A Posteriori Error Estimation of Torsion Constant

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Goal of research

The goal of my research was to calculate a torsion constant of cross section using the finite element method and then to estimate how large error was made during calculation.

Torsion constant

$$I_k = \int_\Omega x^2 + y^2 + \frac{\partial \psi}{\partial y} x - \frac{\partial \psi}{\partial x} y \, dxdy.$$

Laplace's equation for ψ

$$\frac{\partial^2 \psi(x,y)}{\partial x^2} + \frac{\partial^2 \psi(x,y)}{\partial y^2} = 0.$$

Boundary condition

$$\frac{\partial \psi}{\partial n} = n_x y - n_y x.$$

Let's have variation formula of Laplace's equation with the boundary condition. We are searching for such $u \in V$ that

$$B(u,v) = F(v), \qquad v \in V, \tag{1}$$

where $V = \{v;\ v \in H^1(\Omega),\ \int_\Omega v\,dx = 0\}$. F is continuous linear functional in V a B is positive-definite bilinear form on V,

$$B(u,v) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dxdy,$$

$$F(u) = \int_{\partial\Omega} (yn_x - xn_y)u\,ds,$$

where (n_x, n_y) is vector of unit normal line to boundary $\partial\Omega$ of area Ω.

Let L be linear functional in V,

$$L(u) = \int_\Omega \frac{\partial u}{\partial y}x - \frac{\partial u}{\partial x}y\,dxdy.$$

Our goal is to find an approximate solution u_h of equation (1) and to calculate difference $L(u) - L(u_h)$.

$$I_k = \int_\Omega x^2 + y^2 + \frac{\partial \psi}{\partial y}x - \frac{\partial \psi}{\partial x}y\,dxdy.$$

Let's have $V_h \subset V$, $\dim V_h = N_h < \infty$. We are searching for such $u_h \in V_h$ that

$$B(u_h, v) = F(v), \qquad v \in V_h. \tag{2}$$

One of the possible formulations of our problem:

$$u_h = \text{argmin}_{v \in V_h} \left(B(v, v) - 2F(v) \right).$$

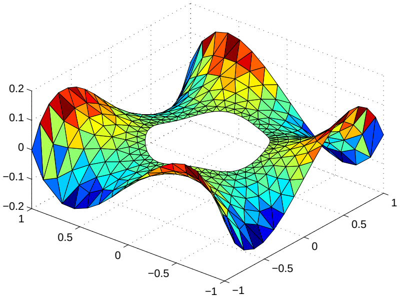
We use the method of Lagrange multipliers to find the extreme. Let's denote energetic norm of operator B

$$|||\cdot||| = \sqrt{B(\cdot,\cdot)}.$$

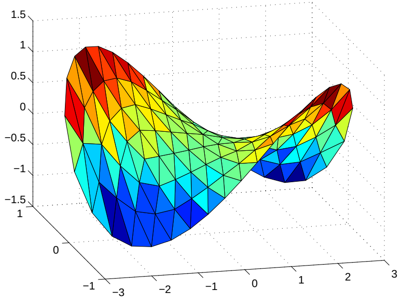
- Triangulation of the elliptic area (on the left) and of the square area with a circle hole (on the right).

Persson, P.-O.; Strang, G., *A Simple Mesh Generator in MATLAB*, June 2004
`http://persson.berkeley.edu/distmesh/`

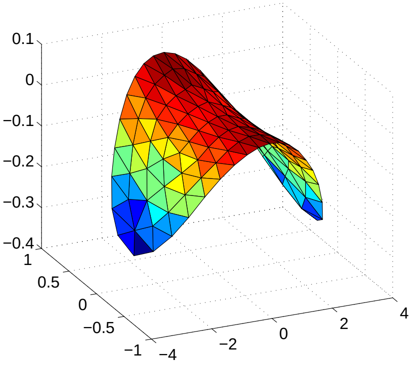
- Solution u_h on the square area with a circle hole.



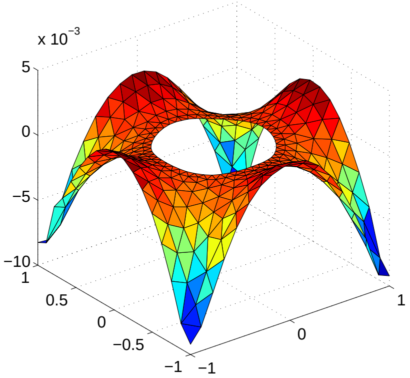
- Solution u_h on the elliptic area.



- Contributions of elements to the function $L(u_h)$ on the elliptic area.



- Contributions of elements to the function $L(u_h)$ on the square area with a circle hole.



Let's call $e \in V$ difference between approximate solution u_h and exact solution,

$$e = u - u_h.$$

As L is linear, we have

$$L(e) = L(u) - L(u_h)$$

and thus

$$B(e, v) = B(u - u_h, v) = 0, \qquad v \in V_h.$$

As $dim\ V = \infty$ we can't calculate neither e nor $L(e)$ exactly.

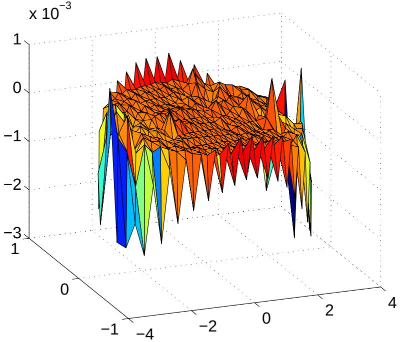
Heuristic estimate of $L(e)$

We choose appropriate space $V_1, V_h \subset V_1 \subset V, dim\ V_1 < \infty$. Let's denote

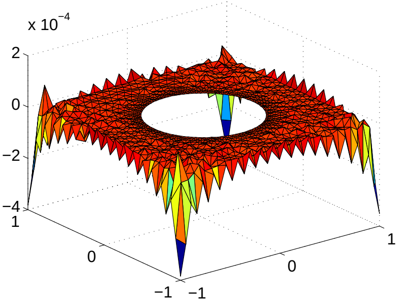
$$\tilde{e}_{V1} = u_1 - u_h.$$

Function $\tilde{e}_{V1}$ can be considered an estimate of exact error e, $\tilde{e}_{V1} \approx e$.

- Contributions of elements to the function $L(\tilde{e}_{V1})$ on the elliptic area.



- Contributions of elements to the function $L(\tilde{e}_{V1})$ on the square area with a circle hole.



Estimate of $L(e)$ using dual problem

Let's denote residue

$$R_h(v) = F(v) - B(u_h, v) = B(u - u_h, v) = B(e, v).$$

For each $v \in V_h$

$$R_h(v) = 0.$$

Relation between residue $R_h(v)$ and value of $L(e)$ is linear in V, because both depends linearly on $e = u - u_h$. It means there is such a linear functional ω that

$$L(e) = \omega(R_h).$$

Because of features of residue we can say that $\omega(R_h) = R_h(\omega)$, thus

$$B(u - u_h, \omega) = L(e),$$

$$B(e, \omega) = L(e).$$

Oden, J.T.; Prudhomme, S., *Goal-Oriented Error Estimation and Adaptivity for the Finite Element Method*, 1999

We have

$$B(e, \omega) = L(e).$$

It is true for example if

$$B(v, \omega) = L(v),$$

for each $v \in V$. This is so called **dual problem**. Let's denote approximate solution of dual problem $\omega_h \in V_h$ then

$$B(v, \omega_h) = L(v), \qquad v \in V_h.$$

We can calculate error analogically as in original problem $\varepsilon = \omega - \omega_h \in V$. We have $B(e, \omega_h) = 0$. It means

$$L(e) = B(e, \omega - \omega_h) = B(e, \varepsilon),$$

then

$$|L(e)| = |B(e, \varepsilon)| \leq \sum_K |B(e, \varepsilon)_K| \leq \sum_K |||e|||_K |||\varepsilon|||_K.$$

Curiosity

Using Green theorem we can prove that for functional F a L is

$$F(v) = -L(v)$$

for each $v \in V$. This is a special feature of our primal problem. Therefore $\omega = -u$ a $\omega_h = -u_h$ and thus $e = -\varepsilon$. Then

$$|L(e)| = |B(e,\varepsilon)| = |B(e,e)| = |||e|||^2 = \sum_K |||e|||_K^2.$$

Therefore we can replace calculation of $|L(e)|$ by calculation of energetic norm of error e or by estimate of energetic norm of error.

Heuristic estimate of $|||e|||$

Let's remind that u_1 is solution in space V_1 and thus

$$B(u_1, v) = F(v), \qquad v \in V_1.$$

Difference $\tilde{e}_{V_1} = u_1 - u_h$ can be considered an approximate error e.

Projection of exact error e to V_1

Let's call e_{V_1} orthogonal projection of error e to the space V_1. Then we have

$$B(e - e_{V_1}, v) = 0$$

for each $v \in V_1$. Then we can prove

$$|||e_{V_1}|||^2 = B(e_{V_1}, e_{V_1}) = B(e, e_{V_1}) \leq |||e|||\,|||e_{V_1}|||$$

and thus

$$|||e_{V_1}||| \leq |||e|||.$$

We can prove that $\tilde{e}_{V_1} = e_{V_1}$, therefore we don't need to calculate heuristic estimate of $\tilde{e}_{V_1}$.

Let's define such a new space W_h that $V_h \oplus W_h = V_1$ a $V_h \cap W_h = \{0\}$. Let's remind that $V_1 \subset V$, and thus $W_h \subset V$.

Relations of norms of errors in V_h and V_1

Let's call β saturation constant for spaces V_1 a $V_h \subset V_1$ and operator $B(\cdot,\cdot)$ and thus

$$|||u - u_1||| \leq \beta |||u - u_h|||.$$

We can prove that

$$|||u_1 - u_h|||^2 \leq |||u - u_h|||^2 \leq \frac{1}{1-\beta^2} |||u_1 - u_h|||^2.$$

Relations of norms of errors in W_h and V_1

Let's remind that $e_{W_h} \in W_h$ is orthogonal projection of e to the space W_h, thus

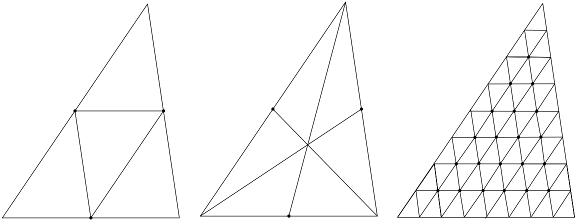
$$B(u - u_h - e_{W_h}, w) = 0$$

for each $w \in W_h$. We have

$$|||e_{W_h}||| \leq |||u_1 - u_h|||.$$

Space V_h is created by dividing of area Ω into triangles. Space V_h consists of linear functions continuous in triangles. We distinguish three types of space W_h , space V_1 is direct sum of V_h and W_h, $V_1 = V_h \oplus W_h$.

- Triangulation for space W_h: type A (on the left), type B (in the middle), type C (on the right).

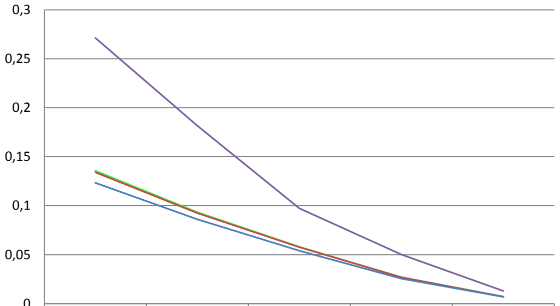


For the eliptic area there is known an exact solution,

$$\psi(x,y) = \frac{b^2 - a^2}{b^2 + a^2} xy,$$

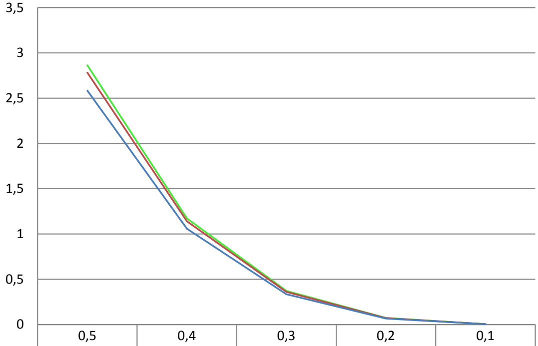
where a and b are sizes of semi-axises.

- Graph of error estimates of $L(u_h)$ for projections of error of u_h to the spaces on the elliptic area.



	0,5	0,4	0,3	0,2	0,1
V	0,2713	0,1817	0,0973	0,0502	0,0129
V1(Typ A)	0,1356	0,0933	0,0580	0,0270	0,0071
Wh(Typ A)	0,1341	0,0925	0,0576	0,0269	0,0071
Wh(Typ B)	0,1233	0,0861	0,0540	0,0256	0,0069

- Graph of error estimates of $L(u_h)$ for projections of error of u_h to the spaces on the cross area.



	0,5	0,4	0,3	0,2	0,1
V1(Typ A)	2,8637	1,173	0,3711	0,0733	0,0046
Wh(Typ A)	2,7829	1,1399	0,3607	0,0712	0,0045
Wh(Typ B)	2,5843	1,0585	0,3349	0,0662	0,0041

CONCLUSION:

- Thanks to special features of our problem we could replace calculation of $L(e)$ by calculation of $|||e|||$.
- Each of used estimates of $|||e|||$ gives us guaranteed lower bound of $|||e|||$.
- Both types of estimates give us similar results they depends on choice of spaces. We should think about difficulty of calculation.

Thank you for your attention