depicted on the diagram in Fig. 8.4.

The following definition will be useful.

Definition 45 *A fuzzy subset $A = \{A_\alpha\}_{\alpha\in[0,1]}$ of $\mathbf{R}^m$ is called* closed, bounded, compact *or* convex *if A_α is a closed, bounded, compact or convex subset of $\mathbf{R}^m$ for every $\alpha \in (0, 1]$, respectively.*

If a fuzzy subset A of $\mathbf{R}^m$ given by the membership function $\mu_A : \mathbf{R}^m \to [0, 1]$ is closed, bounded, compact or convex, then $[A]_\alpha$ is a closed, bounded, compact or convex subset of $\mathbf{R}^m$ for every $\alpha \in (0, 1]$, respectively. Notice that A is convex if and only if its membership function μ_A is quasiconcave on $\mathbf{R}^m$.

In what follows we shall use the following important condition requiring that a special class of optimization problems always posses some optimal solution. Some sufficient conditions securing this requirement will be investigated later.

Figure 8.4:

Definition 46 Condition (C):
Let $f : \mathbf{R}^m \to \mathbf{R}$*,* $\mu : \mathbf{R}^m \to [0,1]$*. We say that* condition (C) is satisfied for f and μ*, if for every* $y \in Ran(f)$ *there exists* $x_y \in \mathbf{R}^m$ *such that* $f(x_y) = y$ *and*

$$\mu(x_y) = \sup\{\mu(x) | x \in \mathbf{R}^m, f(x) = y\}. \tag{8.74}$$

Theorem 47 *Let* $A \in \mathcal{F}(\mathbf{R}^m)$ *be a fuzzy quantity, let* μ_A *be upper-quasiconnected on* $\mathbf{R}^m$*, let* $f : \mathbf{R}^m \to \mathbf{R}$ *be continuous on* $\mathbf{R}^m$ *and let condition (C) be satisfied for* f *and* μ_A*. Then the membership function of* $\tilde{f}(A)$ *is quasiconcave on* $\mathbf{R}$*.*

Proof. Let $\alpha \in (0,1]$ we show that $\left[\tilde{f}(A)\right]_\alpha$ is convex. Let $y_i \in \left[\tilde{f}(A)\right]_\alpha$, $i = 1,2$, with $y_1 < y_2$ and $\lambda \in (0,1)$. Putting $y_0 = \lambda y_1 + (1-\lambda) y_2$, we have $y_1 < y_0 < y_2$. (If $y_1 = y_2$, then there is nothing to prove.)

By Condition (C) there exists $x_i \in \mathbf{R}^m$, $i = 1,2$, with $f(x_i) = y_i$ such that by (8.74) and (8.19) we get $\mu_A(x_i) = \mu_{\tilde{f}(A)}(y_i) \geq \alpha$, therefore, $x_i \in [A]_\alpha$. Since μ_A is upper quasiconnected on $\mathbf{R}^m$, $[A]_\alpha$ is path-connected, therefore there exists a path P belonging to $[A]_\alpha$, i.e.

$$P \subset [A]_\alpha. \tag{8.75}$$

Since P is connected and f is continuous on P with $f(x_i) = y_i$ and $y_1 < y_0 < y_2$, then $f(P)$ is also connected, $y_1, y_0, y_2 \in f(P)$ and it follows that there exists $x_0 \in P$ such that $f(x_0) = y_0$. By (8.75) we have $x_0 \in [A]_\alpha$, i.e. $\mu_A(x_0) \geq \alpha$, which implies $\mu_{\tilde{f}(A)}(y_0) = \sup\{\mu_A(x) | x \in \mathbf{R}^m,\ f(x) = y_0\} \geq \mu_A(x_0) \geq \alpha$. Consequently, $y_0 \in \left[\tilde{f}(A)\right]_\alpha$, thus $\left[\tilde{f}(A)\right]_\alpha$ is convex. ∎

Now, we return back to the question concerning sufficient conditions under which condition (C) is satisfied.

Proposition 48 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a compact fuzzy quantity and let $f : \mathbf{R}^m \to \mathbf{R}$ be a continuous function. Then condition (C) is satisfied for f and μ_A.*

Proof. Let $y \in Ran(f)$ and denote $X_y = \{x \in \mathbf{R}^m | f(x) = y\}$. Then X_y is nonempty and closed. Put

$$\alpha = \sup\{\mu_A(x) | x \in X_y\}. \tag{8.76}$$

Without loss of generality we assume that $\alpha > 0$. Take a number β, $0 < \beta < \alpha$ such that $\alpha - \frac{\beta}{k} > 0$ for all $k = 1, 2, ...$, and denote

$$U_k = \{x \in \mathbf{R}^m | \mu_A(x) \geq \alpha - \frac{\beta}{k}\},\ k = 1, 2, ... \tag{8.77}$$

By the compactness of $[A]_\delta$ for all $\delta \in (0, 1]$ we know that all U_k are compact and $U_{k+1} \subset U_k$ for all $k = 1, 2, ...$ Putting $V_k = U_k \cap X_y$ we obtain by (8.76) and (8.77) that V_k is nonempty, compact and $V_{k+1} \subset V_k$ for all $k = 1, 2, ...$ From the well known property of compact spaces it follows that $\bigcap\limits_{k=1}^{\infty} V_k$ is nonempty. Hence, for any $x_y \in \bigcap\limits_{k=1}^{\infty} V_k$ it holds: $f(x_y) = y$ and $\mu_A(x_y) \geq \alpha$. ■

Clearly, the fuzzy set A is compact, if the α-cuts $[A]_\alpha$ are compact for all $\alpha \in (0, 1]$, or the α-cuts $[A]_\alpha$ are bounded for all $\alpha \in (0, 1]$ and the membership function μ_A is upper semicontinuous on $\mathbf{R}^m$.

Returning back to the problem formulated at the end of the last section, namely, the problem of the existence of sufficient conditions under which the membership function of $\tilde{f}(A)$ is quasiconcave on $\mathbf{R}$ with μ_A defined by (8.58), we have the following result.

Theorem 49 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be compact fuzzy intervals. Let T be a continuous t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0, 1]$ as*

$$\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_m}(x_m)),$$

for all $x = (x_1, ..., x_m) \in \mathbf{R}^m$. Moreover, let $f : \mathbf{R}^m \to \mathbf{R}$ be continuous on $\mathbf{R}^m$. Then the membership function of $\tilde{f}(A)$ given by (8.59) is quasiconcave on $\mathbf{R}$.

Proof. It is sufficient to show that $\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_m}(x_m))$ is upper-quasiconnected and $[A]_\alpha$ are compact for all $\alpha \in (0, 1]$. Having this, the result follows from Proposition 48 and Theorem 47.

By Proposition 42, μ_A is upper starshaped on $\mathbf{R}^m$, hence μ_A is upper connected.

As T is continuous, $[A]_\alpha$ is closed for all $\alpha \in (0, 1]$.

It is also supposed that $[A_i]_\alpha$ are compact for all $\alpha \in (0, 1]$, $i = 1, 2, ..., m$, therefore the same holds for a Cartesian product $[A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$. Applying (8.71), we obtain that all $[A]_\alpha$ are bounded, hence compact.

Finally, all assumptions of Proposition 48 are satisfied, thus Condition (C) is satisfied and by applying Theorem 47 we obtain the required result. ■

Now, we resolve the former problem of commuting of the diagrams in Fig. 8.2 and Fig. 8.4.

Proposition 50 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy quantity, let μ_A be upper-quasiconnected on $\mathbf{R}^m$, let $f : \mathbf{R}^m \to \mathbf{R}$ be continuous on $\mathbf{R}^m$. Then $f([A]_\alpha)$ is convex for all $\alpha \in [0,1]$.*

Proof. Let $\alpha \in [0,1]$. As $[A]_\alpha$ is path-connected and f is continuous, therefore $f([A]_\alpha)$ is a connected subset of $\mathbf{R}$, thus it is a convex subset of $\mathbf{R}$. ■

Proposition 51 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy set, let $f : \mathbf{R}^m \to \mathbf{R}$. Then*

$$f([A]_\alpha) \subset \left[\tilde{f}(A)\right]_\alpha, \tag{8.78}$$

for all $\alpha \in (0,1]$.

Proof. Let $\alpha \in (0,1]$ and $y \in f([A]_\alpha)$. Then there exists $x_y \in [A]_\alpha$, such that $f(x_y) = y$. By (8.59) we obtain $\mu_{\tilde{f}(A)}(y) = \sup\{\mu_A(x) | x \in \mathbf{R}^m, f(x) = y\} \geq \mu_A(x_y) \geq \alpha$. Hence, $y \in \left[\tilde{f}(A)\right]_\alpha$. ■

The following theorem gives a necessary and sufficient condition for the diagram in Fig. 8.4 to commute.

Theorem 52 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a fuzzy quantity. Condition (C) is satisfied if and only if*

$$\left[\tilde{f}(A)\right]_\alpha = f([A]_\alpha), \tag{8.79}$$

for all $\alpha \in (0,1]$.

Proof. 1. Let Condition (C) be satisfied. We have to prove only $\left[\tilde{f}(A)\right]_\alpha \subset f([A]_\alpha)$, the opposite inclusion holds by Proposition 51.

Let $\alpha \in (0,1]$ and $y \in \left[\tilde{f}(A)\right]_\alpha$. Then $\mu_{\tilde{f}(A)}(y) \geq \alpha$ and by Condition (C) we have

$$\mu_{\tilde{f}(A)}(y) = \sup\{\mu_A(x) | x \in \mathbf{R}^m, f(x) = y\} = \mu_A(x_y)$$

for some $x_y \in \mathbf{R}^m$ with $f(x_y) = y$. Combining these results we obtain $x_y \in [A]_\alpha$, consequently, $f(x_y) = y \in f([A]_\alpha)$.

2. On the contrary, suppose that Condition (C) does not hold. Then there exists y_0 such that for all $z \in \mathbf{R}^m$ with $f(z) = y_0$ we have

$$\sup\{\mu_A(x) | x \in \mathbf{R}^m, f(x) = y_0\} > \mu_A(z). \tag{8.80}$$

Put $\beta = \sup\{\mu_A(x) | x \in \mathbf{R}^m, f(x) = y_0\}$. Then $\mu_{\tilde{f}(A)}(y_0) = \beta$, i.e. $y_0 \in \left[\tilde{f}(A)\right]_\beta$. Suppose that (8.79) holds for $\alpha = \beta$, then it follows that there exists

$x_0 \in [A]_\beta$, i.e. $\mu_A(x_0) \geq \beta$, with $f(x_0) = y_0$. However, this is a contradiction with (8.80). ■

Theorem 52 is a reformulation of the well known Nguyen's result, see [53]. As a consequence of the Theorems 52, 49 and Proposition 48, we resolve the problem of commuting of the diagram in Fig. 8.2.

Theorem 53 *Let $A_i \in \mathcal{F}_I(\mathbf{R})$ be compact fuzzy intervals, $i = 1, 2, ..., m$. Let $f : \mathbf{R}^m \to \mathbf{R}$ be a continuous function, let T be a continuous t-norm and let a fuzzy quantity $A \in \mathcal{F}(\mathbf{R}^m)$ be given by the membership function $\mu_A : \mathbf{R}^m \to [0,1]$ by (8.58). Then*

$$\left[\tilde{f}(A)\right]_\alpha = f([A_1]_\alpha, ..., [A_m]_\alpha), \tag{8.81}$$

for all $\alpha \in (0, 1]$.

Proof. It is sufficient to show that $\mu_A(x) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_m}(x_m))$ is upper starshaped on $\mathbf{R}^m$ with $[A]_\alpha$ compact for all $\alpha \in (0, 1]$. Then the rest of the proof follows from Proposition 48 and Theorem 52.

Indeed, by Proposition 42, μ_A is upper starshaped on $\mathbf{R}^m$.

As μ_{A_i} are upper semicontinuous, $i = 1, 2, ..., m$, T is continuous, it follows that $\mu_A = T(\mu_{A_1}, ..., \mu_{A_m})$ is upper semicontinuous on $\mathbf{R}^m$. Hence, $[A]_\alpha$ is closed for all $\alpha \in (0, 1]$.

It is supposed that $[A_i]_\alpha$ is compact for all $\alpha \in (0, 1]$, $i = 1, 2, ..., m$, the same holds for the Cartesian product $[A_1]_\alpha \times [A_2]_\alpha \times \cdots \times [A_m]_\alpha$ and applying (8.71), we obtain that $[A]_\alpha$ is bounded, thus compact.

Now, all assumptions of Proposition 48 are satisfied, thus Condition (C) holds and by Theorem 52 we obtain the required result. ■

Proposition 54 *Let $A \in \mathcal{F}(\mathbf{R}^m)$ be a compact fuzzy quantity. If $f : \mathbf{R}^m \to \mathbf{R}$ is continuous then $\tilde{f}(A)$ is compact.*

Proof. Let $\alpha \in (0, 1]$. Since $[A]_\alpha$ is compact, then by continuity of f, it follows that $f([A]_\alpha)$ is compact. By Proposition 48, Condition (C) is satisfied and by Theorem 52 we obtain

$$\left[\tilde{f}(A)\right]_\alpha = f([A]_\alpha), \tag{8.82}$$

for all $\alpha \in (0, 1]$. ■

Corollary 55 *If in Proposition 54, μ_A is upper-quasiconnected, then $\left[\tilde{f}(A)\right]_\alpha$ is a compact interval for each $\alpha \in (0, 1]$.*

Proof. By Proposition 54, $\left[\tilde{f}(A)\right]_\alpha$ are compact and by Proposition 50, $f([A]_\alpha)$ are convex for all $\alpha \in (0, 1]$. Using equation (8.82), $\left[\tilde{f}(A)\right]_\alpha$ are convex and compact, i.e. compact intervals in $\mathbf{R}$. ■

Commuting of the diagram in Fig. 8.4 is important when calculating the extensions of aggregation operators in multi-criteria decision making, see Chapter 9.

8.11 Fuzzy Extensions of Valued Relations

In Section 8.6, Definition 27, we have introduced and investigated six types of fuzzy extensions of valued relations on $X \times Y$. In this section we shall deal with fuzzy extensions of valued and binary relations on $\mathbf{R}^m$, where m is a positive integer. Binary relations can be viewed as special valued relations with values from $\{0,1\}$. The usual component-wise equality relation $=$ and inequality relations $\leq, \geq, <$ and $>$ on $\mathbf{R}^m$ are simple examples of binary relations. The results derived in this section will be useful in fuzzy mathematical programming we shall investigate in Chapter 11.

Now, let us turn our attention to fuzzy relations on $\mathbf{R}^m$, i.e. consider $X = Y = \mathbf{R}^m$. We start with three important examples.

Example 56 *Let us consider the usual binary relation $=$ ("equal") on $\mathbf{R}^m$, given by the membership function $\mu_=$ for all $x, y \in \mathbf{R}^m$ as*

$$\mu_=(x,y) = \{ \begin{array}{ll} 1 & \text{if } x_i = y_i \text{ for all } i = 1,2,...,m, \\ 0 & \text{otherwise.} \end{array} \tag{8.83}$$

Let T be a t-norm, A, B be fuzzy subsets of $\mathbf{R}^m$ with the corresponding membership functions $\mu : \mathbf{R}^m \to [0,1]$, $\nu : \mathbf{R}^m \to [0,1]$, respectively. Then by (8.27), the membership function $\mu_{\tilde{=}}$ of the T-fuzzy extension $\tilde{=}$ of relation $=$ can be derived as

$$\begin{aligned} \mu_{\tilde{=}}(A,B) &= \sup \{T(\mu_= (x,y), T(\mu(x), \nu(y))) \,|\, x,y \in \mathbf{R}^m\} \\ &= \sup \{T(1, T(\mu(x), \nu(y)))|\ x,y \in \mathbf{R}^m, x = y\} \\ &= \sup \{T(\mu(x), \nu(x)) \,|\, x \in \mathbf{R}^m\} = Hgt(A \cap_T B). \end{aligned}$$

Example 57 *Let us consider the usual binary relation $\geq$ ("greater or equal") on $\mathbf{R}^m$. The corresponding membership function is defined as*

$$\mu_\geq(x,y) = \{ \begin{array}{ll} 1 & \text{if } x_i \geq y_i \text{ for all } i = 1,2,...,m, \\ 0 & \text{otherwise.} \end{array} \tag{8.84}$$

Let T be a t-norm, A, B be fuzzy subsets of $\mathbf{R}^m$ with the corresponding membership functions $\mu : \mathbf{R}^m \to [0,1]$, $\nu : \mathbf{R}^m \to [0,1]$, respectively. Then by (8.27), the membership function $\mu_{\tilde{\geq}}$ of the T-fuzzy extension $\tilde{\geq}$ of relation $\geq$ can be derived as follows:

$$\begin{aligned} \mu_{\tilde{\geq}}(A,B) &= \sup \left\{T\left(\mu_\geq (x,y), T(\mu(x), \nu(y))\right) \,|\, x,y \in \mathbf{R}^m\right\} \\ &= \sup \{T(1, T(\mu(x), \nu(y)))|\ x,y \in \mathbf{R}^m, x \geq y\} \\ &= \sup \{T(\mu(x), \nu(y)) \,|\, x,y \in \mathbf{R}^m, x \geq y\}. \end{aligned}$$

Example 58 *Let $d > 0$ and let $\varphi_d : \mathbf{R} \to [0,1]$ be a function defined as follows*

$$\varphi_d(t) = \{ \begin{array}{ll} 1 & \text{if } t \geq 0, \\ \frac{d+t}{d} & \text{if } -d \leq t < 0, \\ 0 & \text{otherwise.} \end{array} \tag{8.85}$$

Then the valued relation R_d defined by the membership function μ_{R_d} for all $x, y \in \mathbf{R}$ as

$$\mu_{R_d}(x, y) = \varphi_d(x - y) \tag{8.86}$$

is a "generalized" inequality relation $\geq$ on $\mathbf{R}$. By (8.27), the membership function $\mu_{\tilde{R}_d}$ of the T-fuzzy extension $\tilde{R}_d$ of relation R_d is as follows

$$\mu_{\tilde{R}_d}(A, B) = \sup \{T(\varphi_d(x - y), T(\mu(x), \nu(y))) \, | x, y \in \mathbf{R}\}.$$

Further on, we shall deal with properties of fuzzy extensions of binary relations on $\mathbf{R}^m$. We start with investigation of m-dimensional intervals. Recall that the notation $A \ R \ B$ means $\mu_R(A, B) = 1$. We consider the usual componentwise binary relations in $\mathbf{R}^m$, namely "less or equal" and "equal", i.e. $R \in \{\leq, =\}$.

Theorem 59 *Let A, B be nonempty and closed intervals of $\mathbf{R}^m$, $A = \{a \in \mathbf{R}^m | \underline{a} \leq a \leq \overline{a}\}$, $B = \{b \in \mathbf{R}^m | \underline{b} \leq b \leq \overline{b}\}$. Let T be a t-norm, S be a t-conorm. Let $\leq$ and $=$ be usual binary relations in $\mathbf{R}^m$, $\tilde{\leq}$ and $\tilde{=}$ be the respective fuzzy extensions of relations $\leq$ and $=$, where*

$$\begin{aligned} \tilde{\leq} \ &\in \{\tilde{\leq}^T, \tilde{\leq}_S, \tilde{\leq}^{T,S}, \tilde{\leq}_{T,S}, \tilde{\leq}^{S,T}, \tilde{\leq}_{S,T}\}, \\ \tilde{=} \ &\in \{\tilde{=}^T, \tilde{=}_S, \tilde{=}^{T,S}, \tilde{=}_{T,S}, \tilde{=}^{S,T}, \tilde{=}_{S,T}\}. \end{aligned}$$

Then

(i)

(1) $A \tilde{\leq}^T B$ *if and only if* $\underline{a} \leq \overline{b}$,
(2) $A \tilde{=}^T B$ *if and only if* $\underline{a} \leq \overline{b}$ *and* $\overline{a} \geq \underline{b}$;

(ii)

(1) $A \ \tilde{\leq}_S B$ *if and only if* $\overline{a} \leq \underline{b}$,
(2) $A \tilde{=}_S B$ *if and only if* $\underline{a} = \underline{b} = \overline{b} = \overline{a}$;

(iii)

(1) $A \ \tilde{\leq}^{T,S} B$ *if and only if* $\underline{a} \leq \underline{b}$,
(2) $A \tilde{=}^{T,S} B$ *if and only if* $\underline{a} \leq \underline{b} = \overline{b} \leq \overline{a}$;

(iv)

(1) $A \ \tilde{\leq}_{T,S} B$ *if and only if* $\underline{a} \leq \underline{b}$,
(2) $A \ \tilde{=}_{T,S} B$ *if and only if* $\underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}$;

(v)

(1) $A \tilde{\leq}^{S,T} B$ *if and only if* $\overline{a} \leq \overline{b}$,
(2) $A \tilde{=}^{S,T} B$ *if and only if* $\underline{b} \leq \underline{a} = \overline{a} \leq \overline{b}$;

(vi)

(1) $A \tilde{\leq}_{S,T} B$ *if and only if* $\overline{a} \leq \overline{b}$,
(2) $A \tilde{=}_{S,T} B$ *if and only if* $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$.

Proof. (i) 1. Let $A \tilde{\leq}^T B$. Then by (i) in Proposition 30 there exists $a \in A$ and $b \in B$ such that $a \leq b$, thus $\underline{a} \leq a \leq b \leq \overline{b}$.

Conversely, let $\underline{a} \leq \overline{b}$.

Since $\underline{a} \in A$, $\overline{b} \in B$, by Proposition 30 (i), we immediately obtain $A \tilde{\leq}^T B$.

2. Observe that $\underline{a} \leq \overline{b}$ and $\underline{b} \leq \overline{a}$ is equivalent to $A \cap B$ is nonempty, i.e. there is c such that $c \in A \cap B$. However, by Proposition 30 (i), it is equivalent to $A \tilde{=}^T B$.

(ii) 1. Let $A \tilde{\leq}_S B$. Then by (ii) in Proposition 30, for every $a \in A$ and every $b \in B$ we have $a \leq b$, thus $\overline{a} \leq \underline{b}$.

Conversely, let $\underline{a} \leq \overline{a} \leq \underline{b} \leq \overline{b}$. Then by Proposition 30, (ii), we easily obtain $A \tilde{\leq}_S B$.

2. Let $A \tilde{=}_S B$. Then by (ii) in Proposition 30, for every $a \in A$ and every $b \in B$ we have $a = b$, thus $\underline{a} = \overline{a} = \underline{b} = \overline{b}$.

Conversely, let $\underline{a} = \overline{a} = \underline{b} = \overline{b}$. Then by Proposition 30, (ii), we obtain $A \tilde{=}_S B$.

(iii) 1. Let $A \tilde{\leq}^{T,S} B$. Then by (iii) in Proposition 30, there exists $a \in A$ such that for every $b \in B$ we have $a \leq b$, thus $\underline{a} \leq a \leq \underline{b}$.

Conversely, let $\underline{a} \leq \underline{b}$. Then by Proposition 30, (iii), we take $a = \underline{a}$ and since $\underline{b} \leq b$, we easily obtain $A \tilde{\leq}^{T,S} B$.

2. Let $A \tilde{=}^{T,S} B$. Then by (iii) in Proposition 30, there exists $a \in A$ such that for every $b \in B$ we have $a = b$, thus $\underline{a} \leq \underline{b} = \overline{b} \leq \overline{a}$.

Conversely, let $\underline{a} \leq \underline{b} = \overline{b} \leq \overline{a}$. Then by Proposition 30 (iii), we take $a = \underline{b}$ and immediately obtain $A \tilde{=}^{T,S} B$.

(iv) 1. Let $A \tilde{\leq}_{T,S} B$. Then by (iv) in Proposition 30, for every $b \in B$ there exists $a \in A$ such that $a \leq b$, thus $\underline{a} \leq a \leq \underline{b}$.

Conversely, let $\underline{a} \leq \underline{b}$. Then by Proposition 30 (iv), we take $a = \underline{a}$ and since $\underline{b} \leq b$, we obtain $A \tilde{\leq}_{T,S} B$.

2. Let $A \tilde{=}_{T,S} B$. Then (iv) in Proposition 30, for every $b \in B$ there exists $a \in A$ such that $a = b$, hence $\underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}$.

Conversely, let $\underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}$. Then by Proposition 30 (iv), we take $a = \underline{b}$ and obtain $A \tilde{=}_{T,S} B$.

(v) 1. Let $A \tilde{\leq}^{S,T} B$. Then by (v) in Proposition 30, there exists $b \in B$ such that for every $a \in A$ we have $a \leq b$, thus $\overline{a} \leq b \leq \overline{b}$.

Conversely, let $\overline{a} \leq \overline{b}$. Then by Proposition 30 (v), we take $b = \overline{b}$ and since $a \leq \overline{a}$ for every $a \in A$, we easily obtain $A \tilde{\leq}^{S,T} B$.

2. Let $A \tilde{=}^{S,T} B$. Then by (v) in Proposition 30, there exists $b \in B$ such tat for every $a \in A$ we have $a = b$, thus $\underline{b} \leq \underline{a} = \overline{a} \leq \overline{b}$.

Conversely, let $\underline{b} \leq \underline{a} = \overline{a} \leq \overline{b}$. Then by Proposition 30 (v), we take $b = \underline{a}$ and obtain $A \tilde{=}^{S,T} B$.

(vi) 1. Let $A \tilde{\leq}_{S,T} B$. Then by (vi) in Proposition 30, for every $a \in A$ there exists $b \in B$ such that $a \leq b$, thus $\overline{a} \leq b \leq \overline{b}$.

Conversely, let $\overline{a} \leq \overline{b}$. Then by Proposition 30 (vi), we take $b = \overline{b}$ and since $a \leq \overline{a}$, we therefore obtain $A \tilde{\lesssim}_{S,T} B$.

2. Let $A \tilde{=}_{S,T} B$. Then by (vi) in Proposition 30, for every $a \in A$ there exists $b \in B$ such that $a = b$, thus $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$.

Conversely, let $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$. Then by Proposition 30, (vi), we take $a = b$ and finally obtain $A \tilde{=}_{S,T} B$. ■

If A, B are nonempty and closed intervals of $\mathbf{R}^m$, then by comparing (iii) and (iv) in Proposition 59, we can see that

$$A \tilde{\lesssim}^{T,S} B \text{ if and only if } A \tilde{\lesssim}_{T,S} B. \tag{8.87}$$

Likewise,

$$A \tilde{\lesssim}^{S,T} B \text{ if and only if } A \tilde{\lesssim}_{S,T} B, \tag{8.88}$$

as is clear from (v) and (vi), in the same proposition.

The following proposition shows that (8.87) and (8.88) can be presented in a stronger form.

Proposition 60 *Let $A, B \in \mathcal{F}(\mathbf{R})$ be compact fuzzy sets, $T = \min$, $S = \max$. Then*

$$\mu_{\tilde{\lesssim}^{T,S}}(A, B)) = \mu_{\tilde{\lesssim}_{S,T}}(A, B), \mu_{\tilde{\lesssim}_{T,S}}(A, B) = \mu_{\tilde{\lesssim}^{S,T}}(A, B). \tag{8.89}$$

The proof of Proposition 60 is given in [88] in a more generalized setting.

The following two propositions hold for a binary relation R on $X \times Y$, where X, Y are arbitrary sets, and the T-fuzzy extension $\tilde{\lesssim}^{T}$ of R.

Proposition 61 *Let X, Y be sets, let $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ be fuzzy sets given by the membership functions $\mu_A : X \to [0,1]$, $\mu_B : Y \to [0,1]$, respectively. Let T be a t-norm and R be a binary relation on $X \times Y$, $\tilde{R}^T$ be a T-fuzzy extension of R, let $\alpha \in (0,1)$.*
(i) *If $\mu_{\tilde{R}^T}(A, B) > \alpha$, then $\mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1$.*
(ii) *Let $T = \min$. If $\mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1$, then $\mu_{\tilde{R}^T}(A, B) \geq \alpha$.*

Proof. (i) Let $\alpha \in (0,1)$, $\mu_{\tilde{R}^T}(A, B) > \alpha$. Then by (8.27) we obtain

$$\mu_{\tilde{R}^T}(A, B) = \sup\{T(\mu_A(u), \mu_B(v)) \,|uRv\}. \tag{8.90}$$

Since $\sup\{T(\mu_A(u), \mu_B(v)) \,|u \in X, v \in Y, uRv\} > \alpha$, there exist $u', v' \in \mathbf{R}^m$ such that $u'Rv'$ and $T(\mu_A(u'), \mu_B(v')) \geq \alpha$. The minimum is a maximal t-norm, therefore

$$\min\{\mu_A(u'), \mu_B(v')\} \geq T(\mu_A(u'), \mu_B(v')) \geq \alpha,$$

hence

$$\mu_A(u') \geq \alpha \text{ and } \mu_B(v') \geq \alpha, \tag{8.91}$$

in other words, $u' \in [A]_\alpha$, $v' \in [B]_\alpha$ and $u'Rv'$. By Proposition 30, (i), we obtain

$$\mu_{\tilde{R}}([A]_\alpha, [B]_\alpha) = 1. \tag{8.92}$$

(ii) Let $T = \min$, $\mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1$. By Proposition 30 there exist $u'' \in [A]_\alpha$, $v'' \in [B]_\alpha$ such that $u''Rv''$. Then $\mu_A(u'') \geq \alpha$ and $\mu_B(v'') \geq \alpha$, therefore $\min\{\mu_A(u''), \mu_B(v'')\} \geq \alpha$. Consequently,

$$\mu_{\tilde{R}}(A, B) = \sup\{\min\{\mu_A(u), \mu_B(v)\} | u \in X, v \in Y, uRv\} \geq \alpha.$$

■

If we replace the strict inequality $>$ in (i) by $\geq$, then the conclusion of (i) is clearly no longer true. To prove the result with $\geq$ instead of $>$, we shall make an assumption similar to condition (C) from the preceding section.

Proposition 62 *Let X, Y be sets, let $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ be fuzzy sets given by the membership functions $\mu_A : X \to [0, 1]$, $\mu_B : Y \to [0, 1]$, respectively. Let T be a t-norm and R be a binary relation on $X \times Y$, $\tilde{R}^T$ be a T-fuzzy extension of R, let $\alpha \in (0, 1]$.*
Suppose that there exist $u^* \in X, v^* \in Y$ such that u^*Rv^* and

$$T\left(\mu_A(u^*), \mu_B(v^*)\right) = \sup\{T\left(\mu_A(u), \mu_B(v)\right) | u \in X, v \in Y, uRv\}. \tag{8.93}$$

If $\mu_{\tilde{R}^T}(A, B) \geq \alpha$, then $\mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1$.

Proof. Let $\alpha \in (0, 1]$, $\mu_{\tilde{R}^T}(A, B) \geq \alpha$. Then by (8.27) we obtain

$$\mu_{\tilde{R}^T}(A, B) = \sup\{T\left(\mu_A(u), \mu_B(v)\right) | u \in X, v \in Y, uRv\}. \tag{8.94}$$

Applying (8.93) and (8.94), we obtain$T\left(\mu_A(u^*), \mu_B(v^*)\right) \geq \alpha$. Since min is the maximal t-norm, we have

$$\min\{\mu_A(u^*), \mu_B(v^*)\} \geq T\left(\mu_A(u^*), \mu_B(v^*)\right) \geq \alpha,$$

hence

$$\mu_A(u^*) \geq \alpha \text{ and } \mu_B(v^*) \geq \alpha,$$

in other words, u^*Rv^* and $u^* \in [A]_\alpha$, $v^* \in [B]_\alpha$. By Proposition 30 we obtain

$$\mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1.$$

■

The following proposition gives some sufficient conditions for (8.93).

Proposition 63 *Let A, B be compact fuzzy quantities with the membership functions $\mu_A : \mathbf{R}^m \to [0, 1]$, $\mu_B : \mathbf{R}^m \to [0, 1]$. Let T be a continuous t-norm and R be a closed binary relation on $\mathbf{R}^m$, moreover, let $\tilde{R}^T$ be a T-fuzzy extension of R. Then there exist $u^*, v^* \in \mathbf{R}^m$ such that u^*Rv^* and*

$$T\left(\mu_A(u^*), \mu_B(v^*)\right) = \sup\{T\left(\mu_A(u), \mu_B(v)\right) | u, v \in \mathbf{R}^m, uRv\}. \tag{8.95}$$

Proof. We show that $\varphi(u,v) = T(\mu_A(u), \mu_B(v))$ attains its maximum on the set $Z = \{(u,v) \in \mathbf{R}^{2m} | uRv\}$. Since R is closed binary relation, Z is a closed set. Further, since for all $\beta \in (0,1]$, the upper level sets $U(\varphi, \beta)$ are compact, it follows that either $U(\varphi,\beta) \cap Z$ is empty for all $\beta \in (0,1]$, or there exists $\beta_0 \in (0,1]$ such that $U(\varphi, \beta_0) \cap Z$ is nonempty.

In the former case, (8.93) holds for any $u^*, v^* \in Z$ with

$$T(\mu_A(u^*), \mu_B(v^*)) = 0.$$

In the latter case, there exists $(u^*, v^*) \in \mathbf{R}^{2m}$, such that φ attains its maximum on $U(\varphi, \beta_0) \cap Z$ in (u^*, v^*), which is a global maximizer of φ on Z. ■

Corollary 64 *Let $A, B \in \mathcal{F}(\mathbf{R}^m)$ be compact fuzzy quantities with the membership functions $\mu_A : \mathbf{R}^m \to [0,1]$, $\mu_B : \mathbf{R}^m \to [0,1]$. Let $T = \min$ and R be a closed binary relation on $\mathbf{R}^m$, $\tilde{R}^T$ be a T-fuzzy extension of R.*
For $\alpha \in (0,1]$,

$$\mu_{\tilde{R}^T}(A,B) \geq \alpha \text{ if and only if } \mu_{\tilde{R}^T}([A]_\alpha, [B]_\alpha) = 1.$$

Proof. By Proposition 63, condition (8.93) is satisfied. Then by Proposition 62, we obtain the "if" part of the statement. The opposite part follows from Proposition 61. ■

Notice that the usual binary relations "=", "$\leq$" and "$\geq$" are closed binary relations.

Propositions 62, 61 and 63 hold for the T-fuzzy extension $\tilde{R}^T$of the valued relation R. Similar results can be derived also for the other fuzzy extensions, particularly $\tilde{R}_S, \tilde{R}^{T,S}, \tilde{R}_{T,S}, \tilde{R}^{S,T}$ and $\tilde{R}_{S,T}$. Here, we present an important result for a particular case $m = 1$, i.e. $\mathbf{R}^m = \mathbf{R}$. The following theorem is a parallel to Theorem 59. Similar results to Theorem 59, we obtain the following proposition.

Theorem 65 *Let $A, B \in \mathcal{F}(\mathbf{R})$ be strictly convex and compact fuzzy sets, $T = \min$, $S = \max$, $\alpha \in (0,1)$.*
Then
(i) $\mu_{\tilde{\leq}^T}(A,B) \geq \alpha$ *if and only if* $\inf[A]_\alpha \leq \sup[B]_\alpha$,
(ii) $\mu_{\tilde{\leq}_S}(A,B) \geq \alpha$ *if and only if* $\sup[A]_{1-\alpha} \leq \inf[B]_{1-\alpha}$,
(iii) $\mu_{\tilde{\leq}^{T,S}}(A,B) \geq \alpha$ *iff* $\mu_{\tilde{\leq}_{T,S}}(A,B) \geq \alpha$ *iff* $\sup[A]_{1-\alpha} \leq \sup[B]_\alpha$,
(iv) $\mu_{\tilde{\leq}^{S,T}}(A,B) \geq \alpha$ *iff* $\mu_{\tilde{\leq}_{S,T}}(A,B) \geq \alpha$ *iff* $\inf[A]_\alpha \leq \inf[B]_{1-\alpha}$.

From the practical point of view, the last theorem is important for calculating the membership function of both fuzzy feasible solutions and fuzzy optimal solutions of fuzzy mathematical programming problem in Chapter 10. Some related results to this problem can be found also in [21].

Chapter 9

Fuzzy Multi-Criteria Decision Making

9.1 Introduction

When dealing with practical decision problems, we often have to take into consideration uncertainty in the problem data. It may arise from errors in measuring physical quantities, from errors caused by representing some data in a computer, from the fact that some data are approximate solutions of other problems or estimations by human experts, etc. In some of these situations, the fuzzy set approach may be applicable. In the context of multicriteria decision making, functions mapping the set of feasible alternatives into the unit interval $[0,1]$ of real numbers representing normalized utility functions can be interpreted as membership functions of fuzzy subsets of the underlying set of alternatives. However, functions with the range in $[0,1]$ arise in more contexts.

In this chapter, we consider a *decision problem in* X, i.e. the problem to find a "best" decision in the set of feasible alternatives X with respect to several (i.e. more than one) criteria functions, see [81], [90], [80], [76], [77], [78]. Within the framework of such a decision situation, we deal with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. Here, they are commonly called Pareto-optimal decisions.

We study also the compromise decisions $x^* \in X$ maximizing some aggregation of the criteria μ_i, $i \in I = \{1, 2, ..., m\}$. The criteria μ_i considered here will be functions defined on the set X of feasible alternatives with the values in the unit interval $[0,1]$, i.e. $\mu_i : X \to [0,1]$, $i \in I$. Such functions can be interpreted as membership functions of fuzzy subsets of X and will be called here *fuzzy criteria*. Later on, in chapters 8 and 9, each constraint or objective function of the fuzzy mathematical programming problem will be naturally apointed to the unique fuzzy criterion. From this point of view this chapter should follow the chapters 10 and 11 dealing with fuzzy mathematical programming. Our approach here is, however, more general and can be adopted to a more general class of decison problems.

The set X of feasible alternatives is supposed to be a convex subset, or a generalized convex subset of the n-dimensional Euclidean space $\mathbf{R}^n$, frequently we consider $X = \mathbf{R}^n$. The main subject of our interest is to derive some important relations between Pareto-optimal decisions and compromise decisions. Moreover, we generalize the concept of the compromise decision by adopting aggregation operators which were investigated in [79] and we also extend the results derived for max-min decisions. The results will be derived for the n-dimensional Euclidean vector space $\mathbf{R}^n$ with $n \geq 1$. However, some results can be derived only for $\mathbf{R}^1$ denoted here simply by $\mathbf{R}$.

9.2 Fuzzy Criteria

Since no function mapping $\mathbf{R}$ into $[0, 1]$ is strictly concave, and each concave function mapping $\mathbf{R}$ into $[0, 1]$ is constant on $\mathbf{R}$, we take advantage of the definition presented in Chapter 3, [79], where we defined (quasi)concave and (quasi)convex functions on arbitrary subsets X of $\mathbf{R}^n$. Now, for membership functions of fuzzy subsets of $\mathbf{R}^n$, the concavity concepts will be applied to the supports of the membership functions, that is, for a fuzzy subset A of $\mathbf{R}^n$ with the membership function $\mu : \mathbf{R}^n \to [0, 1]$, we consider $X = Supp(A)$. Here, we use the notation and nomenclature of Chapter 8.

It is evident that a function $\mu : \mathbf{R}^n \to [0, 1]$ quasiconcave on $\mathbf{R}^n$ is quasiconcave on $Supp(\mu)$, and vice versa: any function μ quasiconcave on $Supp(\mu)$ is quasiconcave on $\mathbf{R}^n$. However, this is no longer true that the membership functions strictly (quasi)concave on $\mathbf{R}^n$ and membership functions strictly (quasi)concave on their supports coincide.

Example 66 *Let $\mu : \mathbf{R} \to [0, 1]$ be defined by $\mu(x) = \max\{0, 1 - x^2\}$, see Fig. 9.1(a). It can be easily shown that $Supp(\mu) = [-1, 1]$ and μ is strictly concave and strictly quasiconcave on $Supp(\mu)$. However, μ is neither strictly concave nor strictly quasiconcave on $\mathbf{R}$. In Fig.9.1 (b), a semistrictly quasiconcave function on $Supp(\mu)$ which is not semistrictly quasiconcave on $\mathbf{R}$ is depicted.*

As mentioned in the introduction, we are interested in the properties of solution concepts of optimization problems whose objectives are expressed in the terms of fuzzy criteria. A particular interest will be given to fuzzy criteria defined as follows.

Definition 67 *A fuzzy subset of $\mathbf{R}^n$ given by its membership function $\mu : \mathbf{R}^n \to [0, 1]$ is called a* fuzzy criterion *on $\mathbf{R}^n$ if μ is upper normalized.*

By Definition 67 any fuzzy criterion is given by the membership function attaining the maximal membership value 1. Sometimes in this chapter, the

Figure 9.1:

results will be derived for membership functions of fuzzy subsets of $\mathbf{R}^n$ not necessarily upper normalized. This fact will be distinctly stressed if necessary.

Fuzzy criteria in one-dimensional Euclidean space $\mathbf{R}$ with additional concavity properties can be characterized by the following simple propositions. The corresponding proofs follow easily from the definition.

Proposition 68 *If the membership function μ of a fuzzy criterion on $\mathbf{R}$ is quasiconcave on $\mathbf{R}$, then there exist $\alpha, \beta \in [0,1]$ and $a, b, c, d \in \mathbf{R} \cup \{-\infty, +\infty\}$, such that $a \leq b \leq c \leq d$ and*
$\mu(x) = \alpha$ for $x < a$,
$\mu(x)$ is non-decreasing for $a \leq x \leq b$,
$\mu(x) = 1$ for $b < x < c$,
$\mu(x)$ is non-increasing for $c \leq x \leq d$,
$\mu(x) = \beta$ for $d < x$.

Proposition 69 *If the membership function μ of a fuzzy criterion on $\mathbf{R}$ is strictly quasiconcave on $Supp(\mu)$ and $Supp(\mu)$ is convex, then there exist $a, b \in \mathbf{R} \cup \{-\infty, +\infty\}$, and $\bar{x} \in \mathbf{R}$, such that $a \leq \bar{x} \leq b$ and*
$\mu(x) = 0$ for $x \leq a$ or $x \geq b$,
$\mu(x)$ is increasing for $a \leq x \leq \bar{x}$,
$\mu(\bar{x}) = 1$,
$\mu(x)$ is decreasing for $\bar{x} \leq x \leq b$.

Later on, we shall take advantage of the above stated properties in case of $\mathbf{R}^n$ for $n > 1$.

9.3 Pareto-Optimal Decisions

Throughout this chapter we suppose that $I = \{1, 2, ..., m\}$, $m > 1$, I is an index set of a given family $F = \{\mu_i |\ i \in I\}$ of membership functions of fuzzy subsets of $\mathbf{R}^n$. Let X be a subset of $\mathbf{R}^n$ such that $Supp(\mu_i) \subset X$ for all $i \in I$. The elements of X are called *decisions*.

Definition 70 *(i) A decision x_{WP} is said to be a* Weak Pareto-Maximizer *(WPM), if there is no $x \in X$ such that*

$$\mu_i(x_{WP}) < \mu_i(x), \text{ for every } i \in I. \tag{9.1}$$

(ii) A decision x_P is said to be a Pareto-Maximizer *(PM), if there is no $x \in X$ such that*

$$\mu_i(x_P) \le \mu_i(x), \text{ for every } i \in I, \tag{9.2}$$
$$\mu_i(x_{WP}) < \mu_i(x), \text{ for some } i \in I.$$

(iii) A decision x_{SP} is said to be a Strong Pareto-Maximizer *(SPM), if there is no $x \in X$, $x \neq x_{SP}$, such that*

$$\mu_i(x_{SP}) \le \mu_i(x), \text{ for every } i \in I. \tag{9.3}$$

Definition 71 *The sets of all WPM, PM and SPM are denoted by X_{WP}, X_P, X_{SP}, respectively. The elements of $X_{WP} \cup X_P \cup X_{SP}$ are called* Pareto-Optimal Decisions*.*

The following property is evident.

Proposition 72 *Any SPM is PM, and any PM is WPM, i.e.*

$$X_{SP} \subset X_P \subset X_{WP}. \tag{9.4}$$

To illustrate the above concepts, let us inspect the following example.

Example 73 *Let μ_1 and μ_2 be as in Fig.9.2. Then*
$X_{WP} = $ [a,g], $X_P = $ [b,c] $\cup$ (d,e) $\cup$ {f}, $X_{SP} = $ (d,e) $\cup$ {f}.

Let $D_j \subset \mathbf{R}^n$ for all $j \in J$, where J is a finite index set. By $Conv\{D_j \mid j \in J\}$ we denote the *convex hull* of all sets D_j, i.e.

$$\begin{aligned} Conv\{D_j | j \in J\} \;=\; & \{z \in \mathbf{R}^n | z = \sum_{j\in J} \lambda_j x_j, \text{ where } x_j \in D_j, \lambda_j \ge 0 \\ & \text{and } \sum_{j \in J} \lambda_j = 1 \}. \end{aligned}$$

In the following propositions, we obtain a transparent characterization of all WPM, PM and SPM for strictly quasiconcave fuzzy criteria in one-dimensional space $\mathbf{R}$. Unfortunately, we cannot obtain parallel results in $\mathbf{R}^n$ for $n > 1$, as is demonstrated by Example 75, see also [81].

Proposition 74 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on $\mathbf{R}$, quasiconcave on $\mathbf{R}$. Then*

$$Conv\{Core(\mu_i) | i \in I\} \subset X_{WP}.$$

Figure 9.2:

Proof. Let $x_i' = \inf Core(\mu_i)$, $x_i'' = \sup Core(\mu_i)$ and set $x' = \min\{x_i'|i \in I\}, x'' = \max\{x_i''|i \in I\}$, then $Cl(Conv\{Core(\mu_i)|i \in I\}) = [x', x'']$.
Let $x \in Conv\{Core(\mu_i)|i \in I\}$ and suppose that $x \notin X_{WP}$. Then there exists y with $\mu_i(y) > \mu_i(x)$, for all $i \in I$.
Assume that $y < x$, then there exists $k \in I$ and $y' \in Core(\mu_k)$ with $y < x \leq y'$ such that by Proposition 68 we obtain $\mu_k(y) \leq \mu_k(x)$, a contradiction. Otherwise, if $x < y$, then again by Proposition 68 we have $\mu_j(x) \geq \mu_j(y)$, again a contradiction. ■

Example 75 *This example demonstrates that Proposition 74 is not true for* $\mathbf{R}^n$*, where* $n > 1$*, particularly for* $n = 2$*. Set*

$$\mu_1(x_1, x_2) = \max\{0, 1 - \tfrac{1}{4}x_1^2 - x_2^2\},$$
$$\mu_2(x_1, x_2) = \max\{0, 1 - (x_1 - 1)^2 - (x_2 - 1)^2\}.$$

Notice that μ_1, μ_2 *are continuous membership functions of fuzzy criteria, strictly concave on their supports, hence quasiconcave on* $\mathbf{R}^2$*. Here,* $Core(\mu_1) = \bar{\mathbf{x}}_1 = (0;0)$*,* $Core(\mu_2) = \bar{\mathbf{x}}_2 = (1;1)$ *are the end points of the segment* $Conv\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$ *in* $\mathbf{R}^2$*.*
It is easy to calculate that

$$\mu_1(0.5, 0.5) = 0.6875 \text{ and } \mu_2(0.5, 0.5) = 0.5.$$

On the other hand, $\mu_1(0.7, 0.4) = 0.7175$*,* $\mu_2(0.7, 0.4) = 0.55$*. We obtain* $\mu_1(0.5, 0.5) < \mu_1(0.7, 0.4)$ *and* $\mu_2(0.5, 0.5) < \mu_2(0.7, 0.4)$*.*
As $(0.5; 0.5) \in Conv\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$ *and* $(0.7; 0.4) \notin Conv\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\}$*, it follows that*

$$Conv\{Core(\mu_i)|i = 1, 2\} = Conv\{\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2\},$$

which is not a subset of X_{WP}*.*

Example 76 *Let μ_1, μ_2 be as in Fig. 9.3. Here, μ_1, μ_2 are continuous and evidently $X_{WP} = $ [a,d], $X_P = $ [b,c], however, $X_{SP} = \emptyset$.*

Figure 9.3:

Proposition 77 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on* **R**, *strictly quasiconcave on their convex supports. Then*

$$X_P \subset Conv\{Core(\mu_i)|i \in I\}.$$

Proof. By Proposition 69, each $Core(\mu_i)$ contains exactly one element, i.e. $x_i = Core(\mu_i)$. Setting $x' = \min\{x_i|i \in I\}$, $x'' = \max\{x_i|i \in I\}$, then

$$Conv\{Core(\mu_i)|i \in I\}) = [x', x''].$$

Let $x \notin Conv\{Core(\mu_i)|i \in I\}$ and suppose that $x < x'$. By monotonicity of μ_i we get $\mu_i(x) \leq \mu_i(x')$ for all $i \in I$. Moreover, by Proposition 69, $\mu_j(x) < \mu_j(x')$ for j satisfying $x_j = x'$, consequently, $x \notin X_P$.
On the other hand, suppose that $x'' < x$. Again by monotonicity of μ_i we get $\mu_i(x'') \geq \mu_i(x)$ for all $i \in I$, and by Proposition 69, $\mu_k(x'') > \mu_k(x)$ for k satisfying $x_k = x''$. Hence, again $x \notin X_P$, which gives the required result. ■

Proposition 78 *Let μ_i, $i \in I$, be membership functions of fuzzy criteria on* **R**, *strictly quasiconcave on their convex supports. If*

$$Conv\{Core(\mu_i)|\ i \in I\} \subset \bigcap_{i \in I} Supp(\mu_i), \tag{9.5}$$

then

$$X_{WP} = X_P = X_{SP} = Conv\{Core(\mu_i)|\ i \in I\}. \tag{9.6}$$

Proof. By Proposition 69, each $Core(\mu_i)$ contains exactly one element, i.e. $x_i = Core(\mu_i)$. Setting $x' = \min\{x_i | i \in I\}$, $x'' = \max\{x_i | i \in I\}$, we obtain

$$Conv\{Core(\mu_i)|i \in I\}) = [x', x''].$$

First, we prove that $[x', x''] \subset X_{SP}$.

Let $x \in [x', x'']$ and suppose by contrary that $x \notin X_{SP}$. Then there exists y with $y \neq x$ and $\mu_i(y) \geq \mu_i(x)$, for all $i \in I$.
Further, suppose that $y < x$, then by strict quasiconcavity of μ_k, for k satisfying $x_k = x''$, and by Proposition 69, we get $\mu_k(y) < \mu_k(x)$, a contradiction.

On the other hand, if $x < y$, then by strict quasiconcavity of μ_j, for j satisfying $x_j = x'$, we get $\mu_j(x) > \mu_j(y)$, again a contradiction. Hence

$$Conv\{Core(\mu_i)|i \in I\}) \subset X_{SP}.$$

Second, we prove that $X_{WP} \subset Conv\{Core(\mu_i)|i \in I\})$.
Suppose that $y \notin [x', x'']$. Let $y < x'$, then by (9.5) $\mu_i(x') > 0$ for all $i \in I$. Applying strict monotonicity of μ_i, we get $\mu_i(y) < \mu_i(x')$, for all $i \in I$, hence, $y \notin X_{WP}$. Assuming $y > x''$ we obtain by analogy the same result.
Combining the first and the second result, we obtain the chain of inclusions

$$X_{WP} \subset Conv\{Core(\mu_i)|i \in I\}) \subset X_{SP}.$$

However, by (9.4) we have $X_{SP} \subset X_P \subset X_{WP}$, consequently, we obtain the required equalities (9.6). ■

Notice that inclusion (9.5) is satisfied if all $Supp(\mu_i), i \in I$, are identical.

9.4 Compromise Decisions

In the theory of multi-objective optimization, the "compromise decision", or, "compromise solution" is obtained as the solution of single-objective problem with the objective being a combination of all criteria in question, see e.g. [37].

In this section we investigate a concept and some properties of compromise decision $x^* \in X$, maximizing the aggregation of all criteria, e.g. $\min\{\mu_i(x)| i \in I\}$, where $X \subset \mathbf{R}^n$ is a convex set, $I = \{1, 2, ..., m\}, m > 1$. The original idea belongs to Bellman and Zadeh in [9], who proposed its use in decision analysis by using the following definition.

Definition 79 *Let μ_i, $i \in I$, be the membership functions of fuzzy subsets of X, X be a convex subset of $\mathbf{R}^n$. A decision $x^* \in X$ is called a* max-min decision*, if*

$$\min\{\mu_i(x^*)| \ i \in I\} = \max\{\min\{\mu_i(x)| \ i \in I\}| \ x \in X\}. \tag{9.7}$$

The set of all max-min decisions in X is denoted by X_M.

We start with two propositions that are concerned with the existence of max-min decision. In $\mathbf{R}$, the requirement of compactness of the α-cuts of the criteria is not necessary for the existence of nonempty X_M, whereas the same result is no longer true in $\mathbf{R}^n$ with $n > 1$, as will be demonstrated on an example.

Proposition 80 *Let μ_i , $i \in I$, be the membership functions of fuzzy criteria on* $\mathbf{R}$*, quasiconcave on* $\mathbf{R}$*. If all μ_i are upper semicontinuous (USC) on* $\mathbf{R}$*, then $X_M \neq \emptyset$.*

Proof. For each $i \in I$ there exists $x_i \in Core(\mu_i)$. Put $x' = \min\{x_i | i \in I\}$ and $x'' = \max\{x_i | i \in I\}$. By Proposition 68, μ_i are non-decreasing in $(-\infty, x']$ and non-increasing in $[x'', +\infty)$ for all $i \in I$. Then $\varphi = \min\{\mu_i | \ i \in I\}$ is also non-decreasing in $(-\infty, x']$ and non-increasing in $[x'', +\infty)$. As μ_i, $i \in I$, are upper semicontinuous, φ is also USC on $\mathbf{R}$, particularly on the compact interval $[x', x'']$. Hence $\varphi = \min\{\mu_i | \ i \in I\}$ attains its maximum on $[x', x'']$, in a global maximizer on $\mathbf{R}$. This maximizer is a max-min decision, i.e. $X_M \neq \emptyset$. ■

Example 81 *This example demonstrates that semicontinuity is essential in the above proposition. Let*

$$\mu_1(x) = \{ \begin{array}{l} 1 \textit{ for } x < 0, \\ 0 \textit{ for } x \geq 0, \end{array}$$

$$\mu_2(x) = \{ \begin{array}{l} e^x \textit{ for } x < 0, \\ 1 \ \ \textit{ for } x \geq 0. \end{array}$$

Here, μ_1, μ_2 are the membership functions of fuzzy criteria, μ_2 is continuous, μ_1 is not upper semicontinuous on $\mathbf{R}$*. It is easy to see that* $\psi(x) = \min\{\mu_1(x), \mu_2(x)\}$ *does not attain its maximum on* $\mathbf{R}$ *, i.e. $X_M = \emptyset$.*

Example 82 *This example demonstrates that Proposition 80 does not hold in* $\mathbf{R}^n$ *with $n > 1$, particularly for $n = 2$. Set*

$$\mu_1(x_1, x_2) = \max \left\{ 0, \min\{x_2, 1 - (\frac{x_1 + 1}{x_2 + 1})^2\} \right\},$$

$$\mu_2(x_1, x_2) = \max \left\{ 0, \min\{x_2, 1 - (\frac{x_1 - 1}{x_2 + 1})^2\} \right\}.$$

It can be easily verified that μ_1 and μ_2 are continuous fuzzy criteria on $\mathbf{R}^2$*, quasiconcave on* $\mathbf{R}^2$*. Let*

$$\varphi(x_1, x_2) = \min\{\mu_1(x_1, x_2), \mu_2(x_1, x_2)\}.$$

It is not difficult to show that $\varphi(x_1, x_2) < 1$ on $\mathbf{R}^2$ *and* $\varphi(0, x_2) = 1 - \frac{1}{(x_2+1)^2}$ *for $x_2 > 1$.*
Since $\lim \varphi(0, x_2) = 1$ *for $x_2 \to +\infty$, φ does not attain its maximum on X, i.e.* $X_M(\mu_1, \mu_2) = \emptyset$.

Remember that a fuzzy subset A given by the membership function $\mu : \mathbf{R}^n \to [0, 1]$ is compact if and only if μ is USC on $\mathbf{R}^n$ and $[A]_\alpha$ is a bounded subset of $\mathbf{R}^n$ for every $\alpha \in (0, 1]$, or, $[A]_\alpha$ is a compact subset of $\mathbf{R}^n$ for every $\alpha \in (0, 1]$.

Proposition 83 *Let A be a compact fuzzy subset of $\mathbf{R}^n$ given by the membership function $\mu : \mathbf{R}^n \to [0,1]$. Then μ attains its maximum on $\mathbf{R}^n$.*

Proof. Let $\alpha^* = \sup\{\mu(x) | x \in \mathbf{R}^n\} > 0$ and let $\{\alpha_k\}_{k=1}^{\infty}$ is an increasing sequence of numbers such that $\alpha_k \in (0,1)$ and $\alpha_k \to \alpha^*$. (If $\alpha^* = 0$, then there is nothing to prove.) Then $[A]_{\alpha_k}$ is compact and $[A]_{\alpha_{k+1}} \subset [A]_{\alpha_k}$ for all $k = 1, 2, ...$ By the well known property of compact sets there exists $x^* \in \mathbf{R}^n$ with

$$x^* \in \bigcap_{k=1}^{\infty} [A]_{\alpha_k}. \tag{9.8}$$

It remains to show that

$$\mu(x^*) = \alpha^*. \tag{9.9}$$

On contrary, suppose that $\mu(x^*) < \alpha^*$. Then there exists k_0 such that

$$\mu(x^*) < \alpha_{k_0} \leq \alpha^*. \tag{9.10}$$

By (9.8), $x^* \in [A]_{\alpha_{k_0}}$, hence, $\mu(x^*) \geq \alpha_{k_0}$, a contradiction to (9.10). Consequently, (9.9) is true, which completes the proof. ■

Proposition 84 *Let μ_i , $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$. If A_i is compact for every $i \in I$, then $X_M \neq \emptyset$.*

Proof. Let $\varphi = \min\{\mu_i | \ i \in I\}$. Observe that $[\varphi]_\alpha = \bigcap_{i \in I} [\mu_i]_\alpha$ for all $\alpha \in (0,1]$. Since all $[\mu_i]_\alpha$ are compact, the same holds for $[\varphi]_\alpha$. By Proposition 83, φ attains its maximum on $\mathbf{R}^n$, i.e. $X_M \neq \emptyset$. ■

In what follows we investigate some relationships between Pareto-Optimal decisions and max-min decisions. In the following two propositions, normality of μ_i is not required.

Proposition 85 *Let $\mu_i, i \in I$, be membership functions of fuzzy subsets of $\mathbf{R}^n$. Then*

$$X_M \subset X_{WP}.$$

Proof. Let $x^* \in X_M$. Suppose that x^* is not a Weak Pareto-Maximizer, then by Definition 70 there exists x' such that $\mu_i(x^*) < \mu_i(x')$, for all $i \in I$. Then

$$\min\{\mu_i(x^*) | \ i \in I\} < \min\{\mu_i(x') | \ i \in I\},$$

showing that x^* is not a max-min decision, a contradiction. ■

Proposition 86 *Let μ_i, $i \in I$, be the membership functions of fuzzy sets of $\mathbf{R}^n$, let $X_M = \{x^*\}$, i.e. $x^* \in X$ be a unique max-min decision. Then*

$$X_M \subset X_{SP}.$$

Proof. Suppose that x^* is a unique max-min decision and suppose that x^* is not a SPM. Then there exists $x^+ \in X$, $x^* \neq x^+$, $\mu_i(x^*) \leq \mu_i(x^+)$, for all $i \in I$. Then

$$\min\{\mu_i(x^*)|\ i \in I\} \leq \min\{\mu_1(x^+)|\ i \in I\},$$

which is a contradiction with the uniqueness of $x^* \in X_M$. ■

In the following proposition sufficient conditions for the uniqueness of a compromise decision are given.

Proposition 87 *Let μ_i, $i \in I$, be membership functions of fuzzy subsets of $\mathbf{R}^n$ strictly quasiconcave on their convex supports Supp(μ_i). Let $x^* \in X_M$ such that*

$$\min\{\mu_i(x^*)|\ i \in I\} > 0. \tag{9.11}$$

Then

$$X_M = \{x^*\},$$

i.e. the max-min decision $x^*$is unique.

Proof. Let $\varphi(x) = \min\{\mu_i(x)|\ i \in I\}$, where $x \in \mathbf{R}^n$, and suppose that there exists $x' \in X_M$, $x^* \neq x'$. Then by (9.11)

$$\varphi(x') = \varphi(x^*) > 0.$$

As by (9.11) we have also $x^*, x' \in X = \bigcap_{i \in I} Supp(\mu_i)$, where X is convex. By strict quasiconcavity of φ on X we obtain for $x^+ = \lambda x' + (1-\lambda)x^*$ and $0 < \lambda < 1$:

$$\varphi(x^+) > \min\{\varphi(x'), \varphi(x^*)\},$$

which contradicts the fact that $\varphi(x^*) = \max\{\varphi(x)|\ x \in \mathbf{R}^n\}$. Consequently, X_M consists of the unique element. ■

Corollary 88 *If μ_i, $i \in I$, are membership functions of fuzzy subsets of $\mathbf{R}^n$, strictly quasiconcave on their convex supports and $x^* \in X_M$ satisfying (9.11), then $x^* \in X_{SP}$.*

9.5 Generalized Compromise Decisions

In this section we generalize the concept of the max-min decision by adopting aggregation operators investigated in [79].

Definition 89 *Let $\mu_i, i \in I$, be the membership functions of fuzzy subsets of X, X be a convex subset of $\mathbf{R}^n$. Let $G = \{G_m\}_{m=1}^{\infty}$ be an aggregation operator. A decision $x^* \in X$ is called a max-G* decision, *if*

$$G_m(\mu_1(x^*), ..., \mu_m(x^*)) = \max\{G_m(\mu_1(x), ..., \mu_m(x))|\ x \in X\}. \tag{9.12}$$

The set of all max-G decisions on X is denoted by $X_G(\mu_1, ..., \mu_m)$, or, shortly, X_G.

If there is no danger of misunderstanding, we omit the subscript m in the aggregation mapping G_m, writing shortly G. In the previous section we have investigated some properties of the compromise decisions considering a particular aggregation operator, namely the t-norm, T_M. In what follows we extend the results from the previous section to some more general aggregation operators. The following propositions generalize Propositions 84, 85, 86 and 87.

Proposition 90 *Let μ_i, $i \in I$, be the USC membership functions of fuzzy subsets of $\mathbf{R}^n$, G be an USC aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\psi(x) = G(\mu_1(x), ..., \mu_m(x)) \tag{9.13}$$

is USC on $\mathbf{R}^n$.

Proof. Let $x_0 \in R^n$ and $\varepsilon > 0$. It is sufficient to prove that there exist $\delta > 0$, such that $\psi(x) \leq \psi(x_0) + \varepsilon$ for every $x \in B(x_0, \delta) = \{x \in R^n | \, \|x - x_0\| < \delta\}$.

Let $y_{0i} = \mu_i(x_0)$ for $i \in I$, put $y_0 = (y_{01}, ..., y_{0m})$. Since G is USC on $[0,1]^m$, there exists $\eta > 0$, such that $y \in B(y_0, \eta) = \{y \in [0,1]^m | \, \|y - y_0\| < \eta\}$ implies

$$G(y_1, ..., y_m) \leq G(y_{01}, ..., y_{0m}) + \varepsilon. \tag{9.14}$$

By USC of all μ_i, $i \in I$, there exists $\delta > 0$, such that $\mu_i(x) \leq \mu_i(x_0) + \frac{\eta}{2}$ for every $x \in B(x_0, \delta) = \{x \in R^n | \, \|x - x_0\| < \delta\}$. By monotonicity property of G, we obtain

$$G(\mu_1(x), ..., \mu_m(x)) \leq G(z_1, ..., z_m), \tag{9.15}$$

where $z_i = \min\{1, \mu_i(x_0) + \frac{\eta}{2}\}$, and also $(z_1, ..., z_m) \in B(y_0, \eta)$. Moreover, by (9.13) we have $\psi(x_0) = G(y_{01}, ..., y_{0m})$. Combining inequalities (9.14) and (9.15), we obtain the required result $\psi(x) \leq \psi(x_0) + \varepsilon$. ■

The next two proposition give some sufficient conditions for the existence of max-G decisions.

Proposition 91 *Let μ_i , $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$, G be an USC and idempotent aggregation operator. If A_i is compact for every $i \in I$, then $X_G \neq \emptyset$.*

Proof. Let $\alpha \in (0,1]$, $\psi(x) = G(\mu_1(x), ..., \mu_m(x))$. We prove that $[\psi]_\alpha = \{x \in R^n | G(\mu_1(x), ..., \mu_m(x)) \geq \alpha\}$ is a compact subset of $\mathbf{R}^n$. First, we prove that $[\psi]_\alpha$ is bounded. Assume the contrary; then there exist $x_k \in [\psi]_\alpha$, $k = 1, 2, ...$, with $\lim \|x_k\| = +\infty$ for $k \to +\infty$. Take an arbitrary β, with $0 < \beta < \alpha$. Since all $[\mu_i]_\beta$ are bounded, then there exists k_0 such that for all $i \in I$ and $k > k_0$ we obtain $x_k \notin [\mu_i]_\beta$, i.e. $\mu_i(x_k) < \beta$. By monotonicity and idempotency of A it follows that for $k > k_0$ we have

$$G(\mu_1(x_k), ..., \mu_m(x_k)) \leq G(\beta, ..., \beta) = \beta < \alpha.$$

But this is a clear contradiction, consequently, $[\psi]_\alpha$ is bounded.

By Proposition 90 $\psi(x) = G(\mu_1(x), ..., \mu_m(x))$ is USC on $\mathbf{R}^n$, hence $[\psi]_\alpha$ is closed. Consequently, $[\psi]_\alpha$ is compact. Then ψ is a membership function of a compact fuzzy subset of $\mathbf{R}^n$, therefore, by Proposition 83, ψ attains its maximum on $\mathbf{R}^n$, i.e. $X_G \neq \emptyset$. ■

Notice that the mean aggregation operators are idempotent. However, the only idempotent t-norm is the minimum t-norm T_M. The following proposition extends the statement of Proposition 91 to all other USC t-norms.

Proposition 92 *Let μ_i , $i \in I$, be membership functions of fuzzy subsets A_i of $\mathbf{R}^n$, T be an USC t-norm. If A_i is compact for every $i \in I$, then $X_T \neq \emptyset$.*

Proof. Let $\phi : \mathbf{R}^n \to [0, 1]$ be defined for $x \in \mathbf{R}^n$ as

$$\phi(x) = T(\mu_1(x), ..., \mu_m(x)). \tag{9.16}$$

Let $\alpha \in (0, 1]$, we prove that $[\phi]_\alpha$ is a bounded subset of $\mathbf{R}^n$. By definition we have $[\phi]_\alpha = \{x \in R^n | T(\mu_1(x), ..., \mu_m(x)) \geq \alpha\}$. Since $[\mu_i]_\alpha$ are bounded for all $i \in I$, it follows that

$$\bigcap_{i \in I} [\mu_i]_\alpha = \{x \in R^n |\, \min\{\mu_1(x), ..., \mu_m(x)\} \geq \alpha\}$$

is also bounded. We show that $[\phi]_\alpha \subset \bigcap_{i \in I} [\mu_i]_\alpha$.

Let $x \in [\phi]_\alpha$. Then by definition of α-cut we have

$$T(\mu_1(x), ..., \mu_m(x)) \geq \alpha. \tag{9.17}$$

Since T is dominated by the minimum t-norm T_M, it follows that

$$T(\mu_1(x), ..., \mu_m(x)) \leq \min\{\mu_1(x), ..., \mu_m(x)\}. \tag{9.18}$$

From (9.17) and (9.18) we obtain

$$\alpha \leq \min\{\mu_1(x), ..., \mu_m(x)\},$$

thus $x \in \bigcap_{i \in I} [\mu_i]_\alpha$, proving that $[\phi]_\alpha \subset \bigcap_{i \in I} [\mu_i]_\alpha$. Consequently, $[\phi]_\alpha$ is bounded for all $\alpha \in (0, 1]$.

By Proposition 90 $\phi(x) = T(\mu_1(x), ..., \mu_m(x))$ is USC on $\mathbf{R}^n$ and by Proposition 83 ϕ attains its maximum on $\mathbf{R}^n$, i.e. $X_T \neq \emptyset$. ■

The following proposition is an extension of Proposition 85.

Proposition 93 *Let μ_i , $i \in I$, be the membership functions of fuzzy subsets of $\mathbf{R}^n$, G be a strictly monotone aggregation operator. Then*

$$X_G \subset X_{WP}.$$

Proof. Let $x^* \in X_G$. Suppose that x^* is not a Weak Pareto-Maximizer, then there exists x' such that $\mu_i(x^*) < \mu_i(x')$, for all $i \in I$. Then by the strict monotonicity of G we obtain

$$G(\mu_1(x^*), ..., \mu_m(x^*)) < G(\mu_1(x'), ..., \mu_m(x')),$$

showing that x^* is not a max-G decision, a contradiction. ■

The following proposition is a generalization of Proposition 86.

Proposition 94 *Let μ_i , $i \in I$, be the membership functions of fuzzy subsets of $\mathbf{R}^n$, let $X_G = \{x^*\}$, i.e. $x^* \in \mathbf{R}^n$ be a unique max-G decision, let G be an aggregation operator. Then*

$$X_G \subset X_{SP}.$$

Proof. Suppose that x^* is a unique max-G decision and suppose that x^* is not a SPM. Then there exists $x^+ \in \mathbf{R}^n$, $x^* \neq x^+$, $\mu_i(x^*) \leq \mu_i(x^+)$, for all $i \in I$. Then by monotonicity of G we obtain

$$G(\mu_1(x^*), ..., \mu_m(x^*)) \leq G(\mu_1(x^+), ..., \mu_m(x^+)),$$

which is a contradiction with the uniqueness of $x^* \in X_G$. ■

The proof of the following proposition is a slight adaptation of that of Proposition 87 and requires that G dominates T_M, i.e. $G \gg T_M$, see [79].

Proposition 95 *Let μ_i , $i \in I$, be the membership functions of fuzzy criteria on $\mathbf{R}^n$, strictly quasiconcave on their convex supports $Supp(\mu_i)$. Let G be a strictly monotone aggregation operator such that G dominates T_M and let $x^* \in X_G$ with*

$$\min\{\mu_i(x^*) |\ i \in I\} > 0. \tag{9.19}$$

Then

$$X_G = \{x^*\},$$

i.e. $x^*$is a unique max-G decision.

Proof. Let $\varphi(x) = G(\mu_1(x), ..., \mu_m(x))$, where $x \in \mathbf{R}^n$, and suppose that there exists $x' \in X_M$, $x^* \neq x'$. Then by (9.19)

$$\varphi(x') = \varphi(x^*) > 0. \tag{9.20}$$

By (9.19) we have also $x^*, x' \in X = \bigcap_{i \in I} Supp(\mu_i)$, where X is convex. Since G is a strictly monotone aggregation operator and μ_i are strictly quasiconcave, it follows that φ is strictly quasiconcave on X. Then we obtain for $x^+ = \lambda x' + (1-\lambda)x^*$ and $0 < \lambda < 1$:

$$\begin{aligned} \varphi(x^+) = &\ G(\mu_1(\lambda x' + (1-\lambda)x^*), ..., \mu_m(\lambda x' + (1-\lambda)x^*)) \\ > &\ G(\min\{\mu_1(x'), \mu_1(x^*)\}, ..., \min\{\mu_m(x'), \mu_m(x^*)\}). \end{aligned}$$

As G dominates T_M, we obtain

$$\begin{aligned} &G(\min\{\mu_1(x'), \mu_1(x^*)\}, ..., \min\{\mu_m(x'), \mu_m(x^*)\}) \\ &\geq \min\{G(\mu_1(x'), ..., \mu_m(x')), G(\mu_1(x^*), ..., \mu_m(x^*))\}. \end{aligned}$$

Combining the last two inequalities with (9.20), we obtain

$$\varphi(x^+) > \min\{\varphi(x'), \varphi(x^*)\} = \varphi(x^*),$$

which contradicts the fact that $\varphi(x^*) = \max\{\varphi(x) |\ x \in X\}$. Consequently, X_G consists of the unique element. ■

For fuzzy criteria μ_i, $i \in I$, an aggregation operator G and $x^* \in X_G$ satisfying the assumptions of Proposition 95, it follows that $x^* \in X_{SP}$.

9.6 Aggregation of Fuzzy Criteria

In this section we shall investigate the problem of aggregation of several fuzzy criteria μ_i on $\mathbf{R}^n$ with additional property of some generalized concavity property (e.g. upper-connectedness, T-quasiconcavity, or quasiconcavity). We will look for some sufficient conditions which secure some attractive properties. The proofs of the following propositions are omitted as they can be obtained by slight modifications of the corresponding propositions in [79].

Proposition 96 *Let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T_D-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$\bigcap_{i\in I} Core(\mu_i) \neq \emptyset. \tag{9.21}$$

Let $A : [0,1]^m \to [0,1]$ be an aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by

$$\psi(x) = A(\mu_1(x), ..., \mu_m(x)) \tag{9.22}$$

is upper-starshaped on $\mathbf{R}^n$.

Proposition 97 *Let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T_D-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$Core(\mu_1) = \cdots = Core(\mu_m) \neq \emptyset. \tag{9.23}$$

Let $A : [0,1]^m \to [0,1]$ be a strictly monotone aggregation operator. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (9.22) is T_D-quasiconcave on $\mathbf{R}^n$.

Conditions (9.21) and (9.23) in Propositions 96 and 97 are essential for validity of the statements of Propositions 97 and 97 as has been demonstrated on Examples in [79].

Proposition 98 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria. Let A be an aggregation operator, and let A dominates T. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (9.22) is T-quasiconcave on $\mathbf{R}^n$.*

Obviously, any t-norm T dominates T (reflexivity) and the minimum t-norm T_M dominates any other t-norm T. Accordingly, we obtain the following consequence of Proposition 98.

Corollary 99 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria. Then $\varphi_j : \mathbf{R}^n \to [0,1]$, $j = 1,2$, defined by*

$$\varphi_1(x) = T(\mu_1(x), ..., \mu_m(x)), \; x \in \mathbf{R}^n, \tag{9.24}$$

$$\varphi_2(x) = T_M(\mu_1(x), ..., \mu_m(x)), \; x \in \mathbf{R}^n, \tag{9.25}$$

are also T-quasiconcave on $\mathbf{R}^n$.

9.7 Extremal Properties

In this section we derive several results concerning relations between local and global maximizers (i.e. max-A decisions) of some aggregations of fuzzy criteria. For this purpose we apply the local-global properties of generalized concave functions from [79].

Theorem 100 *Let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$ such that*

$$\bigcap_{i \in I} Core(\mu_i) \neq \emptyset.$$

Let $A : [0,1]^m \to [0,1]$ be an aggregation operator. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by

$$\psi(x) = A(\mu_1(x), ..., \mu_m(x)) \tag{9.26}$$

attains its strict local maximizer at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a strict global maximizer of ψ on $\mathbf{R}^n$.

Proof. Observe that ψ is upper-starshaped on $\mathbf{R}^n$. Now, we easily obtain the required result. ∎

Theorem 101 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. Let A be an aggregation operator and let A dominates T. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (9.26) attains its strict local maximum at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a strict global maximizer of ψ on $\mathbf{R}^n$.*

Proof. Obviously, $\psi = A(\mu_1, \ldots, \mu_m)$ is T-quasiconcave on $\mathbf{R}^n$.

Any T-quasiconcave function on $\mathbf{R}^n$ is upper-quasiconnected on $\mathbf{R}^n$. Then the statement follows from the corresponding theorem in [79]. ∎

Theorem 102 *Let T be a t-norm, let $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be semistrictly T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. Let A be a strictly monotone aggregation operator and let A dominates T. If $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (9.26) attains its local maximizer at $\bar{x} \in \mathbf{R}^n$, then $\bar{x}$ is a global maximizer of ψ on $\mathbf{R}^n$.*

Proof. Again, ψ is T-quasiconcave on $\mathbf{R}^n$. Any T-quasiconcave function on $\mathbf{R}^n$ is upper-quasiconnected on $\mathbf{R}^n$. Then the statement follows from the corresponding theorem in [79]. ∎

Since any t-norm T dominates T and the minimum t-norm T_M dominates any other t-norm T, we obtain the following results.

Corollary 103 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_1 : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by (9.24) attains its strict local maximum at $\bar{x}_1 \in \mathbf{R}^n$, then $\bar{x}_1$ is a strict global maximizer of φ_1 on $\mathbf{R}^n$.*

Corollary 104 *Let T be a strict t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be semistrictly T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_1 : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\varphi_1(x) = T(\mu_1(x), \ldots, \mu_m(x)),$$

attains its local maximum at $\bar{x}_1 \in \mathbf{R}^n$, then $\bar{x}_1$ is a global maximizer of φ_1 on $\mathbf{R}^n$.

Corollary 105 *Let T be a t-norm, $\mu_i : \mathbf{R}^n \to [0,1]$, $i \in I$, be T-quasiconcave membership functions of fuzzy criteria on $\mathbf{R}^n$. If $\varphi_2 : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by*

$$\varphi_2(x) = T_M(\mu_1(x), \ldots, \mu_m(x)),$$

attains its strict local maximum at $\bar{x}_2 \in \mathbf{R}^n$, then $\bar{x}_2$ is a strict global maximizer of φ_2 on $\mathbf{R}^n$.

9.8 Application to Location Problem

A classical problem in location theory consists in location p suppliers to cover given demands of q consumers in such a way that total shipping costs are minimized, see e.g. [19]. Consider the following mathematical model:

Let $I = \{1, 2, \ldots, q\}$ be a set of q consumers located on the plane $\mathbf{R}^2$ in the points $\mathbf{c}_i$ with coordinates $\mathbf{c}_i = (u_i; v_i) \in \mathbf{R}^2$, $i \in I$. Each consumer $i \in I$ is characterized by a given nonnegative demand b_i - an amount of products, goods, services, etc. The demands of consumers are to be satisfied by a given set of p suppliers denoted by $J = \{1, 2, \ldots, p\}$. The distance of consumer $i \in I$ located at $(u_i; v_i)$ and supplier $j \in J$ at $(x_j; y_j)$ is denoted by $d_{ij}(x_j, y_j)$ and defined by

$$d_{ij}(x_j, y_j) = d((u_i; v_i), (x_j; y_j)),$$

where d is an appropriate distance function (e.g. Euclidean distance). The shipping cost between i and j depends on the location of consumer i, on the distance $d_{ij}(x_j, y_j)$ of consumer i to supplier j, and, on the amount of goods z_{ij} transported from j to i. These characteristics can be expressed by the value $f_i(d_{ij}(x_j, y_j), z_{ij})$ of a cost function $f_i : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbf{R}^1$, $i \in I$, that is nondecreasing in both variables. The total cost of the shipment of goods from all producers to all consumers is defined as the sum of the individual cost functions

$$f((x_1; y_1), \ldots, (x_p; y_p), z_{11}, \ldots, z_{qp}) = \sum_{i=1}^{q} \sum_{j=1}^{p} f_i(d_{ij}(x_j, y_j), z_{ij}).$$

The problem is to find locations of suppliers $(x_j; y_j)$ and transported amounts z_{ij} for all consumers and suppliers such that the requirements of the consumers are covered and total shipping cost is minimal. The mathematical model of the above location problem can be formulated as follows:

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{q}\sum_{j=1}^{p} f_i\left(d_{ij}\left(x_j, y_j\right), z_{ij}\right) \\ &\text{subject to} \sum_{j=1}^{p} z_{ij} \geq b_i,\ i \in I, z_{ij} \geq 0, (x_j, y_j) \in R^2,\ i \in I,\ j \in J. \end{aligned} \tag{9.27}$$

Problem (9.27) is a constrained nonlinear optimization problem with $2p+pq$ variables x_j, y_j, z_{ij}. We assume that functions f_i have the following simple form:

$$f_i\left(d_{ij}\left(x_j, y_j\right), z_{ij}\right) = \alpha_i.d_{ij}\left(x_j, y_j\right).z_{ij},$$

where $\alpha_i > 0$ are constant coefficients, and distance function d_{ij} is defined as

$$d_{ij}\left(x_j, y_j\right) = \left(\left(u_i - x_j\right)^{\beta} + \left(v_i - y_j\right)^{\beta}\right)^{\frac{1}{\beta}},$$

where $i \in I$, $j \in J$ and $\beta > 0$. Problem (9.27), even in the above simple form, is difficult to solve numerically because of its possible nonlinearities which bring numerous local optima.

In order to transform problem (9.27) in a moretractable form, we consider the objective function as a utility or satisfaction function μ, such that $\mu : \mathbf{R}^{2p} \times \mathbf{R}^q \to [0,1]$. In such a case

$$\mu\left((x_1; y_1), \ldots, (x_p; y_p), b_1, \ldots, b_q\right) = 0$$

denotes the total dissatisfaction (or, zero utility) with location $(x_j, y_j) \in \mathbf{R}^2$, $j \in J$, and supplied amounts $b_i, i \in I$. On the other hand,

$$\mu((x_1; y_1), \ldots, (x_p; y_p), b_1, \ldots, b_q) = 1$$

denotes the maximal total satisfaction (or, maximal utility) with location $(x_j; y_j) \in \mathbf{R}^2$, $j \in J$ and supplied amounts.

Depending on the required amount b_i, an individual consumer $i \in I$ may express his satisfaction with the supplier $j \in J$ located at (x_j, y_j) by membership grade $\mu_{ij}\left(x_j, y_j, b_i\right)$, where membership function $\mu_{ij} : \mathbf{R}^2 \times \mathbf{R}^1 \to [0,1]$ satisfies condition $\mu_{ij}\left(u_i, v_i, b_i\right) = 1$, i.e. the maximal satisfaction is equal to 1, provided that the facility $j \in J$ is located at the same place as the consumer $i \in I$.

The individual satisfaction expressed by the function $\mu_i : \mathbf{R}^{2p} \times \mathbf{R}^1 \to [0,1]$ of the consumer $i \in I$ with the amount b_i, and with suppliers located at $(x_1; y_1), (x_2; y_2), \ldots, (x_p; y_p)$, is defined by the satisfaction of the location of the facility with maximal value, i.e.:

$$\mu_i\left((x_1; y_1), \ldots, (x_p; y_p), b_i\right) = \max\{\mu_{i1}\left(x_1, y_1, b_i\right), \ldots, \mu_{ip}\left(x_p, y_p, b_i\right)\}.$$

More generaly, we can apply a compensative aggregation operator with the values inbetween max and min. The supplier $j \in J$ with the maximal grade of satisfaction $\mu_{ij}\left(x_j, y_j, b_i\right)$ will cover the required amount b_i. If there are more such suppliers, then they share amount b_i equally. The above formula can be generalized by using an aggregation operator A for all $i \in I$ as follows

$$\mu_i\left((x_1, y_1), \ldots, (x_p, y_p), b_i\right) = A(\mu_{i1}\left(x_1, y_1, b_i\right), \ldots, \mu_{ip}\left(x_p, y_p, b_i\right)). \quad (9.28)$$

Moving the location point of a supplier along the path from a location (x, y) toward the location site of consumer i at $\mathbf{c}_i = (u_i; v_i)$, it is natural to assume that satisfaction grade of consumer i is increasing, or, at least non-decreasing, provided that b_i is constant. This assumption results in (Φ,Ψ)-concavity (e.g. T-quasiconcavity) requirement of membership functions μ_{ij} on $\mathbf{R}^2$.

On the other hand, with the given location (x_j, y_j) of supplier j, satisfaction grade $\mu_{ij}\left(x_j, y_j, b\right)$ is nonincreasing in variable b.

We obtain $\mathbf{c}_i = (u_i; v_i) \in \bigcap_{j\in J} Core(\mu_{ij})$. If all μ_{ij} are T-quasiconcave on $\mathbf{R}^2$, then by Proposition 96, individual satisfaction μ_i of consumer i, is upper starshaped on $\mathbf{R}^{2p}$.

The total satisfaction with (or, utility of) locations $(x_j; y_j) \in \mathbf{R}^2$, $j \in J$, and required amounts b_i, $i \in I$, is defined as an aggregation of the individual satisfaction grades, e.g. a minimal satisfaction, or, more generally, the value of an aggregation operator G, e.g. a t-norm T.

$$\begin{aligned}\mu\left((x_1; y_1), \cdots, (x_p; y_p), b_1, \cdots, b_q\right) &= G(A(\mu_{11}\left(x_1, y_1, b_1\right), \ldots, \mu_{1p}(x_p, y_p, b_1)),\\ \ldots, A(\mu_{q1}\left(x_1, y_1, b_q\right), \ldots, \mu_{qp}(x_p, y_p, b_q))). & \qquad (9.29)\end{aligned}$$

Now, we have the transformed unconstrained problem of optimal location:

$$\text{maximize } (9.29), \text{ subject to } (x_j; y_j) \in \mathbf{R}^2, j \in J. \quad (9.30)$$

As a problem of unconstrained optimization, (9.30) can be numerically more easily tractable than the original problem (9.27). Notice also that problem (9.30) has only $2p$ variables, whereas problem (9.27) has $2p + pq$ variables.

We illustrate the above approach in the following two numerical examples.

Example 106

Consider the location problem with $q = 3$ consumers and $p = 1$ supplier given by $(u_1, v_1) = (0, 1)$, $(u_2, v_2) = (0, 2)$, $(u_3, v_3) = (3, 0)$, $b_1 = 10$, $b_2 = 20$, $b_3 = 30$.

First, let us deal with the classical problem (9.27) with $\alpha_i = 1$, $i = 1, 2, 3$, $\beta = 2$. Then the problem becomes that of minimizing:

$$f(x, y, z_1, z_2, z_3) = z_1(x^2 + (1-y)^2)^{\frac{1}{2}} + z_2(x^2 + (2-y)^2)^{\frac{1}{2}} + z_3\left((x-3)^2 + y^2\right)^{\frac{1}{2}},$$

subject to $z_1 \geq 10, z_2 \geq 20, z_3 \geq 30, (x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found as $(x^C, y^C, z_1^C, z_2^C, z_3^C) = (3, 0, 10, 20, 30)$, the minimal cost is

$$f\left(x^C, y^C, z_1^C, z_2^C, z_3^C\right) = 103.73.$$

Applying the alternative approach, we assume that the individual satisfaction of consumer i, located at $\mathbf{c}_i = (u_i, v_i)$ with location of the supplier at (x, y) and with demand b_i, is given by the following membership (satisfaction) function:

$$\mu_i(x, y, b_i) = \frac{1}{1 + b_i((x - u_i)^2 + (y - v_i)^2)^{\frac{1}{2}}}.$$

We shall investigate the problem with two different aggregation operators, particularly, t-norms.

If we consider the minimum t-norm, i.e. $G(u, v) = T_M(u, v) = \min\{u, v\}$, then for

$$\begin{aligned}
\mu_1(x, y, 10) &= \tfrac{1}{1+10(x^2+(y-1)^2)^{\frac{1}{2}}}, \\
\mu_2(x, y, 20) &= \tfrac{1}{1+20(x^2+(y-2)^2)^{\frac{1}{2}}}, \\
\mu_3(x, y, 30) &= \tfrac{1}{1+30((x-3)^2+y^2)^{\frac{1}{2}}},
\end{aligned}$$

we solve the maximum satisfaction problem:
maximize

$$\mu_M(x, y, 10, 20, 30) = \min\{\mu_1(x, y, 10), \mu_2(x, y, 20), \mu_3(x, y, 30)\},$$

subject to $(x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found as $(x^M, y^M) = (1.8, 0.8)$ with the optimal membership function value

$$\mu_M(1.8, 0.8, 10, 20, 30) = 0.023,$$

see Fig. 9.4.

Figure 9.4:

Comparing with the optimal solution of the classical problem, here the membership function value is

$$\mu_M(x^C, y^C, 10, 20, 30) = 0.013.$$

On the other hand, the cost is $f\left(x^M, y^M, 10, 20, 30\right) = 104.64$.

Now, as an aggregation operator we consider the product t-norm, i.e. $G(u,v) = T_P(u,v) = u \cdot v$. We solve the maximum satisfaction problem:

maximize

$$\mu_P(x, y, 10, 20, 30) = \mu_1(x, y, 10) \cdot \mu_2(x, y, 20) \cdot \mu_3(x, y, 30),$$

subject to $(x, y) \in \mathbf{R}^2$.

The optimal location of the supplier has been found as $(x^P, y^P) = (0, 1)$ with the optimal membership function value

$$\mu_P(0, 1, 10, 20, 30) = 0.0005,$$

see Fig. 9.5.

Figure 9.5:

Moreover, we get

$$\mu_p(x^C, y^C, 10, 20, 30) = 0.0004$$

and

$$\mu_P(x^M, y^M, 10, 20, 30) = 0.00002.$$

The cost of this solution is

$$f\left(x^P, y^P, 10, 20, 30\right) = 114.87.$$

The obtained results are summarized in the following table.

	x	y	f	μ_M	μ_P
1.	3.0	0.0	103.7	0.01	0.00041
2.	1.8	0.8	104.6	0.02	0.00002
3.	0.0	1.0	114.9	0.01	0.00050

In the above table we can see differences between the results of the individual approaches. In Row 1., the results of solving the classical problem (9.27) are displayed. In Row 2., the solution of maximum satisfaction problem with the aggregation operators being minimum t-norm and maximum t-conorm is presented. In Row 3., the results of the same problem with the aggregation operators being product t-norm and t-conorm are given. Depending both on input information and decision-making requirements, various locations of the supplier may be optimal.

Example 107

Consider the same problem as in Example 106 with $p = 2$, i.e. with two suppliers to be optimally located.

Again, we begin with the classical problem (9.27) with $a_i = 1$, $i = 1, 2, 3$, $\beta = 2$. Then the problem to solve is to minimize the cost function:

$$f\left(x_1, y_1, x_2, y_2, z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\right) \tag{9.31}$$

$$= z_{11}(x_1^2 + (1 - y_1)^2)^{\frac{1}{2}} + z_{12}(x_2^2 + (1 - y_2)^2)^{\frac{1}{2}} + z_{21}(x_1^2 + (2 - y_1)^2)^{\frac{1}{2}}$$

$$+z_{22}(x_2^2 + (2 - y_2)^2)^{\frac{1}{2}} + z_{31}((x_1 - 3)^2 + y_1^2)^{\frac{1}{2}} + z_{32}((x_2 - 3)^2 + y_2^2)^{\frac{1}{2}},$$

$$\text{subject to} \qquad z_{11} + z_{12} \geq 10, z_{21} + z_{22} \geq 20, z_{31} + z_{32} \geq 30,$$

$$z_{ij} \geq 0, (x_j, y_j) \in \mathbf{R}^2, i = 1, 2, 3, j = 1, 2.$$

The optimal solution, i.e. the locations of the facilities and shipment amounts have been found as

$$\left(x_1^C, y_1^C, x_2^C, y_2^C, z_{12}^C, z_{21}^C, z_{22}^C, z_{31}^C, z_{32}^C\right) = (0, 2, 3, 0, 10, 0, 20, 0, 0, 30),$$

whereas the minimal cost is

$$f\left(0, 2, 3, 0, 10, 0, 20, 0, 0, 30\right) = 10.0.$$

Applying our approach, the individual satisfaction of consumer $i \in I$ located at (u_i, v_i) with location of the facility at (x_j, y_j), $j \in J = \{1, 2\}$, and with demand b_i, is given by the membership (satisfaction) function:

$$\mu_{ij}(x_j, y_j, b_i) = \frac{1}{1+b_i((x_j-u_i)^2+(y_j-v_i)^2)^{\frac{1}{2}}}. \tag{9.32}$$

We investigate the problem again with two different aggregation operators, particularly, t-norms and t-conorms.

First, the aggregation operator $S_M = \max$ is used for aggregating the suppliers, $T_M = \min$ is used for combining consumers. According to (9.30) we solve the optimization problem:

maximize

$$\mu_M((x_1, y_1), (x_2, y_2), 10, 20, 30) = \min\{\max_{j\in J}\{\mu_{1j}(x_j, y_j, 10)\},$$
$$\max_{j\in J}\{\mu_{2j}(x_j, y_j, 20)\}, \max_{j\in J}\{\mu_{3j}(x_j, y_j, 30)\}\},$$

subject to $(x_1, y_1, x_2, y_2) \in \mathbf{R}^4$.

The optimal locations of the facilities has been computed as

$$(x_1^M, y_1^M, x_2^M, y_2^M) = (3, 0, 0, 5/3)$$

with the optimal membership value

$$\mu_M((3, 0), (0, 5/3), 10, 20, 30) = 0.130.$$

Notice that membership functions (9.32) are all T_M-quasiconcave on $\mathbf{R}^4$ and, by Proposition 98, μ_M is also T_M-quasiconcave on $\mathbf{R}^4$.

Second, the operator $S_P(u, v) = u + v - u \cdot v$ is used for aggregating the suppliers, $T_P(u, v) = u \cdot v$ is used for combining consumers. Again, by (9.30) we solve the maximum satisfaction problem:

maximize

$$\mu_P((x_1, y_1), (x_2, y_2), b_1, b_2, b_3)$$
$$= \prod(\mu_{i1}(x_1, y_1, b_i) + \mu_{i2}(x_2, y_2, b_i) - \mu_{i1}(x_1, y_1, b_i) \cdot \mu_{i2}(x_2, y_2, b_i)),$$

subject to $(x_1, y_1, x_2, y_2) \in \mathbf{R}^4$.

The optimal locations of the facilities have been found as

$$(x_1^P, y_1^P, x_2^P, y_2^P) = (3, 0, 0, 2)$$

with the optimal membership value

$$\mu_P((3, 0), (0, 2), 10, 20, 30) = 0.119,$$

being the same as the optimal solution of classical problem (9.31). The results are summarized in the following table.

	x_1	y_1	x_2	y_2	f	μ_M	μ_P
1.	3.0	0.0	0.0	2.0	10.0	0.090	0.119
2.	3.0	0.0	0.0	1.67	13.3	0.130	0.022
3.	3.0	0.0	0.0	2.0	10.0	0.090	0.119

In the above table we can see the differences between the results of the individual problems. Solving classical problem we obtain the same results as in the maximum satisfaction problem with the aggregation operators being the product t-norm T_P and t-conorm S_P. Notice again, that membership functions (9.32) are T_P-quasiconcave on $\mathbf{R}^4$ and by Proposition 98 μ_P is T_P-quasiconcave on $\mathbf{R}^4$.

9.9 Application in Engineering Design

Innovative product development requires high quality and resource-efficient engineering design. Traditionally, it is common for engineers to evaluate promising design alternatives one by one. Such an approach does not consider the nature of imprecision of the design process and leads to expensive design computations. At the stage where technical solution concepts are being generated, the description of a design is largely vague or imprecise. The need for a methodology to represent and manipulate imprecision is greatest in the early, preliminary phases of engineering design, where the designer is most unsure of the final dimensions and shapes, material and properties and performance of the completed design, see [2].

Because design imprecision concerns the choice of design variable values used to describe an product or process, the designer's preference is used to quantify the imprecision with which design variables are known. Preferences, modelled here by quasiconcave membership functions, denote either subjective or objective information that may be quantified and included in the evaluation of design alternatives.

Each design variable is characterized by the membership function

$$\mu_i : \mathbf{R}^{k_i} \to [0,1], i \in I = \{1,2,...,n\},$$

where the value of the membership function $\mu_i(x_i)$ specifies the design preference of the design parameter x_i, its nature is possibly multi-dimensional, n is a number of variables. This preference function, which may arise objectively (e.g. cost of the parameter), or subjectively (e.g. from experience), is used to quantify the imprecision associated with the design variable. Thus the designer's experience and judgement are incorporated into the design evaluations. In practice, the design variables-preferences are divided into two groups: individual design preferences ($D = \{1,2,...,m\}$) and individual customers preferences - functional requirements ($P = \{m+1,...,n\}$).

In order to evaluate a design, the various individual preferences must be combined or aggregated to give a single, overall measure. This aggregation, in practice, occurs in two stages, see [2].

First, the individual design preferences μ_i, $i \in D$, are aggregated into the combined design preference μ_D by an aggregation operator A_D, and the customer individual preferences μ_i, $i \in P$, are aggregated by an aggregating operator A_P into the combined design preference μ_P.

Second, once the all preferences are aggregated according to its nature into two resulting preferences μ_D and μ_P, we combine them by an aggregating operator A_O to obtain an overall preference μ_O from the following formula:

$$\mu_O = A_O(\mu_D, \mu_P) = A_O(A_D(\mu_1, ..., \mu_m), A_P(\mu_{m+1}, ..., \mu_n)). \tag{9.33}$$

where $A_D : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \times \cdots \times \mathbf{R}^{k_m} \to [0,1]$, $A_P : \mathbf{R}^{k_{m+1}} \times \mathbf{R}^{k_{m+2}} \times \cdots \times \mathbf{R}^{k_n} \to [0,1]$, $A_O : \mathbf{R} \times \mathbf{R} \to [0,1]$ are aggregation operators. In the definition it is required that each aggregation operator A is monotone and satisfies the boundary conditions $A(0,0,...,0) = 0$, and $A(1,1,...,1) = 1$. In engineering design it is often required that aggregating operators are continuous and idempotent. The last condition restricts the class of feasible aggregation operators to the operators between the t-norm T_M and t-conorm S_M.

For some preferences of the system of engineering design, for example, where the failure of one component results in the failure of the whole system, the non-compensating aggregation operators such as minimum T_M should be applied. On the other hand, a better performance of some component can compensate some worse performance of another one. In other words, a lower membership value of some design variable can be compensated by a higher value of some other one. Such preferences can be aggregated by compensative aggregation operators, e.g. averaging operators, see [79]. Notice that by the definition, the t-norm T_M is considered also as a compensative operator, however in some other sense.

The problem of engineering design is to find an optimal configuration of the design parameters, i.e.:

Maximize

$$A_O(A_D(\mu_1(x_1), ..., \mu_m(x_m)), A_P(\mu_{m+1}(x_{m+1}), ..., \mu_n(x_n))). \tag{9.34}$$

We illustrate this approach on a simple example.

Example 108 Car design

Consider 4 design variables, 2 of them areindividual preferences:

μ_1- maximal speed,

μ_2 - time to reach 100km/hour,

defined as follows, see Fig. 9.6 and Fig. 9.7:

$$\mu_1(x) = \{ \begin{array}{ll} \frac{1}{1+0.25(160-x)} & \text{for } 0 \le x \le 160, \\ 1 & \text{for } x > 160. \end{array}$$

$$\mu_2(y) = \{ \begin{array}{ll} 1 & \text{for } 0 \le y \le 7, \\ \frac{1}{1+0.3(7-y)} & \text{for } y > 7. \end{array}$$

We consider compensative aggregation operator - Product t-norm T_P:

$$\mu_D(x,y) = A_D(\mu_1(x), \mu_2(y)) = T_P(\mu_1(x), \mu_2(y)),$$

particularly,

$$\mu_D(x,y) = \{ \begin{array}{ll} \frac{1}{1+0.25(160-x)} \cdot \frac{1}{1+0.3(y-7)} & \text{for } 0 \le x \le 160,\ y \ge 7, \\ 1 & \text{for } x \ge 160,\ 0 \le y \le 7, \end{array}$$

see Fig. 9.8.

Notice, that μ_D is a starshaped function which is not quasiconcave on $\mathbf{R}^2_+$.

Further, we consider 2 customer preferences:

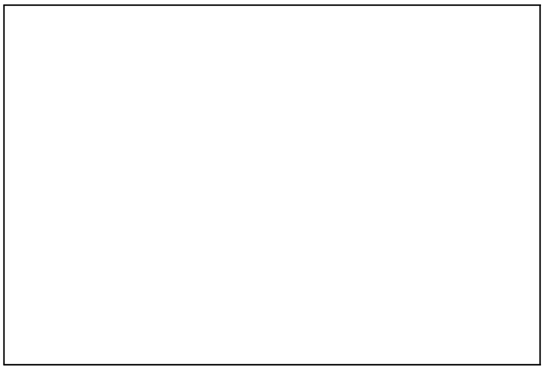

Figure 9.6:

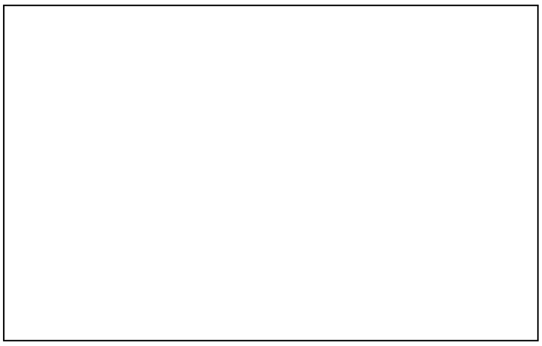

Figure 9.7:

μ_3- price of the car in 1000 \$,
μ_4 - fuel consumption in liter/100 km,
defined as follows:

$$\mu_3(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 8, \\ \frac{1}{1+0.2(u-8)} & \text{for } u > 8. \end{cases}$$

$$\mu_4(v) = \begin{cases} 1 & \text{for } 0 \leq v \leq 4, \\ \frac{1}{1+0.1(v-4)} & \text{for } v > 4. \end{cases}$$

Moreover, technological and economical functional dependences are given by the following (regression) model:

$$\begin{aligned} x \leq & \ \frac{400}{y} + 120, \\ u = & \ 0.03x - 0.3y + \frac{175}{y}, \\ v = & \ 0.05x - \frac{10}{y} - 2.5. \end{aligned}$$

Figure 9.8:

For an aggregation operator A_P we apply a compensative operator geometric average G, i.e.

$$\mu_P(u, v) = A_P(\mu_3(u), \mu_4(v)) = G(\mu_3(u), \mu_4(v)).$$

Finally, μ_D and μ_P, have been combined by an aggregating operator

$$A_O = T_M,$$

to obtain the overall aggregation:

$$\mu_O(x, y, u, v) = T_M(\mu_D(x, y), \mu_P(u, v)).$$

The optimal configuration of the parameters has been found by Conjugate Gradient optimization method as follows:

$$x^* = 159.8,\ y^* = 10.1,\ u^* = 19.2,\ v^* = 6.5,\ A_O^* = 0.497.$$

Chapter 10

Fuzzy Mathematical Programming

10.1 Introduction

Mathematical programming problems (MP) form a subclass of decision - making problems where preferences between alternatives are described by means of objective function(s) defined on the set of alternatives in such a way that greater values of the function(s) correspond to more preferable alternatives (if "higher value" is "better"). The values of the objective function describe effects from choices of the alternatives. In economic problems, for example, these values may reflect profits obtained when using various means of production. The set of feasible alternatives in MP problems is described implicitly by means of constraints - equations or inequalities, or both - representing relevant relationships between alternatives. In any case the results of the analysis using given formulation of the MP problem depend largely upon how adequately various factors of the real system are reflected in the description of the objective function(s) and of the constraints.

Descriptions of the objective function and of the constraints in a MP problem usually include some parameters. For example, in problems of resources allocation such parameters may represent economic parameters like costs of various types of production, labor costs requirements, shipment costs, etc. The nature of these parameters depends, of course, on the detailization accepted for the model representation, and their values are considered as data that should be exogenously used for the analysis.

Clearly, the values of such parameters depend on multiple factors not included into the formulation of the problem. Trying to make the model more representative, we often include the corresponding complex relations into it, causing that the model becomes more cumbersome and analytically unsolvable. Moreover, it can happen that such attempts to increase "the precision" of the model will be of no practical value due to the impossibility of measuring the parameters accurately. On the other hand, the model with some fixed values of its parameters may be too crude, since these values are often chosen in a quite arbitrary way.

An intermediate approach is based on introduction into the model the means of a more adequate representation of experts´understanding of the nature of the parameters in the form of fuzzy sets of their possible values. The resultant model, although not taking into account many details of the real system in question could be a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. In this way we obtain a new type of MP problems containing fuzzy parameters. Treating such problems requires the application of fuzzy-set-theoretic tools in a logically consistent manner. Such treatment forms an essence of *fuzzy mathematical programming* (FMP) investigated in this chapter.

FMP and related problems have been extensively analyzed and many papers have been published displaying a variety of formulations and approaches. Most approaches to FMP problems are based on the straightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function. This approach has been mentioned by Bellman and Zadeh already in their paper [9] published in the early seventies. Later on many papers have been devoted to the problem of mathematical programming with fuzzy parameters, known under different names, mostly as fuzzy mathematical programming, but sometimes as possibilistic programming, flexible programming, vague programming, inexact programming etc. For an extensive bibliography see the overview paper [33].

Here we present a general approach based on a systematic extension of the traditional formulation of the MP problem. This approach is based on the numerous former works of the author of this study, see [60] - [81] and also on the works of many other authors, e.g. [38, 23, 19, 2, 43, 41, 69, 70], and others.

FMP is one of more possible approaches how to treat uncertainty in MP problems. Much space has been devoted to similarities and dissimilarities of FMP and stochastic programming (SP), see e.g. [89] and [90]. In Chapters 10 and 11 we demonstrate that FMP (in particular, fuzzy linear programming - FLP) essentially differes from SP; FMP has its own structure and tools for investigating a broad class of optimization problems.

FMP is also different to parametric programming (PP). PP problems are in essence deterministic optimization problems with a special variable called a parameter. The main interest in PP is focused on finding relationships between the values of parameters and optimal solutions of MP problem.

In FMP some methods and approaches motivated by SP and PP are utilized, see e.g. [85, 13]. In this book, however, algorithms and solution procedures for MP problems are not studied, they can be found elsewhere, see e.g. the overview paper [83].

10.2 Modelling Reality by FMP

An alternative approach to classical MP is based on the introduction into the model the means of a more adequate representation of the parameters in the form of fuzzy sets was named FMP. By applying FMP on real problems, one obtains a mathematical model which, although not taking into account many details of the real system in question, could be a more adequate representation of the reality than that with more or less arbitrarily chosen values of its parameters. In this way we obtain a new type of MP problems containing fuzzy parameters.

As it was mentioned in [83], the use of FMP models does not only avoid unrealistic modeling, but also offers a chance for reducing information costs. Then, in the first step of the interactive solution process, the fuzzy system is modeled by using ony the information which the decision maker can provide without any expensive acquisition so as to obtain an initial compromise solution. Then the decion maker can percieve which further information would be required and is able to justify additional information costs.

An appropriate treatment of such problems requires proper application of fuzzy-set-theoretic tools in a logically consistent manner. An important role in this treatment is played by *generalized concave membership functions*. Such approach forms the essence of FMP investigated in this chapter. The following explanation is based on the substance investigated formerly in [79] and, particularly, in Chapter 8.

In this chapter we begin with the formulation a FMP problem associated with the classical MP problem. After that we define a feasible solution of FMP problem and optimal solution of FMP problem as special fuzzy sets. From practical point of view, α-cuts of these fuzzy sets are important, particularly the α-cuts with the maximal α. The main result of this part says that the class of all MP problems with (crisp) parameters can be naturally embedded into the class of FMP problems with fuzzy parameters.

10.3 MP Problem with Parameters

The classical constrained optimization problem is given as follows

$$\begin{aligned} &\text{maximize } \ f(x) \\ &\text{subject to } x \in X, \end{aligned} \tag{10.1}$$

where we assume that:

(i) The set X is a nonempty subset of $\mathbf{R}^n$, n is a positive integer, and is called the *set of feasible solutions (set of alternatives)*.

(ii) The function f is a real function, $f : \mathbf{R}^n \to \mathbf{R}$, called the *objective function (criterion function)*.

An *optimal solution* of problem (10.1) is a vector $x^* \in X$ which maximizes the function f on X. The set of all optimal solution is denoted by X^*. We have

$$X^* = \{x^* \in X | f(x^*) = \sup\{f(x)|x \in X\}\}. \tag{10.2}$$

Observe that

$$X^* = \bigcap_{y \in X} \{x \in X | f(x) \geq f(y)\}. \tag{10.3}$$

Usually, the set X has a structure specified by equalities and inequalities including some parameters. We distinguish the individual constraints according to the sense of inequality, i.e. we set $\mathcal{M}_1 = \{1, ..., m_1\}$, $\mathcal{M}_2 = \{m_1 + 1, ..., m_2\}$, $\mathcal{M}_3 = \{m_2 + 1, ..., m\}$ with $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_1 \cup \mathcal{M}_1$. Here, m_1, m_2 and m are nonnegative integers with $0 \leq m_1 \leq m_2 \leq m$. If $m_1 = 0$, then $\mathcal{M}_1 = \emptyset$. Similarly, if $m_1 = m_2$ or $m_2 = m$, then $\mathcal{M}_2 = \emptyset$ or $\mathcal{M}_3 = \emptyset$, respectively. Moreover, denote $\mathcal{N} = \{1, 2, ..., n\}$.

Consider the following MP problem:

$$\begin{array}{ll} \text{maximize} & f(x; c) \\ \text{subject to} & \\ & g_i(x; a_i) = b_i,\ i \in \mathcal{M}_1, \\ & g_i(x; a_i) \leq b_i,\ i \in \mathcal{M}_2, \\ & g_i(x; a_i) \geq b_i,\ i \in \mathcal{M}_3. \end{array} \tag{10.4}$$

In (10.4) f, g_i are real functions, $\mathbf{C}$ and $\mathbf{P}_i$ are sets of parameters, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, $i \in \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$, $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$, $b_i \in \mathbf{R}$, $x \in \mathbf{R}^n$.

The maximization in (10.4) is understood in the usual sense of finding a maximizer of the objective function on the set of feasible solutions (10.4).

The sets of parameters $\mathbf{C}$ and $\mathbf{P}_i$ are some subsets of finite dimensional vector spaces, depending on the specification of the problem. Particularly, $\mathbf{C} = \mathbf{P}_i = \mathbf{R}^n$ for all $i \in \mathcal{M}$, but here we consider also a more general case. The right-hand sides $b_i \in \mathbf{R}$, for $i \in \mathcal{M}$ in (10.4) are also considered as parameters. By parameters $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$, $b_i \in \mathbf{R}$, taken from the parameter sets, a flexible structure of MP problem (10.4) is modelled. The subject of *parametric programming* is to investigate relations and dependences between parameters and optimal solutions of MP problem (10.4). This problem is, however, not studied here.

A *linear programming problem* (LP) is a particular case of the above formulated MP problem (10.4), where $c \in \mathbf{C} \subset \mathbf{R}^n$, $a_i \in \mathbf{P}_i \subset \mathbf{R}^n$ and $b_i \in \mathbf{R}$ for all $i \in \mathcal{M}$, that is

$$f(x; c) = c^T x = c_1 x_1 + \cdots + c_n x_n, \tag{10.5}$$

$$g_i(x; a_i) = a_i^T x = a_{i1} x_1 + \cdots + a_{in} x_n, i \in \mathcal{M}. \tag{10.6}$$

As a special case of this problem, we have the standard linear programming problem:

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & \\ & a_{i1} x_1 + \cdots + a_{in} x_n = b_i,\ i \in \mathcal{M}, \\ & x_j \geq 0,\ j \in \mathcal{N}. \end{array} \tag{10.7}$$

Problem (10.7) in a more general setting will be investigated in Chapter 11.

10.4 Formulation of FMP Problem

Before formulating a fuzzy mathematical programming problem as an optimization problem associated with the MP problem (10.4), we make a few assumptions and remarks. We use notation introduced in Chapter 8.

Let f, g_i be functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{C}$ and $\mathbf{P}_i$ are sets of parameters. Let $\mu_{\tilde{c}} : \mathbf{C} \to [0,1]$, $\mu_{\tilde{a}_i} : \mathbf{P}_i \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, be membership functions of fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations with the corresponding membership functions $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1]$. They will be used for comparing the left and right sides of the constraints in (10.7).

The maximization of objective function (10.5) needs, however, a special treatment. Generally, fuzzy values of the objective function are not linearly ordered and to maximize the objective function we have to define a suitable ordering on $\mathcal{F}(\mathbf{R})$ which allows for "maximization" of the objective. In our approach it will be done by an exogenously given fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and another fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$. There exist some other approaches, see e.g. [18], [23], [60].

The *fuzzy mathematical programming problem* (FMP problem) associated with MP problem (10.4) is denoted by:

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{f}(x;\tilde{c}) \\ \text{subject to} & \\ & \tilde{g}_i(x;\tilde{a}_i) \;\; \tilde{R}_i \; \tilde{b}_i, \, i \in \mathcal{M} = \{1,2,...,m\}. \end{array} \tag{10.8}$$

Here, in comparison with classical MP problem (10.4), fuzzy parameters c, a_i and b_i are denoted with the upper "wavelet". Formulation (10.8) is not an optimization problem in a classical sense, as it is not yet defined, how the objective function $\tilde{f}(x;\tilde{c})$ is "maximized", and how the constraints $\tilde{g}_i(x;\tilde{a}_i)$ $\tilde{R}_i$ $\tilde{b}_i$ are satisfied. In fact, we need a new concept of a "feasible solution" and also that of "optimal solution", some counterparts to the concepts used in classical MP.

Let us clarify the elements of (10.8).

Remember that for given $x \in \mathbf{R}^n, \tilde{a}_i \in \mathcal{F}(\mathbf{P}_i)$ by extension principle (8.17), $\tilde{g}_i(x;\tilde{a}_i)$ is a fuzzy extension of $g_i(x;\cdot)$ with the membership function defined by

$$\mu_{\tilde{g}_i(x;\tilde{a}_i)}(t) = \{ \begin{array}{ll} \sup\{\mu_{\tilde{a}_i}(a) | a \in \mathbf{P}_i, g_i(x;a) = t\} & \text{if } g_i^{-1}(x;t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{array} \tag{10.9}$$

for each $t \in \mathbf{R}$, where $g_i^{-1}(x;t) = \{a \in \mathbf{P}_i | g_i(x;a) = t\}$.

The fuzzy relations $\tilde{R}_i$ for comparing the constraints of (10.8) will be considered as extensions of valued relations on $\mathbf{R}$, particularly, the usual inequality relations "$\leq$" and "$\geq$". In a special case, namely, if T is a t-norm and $\tilde{R}_i$ is a T-fuzzy extension of relation R_i, then by (8.27) we obtain the membership function of the i-th constraint of (10.8) as follows

$$\begin{aligned}\mu_{\tilde{R}_i}\left(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i\right) &= \sup\{T\left(\mu_{R_i}(u,v),T(\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u),\mu_{\tilde{b}_i}(v))\right)|u,v\in\mathbf{R}\}\\ &= \sup\{T\left(\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u),\mu_{\tilde{b}_i}(v)\right)|uR_iv\}. \qquad (10.10)\end{aligned}$$

Likewise, for a given $x\in\mathbf{R}^n,\tilde{c}\in\mathcal{F}(\mathbf{C})$, $\tilde{f}(x;\tilde{c})$ is a fuzzy extension of $f(x;\cdot)$ given by the membership function defined as follows:

$$\mu_{\tilde{f}(x;\tilde{c})}(t)=\begin{cases}\sup\{\mu_{\tilde{c}}(c)|c\in\mathbf{C},f(x;c)=t\} & \text{if } f^{-1}(x;t)\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases} \qquad (10.11)$$

for each $t\in\mathbf{R}$, where $f^{-1}(x;t)=\{c\in\mathbf{C}|f(x;c)=t\}$.

Therefore, $\tilde{g}_i(x;\tilde{a}_i)\in\mathcal{F}(\mathbf{R})$ and this fuzzy quantity is "compared" with the fuzzy quantity $\tilde{b}_i\in\mathcal{F}(\mathbf{R})$ by a fuzzy relation $\tilde{R}_i$, $i\in\mathcal{M}$. For $x,y\in\mathbf{R}^n$ we obtain $\tilde{f}(x;\tilde{c})\in\mathcal{F}(\mathbf{R})$, $\tilde{f}(y;\tilde{c})\in\mathcal{F}(\mathbf{R})$. However, as has been mentioned earlier, fuzzy values are not linearly ordered. To deal with this problem, we assume the existence of a given additional goal $\tilde{b}_0\in\mathcal{F}(\mathbf{R})$ - a fuzzy set of the real line, which the fuzzy values $\tilde{f}(x;\tilde{c})$ of the objective function are compared to, by means of a fuzzy relation $\tilde{R}_0$, which is also assumed to be given exogenously. This approach is frequently used in the literature, see the overview paper [83]. The fuzzy objective is then treated as another constraint $\tilde{f}(x;\tilde{c})\ \tilde{R}_0\ \tilde{b}_0$, and $\widetilde{\text{maximize}}$ the objective function denotes finding the maximal membership degree of $\mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}),\tilde{b}_0\right)$.

The concepts of the feasible solution and optimal solution of FLP problem (10.8) need a more detailed explanation. Let us begin with the concept of the feasible solution.

10.5 Feasible Solutions of the FMP Problem

The fuzzy relation $\tilde{R}_i$ is considered to be either an extension of the usual equality relation "=" or inequality relations "$\leq$" and "$\geq$", see Examples 56, 57 in Chapter 8. Many authors, however, pointed out some disadvantages of T-fuzzy extensions of equality and inequality relations. Therefore, numerous special fuzzy relations for comparing left and right sides of constraints (10.8) have been proposed, see e.g. [2, 19, 23, 38, 43, 41, 69, 70, 54, 55, 44, 45]. Here, we shall use some extensions of the usual equality and inequality relations in the constraints of FMP (10.8), being not necessarily T-fuzzy extensions. In the following definition, for the sake of generality of the presentation, we consider fuzzy relations.

Definition 109 *Let g_i, $i\in\mathcal{M}=\{1,2,...,m\}$, be functions, $g_i:\mathbf{R}^n\times\mathbf{P}_i\to\mathbf{R}$, where $\mathbf{P}_i$ are given sets of parameters. Let $\mu_{\tilde{a}_i}:\mathbf{P}_i\to[0,1]$ and $\mu_{\tilde{b}_i}:\mathbf{R}\to[0,1]$ be membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{R}_i$, $i\in\mathcal{M}$, be fuzzy relations with the corresponding membership functions $\mu_{\tilde{R}_i}:\mathcal{F}(\mathbf{R})\times\mathcal{F}(\mathbf{R})\to[0,1]$, let A be an aggregation operator.*

A fuzzy subset $\tilde{X}$ of $\mathbf{R}^n$ given by the membership function $\mu_{\tilde{X}}$, for all $x \in \mathbf{R}^n$ defined as

$$\mu_{\tilde{X}}(x) = A\left(\mu_{\tilde{R}_1}\left(\tilde{g}_1(x;\tilde{a}_1),\tilde{b}_1\right),...,\mu_{\tilde{R}_m}\left(\tilde{g}_m(x;\tilde{a}_m),\tilde{b}_m\right)\right) \tag{10.12}$$

is called the feasible solution *of the FMP problem (10.8).*
For $\alpha \in (0,1]$, a vector $x \in [\tilde{X}]_\alpha$ is called the α-feasible solution *of the FMP problem (10.8).*
A vector $\bar{x} \in \mathbf{R}^n$ such that $\mu_{\tilde{X}}(\bar{x}) = Hgt(\tilde{X})$ is called the max-feasible solution.

Notice that the feasible solution of a FMP problem is a fuzzy set. For $x \in \mathbf{R}^n$ the interpretation of $\mu_{\tilde{X}}(x)$ depends on the interpretation of uncertain parameters of the FMP problem. For instance, within the framework of possibility theory, the membership functions of the parameters are explained as possibility degrees and $\mu_{\tilde{X}}(x)$ denotes the possibility that $x \in \mathbf{R}^n$ belongs to the set of feasible solutions of the corresponding FMP problem. Some other interpretations were also applied, see e.g. [14], [30] or [83].

On the other hand, α-feasible solution is a vector belonging to an α-cut of the feasible solution $\tilde{X}$ and the same holds for the max-feasible solution, which is a special α-feasible solution with $\alpha = Hgt(\tilde{X})$. If a decision maker specifies the grade of feasibility $\alpha \in [0,1]$ (the grade of possibility, satisfaction etc.), then a vector $x \in \mathbf{R}^n$ with $\mu_{\tilde{X}}(x) \geq \alpha$ is an α-feasible solution of the corresponding FMP problem.

Considering the i-th constraint of problem (10.8), for given $x, \tilde{a}_i$ and $\tilde{b}_i$, the value $\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i)$ from interval $[0,1]$ can be interpreted as the degree of satisfaction of this constraint.

For $i \in \mathcal{M}$, we use the following notation: by $\tilde{X}_i$ we denote a fuzzy set given by the membership function $\mu_{\tilde{X}_i}$, which is defined for all $x \in \mathbf{R}^n$ as

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i). \tag{10.13}$$

The fuzzy set $\tilde{X}_i$ can be interpreted as an i-th fuzzy constraint. All fuzzy constraints are aggregated into the feasible solution (10.12) by the aggregation operator A, usually, A is a t-norm, or $A = \min$. The aggregation operators have been thouroughly investigated in [79].

10.6 Properties of Feasible Solution

In this section we suppose that T is a t-norm, A is an aggregation operator, $\mathbf{P}_i$ are given sets of parameters, $\tilde{R}_i$ are fuzzy relations, $i \in \mathcal{M}$. In Theorem 110, stated below, the fuzzy relations $\tilde{R}_i$ are supposed to be fuzzy extensions of the usual binary relations on $\mathbf{R}$, but in the other theorems and propositions which will follow, we suppose a stronger condition, namely that $\tilde{R}_i$ are T-fuzzy extensions. For the sake of simplicity, we denote the relations only by $\tilde{R}_i$ and not by $\tilde{R}_i^T$ as it was originally introduced in Definition 27. The other five fuzzy extensions of the usual binary relations on $\mathbf{R}$ defined earlier in Definition 27 shall not be studied in this chapter. However, we shall use them again in Chapter 11.

Investigating the concept of the feasible solution (10.12) of the FMP problem (10.8), we first show that in case of crisp parameters a_i and b_i, the feasible solution is also crisp.

Theorem 110 *Let g_i be real functions, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, let $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be crisp parameters. For $i \in \mathcal{M}_1$, let $\tilde{R}_i$ be a fuzzy extension of the equality relation "$=$"; for $i \in \mathcal{M}_2$, let $\tilde{R}_i$ be a fuzzy extension of the inequality relation "$\leq$"; and for $i \in \mathcal{M}_3$, let $\tilde{R}_i$ be a fuzzy extension of the inequality relation "$\geq$". Let A be a t-norm.*
Then the feasible solution $\tilde{X}$ is a crisp set and coincides with the set of feasible solutions X of MP problem (10.4).

Proof. Let $x \in \mathbf{R}^n$ be an arbitrary vector, we show that

$$\mu_{\tilde{X}}(x) = \chi_X(x)$$

.

Observe first that by extension principle (8.17) we obtain

$$\mu_{\tilde{g}_i(x;a_i)} = \chi_{g_i(x;a_i)}$$

for all $i \in \mathcal{M}$.

Next, for all $i \in \mathcal{M}$ we obtain by (8.25)

$$\mu_{\tilde{R}_i}\left(g_i(x;a_i), b_i\right) = 1. \tag{10.14}$$

Notice that $X = \{x \in \mathbf{R}^n | g_i(x;a_i)R_ib_i, i \in \mathcal{M}\}$, where we write $g_i(x;a_i)R_ib_i$ instead of (10.14). Applying the t-norm A on (10.14), we obtain

$$\mu_{\tilde{X}}(x) = A\left(\mu_{\tilde{R}_1}\left(g_1(x;a_1), b_1\right), ..., \mu_{\tilde{R}_m}\left(g_m(x;a_m), b_m\right)\right) = \chi_X(x),$$

which is the desired result. ■

Let us remind that for two fuzzy subsets $\tilde{a}', \tilde{a}'' \in \mathcal{F}(\mathbf{R}^n)$, $\tilde{a}' \subset \tilde{a}''$ if and only if $\mu_{\tilde{a}'}(x) \leq \mu_{\tilde{a}''}(x)$ for all $x \in \mathbf{R}^n$, see Proposition 6. The following theorem proves some monotonicity of the feasible solution depending on the parameters of the FMP problem. In Theorem 110, we assumed that $\tilde{R}_i$ have been fuzzy extensions of the usual binary relations "$=$", "$\leq$" and "$\geq$". Here, we allow $\tilde{R}_i$ to be fuzzy extensions of more general valued relations.

Theorem 111 *Let g_i be real functions, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{P}_i$ are sets of parameters. Let $\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{a}''_i$, $\tilde{b}''_i$ be two collections of fuzzy parameters of the FMP problem. Let $\tilde{R}_i$ be T-fuzzy extensions of valued relations R_i on $\mathbf{R}$, $i \in \mathcal{M}$. Let T be a t-norm, A be an aggregation operator.*
If $\tilde{X}'$ is a feasible solution of the FMP problem with the collection of parameters

$\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{X}''$ is a feasible solution of the FMP problem with the collection of parameters $\tilde{a}''_i$, $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$

$$\tilde{a}'_i \subset \tilde{a}''_i \text{ and } \tilde{b}'_i \subset \tilde{b}''_i, \tag{10.15}$$

then

$$\tilde{X}' \subset \tilde{X}''. \tag{10.16}$$

Proof. In order to prove $\tilde{X}' \subset \tilde{X}''$, we first show that

$$\tilde{g}_i(x; \tilde{a}'_i) \subset \tilde{g}_i(x; \tilde{a}''_i)$$

for all $i \in \mathcal{M}$.

Indeed, by (8.17), for each $u \in \mathbf{R}$ and $i \in \mathcal{M}$,

$$\mu_{\tilde{g}_i(x;\tilde{a}'_i)}(u) = \max\{0, \sup\{\mu_{\tilde{a}'_i}(a) | a \in \mathbf{P}_i, g_i(x; a) = u\}\}$$
$$\leq \max\{0, \sup\{\mu_{\tilde{a}''_i}(a) | a \in \mathbf{P}_i, g_i(x; a) = u\}\} = \mu_{\tilde{g}_i(x;\tilde{a}''_i)}(u).$$

Now, since $\tilde{b}'_i \subset \tilde{b}''_i$, using monotonicity of T-fuzzy extension $\tilde{R}_i$ of R_i, it follows that $\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}'_i), \tilde{b}'_i) \leq \mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}''_i), \tilde{b}''_i)$. Then, applying monotonicity of A in (10.12), we obtain $\tilde{X}' \subset \tilde{X}''$. ■

Corollary 112 *Let $\tilde{a}_i$, $\tilde{b}_i$ be a collection of fuzzy parameters, and let $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be a collection of crisp parameters such that for all $i \in \mathcal{M}$*

$$\mu_{\tilde{a}_i}(a_i) = \mu_{\tilde{b}_i}(b_i) = 1. \tag{10.17}$$

If the set X of all feasible solutions of MP problem (10.4) with the parameters a_i and b_i is nonempty, and $\tilde{X}$ is a feasible solution of FMP problem (10.8) with fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$, then for all $x \in X$

$$\mu_{\tilde{X}}(x) = 1\ . \tag{10.18}$$

Proof. Observe that $a_i \subset \tilde{a}_i$, $b_i \subset \tilde{b}_i$ for all $i \in \mathcal{M}$. Then by Theorem 111 we obtain $X \subset \tilde{X}$, which is nothing else than (10.18). ■

Corollary 112 says that if we "fuzzify" the parameters of the original crisp MP problem, then the feasible solution of the new FMP problem "fuzzifies" the original set of all feasible solutions such that the membership grade of any feasible solution of the MP problem is eaual to 1.

So far, the parameters $\tilde{a}_i$ of the constraint functions g_i have been specified as fuzzy subsets of arbitrary sets $\mathbf{P}_i$, $i \in \mathcal{M}$. From now on, the space of parameters is supposed to be the k-dimensional Euclidean vector space $\mathbf{R}^k$, i.e., $\mathbf{P}_i = \mathbf{R}^k$ for all $i \in \mathcal{M}$, where k is a positive integer. Particularly, $\mu_{\tilde{a}_i} : \mathbf{R}^k \to [0, 1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0, 1]$ are the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$. We shall also require compactness of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ and closedness of the valued relations R_i. For the rest of this section we suppose that A and T are the minimum t-norms, i.e. $A = T = \min$. As a result, we obtain some formulae for α-feasible solutions of FMP problem based on α-cuts of the parameters. Remember that fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ are compact if $[\tilde{a}_i]_\alpha$ and $[\tilde{b}_i]_\alpha$ are compact for all $\alpha \in (0, 1]$.

Theorem 113 *Let g_i be continuous functions, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Let $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy subsets of $\mathbf{R}^k$ and $\mathbf{R}$, respectively. Let $\tilde{R}_i$ be T-fuzzy extensions of closed valued relations R_i on $\mathbf{R}$, $i \in \mathcal{M}$.*
Then for all $\alpha \in (0, 1]$

$$[\tilde{X}]_\alpha = \bigcap_{i=1}^{m} [\tilde{X}_i]_\alpha, \tag{10.19}$$

and, moreover, for all $i \in \mathcal{M}$

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}. \tag{10.20}$$

Proof. 1. Let $\alpha \in (0, 1]$, $i \in \mathcal{M}$, $x \in [\tilde{X}_i]_\alpha$. Then by (10.13) we have

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) \geq \alpha. \tag{10.21}$$

We prove first that (10.21) is valid if and only if

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x; \tilde{a}_i)]_\alpha, [\tilde{b}_i]_\alpha) \geq \alpha. \tag{10.22}$$

By definition we obtain

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) = \sup\{\min\left\{\mu_{R_i}(u, v), \min\left\{\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u), \mu_{\tilde{b}_i}(v)\right\}\right\} | u, v \in \mathbf{R}\}.$$

As $\tilde{g}_i(x; \tilde{a}_i)$ and $\tilde{b}_i$ are compact fuzzy sets and R_i is a closed valued relation, there exist $u^*, v^* \in \mathbf{R}$ such that

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x; \tilde{a}_i), \tilde{b}_i) = \min\left\{\mu_{R_i}(u^*, v^*), \min\left\{\mu_{\tilde{g}_i(x;\tilde{a}_i)}(u^*), \mu_{\tilde{b}_i}(v^*)\right\}\right\} \geq \alpha.$$

Hence,

$$\mu_{R_i}(u^*, v^*) \geq \alpha, \mu_{\tilde{g}_i(x;\tilde{a}_i)}(u^*) \geq \alpha, \mu_{\tilde{b}_i}(v^*) \geq \alpha. \tag{10.23}$$

On the other hand, by definition $\mu_{\tilde{R}_i}([\tilde{g}_i(x; \tilde{a}_i)]_\alpha, [\tilde{b}_i]_\alpha)$

$$= \sup\{\min\left\{\mu_{R_i}(u, v), \min\left\{\chi_{[\tilde{g}_i(x;\tilde{a}_i)]_\alpha}(u), \chi_{[\tilde{b}_i]_\alpha}(v)\right\}\right\} | u, v \in \mathbf{R}\}$$
$$= \sup\{\mu_{R_i}(u, v) | u \in [\tilde{g}_i(x; \tilde{a}_i)]_\alpha, v \in [\tilde{b}_i]_\alpha\}.$$

Therefore, by (10.23) we obtain

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x; \tilde{a}_i)]_\alpha, [\tilde{b}_i]_\alpha) \geq \alpha.$$

The opposite implication can be proved analogously. Hence, (10.21) is equivalent to (10.22).

Now, by Proposition 48 and Theorem 52, it follows that

$$[\tilde{g}_i(x; \tilde{a}_i)]_\alpha = g_i(x; [\tilde{a}_i]_\alpha). \tag{10.24}$$

Substituting (10.24) into (10.22), we obtain

$$\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha),[\tilde{b}_i]_\alpha) \geq \alpha,$$

a desirable result. We have just proven (10.20).

2. To prove (10.19), observe first that with $A = \min$ in (10.12), we have

$$\mu_{\tilde{X}}(x) = \min\left\{\mu_{\tilde{R}_1}\left(\tilde{g}_1(x;\tilde{a}_1),\tilde{b}_1\right),...,\mu_{\tilde{R}_m}\left(\tilde{g}_m(x;\tilde{a}_m),\tilde{b}_m\right)\right\}. \tag{10.25}$$

Let $x \in [\tilde{X}]_\alpha$, that is $\mu_{\tilde{X}}(x) \geq \alpha$. By (10.25) this is equivalent to

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i) \geq \alpha$$

for all $i \in \mathcal{M}$. Using the arguments of the first part of the proof, the last inequality is equivalent to $x \in [\tilde{X}_i]_\alpha$ for all $i \in \mathcal{M}$, or, in other words, $x \in \bigcap_{i=1}^{m}[\tilde{X}_i]_\alpha$. ■

Theorem 113 has some important computational aspects. Assume that in a FMP problem, we can specify a possibility (satisfaction) level $\alpha \in (0,1]$ and determine the α-cuts $[\tilde{a}_i]_\alpha$ and $[\tilde{b}_i]_\alpha$ of the fuzzy parameters. Then the formulae (10.19) and (10.20) will allow us to compute all α-feasible solutions of FMP problem without performing special computations of functions $\tilde{g}_i$.

If the valued relations R_i are binary relations similar to those in Theorem 110, then the statement of Theorem 113 can be strenghtened as follows.

Theorem 114 *Let g_i be continuous functions, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Let $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy subsets of $\mathbf{R}^k$ and $\mathbf{R}$, respectively. For $i \in \mathcal{M}_1$, let $\tilde{R}_i$ be a T-fuzzy extension of the equality relation "="; for $i \in \mathcal{M}_2$, let $\tilde{R}_i$ be a T-fuzzy extension of the inequality relation "$\leq$"; and for $i \in \mathcal{M}_3$, let $\tilde{R}_i$ be a T-fuzzy extension of the inequality relation "$\geq$".*
Then for all $\alpha \in (0,1]$

$$[\tilde{X}]_\alpha = \bigcap_{i=1}^{m}[\tilde{X}_i]_\alpha, \tag{10.26}$$

and, moreover, for all $i \in \mathcal{M}$

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha),[\tilde{b}_i]_\alpha) = 1\}. \tag{10.27}$$

Proof. 1. Let $\alpha \in (0,1]$, $i \in \mathcal{M}$, $x \in [\tilde{X}_i]_\alpha$. Then by (10.10) and (10.13) we have

$$\mu_{\tilde{R}_i}(\tilde{g}_i(x;\tilde{a}_i),\tilde{b}_i) \geq \alpha. \tag{10.28}$$

We prove first that (10.28) is valid if and only if

$$\mu_{\tilde{R}_i}([\tilde{g}_i(x;\tilde{a}_i)]_\alpha,[\tilde{b}_i]_\alpha) = 1. \tag{10.29}$$

In order to apply Corollary 64, for each $x \in \mathbf{R}^n$ and each $\alpha \in (0,1]$, $[\tilde{g}_i(x;\tilde{a}_i)]_\alpha$ should be compact. Indeed, this is true by Proposition 54. Then by Corollary 64, we obtain the equivalence between (10.28) and (10.29). Moreover, by Proposition 48 and Theorem 52, it follows that

$$[\tilde{g}_i(x;\tilde{a}_i)]_\alpha = g_i(x;[\tilde{a}_i]_\alpha). \tag{10.30}$$

Substituting (10.30) into (10.29), we obtain $\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha),[\tilde{b}_i]_\alpha) = 1$, a desirable result.

For the rest of the proof we can repeat the arguments of the corresponding part of the proof of Theorem 113. ■

Now, we shall see how the concept of generalized concavity introduced in [79] is utilized in the FMP problem. Particularly, we show that all α-feasible solutions (10.19), (10.20) are solutions of the system of inequalities on condition that the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ are upper-quasiconnected for all $i \in \mathcal{M}$.

For given $\alpha \in (0,1]$, $i \in \mathcal{M}$, we introduce the following notation

$$\begin{aligned}
\underline{G}_i(x;\alpha) &= \inf\{g_i(x;a) | a \in [\tilde{a}_i]_\alpha\}, && (10.31)\\
\overline{G}_i(x;\alpha) &= \sup\{g_i(x;a) | a \in [\tilde{a}_i]_\alpha\}, && (10.32)\\
\underline{b}_i(\alpha) &= \inf\{b \in \mathbf{R} | b \in [\tilde{b}_i]_\alpha\}, && (10.33)\\
\overline{b}_i(\alpha) &= \sup\{b \in \mathbf{R} | b \in [\tilde{b}_i]_\alpha\}. && (10.34)
\end{aligned}$$

Theorem 115 *Let all assumptions of Theorem 114 be satisfied. Moreover, let the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$ be upper-quasiconnected for all $i \in \mathcal{M}$.*
Then for all $\alpha \in (0,1]$, we have $x \in [\tilde{X}]_\alpha$ if and only if

$$\begin{aligned}
&\underline{G}_i(x;\alpha) \le \overline{b}_i(\alpha),\ i \in \mathcal{M}_1 \cup \mathcal{M}_2, && (10.35)\\
&\overline{G}_i(x;\alpha) \ge \underline{b}_i(\alpha), i \in \mathcal{M}_1 \cup \mathcal{M}_3, && (10.36)
\end{aligned}$$

Proof. Let $x \in [\tilde{X}]_\alpha$. By Theorem 114, this is equivalent to

$$\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha),[\tilde{b}_i]_\alpha) = 1, \tag{10.37}$$

for all $i \in \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. Moreover, by Proposition 48 and Theorem 52, it follows that

$$[\tilde{g}_i(x;\tilde{a}_i)]_\alpha = g_i(x;[\tilde{a}_i]_\alpha).$$

Since the membership functions of fuzzy parameters $\tilde{a}_i$ and $\tilde{b}_i$, $i \in \mathcal{M}$, are upper-quasiconnected, by Propositions 50 and 54, $[\tilde{g}_i(x;\tilde{a}_i)]_\alpha$ is closed and convex, i.e., it is a closed interval in $\mathbf{R}$. The rest of the proof follows from (10.31) - (10.34) and Theorem 111. ■

If we assume that the functions g_i satisfy some convexity and concavity requirements, then we can prove that the membership function $\mu_{\tilde{X}}$ of the feasible solution $\tilde{X}$ is quasiconcave, or, in other words, that $\tilde{X}$ is convex.

Theorem 116 *Let all assumptions of Theorem 114 be satisfied. Moreover, let g_i be quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, and g_i be quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_3$.*
Then for all $i \in \mathcal{M}$, $\tilde{X}_i$ are convex and therefore the feasible solution $\tilde{X}$ of FMP problem (10.8) is also convex.

Proof. 1. Let $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\alpha \in (0,1]$. We show that $[\tilde{X}_i]_\alpha$ is convex.
Let $x_1, x_2 \in [\tilde{X}_i]_\alpha$, $\lambda \in (0,1)$, put $y = \lambda x_1 + (1-\lambda)x_2$. Since g_i is quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$, then for all $(x_1, a_1) \in \mathbf{R}^n \times \mathbf{R}^k$, $(x_2, a_2) \in \mathbf{R}^n \times \mathbf{R}^k$, we have

$$g_i(\lambda x_1 + (1-\lambda)x_2, \lambda a_1 + (1-\lambda)a_2) \le \max\{g_i(x_1, a_1), g_i(x_2, a_2)\}. \tag{10.38}$$

By (10.20) we get

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}.$$

Hence, it remains only to show that

$$\mu_{\tilde{R}_i}(g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1. \tag{10.39}$$

Apparently, by Proposition 54 and Corollary 55, it follows that $g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha)$ is a compact interval in $\mathbf{R}$. However, $[\tilde{b}_i]_\alpha$ is also a compact interval, therefore for $x \in \mathbf{R}^n$

$$\begin{aligned} \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) &= \sup\{\min\{\chi_{g_i(x;[\tilde{a}_i]_\alpha)}(u), \chi_{[\tilde{b}_i]_\alpha}(v)\} | u \le v\} \\ &= \begin{cases} 1 \text{ if } \underline{G}_i(x;\alpha) \le \overline{b}_i(\alpha) , \\ 0 \text{ otherwise.} \end{cases} \end{aligned} \tag{10.40}$$

To prove (10.39), we have to show that for $y = \lambda x_1 + (1-\lambda)x_2$

$$\underline{G}_i(y;\alpha) \le \overline{b}_i(\alpha), \tag{10.41}$$

Observe that $[\tilde{a}_i]_\alpha$ is convex and compact subset of $\mathbf{R}^k$. By continuity of g_i, there exists $a_j \in [\tilde{a}_i]_\alpha$, $j = 1, 2$, such that

$$\underline{G}_i(x_j;\alpha) = g_i(x_j; a_j). \tag{10.42}$$

Since $x_1, x_2 \in [\tilde{X}_i]_\alpha$, we get from (10.40) $\underline{G}_i(x_j;\alpha) \le \overline{b}_i(\alpha)$, $j = 1, 2$, or

$$\max\{\underline{G}_i(x_1;\alpha), \underline{G}_i(x_2;\alpha)\} \le \overline{b}_i(\alpha). \tag{10.43}$$

Then by (10.43) and (10.42) we immediately obtain

$$\max\{g_i(x_1; a_1), g_i(x_2; a_2)\} \le \overline{b}_i(\alpha). \tag{10.44}$$

Considering inequality (10.38), we get

$$g_i(y; \lambda a_1 + (1-\lambda)a_2) \le \max\{g_i(x_1; a_1), g_i(x_2; a_2)\}. \tag{10.45}$$

Apparently, as $\lambda a_1 + (1-\lambda)a_2 \in [\tilde{a}_i]_\alpha$, we get

$$\underline{G}_i(y;\alpha) \leq g_i(y;\lambda a_1 + (1-\lambda)a_2). \tag{10.46}$$

Then inequalities (10.44) - (10.46) give the required result (10.41).

2. Let $i \in \mathcal{M}_1 \cup \mathcal{M}_3$, $\alpha \in (0,1]$. Again, we show that $[\tilde{X}_i]_\alpha$ is convex.

Let $x_1, x_2 \in [\tilde{X}_i]_\alpha$, $\lambda \in (0,1)$, put $y = \lambda x_1 + (1-\lambda)x_2$. Since g_i is quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$, then for all $(x_1, a_1) \in \mathbf{R}^n \times \mathbf{R}^k$, $(x_2, a_2) \in \mathbf{R}^n \times \mathbf{R}^k$, we have

$$g_i(\lambda x_1 + (1-\lambda)x_2, \lambda a_1 + (1-\lambda)a_2) \geq \min\{g_i(x_1,a_1), g_i(x_2,a_2)\}. \tag{10.47}$$

By (10.20) we get

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}.$$

Hence, it remains to show that

$$\mu_{\tilde{R}_i}(g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1. \tag{10.48}$$

By Proposition 54 and Corollary 55, it follows that $g_i(\lambda x_1 + (1-\lambda)x_2; [\tilde{a}_i]_\alpha)$ is a compact interval in $\mathbf{R}$. However, $[\tilde{b}_i]_\alpha$ is also a compact interval, therefore for $x \in \mathbf{R}^n$

$$\begin{aligned}\mu_{\tilde{R}_i}(g_i(x;[\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) &= \sup\{\min\{\chi_{g_i(x;[\tilde{a}_i]_\alpha)}(u), \chi_{[\tilde{b}_i]_\alpha}(v)\} | u \geq v\} \\ &= \begin{cases} 1 \text{ if } \overline{G}_i(x;\alpha) \geq \underline{b}_i(\alpha), \\ 0 \text{ otherwise.} \end{cases}\end{aligned}$$

To prove (10.48), we have to show that for $y = \lambda x_1 + (1-\lambda)x_2$

$$\overline{G}_i(y;\alpha) \geq \underline{b}_i(\alpha). \tag{10.49}$$

The proof of (10.49) can be conducted in an analogous way to part 1.

Observe first that $[\tilde{a}_i]_\alpha$ is a convex and compact subset of $\mathbf{R}^k$. By continuity of g_i, there exists $a'_j \in [\tilde{a}_i]_\alpha$, $j = 1, 2$, such that

$$\overline{G}_i(x_j;\alpha) = g_i(x_j; a'_j). \tag{10.50}$$

Since $x_1, x_2 \in [\tilde{X}_i]_\alpha$, we get from (10.40)

$$\overline{G}_i(x_j;\alpha) \geq \underline{b}_i(\alpha), j = 1, 2,$$

or

$$\min\{\overline{G}_i(x_1;\alpha), \overline{G}_i(x_2;\alpha)\} \geq \underline{b}_i(\alpha). \tag{10.51}$$

Then by (10.43) and (10.43) we immediately obtain

$$\min\{g_i(x_1;a_1), g_i(x_2;a_2)\} \geq \underline{b}_i(\alpha). \tag{10.52}$$

Considering inequality (10.38), we get

$$g_i(y; \lambda a_1 + (1-\lambda)a_2) \geq \min\{g_i(x_1; a_1), g_i(x_2; a_2)\}. \tag{10.53}$$

Since $\lambda a_1 + (1-\lambda)a_2 \in [\tilde{a}_i]_\alpha$, we have

$$\overline{G}_i(y; \alpha) \geq g_i(y; \lambda a_1 + (1-\lambda)a_2). \tag{10.54}$$

Then inequalities (10.52) - (10.54) give the required result (10.49). ■

The main results of this section are schematically summarized in Table 1.

Table 1.

Constraint functions: g_i	Parameters: $\tilde{a}_i, \tilde{b}_i$	Relations: $R/\tilde{R}_i$	t-norm/ agr. op.	Results:	Theorem:
—	crisp	$=, \leq, \geq$ fuzzy extension	T/T	$\tilde{X} = [\tilde{X}_i]_\alpha = \bar{X}$	T110
—	$\tilde{a}'_i \subset \tilde{a}''_i$, $\tilde{b}'_i \subset \tilde{b}''_i$	valued relat./ T-f. extension	T/A	$\tilde{X}' \subset \tilde{X}''$	T111
continuous	compact	valued rel./ T-f. extension	min / min	$[\tilde{X}]_\alpha = \bigcap_{i=1}^m [\tilde{X}_i]_\alpha$ $[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}$	T113
continuous	compact	$=, \leq, \geq$ / T-f. extension	min / min	$[\tilde{X}]_\alpha = \bigcap_{i=1}^m [\tilde{X}_i]_\alpha$ $[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n \mid \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}$	T114
continuous	compact UQCN	$=, \leq, \geq$ / T-f. extension	min / min	$\underline{G}_i(x; \alpha) \leq \overline{b}_i(\alpha)$, $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\overline{G}_i(x; \alpha) \geq \underline{b}_i(\alpha)$, $i \in \mathcal{M}_1 \cup \mathcal{M}_3$	T115
continuous QCV/ QCA	compact UQCN	$=, \leq, \geq$ / T-f. extension	min / min	$[\tilde{X}_i]_\alpha$- convex	T116

10.7 Optimal Solutions of the FMP Problem

For convenience of the reader we first recall FMP problem (10.8). Let us consider an optimization problem associated with the MP problem (10.4), particularly, let f, g_i, $i \in \mathcal{M}$, be functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{C}$ and $\mathbf{P}_i$ are sets of parameters. Let $\mu_{\tilde{c}} : \mathbf{C} \to [0,1]$, $\mu_{\tilde{a}_i} : \mathbf{P}_i \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$ be membership functions of fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be fuzzy relations with the corresponding membership functions $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1]$.

We assume the existence of an additional goal $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ - a fuzzy subset of the real line, which the fuzzy values of the objective function are compared to, by means of a fuzzy relation $\tilde{R}_0$, which is also assumed to be given exogenously. The fuzzy objective is then treated as another constraint $\tilde{f}(x;\tilde{c})\ \tilde{R}_0\ \tilde{b}_0$, and $\widetilde{\text{maximize}}$ the objective function denotes finding the maximal membership degree of $\mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right)$.

The FMP problem associated with MP problem (10.4) is formulated as follows

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{f}(x;\tilde{c}) \\ \text{subject to} & \\ & \tilde{g}_i(x;\tilde{a}_i)\ \ \tilde{R}_i\ \tilde{b}_i,\ i \in \mathcal{M} = \{1,2,...,m\}. \end{array} \tag{10.55}$$

We obtain a modification of Definition 109.

Definition 117 *Let f, g_i, $i \in \mathcal{M}$, be functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, where $\mathbf{C}$, $\mathbf{P}_i$ are sets of parameters. Let $\mu_{\tilde{c}} : \mathbf{C} \to [0,1]$, $\mu_{\tilde{a}_i} : \mathbf{P}_i \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$ be membership functions of fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, respectively. Let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy subset of $\mathbf{R}$ called a* fuzzy goal. *Furthermore, let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be fuzzy relations given by the membership functions $\mu_{\tilde{R}_i} : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to [0,1]$, let A and A_G be aggregation operators. A fuzzy set $\tilde{X}^*$ given by the membership function $\mu^*_{\tilde{X}}$ for all $x \in \mathbf{R}^n$ as*

$$\mu^*_{\tilde{X}}(x) = A_G\left(\mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right), \mu_{\tilde{X}}(x)\right), \tag{10.56}$$

where $\mu_{\tilde{X}}(x)$ is the membership function of the feasible solution given by (10.12), is called an optimal solution *of the FMP problem (10.55).*
For $\alpha \in (0,1]$ a vector $x \in [\tilde{X}^*]_\alpha$ is called an α-optimal solution *of the FMP problem (10.55).*
A vector $x^* \in \mathbf{R}^n$ with the property

$$\mu^*_{\tilde{X}}(x^*) = Hgt(\tilde{X}^*) \tag{10.57}$$

is called the max-optimal solution.

Notice that the optimal solution $\tilde{X}^*$ of a FMP problem is a fuzzy subset of $\mathbf{R}^n$. Moreover, $\tilde{X}^* \subset \tilde{X}$, where $\tilde{X}$ is the feasible solution. On the other hand, the α-optimal solution is a vector, as well as the max-optimal solution, which is, in fact, the α-optimal solution with $\alpha = Hgt(\tilde{X}^*)$. Notice that in view of Chapter 9 a max-optimal solution is in fact a max-A_G decision on $\mathbf{R}^n$.

In Definition 117 of optimal solution, two aggregation operators A and A_G are used. The former aggregation operator is used for aggregating the individual constraints into the feasible solution by Definitions 109, the latter one is used for the purpose of aggregating the fuzzy set of feasible solution given by the membership function

$$\mu_{\tilde{X}}(x) = A\left(\mu_{\tilde{R}_1}\left(\tilde{g}_1(x;\tilde{a}_1), \tilde{b}_1\right), ..., \mu_{\tilde{R}_m}\left(\tilde{g}_m(x;\tilde{a}_m), \tilde{b}_m\right)\right) \tag{10.58}$$

with the fuzzy set "of the objective" $\tilde{X}_0$ defined by the membership function

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right), \tag{10.59}$$

where $\tilde{b}_0$ is a given fuzzy goal. As a result, we obtain the membership function of optimal solution $\tilde{X}^*$ as

$$\mu^*_{\tilde{X}}(x) = A_G\left(\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\right) \tag{10.60}$$

for all $x \in \mathbf{R}^n$. In particular, if $A = A_G$, then by commutativity and associativity we obtain (10.56) in a simple form

$$\mu^*_{\tilde{X}}(x) = A\left(\mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right), \mu_{\tilde{R}_1}\left(\tilde{g}_1(x;\tilde{a}_1), \tilde{b}_1\right), ..., \mu_{\tilde{R}_m}\left(\tilde{g}_m(x;\tilde{a}_m), \tilde{b}_m\right)\right). \tag{10.61}$$

Since the problem (10.55) is a maximization problem, i.e. "the higher value is better", the membership function $\mu_{\tilde{b}_0}$ of $\tilde{b}_0$ should be increasing, or nondecreasing. By the same reason, the fuzzy relation $\tilde{R}_0$ for comparing $\tilde{f}(x;\tilde{c})$ and $\tilde{b}_0$ should be of the "greater or equal" type. In this section, we consider $\tilde{R}_0$ as a T-fuzzy extension of the usual binary operation $\geq$, where T is a t-norm.

Notice that an extension to a multi-objective MP problem with more than one objective functions and more fuzzy goals and corresponding fuzzy relation are considered, is straightforward, however, it is not followed here.

Formally, in Definitions 109 and 117, the concepts of feasible solution and optimal solution, α-feasible solution and α-optimal solution, respectively, are similar to each other. Therefore, we can take advantage of the results already derived in the preceding section for some characterization of the optimal solutions of the FMP problem. We first show that in case of crisp parameters c, a_i and b_i, the max-optimal solution given by (10.57) coincides with the optimal solution of the crisp problem.

Theorem 118 *Let $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$, let $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be crisp parameters, $i \in \mathcal{M}$. Let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the strictly increasing membership function $\mu_{\tilde{b}_0} : \mathbf{R} \to [0,1]$, let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$,*

be fuzzy relations such that for $i \in \mathcal{M}_1$, $\tilde{R}_i$ is a fuzzy extension of the equality relation "=", for $i \in \mathcal{M}_2$, $\tilde{R}_i$ is a fuzzy extension of the inequality relation "$\leq$", and for $i \in \{0\} \cup \mathcal{M}_3$, $\tilde{R}_i$ is a fuzzy extension of the inequality relation "$\geq$". Let T, A and A_G be t-norms.
Then the set of all max-optimal solution coincides with the set of all optimal solutions X^* of MP problem (10.4).

Proof. By Theorem 110, the feasible solution of (10.55) is crisp, i.e.,

$$\mu_{\tilde{X}}(x) = \chi_X(x) \tag{10.62}$$

for all $x \in \mathbf{R}^n$, where X is the set of all feasible solutions of the crisp MP problem.

Moreover, by (10.59) we obtain for crisp $c \in \mathbf{C}$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right) = \sup\{T\left(\chi_{f(x,c)}(u), \mu_{\tilde{b}_0}(v)\right) | u \geq v\} \\ = \mu_{\tilde{b}_0}(f(x,c)). \tag{10.63}$$

Substituting (10.62) and (10.63) into (10.60) we obtain

$$\mu^*_{\tilde{X}}(x) = A_G\left(\mu_{\tilde{b}_0}(f(x,c)), \chi_X(x)\right) = \{ \begin{array}{l} \mu_{\tilde{b}_0}(f(x,c)) \text{ if } x \in X, \\ 0 \text{ otherwise.} \end{array} \tag{10.64}$$

Since $\mu_{\tilde{b}_0}$is strictly increasing function, by (10.64) it follows that $\mu^*_{\tilde{X}}(x^*) = Hgt(\tilde{X}^G)$, if and only if $\mu^*_{\tilde{X}}(x^*) = \sup\{\mu_{\tilde{b}_0}(f(x,c))|\ x \in X\}$, which is the desired result. ■

For fuzzy subsets $\tilde{a}', \tilde{a}'' \in \mathcal{F}(\mathbf{R}^n)$, we have $\tilde{a}' \subset \tilde{a}''$, if and only if $\mu_{\tilde{a}'}(x) \leq \mu_{\tilde{a}''}(x)$ for all $x \in \mathbf{R}^n$.

Theorem 119 *Let f, g_i, $i \in \mathcal{M}$, be real functions, $f : \mathbf{R}^n \times \mathbf{C} \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \to \mathbf{R}$. Let $\tilde{c}'$, $\tilde{a}'_i$, $\tilde{b}'_i$, and $\tilde{c}''$, $\tilde{a}''_i$, $\tilde{b}''_i$ be two collections of fuzzy parameters of the FMP problem. Let T, A and A_G be t-norms. Let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal, let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be T-fuzzy extensions of valued relations R_i on $\mathbf{R}$.*
If $\tilde{X}^{*\prime}$ is an optimal solution of FMP problem (10.55) with the parameters $\tilde{c}'$, $\tilde{a}'_i$ and $\tilde{b}'_i$, and $\tilde{X}^{*\prime\prime}$ is an optimal solution of the FMP problem with the parameters $\tilde{c}''$, $\tilde{a}''_i$ and $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$

$$\tilde{c}' \subset \tilde{c}'',\ \tilde{a}'_i \subset \tilde{a}''_i \text{ and } \tilde{b}'_i \subset \tilde{b}''_i, \tag{10.65}$$

then

$$\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}. \tag{10.66}$$

Proof. By Theorem 111, for the corresponding feasible solutions it holds $\tilde{X}' \subset \tilde{X}''$. It remains to show that $\tilde{X}'_0 \subset \tilde{X}''_0$, where

$$\mu_{\tilde{X}'_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}'), \tilde{b}_0\right), \mu_{\tilde{X}''_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}''), \tilde{b}_0\right).$$

First, we show that $\tilde{f}(x;\tilde{c}') \subset \tilde{f}(x;\tilde{c}'')$.

Indeed, since $\mu_{\tilde{c}'}(c) \leq \mu_{\tilde{c}''}(c)$ for all $c \in \mathbf{C}$, by (10.11), we obtain for all $u \in \mathbf{R}$

$$\begin{aligned} \mu_{\tilde{f}(x;\tilde{c}')}(u) &= \max\{0, \sup\{\mu_{\tilde{c}'}(c) | c \in \mathbf{C}, f(x;c) = u\}\} \\ &\leq \max\{0, \sup\{\mu_{\tilde{c}''}(c) | c \in \mathbf{C}, f(x;c) = u\}\} = \mu_{\tilde{f}(x;\tilde{c}'')}(u). \end{aligned}$$

Now, using monotonicity of T-fuzzy extension $\tilde{R}_0$, it follows that

$$\mu_{\tilde{R}_i}(\tilde{f}(x;\tilde{c}'), \tilde{b}_0) \leq \mu_{\tilde{R}_i}(\tilde{f}(x;\tilde{c}''), \tilde{b}_0).$$

Applying monotonicity of A_G in (10.60), we obtain $\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}$. ■

Corollary 120 *Let $\tilde{c}$, $\tilde{a}_i$, $\tilde{b}_i$ be a collection of fuzzy parameters, and let $c \in \mathbf{C}$, $a_i \in \mathbf{P}_i$ and $b_i \in \mathbf{R}$ be a collection of crisp parameters such that for all $i \in \mathcal{M}$*

$$\mu_{\tilde{c}}(c) = \mu_{\tilde{a}_i}(a_i) = \mu_{\tilde{b}_i}(b_i) = 1. \tag{10.67}$$

If X^* is a nonempty set of all optimal solutions of MP problem (10.4) with the parameters c, a_i and b_i, $\tilde{X}^*$ is an optimal solution of FMP problem (10.55) with fuzzy parameters $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$, then for all $x \in X^*$

$$\mu^*_{\tilde{X}}(x) = 1. \tag{10.68}$$

Proof. Observe that $c \subset \tilde{c}$, $a_i \subset \tilde{a}_i$, $b_i \subset \tilde{b}_i$ for all $i \in \mathcal{M}$. Then by Theorem 119 we obtain $X^* \subset \tilde{X}^*$, which is equivalent to (10.68). ■

Notice that the optimal solution $\tilde{X}^*$ of FMP problem (10.8) always exists, even if the MP problem with crisp parameters has no crisp optimal solution. Corollary 120 states that if the MP problem with crisp parameters has a crisp optimal solution, then the membership grade of the optimal solution (of the associated FMP problem with fuzzy parameters) is equal to one. This fact enables a natural embedding of the class of (crisp) MP problems into the class of FMP problems.

From now on, the space of parameters is supposed to be the k-dimensional Euclidean vector space $\mathbf{R}^k$, where k is a positive integer, i.e. $\mathbf{C} = \mathbf{P}_i = \mathbf{R}^k$ for all $i \in \mathcal{M}$. For the remaining part of this section we suppose that T , A and A_G are the minimum t-norms, that is $T = A = A_G = T_M$. We shall find some formulae based on α-cuts of the parameters, analogous to those given by Theorem 113 and 115, however, for α-optimal solutions of the FMP problem.

Theorem 121 *Let f, g_i be continuous functions, $f : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Let $\tilde{c}$, $\tilde{a}_i$ and $\tilde{b}_i$ be compact fuzzy parameters, $i \in \mathcal{M}$, let $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal. Let $T = A = A_G = T_M$. Let $\tilde{R}_i$ be T-fuzzy extensions of closed valued relations R_i on $\mathbf{R}$, $i \in \{0\} \cup \mathcal{M}$.*
Then for all $\alpha \in (0, 1]$

$$[\tilde{X}^*]_\alpha = \bigcap_{i=0}^{m} [\tilde{X}_i]_\alpha, \tag{10.69}$$

and, moreover, for all $i \in \mathcal{M}$

$$[\tilde{X}_0]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_0}(f(x; [\tilde{c}]_\alpha), [\tilde{b}_0]_\alpha) \geq \alpha\}, \tag{10.70}$$

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) \geq \alpha\}. \tag{10.71}$$

Proof. The proof is omitted since it is analogous to the proof of Theorem 113. ■

If the valued relations R_i are usual equality and inequality relations, then the stronger statement of Theorem 113 can be proven.

Theorem 122 *Let* f, g_i, $i \in \mathcal{M}$, *be continuous functions,* $f : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. *Let* $\tilde{c}$, $\tilde{a}_i$ *and* $\tilde{b}_i$ *be compact fuzzy parameters. Let* $T = A = A_G = T_M$. *Let* $\tilde{R}_i$ *be the same as in Theorem 118,* $i \in \{0\} \cup \mathcal{M}$.
Then for all $\alpha \in (0, 1]$

$$[\tilde{X}^*]_\alpha = \bigcap_{i=0}^{m} [\tilde{X}_i]_\alpha, \tag{10.72}$$

and, moreover, for all $i \in \mathcal{M}$

$$[\tilde{X}_0]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_0}(f(x; [\tilde{c}]_\alpha), [\tilde{b}_0]_\alpha) = 1\}, \tag{10.73}$$

$$[\tilde{X}_i]_\alpha = \{x \in \mathbf{R}^n | \mu_{\tilde{R}_i}(g_i(x; [\tilde{a}_i]_\alpha), [\tilde{b}_i]_\alpha) = 1\}. \tag{10.74}$$

Proof. The proof is omitted since it is analogous to the proof of Theorem 114. ■

Next, we shall derive an analogue to Theorem 115, for this purpose we extend the notation from the previous section as follows. Given $\alpha \in (0, 1]$, $i \in \mathcal{M}$, let

$$\begin{aligned} \overline{F}(x; \alpha) &= \sup \{f(x; c) | c \in [\tilde{c}]_\alpha\}, & (10.75)\\ \underline{F}(x; \alpha) &= \inf \{f(x; c) | c \in [\tilde{c}]_\alpha\}, & (10.76)\\ \underline{b}_0(\alpha) &= \inf\{b \in \mathbf{R} | b \in [\tilde{b}_0]_\alpha\}, & (10.77)\\ \overline{b}_0(\alpha) &= \sup\{b \in \mathbf{R} | b \in [\tilde{b}_0]_\alpha\}. & (10.78) \end{aligned}$$

Theorem 123 *Let all assumptions of Theorem 122 be satisfied. Moreover, let the membership functions of fuzzy parameters* $\tilde{c}, \tilde{a}_i$ *and* $\tilde{b}_i$ *be upper-quasiconnected for all* $i \in \mathcal{M}$. *Let* $\tilde{b}_0 \in \mathcal{F}(\mathbf{R})$ *be a fuzzy goal with the membership function* $\mu_{\tilde{b}_0}$ *satisfying the following conditions*

$$\begin{array}{l} \mu_{\tilde{b}_0} \textit{ is upper semicontinuous,} \\ \mu_{\tilde{b}_0} \textit{ is strictly increasing,} \\ \lim_{t \to -\infty} \mu_{\tilde{b}_0}(t) = 0. \end{array} \tag{10.79}$$

Then for all $\alpha \in (0, 1]$, *we have* $x \in [\tilde{X}^*]_\alpha$ *if and only if*

$$\overline{F}(x; \alpha) \geq \underline{b}_0(\alpha), \tag{10.80}$$

$$\underline{G}_i(x; \alpha) \leq \overline{b}_i(\alpha),\ i \in \mathcal{M}_1 \cup \mathcal{M}_2, \tag{10.81}$$

$$\overline{G}_i(x; \alpha) \geq \underline{b}_i(\alpha), i \in \mathcal{M}_1 \cup \mathcal{M}_3, \tag{10.82}$$

Proof. The proof is omitted since it is analogous to the proof of Theorem 115 with the only modification that instead of compactness of $\tilde{b}_0$, we have assumptions (10.79). ■

Theorem 124 *Let all assumptions of Theorem 122 be satisfied. Moreover, let g_i be quasiconvex on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, f and g_i be quasiconcave on $\mathbf{R}^n \times \mathbf{R}^k$ for $i \in \mathcal{M}_1 \cup \mathcal{M}_3$.*
Then for all $i \in \{0\} \cup \mathcal{M}$, $\tilde{X}_i$ are convex and the optimal solution $\tilde{X}^*$ of FMP problem (10.55) is convex, too.

Proof. Again, the proof is omitted since it can be performed in an analogous way as the proof of Theorem 116. ■

If the individual membership functions of the fuzzy objective and fuzzy constraints can be expressed in an explicit form, then the max-optimal solution can be found as the optimal solution of some crisp MP problem.

Theorem 125 *Let*
$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}), \tilde{b}_0\right)$$
and
$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}\left(\tilde{g}_i(x;\tilde{a}_i), \tilde{b}_i\right),$$
$x \in \mathbf{R}^n, i \in \mathcal{M}$, be the membership functions of the fuzzy objective and fuzzy constraints of the FMP problem (10.55), respectively. Let $T = A = A_G = T_M$ and (10.79) holds for $\tilde{b}_0$.
The vector $(t^*, x^*) \in \mathbf{R}^{n+1}$ is an optimal solution of the problem

$$\begin{array}{ll} \textit{maximize} & t \\ \textit{subject to} & \mu_{\tilde{X}_0}(x) \geq t, \\ & \mu_{\tilde{X}_i}(x) \geq t,\ i \in \mathcal{M}, \end{array} \tag{10.83}$$

if and only if x^* is a max-optimal solution of the problem (10.55).

Proof. Let $(t^*, x^*) \in \mathbf{R}^{n+1}$ be an optimal solution of the MP problem (10.83). By (10.57) and (10.60) we obtain

$$\mu^*_{\tilde{X}}(x^*) = \sup\{\min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}_i}(x)\} | x \in \mathbf{R}^n\} = Hgt(\tilde{X}^*).$$

Hence, x^* is an optimal solution with the maximal height. The proof of the opposite statement is straightforward. ■

Chapter 11

Fuzzy Linear Programming

11.1 Introduction

Most important mathematical programming problems (10.4) are those where the functions f and g_i are linear.

Let $\mathcal{M} = \{1, 2, ..., m\}$, $\mathcal{N} = \{1, 2, ..., n\}$, m, n be positive integers. Let f, g_i be linear functions, $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $c, a_i \in \mathbf{R}^n$, $i \in \mathcal{M}$, be the parameters such that

$$f(x; c_1, ..., c_n) = c^T x = c_1 x_1 + \cdots + c_n x_n, \tag{11.1}$$

$$g_i(x; a_{i1}, ..., a_{in}) = a_i^T x = a_{i1} x_1 + \cdots + a_{in} x_n, i \in \mathcal{M}, \tag{11.2}$$

where $x \in \mathbf{R}^n$. We consider the following *linear programming problem* (LP problem)

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & \\ & a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i, \ i \in \mathcal{M}, \\ & \qquad\qquad\qquad\quad x_j \geq 0, \ j \in \mathcal{N}. \end{array} \tag{11.3}$$

The set of all feasible solutions X of (11.3) is defined as follows

$$X = \{x \in \mathbf{R}^n | a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i, i \in \mathcal{M}, x_j \geq 0, j \in \mathcal{N}\}. \tag{11.4}$$

11.2 Formulation of FLP problem

Before formulating a fuzzy linear problem as an optimization problem associated with the LP problem (11.3), we make a few assumptions and remarks.

Let f, g_i be linear functions defined by (11.1), (11.2), respectively. From now on, the parameters c_j, a_{ij} and b_i will be considered as normal *fuzzy quantities,* that is, normal fuzzy subsets of the Euclidean space $\mathbf{R}$. The fuzzy quantities will be denoted by symbols with the wavelets above. Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0, 1]$,

$\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively.

Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. They will be used for comparing the left and right sides of the constraints in (11.3).

The maximization of objective function (11.1) needs, however, a special treatment, similar to that of FMP problem. As it was stated in Chapter 8, fuzzy values of the objective function are not linearly ordered and to maximize the objective function we have to define a suitable ordering on $\mathcal{F}(\mathbf{R})$ which allows for "maximization" of the objective. Again it shall be done by an exogenously given fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and another fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$. There exist some other approaches, see [18], [23], [60].

The *fuzzy linear programming problem* (FLP problem) associated with LP problem (11.3) is denoted as

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \\ & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \ \tilde{R}_i \ \tilde{b}_i, \ i \in \mathcal{M}, \\ & \qquad\qquad x_j \geq 0, \ j \in \mathcal{N}. \end{array} \tag{11.5}$$

Let us clarify the elements of (11.5).

The objective function values and the left hand sides values of the constraints of (11.5) have been obtained by the extension principle (8.17) as follows. A membership function of $\tilde{g}_i(x; \tilde{a}_{i1}, ..., \tilde{a}_{i1})$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{g}_i}(t) = \begin{cases} \sup\{T(\mu_{\tilde{a}_{i1}}(a_1), ..., \mu_{\tilde{a}_{in}}(a_n)) | a_1, ..., a_n \in \mathbf{R}, a_1 x_1 + \cdots + a_n x_n = t\} \\ \qquad\qquad \text{if } g_i^{-1}(x;t) \neq \emptyset, \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{11.6}$$

where $g_i^{-1}(x;t) = \{(a_1, ..., a_n) \in \mathbf{R}^n | a_1 x_1 + \cdots + a_n x_n = t\}$. Here, the fuzzy set $\tilde{g}_i(x; \tilde{a}_{i1}, ..., \tilde{a}_{i1})$ is denoted as $\tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n$, i.e.

$$\tilde{g}_i(x; \tilde{a}_{i1}, ..., \tilde{a}_{i1}) = \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \tag{11.7}$$

for every $i \in \mathcal{M}$ and for each $x \in \mathbf{R}^n$.

Also, for given $\tilde{c}_1, ..., \tilde{c}_n \in \mathcal{F}(\mathbf{R})$, $\tilde{f}(x; \tilde{c}_1, ..., \tilde{c}_n)$ is a fuzzy extension of $f(x; c_1, ..., c_n)$ with the membership function defined for each $t \in \mathbf{R}$ as

$$\mu_{\tilde{f}}(t) = \begin{cases} \sup\{T(\mu_{\tilde{c}_1}(c_1), ..., \mu_{\tilde{c}_n}(c_n)) | c_1, ..., c_n \in \mathbf{R}, c_1 x_1 + \cdots + c_n x_n = t\} \\ \qquad\qquad \text{if } f^{-1}(x;t) \neq \emptyset, \\ 0 \qquad\qquad \text{otherwise,} \end{cases} \tag{11.8}$$

where $f^{-1}(x;t) = \{(c_1, ..., c_n) \in \mathbf{R}^n | f(x; c_1, ..., c_n) = t\}$.

The fuzzy set $\tilde{f}(x; \tilde{c}_1, ..., \tilde{c}_n)$ will be denoted as $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$, i.e.

$$\tilde{f}(x; \tilde{c}_1, ..., \tilde{c}_n) = \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n. \tag{11.9}$$

In (11.7) the value $\tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \in \mathcal{F}(\mathbf{R})$ is "compared" with the fuzzy quantity $\tilde{b}_i \in \mathcal{F}(\mathbf{R})$ by a fuzzy relation $\tilde{R}_i$, $i \in \mathcal{M}$.

For x and $y \in \mathbf{R}^n$ we calculate $\tilde{f}(x; \tilde{c}_1, ..., \tilde{c}_n) \in \mathcal{F}(\mathbf{R})$ and $\tilde{f}(y; \tilde{c}_1, ..., \tilde{c}_n) \in \mathcal{F}(\mathbf{R})$, respectively. Such values of the objective function are not linearly ordered and to maximize the objective function we have to define a suitable ordering on $\mathcal{F}(\mathbf{R})$ which allows for "maximization" of the objective. Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be an exogenously given fuzzy goal with an associated fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$.

The fuzzy relations $\tilde{R}_i$ for comparing the constraints of (11.5) are usually extensions of a valued relation on $\mathbf{R}$, particularly, the usual inequality relations "$\leq$" and "$\geq$".

If $\tilde{R}_i$ is a fuzzy extension of relation R_i, then by (8.27) we obtain the membership function of the i-th constraint as

$$\mu_{\tilde{R}_i}\left(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i\right) = \sup\{T\left(\mu_{\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n}(u), \mu_{\tilde{b}_i}(v)\right) | uR_i v\}. \tag{11.10}$$

Apparently, for the feasible solution and also for the optimal solution of a FLP problem, the concepts which have been already defined in the preceding chapter for FMP problem (10.4), can be adopted here. Of course, for FLP problems they have some special features. Let us begin with the concept of feasible solution.

Definition 126 *Let g_i, $i \in \mathcal{M}$, be linear functions defined by (11.2). Let $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M} = \{1, 2, ..., m\}$, $j \in \mathcal{N} = \{1, 2, ..., n\}$, be membership functions of fuzzy quantities $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. Let G_A be an aggregation operator and T be a t-norm. Here T is used for extending arithmetic operations.*
A fuzzy set $\tilde{X}$, a membership function $\mu_{\tilde{X}}$ of which is defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}}(x) = \begin{cases} G_A\left(\mu_{\tilde{R}_1}(\tilde{a}_{11}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{1n}x_n, \tilde{b}_1), \ldots, \mu_{\tilde{R}_m}(\tilde{a}_{m1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mn}x_n, \tilde{b}_m)\right) \\ \qquad\qquad\qquad\qquad\qquad \text{if } x_j \geq 0 \text{ for all } j \in \mathcal{N}, \\ 0 \qquad\qquad\qquad\qquad\quad\;\; \text{otherwise,} \end{cases} \tag{11.11}$$

is called the feasible solution *of the FLP problem (11.5).*
For $\alpha \in (0,1]$, a vector $x \in [\tilde{X}]_\alpha$ is called the α-feasible solution *of the FLP problem (11.5).*
A vector $\bar{x} \in \mathbf{R}^n$ such that $\mu_{\tilde{X}}(\bar{x}) = Hgt(\tilde{X})$ is called the max-feasible solution.

Notice that the feasible solution $\tilde{X}$ of a FLP problem is a fuzzy set. On the other hand, α-feasible solution is a vector belonging to the α-cut of the feasible solution $\tilde{X}$ and the same holds for the max-feasible solution, which is a special α-feasible solution with $\alpha = Hgt(\tilde{X})$.

If a decision maker specifies the grade of membeship $\alpha \in (0,1]$ (the grade of possibility, feasibility, satisfaction etc.), then a vector $x \in \mathbf{R}^n$ satisfying $\mu_{\tilde{X}}(x) \geq \alpha$ is an α-feasible solution of the corresponding FLP problem.

For $i \in \mathcal{M}$ we introduce the following notation: $\tilde{X}_i$ will denote a fuzzy subset of $\mathbf{R}^n$ with the membership function $\mu_{\tilde{X}_i}$ defined for all $x \in \mathbf{R}^n$ as

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i). \tag{11.12}$$

Fuzzy set (11.12) is interpreted as an i-th fuzzy constraint. All fuzzy constraints are aggregated into the feasible solution (11.11) by the aggregation operator G_A. Particularly, $G_A = \min$, the t-norm T is used for extending arithmetic operations. Notice, that the fuzzy solution depends also on the fuzzy relations used in definitions of the constraints of the FLP problem.

11.3 Properties of Feasible Solution

Considering crisp parameters a_{ij} and b_i, clearly, the feasible solution is also crisp. Moreover, it is not difficult to show that if the fuzzy parameters of two FLP problems are ordered by fuzzy inclusion, that is $\tilde{a}'_{ij} \subset \tilde{a}''_{ij}$ and $\tilde{b}'_i \subset \tilde{b}''_i$, then the same inclusion holds for the feasible solutions, i.e. $\tilde{X}' \subset \tilde{X}''$, on condition $\tilde{R}_i$ are T-fuzzy extensions of valued relations, see also below Proposition 130.

Now, we derive special formulae which will allow for computing an α-feasible solution $x \in [\tilde{X}]_\alpha$ of the FLP problem (11.5). For this purpose, the following notation will be useful. Given $\alpha \in (0, 1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, let

$$\underline{a}_{ij}(\alpha) = \inf \{a \in \mathbf{R} | a \in [\tilde{a}_{ij}]_\alpha\}, \tag{11.13}$$

$$\overline{a}_{ij}(\alpha) = \sup \{a \in \mathbf{R} | a \in [\tilde{a}_{ij}]_\alpha\}, \tag{11.14}$$

$$\underline{b}_i(\alpha) = \inf\{b \in \mathbf{R} | b \in [\tilde{b}_i]_\alpha\}, \tag{11.15}$$

$$\overline{b}_i(\alpha) = \sup\{b \in \mathbf{R} | b \in [\tilde{b}_i]_\alpha\}. \tag{11.16}$$

Theorem 127 *Let for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities and let $x_j \geq 0$. Let $T = \min$, $S = \max$, and $\alpha \in (0, 1)$. Then for $i \in \mathcal{M}$*
(i)

$$\mu_{\tilde{\leq}^T}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ if and only if } \sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j \leq \overline{b}_i(\alpha), \tag{11.17}$$

(ii)

$$\mu_{\tilde{\leq}_S}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ if and only if } \sum_{j=1}^{n} \overline{a}_{ij}(1-\alpha)x_j \leq \underline{b}_i(1-\alpha). \tag{11.18}$$

(iii) *Moreover, if $\tilde{a}_{ij}$ and $\tilde{b}_i$ are strictly convex fuzzy quantities, then*

$$\mu_{\tilde{\leq}^{T,S}}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ if and only if } \sum_{j=1}^{n} \overline{a}_{ij}(1-\alpha)x_j \leq \overline{b}_i(\alpha), \tag{11.19}$$

where $\mu_{\tilde{\leq}^{T,S}} = \mu_{\tilde{\leq}^{T,S}} = \mu_{\tilde{\leq}_{T,S}}$, and
(iv)

$$\mu_{\tilde{\leq}_{S,T}}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha \text{ if and only if } \sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j \leq \underline{b}_i(1-\alpha), \tag{11.20}$$

where $\mu_{\tilde{\leq}S,T} = \mu_{\tilde{\leq}^{S,T}} = \mu_{\tilde{\leq}_{S,T}}$.

Proof. We present here only the proof of part (i). The other parts follow analogically by Theorem 65.

Let $i \in \mathcal{M}$, $\mu_{\tilde{\leq}^T}(\tilde{a}_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in}x_n, \tilde{b}_i) \geq \alpha$. By Theorem 65 , this is equivalent to

$$\inf\left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j\right]_\alpha \leq \sup[\tilde{b}_i]_\alpha.$$

By the well known Nguyen's result, see [41], or [64], it follows that

$$\left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j\right]_\alpha = \sum_{j=1}^{n} [\tilde{a}_{ij}]_\alpha x_j.$$

Since $[\tilde{a}_{ij}]_\alpha$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, are compact and convex intervals in $\mathbf{R}$ and $x_j \geq 0$, $j \in \mathcal{N}$, the rest of the proof follows easily from definitions (11.13), (11.14) and Proposition 111. ■

Let $l, r, \in \mathbf{R}$ with $l \leq r$, let $\gamma, \delta \in [0, +\infty)$ and let L, R be non-increasing, non-constant, upper-semicontinuous functions mapping interval $(0, 1]$ into $[0, +\infty)$, i.e. $L, R : (0, 1] \to [0, +\infty)$. Moreover, assume that $L(1) = R(1) = 0$, define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$.

Let A be an (L, R)-fuzzy interval given by the membership function defined for each $x \in \mathbf{R}$

$$\mu_A(x) = \begin{cases} L^{(-1)}\left(\frac{l-x}{\gamma}\right) & \text{if } x \in (l-\gamma, l),\ \gamma > 0, \\ 1 & \text{if } x \in [l, r], \\ R^{(-1)}\left(\frac{x-r}{\delta}\right) & \text{if } x \in (r, r+\delta),\ \delta > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{11.21}$$

where $L^{(-1)}, R^{(-1)}$ are pseudo-inverse functions of L, R, respectively. As it was mentioned already in Chapter 9, the class of (L, R)-fuzzy intervals extends the class of crisp closed intervals $[a, b] \subset \mathbf{R}$ including the case $a = b$, i.e. crisp numbers. Particularly, if the membership functions of $\tilde{a}_{ij}$ and $\tilde{b}_i$ are given analytically by

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} L^{(-1)}\left(\frac{l_{ij}-x}{\gamma_{ij}}\right) & \text{if } x \in [l_{ij}-\gamma_{ij}, l_{ij}),\ \gamma_{ij} > 0, \\ 1 & \text{if } x \in [l_{ij}, r_{ij}], \\ R^{(-1)}\left(\frac{x-r_{ij}}{\delta_{ij}}\right) & \text{if } x \in (r_{ij}, r_{ij}+\delta_{ij}],\ \delta_{ij} > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{11.22}$$

and

$$\mu_{\tilde{b}_i}(x) = \begin{cases} L^{(-1)}\left(\frac{l_i-x}{\gamma_i}\right) & \text{if } x \in [l_i-\gamma_i, l_i),\ \gamma_i > 0, \\ 1 & \text{if } x \in [l_i, r_i], \\ R^{(-1)}\left(\frac{x-r_i}{\delta_i}\right) & \text{if } x \in (r_i, r_i+\delta_i],\ \delta_i > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{11.23}$$

for each $x \in \mathbf{R}$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, then the values of (11.13) - (11.16) can be computed as

$$\begin{aligned} \underline{a}_{ij}(\alpha) &= l_{ij} - \gamma_{ij} L(\alpha), \ \ \overline{a}_{ij}(\alpha) = r_{ij} + \delta_{ij} R(\alpha), \\ \underline{b}_i(\alpha) &= l_i - \gamma_i L(\alpha), \ \ \overline{b}_i(\alpha) = r_i + \delta_i R(\alpha). \end{aligned}$$

Let $G_A = \min$. By Proposition 127, α-cuts $[\tilde{X}]_\alpha$ of the feasible solution of (11.5) can be computed by solving the system of inequalities from (11.17) - (11.20). Moreover, $[\tilde{X}]_\alpha$ is an intersection of a finite number of halfspaces, hence a convex polyhedral set.

11.4 Properties of Optimal Solutions

As the FLP problem is a particular case of the FMP problem, all properties and results which have been derived in Chapter 10 are applicable to any FLP problem.

We assume the existence of an exogenously given additional goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ - a fuzzy set of the real line. The fuzzy value $\tilde{d}$ is compared to fuzzy values $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ of the objective function by a given fuzzy relation $\tilde{R}_0$. In this way the fuzzy objective is treated as another constraint

$$\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \tilde{R}_0 \tilde{d}.$$

We obtain a modification of Definition 126.

Definition 128 *Let f, g_i be linear functions defined by (11.1), (11.2). Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and let $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of normal fuzzy quantities $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively. Moreover, let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy set of the real line called the* fuzzy goal. *Let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be fuzzy relations on $\mathbf{R}$ and let T be a t-norm, let A and A_G be aggregation operators.*
A fuzzy set $\tilde{X}^*$ with the membership function $\mu_{\tilde{X}^*}$ defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}^*}(x) = A_G\left(\mu_{\tilde{R}_0}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}), \mu_{\tilde{X}}(x)\right), \tag{11.24}$$

where $\mu_{\tilde{X}}(x)$ is the membership function of the feasible solution, is called the optimal solution *of FLP problem (11.5).*
*For $\alpha \in (0,1]$ a vector $x \in [\tilde{X}^*]_\alpha$ is called the α-*optimal solution *of FLP problem (11.5).*
A vector $x^* \in \mathbf{R}^n$ with the property

$$\mu_{\tilde{X}^*}(x^*) = Hgt(\tilde{X}^*) \tag{11.25}$$

is called the max-optimal solution.

Notice that the optimal solution of the FLP problem is a fuzzy set. On the other hand, the α-optimal solution is a vector belonging to the α-cut $[\tilde{X}^*]_\alpha$. Likewise, the max-optimal solution is an α-optimal solution with $\alpha = Hgt(\tilde{X}^*)$. Notice that in view of Chapter 9 a max-optimal solution is in fact a max-A_G decision on $\mathbf{R}^n$.

In Definition 128, the t-norms T and the aggregation operators A and A_G have been used. The former t-norm T has been used for extending arithmetic operations, the aggregation operator A for aggregating the individual constraints into the single feasible solution and A_G has been applied for aggregating the fuzzy set of the feasible solution with the fuzzy set of the objective $\tilde{X}_0$ defined by the membership function

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}), \tag{11.26}$$

$x \in \mathbf{R}^n$. As a result, we have obtained the membership function of optimal solution $\tilde{X}^*$ defined for all $x \in \mathbf{R}^n$ by

$$\mu_{\tilde{X}^*}(x) = A_G\left(\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\right). \tag{11.27}$$

Since problem (11.5) is a maximization problem "the higher value is better", the membership function $\mu_{\tilde{d}}$ of $\tilde{d}$ is supposed to be increasing, or non-decreasing. The fuzzy relation $\tilde{R}_0$ for comparing $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ and $\tilde{d}$ is supposed to be a fuzzy extension of $\geq$.

Formally, in Definitions 126 and 128, the concepts of feasible solution and optimal solution, etc., correspond to each other. Therefore, we can derive some properties of optimal solutions of the FLP problem by the case of the feasible solution studied in the preceding section.

We first observe that in case of crisp parameters c_j, a_{ij} and b_i, the set of all max-optimal solutions given by (11.25) coincides with the set of all optimal solutions of the crisp problem. We have the following theorem.

Proposition 129 *Let c_j, a_{ij}, $b_i \in \mathbf{R}$ be crisp parameters of (11.5) for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with a strictly increasing membership function $\mu_{\tilde{d}}$. Let for $i \in \mathcal{M}$, $\tilde{R}_i$ be a fuzzy extension of the relation "$\leq$" on $\mathbf{R}$, and $\tilde{R}_0$ be a T-fuzzy extension of the relation "$\geq$". Let T, A and A_G be t-norms.*
Then the set of all max-optimal solutions coincides with the set of all optimal solutions X^* of LP problem (11.3).

Proof. Clearly, the feasible solution of (11.5) is crisp, i.e.

$$\mu_{\tilde{X}}(x) = \chi_X(x)$$

for all $x \in \mathbf{R}^n$, where X is the set of all feasible solutions (11.4) of the crisp LP problem (11.3). Moreover, by (11.26) we obtain for crisp $c \in \mathbf{R}^n$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}\left(f(x;c), \tilde{d}\right) = \mu_{\tilde{d}}(c_1 x_1 + \cdots + c_n x_n).$$

Substituting (10.62) and (10.63) into (11.27) we obtain

$$\mu_{\tilde{X}^*}(x) = A_G\left(\mu_{\tilde{d}}(f(x,c)), \chi_X(x)\right) = \begin{cases} \mu_{\tilde{d}}(c_1x_1 + \cdots + c_nx_n) \text{ if } x \in X, \\ 0 \text{ otherwise.} \end{cases}$$

Since $\mu_{\tilde{d}}$ is strictly increasing, by (10.64) it follows that $\mu_{\tilde{X}^*}(x^*) = Hgt(\tilde{X}^*)$ if and only if

$$\mu_{\tilde{X}^*}(x^*) = \sup\{\mu_{\tilde{d}}(c_1x_1 + \cdots + c_nx_n) | x \in X\},$$

which is the desired result. ■

Proposition 130 *Let $\tilde{c}'_j$, $\tilde{a}'_{ij}$ and $\tilde{b}'_i$ and $\tilde{c}''_j$, $\tilde{a}''_{ij}$ and $\tilde{b}''_i$ be two collections of fuzzy parameters of FLP problem (11.5), $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let T, A, A_G be t-norms. Let $\tilde{R}_i$, $i \in \{0\} \cup \mathcal{M}$, be T-fuzzy extensions of valued relations R_i on $\mathbf{R}$. If $\tilde{X}^{*\prime}$ is the optimal solution of FLP problem (11.5) with the parameters $\tilde{c}'_j$, $\tilde{a}'_{ij}$ and $\tilde{b}'_i$, $\tilde{X}^{*\prime\prime}$ is the optimal solution of the FLP problem with the parameters $\tilde{c}''_j$, $\tilde{a}''_{ij}$ and $\tilde{b}''_i$ such that for all $i \in \mathcal{M}$, $j \in \mathcal{N}$,*

$$\tilde{c}'_j \subset \tilde{c}''_j,\ \tilde{a}'_{ij} \subset \tilde{a}''_{ij} \text{ and } \tilde{b}'_i \subset \tilde{b}''_i, \tag{11.28}$$

then

$$\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}. \tag{11.29}$$

Proof. First we show that for the feasible solutions it holds $\tilde{X}' \subset \tilde{X}''$. Let $x \in \mathbf{R}^n$, $i \in \mathcal{M}$. Now we show that

$$\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n \subset \tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n.$$

Indeed, by (8.17), for each $u \in \mathbf{R}$

$$\begin{aligned} \mu_{\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n}(u) &= \sup\{T(\mu_{\tilde{a}'_{i1}}(a_1), ..., \mu_{\tilde{a}'_{in}}(a_n)) | a_{i1}x_1 + \cdots + a_{in}x_n = u\} \\ &\le \sup\{T(\mu_{\tilde{a}''_{i1}}(a_1), ..., \mu_{\tilde{a}''_{in}}(a_n)) | a_{i1}x_1 + \cdots + a_{in}x_n = u\} \\ &= \mu_{\tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n}(u). \end{aligned}$$

Now, as $\tilde{b}'_i \subset \tilde{b}''_i$, using monotonicity of T-fuzzy extension $\tilde{R}_i$ of R_i, it follows that

$$\mu_{\tilde{R}_i}(\tilde{a}'_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}'_{in}x_n, \tilde{b}'_i) \le \mu_{\tilde{R}_i}(\tilde{a}''_{i1}x_1 \tilde{+} \cdots \tilde{+} \tilde{a}''_{in}x_n, \tilde{b}''_i).$$

Then, applying again monotonicity of A in (11.11), we obtain $\tilde{X}' \subset \tilde{X}''$.

It remains to show that $\tilde{X}'_0 \subset \tilde{X}''_0$, where

$$\mu_{\tilde{X}'_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}'), \tilde{d}\right),\ \mu_{\tilde{X}''_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{f}(x;\tilde{c}''), \tilde{d}\right).$$

We show that $\tilde{f}(x;\tilde{c}') \subset \tilde{f}(x;\tilde{c}'')$. Indeed, since for all $j \in \mathcal{N}$, $\mu_{\tilde{c}'_j}(c) \le \mu_{\tilde{c}''_j}(c)$ for all $c \in \mathbf{R}$, by (11.8), we obtain for all $u \in \mathbf{R}$

$$\begin{aligned} \mu_{\tilde{c}'_1x_1 \tilde{+} \cdots \tilde{+} \tilde{c}'_nx_n}(u) &= \sup\{T(\mu_{\tilde{c}'_1}(c_1), ..., \mu_{\tilde{c}'_n}(c_n)) | c_1x_1 + \cdots + c_nx_n = u\} \\ &\le \sup\{T(\mu_{\tilde{c}''_1}(c_1), ..., \mu_{\tilde{c}''_n}(c_n)) | c_1x_1 + \cdots + c_nx_n = u\} \\ &= \mu_{\tilde{c}''_1x_1 \tilde{+} \cdots \tilde{+} \tilde{c}''_nx_n}(u). \end{aligned}$$

Again, using monotonicity of $\tilde{R}_0$ it follows that

$$\mu_{\tilde{R}_0}(\tilde{c}'_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}'_n x_n, \tilde{d}) \leq \mu_{\tilde{R}_0}(\tilde{c}''_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}''_n x_n, \tilde{d}).$$

Applying monotonicity of A_G in (11.27), we obtain $\tilde{X}^{*\prime} \subset \tilde{X}^{*\prime\prime}$. ■

Next, we extend Proposition 127 to the case of an optimal solution of a FLP problem. For this purpose we add some new notation as follows. Given $\alpha \in (0,1]$, $j \in \mathcal{N}$, let

$$\begin{aligned} \underline{c}_j(\alpha) &= \inf \{c | c \in [\tilde{c}_j]_\alpha\}, && (11.30)\\ \overline{c}_j(\alpha) &= \sup \{c | c \in [\tilde{c}_j]_\alpha\}, && (11.31)\\ \underline{d}(\alpha) &= \inf\{d | d \in [\tilde{d}]_\alpha\}, && (11.32)\\ \overline{d}(\alpha) &= \sup\{d | d \in [\tilde{d}]_\alpha\}. && (11.33) \end{aligned}$$

Proposition 131 *Let for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities, $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the membership function $\mu_{\tilde{d}}$ satisfying the following conditions*

$$\begin{array}{l} \mu_{\tilde{d}} \text{ is upper semicontinuous,} \\ \mu_{\tilde{d}} \text{ is strictly increasing,} \\ \lim_{t \to -\infty} \mu_{\tilde{d}}(t) = 0. \end{array} \qquad (11.34)$$

Let $\tilde{R}_i = \tilde{\leq}^T$, $i \in \mathcal{M}$, $\tilde{\leq}$ be the T-fuzzy extensions of the binary relation $\leq$ on $\mathbf{R}$, $\tilde{R}_0 = \tilde{\geq}^T$ be the T-fuzzy extension of the binary relation $\geq$ on $\mathbf{R}$. Let $T = A = A_G = \min$. Let $\tilde{X}^*$ be an optimal solution of FLP problem (11.5) and $\alpha \in (0,1)$.
A vector $x = (x_1, ..., x_n) \geq 0$ belongs to $[\tilde{X}^*]_\alpha$ if and only if

$$\sum_{j=1}^{n} \overline{c}_j(\alpha) x_j \geq \underline{d}(\alpha), \qquad (11.35)$$

$$\sum_{j=1}^{n} \underline{a}_{ij}(\alpha) x_j \leq \overline{b}_i(\alpha),\ i \in \mathcal{M}. \qquad (11.36)$$

Proof. The proof is omitted since it is analogous to the proof of Proposition 127, part (i), with a simple modification that instead of compactness of $\tilde{d}$, we have assumptions (11.34). ■

For the sake of simplicity we confined ourselves in Proposition 131 only to the case of T-fuzzy extension of valued relations $\leq$ on $\mathbf{R}$, i.e. for $i \in \mathcal{M}$, $\tilde{R}_i = \tilde{\leq}^T$ and $\tilde{R}_0 = \tilde{\geq}^T$. Evidently, similar results could be obtained for some other fuzzy extensions, e.g. $\tilde{R}_i \in \{\tilde{\leq}^T, \tilde{\leq}_S, \tilde{\leq}^{T,S}, \tilde{\leq}_{T,S}, \tilde{\leq}^{S,T}, \tilde{\leq}_{S,T}\}$ as in [28].

If the membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ can be formulated in an explicit form, e.g. similar to that of (8.18), (11.23), then we can find an optimal solution with maximal height as the (crisp) optimal solution of some optimization problem.

Proposition 132 *Consider FLP problem (11.5), where for each* $x = (x_1, ..., x_n) \in \mathbf{R}^n$ *and every* $i \in \mathcal{M}$

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{R}_0}\left(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}\right)$$

and

$$\mu_{\tilde{X}_i}(x) = \mu_{\tilde{R}_i}\left(\tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n, \tilde{b}_i\right)$$

are the membership functions of the fuzzy objective and fuzzy constraints, respectively. Let $T = A = A_G = \min$ *and (11.34) holds for* $\tilde{d}$.
A vector $(t^*, x^*) \in \mathbf{R}^{n+1}$ *is an optimal solution of the optimization problem*

$$\begin{array}{ll} \textit{maximize} & t \\ \textit{subject to} & \mu_{\tilde{X}_i}(x) \geq t,\ i \in \{0\} \cup \mathcal{M}, \\ & x_j \geq 0,\ j \in \mathcal{N} \end{array} \tag{11.37}$$

if and only if $x^* \in \mathbf{R}^n$ *is a max-optimal solution of FLP problem (11.5).*

Proof. Let $(t^*, x^*) \in \mathbf{R}^{n+1}$ be an optimal solution of problem (11.37). By (11.24) and (11.25) we obtain

$$\mu_{\tilde{X}^*}(x^*) = \sup\{\min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}_i}(x)\} | x \in \mathbf{R}^n\} = Hgt(\tilde{X}^*).$$

Hence, x^* is a max-optimal solution.

The proof of the opposite statement is omitted. ■

11.5 Extended Addition in FLP

In Theorems 127 and 131, formulae (11.17) and (11.36) hold only on condition the special case of $T = T_M$ is assumed. This t-norm has been used not only for the T-fuzzy extensions of the binary relations on $\mathbf{R}$, but also for extending linear functions, that is, the objective function and constraints of the FLP problem. In this section we shall investigate the problem of making summation

$$\tilde{f}(x; \tilde{c}_1, ..., \tilde{c}_n) = \tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n, \tag{11.38}$$

and

$$\tilde{g}_i(x; \tilde{a}_{i1}, ..., \tilde{a}_{in}) = \tilde{a}_{i1} x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{a}_{in} x_n \tag{11.39}$$

for each $x \in \mathbf{R}^n$, where $\tilde{c}_j$, $\tilde{a}_{ij} \in \mathcal{F}(\mathbf{R})$, for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Formulae (11.38) and (11.39) are defined by (11.8) and (8.20), respectively, that is by using of the extension principle. Here $\tilde{+}_T$ denotes that the extended summation is performed by the t-norm T. Note that for arbitrary t-norms T exact formulae for (11.38) and (11.39) can be either complicated or even inaccessible. However, in some special cases such formulae exist, some of which will be given bellow.

For the sake of brevity we shall deal only with (11.38), for (11.39) the results can be obtained analogously.

Let $F, G : (0, +\infty) \to [0, 1]$ be non-increasing left continuous functions. For $\gamma, \delta \in (0, +\infty)$, define functions $F_\gamma, G_\delta : (0, +\infty) \to [0, 1]$ by

$$\begin{aligned} F_\gamma(x) &= F(\tfrac{x}{\gamma}), \\ G_\delta(x) &= G(\tfrac{x}{\delta}), \end{aligned} \tag{11.40}$$

where $x \in (0, +\infty)$. Let $l_j, r_j \in \mathbf{R}$ such that $l_j \leq r_j$, let $\gamma_j, \delta_j \in (0, +\infty)$ and let

$$\tilde{c}_j = \left(l_j, r_j, F_{\gamma_j}, G_{\delta_j}\right), j \in \mathcal{N}, \tag{11.41}$$

be closed fuzzy intervals, with the membership functions given by

$$\mu_{\tilde{c}_j}(x) = \{ \begin{array}{ll} F_{\gamma_j}(l_j - x) & \text{if } x \in (-\infty, l_j), \\ 1 & \text{if } x \in [l_j, r_j], \\ G_{\delta_j}(x - r_j) & \text{if } x \in (r_j, +\infty), \end{array} \tag{11.42}$$

see also Chapter 8. In the following proposition we prove that $\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n$ is also closed fuzzy interval of the same type. The proof is omitted as it is a straightforward application of the extension principle. For the references, see [38].

Proposition 133 *Let* $\tilde{c}_j = \left(l_j, r_j, F_{\gamma_j}, G_{\delta_j}\right), j \in \mathcal{N}$, *be closed fuzzy intervals with the membership functions given by (11.42) and let* $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ *for all* $j \in \mathcal{N}$, *denote*

$$I_x = \{j | x_j > 0, j \in \mathcal{N}\}.$$

Then

$$\tilde{c}_1 x_1 \tilde{+}_{T_M} \cdots \tilde{+}_{T_M} \tilde{c}_n x_n = \left(l, r, F_{l_M}, G_{r_M}\right), \tag{11.43}$$

$$\tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n = \left(l, r, F_{l_D}, G_{r_D}\right), \tag{11.44}$$

where T_M *is the minimum t-norm,* T_D *is the drastic product and*

$$l = \sum_{j \in I_x} l_j x_j, r = \sum_{j \in I_x} r_j x_j, \tag{11.45}$$

$$l_M = \sum_{j \in I_x} \frac{\gamma_j}{x_j}, r_M = \sum_{j \in I_x} \frac{\delta_j}{x_j}, \tag{11.46}$$

$$l_D = \max\{\frac{\gamma_j}{x_j} | j \in I_x\}, r_D = \max\{\frac{\delta_j}{x_j} | j \in I_x\}. \tag{11.47}$$

If all $\tilde{c}_j$ are (L, R)-fuzzy intervals, then we can obtain an analogous and more specific result. Let $l_j, r_j \in \mathbf{R}$ with $l_j \leq r_j$, let $\gamma_j, \delta_j \in [0, +\infty)$ and let L, R be non-increasing, non-constant, upper-semicontinuous functions mapping the interval $(0, 1]$ into $[0, +\infty)$, i.e. $L, R : (0, 1] \to [0, +\infty)$. Moreover, assume that $L(1) = R(1) = 0$, and define $L(0) = \lim_{x \to 0} L(x)$, $R(0) = \lim_{x \to 0} R(x)$.

Let for every $j \in \mathcal{N}$,

$$\tilde{c}_j = \left(l_j, r_j, \gamma_j, \delta_j\right)_{LR} \tag{11.48}$$

be an (L, R)-fuzzy interval given by the membership function defined for each $x \in \mathbf{R}$ by

$$\mu_{\tilde{c}_j}(x) = \begin{cases} L^{(-1)}\left(\frac{l_j - x}{\gamma_j}\right) & \text{if } x \in (l_j - \gamma_j, l_j),\ \gamma_j > 0, \\ 1 & \text{if } x \in [l_j, r_j], \\ R^{(-1)}\left(\frac{x - r_j}{\delta_j}\right) & \text{if } x \in (r_j, r_j + \delta_j),\ \delta_j > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{11.49}$$

where $L^{(-1)}, R^{(-1)}$ are pseudo-inverse functions of L, R, respectively.

Proposition 134 *Let* $\tilde{c}_j = \left(l_j, r_j, \gamma_j, \delta_j\right)_{LR}, j \in \mathcal{N}$, *be* (L, R)*-fuzzy intervals with the membership functions given by (11.49) and let* $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ *for all* $j \in \mathcal{N}$*. Then*

$$\tilde{c}_1 x_1 \tilde{+}_{T_M} \cdots \tilde{+}_{T_M} \tilde{c}_n x_n = (l, r, A_M, B_M)_{LR}\,, \tag{11.50}$$

$$\tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n = (l, r, A_D, B_D)_{LR}\,, \tag{11.51}$$

where T_M *is the minimum t-norm,* T_D *is the drastic product and*

$$l = \sum_{j \in \mathcal{N}} l_j x_j, r = \sum_{j \in \mathcal{N}} r_j x_j, \tag{11.52}$$

$$A_M = \sum_{j \in \mathcal{N}} \gamma_j x_j, B_M = \sum_{j \in \mathcal{N}} \delta_j x_j, \tag{11.53}$$

$$A_D = \max\{\gamma_j | j \in \mathcal{N}\}, B_D = \max\{\delta_j | j \in \mathcal{N}\}. \tag{11.54}$$

The results (11.44) and (11.51) in Proposition 133 and 134, respectively, can be extended as follows, see [38].

Proposition 135 *Let* T *be a continuous Archimedean t-norm with an additive generator* f*. Let* $F : (0, +\infty) \to [0, 1]$ *be defined for each* $x \in (0, +\infty)$ *as*

$$F(x) = f^{(-1)}(x).$$

Let $\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, F_{\delta_j}), j \in \mathcal{N}$, *be closed fuzzy intervals with the membership functions given by (11.42) and let* $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ *for all* $j \in \mathcal{N}$, $I_x = \{j | x_j > 0, j \in \mathcal{N}\}$*. Then*

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = (l, r, F_{l_D}, F_{r_D})\,,$$

where

$$l = \sum_{j \in I_x} l_j x_j, r = \sum_{j \in I_x} r_j x_j,$$

$$l_D = \max\{\frac{\gamma_j}{x_j} | j \in I_x\}, r_D = \max\{\frac{\delta_j}{x_j} | j \in I_x\}.$$

Note that for a continuous Archimedean t-norm T and closed fuzzy intervals $\tilde{c}_j$ satisfying the assumptions of Proposition 135, we have

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = \tilde{c}_1 x_1 \tilde{+}_{T_D} \cdots \tilde{+}_{T_D} \tilde{c}_n x_n,$$

which means that we obtain the same fuzzy linear function based on an arbitrary t-norm T' such that $T' \leq T$.

The following proposition generalizes several results concerning the addition of closed fuzzy intervals based on continuous Archimedean t-norms, see [38].

Proposition 136 *Let T be a continuous Archimedean t-norm with an additive generator f. Let $K : [0, +\infty) \to [0, +\infty)$ be continuous convex function with $K(0) = 0$. Let $\alpha \in (0, +\infty)$ and*

$$F_\alpha(x) = f^{(-1)}(\alpha K(\frac{x}{\alpha}))$$

for all $x \in [0, +\infty)$. Let $\tilde{c}_j = (l_j, r_j, F_{\gamma_j}, F_{\delta_j}), j \in \mathcal{N}$, be closed fuzzy intervals with the membership functions given by (11.42) and let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $x_j \geq 0$ for all $j \in \mathcal{N}$, $I_x = \{j | x_j > 0, j \in \mathcal{N}\}$. Then

$$\tilde{c}_1 x_1 \tilde{+}_T \cdots \tilde{+}_T \tilde{c}_n x_n = (l, r, F_{l_K}, F_{r_K}), \tag{11.55}$$

where

$$l = \sum_{j \in I_x} l_j x_j, r = \sum_{j \in I_x} r_j x_j, \tag{11.56}$$

$$l_K = \sum_{j \in I_x} \frac{\gamma_j}{x_j}, r_K = \sum_{j \in I_x} \frac{\delta_j}{x_j}. \tag{11.57}$$

Two immediate consequences can be obtained from Proposition 135:

(i) The sum based on the product t-norm T_P of Gaussian fuzzy numbers, see Example 33, is again a Gaussian fuzzy number. Indeed, the additive generator f of the product t-norm T_P is given by $f(x) = -\log(x)$. Let $K(x) = x^2$. Then

$$F_\alpha(x) = f^{(-1)}\left(\alpha K(\frac{x}{\alpha})\right) = e^{-\frac{x^2}{\alpha}}.$$

By Proposition 136 we obtain the required result.

(ii) The sum based on Yager t-norm T^Y_λ, see [38], of closed fuzzy intervals generated by the same K, is again a closed fuzzy interval of the same type. Observe that an additive genrator f^Y_λ of the Yager t-norm T^Y_λ is given by $f^Y_\lambda(x) = (1-x)^\lambda$. For $\lambda \in (0, +\infty)$ we obtain

$$F_\alpha(x) = \max\{0, 1 - \frac{x}{\alpha^{\frac{\lambda-1}{\lambda}}}\}.$$

This means that each piecewise linear fuzzy number (l, r, γ, δ) can be written as

$$(l, r, \gamma, \delta) = \left(l, r, F_{\gamma^{\frac{\lambda}{\lambda-1}}}, F_{\delta^{\frac{\lambda}{\lambda-1}}}\right),$$

and the sum of piecewise linear fuzzy numbers $\tilde{c}_j = (l_j, r_j, \gamma_j, \delta_j), j \in \mathcal{N}$, is again a piecewise linear fuzzy number

$$(l, r, \gamma, \delta),$$

where l and r are given by (11.56), and γ and δ are given as

$$\gamma = \sum_{j \in \mathcal{N}} \gamma_j^{\frac{\lambda}{\lambda-1}}, \delta = \sum_{j \in \mathcal{N}} \delta_j^{\frac{\lambda}{\lambda-1}}.$$

The extensions can be obtained also for some other t-norms, see e.g. [38], [83].

An alternative approach based on centered fuzzy numbers will be mentioned later in this chapter, see also [42], [43].

11.6 Duality

In this section we generalize the well known concept of duality in LP for FLP problems. The results of this section, in a more general setting, can be found in [64]. We derive some weak and strong duality results which extend the known results for LP problems.

Consider the following FLP problem

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \\ & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \ \tilde{R} \ \tilde{b}_i, \ i \in \mathcal{M}, \\ & x_j \geq 0, \ j \in \mathcal{N}. \end{array} \tag{11.58}$$

Here, the parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are considered as normal fuzzy quantities, i.e. $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let $\tilde{R}$ be a fuzzy extension of a valued relation R on $\mathbf{R}$. FLP problem (11.58) will be called the *primal FLP problem* (P).

The *dual FLP problem* (D) is defined as

$$\begin{array}{ll} \widetilde{\text{minimize}} & \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\ \text{subject to} & \\ & \tilde{a}_{1j} y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj} y_m \ {}^*\tilde{R} \ \tilde{c}_j, \ j \in \mathcal{N}, \\ & y_i \geq 0, \ i \in \mathcal{M}. \end{array} \tag{11.59}$$

Here, ${}^*\tilde{R}$ is the dual fuzzy relation to $\widetilde{{}^cR}$ of ${}^cR$ on $\mathbf{R}$ as in Definition 20 in Part I . Recall the properties of the dual fuzzy extensions discussed in Proposition 31. Taking combinations of "primal" and dual fuzzy relations from (i)-(iv), Proposition 31, we can create a number of primal - dual pairs of FLP problems.

Further on, we shall investigate the primal - dual pair of FLP problems with the fuzzy relation $\tilde{\leq}^T$ and the corresponding dual fuzzy relation $\tilde{\geq}_S$. Remember that S should be the dual t-conorm to the t-norm T, e.g. $S = \max$, $T = \min$.

Now, consider the following pair of FLP problems
(P):

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\ \text{subject to} & \\ & \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \ \tilde{\le}^T \ \tilde{b}_i, \ i \in \mathcal{M}, \\ & x_j \ge 0, \ j \in \mathcal{N}. \end{array} \tag{11.60}$$

(D):

$$\begin{array}{ll} \widetilde{\text{minimize}} & \tilde{b}_1 y_1 \tilde{+} \cdots \tilde{+} \tilde{b}_m y_m \\ \text{subject to} & \\ & \tilde{a}_{1j} y_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{mj} y_m \ \tilde{\ge}_S \ \tilde{c}_j, \ j \in \mathcal{N}, \\ & y_i \ge 0, \ i \in \mathcal{M}. \end{array} \tag{11.61}$$

Let the feasible solution of the primal FLP problem (P) be denoted by $\tilde{X}$, the feasible solution of the dual FLP problem (D) by $\tilde{Y}$. Clearly, $\tilde{X}$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}$ is a fuzzy subset of $\mathbf{R}^m$.

Notice that in the crisp case, i.e. when the parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are crisp real numbers, the relation $\tilde{\le}^T$coincides with $\le$ and $\tilde{\ge}_S$ coincides with $\ge$, hence (P) and (D) is a primal - dual couple of LP problems in the usual sense.

In the following proposition we prove the weak form of the duality theorem for FLP problems.

Proposition 137 *Let for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities. Let $\tilde{\le}^T$ be the T-fuzzy extension of the binary relation $\le$ on $\mathbf{R}$ defined by (8.27) and $\tilde{\ge}_S$ be the fuzzy extension of the relation $\ge$ on $\mathbf{R}$ defined by (8.37). Let $A = T = \min$, $S = \max$. Let $\tilde{X}$ be a feasible solution of FLP problem (11.60), $\tilde{Y}$ be a feasible solution of FLP problem (11.61) and let $\alpha \in [0.5, 1)$.*
If a vector $x = (x_1, ..., x_n) \ge 0$ belongs to $[\tilde{X}]_\alpha$ and $y = (y_1, ..., y_m) \ge 0$ belongs to $[\tilde{Y}]_\alpha$, then

$$\sum_{j \in \mathcal{N}} \bar{c}_j (1-\alpha) x_j \le \sum_{i \in \mathcal{M}} \bar{b}_i (1-\alpha) y_i. \tag{11.62}$$

Proof. Let $x \in [\tilde{X}]_\alpha$ and $y \in [\tilde{Y}]_\alpha$, $x_j \ge 0, y_i \ge 0$ for all $i \in \mathcal{M}$, $j \in \mathcal{N}$. Then by Proposition 127 we obtain

$$\sum_{i=1}^{m} \underline{a}_{ij} (1-\alpha) y_i \ge \overline{c}_j (1-\alpha). \tag{11.63}$$

Since $\alpha \ge 0.5$, it follows that $1-\alpha \le \alpha$, hence $[\tilde{X}]_\alpha \subset [\tilde{X}]_{1-\alpha}$. Again by Proposition 127 we obtain for all $i \in \mathcal{M}$

$$\sum_{j=1}^{n} \underline{a}_{ij} (1-\alpha) x_j \le \overline{b}_i (1-\alpha). \tag{11.64}$$

Multiplying both sides of (11.63) and (11.64) by x_j and y_i, respectively, and summing up the results, we obtain

$$\sum_{j=1}^{n} \overline{c}_j(1-\alpha)x_j \leq \sum_{j=1}^{n}\sum_{i=1}^{m} \underline{a}_{ij}(1-\alpha)y_i x_j \leq \sum_{i=1}^{m} \overline{b}_i(1-\alpha)y_i,$$

which is the desired result. ■

Notice that in the crisp case, (11.62) is nothing else than the standard *weak duality*. Let us turn to the *strong duality*.

For this purpose, we assume the existence of an exogenously given additional goals $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and $\tilde{h} \in \mathcal{F}(\mathbf{R})$ - fuzzy sets of the real line. The fuzzy goal $\tilde{d}$ is compared to fuzzy values $\tilde{c}_1x_1\tilde{+}\cdots\tilde{+}\tilde{c}_nx_n$ of the objective function of the primal FLP problem (P) by a given fuzzy relation $\tilde{\geq}^T$. On the other hand, the fuzzy goal $\tilde{h}$ is compared to fuzzy values $\tilde{b}_1y_1\tilde{+}\cdots\tilde{+}\tilde{b}_my_m$ of the objective function of the dual FLP problem (D) by a given fuzzy relation $\tilde{\leq}_S$. In this way the fuzzy objectives are treated as constraints

$$\tilde{c}_1x_1\tilde{+}\cdots\tilde{+}\tilde{c}_nx_n\tilde{\geq}^T\tilde{d}, \ \ \tilde{b}_1y_1\tilde{+}\cdots\tilde{+}\tilde{b}_my_m\tilde{\leq}_S\tilde{h}.$$

Let the optimal solution of the primal FLP problem (P), defined by Definition 128, be denoted by $\tilde{X}^*$, the optimal solution of the dual FLP problem (D), defined also by Definition 128, by $\tilde{Y}^*$. Clearly, $\tilde{X}^*$ is a fuzzy subset of $\mathbf{R}^n$, $\tilde{Y}^*$ is a fuzzy subset of $\mathbf{R}^m$, moreover, $\tilde{X}^* \subset \tilde{X}$ and $\tilde{Y}^* \subset \tilde{Y}$.

Proposition 138 *Let for all $i \in \mathcal{M}$, $j \in \mathcal{N}$, $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ be compact, convex and normal fuzzy quantities. Let $\tilde{d}, \tilde{h} \in \mathcal{F}(\mathbf{R})$ be fuzzy goals with the membership function $\mu_{\tilde{d}}, \mu_{\tilde{h}}$ satisfying the following conditions*

$$\begin{array}{ll} (i) & \mu_{\tilde{d}}, \mu_{\tilde{h}} \text{ are upper semicontinuous,} \\ (ii) & \mu_{\tilde{d}} \text{ is strictly increasing, } \mu_{\tilde{h}} \text{ is strictly decreasing,} \\ (iii) & \lim\limits_{t\to-\infty} \mu_{\tilde{d}}(t) = \lim\limits_{t\to+\infty} \mu_{\tilde{h}}(t) = 0. \end{array} \tag{11.65}$$

Let $\tilde{\leq}^T$ be the T-fuzzy extension of the relation $\leq$ on $\mathbf{R}$ and $\tilde{\geq}_S$ be the fuzzy extension of the relation $\geq$ on $\mathbf{R}$. Let $A = T = \min$, $S = \max$. Let $\tilde{X}^*$ be an optimal solution of FLP problem (11.60), $\tilde{Y}^*$ be an optimal solution of FLP problem (11.61) and $\alpha \in (0,1)$.
If a vector $x^* = (x_1^*, ..., x_n^*) \geq 0$ belongs to $[\tilde{X}^*]_\alpha$, then there exists a vector $y^* = (y_1^*, ..., y_m^*) \geq 0$ which belongs to $[\tilde{Y}^*]_{1-\alpha}$, and

$$\sum_{j\in\mathcal{N}} \overline{c}_j(\alpha)x_j^* = \sum_{i\in\mathcal{M}} \overline{b}_i(\alpha)y_i^*. \tag{11.66}$$

Proof. Let $x^* = (x_1^*, ..., x_n^*) \geq 0$, $x^* \in [\tilde{X}^*]_\alpha$. By Proposition 131

$$\sum_{j=1}^{n} \overline{c}_j(\alpha)x_j^* \geq \underline{d}(\alpha), \tag{11.67}$$

$$\sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j^* \leq \overline{b}_i(\alpha),\ i \in \mathcal{M}. \tag{11.68}$$

Consider the following LP problem:

$$\begin{aligned}
\text{(P1) maximize} \quad & \sum_{j=1}^{n} \overline{c}_j(\alpha)x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} \underline{a}_{ij}(\alpha)x_j \le \overline{b}_i(\alpha),\ i \in \mathcal{M}, \\
& x_j \ge 0,\ j \in \mathcal{N}.
\end{aligned}$$

By conditions (11.65) concerning the fuzzy goal $\tilde{d}$, the system of inequalities (11.67),(11.68) is satisfied if and only if x^* is the optimal solution of (P1). By the standard strong duality theorem for LP, there exists $y^* \in \mathbf{R}^*$ being an optimal solution of the dual problem

$$\begin{aligned}
\text{(D1) minimize} \quad & \sum_{i=1}^{m} \overline{b}_i(\alpha)y_i \\
\text{subject to} \quad & \sum_{i=1}^{m} \underline{a}_{ij}(\alpha)y_i \ge \overline{c}_j(\alpha),\ j \in \mathcal{N}, \\
& y_i \ge 0,\ i \in \mathcal{M},
\end{aligned}$$

such that (11.66) holds.

It remains only to prove that $y^* \in [\tilde{Y}^*]_{1-\alpha}$. This fact follows, however, from conditions (11.65) concerning the fuzzy goal $\tilde{h}$, and from (11.18). ■

Notice that in the crisp case, (11.66) is the standard strong duality result for LP.

11.7 Special Models of FLP

Several models of FLP problem known from the literature are investigated in this section.

11.7.1 Interval Linear Programming

In this subsection we apply the previous results for a special case of the FLP problem - interval linear programming problem (ILP problem). By *interval linear programming problem* (ILP) we understand the following FLP problem

$$\begin{aligned}
&\widetilde{\text{maximize}} \quad \tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \\
&\text{subject to} \\
&\qquad \tilde{a}_{i1} x_1 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \ \tilde{R}\ \tilde{b}_i,\ i \in \mathcal{M}, \\
&\qquad\qquad x_j \ge 0,\ j \in \mathcal{N}.
\end{aligned} \tag{11.69}$$

Here, the parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are considered to be compact intervals in $\mathbf{R}$, i.e. $\tilde{c}_j = [\underline{c}_j, \overline{c}_j]$, $\tilde{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$ and $\tilde{b}_i = [\underline{b}_i, \overline{b}_i]$, where $\underline{c}_j, \overline{c}_j$, $\underline{a}_{ij}, \overline{a}_{ij}$ and $\underline{b}_i, \overline{b}_i$ are lower and upper bounds of the corresponding intervals, respectively. The membership functions of $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are the characteristic functions of the intervals, i.e. $\chi_{[\underline{c}_j, \overline{c}_j]} : \mathbf{R} \to [0,1]$, $\chi_{[\underline{a}_{ij}, \overline{a}_{ij}]} : \mathbf{R} \to [0,1]$ and $\chi_{[\underline{b}_i, \overline{b}_i]} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$. The fuzzy relation $\tilde{R}$ is considered as a fuzzy extension of a valued relation R on $\mathbf{R}$. We assume that R is the usual binary relation $\leq$, $A = T = \min$, $S = \max$, and

$$\tilde{R} \in \{\tilde{\leq}^{T}, \tilde{\leq}_{S}, \tilde{\leq}^{T,S}, \tilde{\leq}_{T,S}, \tilde{\leq}^{S,T}, \tilde{\leq}_{S,T}\}.$$

Then by Proposition 127 we obtain 6 types of feasible solutions of ILP problem (11.69):

(i)

$$X_{\tilde{\leq}^{T}} = \left\{ x \in \mathbf{R}^n |\ \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i, x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{11.70}$$

(ii)

$$X_{\tilde{\leq}_{S}} = \left\{ x \in \mathbf{R}^n |\ \sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \underline{b}_i, x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{11.71}$$

(iii)

$$X_{\tilde{\leq}^{T,S}} = X_{\tilde{\leq}_{T,S}} = \left\{ x \in \mathbf{R}^n |\ \sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \overline{b}_i, x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{11.72}$$

(iv)

$$X_{\tilde{\leq}^{S,T}} = X_{\tilde{\leq}_{S,T}} = \left\{ x \in \mathbf{R}^n |\ \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \underline{b}_i, x_j \geq 0,\ j \in \mathcal{N} \right\}. \tag{11.73}$$

Clearly, feasible solutions (11.70) - (11.73) are crisp subsets of $\mathbf{R}^n$, moreover, they all are polyhedral.

In order to find an optimal solution of ILP problem (11.69), we consider a fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and $\tilde{R}_0$, a fuzzy extension of the usual binary relation $\geq$ for comparing the objective with the fuzzy goal.

In the following proposition we show that if the feasible solution of ILP problem is crisp then its max-optimal solution is the same as the set of all optimal solution of the problem of maximizing a particular crisp objective over the set of feasible solutions.

Proposition 139 *Let X be a crisp feasible solution of ILP problem (11.69). Let $\tilde{d} \in \mathcal{F}(\mathbf{R})$ be a fuzzy goal with the membership function $\mu_{\tilde{d}}$ satisfying conditions (11.34). Let $A_G = A = T = \min$, $S = \max$.*

(i) If $\tilde{R}_0$ is $\tilde{\geq}^{T}$, then the set of all max-optimal solutions of ILP problem (11.69) coincides with the set of all optimal solution of the problem

$$\begin{array}{c} \textit{maximize } \sum_{j=1}^{n} \overline{c}_j x_j \\ \textit{subject to } x \in X; \end{array} \tag{11.74}$$

(ii) If $\tilde{R}_0$ is $\tilde{\geq}_S$, then the set of all max-optimal solutions of ILP problem (11.69) coincides with the set of all optimal solution of the problem

$$\begin{array}{c} \textit{maximize } \sum_{j=1}^{n} \underline{c}_j x_j \\ \textit{subject to } x \in X. \end{array} \tag{11.75}$$

Proof. (i) Let $x \in X$ be a max-optimal solution of ILP problem (11.69), $\underline{c} = \sum_{j=1}^{n} \underline{c}_j x_j$, $\overline{c} = \sum_{j=1}^{n} \overline{c}_j x_j$. By our assumptions, (8.37) and (11.34) give

$$\begin{aligned} \mu_{\tilde{\geq}^T}\left(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}\right) &= \sup\{\min\{\mu_{\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n}(u), \mu_{\tilde{d}}(v)\} | u \geq v\} \\ &= \sup\{\min\{\chi_{[\underline{c},\overline{c}]}(u), \mu_{\tilde{d}}(v)\} | u \geq v\} \\ &= \mu_{\tilde{d}}\left(\sum_{j=1}^{n} \overline{c}_j x_j\right). \end{aligned}$$

Hence, x is an optimal solution of (11.74). Conversely, if $x \in X$ is an optimal solution of (11.74), then by Definition 128 and by (11.34), x is a max-optimal solution of problem (11.69).

(ii) Analogously to the proof of (i), we have

$$\begin{aligned} \mu_{\tilde{\geq}_S}\left(\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n, \tilde{d}\right) &= \inf\{\max\{1 - \mu_{\tilde{c}_1 x_1 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n}(u), 1 - \mu_{\tilde{d}}(v)\} | u \geq v\} \\ &= \inf\{1 - \min\{\chi_{[\underline{c},\overline{c}]}(u), \mu_{\tilde{d}}(v)\} | u \leq v\} \\ &= \mu_{\tilde{d}}\left(\sum_{j=1}^{n} \underline{c}_j x_j\right). \end{aligned}$$

By the same arguments as in (i) we conclude the proof. ■

We close this section with several observations about duality in ILP problems.

Let the primal ILP problem (P) be problem (11.69) with $\tilde{R} = \tilde{\leq}^T$, i.e. (11.60). Then the dual ILP problem (D) is (11.61). Clearly, the feasible solution $X_{\tilde{\leq}^T}$ of (P) is defined by (11.70) and the feasible solution $Y_{\tilde{\geq}_S}$ of the dual problem (D) can be derived from (11.71) as

$$Y_{\tilde{\geq}_S} = \{y \in \mathbf{R}^m | \sum_{i=1}^{m} \underline{a}_{ij} y_i \geq \overline{c}_j, y_i \geq 0, i \in \mathcal{M}\}. \tag{11.76}$$

Notice that the problems

$$\begin{array}{c} \text{maximize } \sum_{j=1}^{n} \overline{c}_j x_j \\ \text{subject to } x \in X_{\tilde{\leq}^T} \end{array}$$

and

$$\begin{array}{rl} \text{minimize} & \sum_{i=1}^{m} \underline{b}_i y_i \\ \text{subject to} & y \in Y_{\tilde{\geq}_S} \end{array}$$

are dual to each other in the usual (crisp) sense if and only if $\underline{c}_j = \overline{c}_j$ and $\underline{b}_i = \overline{b}_i$ for all $i \in \mathcal{M}$ and $j \in \mathcal{N}$.

For ILP problems our results correspond to that of [24], [49], [82].

11.7.2 Flexible Linear Programming

The term *flexible linear programming* is referred to an original approach to LP problems, see e.g. [100], allowing for a kind of flexibility of the objective function and constraints in standard LP problem (11.3), that is

$$\begin{array}{ll} \text{maximize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & \\ & a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i,\ i \in \mathcal{M}, \\ & x_j \geq 0,\ j \in \mathcal{N}, \end{array} \tag{11.77}$$

see also [83]. In (11.77) the values of parameters c_j, a_{ij} and b_i are known, they are, however, uncertain, not confident, etc. That is why nonnegative values p_i, $i \in \{0\} \cup \mathcal{M}$, of admissible violations of the objective and constraints are (subjectively) chosen and supplemented to the original model (11.77).

For the objective function, an aspiration value $d_0 \in \mathbf{R}$ is also (subjectively) determined such that if the objective function attains this value, or if it is greater, then the decision maker (DM) is fully satisfied. On the other hand, if the objective function attains a value smaller than $d_0 - p_0$, then (DM) is fully dissatisfied. Within the interval $(d_0 - p_0, d_0)$, the satisfaction of DM increases linearly from 0 to 1. By these considerations a membership function $\mu_{\tilde{d}}$ of the fuzzy goal $\tilde{d}$ is defined as follows

$$\mu_{\tilde{d}}(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \geq d_0, \\ 1 + \frac{t-d_0}{p_0} & \text{if } d_0 - p_0 \leq t < d_0, \\ 0 & \text{otherwise.} \end{array} \right. \tag{11.78}$$

Similarly, let for the i-th constraint function of (11.77), $i \in \mathcal{M}$, a right hand side $b_i \in \mathbf{R}$ is known such that if the left hand side attains this value, or if it is smaller, then the decision maker (DM) is fully satisfied. On the other side, if the objective function attains its value greater than $b_i + p_i$, then (DM) is fully dissatisfied. Within the interval $(b_i, b_i + p_i)$, the satisfaction of DM decreases linearly from 1 to 0. By these considerations a membership function $\mu_{\tilde{b}_i}$ of the fuzzy right hand side $\tilde{b}_i$ is defined as

$$\mu_{\tilde{b}_i}(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \leq b_i, \\ 1 - \frac{t-b_i}{p_i} & \text{if } b_i \leq t < b_i + p_i, \\ 0 & \text{otherwise.} \end{array} \right. \tag{11.79}$$

The relationship between the objective function and constraints in the flexible LP problem is considered as fully symetric; i.e. there is no longer a difference between the former and latter. "Maximization" is then understood as finding a vector $x \in \mathbf{R}^n$ such that the membership grade of the intersection of all fuzzy sets (11.78) and (11.79) is maximal. In other words, we have to solve the following optimization problem:

$$\begin{array}{lrcl} \text{maximize} & \lambda & & \\ \text{subject to} & & & \\ & \mu_{\tilde{d}}(\sum\limits_{j\in\mathcal{N}} c_j x_j) & \geq & \lambda, \\ & \mu_{\tilde{b}_i}(\sum\limits_{j\in\mathcal{N}} a_{ij} x_j) & \geq & \lambda, \quad i \in \mathcal{M}, \\ & 0 \leq \lambda \leq 1, x_j & \geq & 0, \quad j \in \mathcal{N}. \end{array} \tag{11.80}$$

Problem (11.80) can be easily transformed to the equivalent LP problem:

$$\begin{array}{llcl} \text{maximize} & \lambda & & \\ \text{subject to} & & & \\ & \sum\limits_{j\in\mathcal{N}} c_j x_j & \geq & d_0 + \lambda p_0, \\ & \sum\limits_{j\in\mathcal{N}} a_{ij} x_j & \leq & b_i + (1-\lambda) p_i, \quad i \in \mathcal{M}, \\ & 0 \leq \lambda \leq 1, x_j & \geq & 0, j \in \mathcal{N}. \end{array} \tag{11.81}$$

In Section 2, we introduced FLP problem (11.5). Now, consider the following, a more specific FLP problem:

$$\begin{array}{ll} \widetilde{\text{maximize}} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & \\ & a_{i1} x_1 + \cdots + a_{in} x_n \mathrel{\tilde{\leq}^T} \tilde{b}_i, \; i \in \mathcal{M}, \\ & \qquad\qquad x_j \geq 0, \; j \in \mathcal{N}, \end{array} \tag{11.82}$$

where c_j, a_{ij} and b_i are the same as above, that is crisp numbers, whereas $\tilde{d}$ and $\tilde{b}_i$ are fuzzy quantities defined by (11.78) and (11.79). Moreover, $\tilde{\leq}^T$ is a T-fuzzy extension of the usual inequality relation $\leq$, with $T = \min$. It turns out that the vector $x \in \mathbf{R}^n$ is an optimal solution of flexible LP problem (11.81) if and only if it is a max-optimal solution of FLP problem (11.82). This statement follows directly from Theorem 132.

Notice that piecewise linear membership functions (11.78) and (11.79) can be replaced by more general nondecreasing and nonincreasing functions, respectively. In general, problem (11.80) cannot be then equivalently transformed to the LP problem (11.81). Such transformation is, however, sometimes possible, e.g. if all membership functions are generated by the same strictly monotone function.

11.7.3 FLP Problems with Interactive Fuzzy Parameters

In this subsection we shall deal with a fuzzy linear programming problem with the parameters being interacitive fuzzy quantities as introduced in Chapter 8.

Let f, g_i be linear functions defined by (11.1), (11.2), i.e.

$$f(x; c_1, ..., c_n) = c_1x_1 + \cdots + c_nx_n,$$

$$g_i(x; a_{i1}, ..., a_{in}) = a_{i1}x_1 + \cdots + a_{in}x_n, i \in \mathcal{M},$$

The parameters c_j, a_{ij} and b_i will be considered as normal fuzzy quantities, that is normal fuzzy subsets of the Euclidean space $\mathbf{R}$. Let $\mu_{\tilde{c}_j} : \mathbf{R} \to [0,1]$, $\mu_{\tilde{a}_{ij}} : \mathbf{R} \to [0,1]$ and $\mu_{\tilde{b}_i} : \mathbf{R} \to [0,1]$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, be membership functions of the fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively.

Let $\tilde{R}_i$, $i \in \mathcal{M}$, be fuzzy relations on $\mathcal{F}(\mathbf{R})$. Similarly to section 11.2. we have an exogenously given fuzzy goal $\tilde{d} \in \mathcal{F}(\mathbf{R})$ and another fuzzy relation $\tilde{R}_0$ on $\mathbf{R}$. Moreover, let $D_i = (d_{i1i}, d_{i2}, \ldots, d_{im})$ be a nonsingular $n \times n$ matrices - obliquity matrices, where all $d_{ij} \in \mathbf{R}^n$ are columns of matrices D_i, $i = \{0\} \cup \mathcal{M}$.

The *fuzzy linear programming problem with interacive parameters* (FLP problem with IP) associated with LP problem (11.3) is denoted as

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}_1x_1 \tilde{+}^{D_0} \cdots \tilde{+}^{D_0} \tilde{c}_nx_n \\ \text{subject to} & \\ & \tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n \ \tilde{R}_i \ \tilde{b}_i, \ i \in \mathcal{M}, \\ & \qquad\qquad x_j \geq 0, \ j \in \mathcal{N}. \end{array} \tag{11.83}$$

Let us clarify the elements of (11.83).

The objective function values and the left hand sides values of the constraints of (11.83) have been obtained by the extension principle (8.17) as follows. By (8.60) we obtain

$$\mu_{\tilde{a}_i}(a) = T(\mu_{\tilde{a}_{i1}}(\langle d_{i1}, a\rangle), \mu_{\tilde{a}_{i2}}(\langle d_{i2}, a\rangle), ..., \mu_{\tilde{a}_{in}}(\langle d_{in}, a\rangle)). \tag{11.84}$$

A membership function of $\tilde{g}_i(x; \tilde{a}_i)$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{g}_i}(t) = \begin{cases} \sup\{\mu_{\tilde{a}_i}(a) | & a = (a_1, ..., a_n) \in \mathbf{R}^n, a_1x_1 + \cdots + a_nx_n = t\} \\ & \text{if } g_i^{-1}(x; t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{11.85}$$

where $g_i^{-1}(x; t) = \{(a_1, ..., a_n) \in \mathbf{R}^n | a_1x_1 + \cdots + a_nx_n = t\}$. Here, the fuzzy set $\tilde{g}_i(x; \tilde{a}_i)$ is denoted as $\tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n$, i.e.

$$\tilde{g}_i(x; \tilde{a}_i) = \tilde{a}_{i1}x_1 \tilde{+}^{D_i} \cdots \tilde{+}^{D_i} \tilde{a}_{in}x_n \tag{11.86}$$

for every $i \in \mathcal{M}$ and for each $x \in \mathbf{R}^n$.

Also, for given interactive $\tilde{c}_1, ..., \tilde{c}_n \in \mathcal{F}(\mathbf{R})$, by (8.60) we obtain

$$\mu_{\tilde{c}}(c) = T(\mu_{\tilde{c}_1}(\langle d_{01}, c\rangle), \mu_{\tilde{c}_2}(\langle d_{02}, c\rangle), ..., \mu_{\tilde{c}_n}(\langle d_{0n}, c\rangle)). \tag{11.87}$$

A membership function of $\tilde{f}(x;\tilde{c})$ is defined for each $t \in \mathbf{R}$ by

$$\mu_{\tilde{f}}(t) = \begin{cases} \sup\{\mu_{\tilde{c}}(c) | & c = (c_1, ..., c_n) \in \mathbf{R}^n, c_1x_1 + \cdots + c_nx_n = t\} \\ & \text{if } f^{-1}(x;t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{11.88}$$

where $f^{-1}(x;t) = \{(c_1, ..., c_n) \in \mathbf{R}^n | c_1x_1 + \cdots + c_nx_n = t\}$. Here, the fuzzy set $\tilde{f}(x;\tilde{c})$ is denoted as $\tilde{c}_1x_1\tilde{+}^{D_0}\cdots\tilde{+}^{D_0}\tilde{c}_nx_n$, i.e.

$$\tilde{f}(x;\tilde{c}) = \tilde{c}_1x_1\tilde{+}^{D_0}\cdots\tilde{+}^{D_0}\tilde{c}_nx_n \tag{11.89}$$

for each $x \in \mathbf{R}^n$.

The treatment of FLP problem (11.83) is analogical to that of (11.83). The following proposition demonstrates how the α-cuts of (11.86) and (11.89) can be calculated.

Let D_0 be the non-singular obliquity matrix, denote $D_0^{-1} = \{d^*_{ij}\}^n_{i,j=1}$. For $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ we denote

$$\begin{aligned} I^+_x &= \{i \in \mathcal{N} | x_i \geq 0\}, \\ I^-_x &= \{i \in \mathcal{N} | x_i < 0\} \end{aligned}$$

and for all $i \in \mathcal{N}$

$$x^*_i = \sum_{j=1}^{n} d^*_{ij}x_j. \tag{11.90}$$

Given $\alpha \in (0,1]$, $j \in \mathcal{N}$, let

$$\underline{c}_j(\alpha) = \inf\{c | c \in [\tilde{c}_j]_\alpha\}, \tag{11.91}$$

$$\overline{c}_j(\alpha) = \sup\{c | c \in [\tilde{c}_j]_\alpha\}, \tag{11.92}$$

Proposition 140 *Let $\tilde{c}_1, ..., \tilde{c}_n \in \mathcal{F}_I(\mathbf{R})$ be compact interactive fuzzy intervals with an obliquity matrix D_0, $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$. Let T be a continuous t-norm and $\tilde{f}(x;\tilde{c}) = \tilde{c}_1x_1\tilde{+}^{D_0}\cdots\tilde{+}^{D_0}\tilde{c}_nx_n$ be defined by (11.88), $\alpha \in (0,1]$. Then*

$$\left[\tilde{f}(x;\tilde{c})\right]_\alpha = \left[\sum_{j\in I^+_{x^*}} \underline{c}_j(\alpha)x^*_j + \sum_{j\in I^-_{x^*}} \overline{c}_j(\alpha)x^*_j, \sum_{j\in I^+_{x^*}} \overline{c}_j(\alpha)x^*_j + \sum_{j\in I^-_{x^*}} \underline{c}_j(\alpha)x^*_j\right]. \tag{11.93}$$

Proof. Observe that $[\tilde{c}_j]_\alpha = \left[\underline{c}_j(\alpha), \overline{c}_j(\alpha)\right]$. The proof follows directly from (11.87), (11.88), (11.90) and Theorem 53. ∎

Analogical result can be formulated and proved for interactive fuzzy parameters in the constraints of (11.83), i.e. if $\tilde{a}_{i1}, ..., \tilde{a}_{in} \in \mathcal{F}_I(\mathbf{R})$ are compact interactive fuzzy intervals with an obliquity matrix D_i, $i \in \mathcal{M}$. Then we can take advatage of Theorem 127.

The practical difficulty of FLP with interactive parameters is that the membership functions of interactive parameters $\tilde{c}_j, \tilde{a}_{ij}$ are not observable. Instead, marginal fuzzy parameters can be measured or estimated. The problem of a unique representation of interactive fuzzy parameters by their marginals has been resolved in [35] and [70].

11.7.4 FLP Problems with Centered Parameters

Interesting FLP models can be obtained if the parameters of the FLP problem are supposed to be centered fuzzy numbers called $\mathcal{B}$-fuzzy intervals, see Definition 36 in Chapter 8, or [41].

Let $\mathcal{B}$ be a basis of generators ordered by $\subset$. Let $\leq_{\mathcal{B}}$be a partial ordering on the set $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$ of all $\mathcal{B}$-fuzzy intervals on $\mathbf{R}$, defined by (8.57) in Definition 36. Obviously, if $\mathcal{B}$ is completely ordered by $\subset$, then $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$ is completely ordered by $\leq_{\mathcal{B}}$. By Definition 36, each $\tilde{c} \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})$ can be uniquely represented by a couple (c, μ), where $c \in \mathbf{R}$ and $\mu \in \mathcal{B}$ such that

$$\mu_{\tilde{c}}(t) = \mu(c - t),$$

we can write $\tilde{c} = (c, \mu)$.

Let $\circ$ be either addition $+$ or multiplication $\cdot$ arithmetic operations on $\mathbf{R}$ and $\star$ be either min or max operations on $\mathcal{B}$. Let us introduce on $\mathcal{F}_{\mathcal{B}}(\mathbf{R})$ the following four operations:

$$(a, f) \circ^{(\star)} (b, g) = (a \circ b, f \star g) \tag{11.94}$$

for all $(a, f), (b, g) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})$. It can be easily proved that the pairs of operations $\left(+^{(\min)}, \cdot^{(\min)}\right)$, $\left(+^{(\min)}, \cdot^{(\max)}\right)$, $\left(+^{(\max)}, \cdot^{(\min)}\right)$, and $\left(+^{(\max)}, \cdot^{(\max)}\right)$, are distributive. For more properties of these information, see [43].

Now, let $\tilde{c}_j = (c_j, f_j), \tilde{a}_{ij} = (a_{ij}, g_{ij})$, $\tilde{b}_i = (b_i, h_i)$, $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})$ be $\mathcal{B}$-fuzzy intervals, $i \in \mathcal{M}$, $j \in \mathcal{N}$. Let $\diamond$ and $\star$ be either min or max operations on $\mathcal{B}$. Consider the following optimization problem:

$$\begin{aligned} &\text{maximize } \ \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n \\ &\text{subject to} \\ &\qquad \tilde{a}_{i1} \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{a}_{in} \cdot^{(\diamond)} \tilde{x}_n \ \leq_{\mathcal{B}} \ \tilde{b}_i,\ i \in \mathcal{M}, \\ &\qquad \tilde{x}_j \geq_{\mathcal{B}} \tilde{0},\ j \in \mathcal{N}. \end{aligned} \tag{11.95}$$

Here, maximization is performed with respect to the ordering $\leq_{\mathcal{B}}$. Moreover, $\tilde{x}_j = (x_j, \xi_j)$, where $x_j \in \mathbf{R}$ and $\xi_j \in \mathcal{B}$, $\tilde{0} = (0, \chi_{\{0\}})$. The constraints $\tilde{x}_j \geq_{\mathcal{B}} \tilde{0}$, $j \in \mathcal{N}$, is equivalent to $x_j \geq 0$, $j \in \mathcal{N}$. Comparing to the previous approach, we consider a different concept of feasible and optimal solution.

A *feasible solution* of the optimization problem (11.95) is a vector $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \cdots \times \mathcal{F}_{\mathcal{B}}(\mathbf{R})$, satisfying the constraints

$$\tilde{a}_{i1} \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{a}_{in} \cdot^{(\diamond)} \tilde{x}_n \ \leq \ {}_{\mathcal{B}}\tilde{b}_i, i \in \mathcal{M}, \tag{11.96}$$

$$\tilde{x}_j \ \geq \ {}_{\mathcal{B}}\tilde{0}, j \in \mathcal{N}. \tag{11.97}$$

The set of all feasible solutions of (11.95) is denoted by $X_{\mathcal{B}}$.

An *optimal solution* of the optimization problem (11.95) is a vector $(\tilde{x}_1^*, \tilde{x}_2^*, \cdots, \tilde{x}_n^*) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \mathcal{F}_{\mathcal{B}}(\mathbf{R}) \times \cdots \times \mathcal{F}_{\mathcal{B}}(\mathbf{R})$ such that

$$\tilde{z}^* = \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1^* +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n^* \tag{11.98}$$

is the maximal element of the set

$$X_{\mathcal{B}}^* = \{\tilde{z} | \tilde{z} = \tilde{c}_1 \cdot^{(\diamond)} \tilde{x}_1 +^{(\star)} \cdots +^{(\star)} \tilde{c}_n \cdot^{(\diamond)} \tilde{x}_n, (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in X_{\mathcal{B}}\}. \tag{11.99}$$

Notice that for each of four possible combinations of min and max in the operations $\cdot^{(\diamond)}$ and $+^{(\star)}$, (11.95) defines in fact an individual optimization problem.

Proposition 141 *Let $\mathcal{B}$ be a completely ordered basis of generators. Let $(\tilde{x}_1^*, \tilde{x}_2^*, \cdots, \tilde{x}_n^*) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})^n$ be an optimal solution of (11.95), where $\tilde{x}_j^* = (x_j^*, \xi_j^*)$, $j \in \mathcal{N}$. Then the vector $x^* = (x_1^*, \cdots, x_n^*)$ is the optimal solution of the following LP problem:*

$$\begin{array}{ll} \textit{maximize} & c_1 x_1 + \cdots + c_n x_n \\ \textit{subject to} & \\ & a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i,\ i \in \mathcal{M}, \\ & \qquad\qquad x_j \geq 0,\ j \in \mathcal{N}. \end{array} \tag{11.100}$$

Proof. The proof immediatelly follows from the definition (11.94) of the extended operations and from (8.57). ■

By A_x we denote the set of indices of all active constraints of (11.100) at $x = (x_1, \cdots, x_n)$, i.e.

$$A_x = \{i \in \mathcal{M} | a_{i1} x_1 + \cdots + a_{in} x_n = b_i\}.$$

The following proposition gives a necessary condition for the existence of a feasible solution of (11.95). The proof can be found in [41].

Proposition 142 *Let $\mathcal{B}$ be a completely ordered basis of generators. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \mathcal{F}_{\mathcal{B}}(\mathbf{R})^n$ be a feasible solution of (11.95), where $\tilde{x}_j = (x_j, \xi_j)$, $j \in \mathcal{N}$. Then the vector $x = (x_1, \cdots, x_n)$ is the feasible solution of the LP problem (11.100) and it holds*
(i) if $\diamond = \max$ and $\star = \min$, then

$$\min\{a_{ij} | j \in \mathcal{N}\} \leq_{\mathcal{B}} b_i \textit{ for all } i \in A_x;$$

(ii) if $\diamond = \max$ and $\star = \max$, then

$$\max\{a_{ij} | j \in \mathcal{N}\} \leq_{\mathcal{B}} b_i \textit{ for all } i \in A_x.$$

In this subsection we have presented an alternative approach to LP problems with fuzzy parameters. Comparing to the approach presented in the previous sections, the decision variables x_j considered here have not been taken as crisp numbers, they have been considered as fuzzy intervals of the same type as the corresponding coefficients - parameters of the optimization problem. From computational point of view this approach is simple as it requires to solve only a classical LP problem.

11.8 Illustrative Examples

In this section we present two "one-dimensional examples" illustrating the basic concepts. The examples below could be, in many aspects, extended from $\mathbf{R}^1$ to $\mathbf{R}^n$.

Example 143 *Consider the following simple FLP problem in* $\mathbf{R}^1$.

$$\begin{array}{ll} \widetilde{maximize} & \tilde{c}x \\ subject\ to & \\ & \tilde{a}x \mathrel{\tilde{\leq}} \tilde{b}, \\ & x \geq 0. \end{array} \tag{11.101}$$

Here, $\tilde{c}, \tilde{a}$ and $\tilde{b}$ are supposed to be crisp subsets of $\mathbf{R}$, particularly, closed bounded intervals: $\tilde{c} = [\underline{c}, \overline{c}]$, $\tilde{a} = [\underline{a}, \overline{a}]$, $\tilde{b} = [\underline{b}, \overline{b}]$, with $\underline{c}, \underline{b} > 0$. Let $T = \min$. Remember that the membership functions of $\tilde{c}, \tilde{a}$ and $\tilde{b}$ are their characteristic functions. The fuzzy relation $\tilde{\leq}$ and $\tilde{\geq}$ is assumed to be a T-fuzzy extension of the binary relation $\leq$ and $\geq$, respectively.

(I) *Membership functions of $\tilde{c}x$ and $\tilde{a}x$.*
By (8.17) we obtain for every $t \in \mathbf{R}$:

$$\mu_{\tilde{c}x}(t) = \sup\{\chi_{[\underline{c},\overline{c}]}(c) | c \in \mathbf{R}, cx = t\} = \{ \begin{array}{l} 1 \text{ if } \underline{c}x \leq t \leq \overline{c}x, \\ 0 \text{ otherwise.} \end{array} \tag{11.102}$$

Similarly, we obtain the membership function of $\tilde{a}x$ as

$$\mu_{\tilde{a}x}(t) = \sup\{\chi_{[\underline{a},\overline{a}]}(a) | a \in \mathbf{R}, ax = t\} = \{ \begin{array}{l} 1 \text{ if } \underline{a}x \leq t \leq \overline{a}x, \\ 0 \text{ otherwise.} \end{array} \tag{11.103}$$

Now, we derive the membership function $\mu_{\tilde{\leq}}$, $\mu_{\tilde{\geq}}$ of the fuzzy relations $\tilde{\leq}$, $\tilde{\geq}$, respectively.
By (8.37) we obtain

$$\mu_{\tilde{\geq}}(\tilde{c}y, \tilde{c}x) = \sup\{\min\{\mu_{\tilde{c}y}(u), \mu_{\tilde{c}x}(v)\} | u \geq v\},$$

$$\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) = \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} | u \leq v\}. \tag{11.104}$$

A feasible solution can be calculated as follows.
By (11.11) a feasible solution $\tilde{X}$ of the FLP problem (11.101) is given by the membership function

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}), \chi_{[0,+\infty)}(x)\} \tag{11.105}$$

By (11.104) and (11.103), we get

$$\begin{aligned} \mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) &= \sup\{\min\{\chi_{[\underline{a}x,\overline{a}x]}(u), \chi_{[\underline{b},\overline{b}]}(v)\} | u \leq v\} \\ &= \{ \begin{array}{l} 1 \text{ if } \underline{a}x \leq \overline{b}, \\ 0 \text{ otherwise.} \end{array} \end{aligned} \tag{11.106}$$

Consider 3 cases of the value of $\underline{a}$:

Case 1: $\underline{a} > 0$. *From (11.106) it follows that*

$$\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) = \chi_{[0,\frac{\overline{b}}{\underline{a}}]}(x). \tag{11.107}$$

By (11.105) and (11.107) we get

$$\mu_{\tilde{X}}(x) = \min\{\chi_{[0,\frac{\overline{b}}{\underline{a}}]}(x), \chi_{[0,+\infty)}(x)\} = \chi_{[0,\frac{\overline{b}}{\underline{a}}]}(x), \tag{11.108}$$

or, in other words

$$\tilde{X} = \left[0, \frac{\overline{b}}{\underline{a}}\right].$$

Case 2: $\underline{a} = 0$. *Since $\overline{b} > 0$, apparently by (11.105) and (11.106) we get*

$$\mu_{\tilde{X}}(x) = \chi_{[0,+\infty)}(x), \tag{11.109}$$

or

$$\tilde{X} = [0, +\infty).$$

Case 3: $\underline{a} < 0$, *then $\frac{\overline{b}}{\underline{a}} < 0$. From (11.110) and (11.106) it follows that*

$$\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) = \chi_{[\frac{\overline{b}}{\underline{a}},+\infty)}(x). \tag{11.110}$$

By (11.105) and (11.110) we get for all $x \geq 0$

$$\mu_{\tilde{X}}(x) = \chi_{[0,+\infty)}(x),$$

or

$$\tilde{X} = [0, +\infty).$$

(II) *Optimal solution $\tilde{X}^*$ of FLP problem (11.101).*

Consider a fuzzy goal $\tilde{d}$ given by the membership function

$$\mu_{\tilde{d}}(t) = \max\{0, \min\{\beta t, 1\}\},$$

where β is sufficiently small positive number, e.g. $\beta \leq \underline{a}/\overline{b}$, to secure that $\mu_{\tilde{d}}$ is strictly increasing function in a sufficiently large interval. By (11.26) and (11.27) we obtain

$$\mu_{\tilde{X}^*}(x) = \min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\}, \tag{11.111}$$

$$\begin{aligned}\mu_{\tilde{X}_0}(x) = \mu_{\tilde{\geq}}(\tilde{a}x\ ,\ \tilde{b}) = \sup\{\min\{\chi_{[\underline{c}x,\overline{c}x]}(u), \mu_{\tilde{d}}(v)\} | u \geq v\}& \\ = \mu_{\tilde{d}}(\overline{c}x).&\end{aligned} \tag{11.112}$$

Consider 2 cases corresponding to the value of $\underline{a}$*:*

Case 1: $\underline{a} > 0$. *Then by (11.108),* $\tilde{X} = [0, \frac{\overline{b}}{\underline{a}}]$ *and by (11.111) and (11.112) we obtain*

$$\mu^*_{\tilde{X}}(x) = \begin{cases} \mu_{\tilde{d}}(\overline{c}x) & \text{if } x \in [0, \frac{\overline{b}}{\underline{a}}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in (0, 1]$. *By Proposition 131, it is easy to verify that*

$$[\tilde{X}^*]_\alpha = \left[\frac{\alpha}{\beta}, \frac{\overline{b}}{\underline{a}}\right]. \tag{11.113}$$

By Proposition 132 we obtain the unique optimal solution with maximal height

$$x^* = \frac{\overline{b}}{\underline{a}}.$$

Case 2: $\underline{a} \leq 0$. *Then by (11.109),* $\tilde{X} = [0, +\infty)$ *and*

$$\mu^*_{\tilde{X}}(x) = \mu_{\tilde{b}_0}(\overline{c}x).$$

for all $x \in \mathbf{R}$.
Again, by Proposition 131 we obtain the α*-cut of the optimal solution*

$$[\tilde{X}^*]_\alpha = \left[\frac{\alpha}{\overline{c}\beta}, +\infty\right).$$

The set of all optimal solution with maximal height is the interval

$$\left[\frac{1}{\overline{c}\beta}, +\infty\right).$$

Example 144 *Consider the same FLP problem as in Example 143, but with different fuzzy parameters. The problem is as follows*

$$\begin{array}{ll} \widetilde{\text{maximize}} & \tilde{c}x \\ \text{subject to} & \\ & \tilde{a}x \,\tilde{\leq}\, \tilde{b}, \\ & x \geq 0. \end{array} \tag{11.114}$$

Here, the parameters $\tilde{c}, \tilde{a}$ *and* $\tilde{b}$ *are supposed to be triangular fuzzy numbers. To restrict a large amount of particular cases, we suppose that*

$$0 < \gamma < c, 0 < \alpha < a, 0 < \beta < b. \tag{11.115}$$

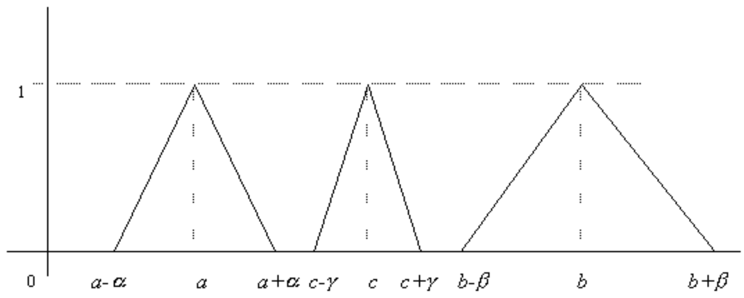

Figure 11.1:

Piecewise linear membership functions $\mu_{\tilde{c}}$, $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$ are defined for each $x \in \mathbf{R}$ as follows:

$$\mu_{\tilde{c}}(x) = \max\left\{0, \min\left\{1 - \frac{c-x}{\gamma}, 1 + \frac{c-x}{\gamma}\right\}\right\}, \tag{11.116}$$

$$\mu_{\tilde{a}}(x) = \max\left\{0, \min\left\{1 - \frac{a-x}{\alpha}, 1 + \frac{a-x}{\alpha}\right\}\right\}, \tag{11.117}$$

$$\mu_{\tilde{b}}(x) = \max\left\{0, \min\left\{1 - \frac{b-x}{\beta}, 1 + \frac{b-x}{\beta}\right\}\right\}, \tag{11.118}$$

see Fig. 11.1.
Let $T = \min$. The fuzzy relation $\tilde{\leq}$ is assumed to be a T-fuzzy extension of the binary relation $\leq$.

(I) *Membership functions of $\tilde{c}x$ and $\tilde{a}x$.*
Let $x > 0$. Then by (8.17) we obtain for every $t \in \mathbf{R}$:

$$\begin{aligned} \mu_{\tilde{c}x}(t) &= \sup\{\mu_{\tilde{c}}(c) | c \in \mathbf{R}, cx = t\} = \mu_{\tilde{c}}(\frac{t}{x}) \\ &= \max\left\{0, \min\left\{1 - \frac{cx-t}{\gamma x}, 1 + \frac{cx-t}{\gamma x}\right\}\right\}. \end{aligned} \tag{11.119}$$

In the same way, we obtain the membership function of $\tilde{a}x$ as

$$\mu_{\tilde{a}x}(t) = \max\left\{0, \min\left\{1 - \frac{ax-t}{\alpha x}, 1 + \frac{ax-t}{\alpha x}\right\}\right\} \tag{11.120}$$

Let $x = 0$. Then

$$\mu_{\tilde{c}x}(t) = \chi_0(t) \textit{ and } \mu_{\tilde{a}x}(t) = \chi_0(t)$$

for every $t \in \mathbf{R}$.

Figure 11.2:

Second, we calculate the membership function $\mu_{\tilde{\leq}}$ *of the fuzzy relation* $\tilde{\leq}$.
Let $x > 0$. *Then, see Fig. 11.2,*

$$\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) = \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} | u \leq v\},$$

For $x = 0$ *we calculate*

$$\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) = \sup\{\min\{\chi_{\{0\}}(u), \mu_{\tilde{b}}(v)\} | u \leq v\} = 1. \tag{11.121}$$

(II) *Feasible solution.*
By (11.11) a feasible solution $\tilde{X}$ *of the FLP problem (11.114) is given by the membership function*

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}), \chi_{[0,+\infty)}(x)\}. \tag{11.122}$$

Suppose that $x \geq 0$. *Using (11.116), (11.117) and (11.120), we calculate*

$$\begin{aligned} \mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}) &= \sup\{\min\{\mu_{\tilde{a}x}(u), \mu_{\tilde{b}}(v)\} | u \leq v\} \qquad (11.123) \\ &= \{ \begin{array}{ll} 1 & \textit{if } 0 < x, ax \leq b, \\ \dfrac{b + \beta - (a-\alpha)x}{\alpha x + \beta} & \textit{if } \ b < ax, (a-\alpha)x \leq b + \beta, \\ 0 & \textit{otherwise.} \end{array} \end{aligned}$$

By (11.11) a feasible solution $\tilde{X}$ *of the FLP problem (11.114) is given by the membership function*

$$\mu_{\tilde{X}}(x) = \min\{\mu_{\tilde{\leq}}(\tilde{a}x\ ,\ \tilde{b}), \chi_{[0,+\infty)}(x)\}.$$

Recalling (11.122), (11.126) we summarize, see Fig. 11.3.

Figure 11.3:

$$\mu_{\tilde{X}}(x) = \begin{cases} 1 & \textit{if} \quad 0 \leq x, ax \leq b, \\ \dfrac{b+\beta-(a-\alpha)x}{\alpha x+\beta} & \textit{if} \;\; b < ax, (a-\alpha)x \leq b+\beta, \\ 0 & \textit{otherwise.} \end{cases} \tag{11.124}$$

Let $\varepsilon \in (0,1]$. *From (11.124) it follows that*

$$\mu_{\tilde{X}}(x) \geq \varepsilon$$

if and only if

$$\frac{b+\beta-(a-\alpha)x}{\alpha x+\beta} \geq \varepsilon \text{ and } x \geq 0,$$

or equivalently,

$$0 \leq x \leq \frac{b+(1-\varepsilon)\beta}{a-(1-\varepsilon)\alpha}.$$

In other words,

$$[\tilde{X}]_\varepsilon = \left[0, \frac{b+(1-\varepsilon)\beta}{a-(1-\varepsilon)\alpha}\right]. \tag{11.125}$$

(III) *Optimal solution of FLP problem (11.114).*
Consider a fuzzy goal $\tilde{d}$ *given by the membership function*

$$\mu_{\tilde{d}}(x) = \min\{\beta x, 1\}, \tag{11.126}$$

for all $x \geq 0$, *where* β *is sufficiently small positive number, e.g.* $\beta \leq a/b$. *By (11.26) and (11.27) we have*

$$\mu_{\tilde{X}^*}(x) = \min\{\mu_{\tilde{X}_0}(x), \mu_{\tilde{X}}(x)\}, \tag{11.127}$$

Figure 11.4:

$$\mu_{\tilde{X}_0}(x) = \mu_{\tilde{\geq}}(\tilde{c}x \ , \ \tilde{d}) = \sup\{\min\{\mu_{\tilde{c}x}(u), \mu_{\tilde{d}}(v)\} | u \geq v\}. \tag{11.128}$$

By (11.119) and (11.126) we calculate for all $x \geq 0$

$$\mu_{\tilde{X}_0}(x) = \min\{\delta x, 1\}, \tag{11.129}$$

where

$$\delta = \frac{\beta(c+\gamma)}{1+\beta\gamma}.$$

The membership function of optimal solution given by (11.127) is depicted in Fig. 11.4. Combining (11.125) and (11.129) we obtain the set of all max-optimal solution $\bar{X}$ *from formula (11.127) as*

$$\bar{X} = \begin{cases} \dfrac{\sqrt{D} - [\beta\delta + (a-\alpha)]}{2\alpha\delta} & \textit{if } \dfrac{a}{b} < \dfrac{1}{\delta}, \\ \left[\dfrac{1}{\delta}, \dfrac{a}{b}\right] & \textit{otherwise}, \end{cases}$$

where

$$D = [\beta\delta + (a-\alpha)]^2 + 4\alpha\delta(b+\beta),$$

see Fig. 11.4.

Chapter 12

Conclusion

In this work, we focused primarily on fuzzy methodologies and fuzzy systems, as they bring basic ideas to the area of Soft Computing. The other constituents of SC are also surveyed here but for details we refere to the existing vast literature.

In the first part of this study we presented an overview of developments in the individual parts of SC. For each constituent of SC we briefly overviewed its background, main problems, methodologies and recent developments. Also the main literature, professional journals and technical newsletters, professional organizations and other relevant information are mentioned.

In the second part of the study we extensively studied the subject of fuzzy optimization, being an important part of SC. Here we presented original results of our research conducted by during the author's research stay at the School of Knowledge Science, JAIST, Hokuriku, Japan, in the first three months of 2001.

Already in the early stages of the development of fuzzy set theory, it has been recognized that fuzzy sets can be defined and represented in several different ways. Here we defined fuzzy sets within the classical set theory by nested families of sets, and then we discussed how this concept is related to the usual definition by membership functions. Binary and valued relations were extended to fuzzy relations and their properties were extensively investigated. Moreover, fuzzy extensions of real functions were studied, particularly the problem of the existence of sufficient conditions under which the membership function of the function value is quasiconcave.

We brought also some important applications of the theory, namely, we considered a decision problem, i.e. the problem to find a "best" decision in the set of feasible alternatives with respect to several (i.e. more than one) criteria functions. Within the framework of such a decision situation, we dealt with the existence and mutual relationships of three kinds of "optimal decisions": Weak Pareto-Maximizers, Pareto-Maximizers and Strong Pareto-Maximizers - particular alternatives satisfying some natural and rational conditions. We studied also the compromise decisions maximizing some aggregation of the criteria. The criteria considered here are functions defined on the set of feasible alternatives with the values in the unit interval. Such functions can be interpreted as membership functions of fuzzy subsets and will be called here fuzzy criteria. Each constraint or objective function of the fuzzy mathematical programming problem has been naturally appointed to the unique fuzzy criterion.

Fuzzy mathematical programming problems form a subclass of decision - making problems where preferences between alternatives are described by means of objective function(s) defined on the set of alternatives in such a way that greater values of the function(s) correspond to more preferable alternatives (if "higher value" is "better"). The values of the objective function describe effects from choices of the alternatives. First we presented a general formulation FMP problem associated with the classical MP problem, then we defined a feasible solution of FMP problem and optimal solution of FMP problem as special fuzzy sets. Among others we have shown that the class of all MP problems with (crisp) parameters can be naturally embedded into the class of FMP problems with fuzzy parameters.

We dealt also with a class of fuzzy linear programming problems. and again investigated feasible and optimal solutions - the necessary tools for dealing with such problems. In this way we showed that the class of crisp (classical) LP problems can be embedded into the class of FLP ones. Moreover, for FLP problems we defined the concept of duality and proved the weak and strong duality theorems. Further, we investigated special classes of FLP - interval LP problems, flexible LP problems, LP problems with interactive coefficients and LP problems with centered coefficients.

In the study we introduced an original unified approach by which a number of new and yet unpublished results have been acquired.

Our approach to SC presented in this work is mathematically oriented as the author is a mathematician. There exist, however, other approaches to SC, e.g. human-science approach and also computer approach, putting more stress on other aspects of he subject.

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