On a functional equation related to distributivity of fuzzy implications

Michał Baczyński, Wanda Niemyska, Tomasz Szostok

Institute of Mathematics
University of Silesia
Poland

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Content

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- The solutions of equation we have obtained.

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- Some intuition what „fuzzy logic” is.
- What do we study this equation for?
- How have we come to this equation?
- The solutions of equation we have obtained.
- What does still remain to do?

Classical formulas

I'm Human, so I'm Female.

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I'm Human, so I'm Female.

I'm Handsome and Smart,
so I'm Happy.

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$$(\text{Handsome} \wedge \text{Smart}) \rightarrow \text{Happy ?}$$

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$$(\text{Handsome} \wedge \text{Smart}) \to \text{Happy ?}$$

$$(0.9 \wedge 0.5) \to 0.6 \ ?$$

Classical formulas

I'm Human, so I'm Female.

I'm Handsome and Smart,
so I'm Happy.

$$(\text{Handsome} \wedge \text{Smart}) \to \text{Happy ?}$$

$$(0.9 \wedge 0.5) \to 0.6 \text{ ?}$$

$$I(T(0.9, 0.5), 0.6) = ?$$

Introduction of fuzzy operators - Implication

$$x \to y$$

$$" \to " : \{0,1\}^2 \to \{0,1\}$$

Introduction of fuzzy operators - Implication

$x \to y$

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$$\begin{aligned} 0 \to 0 &= 1 \\ 0 \to 1 &= 1 \\ 1 \to 0 &= 0 \\ 1 \to 1 &= 1 \end{aligned}$$

Introduction of fuzzy operators - Implication

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$$" \to " : \{0,1\}^2 \to \{0,1\}$$

$$0 \to 0 = 1$$
$$0 \to 1 = 1$$
$$1 \to 0 = 0$$
$$1 \to 1 = 1$$

$$I(x,y)$$

$$I : [0,1]^2 \to [0,1]$$

Introduction of fuzzy operators - Implication

$$x \to y \qquad\qquad I(x, y)$$

$$" \to " : \{0,1\}^2 \to \{0,1\} \qquad\qquad I : [0,1]^2 \to [0,1]$$

$$\rightsquigarrow$$

$$\begin{aligned} 0 \to 0 &= 1 \\ 0 \to 1 &= 1 \\ 1 \to 0 &= 0 \\ 1 \to 1 &= 1 \end{aligned} \qquad\qquad \begin{aligned} I(0,0) &= 1 \\ I(0,1) &= 1 \\ I(1,0) &= 0 \\ I(1,1) &= 1 \end{aligned}$$

Introduction of fuzzy operators - Implication

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$$I(x,y)$$

$$I\colon [0,1]^2 \to [0,1]$$

$$\begin{aligned} I(0,0) &= 1 \\ I(0,1) &= 1 \\ I(1,0) &= 0 \\ I(1,1) &= 1 \end{aligned}$$

Definition 1

A function $I : [0,1]^2 \to [0,1]$ is called a FUZZY IMPLICATION if it satisfies, for all $x, y \in [0,1]$, the following conditions

1. $I(\cdot, y)$ *is decreasing,*
2. $I(x, \cdot)$ *is increasing,*
3. $I(0,0) = 1,$
4. $I(1,1) = 1,$
5. $I(1,0) = 0.$

Introduction of fuzzy operators - Conjunction

$$x \wedge y$$

$$" \wedge ": \{0,1\}^2 \to \{0,1\}$$

$$\begin{aligned} 0 \wedge 0 &= 0 \\ 0 \wedge 1 &= 0 \\ 1 \wedge 0 &= 0 \\ 1 \wedge 1 &= 1 \end{aligned}$$

$$T(x,y)$$

$$T : [0,1]^2 \to [0,1]$$

$$\begin{aligned} T(0,0) &= 0 \\ T(0,1) &= 0 \\ T(1,0) &= 0 \\ T(1,1) &= 1 \end{aligned}$$

Definition 2

A function $T : [0,1]^2 \to [0,1]$ is called a TRIANGULAR NORM (t-norm) if it satisfies, for all $x, y, z \in [0,1]$, the following conditions

1. $T(x,y) = T(y,x)$,
2. $T(x, T(y,z)) = T(T(x,y),z)$,
3. $T(x,\cdot)$ *is decreasing,*
4. $T(x,1) = x$.

Introduction of fuzzy operators - Disjunction

$x \vee y$

$$" \vee ": \{0,1\}^2 \to \{0,1\}$$

$$\begin{aligned} 0 \vee 0 &= 0 \\ 0 \vee 1 &= 1 \\ 1 \vee 0 &= 1 \\ 1 \vee 1 &= 1 \end{aligned}$$

$S(x,y)$

$$S : [0,1]^2 \to [0,1]$$

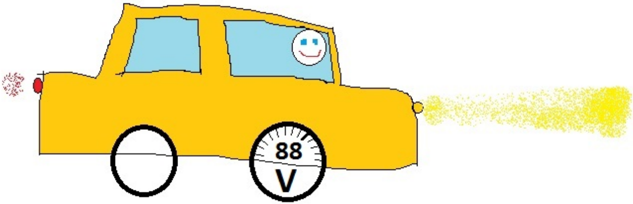
$$\begin{aligned} S(0,0) &= 0 \\ S(0,1) &= 1 \\ S(1,0) &= 1 \\ S(1,1) &= 1 \end{aligned}$$

Definition 3

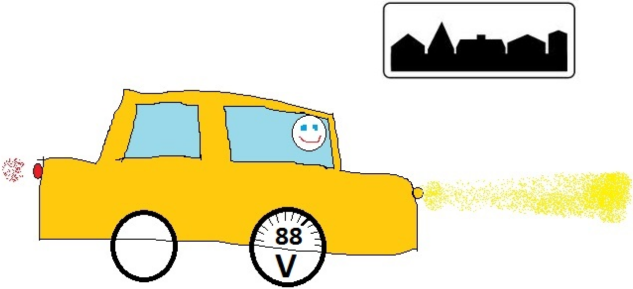
A function $S : [0,1]^2 \to [0,1]$ is called a TRIANGULAR CONORM (t-conorm) if it satisfies, for all $x, y, z \in [0,1]$, the following conditions

1. $S(x,y) = S(y,x)$,
2. $S(x, S(y,z)) = S(S(x,y),z)$,
3. $S(x,\cdot)$ *is increasing,*
4. $S(x,0) = x$.

Fuzzy thinking



Fuzzy thinking



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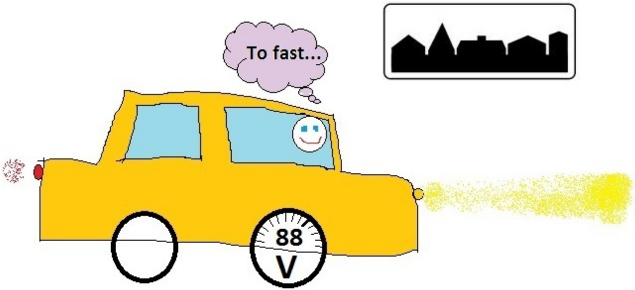

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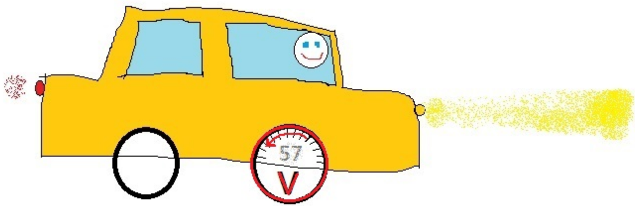
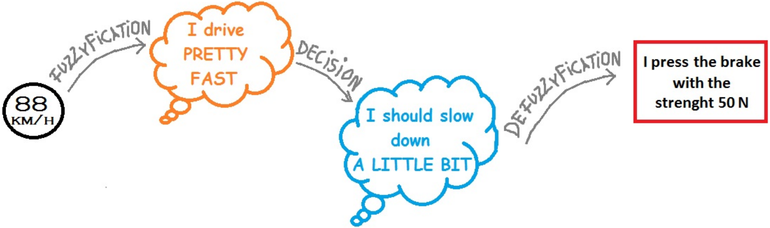


Fuzzy thinking



Compositional Rule of Inference (CRI): Laundry

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- **D** - the laundry is Dirty;

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- **D** - the laundry is Dirty;
- **VD** - the laundry is Very Dirty;

Compositional Rule of Inference (CRI): Laundry

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- **VD** - the laundry is Very Dirty;
- **S** - the laundry is Small;

Compositional Rule of Inference (CRI): Laundry

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- **VD** - the laundry is Very Dirty;
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- **B** - the laundry is Big;

Compositional Rule of Inference (CRI): Laundry

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- $\mathbf{T}_{>x}$ - at least x minutes of laundering;

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$$(D \wedge S) \to T_{>45}$$
$$(VD \wedge S) \to T_{>60}$$
$$(D \wedge B) \to T_{>70}$$
$$(VD \wedge B) \to T_{>90}$$

Compositional Rule of Inference (CRI): Laundry

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$$(D \wedge S) \rightarrow T_{>45} \qquad (D \rightarrow T_{>45}) \vee (S \rightarrow T_{>45})$$

$$(VD \wedge S) \rightarrow T_{>60} \qquad (VD \rightarrow T_{>60}) \vee (S \rightarrow T_{>60})$$

$$(D \wedge B) \rightarrow T_{>70} \qquad (D \rightarrow T_{>70}) \vee (B \rightarrow T_{>70})$$

$$(VD \wedge B) \rightarrow T_{>90} \qquad (VD \rightarrow T_{>90}) \vee (B \rightarrow T_{>90})$$

$$(x \wedge y) \rightarrow z \equiv (x \rightarrow z) \vee (y \rightarrow z) \qquad (T1)$$

Compositional Rule of Inference (CRI): Laundry

$$x \to (y \vee z) \equiv (x \to y) \vee (x \to z) \qquad (T2)$$

$$D \to T_{>45}, \quad S \to T_{>45},$$
$$VD \to T_{>60}, \quad S \to T_{>60},$$
$$D \to T_{>70}, \quad B \to T_{>70},$$
$$VD \to T_{>90}, \quad B \to T_{>90}$$

Compositional Rule of Inference (CRI): Laundry

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$$D \to (T_{>45} \vee T_{>70})$$

Compositional Rule of Inference (CRI): Laundry

$$x \to (y \vee z) \equiv (x \to y) \vee (x \to z) \qquad (T2)$$

$$\begin{array}{ll} D \to T_{>45}, & S \to T_{>45}, \\ VD \to T_{>60}, & S \to T_{>60}, \\ D \to T_{>70}, & B \to T_{>70}, \\ VD \to T_{>90}, & B \to T_{>90} \end{array} \qquad \begin{array}{c} D \to (T_{>45} \vee T_{>70}) \\ VD \to (T_{>60} \vee T_{>90}) \\ S \to (T_{>45} \vee T_{>60}) \\ B \to (T_{>70} \vee T_{>90}) \end{array}$$

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IF the laundry is Big, THEN ...

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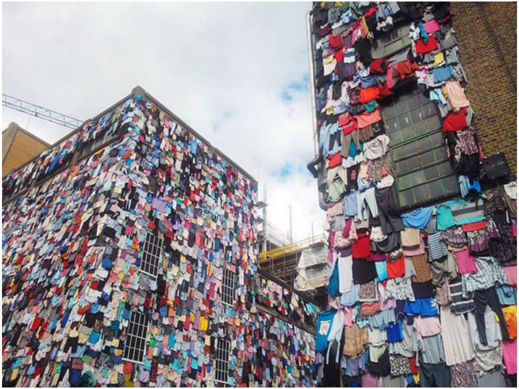

Compositional Rule of Inference (CRI): Laundry

IF the laundry is Big, THEN ....

Compositional Rule of Inference (CRI): Laundry

IF the laundry is Big, THEN ....

$Big(Laundry1) = 1$

$Big(Laundry2) = 0$

$Big(Laundry3) = 0.27$

$Big(Laundry4) = 0.62$

Compositional Rule of Inference (CRI) - Laundry

The general schema of multiconditional approximate reasoning has the form:

Rule 1: IF $\tilde{x}$ is A_1, THEN $\tilde{y}$ is B_1
Rule 2: IF $\tilde{x}$ is A_2, THEN $\tilde{y}$ is B_2
. . . .
Rule n: IF $\tilde{x}$ is A_n, THEN $\tilde{y}$ is B_n

Fact: $\tilde{x}$ is A

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Fact: $\tilde{x}$ is A

Conclusion: $\tilde{y}$ is B

$$B(y) = \sup_{j\in N_n} \sup_{x\in X} T(A(x), I(A_j(x), B_j(y)))$$

Compositional Rule of Inference (CRI) - Laundry

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Conclusion: $\tilde{y}$ is B

$$B(y) = \sup_{j \in N_n} \sup_{x \in X} T(A(x), I(A_j(x), B_j(y)))$$

IF $\tilde{x}$ is **D**, THEN $\tilde{y}$ is $\mathbf{T}_{>45}$

IF $\tilde{x}$ is **VD**, THEN $\tilde{y}$ is $\mathbf{T}_{>60}$

IF $\tilde{x}$ is **S**, THEN $\tilde{y}$ is $\mathbf{T}_{>45}$

IF $\tilde{x}$ is **B**, THEN $\tilde{y}$ is $\mathbf{T}_{>70}$

Derivation of the functional equation

$$(x \wedge y) \to z \equiv (x \to z) \vee (y \to z) \qquad (T1)$$

$$x \to (y \vee z) \equiv (x \to y) \vee (x \to z) \qquad (T2)$$

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$$\downarrow$$

$\to \rightsquigarrow I$

$\wedge \rightsquigarrow T$

$\vee \rightsquigarrow S$

$$I(T(x,y),z) = S(I(x,z), I(y,z)) \qquad (D1)$$

$$I(x, S(y,z)) = S(I(x,y), I(x,z)) \qquad (D2)$$

Derivation of the functional equation

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$$\Downarrow$$

$$\to \rightsquigarrow I$$
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TAUTOLOGY

Derivation of the functional equation

$$x \to (y \vee z) \equiv (x \to y) \vee (x \to z) \qquad (T2)$$

TAUTOLOGY

x	y	z	$x \to (y \vee z)$		$(x \to y) \vee (x \to z)$			$L \equiv R$
			$y \vee z$	L	$x \to y$	$x \to z$	R	
0	0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1
1	0	0	0	0	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	1	1	1	0	1	1	1
1	1	0	1	1	1	0	1	1
1	1	1	1	1	1	1	1	1

Derivation of the functional equation

$$I(x, S(y, z)) = S(I(x, y), I(x, z)) \qquad (D2)$$

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TAUTOLOGY?

The question: When the above equation is true? For which S and I?

Derivation of the functional equation

$$I(x, S(y, z)) = S(I(x, y), I(x, z)) \qquad (D2)$$

TAUTOLOGY?

The question: When the above equation is true? For which S and I?

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$$

Derivation of the functional equation

$$I(x, S(y, z)) = S(I(x, y), I(x, z)) \qquad (D2)$$

TAUTOLOGY?

The question: When the above equation is true? For which S and I?

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$$

Lemma 1

When S is continuous and Archimedean triangular conorm, then S is of the form

$$S(x, y) = s^{-1}(\min(s(x) + s(y), s(1))), \quad x, y \in [0, 1]$$

where $s : [0, 1] \to [0, \infty]$ is a continuous, strictly increasing function with $s(0) = 0$.

Derivation of the functional equation

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$$

Derivation of the functional equation

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$$

$$\downarrow$$

Lemma 1 *and some simple calculations*

$$\downarrow$$

Derivation of the functional equation

$$I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$$

$$\downarrow$$

Lemma 1 *and some simple calculations*

$$\downarrow$$

$$f(\min(u+v, r_1)) = \min(f(u)+f(v), r_2),$$

where $u, v \in [0, r_1], \quad r_1 := s_1(1), r_2 := s_2(1) \in (0, \infty)$

Derivation of the functional equation

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$$

$$\downarrow$$

***Lemma 1** and some simple calculations*

$$\downarrow$$

$$f(\min(u + v, r_1)) = \min(f(u) + f(v), r_2),$$

where $u, v \in [0, r_1], \quad r_1 := s_1(1), r_2 := s_2(1) \in (0, \infty)$

- For S_1, S_2 both nilpotent or both strict t-conorms we know the form of implication I,
- For a R-implication I generated from a strict t-norm T equation holds if and only if t-conorms $S_1 = S_2$ are Φ-conjugate with the Łukasiewicz t-conorm for some increasing bijection Φ, which is a multiplicative generator of the strict t-norm T.

Derivation of the functional equation

$$f(\min(u+v, r_1)) = \min(f(u)+f(v), r_2),$$

where $u, v \in [0, r_1], \quad r_1 := s_1(1), r_2 := s_2(1) \in (0, \infty)$

Derivation of the functional equation

$$f(\min(u+v, r_1)) = \min(f(u)+f(v), r_2),$$

where $u, v \in [0, r_1]$, $\quad r_1 := s_1(1), r_2 := s_2(1) \in (0, \infty)$

$$\downarrow$$

$$\boxed{f(m_1(x+y)) = m_2(f(x)+f(y))} \qquad (1),$$

where for fixed $r_1, r_2 \in (0, \infty)$ functions $m_1 : [0, 2r_1] \to [0, r_1]$, $m_2 : [0, 2r_2] \to [0, r_2]$ and $f : [0, r_1] \to [0, r_2]$.

m_2 - injective

Theorem 1

Let $r_1, r_2 \in (0, \infty)$ *be some numbers and let* $m_1\colon [0, 2r_1] \to [0, r_1]$, $m_2\colon [0, 2r_2] \to [0, r_2]$, $f : [0, r_1] \to [0, r_2]$ *be given functions. Further, let* m_2 *be injective. Then the following sentences are equivalent:*

m_2 - injective

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1. *The triple of functions m_1, m_2, f satisfies the equation (1).*

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1. *The triple of functions m_1, m_2, f satisfies the equation (1).*
2. $f(x) = ax + b$ *for some* $a, b \in \mathbb{R}$.

m_2 - injective

Theorem 1

Let $r_1, r_2 \in (0, \infty)$ be some numbers and let $m_1 \colon [0, 2r_1] \to [0, r_1]$, $m_2 \colon [0, 2r_2] \to [0, r_2]$, $f \colon [0, r_1] \to [0, r_2]$ be given functions. Further, let m_2 be injective. Then the following sentences are equivalent:

1. *The triple of functions m_1, m_2, f satisfies the equation (1).*
2. *$f(x) = ax + b$ for some $a, b \in \mathbb{R}$.*
   More precisely:
   Either $f = b$ for some $b \in [0, r_2]$ and $m_2(2b) = b$, or $f(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ such that

$$ax + b \in [0, r_2], \qquad \text{for all } x \in [0, r_1] \tag{1}$$

and

$$m_1(x) = \frac{m_2(ax + 2b) - b}{a}. \tag{2}$$

m_2 - NOT injective

Theorem 2

Let $r_1, r_2 \in (0, \infty)$ *be some numbers and let functions* $m_1 \colon [0, 2r_1] \to [0, r_1]$, $m_2 \colon [0, 2r_2] \to [0, r_2]$ *be continuous and strictly increasing on some intervals* $[0, x_1]$, $[0, x_2]$, *respectively, and then be equal to* r_1, r_2, *respectively, where* $x_1 \leq r_1$ *and* $x_2 \leq r_2$. *Further, let* m_1, m_2 *satisfy*

$$m_1(0) = 0, \qquad 2m_1(x) > x, \qquad x \in (0, 2r_1) \tag{3}$$

and

$$m_2(0) = 0, \qquad 2m_2(x) > x, \qquad x \in (0, 2r_2). \tag{4}$$

. *Finally let* f *be a function* $f \colon [0, r_1] \to [0, r_2]$.

m_2 - NOT injective

Theorem 2

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4. *there exists* $x_0 \in (0, r_1]$ *such that* $f(x) \geq x_2$ *for* $x \geq x_0$, $f(x) = r_2$ *for* $x \in [m_1(x_0), r_1]$ *and* $f(x) = \frac{x_2}{x_0} x$ *for* $x < x_0$. *Moreover in this case*

$$m_1(x) = \frac{x_0 m_2(\frac{x_2}{x_0} x)}{x_2} \qquad \text{for } x < y_0, \ \text{such that } m_1(y_0) = x_0.$$

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Conversely, if we add to the case (4) an assumption that $y_0 = x_0$ *or* $f(m_1(x)) = m_2(f(x))$ *for* $x \in [y_0, x_0)$, *then each of the triples of functions described above satisfies the equation (1).*

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Remark 1

We've showed that an additional assumption in the converse to Theorem 2 ($y_0 = x_0$ or $f(m_1(x)) = m_2(f(x))$ for $x \in [y_0, x_0)$) is necessary.

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Let us fix arbitrarily $r_1, r_2 > 0$ *and* $\alpha \geq 1$.

- $m_1(x) = \min(\alpha x, r_1)$ *for* $x \in [0, 2r_1]$
- $m_2(x) = \min(\alpha x, r_2)$ *for* $x \in [0, 2r_2]$.

In this case we obtain the following equation

$$f(\min(\alpha(x+y), r_1)) = \min(\alpha(f(x)+f(y)), r_2).$$

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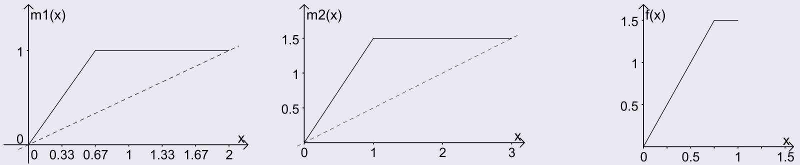
From Theorem 2 we obtain that the only nontrivial continuous solution is $f(x) = \min(kx, r_2)$, *where* $k = \frac{r_2}{\alpha x_0}$.



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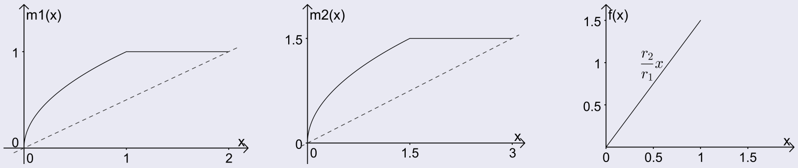

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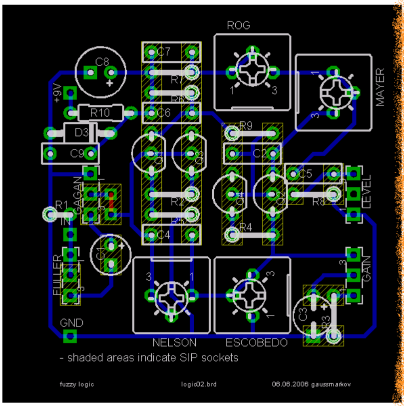

THANK YOU FOR YOUR ATTENTION!