Oscillation of the fourth-order nonlinear difference equations

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Introduction

Consider the fourth-order difference equation

$$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right) \pm d_n x_{n+3}^{\delta} = 0,$$

where α, β, γ, δ are the ratios of odd positive integers and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are positive real sequences defined for all $n \in \mathbb{N}$.

Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

The solution x_n is said to be oscillatory if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_{n+1}x_n \leq 0$.

The equation is said to be oscillatory if all its solutions are oscillatory.

We study the nonoscillatory solutions depending on positive or negative perturbation d_n.
We assume

$$\sum_{n=n_0}^{\infty} a_n^{-\frac{1}{\alpha}} = \sum_{n=n_0}^{\infty} b_n^{-\frac{1}{\beta}} = \sum_{n=n_0}^{\infty} c_n^{-\frac{1}{\gamma}} = \infty$$

(canonical form of the operator L_4).

If we denote $y_n = c_n \left(\Delta x_n\right)^{\gamma} \quad z_n = b_n \left(\Delta y_n\right)^{\beta} \quad w_n = a_n \left(\Delta z_n\right)^{\alpha}$, then the equation can be written as a four-dimensional nonlinear system

$$\boxed{\begin{aligned} \Delta x_n &= C_n \cdot y_n^{\frac{1}{\gamma}} \\ \Delta y_n &= B_n \cdot z_n^{\frac{1}{\beta}} \\ \Delta z_n &= A_n \cdot w_n^{\frac{1}{\alpha}} \\ \Delta w_n &= \mp D_n \cdot x_{n+3}^{\delta}, \end{aligned} \qquad \text{(S)}}$$

where

$$A_n = a_n^{-\frac{1}{\alpha}} \quad B_n = b_n^{-\frac{1}{\beta}} \quad C_n = c_n^{-\frac{1}{\gamma}} \quad D_n = d_n,$$

and the solution of system is

$$\left(x_n, y_n, z_n, w_n\right).$$

Historical survey

Tanigawa [2000]
Thandapani and Selvaraj [2004]
Marini, Matucci, Řehák [2004, 2007]
Agarwal, Grace and Wong [2007]
Agarwal and Manojlovič [2009]
Schmeidel [2010]

Thandapani and Selvaraj; Agarwal and Manojlovič

$$\Delta^2 \left(p_n \left(\Delta^2 y_n \right)^{\alpha} \right) + q_n y_{n+3}^{\beta} = 0$$

Tanigawa; Marini, Matucci, Řehák

Coupled system

Coupled system

$$\boxed{\begin{aligned} \Delta\left(r_n\left(\Delta x_n\right)^{\gamma}\right) &= -\varphi_n z_{n+1}^{\frac{1}{\beta}} \\ \Delta\left(q_n\left(\Delta z_n\right)^{\alpha}\right) &= \psi_n x_{n+1}^{\delta} \end{aligned}}$$

Denote $y_n = r_n\left(\Delta x_n\right)^{\gamma} \quad w_n = q_n\left(\Delta\left(-z_{n+1}\right)\right)^{\alpha} \quad \bar{z}_k = -z_{k+1}$.

$$\boxed{\begin{aligned} \Delta x_n &= r_n^{-\frac{1}{\gamma}} \cdot y_n^{\frac{1}{\gamma}} \\ \Delta y_n &= \varphi_n \cdot \bar{z}_n^{\frac{1}{\beta}} \\ \Delta \bar{z}_n &= q_n^{-\frac{1}{\alpha}} \cdot w_n^{\frac{1}{\alpha}} \\ \Delta w_n &= -\psi_n \cdot x_{n+2}^{\delta} \end{aligned}}$$

$$\Delta\left(q_n\left(\Delta\left(\varphi_n^{-\beta}\left(\Delta\left(r_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right) + \psi_n x_{n+2}^{\delta} = 0$$

$$\sum_{n=n_0}^{\infty} q_n^{-\frac{1}{\alpha}} = \sum_{n=n_0}^{\infty} \varphi_n = \sum_{n=n_0}^{\infty} r_n^{-\frac{1}{\gamma}} = \infty$$

Negative perturbation

$$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right)-d_n x_{n+3}^{\delta}=0 \tag{1}$$

We investigate when the equation has property B, that means that all nonoscillatory solutions are either Kneser solutions or strongly monotone solutions.

Lemma

Any nonoscillatory solution (x,y,z,w) *of system (S) such that* $x_n > 0$ *is one of the following types:*

type(a)	$x_n > 0$	$y_n > 0$	$z_n > 0$	$w_n > 0$	*for all large n,*
type(b)	$x_n > 0$	$y_n > 0$	$z_n > 0$	$w_n < 0$	*for all large n,*
type(c)	$x_n > 0$	$y_n < 0$	$z_n > 0$	$w_n < 0$	*for all large n.*

Theorem 1

If one of the following assumptions holds:

$$\sum_{n=n_0}^{\infty} d_n = \infty,$$

$$\sum_{n=n_0}^{\infty} d_n < \infty \text{ and } \sum_{k=n_0}^{\infty} \left(\frac{1}{b_k} \sum_{n=n_0}^{k-1} \left(\frac{1}{a_n} \sum_{j=n}^{\infty} d_j \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} = \infty,$$

then every bounded nonoscillatory solution (x, z, y, w) *of system (S) is type (c) and satisfies*

$$\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} y_n = 0, \lim_{n\to\infty} z_n = 0, \lim_{n\to\infty} w_n = 0.$$

Kneser solution

Theorem 2

Any type (a) solution (x, y, z, w) *of system (S) satisfies* $\lim_{n\to\infty} |x_n| = \infty, \lim_{n\to\infty} |z_n| = \infty$. *If*

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{r=n_0}^{n-1} \left(\frac{1}{c_r} \sum_{t=n_0}^{r-1} \left(\frac{1}{b_t} \sum_{i=n_0}^{t-1} \left(\frac{1}{a_i} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\gamma}} \right)^{\delta} = \infty,$$

then $\lim_{n\to\infty} |w_n| = \infty$. *If in addition*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{b_n} \sum_{i=n_0}^{n-1} \left(\frac{1}{a_i} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} = \infty,$$

then $\lim_{n\to\infty} |y_n| = \infty$.

Strongly monotone solution

Theorem 3

$$\text{If } \sum_{n=n_0}^{\infty} d_n \left(\sum_{r=n_0}^{n-1} \left(\frac{1}{c_r} \sum_{t=n_0}^{r-1} \left(\frac{1}{b_t} \sum_{i=n_0}^{t-1} \left(\frac{1}{a_i} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\gamma}} \right)^{\delta} = \infty$$

$$\text{and } \sum_{k=n_0}^{\infty} \left(\frac{1}{b_k} \sum_{n=n_0}^{k-1} \left(\frac{1}{a_n} \sum_{j=n}^{\infty} d_j \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} = \infty,$$

then the equation (1) has property B, that means that all nonoscillatory solutions are either type (a) or (c) and solutions satisfy either

$$\lim_{n\to\infty} |x_n| = \infty, \lim_{n\to\infty} |y_n| = \infty, \lim_{n\to\infty} |z_n| = \infty, \lim_{n\to\infty} |w_n| = \infty,$$

$$\text{or } \lim_{n\to\infty} |x_n| = 0, \lim_{n\to\infty} |y_n| = 0, \lim_{n\to\infty} |z_n| = 0, \lim_{n\to\infty} |w_n| = 0.$$

Došlá Z., Krejčová J.:
Nonoscillatory solutions of the four-dimensional difference system,
E. J. Qualitative Theory of Diff. Equ., Proc. 9'th Coll. Qualitative Theory of Diff. Equ., No. 4 (2011), pp. 1-11

Positive perturbation

$$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right)+d_n x_{n+3}^{\delta}=0 \tag{2}$$

We establish necessary and sufficient conditions for equation to have nonoscillatory solutions with specific asymptotic behavior. We give conditions that (2) is oscillatory, has property A, that means there are only oscillatory solutions of the equation.

Lemma

Any nonoscillatory solution (x, y, z, w) *of system (S) such that* $x_n > 0$ *is one of the following types:*

Type(a)	$x_n > 0$	$y_n > 0$	$z_n > 0$	$w_n > 0$	*for all large n,*
Type(b)	$x_n > 0$	$y_n > 0$	$z_n < 0$	$w_n > 0$	*for all large n.*

Theorem 4

A necessary and sufficient condition for equation (2) to have a nonoscillatory solution x_n which satisfies

$$\lim_{n\to\infty} \frac{x_n}{\rho_n} = R,$$

$0 < R < \infty$, *where*

$$\rho_n = \sum_{s=n_0}^{n-1} \left(\frac{1}{c_s} \sum_{r=n_0}^{s-1} \left(\frac{1}{b_r} \sum_{t=n_0}^{r-1} \left(\frac{1}{a_t} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\gamma}}$$

is that

$$\sum_{n=n_0}^{\infty} d_n \rho_{n+3}^{\delta} < \infty.$$

Theorem 5

A necessary and sufficient condition for equation (2) to have a nonoscillatory solution x_n *which satisfies*

$$\lim_{n\to\infty} x_n = x_0,$$

$0 < x_0 < \infty$, *is that*

$$\sum_{s=n_0}^{\infty}\left[\frac{1}{c_s}\sum_{r=s}^{\infty}\left[\frac{1}{b_r}\sum_{l=r}^{\infty}\left[\frac{1}{a_l}\sum_{m=l}^{\infty}d_m\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\gamma}} < \infty.$$

Theorem 6

Let any of the following conditions hold

(i)

$$\sum_{n=n_0}^{\infty} d_n = \infty$$

(ii)

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{k=n}^{\infty} d_k \right)^{\frac{1}{\alpha}} = \infty \textit{ and } \sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+2} \left(\frac{1}{c_i} \right)^{\frac{1}{\gamma}} \right)^{\delta} = \infty,$$

then all solutions of the equation (2) are oscillatory.

Property A

Open problems

- the role of the shift $\tau \in \mathbb{Z}$ in the equation

(3) $$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right)+d_n x_{n+\tau}^{\delta}=0$$

Theorem 7

Equation (3) has no solution of type (a) if any of the following conditions hold:

(i)

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \right)^{\lambda} = \infty;$$

(ii)

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \right)^{1/\gamma} \right)^{\lambda} = \infty;$$

(iii) $\lambda < \alpha\beta\gamma$ *and*

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda} = \infty.$$

Theorem 8

Equation (3) has no solution of type (b) if any of the following conditions hold:

(i)

$$T := \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha} = \infty,$$

(ii) $T < \infty$ *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n}^{\infty} \frac{1}{a_k^{1/\alpha}} \left(\sum_{i=n}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} = \infty,$$

(iii) $\lambda < \alpha\beta\gamma$, $T < \infty$ *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n_0}^{n+\tau-1} \frac{1}{c_k^{1/\gamma}} \right)^{\lambda/(\alpha\beta)} \left(\sum_{k=n}^{\infty} \frac{1}{a_k^{1/\alpha}} \left(\sum_{i=n}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} = \infty.$$

Open problems

- $$\sum_{n=n_0}^{\infty} a_n^{-\frac{1}{\alpha}},\ \sum_{n=n_0}^{\infty} b_n^{-\frac{1}{\beta}},\ \sum_{n=n_0}^{\infty} c_n^{-\frac{1}{\gamma}}$$

$$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right)+d_n x_{n+\tau}^{\delta}=0$$

Cyclic permutation

Lemma

The following statements are equivalent:

(i) x *is a solution of (3).*

(ii) $y = \{y_n\}$, *where* $y_n = c_n\,(\Delta x_n)^{\gamma}$, *is a solution of*

$$\text{(R1)}\qquad \Delta\left(\frac{1}{d_n^{1/\lambda}}\left(\Delta\, a_n\left(\Delta\, b_n\,(\Delta y_n)^{\beta}\right)^{\alpha}\right)^{1/\lambda}\right)+\frac{1}{c_{n+\tau}^{1/\gamma}}\,y_{n+\tau}^{1/\gamma}=0.$$

(iii) $z = \{z_n\}$, *where* $z_n = b_n\,(\Delta y_n)^{\beta}$, *is a solution of*

$$\text{(R2)}\qquad \Delta\left(c_{n+\tau}\left(\Delta\frac{1}{d_n^{1/\lambda}}\left(\Delta\, a_n\,(\Delta z_n)^{\alpha}\right)^{1/\lambda}\right)^{\gamma}\right)+\frac{1}{b_{n+\tau}^{1/\beta}}\,z_{n+\tau}^{1/\beta}=0.$$

(iv) $w = \{w_n\}$, *where* $w_n = a_n\,(\Delta z_n)^{\alpha}$ *is a solution of*

$$\text{(R3)}\qquad \Delta\left(b_{n+\tau}\left(\Delta\, c_{n+\tau}\left(\Delta\frac{1}{d_n}\,(\Delta w_n)^{1/\lambda}\right)^{\gamma}\right)^{\beta}\right)+\frac{1}{a_{n+\tau}^{1/\alpha}}\,w_{n+\tau}^{1/\alpha}=0.$$

Theorem 9

Equation (3) is oscillatory if and only if equation (R_i) is oscillatory for $i \in \{1, 2, 3\}$.

Example 1

Equation

$$\Delta^2 \left(b_n \left(\Delta^2 x_n \right)^{\beta} \right) + d_n x_{n+3}^{\lambda} = 0.$$

is oscillatory if and only if the equation

$$\Delta^2 \left(\frac{1}{d_n^{1/\lambda}} \left(\Delta^2 z_n \right)^{1/\lambda} \right) + \frac{1}{b_{n+3}^{1/\beta}} z_{n+3}^{1/\beta} = 0$$

is oscillatory.

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Thank you for your attention!