# Half-linear Euler differential equations in the critical case

Hana Haladová

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Masaryk University, Brno, Czech republic

Joint research with Ondřej Došlý

# Outline

# Introduction

Term half-linear is motivated by the fact that the solution space of half-linear equation has just one half of the properties which characterize linearity, namely homogenity (but not aditivity).

General second order half-linear differential equation is in the form of

(1) $$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0,$$

where $r, c$ are continuous functions and $r(t) > 0$.

The half-linear Euler differential equation

(2) $$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,$$

$p > 1,\ \gamma \in \mathbb{R}$.

The second order Sturm-Liouville linear differential equation

(3) $$\big(r(t)x'\big)' + c(t)x = 0$$

is a special case for $p = 2$ in (1).

Then, given $t_0, x_0, x_1 \in \mathbb{R}$, there exists the unique solution of (3) satisfying initial conditions $x(t_0) = x_0, x'(t_0) = x_1$, which is extensible over the whole interval where the functions $r, c$ are continuous and $r(t) > 0$.

If we rewrite (2) into the first order system (substituting $u = r\Phi(x')$), we get the system

$$x' = r^{1-q}(t)\Phi^{-1}(u), \quad u' = -c(t)\Phi(x), \tag{4}$$

where $q$ is the conjugate number of $p$ ($\frac{1}{p} + \frac{1}{q} = 1$), and $\Phi^{-1}$ is the inverse funciton of $\Phi$. The right hand-side of (4) is no longer LIpschitzian in $x, u$, hence the standard existence and uniqueness theorems do not apply directly to this system.

The half-linear Euler differential equation (2) can be solved explicitly.

If we look for a solution of (2) in the form $x(t) = t^\lambda$, substituting into (2) we find that $\lambda$ has to be a solution of the algebraic equation

$$(p-1)\Phi(\lambda)(\lambda-1)+\gamma=0.$$

$F(\lambda) := (p-1)\Phi(\lambda)(\lambda-1)$ has a global minimum at $\lambda^* = \frac{p-1}{p}$ and its value is $F(\lambda^*) = -\gamma_p := -\left(\frac{p-1}{p}\right)^p$.

The equation $F(\lambda)+\gamma=0$ has two real roots if $\gamma<\gamma_p$, one double real root if $\gamma=\gamma_p$, and no real root if $\gamma>\gamma_p$.

Equation (2) is a particular case of the general half-linear second order differential equation (1)

(2) is nonoscillatory if and only if $\gamma \leq \gamma_p$.
(2) with the critical coefficient $\gamma = \gamma_p$ serves as a comparison equation for the Kneser-type (non)oscillation test.

(1) with $r(t) = 1$ is oscillatory provided

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^p c(t) < \gamma_p$$

The Kneser test does not apply when $\lim_{t\to\infty} t^p c(t) = \gamma_p$.

# Auxiliary results

We suppose that equation (1) is nonoscillatory and we consider its perturbation

(5) $$[(r(t)+\tilde{r}(t))\Phi(x')]' + (c(t)+\tilde{c}(t))\Phi(x) = 0,$$

$\tilde{r}, \tilde{c}$ are continuous functions such that $r(t)+\tilde{r}(t) > 0$ for large $t$.

Let $x$ be a solution of (1), $x(t) \neq 0$, then $w = r\Phi(x'/x)$ is a solution of the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q := \frac{p}{p-1}.$$

Equation (1) with $\lambda c(t)$ is said to be *conditionally oscillatory* if there exists a constant $\lambda_0$ such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$.

$\lambda_0$ is called the *oscillation constant* of (1).

We consider the equation

$$[(r(t)+\lambda\tilde{r}(t))\Phi(x')]' + (c(t)+\mu\tilde{c}(t))\Phi(x) = 0. \tag{6}$$

We say that (6) is conditionally oscillatory if there exist constants $\alpha, \beta, \omega \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 0$, such that (6) is oscillatory for $\alpha\lambda + \beta\mu > \omega$ and nonoscillatory for $\alpha\lambda + \beta\mu < \omega$.

## Example

$$\left[\left(1+\frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0 \tag{7}$$

(7) is oscillatory if $\mu - \lambda\gamma_p > \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$ and nonoscillatory if $\mu - \lambda\gamma_p < \mu_p$. (7) is nonoscillatory also in the limiting case

$$\mu - \lambda\gamma_p = \mu_p. \tag{8}$$

We derive the modified Riccati equation. Let $h(t) \neq 0$ be a differentiable function, denote

$$G(t) := r(t)h(t)\Phi(h'(t)) \tag{9}$$

and let $\Omega(t) := \left(1 + \frac{r(t)}{\tilde{r}(t)}\right)G(t)$ . Define the function

$$\mathcal{G}(t,z) := |z + \Omega(t)|^q - q\Phi^{-1}(\Omega(t))z + |\Omega(t)|^q \tag{10}$$

and put

$$z := h^p(w - w_h) - \frac{\tilde{r}}{r}G = h^p w - G - \tilde{G},$$

where $w$ is a solution of the Riccati equation associated with (5)

$$w' + c(t) + \tilde{c}(t) + (p-1)(r(t) + \tilde{r}(t))^{1-q}|w|^q = 0,$$

$w_h = r\Phi(h'/h)$, and $\tilde{G} = \tilde{r}h\Phi(h')$. Then $z$ is a solution of the so-called *modified Riccati equation*

$$z' + C(t) + (p-1)(r(t) + \tilde{r}(t))^{1-q}h^{-q}(t)\mathcal{G}(t,z) = 0, \tag{11}$$

where

$$C(t) = h(t)\left[\left((r(t) + \tilde{r}(t))\Phi(h'(t))\right)' + (c(t) + \tilde{c}(t))\Phi(h(t))\right]. \tag{12}$$

## Theorem 1

*Let $h$ be a positive differentiable function such that $h'(t) \neq 0$ for large $t$. Denote*

(13) $$R(t) = \big(r(t) + \tilde{r}(t)\big) h^2(t) |h'(t)|^{p-2},$$

*and suppose that*

$$\int^{\infty} \frac{dt}{R(t)} = \infty, \quad \int^{\infty} C(t)\, dt \text{ is convergent},$$

*where $C$ is given by* (12), *and* $\liminf_{t\to\infty} (r(t) + \tilde{r}(t))\, h(t)|h'(t)|^{p-1} > 0$. *If*

(14) $$\limsup_{t\to\infty} \int^{t} \frac{ds}{R(s)} \int_{t}^{\infty} C(s)\, ds < \frac{1}{2q}$$

*and*

(15) $$\liminf_{t\to\infty} \int^{t} \frac{ds}{R(s)} \int_{t}^{\infty} C(s)\, ds > -\frac{3}{2q}$$

*then equation* (5) *is nonoscillatory.*

# Equation (7) in the limiting case

## Theorem 2

*Suppose that* (8) *holds. Then the perturbed Euler equation with the critical coefficient*

$$\left[\left(1+\frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p}+\frac{\mu}{t^p\log^2 t}\right]\Phi(x)=0.$$

*is nonoscillatory.*

**Proof.**

We rewrite (7) into the form

$$\left[\left(1+\frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p\log^2 t} + \frac{\mu-\mu_p}{t^p\log^2 t}\right]\Phi(x) = 0 \tag{16}$$

and we use the previous computation with $r(t)=1$, $\tilde{r}(t)=\frac{\lambda}{\log^2 t}$, $c(t)=\frac{\gamma_p}{t^p}+\frac{\mu_p}{t^p\log^2 t}$, $\tilde{c}(t)=\frac{\mu-\mu_p}{t^p\log^2 t}$, $h(t)=t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$.

We have

$$h' = \frac{p-1}{p}t^{-\frac{1}{p}}\log^{\frac{1}{p}} t\left(1+\frac{1}{(p-1)\log t}\right),$$

$$\Phi(h') = \left(\frac{p-1}{p}\right)^{p-1} t^{-\frac{p-1}{p}}\log^{\frac{p-1}{p}} t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1},$$

$$\left(\Phi(h')\right)' = t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}} t\left[-\gamma_p - \frac{\mu_p}{\log^2 t} + O\left(\log^{-3} t\right)\right].$$

**Proof.**

Similarly,

$$\left(\frac{\lambda}{\log^2 t}\Phi(h')\right)' = \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[-\frac{p-1}{p} - \frac{2}{\log t} + o\big(\log^{-1} t\big)\right].$$

Hence, in the limiting case (8) it holds

$$[(\tilde{r}\Phi(h'))' + \tilde{c}\Phi(h)] = -\frac{2\gamma_p}{p-1} t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}} t(1+o(1))$$

as $t \to \infty$. Consequently,

$$h[(\tilde{r}\Phi(h'))' + \tilde{c}\Phi(h)] = O\big(t^{-1}\log^{-2} t\big)$$

as $t \to \infty$.

## Proof.

Now we use Theorem 1. In this theorem

$$R = (r + \tilde{r})h^2|h'|^{p-2} = t\log t(1 + o(1)) \sim t\log t$$

(here $f(t) \sim g(t)$ for a pair of functions $f, g$ means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$),

$$G = rh\Phi(h') = \left(\frac{p-1}{p}\right)^{p-1} \log t \left(1 + \frac{1}{(p-1)\log t}\right)^{p-1}$$

and using the previous computations

$$C = h\big[\big((r + \tilde{r})\Phi(h')\big)' + (c + \tilde{c})\Phi(h)\big] = O\big(t^{-1}\log^{-2} t\big)$$

as $t \to \infty$, i.e., there exists a constant $M > 0$ such that $|C(t)| \le M$.
Now, by a direct computation

$$\lim_{t\to\infty} \left| \int^t R^{-1}(s)\, ds \int_t^\infty C(s)\, ds \right| \le M \lim_{t\to\infty} \frac{\log(\log t)}{\log t} = 0,$$

so by Theorem 1 equation (7) with $\lambda$ and $\mu$ satisfying (8) is nonoscillatory. □

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**Thank you for your attention.**

Masaryk University, Brno,
Czech republic