

An Alternative Description of a Fuzzy Metric

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- 7 An alternative description
- 8 Applications

The most significant years

- ① 1906 M. Fréchet *metric space*
- ② 1942 K. Menger *probabilistic (statistical) metric*
- ③ 1960 B. Schweizer and A. Sklar *theory of probabilistic (statistical) metric spaces*
- ④ 1975 I. Kramosil and J. Michalek *fuzzy metric*
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Definition of a metric

Definition

A *metric* on a (nonempty) set X is a function $d : X \times X \rightarrow \mathbb{R}$ satisfying the following conditions $\forall x, y, z \in X$:

A1 $d(x, y) \geq 0$ (nonnegativity);

A2 $d(x, y) = 0 \Leftrightarrow x = y$ (identity);

A3 $d(x, y) = d(y, x)$ (symmetry);

A4 $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition

A *t-norm* is a function $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following properties ($x, y, z \in [0, 1]$):

- A1 $x * y = y * x$ (symmetry);
- A2 $(x * y) * z = x * (y * z)$ (associativity);
- A3 $x_1 \leq x_2 \Rightarrow x_1 * y \leq x_2 * y$ (monotonicity);
- A4 $x * 1 = x$.

(Examples)

- ① **Minimum** $x \wedge y = \min\{x, y\}$
- ② **Product** $x \cdot y = xy$
- ③ **Lukasiewicz** $x L y = \max\{x + y - 1, 0\}$

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Definition of a fuzzy metric

A. George and P. Veeramani: *On some results in fuzzy metric spaces.*
Fuzzy Sets and Systems. **64** (1994) 395-399.

Definition

A *fuzzy metric* on a (nonempty) set X is an ordered pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function $M : X \times X \times (0, +\infty) \rightarrow (0, 1]$ satisfying the following conditions $\forall x, y, z \in X$:

M1 $M(x, y, t) = 1 \Leftrightarrow x = y$ (identity);

M2 $M(x, y, t) = M(y, x, t)$ (symmetry);

M3 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ (triangle inequality);

M4 $M(x, y, \circ) : (0, +\infty) \rightarrow [0, 1]$ is continuous.

From axioms **M3**, **M1** $\Rightarrow M(x, y, \circ)$ is nondecreasing function
 $\forall x, y \in X$.

Some examples of fuzzy metrics

(X, d) is a metric space, $M : X \times X \times (0, +\infty)$ is a function of a fuzzy set M

- $n \in \mathbb{N}$

$$M(x, y, t) = \frac{t^n}{t^n + d(x, y)}$$

$\Rightarrow (M, \wedge)$ is a fuzzy metric

- $M(x, y, t) = \frac{t}{t + d(x, y)}$

$\Rightarrow (M, \cdot)$ is a fuzzy metric (called standard)

- $X = \mathbb{R}^+$

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

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Comparison of axioms

1	$d(x, y) \geq 0$	$M(x, y, t) \leq 1$
2	$d(x, y) < +\infty$	$M(x, y, t) > 0$
3	$d(x, y) = 0 \Leftrightarrow x = y$	$M(x, y, t) = 1 \Leftrightarrow x = y$
4	$d(x, y) = d(y, x)$	$M(x, y, t) = M(y, x, t)$
5	$d(x, z) \leq d(x, y) + d(y, z)$	$M(x, y, t) * M(y, z, s) \leq M(x, z, s + t)$

Д.А. Райков: *Многомерный математический анализ*.(1989)

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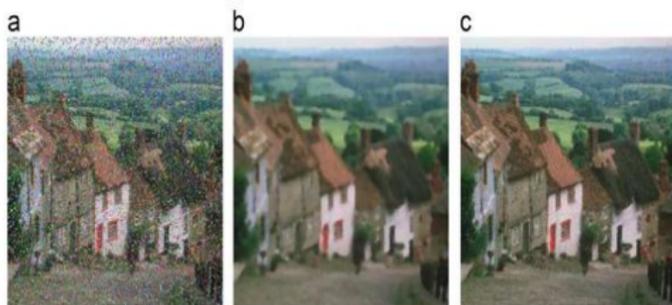
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V. Gregori et al. *Fuzzy Sets and Systems* 170 (2011) 95-111

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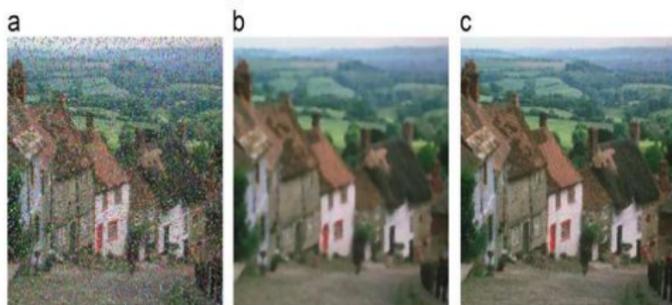
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**Thank you
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