Maps and their exponents

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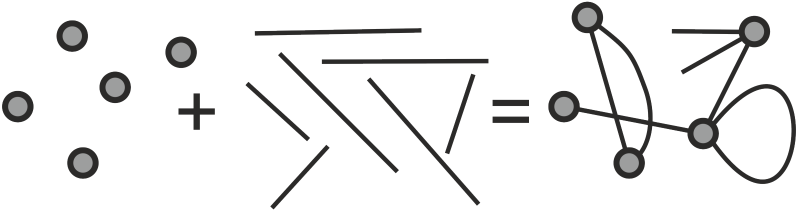
Exponents of maps

Graphs

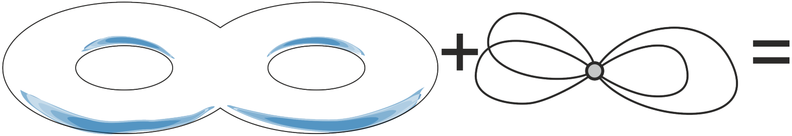
By a **graph** we mean any composition of 0-dimensional and 1-dimensional cells, which are *vertices* and *edges* of the graph

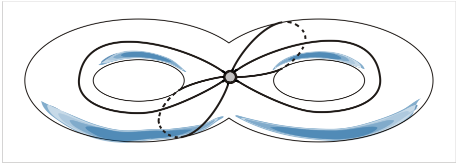


We allow loops and semiedges.

Maps

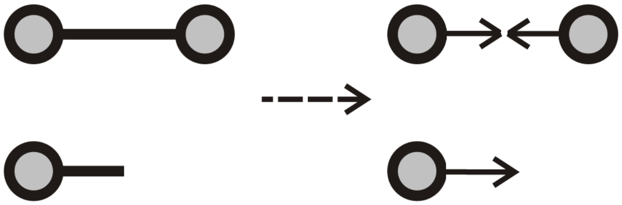
A **map** is a cellular embedding of graph in a surface.





In maps we obtain new 2-dimensional cells call **faces**.
The faces are connected components of surface with out image of edges an vertices.

For formal description of map we will need to subdivide the structure of edges that are not semi-edges.
Every edge that is not a semi-edge we subdivide to two **darts**.
Semi-edge will be considered to be also a **dart**.



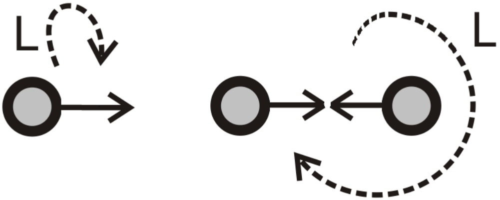
Note, that every dart is incident with just one vertex.
We will call it **initial** vertex of dart.

Edge *consists* of two darts. Two darts *form* edge.

Dart-flip

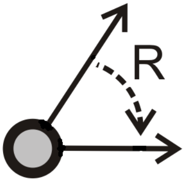
This all allows us to define a natural permutation L on the set D of darts of a graph, called a **dart-flip**,by letting;

- $xL = x$ if x is a semi-edges
- $xL = y$ if the darts x, y form an edge that is not a semi-edge.



The permutation L obviously satisfies $L^2 = id$.

A **local rotation** R_v describe clockwise order of darts around vertex v.
A **rotation** R is the product $\Pi_v R_v$ where v ranges over all vertices.



So, for a dart a the dart aR is the one that follows a on the surface as one moves around its initial vertex in the clockwise orientation of the surface

Monodromy group

So permutation L and R gives us a full information about map.

- cycles of R are vertices
- cycles of L are edges or semi-edges
- cycles of LR are faces

Hence we only need to remember the set of darts and this two permutations and we can reconstruct edges, vertices and faces.
So we can identify map M with triples $(D; R, L)$.

Group generated with R and L is called monodromy group of map $M = (D; R, L)$. Label $Mon(M) = \langle R, L \rangle$.

Exponents

Let $M = (D; R, L)$ be a map and e be some natural number.

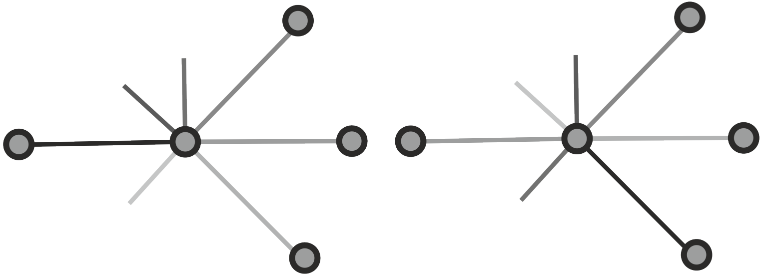
Now consider mapping $M = (D; R, L) \mapsto M^e = (D; R^e, L)$.

If e is coprime with all degree of vertices of M, then this mapping doesn't change structure of underlying graph.

If $M \cong M^e$ we say M has **exponent** e.

Exponents

For example mapping $M = (D; R, L) \mapsto M^4 = (D; R^4, L)$

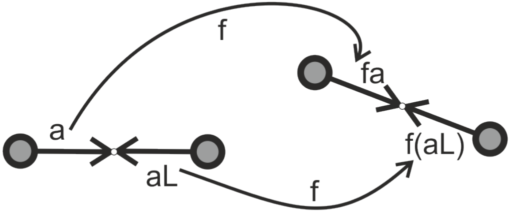


We can see 4 is exponent for this map.

1 and 2 is also exponents
3,5 and 6 are not exponents for now
7 is not coprime with 7

Exponents of regular maps

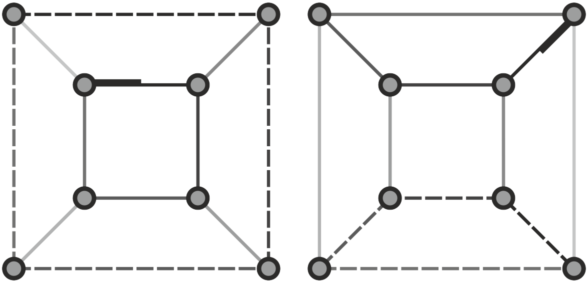
An automorphism of $M = (D; R, L)$ is a permutation f of the dart set D of M such that $f(aR) = (fa)R$ and $f(aL) = (fa)L$ for each dart $a \in D$.



All automorphisms of map form the group $Aut(M)$.

Regular maps

A regular map is a map M such that $Aut(M)$ is regular on D, in such a case $Aut(M) \simeq Mon(M)$ and $|Aut(M)| = |D|$



In a regular map, all vertices have the same degree, say, k, and all faces of the map have the same length, say, m;
the map is then said to have type $(k; m)$.

So e can be exponent of given regular map only if e is coprime with k.

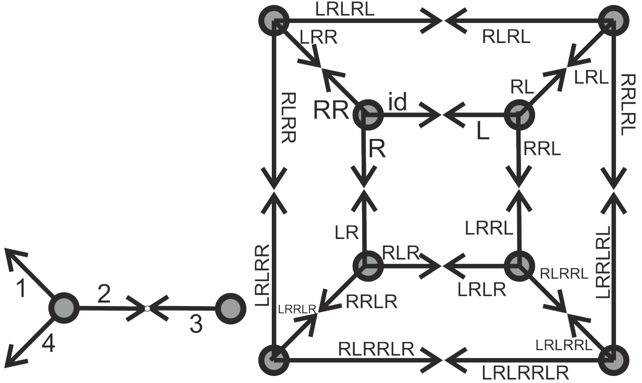
Regular maps and exponents

How to construct regular map with given exponent.

How to construct regular map without any exponent.

Canonical regular cover

Consider map $M = (D; R, L)$ with monodromy group $G = \langle R, L \rangle$. Map $\tilde{M} = (G; R, L)$ (where G is given just like set) is always regular map. Hence $\langle R, L \rangle$ is regular on its underlying set.



Map M we call **base map** and $\tilde{M}$ the **canonical regular cover** of M

Canonical regular cover and exponents

Lemma

Consider base map $M = (D; R, L)$ where $|D| = n$ and $n > 6$. If $\langle R, L \rangle \cong S_n$ or A_n, then its canonical regular cover $\tilde{M}$ has exponent e if and only if M has exponent e.

To satisfy $\langle R, L \rangle \cong S_n$ or A_n we use classical result of Jordan;

Lemma

Let G be a primitive permutation group of degree n that contains a cycle of prime length p. If $p \leq n - 3$, then G is isomorphic to S_n or A_n.