Variolinear Copulas with a Given Diagonal Section

Tarad Jwaid, Bernard De Baets and Hans De Meyer

Ghent University, Belgium

ISCAMI 2012

KERMIT

Outline

- Motivations
- Introduction
- Semilinear copulas
- Variolinear copulas
- Conclusions

The wide range of applications

- Statistics
- Probability theory
- Fuzzy set theory
- ... etc

Why do we need new methods to construct copulas?

New methods to construct copulas are being proposed continuously in the literature

- to increase modelling flexibility
- to extend the known classes of copulas
- ... etc

Copulas

Definition

A bivariate **copula** is a binary operation on the unit interval that satisfies the following conditions

- **the boundary conditions**, i.e. for all $x \in [0,1]$, it holds that

$$C(x,0) = C(0,x) = 0, \quad C(x,1) = C(1,x) = x$$

- **2-increasing property**, i.e. for all $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$V_C([x,x'] \times [y,y']) := C(x,y) + C(x',y') - C(x,y') - C(x',y) \geq 0$$

Copulas

Definition

A bivariate **copula** is a binary operation on the unit interval that satisfies the following conditions

- **the boundary conditions**, i.e. for all $x \in [0,1]$, it holds that

$$C(x,0) = C(0,x) = 0, \quad C(x,1) = C(1,x) = x$$

- **2-increasing property**, i.e. for all $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$V_C([x,x'] \times [y,y']) := C(x,y) + C(x',y') - C(x,y') - C(x',y) \geq 0$$

Well-known examples of copulas

- $\mathrm{M}(\mathrm{x},\mathrm{y}) = \min(\mathrm{x},\mathrm{y})$
- $\mathrm{W}(\mathrm{x},\mathrm{y}) = \max(\mathrm{x}+\mathrm{y}-1,0)$
- $\Pi(x,y) = xy$

The copulas M and W are called the Fréchet-Hoeffding upper and lower bounds: for any copula C it holds that $\mathrm{W} \leq C \leq \mathrm{M}$

The copula Π expresses the independence of two random variables

Well-known examples of copulas

- $\mathrm{M}(\mathrm{x},\mathrm{y}) = \min(\mathrm{x},\mathrm{y})$
- $\mathrm{W}(\mathrm{x},\mathrm{y}) = \max(\mathrm{x}+\mathrm{y}-1,0)$
- $\Pi(x,y) = xy$

The copulas M and W are called the Fréchet-Hoeffding upper and lower bounds: for any copula C it holds that $\mathrm{W} \leq C \leq \mathrm{M}$

The copula Π expresses the independence of two random variables

The diagonal section and the diagonal function

Definition

The **diagonal section** of a copula C is the function $\delta_C : [0,1] \to [0,1]$ defined by $\delta_C(x) = C(x,x)$

A **diagonal function** is a function $\delta : [0,1] \to [0,1]$ satisfying the following conditions:

- $\delta(1) = 1$
- for all $x \in [0,1]$, it holds that $\delta(x) \leq x$
- δ is increasing and 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$ such that $x \leq x'$, it holds that

$$0 \leq \delta(x') - \delta(x) \leq 2(x' - x)$$

The diagonal section and the diagonal function

Definition

The **diagonal section** of a copula C is the function $\delta_C : [0,1] \to [0,1]$ defined by $\delta_C(x) = C(x,x)$

A **diagonal function** is a function $\delta : [0,1] \to [0,1]$ satisfying the following conditions:

- $\delta(1) = 1$
- for all $x \in [0,1]$, it holds that $\delta(x) \leq x$
- δ is increasing and 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$ such that $x \leq x'$, it holds that

$$0 \leq \delta(x') - \delta(x) \leq 2(x' - x)$$

Some properties of the diagonal section of a copula C and the diagonal function

- The diagonal section δ_C of a copula C is a diagonal function
- Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$, for example, the copulas K_δ, B_δ defined by:
  - $K_\delta(x, y) = \min(x, y, \frac{\delta(x)+\delta(y)}{2})$
  - $B_\delta(x, y) = \mathrm{M}(x, y) + \min\{t - \delta(t) \mid t \in [\min(x, y), \max(x, y)]\}$

  are respectively the greatest and smallest symmetric copulas with diagonal section δ

Some properties of the diagonal section of a copula C and the diagonal function

- The diagonal section δ_C of a copula C is a diagonal function
- Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$, for example, the copulas K_δ, B_δ defined by:
  - $K_\delta(x, y) = \min(x, y, \frac{\delta(x)+\delta(y)}{2})$
  - $B_\delta(x, y) = \mathrm{M}(x, y) + \min\{t - \delta(t) \mid t \in [\min(x, y), \max(x, y)]\}$

  are respectively the greatest and smallest symmetric copulas with diagonal section δ

Definition and examples

Definition

A **semilinear** copula is a copula that is linear in at least one direction in any point of the unit square

Examples of semilinear copulas:

- lower (resp. upper) semilinear copulas (Durante et al.)
- vertical (resp. horizontal) semilinear copulas (De Baets et al.)
- piecewise linear copulas based on triangulation (De Baets et al.)
- orbital semilinear copulas (Jwaid et al.)
- conic copulas (Jwaid et al.)
- biconic copulas (Jwaid et al.)
- ortholinear copulas (Jwaid et al.)

Definition and examples

Definition

A **semilinear** copula is a copula that is linear in at least one direction in any point of the unit square

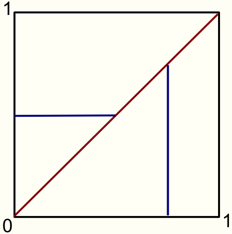
Examples of semilinear copulas:

- lower (resp. upper) semilinear copulas (Durante et al.)
- vertical (resp. horizontal) semilinear copulas (De Baets et al.)
- piecewise linear copulas based on triangulation (De Baets et al.)
- orbital semilinear copulas (Jwaid et al.)
- conic copulas (Jwaid et al.)
- biconic copulas (Jwaid et al.)
- ortholinear copulas (Jwaid et al.)

Definition and characterization

Let δ be a diagonal function and $C^l_\delta : [0,1]^2 \to [0,1]$

- C^l_δ is a lower semilinear function,
- $C^l_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & \text{, if } y \le x\,, \\ x\dfrac{\delta(y)}{y} & \text{, otherwise}\,, \end{cases}$
- C^l_δ is *lower semilinear* copula if and only if
  - $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing
  - $\rho_\delta :]0,1] \to [1,\infty[$, defined by $\rho_\delta(x) = \frac{\delta(x)}{x^2}$, is decreasing

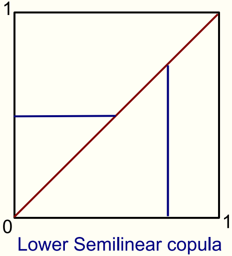


Lower Semilinear copula

Definition and characterization

Let δ be a diagonal function and $C_\delta^l : [0,1]^2 \to [0,1]$

- C_δ^l is a lower semilinear function,
- $C_\delta^l(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & \text{, if } y \leq x\,, \\ x\dfrac{\delta(y)}{y} & \text{, otherwise}\,, \end{cases}$

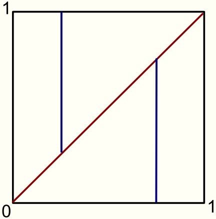


- C_δ^l is *lower semilinear* copula if and only if
  - $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing
  - $\rho_\delta :]0,1] \to [1,\infty[$, defined by $\rho_\delta(x) = \frac{\delta(x)}{x^2}$, is decreasing

Definition and characterization

Let δ be a diagonal function and $C_\delta^v : [0,1]^2 \to [0,1]$

- C_δ^v is a vertical semilinear function
- $$C_\delta^v(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & \text{, if } y \le x\,, \\ x - \dfrac{1-y}{1-x}(x-\delta(x)) & \text{, otherwise} \end{cases}$$



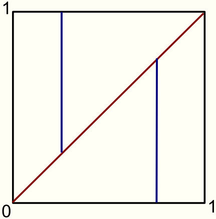
Vertical Semilinear copula

- C_δ^v is a *vertical semilinear* copula if and only if
  - $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing
  - $\mu_\delta : [0,1[\to [0,1]$, defined by $\mu_\delta(x) = \frac{x-\delta(x)}{1-x}$, is increasing
  - $\delta \ge \delta_\Pi$, i.e. for all $x \in [0,1]$, it holds that $\delta(x) \ge x^2$

Definition and characterization

Let δ be a diagonal function and $C_\delta^v : [0,1]^2 \to [0,1]$

- C_δ^v is a vertical semilinear function
- $C_\delta^v(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & \text{, if } y \le x\,, \\ x - \dfrac{1-y}{1-x}(x-\delta(x)) & \text{, otherwise} \end{cases}$
- C_δ^v is a *vertical semilinear* copula if and only if
  - $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing
  - $\mu_\delta : [0,1[\to [0,1]$, defined by $\mu_\delta(x) = \frac{x-\delta(x)}{1-x}$, is increasing
  - $\delta \ge \delta_\Pi$, i.e. for all $x \in [0,1]$, it holds that $\delta(x) \ge x^2$



Vertical Semilinear copula

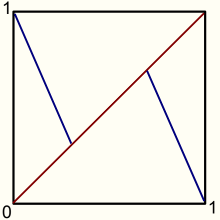
Definition and characterization

For any $(x, y) \in [0,1]^2$, we introduce the following notations

$$u = \frac{x}{1+x-y}, \quad v = \frac{y}{1+y-x}$$

Let δ be a diagonal function and $C_\delta^b : [0,1]^2 \to [0,1]$

- C_δ^b is a biconic function
- $C_\delta^b(x,y) = \begin{cases} y\,\dfrac{\delta(v)}{v} & \text{, if } y \leq x\,, \\ x\,\dfrac{\delta(u)}{u} & \text{, otherwise} \end{cases}$
- C_δ^b is a *biconic* copula if and only if δ is convex



Biconic copula

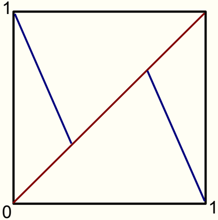
Definition and characterization

For any $(x, y) \in [0,1]^2$, we introduce the following notations

$$u = \frac{x}{1+x-y}, \quad v = \frac{y}{1+y-x}$$

Let δ be a diagonal function and $C_\delta^b : [0,1]^2 \to [0,1]$

- C_δ^b is a biconic function
- $$C_\delta^b(x,y) = \begin{cases} y\,\dfrac{\delta(v)}{v} & \text{, if } y \le x\,, \\ x\,\dfrac{\delta(u)}{u} & \text{, otherwise} \end{cases}$$
- C_δ^b is a *biconic* copula if and only if δ is convex



Biconic copula

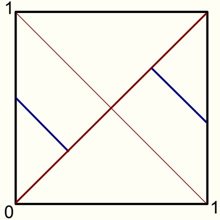
Definition and characterization

For any $(x, y) \in [0,1]^2$, we introduce the following notation

$$z = \frac{x+y}{2}$$

Let δ be a diagonal function and $C_\delta^o : [0,1]^2 \to [0,1]$

- C_δ^o is an ortholinear function
- $C_\delta^o(x,y) = C_\delta^o(y,x) =$
$$\begin{cases} x\dfrac{\delta(z)}{z} & \text{, if } x \le y,\ x+y \le 1\,, \\[2ex] x-(1-y)\dfrac{z-\delta(z)}{1-z} & \text{, if } x \le y,\ x+y \ge 1 \end{cases}$$
- C_δ^o is an *ortholinear* copula if and only if δ is convex



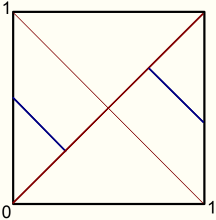
Definition and characterization

For any $(x, y) \in [0,1]^2$, we introduce the following notation

$$z = \frac{x+y}{2}$$

Let δ be a diagonal function and $C^o_\delta : [0,1]^2 \to [0,1]$

- C^o_δ is an ortholinear function
- $C^o_\delta(x,y) = C^o_\delta(y,x) =$
$$\begin{cases} x\dfrac{\delta(z)}{z} & \text{, if } x \le y,\ x+y \le 1\,, \\ x-(1-y)\dfrac{z-\delta(z)}{1-z} & \text{, if } x \le y,\ x+y \ge 1 \end{cases}$$

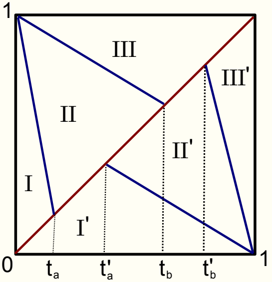


Ortholinear copula

- C^o_δ is an *ortholinear* copula if and only if δ is convex

The idea and the aim

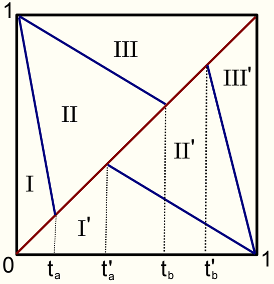
- We want to allow more flexibility in the choice of the segments on which the copula is linear
- We allow for variability of the direction of these segments
- We want also to cover the previous classes of semilinear copulas
- For this purpose we propose to divide the unit square as follows



Variolinear copula

The idea and the aim

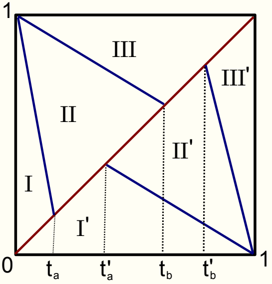
- We want to allow more flexibility in the choice of the segments on which the copula is linear
- We allow for variability of the direction of these segments
- We want also to cover the previous classes of semilinear copulas
- For this purpose we propose to divide the unit square as follows



Variolinear copula

The idea and the aim

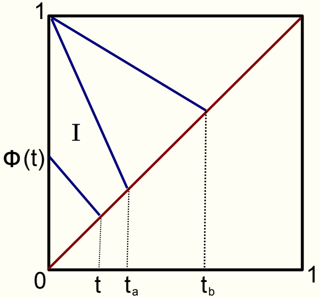
- We want to allow more flexibility in the choice of the segments on which the copula is linear
- We allow for variability of the direction of these segments
- We want also to cover the previous classes of semilinear copulas
- For this purpose we propose to divide the unit square as follows



Variolinear copula

The expression in sector I

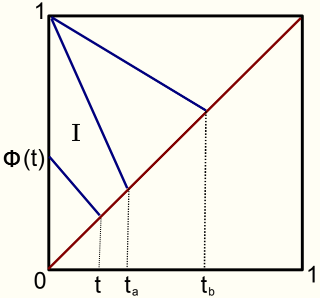
- In sector I, the diagonal extends from $(0,0)$ to (t_a, t_a), with $0 \leq t_a \leq t_b$
- We consider a continuous and strictly increasing function $\phi : [0, t_a] \to [0,1]$
- The function C_{I} defined on I is linear on each segment connecting the point (t,t) to the point $(0, \phi(t))$



The linear Interpolation in sector I

The expression in sector I

- In sector I, the diagonal extends from $(0,0)$ to (t_a, t_a), with $0 \leq t_a \leq t_b$
- We consider a continuous and strictly increasing function $\phi : [0, t_a] \rightarrow [0,1]$
- The function C_{I} defined on I is linear on each segment connecting the point (t,t) to the point $(0, \phi(t))$



The linear Interpolation in sector I

The expression in sector I

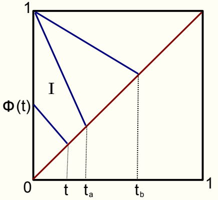
$$C_{\mathrm{I}}(x, y) = x\lambda(t)$$

where t be the unique solution in $[0, t_a]$ of the equation

$$y - x = \varphi(t)(t - x)$$

with

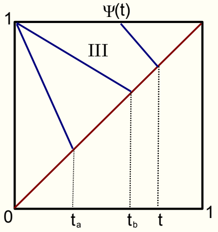
$$\lambda(t) = \frac{\delta(t)}{t} \text{ and } \varphi(t) = \frac{\phi(t)}{t}$$



The linear Interpolation in sector I

The expression in sector III

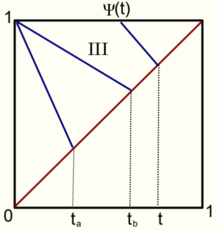
- In sector III, the diagonal extends from (t_b, t_b) to $(1, 1)$, with $t_a \leq t_b \leq 1$
- We consider a continuous and strictly increasing function $\Psi : [t_b, 1] \to [0, 1]$
- The function C_{III} defined on III is linear on each segment connecting the point (t, t) to the point $(\psi(t), 1)$



The linear Interpolation in sectorIII

The expression in sector III

- In sector III, the diagonal extends from (t_b, t_b) to $(1, 1)$, with $t_a \leq t_b \leq 1$
- We consider a continuous and strictly increasing function $\Psi : [t_b, 1] \to [0, 1]$
- The function C_{III} defined on III is linear on each segment connecting the point (t, t) to the point $(\psi(t), 1)$



The linear Interpolation in sector III

The expression in sector III

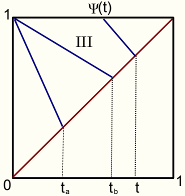
$$C_{\mathrm{III}}(x, y) = x - (1 - y)\overline{\lambda}(t)$$

where t be the unique solution in $[t_b, 1]$ of the equation

$$y - x = 1 - \xi(t)(t - x)$$

with

$$\overline{\lambda}(t) = \frac{1 - \delta(t)}{1 - t} \text{ and } \xi(t) = \frac{1 - \psi(t)}{1 - t}$$

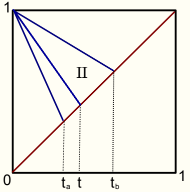


The linear Interpolation in sectorIII

The expression in sector II

- In sector II, the diagonal extends from (t_a, t_a) to (t_b, t_b), with $0 \leq t_a \leq t_b \leq 1$
- The function C_{II} defined on II is linear on each segment connecting the point (t, t) to the point $(0, 1)$
- In other words, C_{II} is biconic and is given by

$$C_{\mathrm{II}}(x, y) = (1 + x - y)\delta\left(\frac{x}{1+x-y}\right)$$

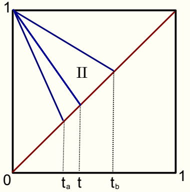


The linear Interpolation in sector II

The expression in sector II

- In sector II, the diagonal extends from (t_a, t_a) to (t_b, t_b), with $0 \le t_a \le t_b \le 1$
- The function C_{II} defined on II is linear on each segment connecting the point (t, t) to the point $(0, 1)$
- In other words, C_{II} is biconic and is given by

$$C_{\mathrm{II}}(x, y) = (1 + x - y)\delta\left(\frac{x}{1+x-y}\right)$$

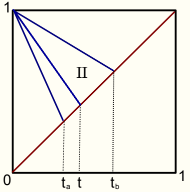


The linear Interpolation in sector II

The expression in sector II

- In sector II, the diagonal extends from (t_a, t_a) to (t_b, t_b), with $0 \leq t_a \leq t_b \leq 1$
- The function C_{II} defined on II is linear on each segment connecting the point (t, t) to the point $(0, 1)$
- In other words, C_{II} is biconic and is given by

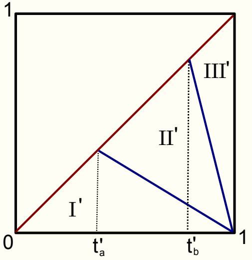
$$C_{\mathrm{II}}(x, y) = (1 + x - y)\delta\left(\frac{x}{1+x-y}\right)$$



The linear Interpolation in sector II

The expression for the functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$

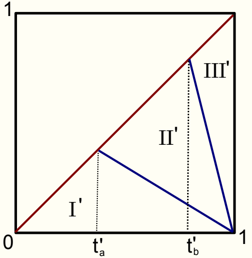
- The segments in sectors I', II' and III' are analogously defined as in I, II and III
- In sector I' (resp. III') the function $\rho : [0, t'_a] \to [0, 1]$(resp. $\sigma : [t'_b, 1] \to [0, 1]$) plays the same role as Φ (resp. Ψ) in sector I (resp. III)
- The expressions for the corresponding functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$ are obtained easily



The sectors I',II' and III'

The expression for the functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$

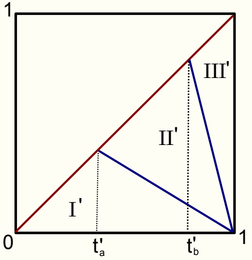
- The segments in sectors I', II' and III' are analogously defined as in I, II and III
- In sector I' (resp. III') the function $\rho : [0, t'_a] \to [0, 1]$(resp. $\sigma : [t'_b, 1] \to [0, 1]$) plays the same role as Φ (resp. Ψ) in sector I (resp. III)
- The expressions for the corresponding functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$ are obtained easily



The sectors I', II' and III'

The expression for the functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$

- The segments in sectors I', II' and III' are analogously defined as in I, II and III
- In sector I' (resp. III') the function $\rho : [0, t'_a] \to [0, 1]$(resp. $\sigma : [t'_b, 1] \to [0, 1]$) plays the same role as Φ (resp. Ψ) in sector I (resp. III)
- The expressions for the corresponding functions $C_{\mathrm{I}'}$, $C_{\mathrm{II}'}$ and $C_{\mathrm{III}'}$ are obtained easily



The sectors I',II' and III'

The variolinear function

- With the six functions C_{I}, C_{II}, C_{III}, $C_{\mathrm{I'}}$ $C_{\mathrm{II'}}$ and $C_{\mathrm{III'}}$
- We compose on the unit square the variolinear function associated with the functions ϕ, ψ, ρ, σ and with the given diagonal function δ

The variolinear function

- With the six functions C_{I}, C_{II}, C_{III}, $C_{\mathrm{I'}}$ $C_{\mathrm{II'}}$ and $C_{\mathrm{III'}}$
- We compose on the unit square the variolinear function associated with the functions ϕ, ψ, ρ, σ and with the given diagonal function δ

Semilinear copulas as special cases

It is easily verified to retrieve

- for $t_a = t'_a = 1$ and $\phi(t) = \rho(t) = t$, the lower semilinear functions with a given diagonal section
- for $t_b = 0, t'_a = 1$ and $\psi(t) = \rho(t) = t$, the vertical semilinear functions with a given diagonal section
- for $t_a = t'_a = 0$ and $t_b = t'_b = 1$, the biconic functions with a given diagonal section
- for $t_a = t'_a = t_b = t'_b = 1/2$, $\phi(t) = \rho(t) = 2t$, and $\psi(t) = \sigma(t) = 2t - 1$, the ortholinear functions with a given diagonal section

Semilinear copulas as special cases

It is easily verified to retrieve

- for $t_a = t'_a = 1$ and $\phi(t) = \rho(t) = t$, the lower semilinear functions with a given diagonal section
- for $t_b = 0, t'_a = 1$ and $\psi(t) = \rho(t) = t$, the vertical semilinear functions with a given diagonal section
- for $t_a = t'_a = 0$ and $t_b = t'_b = 1$, the biconic functions with a given diagonal section
- for $t_a = t'_a = t_b = t'_b = 1/2$, $\phi(t) = \rho(t) = 2t$, and $\psi(t) = \sigma(t) = 2t - 1$, the ortholinear functions with a given diagonal section

Semilinear copulas as special cases

It is easily verified to retrieve

- for $t_a = t'_a = 1$ and $\phi(t) = \rho(t) = t$, the lower semilinear functions with a given diagonal section
- for $t_b = 0, t'_a = 1$ and $\psi(t) = \rho(t) = t$, the vertical semilinear functions with a given diagonal section
- for $t_a = t'_a = 0$ and $t_b = t'_b = 1$, the biconic functions with a given diagonal section
- for $t_a = t'_a = t_b = t'_b = 1/2$, $\phi(t) = \rho(t) = 2t$, and $\psi(t) = \sigma(t) = 2t - 1$, the ortholinear functions with a given diagonal section

Semilinear copulas as special cases

It is easily verified to retrieve

- for $t_a = t'_a = 1$ and $\phi(t) = \rho(t) = t$, the lower semilinear functions with a given diagonal section
- for $t_b = 0, t'_a = 1$ and $\psi(t) = \rho(t) = t$, the vertical semilinear functions with a given diagonal section
- for $t_a = t'_a = 0$ and $t_b = t'_b = 1$, the biconic functions with a given diagonal section
- for $t_a = t'_a = t_b = t'_b = 1/2$, $\phi(t) = \rho(t) = 2t$, and $\psi(t) = \sigma(t) = 2t - 1$, the ortholinear functions with a given diagonal section

Variolinear functions with segments govern by linear functions

Since lower semilinear (resp. biconic, ortholinear) copulas are types of symmetry variolinear copulas for which the functions ϕ, ψ, ρ and σ are linear.

We consider for the characterization that

- $\rho = \phi$ and $\psi = \sigma$ (symmetric variolinear function)
- ϕ and ψ are linear functions, i.e.

$$\phi(t) = at,\ a \geq 1;\ \psi(t) = bt + 1 - b,\ b \geq 1$$

Clearly, $t_a = 1/a$ and $t_b = 1 - 1/b$
It must hold also that

$$\frac{1}{a} + \frac{1}{b} \leq 1$$

Variolinear functions with segments govern by linear functions

Since lower semilinear (resp. biconic, ortholinear) copulas are types of symmetry variolinear copulas for which the functions ϕ, ψ, ρ and σ are linear.
We consider for the characterization that

- $\rho = \phi$ and $\psi = \sigma$ (symmetric variolinear function)
- ϕ and ψ are linear functions, i.e.

$$\phi(t) = at,\ a \geq 1;\ \psi(t) = bt + 1 - b,\ b \geq 1$$

Clearly, $t_a = 1/a$ and $t_b = 1 - 1/b$
It must hold also that

$$\frac{1}{a} + \frac{1}{b} \leq 1$$

Variolinear functions with segments govern by linear functions

Since lower semilinear (resp. biconic, ortholinear) copulas are types of symmetry variolinear copulas for which the functions ϕ, ψ, ρ and σ are linear.
We consider for the characterization that

- $\rho = \phi$ and $\psi = \sigma$ (symmetric variolinear function)
- ϕ and ψ are linear functions, i.e.

$$\phi(t) = at,\ a \geq 1;\ \psi(t) = bt + 1 - b,\ b \geq 1$$

Clearly, $t_a = 1/a$ and $t_b = 1 - 1/b$
It must hold also that

$$\frac{1}{a} + \frac{1}{b} \leq 1$$

Expression and characterization

- Let δ be a twice differentiable diagonal function
- Let the function $C_\delta^{vl} : [0,1]^2 \to [0,1]$ be defined by

$$C_\delta^{vl}(x,y) = C_\delta^{vl}(y,x) = \begin{cases} x\lambda\left(\dfrac{y+(a-1)x}{a}\right) & \text{, if } x \leq y,\ y+(a-1)x \leq 1\,, \\ x-(1-y)\overline{\lambda}\left(\dfrac{x+(b-1)x}{b}\right) & \text{, if } x \leq y,\ (b-1)(1-y) \leq x \end{cases}$$

with $\lambda(t) = \frac{\delta(t)}{t}$ and $\overline{\lambda}(t) = \frac{t-\delta(t)}{1-t}$

Expression and characterization

- Let δ be a twice differentiable diagonal function
- Let the function $C_\delta^{vl} : [0,1]^2 \to [0,1]$ be defined by

$$C_\delta^{vl}(x,y) = C_\delta^{vl}(y,x) = \begin{cases} x\lambda\left(\dfrac{y+(a-1)x}{a}\right) & \text{, if } x \leq y,\ y+(a-1)x \leq 1\,, \\ x-(1-y)\overline{\lambda}\left(\dfrac{x+(b-1)x}{b}\right) & \text{, if } x \leq y,\ (b-1)(1-y) \leq x \end{cases}$$

with $\lambda(t) = \frac{\delta(t)}{t}$ and $\overline{\lambda}(t) = \frac{t-\delta(t)}{1-t}$

The necessary and sufficient condition to have a variolinear copula

C_δ^{vl} is a symmetric copula, called *the variolinear copula* with diagonal section δ if and only if

(1) $\lambda'(t) \geq 0$ for $0 \leq t \leq 1/a$

(1') $\overline{\lambda}'(t) \geq 0$ for $1 - 1/b \leq t \leq 1$

(2) $(a-1)\delta''(t) + (2-a)\lambda'(t) \geq 0$ for $0 \leq t \leq 1/a$

(2') $\delta''(t) \geq 0$ for $1/a \leq t \leq 1 - 1/b$

(2") $(b-1)\delta''(t) + (2-b)\overline{\lambda}' \geq 0$ for $1 - 1/b \leq t \leq 1$

(3) $2\delta(t) - (2-a)t\delta'(t) \geq 0$ for $0 \leq t \leq 1/a$

(3') $2\delta(t) + (2-b)(1-t)\delta'(t) \geq 2(1 - b(1-t))$ for $1 - 1/b \leq t \leq 1$

Special cases and an example

- Lower semilinear copula (a=1, only conditions (1) and (3) apply)
- Biconic copulas (only condition (2) applies)
- Ortholinear copulas (a=b=2, condition (2') is irrelevant)
- As an example consider the diagonal function $\delta(x) = x^{1+\alpha}$ with $\alpha \in [0, 1]$

1. We have introduced the class "variolinear functions with a given diagonal section"
2. Several classes of semilinear functions such as lower (resp. vertical) semilinear functions and biconic (resp. ortholinear) functions turn out to be special classes of this type of functions
3. The class of symmetric variolinear copulas for which the segments are governed by linear functions ϕ and ψ has been characterized

Some relevant references

- B. De Baets, H. De Meyer and R. Mesiar, *Asymmetric semilinear copulas*, Kybernetika **43** (2007), 221–233.
- B. De Baets, H. De Meyer and R. Mesiar, *Piecewise linear aggregation functions based on triangulation*, Information Sciences **181** (2011), 466–478.
- F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, *Semilinear copulas*, Fuzzy Sets and Systems **159** (2008), 63–76.
- T. Jwaid, B. De Baets and H. De Meyer, *Orbital semilinear copulas*, Kybernetika **45** (2009), 1012–1029.
- T. Jwaid, B. De Baets, J. Kalická and R. Mesiar, *Conic aggregation functions*, Fuzzy Sets and Systems **167** (2011), 3–20.
- T. Jwaid, B. De Baets and H. De Meyer, *Biconic aggregation functions*, Information Sciences **187** (2012), 129–150.
- T. Jwaid, B. De Baets and H. De Meyer, *Ortholinear and paralinear semi-copulas*, Fuzzy Sets and Systems, submitted.

- B. De Baets, H. De Meyer and M. Úbeda-Flores, *Opposite diagonal sections of quasi-copulas and copulas*, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems **17** (2009), 481–490.
- B. De Baets, H. De Meyer and M. Úbeda-Flores, *Constructing copulas with given diagonal and opposite diagonal sections*, Communications in Statistics: Theory and Methods **40** (2011), 828–843.
- F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, *Copulas with given diagonal sections, novel constructions and applications*, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems **15** (2007), 397–410.
- F. Durante, R. Mesiar and C. Sempi, *On a family of copulas constructed from the diagonal section*, Soft Computing **10** (2006), 490–494.
- R. Nelsen, *An Introduction to Copulas*, Springer, New York, 2006.
- R. Nelsen, J. Quesada-Molina, J. Rodríguez-Lallena and M. Úbeda-Flores, *On the construction of copulas and quasi-copula with given diagonal sections*, Insurance: Math. Econ. **42** (2008), 473–483.

Thank you for your attention!