Terminal Constraint vs. Risk Minimalization

Stochastic discrete optimal control problems with terminal state constraints

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Outline

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Optimal Fund Selection Problem

- Assume investor wants to divide capital between two funds
  - risk-free fund (bonds) – interest rate r_i known in advance
  - risky fund (stocks, indices) – return z_i is random, distribution is known, returns are independent
- Time horizon: long term, e.g., $k = 40$ years
- Possibility to rebalance every year based on current value
- Investor's objective: risk minimization
- Motivation: long-term investment, pension savings

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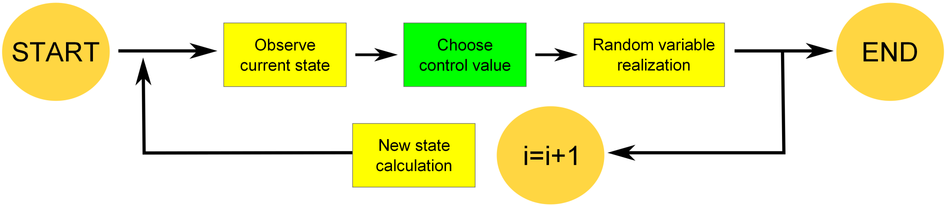
Introduction to Stochastic Optimal Control

- We use discrete stochastic optimal control to solve problem
- There is object managed during several periods
- New state of the object follows difference equation
- We have to choose control value to maximize objective



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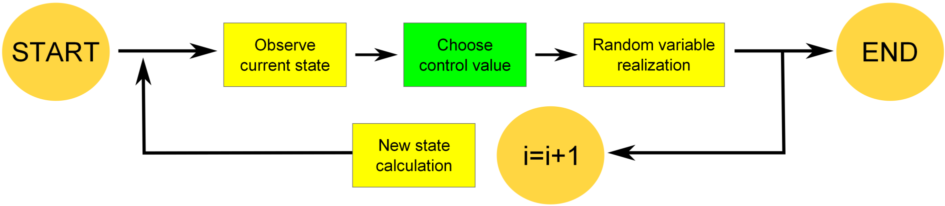

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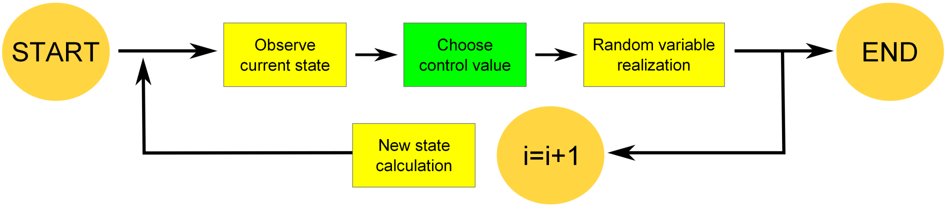

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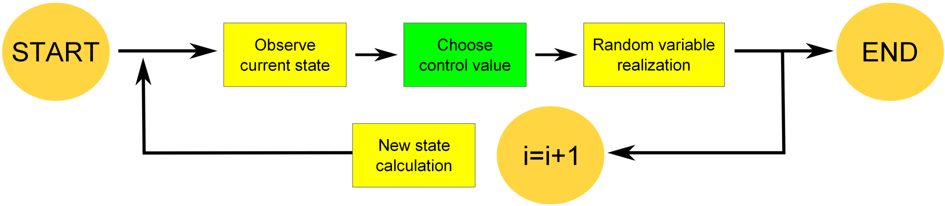

Stochastic Optimal Control

- We use stochastic optimal control to solve problem – assuming discrete time, discrete finite random distribution

  i – time period, $i = 0, \ldots, k-1$,
  x_i – capital at the beginning of year i, x_0 start capital,
  u_i – share in risky fund in year i, $u_i \in \mathbb{U}_i \equiv [0,1]$
  r_i – known risk-free fund return in year i,
  z_i – random risky return (year i), distribution $z_i \sim \mathbb{Z}_i$ is known

- Solution is optimal strategy – set of feedback functions

$$\mathcal{V} = \{v_0, v_1, \ldots, v_{k-1}\}$$

- Optimal control $u_i \equiv v_i(x_i)$ is based on current state $x_i \in \mathbb{X}_i$, it fulfils $v_i(x_i) \in \mathbb{U}_i$ for all i, x_i

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Stochastic Optimal Control – Problem Formulation

- Optimal Fund Selection Problem: Risk minimization

$$\min \; E\left[\sum_{i=0}^{k-1} c_i \, CVaRD_\alpha\left(x_{i+1} \,|\, x_i\right)\right]$$

$$\begin{aligned}
& x_{i+1} = x_i\left[1 + u_i z_i + (1 - u_i)\, r_i\right], && i = 0, \ldots, k-1, \\
& x_0 = a > 0, && \\
& u_i = v_i(x_i) \in \mathbb{U}_i = \left[0, 1\right], && i = 0, \ldots, k-1, \\
& z_i \sim \mathbb{Z}_i, && i = 0, \ldots, k-1.
\end{aligned} \tag{1}$$

Unconstrained Problem Solution

- What would be the solution of problem (1)?
- It is unconstrained problem of risk minimization $CVaRD_\alpha$, expected return is not considered
- Certainty is always better than any risk
- Simple solution: $u_i \equiv 0$ (capital goes to risk-free fund)
- We need to add terminal constraint into problem (1): required level of terminal capital μ

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How To Formulate Terminal Constraint

1. Robust Constraint $x_k \geq \mu$
   - it needs to be fulfilled in **any case**, even extremal
   - certainty in all cases might be very expensive or impossible
   - not very suitable for stochastic program
2. Expected Value Constraint $E\left[x_k \,|\, x_{k-1}\right] \geq \mu$
   - fulfilled in **average** case, outliers ignored
   - this condition respects that x_k is random variable
3. Probabilistic Constraint $P\left[x_k \geq \mu\right] \geq \beta,$
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Dynamic Programming Equation

- Problems with terminal constraints can be solved using dynamic programming equation
- Problem: some strategies $v_i(x_i) \in \mathbb{U}_i$ are not feasible
- We need to define set of feasible control values $W_i(x_i)$ for each time period i and each state x_i
- Then we can find a solution using dynamic program:

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\begin{aligned}
V_j(x) &= \max_{v_j(x) \in W_j(x)} E_j\left[f_j^0(x, v_j(x), z_j) + V_{j+1}(f_j(x, v_j(x), z_j))\right] \\
&= E_j\left[f_j^0(x, \hat{v}_j(x), z_j) + V_{j+1}(f_j(x, \hat{v}_j(x), z_j))\right], \\
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Expected Value Constraint

- Now we focus on expected value constraint in a form:

$$E\,[\,x_k \,|\, x_{k-1}\,] \geq \mu, \quad \text{for all } x_{k-1} \in X_{k-1}$$

- Based on *Brunovský et al. 2012*, we can derive following sets of feasible control values for $i = 0, \ldots, k-1$:

$$\begin{aligned} W_i(x) &= \{u_i \in \mathbb{U}_i \,|\, f_i\,(x, u_i, z_i) \in X_{i+1} \ \textbf{for all}\ z_i^s \sim \mathbb{Z}_i\}, \\ W_{k-1}(x) &= \{u_{k-1} \in \mathbb{U}_{k-1} \,|\, E\, f_{k-1}\,(x, u_{k-1}, z_{k-1}) \geq \mu\}, \\ X_i &= \{x \in \mathbb{X}_i \,|\, W_i(x) \neq \emptyset\} \end{aligned} \tag{2}$$

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Alternative formulation for Expected Value Constraint

- We derived alternative formulation for set of feasible control values
- For $i < k-1$, we use relaxed condition for E_{z_i} (instead of "for all" $z_i^s \sim \mathbb{Z}_i$):

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Probabilistic Constraint

- First we solve special problem suggested by *Doyen 2010* – maximization the probability $P[x_k \geq \mu]$, using dynamic program

$$
\begin{aligned}
\mathcal{P}_k(x) &= \Phi(x),\\
\mathcal{P}_i(x) &= \max_{u_i \in \mathbb{U}_i} E_i\left[\mathcal{P}_{i+1}\left(f_i(x, u_i, z_i)\right)\right],
\end{aligned}
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where $\Phi(x_k) = \begin{cases} 1, & \text{if } x_k \geq \mu,\\ 0, & \text{otherwise.} \end{cases}$

- This leads to feasible sets:

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Problem with Penalization (1)

- We also discussed alternative solution – move constraint into objective function

$$\min\ E\left[\sum_{i=0}^{k-1} CVaRD_\alpha(x_{i+1}\,|\,x_i) + \delta\cdot\Lambda(x_k)\right] \tag{5}$$

  where first term is former objective function and a second one represents penalization, with $\Lambda(x_k)$ as penalization function and δ its weight.

- We used e.g. constant penalization

$$\Lambda_1(x) = \begin{cases} 0, & \text{if } x \geq \mu \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

Problem with Penalization (2)

- Penalization is effective way to solve problem with terminal constraint
- Advantage: all states and all controls are considered feasible (in contrast with terminal condition)
- It this case we search for reasonable compromise solution to fulfill both objectives (risk minimization and terminal value maximization)
- This reflects that problem (1) is a trade-off between risk minimization and terminal value maximization, thus we consider solution as a trade-off as well

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Numerical scheme and Parameters

- We use discrete numerical scheme based on dynamic programming equation
- Feasible control values are calculated before optimization
- We use following parameters:
  - Number of periods $k = 40$ years,
  - Starting value $x_0 = 100$,
  - Requested terminal value $\mu = 300$,
  - Risk-free rate – $r_i = 2\%$ annually,
  - Risk fund random return – discrete approximation of normal distribution with expected value $\bar{z}_i = 6\%$, variance $\sigma = 0.1$

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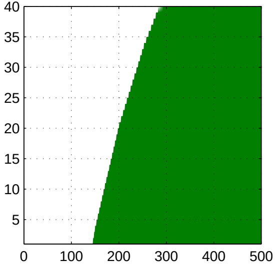
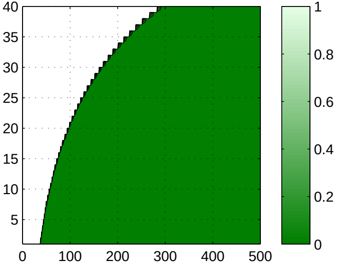
Compare Feasible States for Expected Value



(a) Former formulation (2)



(b) Alternative formulation (3)

Figure: Compare different feasible states for former and alternative formulation. Infeasible states are marked white.
Axes: horizontal – amount of capital (state x_i), vertical – time period i

Compare Feasible States – Notes

- Each expected value formulation have its disadvantages
- Former formulation: too many infeasible states, e.g. $x_0 = 100$ should be feasible intuitively, but in fact it is not
- Alternative formulation: bigger set of feasible states (good), but chance of getting into infeasible state during process
- In both cases, there are many states that are infeasible: if we reach such point, we need to reconsider terminal condition (decrease μ)

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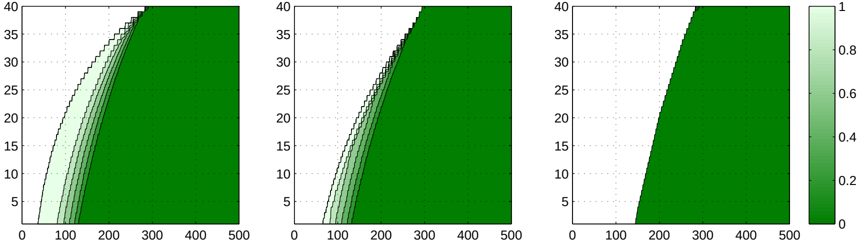
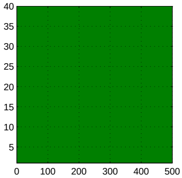
Results with Terminal Constraint



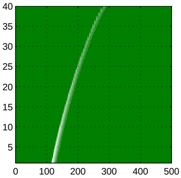
(a) Area $\mathcal{A}$ (b) Area $\mathcal{B}$ (c) Area $\mathcal{C}$

Figure: Risk minimization on restricted area.
Infeasible states are marked white.
$\mathcal{A}$ – expected value alternative formulation (3),
$\mathcal{B}$ – min. probability area with $\beta = 90\%$,
$\mathcal{C}$ – expected value former formulation (2)

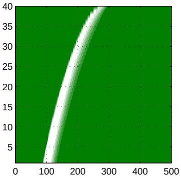
Results with Penalization



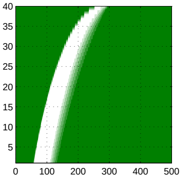
(a) $\delta = 0$



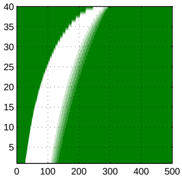
(b) $\delta = 100$



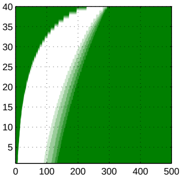
(c) $\delta = 250$



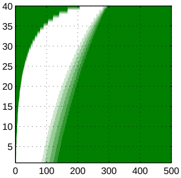
(d) $\delta = 500$



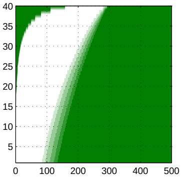
(e) $\delta = 1\,000$



(f) $\delta = 10\,000$



(g) $\delta = 10^6$



(h) $\delta = 10^{20}$

Figure: Risk minimization with penalization – different weight δ

Numerical Results – Notes

- Results with Terminal Constraint
  - Solution more or less the same, only area shape is different, like cropped
  - It is possible to fall down from feasible area
  - Outside feasible area, we don't know what control value shall be used
- Results with Penalization
  - Changing weight δ changes the solution
  - One can choose proper weight that represents his preference regarding terminal condition

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Summary

- We discussed two main questions:
  1. How to formulate terminal constraints in stochastic discrete optimal control problems
  2. How to solve such problems using dynamic programming equation
- We showed some disadvantages of particular constraints:
  - Possibility to reach infeasible state
  - Problem of infeasible starting point
- We also presented effective alternative way of solution using penalization term in objective function

Thank you for your attention! :)

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