Numerical scheme for the level set equation based on discrete duality finite volumes in 3D

Dana Kotorová

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, STU Bratislava

May 2, 2013

Table of contents

Level set equation

- level set equation: $u_t - |\nabla u| \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0$
- unknown function $u(t, x)$ is defined in the domain $Q_T = I x \Omega$, $I = (0, T)$, $\Omega \subset I\!R^3$
- we consider homogeneous Neumann (or Dirichlet) boundary conditions and the initial condition:
- $\partial_\nu u = 0$ on $I \times \partial\Omega$ $\;(u(t, x) = 0$ on $I \times \partial\Omega)$
- $u(0, x) = u^0$

Numerical approximation

time discretization

- we set the unique time step $\tau = \frac{T}{N}$

denote u^n as an approximation of $u(t,x)$ at time $t_n = n\tau$

- first time derivative is replaced by the backward difference $\frac{u^n - u^{n-1}}{\tau}$
- level set equation can be rewritten into the form of semi-implicit scheme: $\frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} = \nabla \cdot \left(\frac{\nabla u^n}{|\nabla u^{n-1}|} \right)$
- Evans - Spruck regularization: $|\nabla u|_\varepsilon = \sqrt{\varepsilon^2 + |\nabla u|^2}$

Numerical approximation

by fully discretization we will divide our domain Ω (prism) into the finite volumes (cubes) and we can denote them as V_{ijk}, with the measure $m(V_{ijk}) = h^3$; e_{ijk}^{pqr} will represent the face between two neighboring finite volumes with the measure $m(e_{ijk}^{pqr}) = h^2$

- by application of the divergence theorem we get the integral formulation $\int\limits_{V_{ijk}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} = \sum\limits_{|p|+|q|+|r|=1} \int\limits_{e_{ijk}^{pqr}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds$
- approximation of the left-hand side is $\int\limits_{V_{ijk}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} dx \approx \frac{h^3}{AQ_{ijk}} \frac{u_{ijk}^n - u_{ijk}^{n-1}}{\tau}$

Original and dual mesh in DDS

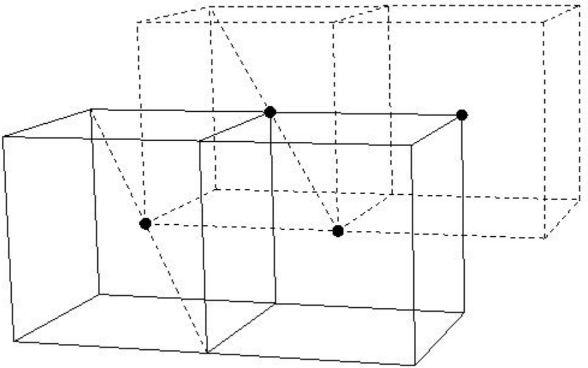


Figure : Original (solid lines cubes) and dual (dashed lines cubes) mesh

Face-edge mesh in DDS

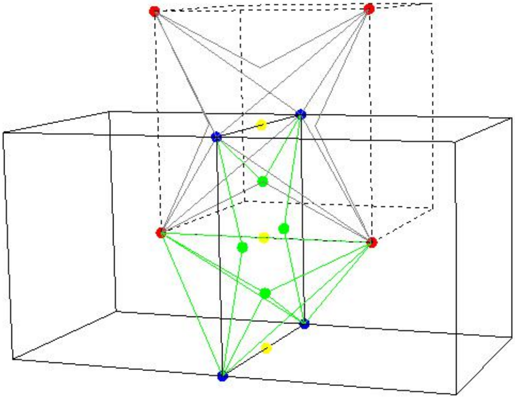
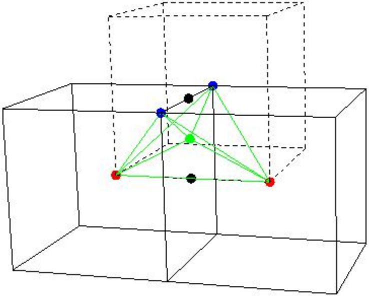


Figure : Face volume (green) and edge volume (gray) in the original mesh

Gradient approximation in DDS



- $\int\limits_{V_{ijk}} \frac{1}{|\nabla u^{n-1}|} \frac{u_{ijk}^n - u_{ijk}^{n-1}}{\tau} dx \approx \frac{h^3}{AQ_{ijk}} \frac{u_{ijk}^n - u_{ijk}^{n-1}}{\tau}$
- $AQ_{ijk} = \frac{1}{6} \sum\limits_{|p|+|q|+|r|=1} Q_{ijk}^{pqr;n-1}$
- $Q_{ijk}^{pqr;n-1} = \sqrt{\varepsilon^2 + |\nabla u_{ijk}^{n-1}|^2}$
- $\nabla D_1 X_{ijk} =$
$$\left(\frac{u_{i+1,j,k}^n - u_{i,j,k}^n}{h}, \frac{v_{i,j,k}^n - v_{i,j-1,k}^n}{h}, \frac{zy_{i,j,k}^n - wx_{i,j,k}^n}{\frac{h}{2}} \right)$$

Linear system of equations in DDS

$$\frac{A^n - A^{n-1}}{\tau} \cdot \frac{1}{AQ^{n-1}} \cdot m(V) + \sum_e \frac{A^n - A_e^n}{d} \cdot \frac{1}{Q^{n-1}} \cdot m(e) = 0$$

$m(V)$ - measure of the finite volume V
$m(e)$ - measure of the face of the finite volume V
d - distance between two neighboring volume centers

Cut paraboloid - Obermann solution

- cut paraboloid with zero Neumann boundary conditions
- numerical solution on the square mesh $n \times n \times n$
- exact solution given by min $\{\frac{1}{2}(x^2 + y^2 + z^2 - 1) - 2t, 0\}$

Cut paraboloid - Obermann solution

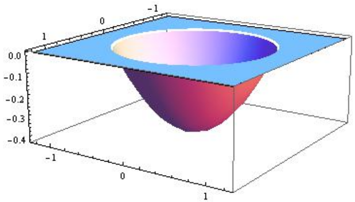


Figure : Exact (left) and numerical (right) solution after 10 time steps

$n = 40$, $\tau = h^2$

Cut paraboloid - Obermann solution

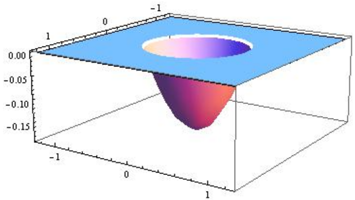


Figure : Exact (left) and numerical (right) solution after 40 time steps

$n = 40,\ \tau = h^2$

Cut paraboloid - Obermann solution

N	L_2 error	EOC L_2 error	L_2 gradient error	EOC L_2 gradient error	CPU
10	$6.5323e^{-2}$	–	$2.6010e^{-1}$	–	$3.7000e^{-1}$
20	$3.4497e^{-2}$	0.9211	$2.1560e^{-1}$	0.2707	$1.3480e^{+1}$
40	$1.7830e^{-2}$	0.9522	$1.7224e^{-1}$	0.3239	$4.6134e^{+2}$
80	$1.0902e^{-2}$	0.7085	$1.3935e^{-1}$	0.3057	$1.5847e^{+4}$

Table : EOC and errors obtained by DDS scheme

Thank you for your attention!