Numerical experiments for discrete duality finite volume scheme for the curvature driven level set equation

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Level set equation

- level set equation: $u_t - |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0$
- unknown function $u(t,x)$ is defined in the domain $Q_T = I x \Omega$
- we consider Neumann (or Dirichlet) boundary conditions and the initial condition:
- $\partial_\nu u = 0$ on $I \times \partial\Omega$ $\;(u(t,x) = 0$ on $I \times \partial\Omega)$
- $u(0,x) = u^0$

Numerical approximation

time discretization

- we set the unique time step $\tau = \frac{T}{N}$

denote u^n as an approximation of $u(t,x)$ at time $t_n = n\tau$

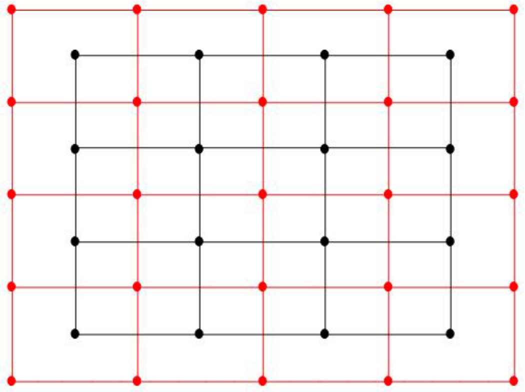
- first time derivative is replaced by the backward difference $\frac{u^n - u^{n-1}}{\tau}$
- level set equation can be rewritten into the form of semi-implicit scheme: $\frac{1}{|\nabla u^{n-1}|}\frac{u^n - u^{n-1}}{\tau} = \nabla \cdot \left(\frac{\nabla u^n}{|\nabla u^{n-1}|}\right)$
- Evans - Spruck regularization: $|\nabla u|_\varepsilon = \sqrt{\varepsilon^2 + |\nabla u|^2}$

Approximation

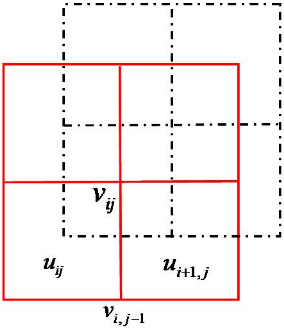
by fully discretization in the finite volume method we can denote V_{ij} as the finite volume with measure of $m(V_{ij}) = h^2$, e_{ij}^{pq} as the edge between two finite volumes p and q

- by application of the divergence theorem we get the integral formulation $\int\limits_{V_{ij}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} = \sum\limits_{|p|+|q|=1} \int\limits_{e_{ij}^{pq}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds$
- approximation of the left-hand side is $\int\limits_{V_{ij}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} dx \approx \frac{h^2}{\overline{Q}_{ij}} \frac{u_{ij}^n - u_{ij}^{n-1}}{\tau}$

Original and dual mesh in DDS



Gradient approximation in DDS



- $\int\limits_{V_{ij}} \frac{1}{|\nabla u^{n-1}|} \frac{u_{ij}^n - u_{ij}^{n-1}}{\tau} dx \approx \frac{h^2}{\overline{Q}_{ij}} \frac{u_{ij}^n - u_{ij}^{n-1}}{\tau}$
- $\overline{Q}_{ij} = \frac{1}{4} \sum\limits_{|p|+|q|=1} Q_{ij}^{pq;n-1}$
- $Q_{ij}^{pq;n-1} = \sqrt{\varepsilon^2 + |\nabla u_{ij}^{n-1}|^2}$
- $|\nabla u_{ij}^{n-1}|^2 = \left(\frac{u_{i+1,j} - u_{ij}}{h}\right)^2 + \left(\frac{v_{ij} - v_{i,j-1}}{h}\right)^2$

Approximation in original mesh in DDS

$$\sum_{|p|+|q|=1} \int_{e_{ij}^{pq}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds \approx \sum_{|p|+|q|=1} h \frac{1}{Q_{ij}^{pq;n-1}} \frac{u_{i+p,j+q}^n - u_{ij}^n}{h}$$

similarly we get the approximation also for the dual mesh

Linear system of equations in DDS

$$\frac{u_{ij}^{n} h^{2}}{\overline{Q}_{ij}^{n-1}} + \tau \sum_{|p|+|q|=1} \frac{u_{ij}^{n} - u_{i+p,j+q}^{n}}{Q_{ij}^{pq;n-1}} = \frac{h^{2} u_{ij}^{n-1}}{\overline{Q}_{ij}^{n-1}}$$

$$\frac{v_{ij}^{n} h^{2}}{\overline{\overline{Q}}_{ij}^{n-1}} + \tau \sum_{|p|+|q|=1} \frac{v_{ij}^{n} - v_{i+p,j+q}^{n}}{\overline{Q}_{ij}^{pq;n-1}} = \frac{h^{2} u_{ij}^{n-1}}{\overline{\overline{Q}}_{ij}^{n-1}}$$

Moving paraboloid

- paraboloid moving in time is the exact solution of the equation $\frac{u_t}{\sqrt{|\nabla u|^2 + \frac{1}{2}}} - \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \frac{1}{2}}} \right) = -\frac{1}{2}(x^2 + y^2 + \frac{1}{2})^{-\frac{3}{2}}$
- non-homogeneous Dirichlet boundary conditions
- numerical solution on the square mesh $n \times n$

Moving paraboloid

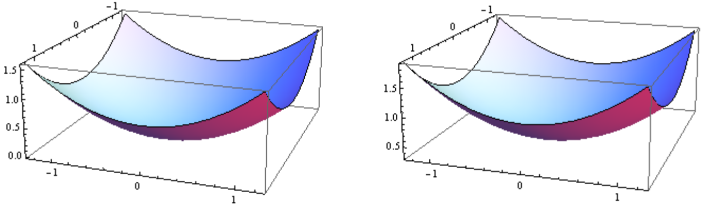


Figure: Initial condition and solution in time $t = 0.3125$

Moving paraboloid

n	L_2 error	EOC L_2 error	L_∞ error	EOC L_∞ error
10	$2.445e^{-3}$	–	$5.806e^{-3}$	–
20	$7.904e^{-4}$	1.6292	$1.779e^{-3}$	1.7065
40	$2.146e^{-4}$	1.8809	$4.605e^{-4}$	1.9498
80	$5.481e^{-5}$	1.9691	$1.159e^{-4}$	1.9903
160	$1.377e^{-5}$	1.9929	$2.904e^{-5}$	1.9968

Moving paraboloid

n	L_2 gradient error	EOC L_2 gradient error	L_∞ gradient error	EOC L_∞ gradient error
10	$5.334e^{-3}$	–	$1.172e^{-2}$	–
20	$2.286e^{-3}$	1.2224	$4.456e^{-3}$	1.3952
40	$7.203e^{-4}$	1.6662	$1.574e^{-3}$	1.5013
80	$1.998e^{-4}$	1.8500	$5.684e^{-4}$	1.4695
160	$5.270e^{-5}$	1.9227	$2.046e^{-4}$	1.4741

Cut paraboloid - Obermann solution

- cut paraboloid with zero Neumann boundary conditions
- numerical solution on the square mesh $n \times n$
- exact solution given by min $\{\frac{1}{2}(x^2 + y^2 + 1) - t, 0\}$

Cut paraboloid - Obermann solution

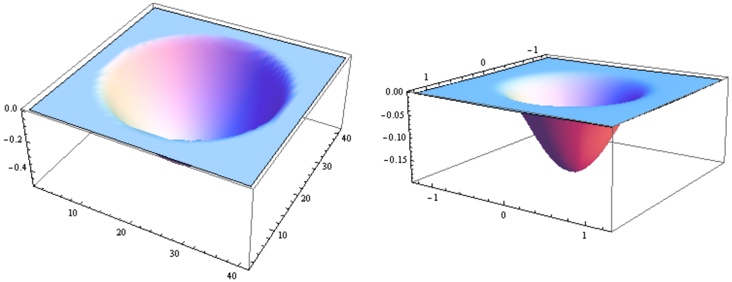


Figure: Initial condition and solution in time $t = 0.3125$

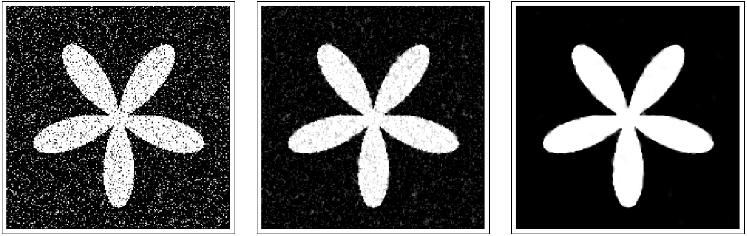
Cut paraboloid - Obermann solution

n	L_2 error	EOC L_2 error	L_∞ error	EOC L_∞ error
10	$6.755e^{-2}$	–	$1.618e^{-1}$	–
20	$3.401e^{-2}$	0.9899	$8.507e^{-2}$	0.9275
40	$1.717e^{-2}$	0.9861	$4.218e^{-2}$	1.0121
80	$8.730e^{-3}$	0.9758	$2.108e^{-2}$	1.0007
160	$4.421e^{-3}$	0.9816	$1.057e^{-2}$	0.9959
320	$2.229e^{-3}$	0.9879	$5.298e^{-3}$	0.9965

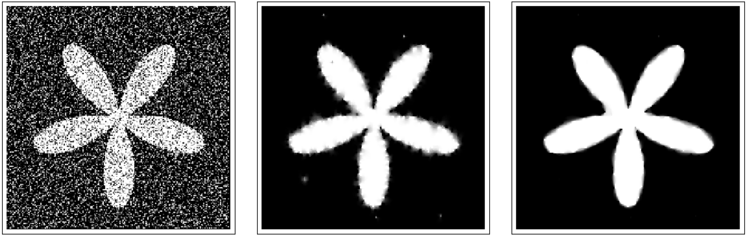
Cut paraboloid - Obermann solution

n	L_2 gradient error	EOC L_2 gradient error	L_∞ gradient error	EOC L_∞ gradient error
10	$2.492e^{-1}$	–	$5.000e^{-1}$	–
20	$2.129e^{-1}$	0.2271	$4.339e^{-1}$	0.2046
40	$1.748e^{-1}$	0.2845	$3.506e^{-1}$	0.3075
80	$1.409e^{-1}$	0.3110	$2.757e^{-1}$	0.3467
160	$1.126e^{-1}$	0.3235	$2.188e^{-1}$	0.3335
320	$8.973e^{-2}$	0.3275	$1.738e^{-1}$	0.3322

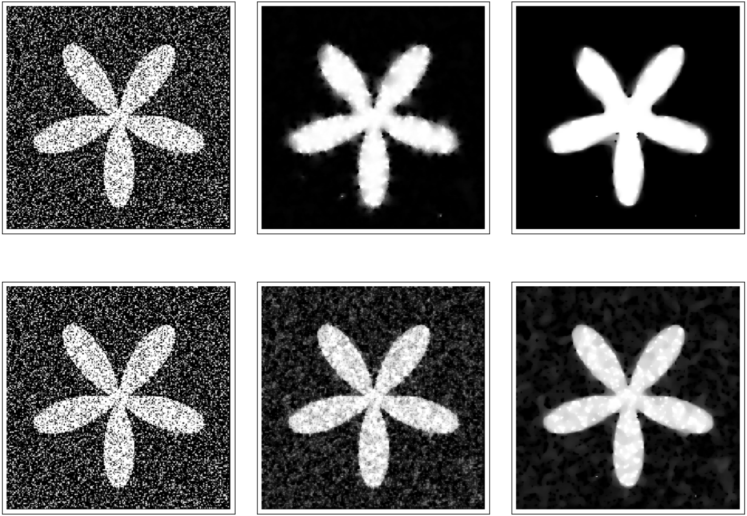
Cinquefoil with 20 percent salt and pepper noise



Cinquefoil with 50 percent salt and pepper noise



Cinquefoil with 50 percent salt and pepper noise



Thank you for your attention!