Optimization of Deterministic Population Dynamics Models

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1 Motivation

2 Optimal control problem

3 Aplication

Biological control

the deliberate use of one organism (natural enemy, bioagent) to regulate the population size of a pest organism

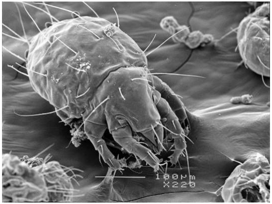
- **classical**

  importation from a native range; the aim is to establish a sustained population of bioagents
- **conservation**

  environmental manipulation
- **augmentation**

  periodic release of a small number of individuals (inoculation); massive release of a vast number of individuals (inundantion)

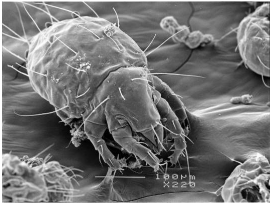
Integrated pest control = biological + chemical



A mite *Acarus siro* is a one of the most important pests of stored products (grain, cereals, oilseeds, cheese). Biological control was developed 40 years ago using a predatory mite *Cheyletus eruditus*:

$$\begin{aligned}\dot{x} &= g(x)\,x - f(x,y)\,y\\ \dot{y} &= \alpha\, f(x,y)\,y - d\,y\end{aligned}$$

- x abundance of prey (Acarus)
- y abundance of predator (Cheyletus)
- $g(x)$ rate of increase
- $f(x,y)$ functional response of the predator
- α conversion efficiency
- d mortality rate



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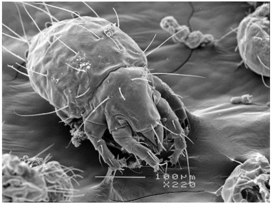

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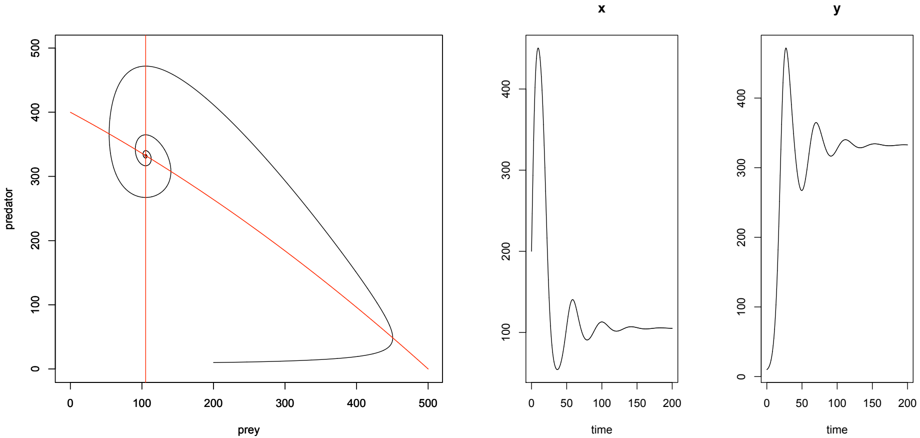
$$\begin{aligned} \dot{x} &= r\left(1-\frac{x}{K}\right)x - \frac{a\,x}{1+a\,x\,T}\,y \\ \dot{y} &= \alpha\,\frac{a\,x}{1+a\,x\,T}\,y - d\,y \end{aligned}$$

- x abundance of prey (Acarus)
- y abundance of predator (Cheyletus)
- α conversion efficiency
- d mortality rate
- r intrinsic growth rate
- K carrying capacity of the environment
- a predation rate (capture efficiency, search rate)
- T handling time (chasing, killing, eating, digesting)

$r = 0.4,\ K = 500,\ d = 0.08,\ f = 0.8,\ a = 0.001,\ T = 0.5$



Questions: Are we able to reduce Acarus population density below economic injury level only by release of bioagent Cheyletus? If it is possible, how many individuals of bioagent should be introduced?

General nonlinear dynamic model of n interacting populations

$$\begin{aligned}\dot{x}_i &= x_i f_i(x_1, x_2, \dots, x_n)\,, \quad i = 1, \dots, n \\ \dot{\mathbf{x}} &= F(\mathbf{x}) \\ \dot{\mathbf{x}} &= A\mathbf{x} + g(\mathbf{x})\end{aligned}$$

Pest control strategy – we are finding optimal control function, such that the system will drive to the desired steady state $\mathbf{x}^\star = [x_1^\star, \dots, x_n^\star]'$ in which the pest density is stable without causing economic damages and the bioagents density is stabilized at a level sufficient to control the pest:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + g(\mathbf{x}) + BU && (1) \\ 0 &= A\mathbf{x}^\star + g(\mathbf{x}^\star) + B\mathbf{u}^\star && (2)\end{aligned}$$

In general the desired steady state can be unstable. In such case $U = \mathbf{u}^\star + \mathbf{u}$. Define

$$\begin{aligned}\mathbf{y} &= \mathbf{x} - \mathbf{x}^\star && (3) \\ \mathbf{u} &= U - \mathbf{u}^\star && (4)\end{aligned}$$

Substituing into (1) we get the error system

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Theorem

If there exist PD matrix R and symmetric PD matrix Q, such as the function

$$l(\mathbf{y}) = \mathbf{y}'Q\mathbf{y} - h'(\mathbf{y})K\mathbf{y} - \mathbf{y}'Kh(\mathbf{y})$$

is positive definite, then the linear feedback control

$$\mathbf{u} = -R^{-1}B'K\mathbf{y} \tag{5}$$

is optimal in order to transfer the nonlinear error system from initial state $\mathbf{y}(0) = \mathbf{y}_0$ to the final state $\mathbf{y}(\infty) = 0$ and minimizing the functional

$$J = \int_0^\infty \left[l(\mathbf{y}) + \mathbf{u}'R\mathbf{u}\right] \mathrm{d}t\,,$$

where symmetric PD matrix K is the solution of the matrix algebraic Riccati equation

$$KA + A'K - KBR^{-1}B'K + Q = 0 \tag{6}$$

Acarus – Cheyletus:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{axy}{1 + aTx} \tag{7}$$

$$\dot{y} = \frac{faxy}{1 + aTx} - dy + U \tag{8}$$

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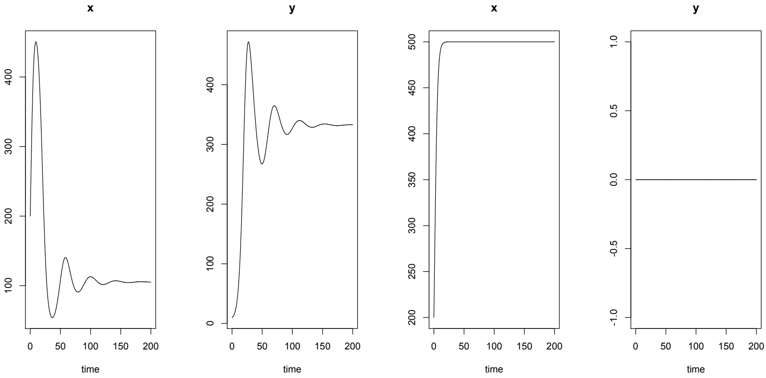

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\dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{axy}{1+aTx} && (7)\\
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\end{aligned}
$$

Substituing parameters and desired state $x^\star = 18$ into (7) and (8) we obtain $y^\star = 389.0704$, $u^\star = 25.57299$:

$$
\begin{aligned}
z &= x - x^\star = x - 18\\
v &= y - y^\star = y - 389.0704\\
u &= U - u^\star = U - 25.57299
\end{aligned}
$$

$$
\begin{aligned}
\dot{z} &= 0.3712z - \frac{4}{5000}z + 6.9408 - v\left(2 - \frac{2}{5\cdot 10^{-4}z + 1.009}\right) - \frac{0.389z + 7.003}{5\cdot 10^{-4}z + 1.009}\\
\dot{v} &= -0.08v + v\left(1.6 - \frac{1.6}{5\cdot 10^{-4}z + 1.009}\right) + \frac{0.3112z + 5.6024}{5\cdot 10^{-4}z + 1.009} + u - 5.55
\end{aligned}
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\end{aligned}
$$

Choosing matrices A, B, Q, R we obtain K as solution of Riccati equation:

$$
A = \begin{pmatrix} 0.3712 & -2 \\ 0 & 0.8 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 \end{pmatrix}
$$

$$
K = \begin{pmatrix} 1.8773 & -1.5472 \\ -1.5472 & 3.5980 \end{pmatrix}
$$

Finally

$$
\begin{aligned}
u &= -R^{-1}B'K\begin{pmatrix} z \\ v \end{pmatrix} = 1.5472z - 3.5980v = \\
&= 1.5472(x-18) - 3.5980(y - 389.0704) \\
U &= u + u^{\star} = 1.5472x - 3.5980y + 1397.599
\end{aligned}
$$

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{axy}{1 + aTx}$$
$$\dot{y} = \frac{faxy}{1 + aTx} - dy + 1.5472x - 3.5980y + 1397.599$$

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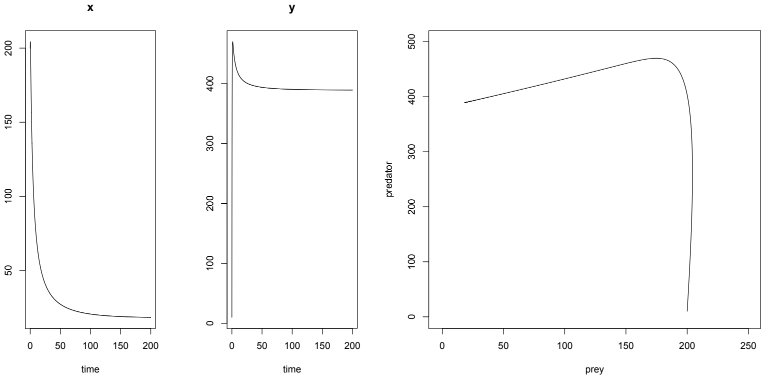

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