Penney's game paradoxes

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Rules of the game

Rules of the classic version of Penney's game:

- 2 players;
- each player selects a three-bit long sequence of heads and tails;
- a coin is tossed until one of those sequences appears as a subsequence of the coin toss outcomes;
- the coin is fair;
- the player whose sequence appears first wins.

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$\binom{8}{2} = 28$ classic games

Example of the game

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Do the players have the same chances of winning in this game?

Stochastic graph's definition

We call a *stochastic graph* an ordered pair (S, Q), where:

1. S - a finite set of vertices;
2. Q - a function from SxS into $\mathbb{R}$ such that the following conditions are satisfied:
   - $Q(i,j) \geq 0 \quad$ for all $i, j \in S$;
   - $\sum_{j \in S} Q(i,j) = 1 \quad$ for all $i \in S$.

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- The set B of all boundary vertices is called the boundary of the graph.

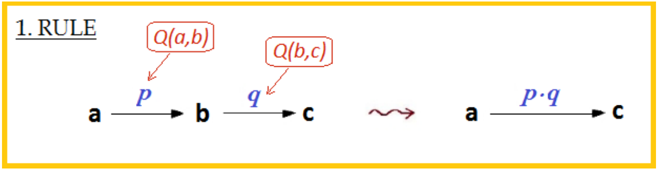
The rules of reduction

1. RULE

Q(a,b) Q(b,c)

$$a \xrightarrow{p} b \xrightarrow{q} c \quad \rightsquigarrow \quad a \xrightarrow{p\cdot q} c$$

The rules of reduction



$$P(A \cap B) = P(A) \cdot P(B), \quad A, B - \text{independent events}$$

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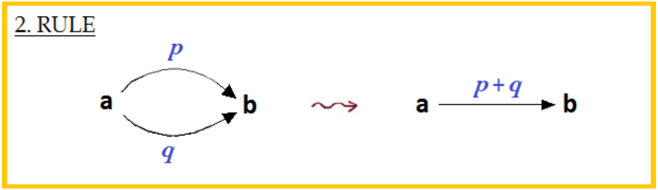

$$P(A \cup B) = P(A) + P(B), \quad A, B - \text{mutually exclusive events}$$

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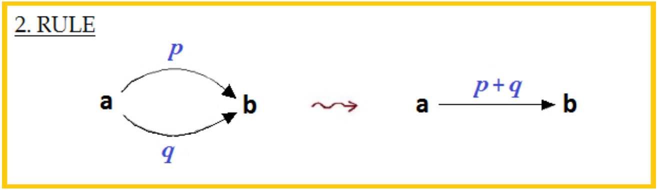

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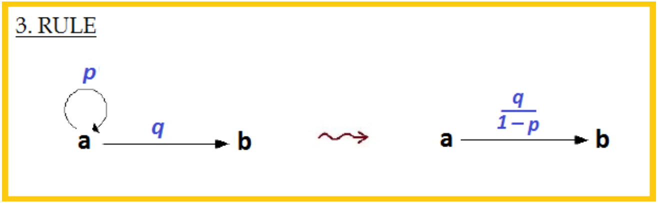
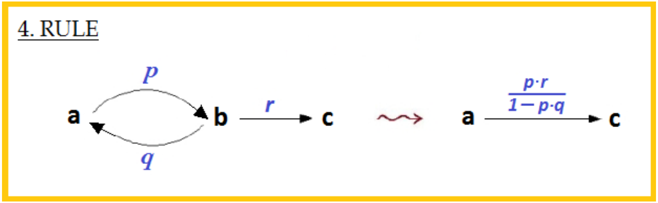
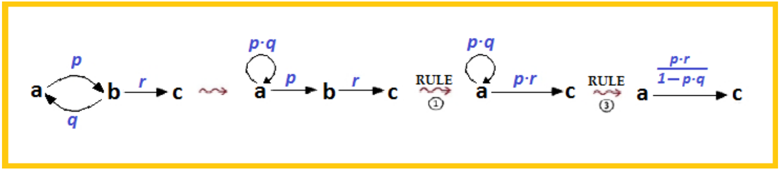


$$a \to b, \qquad a \to a \to b, \qquad a \to a \to a \to b, \ldots \qquad \Rightarrow$$

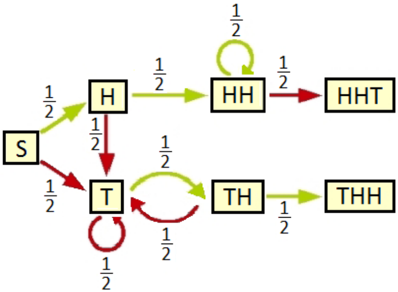
$$q + p \cdot q + p \cdot p \cdot q + \ldots = q \cdot \frac{1}{1-p}$$

The rules of reduction





Game HHT-THH



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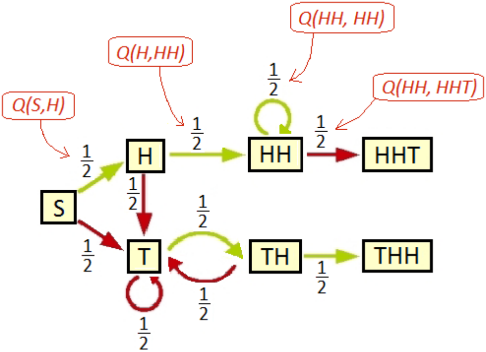

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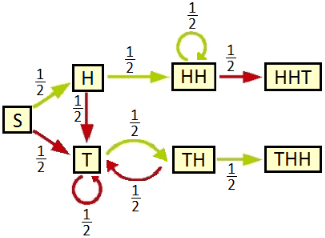

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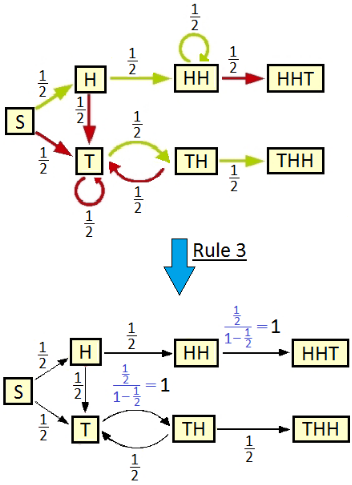

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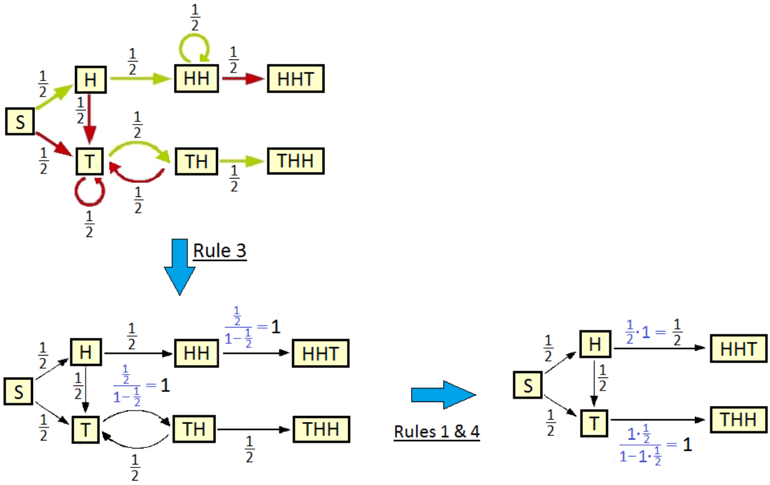
S
H
HH
HHT
T
TH
THH
Rule 3
Rules 1 & 4
$\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$
$\frac{1}{2} \cdot 1 = \frac{1}{2}$
$\frac{1 \cdot \frac{1}{2}}{1 - 1 \cdot \frac{1}{2}} = 1$

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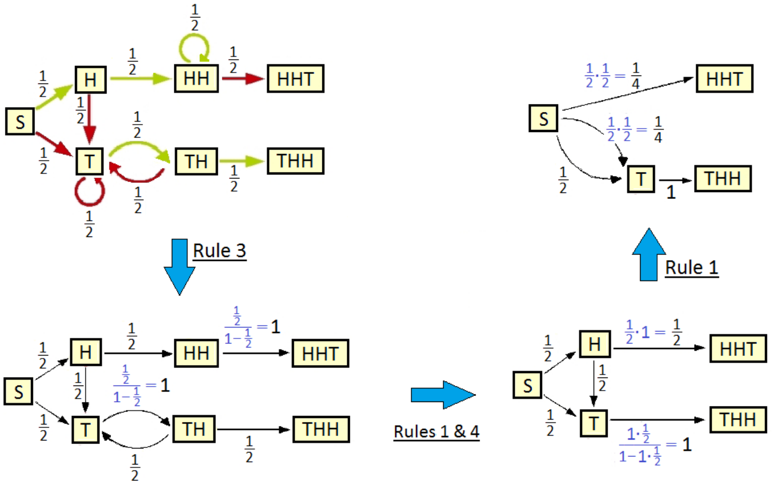
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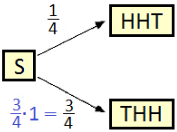
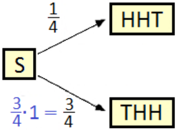


$$P(HHT) = \frac{1}{4}, \quad P(THH) = \frac{3}{4}$$

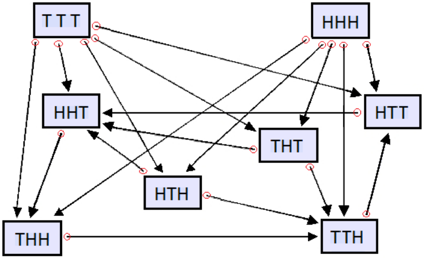
Game HHT-THH



$$P(HHT) = \frac{1}{4}, \quad P(THH) = \frac{3}{4}$$

Among 28 classic Penney's games only 10 are fair!

Relations between 3-bit long sequences



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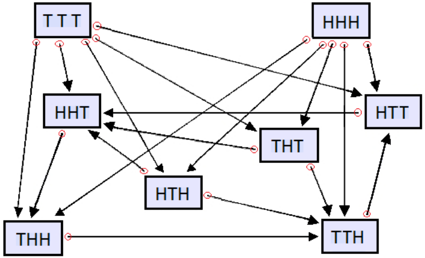

- There is no *the best* sequence in the classic Penney Game. Therefore priority of selection isn't a privilege in this game.

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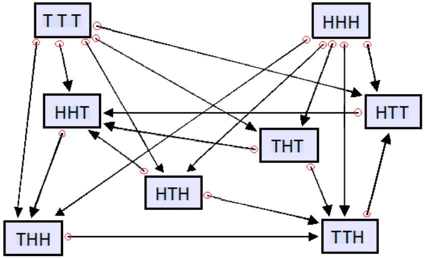
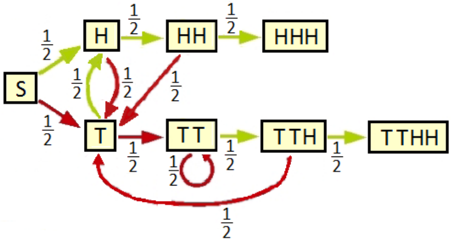


- There is no *the best* sequence in the classic Penney Game. Therefore priority of selection isn't a privilege in this game.
- The property of *being better than* in Penney's game isn't transitive.

Games with sequences of different lengths - Game HHH-TTHH



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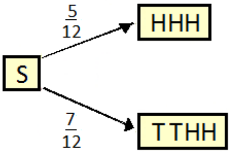

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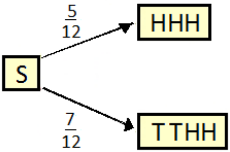

$$P(TTHH) = \frac{7}{12} > \frac{1}{2}$$

Algorithm of calculating mean time of random walk on a stochastic graph with a nonempty boundary

Let (S, Q) be a stochastic graph with a nonempty boundary $B \subseteq S$ and T_j a random variable defined as the time of random walk begun in vertice $j \in S$ (and ending in the boundary B). Let us call:

- $E(T_j) = e_j$
- for $j \notin B$ we define K_j - the set of vertices attained directly from the vertice j $(k \in K_j \Leftrightarrow p_{jk} > 0)$

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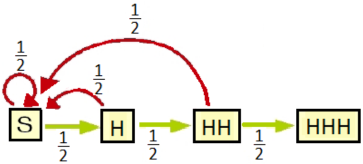
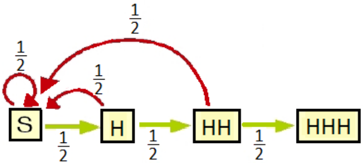
Then:

1. $e_j = 0 \quad$ for $j \in B$
2. $e_j = \sum_{k \in K_j} p_{jk} \cdot e_k + 1 \quad$ for $j \notin B$.

Mean time of waiting for sequence *HHH*



Mean time of waiting for sequence *HHH*



$$\begin{cases} e_S &= 1 + \frac{1}{2}\cdot e_S + \frac{1}{2}\cdot e_H \\ e_H &= 1 + \frac{1}{2}\cdot e_S + \frac{1}{2}\cdot e_{HH} \\ e_{HH} &= 1 + \frac{1}{2}\cdot e_S + \frac{1}{2}\cdot e_{HHH} \\ e_{HHH} &= 0 \end{cases} \qquad \begin{aligned} &\Rightarrow e_S = 14 \\ &\Rightarrow e_H = 12 \\ &\Rightarrow e_{HH} = 8 \\ &\Rightarrow e_{HHH} = 0 \end{aligned}$$

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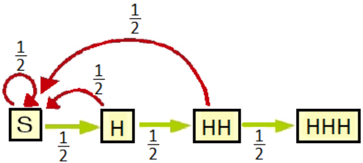

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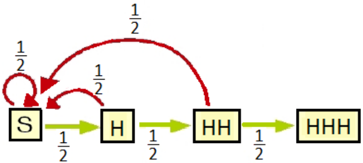

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T_A - time of waiting until pattern A appears the first time

Notation

- A, B - arbitrary sequences of heads and tails, $|A| = k, |B| = m$, for $k, m > 0$;
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- $f_A(n)$ - the number of n-bit long sequences, which include pattern A at the end, and A doesn't appear before;
- for $i = 1, 2, ..., min(k, m)$ we define

$$\delta_i(A, B) = \begin{cases} 2^i & \text{if } A_{(i)} = B^{(i)}, \\ 0 & \text{otherwise.} \end{cases} \qquad = 2^i \cdot [A_{(i)} = B^{(i)}]$$

Derivation of the formula for the mean time of waiting for a sequence

$$\left| \text{the set of } n\text{-bit long sequences without pattern } A \right| = \left| \text{the set of } (n+k)\text{-bit long sequences with pattern } A \text{ at the end, in which } A \text{ doesn't appear before } n\text{-th position} \right|$$

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$\nearrow \quad = f(n)$ $\qquad\qquad$ $\nearrow \quad \geq f_A(n+k)$

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$$P(T_A > n) = P(T_A = n+1) \cdot \delta_1(A,A) + \cdots + P(T_A = n+k) \cdot \delta_k(A,A)$$

Derivation of formula of mean time of waiting for the sequence...

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- This general theory will perhaps find some applications i.e. in:
  - mathematical modeling of gene mutations;
  - game theory and its applications i.e. on stock exchange.

Thank You for your attention!