On Flett's mean value theorem

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Theorem (Rolle, 1691)

If $f \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}(a, b)$ and $f(a) = f(b)$, then

$$(\exists \eta \in (a, b)) \;\; f'(\eta) = 0.$$

Veta (Lagrange, 1797)

If $f \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}(a, b)$, then

$$(\exists \eta \in (a, b)) \;\; f'(\eta) = \frac{f(b) - f(a)}{b - a}.$$

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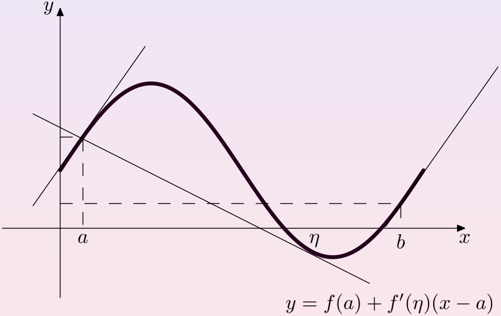

THOMAS MUIRHEAD FLETT (1923–1976)

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$$(\exists \eta \in (a,b)) \;\; f'(\eta) = \frac{f(\eta) - f(a)}{\eta - a}.$$

equivalent forms:

(i) $f(a) = f(\eta) + f'(\eta)(a - \eta) \;\; \Leftrightarrow \;\; f(a) = T_1(f,\eta)(a)$

(ii) $\begin{vmatrix} f'(\eta) & 1 & 0 \\ f(a) & a & 1 \\ f(\eta) & \eta & 1 \end{vmatrix} = 0$

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problems - applications:

(i) if $f \in \mathcal{C}\langle 0, 1\rangle$ and $\int_0^1 f(x)\,\mathrm{d}x = 0$, then

$$(\exists \eta \in (0, 1)) \;\; \int_0^\eta x\, f(x)\,\mathrm{d}x = 0;$$

(ii) if $f \in \mathcal{C}^1\langle 0, 1\rangle$, $f(1) = 0$ and $f'(1) = 1$, then

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Theorem (Riedel-Sahoo, 1998)

If $f \in \mathcal{D}\langle a, b\rangle$, then

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idea of proof:

applying the Flett's theorem to a function

$$F(x) = \begin{vmatrix} f(x) & x^2 & x & 1 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \\ f'(b) & 2b & 1 & 0 \end{vmatrix}, \quad x \in \langle a, b\rangle$$

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Theorem (Wayment, 1970)

If $f \in \mathcal{C}\langle a, b\rangle$ and $f(a) = f(b)$, then

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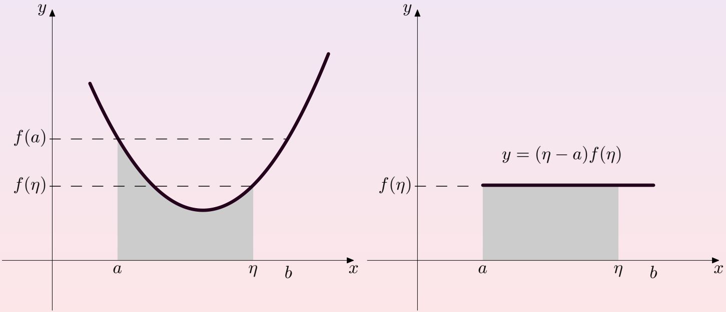

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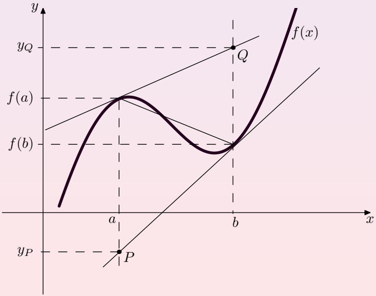
Theorem (Trahan, 1966)

If $f \in \mathcal{D}\langle a, b\rangle$ and

$$\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right)\left(f'(a) - \frac{f(b)-f(a)}{b-a}\right) \geq 0,$$

then $(\exists \eta \in (a, b\rangle)\ f'(\eta) = \dfrac{f(\eta)-f(a)}{\eta - a}$.



$$A_f(a,b) = \frac{f(a)+f(b)}{2}$$
$$I_f(a,b) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}(t)$$

Theorem (Tong, 2004)

If $f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}(a,b)$ and $A_f(a,b) = I_f(a,b)$, then
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Theorem („generalized“ Trahan)

If $f, g \in \mathcal{D}\langle a, b\rangle$, $g(a) \neq g(b)$ and

$$\left(f'(a) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(a)\right)\left(f'(b) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(b)\right) \geq 0,$$

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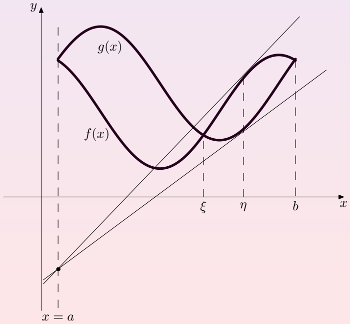

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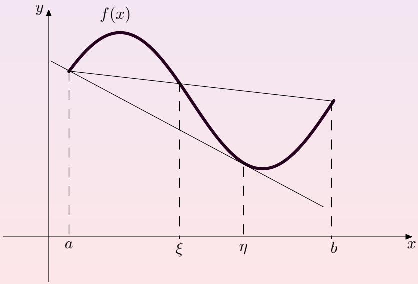

Theorem („new sufficient condition“)

Let $f, g \in \mathcal{D}\langle a, b\rangle$ and f, g be twice differentiable at the point a.
If $g(a) \neq g(b)$ and

$$\left(f'(a) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(a)\right)\left(f''(a) - \frac{f(b)-f(a)}{g(b)-g(a)}g''(a)\right) > 0,$$

then

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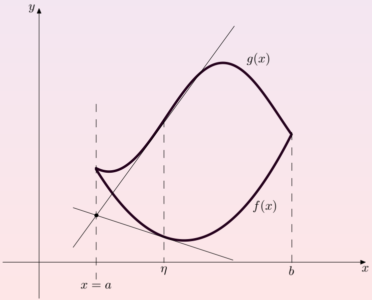


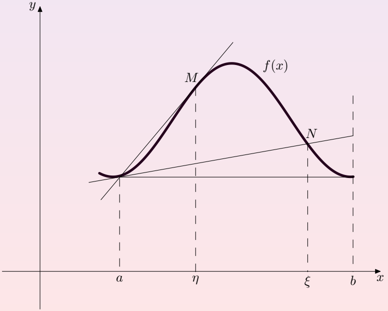
Theorem („on two tangents“)

Let $f \in \mathcal{D}\langle a,b\rangle$ and f be twice differentiable at the point a. If

$$\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] f''(a) > 0,$$

then

$$(\exists \xi \in (a,b))\ f'(a) = \frac{f(\xi)-f(a)}{\xi - a} \text{ and } (\exists \eta \in (a,\xi))\ f'(\eta) = \frac{f(\eta)-f(a)}{\eta - a}.$$



Survey of sufficient conditions:

- $f'(a) = f'(b)$ (Flett's condition);
- $(f'(a) - \frac{f(b)-f(a)}{b-a}) \cdot (f'(b) - \frac{f(b)-f(a)}{b-a}) \geq 0$ (Trahan's condition);
- $A_f(a,b) = I_f(a,b)$ (Tong's condition);
- $(f'(a) - \frac{f(b)-f(a)}{b-a}) \cdot f''(a) > 0$ provided $f''(a)$ exists (Maleševiċ's condition).

Theorem (Pawlikowska, 1999)

If $f \in \mathcal{D}^{(n)}\langle a, b\rangle$ and $f^{(n)}(a) = f^{(n)}(b)$, then

$$(\exists \eta \in (a,b)) \;\; f(\eta) - f(a) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}(\eta - a)^i f^{(i)}(\eta).$$

equivalent form:

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Proof: iteration of Flett's theorem using a function

$$\varphi_k(x) = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!}(k-i)(x-a)^i f^{(n-k+i)}(x) + xf^{(n-k+1)}(a)$$

for $k = 1, 2, \ldots, n$

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- $\varphi_1'(x) = -f^{(n)}(x) + f^{(n)}(a) \quad \Rightarrow \quad \varphi_1'(a) = 0 = \varphi_1'(b)$
- Flett's theorem:
  $(\exists u_1 \in (a,b))\ \varphi_1'(u_1)(u_1 - a) = \varphi_1(u_1) - \varphi_1(a)$
  $$f^{(n-1)}(u_1) - f^{(n-1)}(a) = (u_1 - a) f^{(n)}(u_1)$$
- $\varphi_2(x) = -2f^{(n-2)}(x) + (x-a)f^{(n-1)}(x) + x f^{(n-1)}(a)$
- $$\varphi_2'(x) = -f^{(n-1)}(x) + (x-a)f^{(n)}(x) + f^{(n-1)}(a) \quad \Rightarrow$$
  $$\Rightarrow \quad \varphi_2'(a) = 0 = \varphi_2'(u_1)$$

Theorem („improved“ Pawlikowska)

If $f \in \mathcal{D}^{(n)}\langle a, b\rangle$, then

$$(\exists \eta \in (a,b)) \;\; f(a) = T_n(f,\eta)(a) + \frac{(a-\eta)^{n+1}}{(n+1)!} \cdot \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}.$$

proof 1: (iteration of Flett's theorem)

$$\psi_k(x) = \varphi_k(x) + \frac{(-1)^{k+1}(x-a)^{k+1}}{(k+1)!} \cdot \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}, \quad k = 1, 2 \ldots, n$$

proof 2: (using Pawlikowska's theorem)

$$F(x) = \begin{vmatrix} f(x) & x^{n+1} & x^n & 1 \\ f(a) & a^{n+1} & a^n & 1 \\ f^{(n)}(a) & (n+1)!a & n! & 0 \\ f^{(n)}(b) & (n+1)!b & n! & 0 \end{vmatrix}, \; x \in \langle a, b\rangle$$

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- Hutník, O., Molnárová, J.: On Flett's mean value theorem. (submitted)
- Molnárová, J.: On generalized Flett's mean value theorem. *Internat. J. Math. Math. Sci.* (2012), Article ID 574634.

Thank you for your attention :)