Conditional Value-at-Risk Constrained Portfolio Optimization

Martin Harcek

Department of Applied Mathematics and Statistics
Comenius University in Bratislava

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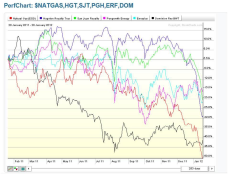
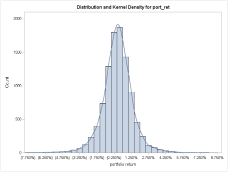
Outline

Portfolio Hedging

- hedging is a process of *insuring* portfolio value against (un)predictable (probably serious) losses
- portfolio can be hedged by several strategies, e.g.:
  1. investment diversification
  2. active buying/selling assets along holding period to achieve a securing result
  3. buying (or replicating) a securing derivative





- as a result of a hedging process we obtain adjusted portfolio payoff with reweighted probabilities of losses

Risk Measures

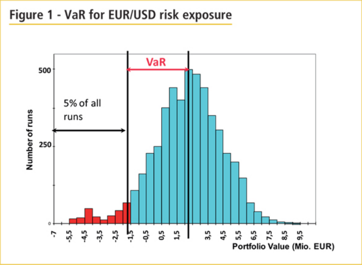
- one of the possible ways of hedging is to constrain portfolio value distribution by a risk measure
- risk measure ρ is a functional which assigns a real number (the risk exposure) to each portfolio represented by its random yield (value) X:

$$\rho : \mathcal{X} \to \mathbb{R}$$

  where $\mathcal{X}$ is a set of all possible portfolios represented by their yields and amount $\rho(X)$ is a risk exposure of portfolio X
- a family of risk measures functionals exhibits a fancy properties motivated by economical background

Main-Stream Risk Measures I

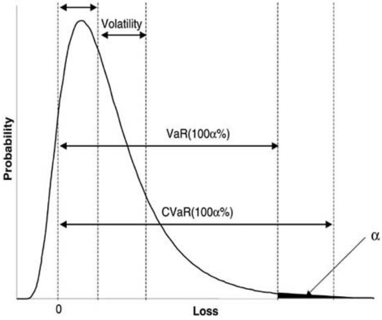
- Value-at-Risk (VaR):
  - defined as $(1-\alpha)$-quantile of a portfolio loss $(-X)$
  - answers the Q:
    What is the loss which is not to be exceeded with probability of $(1-\alpha)$*, e.g.* 99% *?*



  - VaR critism: not concerning the on-tail-shape
  - not a so called "coherent" RM (generally subaditivity property fails)

Main-Stream Risk Measures II

- Conditional Value-at-Risk (CVaR) - "upgraded" ver. of VaR
- defined as a mean extreme loss (greater than VaR)
- good properties (a coherent measure)



General Model Settings

- consider a market (economy) with N risky and 1 risk-free assets
- make assumptions about their price making processes:

$$\begin{aligned} dB(t) &= B(t)r(t)dt \\ dS_j(t) &= S_j(t)\left[\mu_j(t)dt + \sigma_j(t)dw(t)\right] \quad j \in 1, \ldots, N \end{aligned}$$

- assumption of a complete market implies existence of a unique state-price densiy process:

$$d\xi(t) = -\xi(t)\left[r(t)dt + \kappa(t)^\top dw(t)\right],$$

- where market parameters $r(t), \mu_j(t), \sigma_j(t), \kappa(t)$ are subject to calibration process

Problem Statement

- denote W_T a r.v. of T-time portfolio value
- maximize expected utility at investment horizon T
- while controlling risky exposure by CVaR measure (a kind of partial hedging):

$$\max \mathbb{E}\left[u\left(W_T\right)\right]$$
$$\mathbb{E}\left[\xi_T W_T\right] \leq \xi_0 W_0$$
$$CVaR_\alpha(W_T) \leq \delta W_0$$

where

$$CVaR_\alpha(W_T) \equiv \mathbb{E}\left[W_0 - W_T | W_0 - W_T \geq VaR_\alpha(W_T)\right]$$
$$VaR_\alpha(W_T) \equiv -\inf\left\{c \in \mathbb{R} : \mathbb{P}\left(W_0 - W_T \leq c\right) \geq \alpha\right\}$$

The Equivalent Problem

- previous problem formulation is hard to solve
- due to the complex CVaR reprezentation
- as CVaR is substitued by a much simpler functional:

$$\max_{W_T, c} \mathbb{E}\left[u\left(W_T\right)\right]$$
$$\mathbb{E}\left[\xi_T W_T\right] \leq W_0$$
$$c + \frac{1}{\alpha}\mathbb{E}\left[\left(W_0 - W_T - c\right)^+\right] \leq \delta W_0$$
$$c \in \mathbb{R}$$

- the problem statement transforms into two-step optimization
- but a kind of "tricky" CVaR representation disappears :)

Equivalent CVaR Representation I

- in optimization problems CVaR can be represented by a convex function (with respect to c) as stated in the next theorems:

$$G_\alpha(x,c) = c + \frac{1}{\alpha}\int_{y\in\mathbb{R}^n} (l(x,y) - c)^+ \ f(y)\,dy,$$

where $\alpha \in (0,1)$ is exogenous parameter, $c \in \mathbb{R}$, $l(x,y) \in \mathbb{R}$ is a function of deterministic vector $x \in X \subseteq \mathbb{R}^n$ and random vector $y \in \mathbb{R}^m$ s with distribution $f(y)$.

Equivalent CVaR Representation II

Theorem (Main Message)

Minimizing the $CVaR_\alpha(x)$ with respect to $x \in X$ is equivalent to minimizing the function $G_\alpha(x, c)$ with respect to $(x, c) \in X \times \mathbb{R}$ in the sense that:

$$\min_{x \in X} CVaR_\alpha(x) = \min_{(x,c) \in X \times \mathbb{R}} G_\alpha(x, c),$$

- this is motivation for next proposition:

Equivalent CVaR Representation III

Theorem (Main Message)

The two optimization problems:

$$\max_{x\in X} \mathbb{E}\left[u(x)\right], CVaR_\alpha(x) \leq \omega, X \in \{x \in L^2(\mathbb{P}) \,|\, \mathbb{E}\left[\xi x\right] \leq \phi\}$$

$$\max_{(x,c)\in X\times R} \mathbb{E}\left[u(x)\right], G_\alpha(x,c) \leq \omega, X \in \{x \in L^2(\mathbb{P}) \,|\, \mathbb{E}\left[\xi x\right] \leq \phi\}$$

are equivalent in that sense that their objective functions achieve the same minimum values.

Optimal Terminal Wealth I

- solution as a result of a two-stage optimization procedure:
  1. the first-stage solution (with respect to variable $W_T(c)$) solved using Kuhn-Tucker framework
  2. the final solution is solution of the second-stage optimization (with respect to variable c):

$$\max_{c \in \mathbb{R}} \mathbb{E}\left[u\left(\hat{W}_T(c)\right)\right]$$

- where $\hat{W}_T(c)$ T-time policy is defined as follows:

Theorem (First-stage T-time solution)

For each $c \in \mathbb{R}$ the optimal T-time wealth is defined:

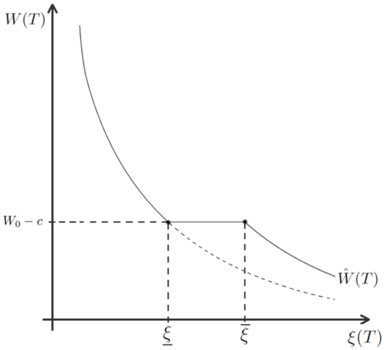
$$\hat{W}_T(c) = \begin{cases} I(y_1\xi_T) & \text{if } \xi_T < \underline{\xi} \\ W_0 - c & \text{if } \underline{\xi} \leq \xi_T < \overline{\xi} \\ I\left(y_1\xi_T - \frac{y_2}{\alpha}\right) & \text{if } \overline{\xi} \leq \xi_T, \end{cases}$$

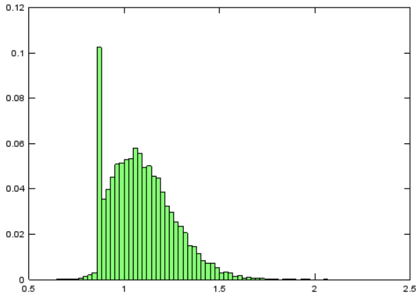
where $\underline{\xi} = u'(W_0 - c)/y_1$, $\overline{\xi} = \left(u'(W_0 - c) + \frac{y_2}{\alpha}\right)/y_1$ a y_1,$y_2 \geq 0$ are solutions of equations array:

$$\mathbb{E}\left[\xi_T \hat{W}_T(y_1, y_2)\right] = W(0)$$

$$c + \frac{1}{\alpha}\mathbb{E}\left[(W_0 - \hat{W}_T(y_1, y_2) - c)^+\right] = \delta W_0 \text{ or } y_2 = 0.$$

- we observe three market-state intervals on which manager exhibits different behavior:





Dynamic Hedging Strategy I

- denote $\hat{W}(t)$ a dynamic investment strategy, which leads to the evaluated optimal T-time wealth
- consider CRRA family utility function and constant market parameters (interest rate r and market price of risk κ)

$$u(x) = \begin{cases} x^p/p & p < 1, p \neq 0 \\ \ln(x) & p = 0 \end{cases}$$

- using martingale risk-neutral pricing framework
- no close-form solution, need of numerical computations

Dynamic Hedging Strategy II

- as T-time distribution of state-variable follows:

$$\ln \xi_T | \mathcal{F}_t \sim \mathcal{N}\left(ln\xi_t - (r + \frac{1}{2}\|\kappa\|^2)(T-t), \|\kappa\|^2(T-t) \right)$$

- and t-time optimal wealth is stated as:

$$\hat{W}(t) = \mathbb{E}\left[\frac{\xi_T}{\xi_t} \hat{W}_T(\cdot) \,|\, \mathcal{F}_t \right],$$

where $\hat{W}_T(\cdot)$ denotes Optimal Terminal Wealth solution

Dynamic Hedging Strategy III

Theorem (t-time solution)

$$\hat{W}(t) \quad = \quad G1 + G2 + G3,$$

where

$$G1 \equiv \frac{y_1^{\frac{1}{p-1}}}{\xi_t} \exp\left\{ \frac{p}{p-1} \left(\ln \xi_t + \left(\frac{\|\kappa\|^2}{2p-2} - r \right) (T-t) \right) \right\} \Phi(d1)$$

$$G2 \equiv \frac{W(0)-c}{\xi_t} \exp\left\{ \ln \xi_t - r(T-t) \right\} \left(\Phi(d2) - \Phi(d3) \right)$$

$$G3 \equiv \frac{1}{\xi_t} \int_{\underline{\xi}}^{\infty} \left(y_1 \xi_T - \frac{y_2}{2} \right)^{1-p} d\mathbb{P}(\xi_T)$$

Dynamic Hedging Strategy IV

Theorem (cont'd)

... and

$$
\begin{aligned}
d1 &\equiv \frac{\ln \underline{\xi} - \ln \xi_t + \left(r - \frac{1}{2}\frac{p+1}{p-1}\|\kappa\|^2 \right)(T-t)}{\|\kappa\|\sqrt{T-t}} \\
d2 &\equiv \frac{\ln \overline{\xi} - \ln \xi_t + \left(r - \frac{1}{2}\|\kappa\|^2 \right)(T-t)}{\|\kappa\|\sqrt{T-t}} \\
d3 &\equiv \frac{\ln \underline{\xi} - \ln \xi_t + \left(r - \frac{1}{2}\|\kappa\|^2 \right)(T-t)}{\|\kappa\|\sqrt{T-t}}
\end{aligned}
$$

- t-payoff profile convergence to T-time payoff

Dynamic Hedging Strategy V

CHART: $\hat{W}(t)$ as a function of $\xi(t)$
LEGEND: $q^B(t)$, $W(0.5)$, $W(0.75)$, $W(0.95)$

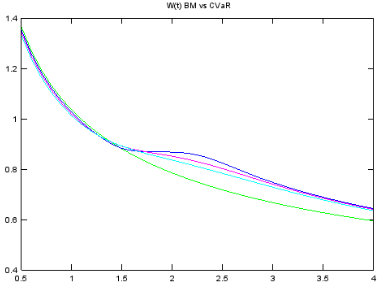
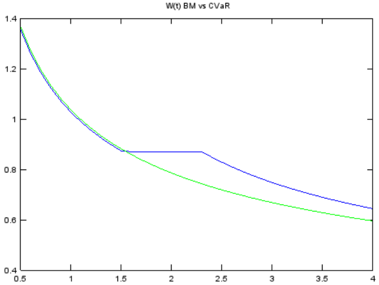


CHART: $\hat{W}(T)$ vs $\hat{W}^B(T)$ as a function of $\xi(t)$
LEGEND: $W^B(T = 1)$, $W(T = 1)$



Dynamic Investment Strategy I

- denote $\hat{\theta}(t)$ the investment strategy (in terms of exposure to risky assets), which leads to the optimal wealth management

$$\hat{\theta}(t) = -\frac{\left(\sigma^T\right)^{-1}\kappa^T}{\hat{W}(t)}\frac{\partial \hat{W}(t)}{\partial \xi(t)}\xi(t)$$
$$= -\frac{1-p}{\hat{W}(t)}\hat{\theta}_B(t)\frac{\partial \hat{W}(t)}{\partial \xi(t)}\xi(t)$$

- where $\hat{\theta}^B(t)$ denotes the Benchmark investor strategy as

$$\hat{\theta}^B(t) = \frac{1}{1-p}\left(\sigma^T\right)^{-1}\kappa^T$$

Dynamic Investment Strategy II

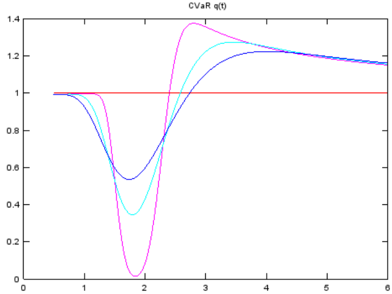
- thus we can define process $q(t)$ as the exposure to risky assets relative to benchmark

$$q(t) \equiv \frac{\hat{\theta}(t)}{\theta^B(t)}$$

- and analyze it...

Dynamic Strategy

CHART: $q(t)$ as a function of $\xi(t)$
LEGEND: $q^B(t)$, $q(0.55)$, $q(0.75)$, $q(0.95)$



- recall that process $q(t)$ express exposition to risky assets
- in good states (left part of chart) investor behaves similar like benchmark agent
- in intermediate states (left-middle part) investor closes his positions in risky assets
- in the worse states (right part) investor exploits a leverage effect to raise his portfolio value

Comparison with other strategies I

- we made a comparison simulation with well known Value-at-Risk based strategy developed by Basak & Shapiro (2001)
- we investigated terminal payoff shape $\hat{W}(T)$ as well as risky assets exposure dynamics $q(t)$
- the results show that CVaR strategy could reasonably over-perform VaR strategy in the worst market states

Comparison with other strategies II

CHART: $q(t)$ as a function of $\xi(t)$
LEGEND: VaR-RM, CVaR-RM

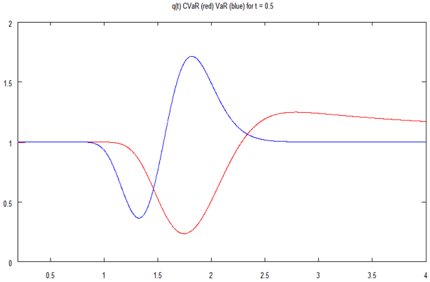
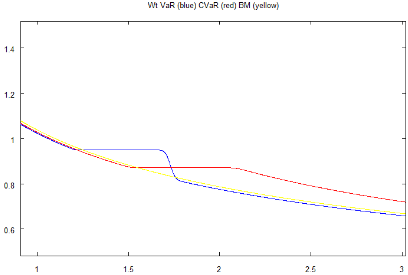


CHART: $\hat{W}(T)$ as a function of $\xi(t)$
LEGEND: VaR-RM, CVaR-RM



Conclusion

- we have defined a CVaR portfolio optimization problem
- we have proposed an equivalent problem definition
- we have developed a new hedging strategy

- Outlook
  - further investigation of stategy props

Financial RM Fancy Properties I

- a financial RM should exhibit special properties
- these are motivated by economical interpretation $$$
- in next statement considering portfolios $X, Y \in \mathcal{X}$ and a constant $\lambda \in \mathbb{R}$:
  - **Monotonicity:**
    $$X \succeq Y \implies \rho(X) \leq \rho(Y)$$
  - **Cash-Invariance:**
    $$\rho(X + \lambda) = \rho(X) - \lambda$$
  - **Convexity:** $\forall \lambda \in [0, 1]$:
    $$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)Y$$

Financial RM Fancy Properties II

- **Positive Homogenity:**
$$\forall \lambda \geq 0 : \quad \rho(\lambda X) = \lambda \rho(X)$$
- **Subaditivity:**
$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$