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2 Main objects of Ergodic Theory

3 Main theorems of Ergodic Theory

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5 Some applications

Personal motivation

Why I have chosen Ergodic Theory?

- It is usually missing in curriculae of universities in our regions.
- I found it interesting. The topic contains nice results with many deep applications.
- In this topic both deterministic and stochastic points of view meet in result of the kind: "For a truly random element the following surely holds".
- Typical result of the theory is educational, it says: "Under stated conditions, almost all elements behave fairly."

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Real motivation

Example from physics

- Statistical mechanics: Consider a system of N particles enclosed in a box. Their positions and momenta determine the system by $6N$ numbers, i.e.the state of the system at each moment can be represented by a point in a bounded subset X of $6N$-dimensional Euclidean space (the so-called *phase space*). The behaviour of the system is then represented by a trajectory in the phase space.
- In classical, deterministic mechanics, the entire trajectory is determined once one of its point is known. In practice we almost never have enough information for such a complete determination.

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- The basic idea of statistical mechanics (Gibbs, Boltzmann): Instead of asking: ”what will the state be at time t?” we should ask ”what is the probability that at time t the state of the system will belong to a specified subset of the phase space?” The questions of greatest interest are the asymptotic ones: ”what will (probably) happen to the system as t tends to infinity”?
- Concentration on asymptotic questions allows us to simplify the model passing from the continuous to the discrete as follows.

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- In classical model the behaviour of the system is determined by the system of $6N$ Hamilton's equations. Solution of this system yields a parametric system of transformations $\{T_t\}_{t\in[0,\infty)}$ of the phase space X into itself where $T_t(x)$ is the state of system at time t from the initial state x. It forms a one parametric semigroup with unit $T_0 = id_X$ (obviously $T_{t+s} = T_t T_s$).
- From the point of view of the asymptotic behaviour ($t \to \infty$) we can simplify the model passing from the continuous to the discrete considering a subgroup $\{T_n \mid n \in \mathbb{Z}\}$ for a suitable choice of the time unit $t_0 = 1$. Moreover, simplifying $T = T_1$, the composed transformation T_n is in fact T^n.

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Real motivation

Example from physics

- Notice that the asymptotic behaviour of both continuous and discrete systems should be identical. In practice the problem is intractable because of enormous number of equations ($N \approx 10^{24}$).
- Nevertheless, Poincaré was able to found important information on asymptotic behaviour: "Almost all points of the phase space are recurrent, i.e. infinitely many of $T^n(x)$ are arbitrarily close to x".
- He used the Liuville's theorem: The Lebesgue measure λ on the phase space X satisfies $\lambda(T_t(E)) = \lambda(E)$ for all t and all measurable $E \subset X$.

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Poincaré's solution

- Fix $\varepsilon > 0$ and denote $W = \{x \in X \mid \forall n \in \mathbb{N}\ d(x, T^n(x)) > \varepsilon\}$. Divide W into finitely many pieces W_i, each of diameter less than ε. For each fixed i, all the sets $T^{-n}(W_i);\ n \in \mathbb{N}$ are pairwise disjoint.
- To see this, assume $x \in T^{-n}(W_i) \cap T^{-(n+k)}(W_i) \neq \emptyset$. Then there exists an $y = T^n(x) \in W_i \cap T^{-k}(W_i)$ and, consequently $d(y, T^k(y)) < \mathrm{diam}\ W_i < \varepsilon$, a contradiction.
- Consequently $\sum\limits_{k=1}^{\infty} \lambda(T^{-k}(W_i)) \le \lambda(X) < \infty$. Using Liuville's theorem we get $\lambda(T^{-k}(W_i)) = 0$ for each $k \in \mathbb{N}$, yielding $\lambda(W) = 0$.

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Location of Ergodic Theory in Mathematics

Relations to other mathematical fields

- Probability and measure. The main object of the theory is a measure preserving mapping.
- Topology. The convergence concept is needed. A typical result says something about some limit, or compares two limits.
- Functional analysis. Most, originally concrete, results are generalized as results on operators in Hilbert spaces.
- Applications in: dynamical systems, measure theory, number theory, physics, combinatorics, ...

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Almost all sets of positive integers have density $\frac{1}{2}$

- Assume that X is the binary uniformly distributed random variable, i.e. $X \in \{0, 1\}$ and $p(\{0\}) = p(\{1\}) = \frac{1}{2}$.
- Consider a sequence $(X_n)_{n\in\mathbb{N}}$ of independent random variables as described above. Then each such (X_n) can be identified with the real number $r = \sum_{n=1}^{\infty} \frac{X_n}{2^n} \in [0, 1]$ as well as with (via characteristic function) a subset $A \subset \mathbb{N}$, i.e. $A_r = \{n \in \mathbb{N} \,|X_n = 1\}$.
- By Borel's law of large numbers: The relation $d(A_r) = \lim_{n\to\infty} \frac{\#\{k\leq n \mid k\in A\}}{n} = \frac{1}{2}$ holds for almost all $r \in [0, 1]$.

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Almost all sequences in $[0,1)$ are uniformly distributed

- A sequence (x_n) in $[0,1]$ is uniformly distributed (u.d.) if $\lim_{N\to\infty} \frac{\#\{n\leq N \mid x_n\in[a,b)\}}{N} = b-a$ holds for every $0\leq a<b\leq 1$.
- Product measure λ^∞ on $[0,1]^\infty$ is the unique measure defined on the σ-algebra generated by all cylinders $C=[a_1,b_1]\times\cdots\times[a_n,b_n]\times[0,1]\times\dots$ such that $\lambda^\infty(C)=\prod_{i=1}^{n}(b_i-a_i)$.
- Almost all sequences in $[0,1]$ are u.d. with respect to the product measure λ^∞ on $[0,1]^\infty$.

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Measure spaces

Basic definitions and examples

- *Measure space* is a triplet $(X, \mathcal{A}, \mu)$ where X is a nonempty set, $\mathcal{A}$ a σ-algebra on X and μ a σ-additive measure on $\mathcal{A}$. A measure is *finite* if $\mu(X) < \infty$ and *normalized* (or *probability measure*) if $\mu(X) = 1$.
- Unit interval, unit circle, unit ball, bounded subset of $\mathbb{R}^n$ with corresponding Borel measurable sets (i.e. σ-algebra generated by open sets) and Lebesgue (often normalized) measure.
- A locally compact topological group with a countable base, with Borel measurability and Haar measure (i.e. shift invariant measure $\mu(gE) = \mu(E)$ for all $g \in G$ and Borel E).

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Measure spaces

Basic examples

- The set of all sequences $x = \{x_n\}$ of 0's and 1's, where n ranges either over the set of all positive integers or over the set of all integers. The measurable sets are the elements of the σ-algebra of generated by sets of the form $\{x \mid x_n = 1\}$. The measure is uniquely determined by the condition that its value on each intersection of k generating sets is always 2^{-k}.

Measurable transformations

Basic definitions

- A *measurable transformation* is a mapping T from a measure space $(X, \mathcal{A}, \mu)$ into a measure space $(Y, \mathcal{B}, \nu)$ such that $T^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}$. It is *measure preserving* if $\mu(T^{-1}(E)) = \nu(E)$ for all $E \in \mathcal{B}$. We always identify all transformations differing on a set of measure 0, thus speaking about a transformation we will always mean its equivalence class. Notice that if a measure preserving transformation is invertible (i.e. its inverse transformation exists), then its inverse is also measure preserving.

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- Transformation of the real line given by $Tx = 2x$ is invertible, measurable, but not measure preserving. In fact, $\lambda(T^{-1}E) = \frac{1}{2}\lambda(E)$ holds for all Borel sets E.
- A closely related transformation of $[0, 1)$ is defined by $T\,x = 2x \pmod 1$. Notice that $Tx = 2x$ if $x \in [0, \frac{1}{2})$ and $Tx = 2x - 1$ otherwise. It is not invertible (it is everywhere two - to - one), it is measurable and also measure preserving. For example $T^{-1}[\frac{1}{4}, \frac{5}{8}) = [\frac{1}{8}, \frac{5}{16}) \cup [\frac{1}{8} + \frac{1}{2}, \frac{5}{16} + \frac{1}{2})$. Similar considerations prove that $\lambda(T^{-1}E) = \lambda(E)$ holds for each half-open interval with dyadically rational endpoints and from there it follows that T is measure preserving. On the other side, notice that $\lambda(T\,E) \neq \lambda(E)$ in general.

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Basic examples

- An isomorphic transform is obtained as follows. Consider the unit circle with Borel measurability and with the normalized Lebesgue measure $\lambda/2\pi$ and define $T\,z = z^2$ for a complex unit z.
- Now consider the two-dimensional Euclid space with the transform given by $T(x, y) = (2x, \frac{1}{2}y)$. It is measure preserving as the inverse image of any rectangle is a rectangle with the same area. This example can be generalized to arbitrary finitely dimensional Euclid spaces and linear mapping with determinant with absolute value equal to 1.

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Basic examples

- Let X be the space of all unilateral sequences $(x_n)_{n=0}^{\infty}$ such that $x_n \in \{0, 1\}$ as described before. Let T be the transformation described by a unit shift on the indices, i.e. $Tx = y = (y_n)$, where $y_n = x_{n+1}$. This transformation is measure preserving, but not invertible.
- Denote by Y the space of all bilateral sequences $(y_n)_{n\in\mathbb{Z}}$ such that $y_n \in \{0, 1\}$ as described before. Let U be the transformation described by a unit shift on the indices, i.e. $Ux = y = (y_n)$, where $y_n = x_{n+1}$. This transformation is measure preserving and invertible.

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Basic examples

- There is a natural mapping S of X onto $[0,1]$ assigning to $x = x_0, x_1, x_2, \ldots$ the number $\sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$. This mapping is measure preserving and essentially one-to-one. Only dyadically rational numbers have two different pre-images. As this set is countable and of measure 0, it plays no role from the point of view of measure theory in further investigations and, consequently, the measure-theoretic structure of both spaces are isomorphic. The isomorphism S carries the unilateral shift T onto a measure preserving transformation T' on the unit interval defined (mod sets of measure 0) by $T' = STS^{-1}$. An examination shows that T' is an old friend: $T'x = 2x (\mathrm{mod}\ 1)$ almost everywhere.

Measurable transformations

Basic examples

- There is a natural correspondence P between the bilateral sequence space Y and $X \times X$ sending $(\dots x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ to the couple $(\{x_0, x_1, x_2, \dots\}, \{x_{-1}, x_{-2}, \dots)\}$. Again, this transformation is measure preserving. Denoting $Q(x, y) = (S\,x, S\,y)$ for $x,\ y \in X$ we have that QP is an isomorphism of Y onto $[0, 1)^2$. This isomorphism sends the bilateral shift onto a invertible and measure preserving transformation T'' of $[0, 1)^2$.

Basic examples

- An examination of the definition shows that the isomorphic image of the bilateral shift is the mapping T'' defined (mod 1) by

$$T''(x,y) = (2x, \frac{1}{2}y) \text{ when } x \in [0, \frac{1}{2})$$

and

$$T''(x,y) = (2x, \frac{1}{2}(y+1)) \text{ when } x \in [\frac{1}{2}, 1).$$

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- The transformation T'' can be geometrically described as follows. It first transforms the unit square by a linear transform onto the rectangle with bottom edge the interval $[0, 2)$ and the left edge $[0, \frac{1}{2})$ and then cut off the right half of this rectangle (with bottom edge $[1, 2)$) and move it, by translation, to the top half of the unit square. Because of its geometric nature, T'' is called Baker's transform.

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Poincaré's theorem

- Let T be a measure preserving transformation on a measure space $(X, \mathcal{A}, \mu)$ and $x \in E \in \mathcal{A}$ be given. The point x is called *recurrent* (with respect to E and T) if $T^n(x) \in E$ for at least one $n \in \mathbb{N}$.
- **Recurrence theorem** (Poincaré). If T is a measure preserving transformation on a space of finite measure, and if E is a measurable set, then almost every point of E is recurrent.
- We will show that a stronger version of this theorem holds. In fact, it is easy to show that for almost every $x \in E$ there are infinitely many values of $n \in \mathbb{N}$ such that $T^n(x) \in E$.

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Stronger version of Poincaré's theorem

- As each T^n is measure preserving, for all sets F_n of non-recurrent points of T^n we have $\mu(F_n) = 0$, consequently $\mu(\bigcup_{n=1}^{\infty} F_n) = 0$.
- Suppose $x \in E \setminus \bigcup_{n=1}^{\infty} F_n$. Then for every $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $T^{kn}(x) \in E$.
- Thus the conclusion of the Recurrence theorem can be formulated in terms of the characteristic function as follows: for almost all $x \in E$ the series $\sum_{n=1}^{\infty} \chi_E(T^n(x))$ diverges.

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Recurrence theorem

Stronger version of Poincaré's theorem

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- Thus the conclusion of the Recurrence theorem can be formulated in terms of the characteristic function as follows: for almost all $x \in E$ the series $\sum_{n=1}^{\infty} \chi_E(T^n(x))$ diverges.

Generalized theorem

- This conclusion can be generalized: if f is an arbitrary non-negative measurable function, then for almost every $x \in \{x \mid f(x) > 0\}$ the series $\sum_{n=1}^{\infty} f(T^n(x))$ diverges.
- The proof is easy: for each $k \in \mathbb{N}$ consider the set $E_k = \{x \mid f(x) > \frac{1}{k}\}$. Then, by Recurrence theorem, the series $\sum_{n=1}^{\infty} \chi_{E_k}(T^n(x))$ diverges except a set of measure zero, say F_n. Consequently, the series $\sum_{n=1}^{\infty} f(T^n(x))$ diverges except the set of measure zero $\bigcup_{n=1}^{\infty} F_n$.

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An addition to Recurrence theorem

Index of recurrence

- Let T be an ergodic transformation on a space X with probability measure μ and let $E \subset X$ be measurable. For $x \in E$ denote by $n(x)$ the smallest integer such that $T^{n(x)} \in E$. Then, by Recurrence theorem, n is defined almost everywhere in E. It is easy to see that n is measurable.
- M.Kac (Bull. A.M.S. 1947, p. 1006) showed that $\int_E n(x)\, dx = 1$. It can be expressed in the form $\frac{1}{\mu(E)} \int_E n(x)\, dx = \frac{1}{\mu(E)}$. The last equation says that the average length of the time that it takes a point of E to return to E is the reciprocal of the measure of E.

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Mean Ergodic Theorem

- We already know that almost all points of E infinitely many times turn back to E under the repeated action of measure preserving transformation T. It is natural to ask for some more precise characteristics of the set of all indices $\{n \in \mathbb{N} \mid T^n(x) \in E\}$.
- There is a natural "measure" on subsets of $\mathbb{N}$ called *asymptotic density*. For $A \subset \mathbb{N}$ it is defined by $d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \chi_A(i)$ provided the limit exists.
- We will be interested in asymptotic behaviour of $S_n(E, T, x) = \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))$.

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- We will set the problem in a more general frame. Given a measure preserving transformation T on X, consider the mapping $U = U(T)$ operating on functions on X by $Uf = f \circ T$ i.e. $Uf(x) = f(T(x))$. Notice that $U^n f(x) = f(T^n(x))$ for all $n \in \mathbb{N}$.
- We will be interested in $S_n(U) = \frac{1}{n} \sum_{i=0}^{n-1} U^i$. Notice that $S_n(U)\chi_E(x) = S_n(E, T, x)$.
- It is easy to see that U is a linear operator and, for measure preserving T, it is not difficult to show that U is an isometry on L_1. It is based on the fact that if E is a set of finite measure, then $U\chi_E(x) = \chi_E(T(x)) = \chi_{T^{-1}E}(x)$.

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Mean convergence

Mean Ergodic Theorem

- **Mean Ergodic Theorem** (von Neumann). If U is an isometry on a complex Hilbert space H and if P is the projection on the space of all vectors invariant under U, then $\frac{1}{n}\sum_{i=0}^{n-1} U^i f$ converges to Pf for every $f \in H$.
- The proof is trivial in dimension one. In this case $Uz = uz$, where u is a complex unit. If $u = 1$, each $S_n(U) = id_H$ and, as $U = id_H$, also $P = id_H$, consequently $\lim_{n\to\infty} S_n(U) = P$. In other case $S_n(U) = \frac{1-u^n}{n(1-u)} \to 0$ and, as 0 is the only invariant element in H, also $Pz = \lim_{n\to\infty} S_n(U)z = 0$.

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Mean convergence

Mean Ergodic Theorem

- In the finite-dimensional case U can be represented by a diagonal matrix with only complex units on the diagonal. It follows that each $S_n(U)$ is also diagonal and, by one-dimensional case, it tends to a diagonal matrix with only 0's and 1's one the diagonal. The limit matrix is therefore a projection, in fact the projection on the space of all vectors f such that $Uf = f$.

Pointwise convergence

Individual Ergodic Theorem

- The adjective ”Mean” in the previous theorem is due to convergence in the L_2 norm. In this section we will be interested in the pointwise convergence.
- **Individual Ergodic Theorem** (Birkhoff). If T is a measure preserving transformation on a space X (with possibly infinity measure) and if $f \in L_1$, then $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))$ converges for almost all $x \in X$. The limit function f^* is integrable and invariant (i.e. $f^*(T(x)) = f^*(x)$ almost everywhere). Moreover, if $\mu(X) < \infty$, then $\int f^*\, d\mu = \int f\, d\mu$.

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Individual Ergodic Theorem

- Notice that the condition $\mu(X) < \infty$ is necessary for $\int f^* \, d\mu = \int f \, d\mu$. Consider the translation $T(x) = x + 1$ in $\mathbb{R}$ and choose $f = \chi_{[0,1)}$. Then $\int f(x) \, dx = 1$, while $f^*(x) = 0$ for all $x \in \mathbb{R}$ as $f(T^i(x)) = 0$ for all $i \in \mathbb{N}$ with one possible exception.

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Decomposability of a mapping

- Suppose that T is a measure preserving transformation on X and X is the union of two disjoint measurable subsets E and F of positive measures, each of which is invariant under T (i.e. $T^{-1}E = E$). Then the study of any property of T on X reduces to the separate studies of the corresponding properties of T on E and F.
- In such a case we call T *decomposable*. The most significant transformations are the indecomposable ones, they are called *ergodic*.
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Equivalent conditions

- It is often useful to work with some conditions equivalent to ergodicity. Trivially, T is ergodic iff it has only trivial invariant subsets, i.e. if E is invariant, then either $\mu(E) = 0$, or $\mu(X \setminus E) = 0$.
- A function f is *invariant under T* iff T does not effect the value of $f(x)$, i.e. iff $f(T(x)) = f(x)$ for (almost) all x.
- A useful reformulation of ergodicity is this: T is ergodic if and only if every measurable invariant function is a constant. To see this, notice that a measurable set is invariant iff its characteristic function is invariant.

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- Consequently, the ”if” statement is trivial.
- To see the ”only if” statement, assume that f is measurable. For every $n \in \mathbb{N}$ consider the system of sets

$$X(k,n) = \left\{ x \in X \mid \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}$$

end observe that all they are invariant. Ergodicity implies that for every n there is a unique $k(n)$ such that $\mu(X(k,n)) = 1$, while all the others are of measure 0. Then $f(x) = c$ almost everywhere, where $c = \lim\limits_{n\to\infty} \frac{k(n)}{2^n}$.

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Translations in $\mathbb{Z}$ and $\mathbb{R}$

- For $a \in \mathbb{Z}$, the translation $T(x) = x + a$ on the space of integers is ergodic if and only if $|a| = 1$.
- If $|a| = 1$, then the only invariant sets are $\emptyset$ and $\mathbb{Z}$.
- If $a = 0$, then all $A \subset \mathbb{Z}$ are invariant.
- If $|a| > 1$, all sets $\{n \in \mathbb{Z} \mid n \equiv b \ (mod\ |a|)\}$ are invariant.
- No translation $T(x) = x + a$ on the space of reals is ergodic.
- More generally, there is no ergodic linear transformation on a finite dimensional real Euclidean space.

Examples

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Examples

Rotation of the unit circle

- Let X be the unit circle (as a subset of the complex plane). If $c \in X$ and T is defined by $T(x) = cx$, then T is ergodic iff c is not a root of unity.
- If $c^n = 1$, then $f(x) = x^n$ is a non-constant measurable invariant function.
- If c is not a root of unity, then no $f_n(x) = x^n,\ n \in \mathbb{Z}$ is ergodic. The rest of proof follows from the fact that $\{f_n \mid n \in \mathbb{Z}\}$ forms a complete orthogonal system in L_2.
- If $f = \sum_{n\in\mathbb{Z}} a_n f_n$, then $f(T(x)) = \sum_{n\in\mathbb{Z}} a_n c^n x^n$ If f is invariant, then $a_n = c^n a_n$ for all $n \in \mathbb{Z}$, hence $a_n = 0$ for all $n \neq 0$, i.e. f is a constant function.

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Multiplication in compact abelian groups

- The preceding example can be generalized as follows. Let X be a compact abelian group with a countable base and $T(x) = cx$ for some $c \in X$. Then T is ergodic if and only if the set $\{c^n \mid n \in \mathbb{Z}\}$ is dense in X.
- The proof is based on the following self-interesting lemma: If the measure space X is a topological space with a countable base, such that each non-empty open set has positive measure, and if T is an ergodic transformation on X, then for almost all $x \in X$ the orbit of x (i.e. the sequence $\{T^n(x) \mid n \in \mathbb{N}\}$ is dense in X.
- Remark: The condition of denseness is not sufficient.

Multiplication in compact abelian groups

- The preceding example can be generalized as follows. Let X be a compact abelian group with a countable base and $T(x) = cx$ for some $c \in X$. Then T is ergodic if and only if the set $\{c^n \mid n \in \mathbb{Z}\}$ is dense in X.
- The proof is based on the following self-interesting lemma: If the measure space X is a topological space with a countable base, such that each non-empty open set has positive measure, and if T is an ergodic transformation on X, then for almost all $x \in X$ the orbit of x (i.e. the sequence $\{T^n(x) \mid n \in \mathbb{N}\}$ is dense in X.
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Rotation on torus and linear transformations in Euclidean spaces

- Torus is the Cartesian product of two circles and it can be represented as $[0,1)^2$, with operations on each coordinate taken (mod 1). For two complex units b, c define the transformation of *rotation on the torus* by $T(x,y) = (bx, cy)$, where x and y are complex units, i.e. (x,y) is a point on the torus.
- A rotation on the torus is ergodic if and only if the numbers b and c are integrally independent, i.e. for any integers m, n the relation $b^m c^n = 1$ implies $m = n = 0$.

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An excursion into Ergodic Theory

Ladislav Mišík

Introduction

Main objects of Ergodic Theory

Main theorems of Ergodic Theory

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Some applications

Shifts

- Both the unilateral and the bilateral shifts are ergodic. Main idea of the proof follows.
- Let E be a measurable invariant set. As the measure on the space $\{0,1\}^{\mathbb{Z}}$ (or $\{0,1\}^{\mathbb{Z}}$) is almost determined by its values on sets that depend on a finite numbers of coordinates, there exists such a "finitely-dimensional" set A that is an arbitrary close approximation of E. For n large enough is the set $B = T^{-n}A$ determined by a disjoint set of coordinates, therefore $\mu(A \cap B) = \mu(A)\mu(B)$. Since all powers of T are measure preserving and E is invariant, B is also very close to E and, consequently also $A \cap B$ is so. Thus $\mu(E) \approx \mu(A \cap B) \approx \mu^2(E)$. Thus $\mu(E)$ is either 0 or 1.

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2 Main objects of Ergodic Theory

3 Main theorems of Ergodic Theory

4 Ergodicity

5 **Some applications**

Space and time means

- Recall that Birkhoff's theorem holds for any measure preserving transformation. Suppose now, that T is also ergodic. Firstly, ergodicity yields that f^*, being invariant under T, has to be constant. Secondly, integrability of f^* and the relation $\int f^* \, d\mu = \int f \, d\mu$ imply that this constant is equal to $\frac{1}{\mu(X)} \int f \, d\mu$ (especially, 0 if $\mu(X) = \infty$). Thus we have:
- A transformation T on X with $\mu(X) < \infty$ is ergodic iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \frac{1}{\mu(X)} \int f \, d\mu.$$

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Space and time means

- For a moment, let us turn back to the continuous version of the model. Remember that the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ has been chosen as a discrete approximation of the continuous model $\{T_t(x)\}_{t\in[0,\infty)}$. In the continuous case the "time term" of the space-time means equation is replaced by $\lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T f(T_t(x))\,dx$. Thus, for ergodic T and integrable f on X, we have:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(T_t(x))\,dt = \frac{1}{\mu(X)}\int f\,d\mu.$$

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Uniform distribution and Birkhoff's theorem

- A sequence (x_n) in $\mathbb{R}$ is said to be *uniformly distributed mod* 1 (u.d) if for every $a < b \in [0,1)$ the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(x_n)$ exists and is equal to $b-a$.
- Informally, the time that (x_n) "spends in $[a,b)$" is asymptotically equal to the relative mass of $[a,b)$ in $[0,1)$ (i.e. the time mean equals to the space mean).
- Remark: For an ergodic T on $[0,1)$ and for almost all $x \in [0,1)$ is the sequence $\{T^n(x)\}$ uniformly distributed.
- Example: The sequence $\{n\alpha\}$ is u.d. mod 1 if and only if α is irrational.

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Uniform distribution mod 1

- The importance of u.d. sequences follows from the following statement. A sequence (x_n) is u.d. mod 1 if and only if

$$\int_0^1 f(x)\,dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$$

holds for all continuous functions $f\colon [0,1) \to \mathbb{R}$

- The right-hand term allows to compute integrals with an arbitrary precision by simple arithmetic operations.
- Analogously are the u.d. mod 1 sequences defined in $\mathbb{R}^d$ and are used for evaluation of multidimensional integrals (Method Quasi Monte Carlo).

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Szemerédi's Theorem

- Szemerédi's theorem is a generalization of the famous Van der Waerden's theorem (1927), one of the fundamental result of Ramsey theory, and it is a milestone of combinatorial mathematics. It proves that if the set of positive integers is partitioned into finitely many subsets, then at least one of them contains arbitrary long arithmetic progression. In 1936 Turán and Erdös conjectured that an arbitrary long arithmetic progression exists in any set of positive integers with positive density.
- Szemerédi (1974) Let k be a positive integer and $\delta > 0$. Then there exists a positive integer $N = N(k, \delta)$, such that every subset of the set $\{1, 2, \ldots, N\}$ of size at least δN contains an arithmetic progression of length k.

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Szemerédi's Theorem

- The original proof by Szemerédi was very intricate and long. In 1977 Furstenberg used the ergodic theory to prove the theorem in much more simple and inspirational way. Furstenberg's techniques have been extended to prove many natural generalizations of the theorem which do not follow from Szemerédi' approach.