Czech Technical University in Prague

Faculty of Civil Engineering

Structures buckling under compressive and tensile dead load described by the equation of the elastica

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Presentation plan

- Theory of Elastica
- Equation
  - derivation of equation describing the beam
  - derivation of equation of motion for mathematical pendulum
  - equations analogy
- Solution
  - analytic solution of linearized equation
  - numerical solution
  - categories of solution
- Application in tension

Theory of Elastica

- James Bernouli 1691
- Leonard Euler 1744
- Equation of Elastica: $x'' + c.\sin(x) = 0$

- Number of different aspects:
  - Mechanical equilibrium
  - Problem of the calculus of variations
  - Solution to elliptic integrals
- Analogies with physical systems:
  - Bending beam
  - Motion of mathematic pendulum
  - Surface of capillary curve

Original purpose

Describe behavior of straight elastic column of length AB=L

End is A fixed and end B is loaded by force F

Assuming:

- bending stiffness EI is constant
- force F is constant and works only in vertical direction
- self weight of beam is insignificant

In contrast to theory of linear elasticity we allow unlimited deformation of beam.

DERIVATION OF EQUATION DESCRIBING THE BEAM

start from definition of derivative of bending moment:

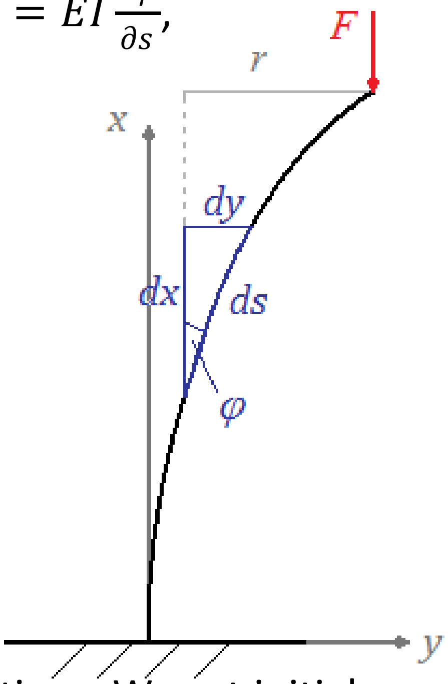
$$\frac{\partial M(s)}{\partial s} = \frac{M(s+ds)-M(s)}{ds} = \frac{F.(r-dy)-Fr}{ds} = -F\frac{\partial y}{\partial s}$$

Where bending moment cam be expressed: $M = EI\kappa = EI\frac{\partial \varphi}{\partial s}$,

from geometrical situation $\frac{\partial y}{\partial s} = \sin(\varphi)$

$$\boxed{\frac{\partial^2 \varphi}{\partial s^2} + c_1 \sin(\varphi) = 0}$$

Where $c_1 = \frac{F}{EI} = const.$



conditions:

$\varphi(0) = 0$ no angular rotation in fixed end

$M(L) = 0 \Rightarrow \left.\frac{\partial \varphi}{\partial s}\right|_{(L)} = 0$ no moment in free end

For numerical solution we need to have initial conditions. We set initial curvature $\left.\frac{\partial \varphi}{\partial s}\right|_{(0)} = \varphi'_0 = \kappa_0$.

The same by using Gâteaux's diferential:

Potential energy:

$$\mathrm{E}_P(\varphi) = \frac{1}{2}\int_0^L EI(\varphi')^2 \mathrm{d}s + \mathrm{F}\int_0^L \cos(\varphi)\,\mathrm{d}s$$

Gâteaux's diferential:

$$\delta\mathrm{E}_P(\varphi, \delta\varphi) = 0 \quad \forall\varphi(s);\ \varphi(0) = 0$$

$$\boxed{\frac{\partial^2\varphi}{\partial s^2} + c_1\sin(\varphi) = 0}$$

Where $c_1 = \frac{F}{EI} = const.$

conditions:

$\varphi(0) = 0$

$\left.\frac{\partial\varphi}{\partial s}\right|_{(L)} = 0$

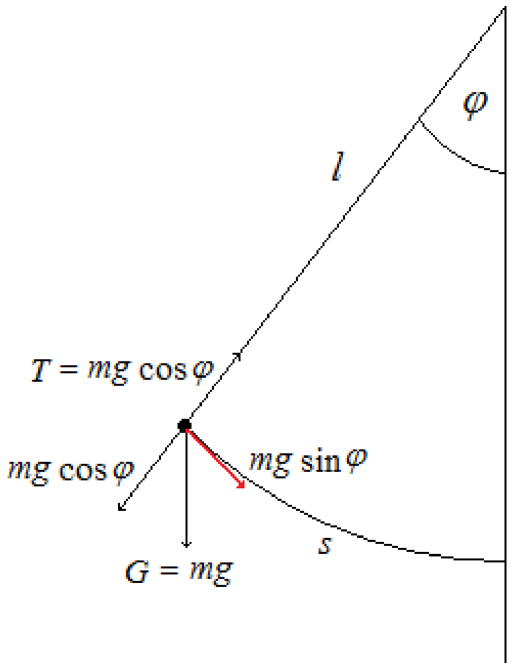
DERIVATION OF EQUATION OF MOTION FOR MATHEMATICAL PENDULUM

Gravity force G

$\longrightarrow$ force in the rope T

$\longrightarrow$ force in the direction of motion F

$$F = -mg\sin(\varphi)$$



Application of 2nd Newton's law:

$$\boxed{\frac{d^2\varphi}{dt^2} + c_2\sin(\varphi) = 0}$$

Where $c_2 = \frac{g}{l} = const.$

Initial conditions:

$$\varphi(0) = \varphi_0 = 0$$

$$v(0) = \left.\frac{\partial\varphi}{\partial t}\right|_{(0)} = v_0$$

The same by energy balance:

$$E_{k\,max} = E_{p\,max}$$

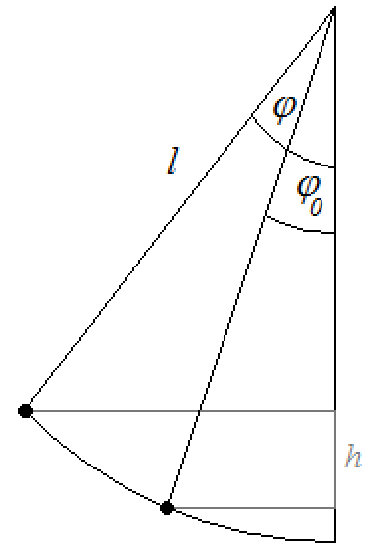
$$\frac{1}{2}mv_{max} = mgh$$

$$\Rightarrow l\frac{d\varphi}{dt} = v = \pm\sqrt{2gh}\,, \quad h = l(\cos\varphi_0 - \cos\varphi)$$

$$\Rightarrow \frac{d\varphi}{dt} = \sqrt{\frac{2g}{l}(\cos\varphi_0 - \cos\varphi)}$$

$$\frac{d^2\varphi}{dt^2} = \frac{-\frac{2g}{l}\sin\varphi}{2\sqrt{\frac{2g}{l}(\cos\varphi_0 - \cos\varphi)}} \cdot \frac{d\varphi}{dt}$$

$$\boxed{\frac{d^2\varphi}{dt^2} + \frac{g}{l}\sin\varphi = 0}$$



$h = l(\cos\varphi_0 - \cos\varphi)$

Initial conditions:

$$\varphi(0) = \varphi_0 = 0$$

$$v(0) = \frac{\partial\varphi}{\partial t}\bigg|_{(0)} = v_0$$

EQUATIONS ANALOGY :

Bending beam:	Mathematical pendulum:
Equation: $\frac{\partial^2\varphi(s)}{\partial s^2} + c_1 \sin(\varphi) = 0$	Equation: $\frac{d^2\varphi(t)}{dt^2} + c_2 \sin(\varphi) = 0$
Constant: $c_1 = \frac{mg}{EI} \quad [m^{-2}]$	Constant: $c_2 = \frac{g}{l} \quad [s^{-2}]$
Initial conditions: $\varphi(0) = 0$ $\left.\frac{\partial\varphi}{\partial s}\right\vert_{(0)} = \kappa_0$	Initial conditions: $\varphi(0) = 0$ $\left.\frac{\partial\varphi}{\partial t}\right\vert_{(0)} = v_0$

Solution of linearized equation:

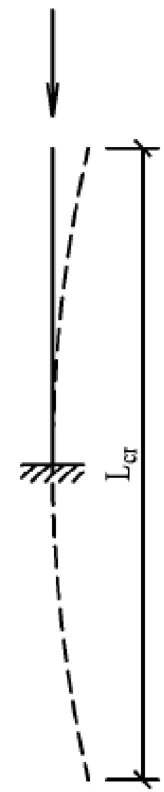
$\varphi \to 0$

$\sin\varphi \cong \varphi$

$\varphi'' + c\,\sin\varphi = 0 \quad \to \quad \varphi'' + c\varphi = 0$

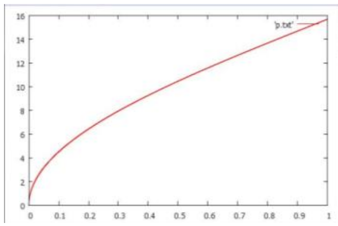
→ Equation of harmonic oscillator

Analytic solution: $\varphi(x) = A_1\cos\left(\sqrt{c}\,x\right) + A_2\sin\left(\sqrt{c}\,x\right).$

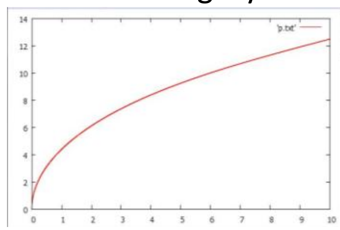


NUMERICAL SOLUTION

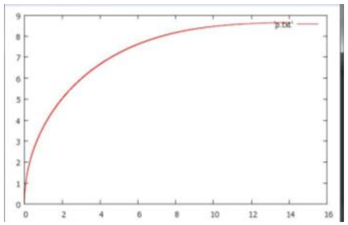
1st Caregory



$\kappa_0 = 0{,}01$

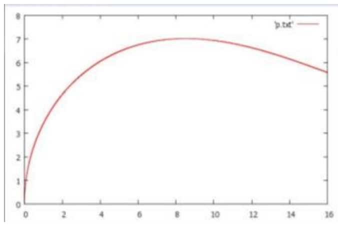


$\kappa_0 = 0{,}1$

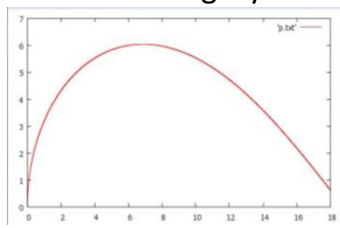


$\kappa_0 = 0{,}14$

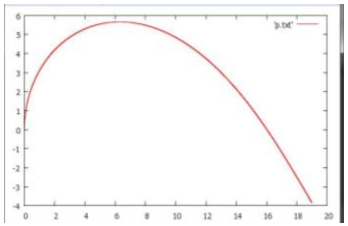
2nd Caregory



$\kappa_0 = 0{,}16$

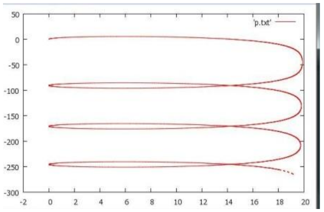


$\kappa_0 = 0{,}18$

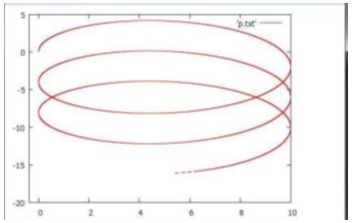


$\kappa_0 = 0{,}19$

3rd Caregory



$\kappa_0 = 0{,}201$



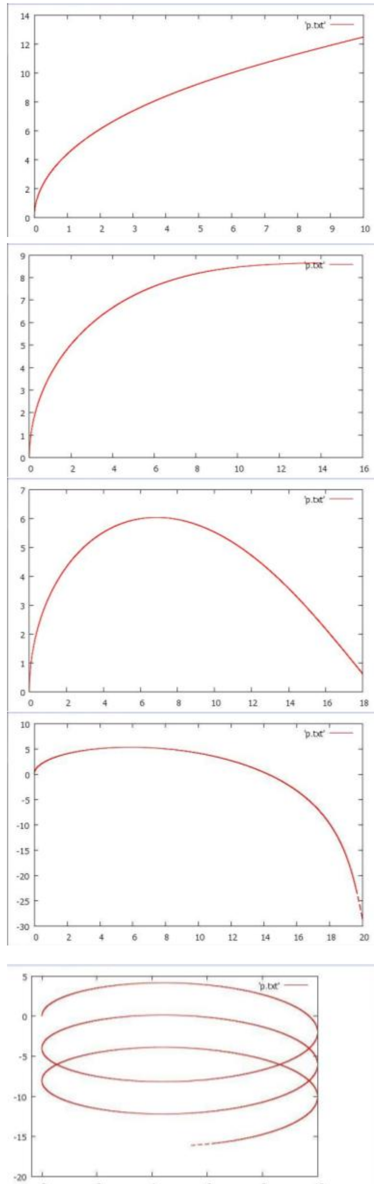
$\kappa_0 = 0{,}25$

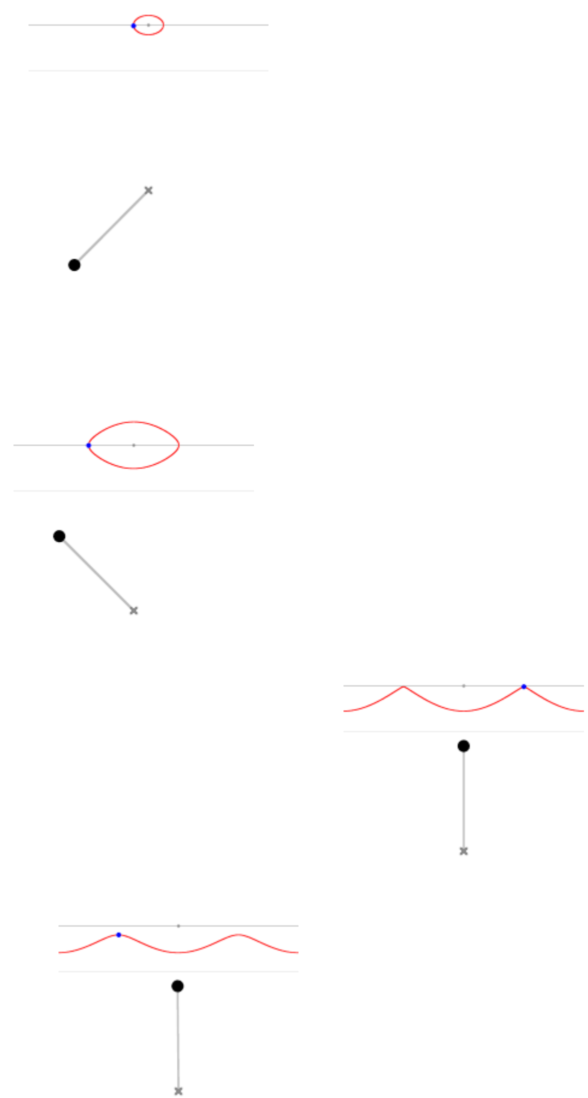
$\kappa_0 > \kappa_{critical}$

beam of infinite length

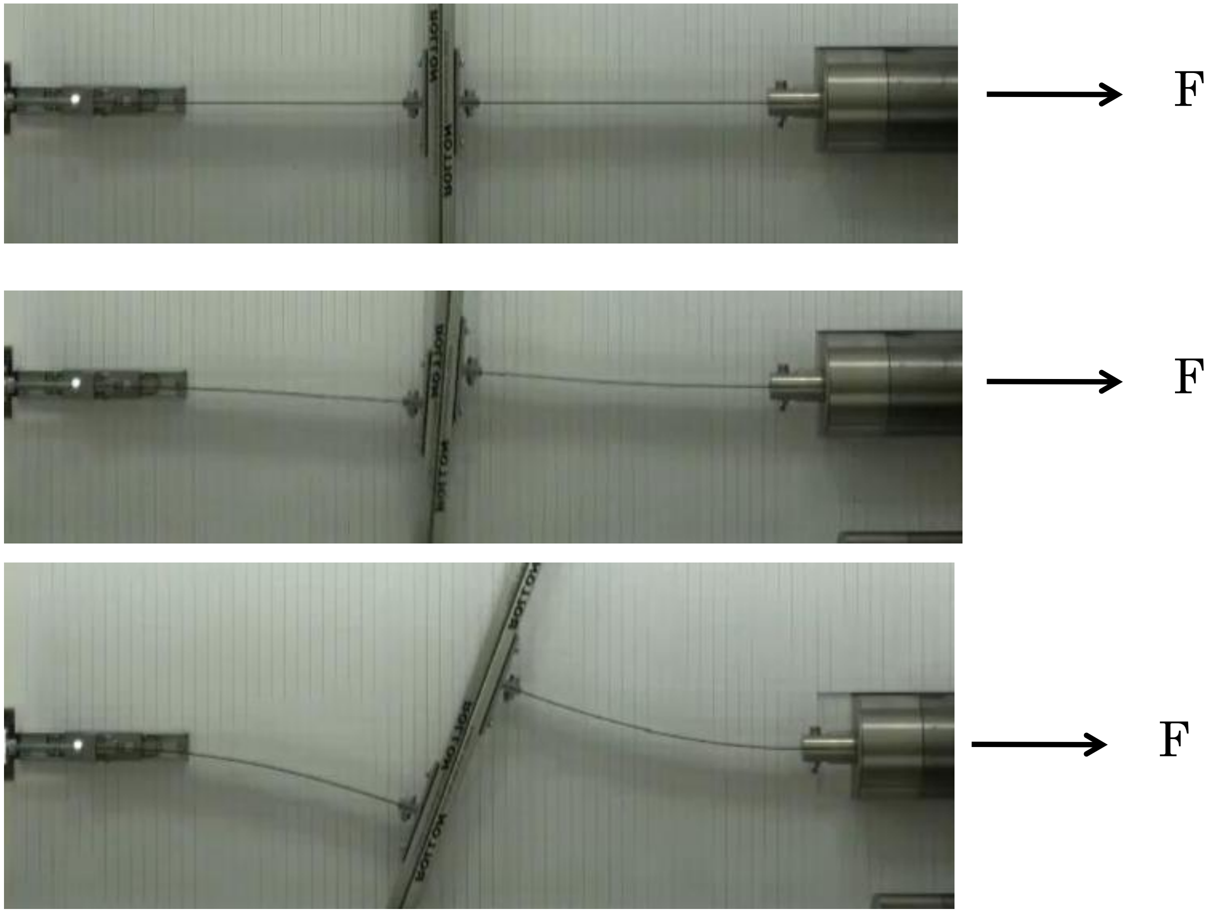
Summary of solution categories

1^{st} Caregory ${\kappa_0}^2 < 2c$ $\varphi(L) < \frac{\pi}{2}$
1-2: ${\kappa_0}^2 = 2c$ $\varphi(L) = \frac{\pi}{2}$
2^{nd} Caregory ${\kappa_0}^2 \in (2c, 4c)$ $\varphi(L) \in \left(\frac{\pi}{2}, \pi\right)$
2-3: ${\kappa_0}^2 = 4c$ $\varphi(L) = \pi$ infinite length
3^{rd} Caregory ${\kappa_0}^2 > 4c$ infinite length





Colaps under tensile load:



[1] Zaccaria, D.; Bigoni, D.; Noselli, G.; Misseroni, D.: *Structures buckling under tensile dead load.* Proceedings of the Royal Society A, 2011

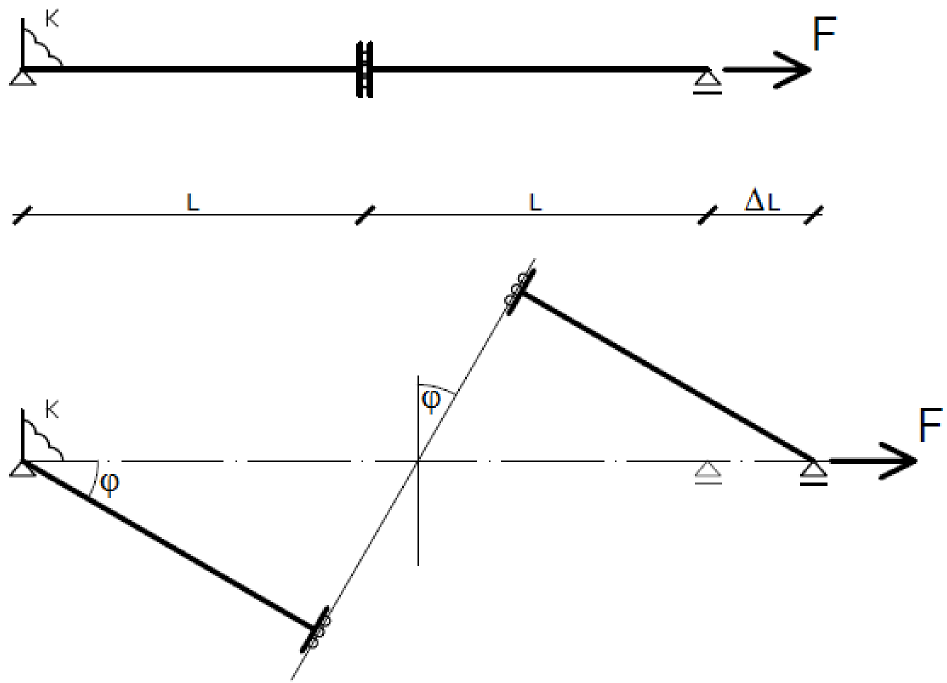
SYSTEM INSTABILITY IN TENSION:

Totally stiff bars with rotation spring:

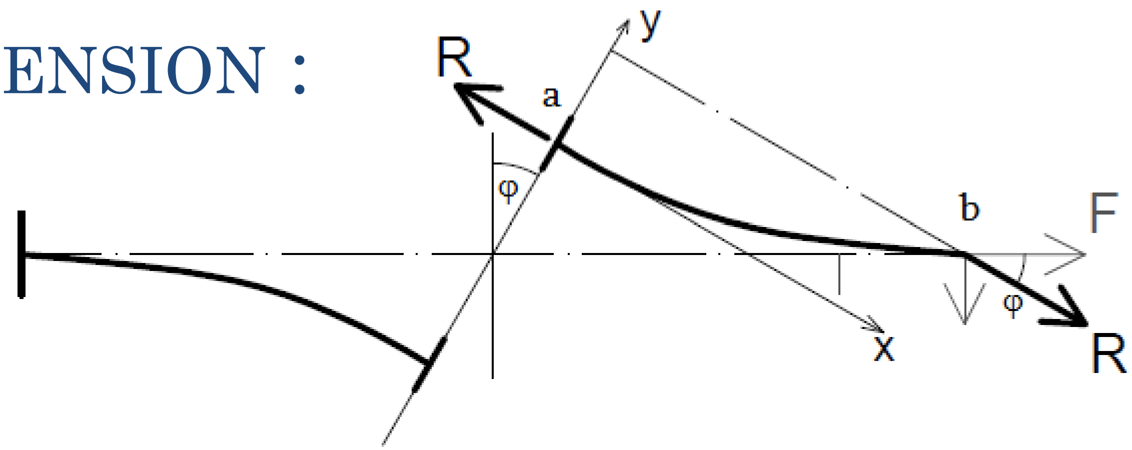
$$E(\varphi) = \frac{1}{2}k\varphi^2 - 2LF\left(\frac{1}{\cos\,\varphi} - 1\right)$$

when $\frac{dE(\varphi)}{d\varphi} = 0 \Rightarrow F = \frac{k}{2L}\frac{\varphi\cos^2\varphi}{\sin\,\varphi}$

Critical force: $F_{CR} = \frac{k}{2L}$



ELASTICA IN TENSION :



Bars with bending stiffness EI

Coordinate origin in **a**

$$\boxed{\frac{\partial^2\varphi(s)}{\partial s^2} - \frac{|R|}{EI}\sin(\varphi)} = 0 \qquad |R| = F\cos(\varphi(l))$$

Conditions: $\varphi(0) = 0$ but $\varphi'(l) \neq 0$

From the moment equilibrium:

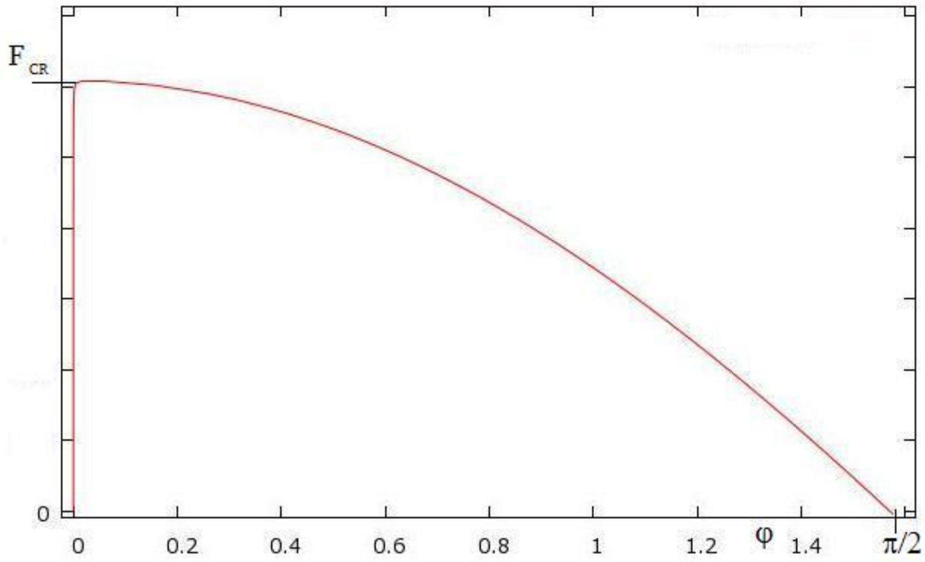
$$M(0) + M(l) + Ry(l) = 0$$

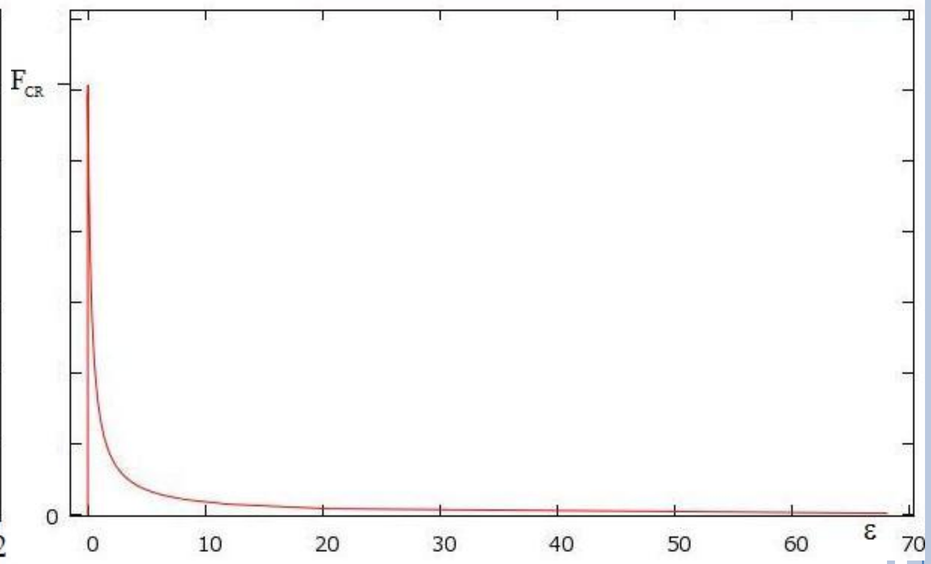
$$\Rightarrow \varphi'(l) = \kappa(l) = -\kappa_0 - \frac{F}{EI}\cos(\varphi(l))$$

SOFTENING:

Rotation of slider $\varphi(l)$

relative displacement $\varepsilon = \frac{\Delta l}{2l} = \frac{x(l)}{l\cos(\varphi)}$





CONCLUSION:

- New phenomenon – instability and collapse in tension
- Theory of elascica can be applied to solve this problem

Thank you for your attention