Chosen Aspects of Polarized Intuitionistic Logic in Theoretical Computer Science

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Outline

- Introduction
- Classical Logic and Intuitionistic Logic
- Polarized Intuitionistic Logic
- Conclusions

Logic and Computer Science

- Curry-Howard Isomorphism
- functional and logic programming
- Girard's linear logic

Classical Propositional Logic (**CPC**)

Language:

- atomic formulas
- connectives: $\wedge, \vee, \rightarrow$
- constants: $\top$ (truth) and 0 (falsum)

- Semantics: truth tables
- **CPC** is decidable
- Computational complexity: coNP-complete

Two-valued logic: statements can be either true or false (*tertium non datur*). How much information does the claim $p \vee \neg p$ contain?

Constructivity = Disjunction Property:
If $(A \vee B)$ is a theorem, then A is a theorem or B is a theorem.

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- different meaning of connectives: notion of construction
  Brouwer-Heyting-Kolmogorow interpretation
- semantics: algebraic, topological
- Kripke semantics - rooted finite structures
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Polarized Intuitionistic Logic [Liang and Miller]

Language:

- atomic formulas
- connectives:
  Red-Polarized: $\vee, \wedge, \rightarrow$
  Green-Polarized: $\wedge^e, \vee^e, \propto$
- constants: $0, 1$ ("intuitionistic"), $\bot, \top$ ("classical")

- Two negations: $\sim A = A \rightarrow 0$ and $\neg A = A \rightarrow \bot$
- all atoms except $\bot$ and $\top$ are red
- intuitionistic formulas are built up from atoms and red connectives
- classical formulas are built up from atoms and green connectives
- duality operator connects green and red operators

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Duality operator $(\cdot)^\perp$

- $(A \vee^e B)^\perp = A^\perp \wedge B^\perp$
- $(A \wedge^e B)^\perp = A^\perp \vee B^\perp$
- $(A \propto B)^\perp = A \to B^\perp$
- $\perp^\perp = 1$
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Examples

- classical formula

$$(p^{\perp} \vee^{e} q) \vee^{e} (q^{\perp} \vee^{e} p)$$

- intuitionistic formula

$$(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$$

- PIL formula

$$((p \propto q^{\perp}) \vee p^{\perp}) \vee^{e} q$$

Kripke semantics for PIL

Hybrid Model: $\langle W, \preceq, C, \Vdash \rangle$

- $\leq$ is a partial ordering on non-empty set W
- $\Vdash$ is a monotonic *forcing* relation between elements of W and sets of atomic formulas
- $C \subseteq W$ is a set of "classical worlds"
- $\Delta_u = \{k \mid k \in C \text{ and } u \preceq k\}$
- $\Delta_k = \{k\}$ for all $k \in C$
- if $\Delta_u = \emptyset$ then u is imaginary or $\perp$-inconsistent

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Kripke semantics for PIL

Forcing of formulas for $u, v \in W; c \in C$:

- $u \Vdash 1$ and $u \nVdash 0$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \rightarrow B$ iff for all $v \geq u$ if $v \Vdash A$ then $v \Vdash B$
- $c \Vdash a^{\perp}$ iff $c \nVdash a$
- $c \Vdash \top$ and $c \nVdash \perp$
- $c \Vdash A \vee^e B$ iff $c \Vdash A$ or $c \Vdash B$
- $c \Vdash A \wedge^e B$ iff $c \Vdash A$ and $c \Vdash B$
- $c \Vdash A \propto B$ iff for some $v \geq c, v \Vdash A$ and $v \nVdash B^{\perp}$

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- $c \Vdash A \propto B$ iff for some $v \geq c$, $v \Vdash A$ and $v \nVdash B^{\perp}$

Properties of PIL

- simple and expressive semantics
- proof system in the form of Gentzen calculus
- simple language
- decidability

What is the computational complexity of PIL?

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Conjecture

Propositional fragment of PIL is PSPACE-complete

PSPACE-completeness - the Path of Wisdom

- Ladner's algorithm for modal logic S4
- Complexity of IPC: Tarski's translation
- Sequent Calculus for IPC by Dyckhoff
- Intuitionistic Control Logic

Intuitionistic Control Logic [Liang nad Miller]

Language:

- countably many atomic formulas
- connectives: $\wedge, \vee, \rightarrow$
- constants: $\top, 0, \perp$

- $C = \{\mathbf{r}\}$
- All worlds properly above $\mathbf{r}$ force $\perp$, but not $\mathbf{r}$ itself
- A formula that does not contain $\perp$ as a subformula is valid in ICL if and only if it is valid in intuitionistic logic

Algorithm for ICL

Procedure of creating a world in a model for ICL:

$$ICL\text{-}W(\mathcal{T}, \mathcal{F}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_{\rightarrow}, \tilde{\mathcal{F}}_{\rightarrow}, \mathcal{L})$$

where

- $\mathcal{T}, \mathcal{F}$ are sets of formulas respectively forced and not forced in current world
- $\tilde{\mathcal{T}}$ is a set of formulas already forced in previous worlds
- $\tilde{\mathcal{T}}_{\rightarrow}$ is a set of pairs of formulas; stores information about forced implication
- $\tilde{\mathcal{F}}_{\rightarrow}$ is a sequence of pairs of subformulas of not forced implication
- $\mathcal{L}$ is a sequence of labels; stores information about forcing in previous worlds

Algorithm for ICL

Test for $A \in$ ICL-SATISFIABLE
read A
$v \leftarrow \neg ICL\text{-W}(\{\top\}, \{A, \bot, 0\}, \emptyset, \emptyset, \emptyset, \emptyset);$
end

Future work

- Details of completeness of the procedure
- Algorithm for Polarized Intuitionistic Logic
- Other aspects of PIL

Q & A

Thank you for your attention!

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