An Alternative Description of a Fuzzy Metric

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Content

The most significant years

1. **1906** M. Frechet *metric space*
2. **1942** K. Menger *probabilistic (statistical) metric*
3. **1960** B. Schweizer and A. Sklar *theory of probabilistic (statistical) metric spaces*
4. **1975** I. Kramosil and J. Michalek *fuzzy metric*
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Definition of a metric

Definition

A *metric* on a (nonempty) set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions $\forall x, y, z \in X$:

A1 $d(x, y) \geq 0$ (nonnegativity);

A2 $d(x, y) = 0 \Leftrightarrow x = y$ (identity);

A3 $d(x, y) = d(y, x)$ (symmetry);

A4 $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

t-norm

Definition

A *t-norm* is a function $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties ($x, y, z \in [0,1]$):

A1 $x * y = y * x$ (symmetry);

A2 $(x * y) * z = x * (y * z)$ (associativity);

A3 $x_1 \leq x_2 \Rightarrow x_1 * y \leq x_2 * y$ (monotonicity);

A4 $x * 1 = x$.

(Examples)

1. Minimum $x \wedge y = \min\{x, y\}$
2. Product $x \cdot y = xy$
3. Lukasiewicz $x\,L\,y = \max\{x + y - 1, 0\}$

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Definition of a fuzzy metric

A. George and P. Veeramani: *On some results in fuzzy metric spaces.* Fuzzy Sets and Systems. **64** (1994) 395-399.

Definition

A *fuzzy metric* on a (nonempty) set X is an ordered pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function $M : X \times X \times (0, +\infty) \to (0, 1]$ satisfying the following conditions $\forall x, y, z \in X$:

M1 $M(x, y, t) = 1 \Leftrightarrow x = y$ (identity);

M2 $M(x, y, t) = M(y, x, t)$ (symmetry);

M3 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ (triangle inequality);

M4 $M(x, y, \circ) : (0, +\infty) \to [0, 1]$ is continuous.

From axioms M3, M1 $\Rightarrow M(x, y, \circ)$ is nondecreasing function $\forall x, y \in X$.

Some examples of fuzzy metrics

(X, d) is a metric space, $M : X \times X \times (0, +\infty)$ is a function of a fuzzy set M

- $n \in \mathbb{N}$

$$M(x, y, t) = \frac{t^n}{t^n + d(x, y)}$$

$$\Rightarrow (M, \wedge) \text{ is a fuzzy metric}$$

- $$M(x, y, t) = \frac{t}{t + d(x, y)}$$

$$\Rightarrow (M, \cdot) \text{ is a fuzzy metric (called standard)}$$

- $X = \mathbb{R}^+$

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

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Comparison of axioms

1	$d(x,y) \geq 0$	$M(x,y,t) \leq 1$
2	$d(x,y) < +\infty$	$M(x,y,t) > 0$
3	$d(x,y) = 0 \Leftrightarrow x = y$	$M(x,y,t) = 1 \Leftrightarrow x = y$
4	$d(x,y) = d(y,x)$	$M(x,y,t) = M(y,x,t)$
5	$d(x,z) \leq d(x,y) + d(y,z)$	$M(x,y,t) * M(y,z,s) \leq$ $\leq M(x,z,s+t)$

Motivation

Д.А. Райков: *Многомерный математический анализ.*(1989)

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Applications

V. Gregori et al. *Fuzzy Sets and Systems* 170 (2011) 95-111

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Thank you for attention!