On the generalizations of the unit sum number problem

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Preliminaries I

Let α be an *algebraic number* of degree n (root of $p \in \mathbb{Z}[x]$ of degree n).

- conjugates
  $\alpha_1, \ldots, \alpha_s \in \mathbb{R},\ \alpha_{s+1}, \overline{\alpha_{s+1}}, \ldots, \alpha_{s+t}, \overline{\alpha_{s+t}} \in \mathbb{C} \setminus \mathbb{R}$,
  $n = s + 2t$
- *algebraic number field* $K = \mathbb{Q}(\alpha) = \mathbb{Q} + \alpha\mathbb{Q} + \ldots + \alpha^{n-1}\mathbb{Q}$
  $\cdots$ the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and α
- *norm* $\cdots$ $N(\beta) = \sigma_1(\beta) \cdots \sigma_s(\beta) |\sigma_{s+1}(\beta)|^2 \cdots |\sigma_{s+t}(\beta)|^2$,
  where σ_i is the *field isomorphism* corresponding to α_i

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Preliminaries II

- $O_K \cdots$ the ring of *algebraic integers* in K (roots of $p \in \mathbb{Z}[x]$ monic)
- one can map $O_K \to$ lattice in $\mathbb{R}^n$, its covolume equals the *discriminant* of the field $D(K)$
- $U_K \cdots$ the group of *units* in O_K ($N(\beta) = \pm 1$)
- logarithmic projection: $U_K \to$ lattice in $\mathbb{R}^{s+t}$, its covolume equals the *regulator* of the field $R(K)$
- $\beta \in U_K$ is a *root of unity*, if $\beta^k = 1$ for some $k \in \mathbb{N}$

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Unit sum number

Unit sum number of ring R:

$$u(R) = \begin{cases} t \in \mathbb{N} & \text{all } \beta \in R \text{ are sums of at most } t \text{ units} \\ \omega & \text{all } \beta \in R \text{ are (possibly infinite) sums of units} \\ \infty & \text{some } \beta \in R \text{ is not a sum of units} \end{cases}$$

Theorem 1 (Jarden, Narkiewicz)

Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field. Then there is no $t \in \mathbb{N}$, such that every element of O_K is a sum of at most t units, i.e. $u(O_K) \geq \omega$.

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Unit sum number $\cdots$ examples

Theorem 2 (Ashrafi, Vámos)

Let $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ squarefree. Then $u(O_K) = \omega$ iff

1. $d \in \{-1, -3\}$, *or*
2. $d > 0, d \not\equiv 1$ *mod* $4, d + 1$ *or* $d - 1$ *is a perfect square, or*
3. $d > 0, d \equiv 1$ *mod* $4, d + 4$ *or* $d - 4$ *is a perfect square.*

Theorem 3 (Tichý, Ziegler)

Let $K = \mathbb{Q}(\sqrt[3]{d})$, $d \in \mathbb{Z}$ cubefree. Then $u(O_K) = \omega$ iff

1. $d = 28$, *or*
2. d *squarefree,* $d \not\equiv \pm 1$ *mod* $9, d + 1$ *or* $d - 1$ *is a perfect cube.*

Note:

$\Rightarrow u(O_K) = \infty$ for infinitely many algebraic number fields K

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Arithmetic progressions I

Useful step to show $u(O_K) \geq \omega$ is to bound the length of APs in U_K:

Theorem 4 (Newman)

Let $K = \mathbb{Q}(\alpha)$. Any nontrivial AP in U_K is of length at most $n = \deg(K)$.

That can be generalized, let:

$$\mathcal{N}_m = \{\beta \in O_K : N(\beta) = m,\ m > 0\},$$

$$\mathcal{N}_m^* = \{\beta \in O_K : |N(\beta)| \leq m,\ m > 0\}$$

and

$$t \times \mathcal{N}_m^* = \{\beta_1 + \ldots + \beta_t : \beta_i \in \mathcal{N}_m^*\}.$$

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Let $K = \mathbb{Q}(\alpha)$. Any nontrivial AP in $\mathcal{N}_m$ is of length at most $n = \deg(K)$.

Theorem 6 (D., Hajdu, Pethő)

Let $K = \mathbb{Q}(\alpha)$. Any nontrivial AP in $t \times \mathcal{N}_m^*$ is of length at most $c_1 = c_1(m, n, t, D(K))$, where c_1 is an explicitly computable constant.

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Sums of small-norm elements I

Possible generalization of the unit sum number problem:

- define the m-sum number $u_m(O_K)$ as $u(O_K)$, but with elements of $\mathcal{N}_m^*$ instead of U_K
- $u_1(O_K) = u(O_K)$

Theorem 7 (D., Hajdu, Pethő)

Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field. For any $m, t \in \mathbb{N}$ there exists $\beta \in O_K$ which cannot be obtained as a sum of at most t terms from $\mathcal{N}_m^*$, i.e. $u_m(O_K) \geq \omega$ for any $m > 0$.

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Sums of small-norm elements II

- $u(O_K) = \infty$ for infinitely many $K = \mathbb{Q}(\alpha)$
- for $u_m(O_K)$, the situation changes:

Theorem 8 (D., Hajdu, Pethő)

For every number field $K = \mathbb{Q}(\alpha)$ there exists a positive integer $m_0 = m_0(n, D(K))$, such that for any $m \geq m_0$ we have $u_m(O_K) = \omega$, i.e. any $\beta \in O_K$ can be obtained as the sum of elements from $\mathcal{N}_m^*$.

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Linear combinations of units

- sum of elements from $\mathcal{N}_m^* \sim$ LC of units with bounded set of coefficients $\in O_K$
- we want to express any $\beta \in O_K$ as LC of units with coefficients $\in \mathbb{Q}$
- necessary to exclude the cases where U_K contained in some proper subfield of K

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Linear combinations of units $\cdots$ main result

- $K = \mathbb{Q}(\alpha)$ is called a CM-field, if it is a totally imaginary quadratic extension of a totally real number field

Theorem 9 (D., Hajdu, Pethő)

Suppose that $K = \mathbb{Q}(\alpha)$ is not a CM-field or it is a CM-field containing a root of unity different from ± 1. Then there exists a positive integer $k = e^{c_2(n)R(K)}$, such that any $\beta \in O_K$ can be obtained as an LC of (not necessarily distinct) units of K with coefficients $\{1, 1/2, 1/3, \ldots, 1/k\}$.

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Example 10

Let $K = \mathbb{Q}(\sqrt{d})$.

- $d = -3$, *K is a CM-field with non-real roots of unity:*

$$O_K = \left\{a+b\left(\frac{1+i\sqrt{3}}{2}\right) : a, b \in \mathbb{Z}\right\},\ U_K = \left\{\pm\left(\frac{1+i\sqrt{3}}{2}\right)^k : k \in \mathbb{N}\right\}$$

$\Rightarrow$ *any* $\beta \in O_K$ *is a sum of units*

- $d = 6$, *K is not a CM-field:*

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- $d < 0, d \notin \{-1, -3\}$, *$K$ is a CM-field, without non-real roots of unity:*
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References

- F. Barroero, C. Frei, R. F. Tichy, *Additive unit representations in rings over global fields - a survey*, Publ. Math. Debrecen **79** (2011), 291–307.
- A. Bérczes, L. Hajdu, A. Pethő, *Arithmetic progressions in the solution sets of norm form equations*, Rocky Mountain Math. J. **40** (2010), 383–396.
- D. Dombek, L. Hajdu, A. Pethő, *Representing algebraic integers as linear combinations of units*, preprint (2012), 10pp.
- L. Hajdu, *Arithmetic progressions in linear combinations of S-units*, Period. Math. Hungar. **54** (2007), 175–181.
- M. Jarden, W. Narkiewicz, *On Sums of Units*, Monatsh. Math. **150** (2007), 327–332.