On the predicate EQ-logic

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Outline

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How did EQ-logic arise?

- Motivation comes from G. W. Leibniz, L. Wittgenstein and F. P. Ramsey. To develop logic on the basis of identity (equality) as the principle connective.

  [H. Castañeda, Leibniz's syllogistico-propositional calculus, *Notre Dame Journal of Formal Logic XVII* (4) (1976) 338–384.]

  [F. Ramsey, The foundations of mathematics, *Proceedings of the London Mathematical Society* 25 (1926) 338–384.]

- Implication-based logic X Equality-based logic
  [D. Gries, F. B. Schneider. Equational propositional logic. *Information Processing Letters*, 53:145-152, 1995.]
  [G. Tourlakis. Mathematical Logic. New York, J.Wiley & Sons, 2008.]

How did EQ-logic arise?

Implication-based fuzzy logic:

Structure of their truth values is extension of the MTL-algebra, implication is interpreted by residuation

X

Equality-based fuzzy logic:

Structure of their values is EQ-algebra, equivalence is interpreted by fuzzy equality not biresiduation!

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How did EQ-logic arise?

[M. Dyba and V. Novák. Non-commutative EQ-logics and their extensions. *Proc. of the Joint 2009 IFSA World Congress and 2009 EUSFLAT Conf., July 20-24, 2009*, 1422–1427, Lisbon, Portugal, 2009. University of Málaga.]

[M. Dyba and V. Novák. EQ-logics: Non-commutative fuzzy logics based on fuzzy equality. *Fuzzy Sets and Systems*, 2011, sv. 172, 13–32.]

[Dyba, M., Novák, V., EQ-logics with delta connective. *Fuzzy Sets and Systems*, submitted.]

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Definition

Non-commutative EQ-algebra is the algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

of type (2, 2, 2, 0)

(E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$) with the ordering: $a \leq b$ iff $a \wedge b = a$

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and $\otimes$ is isotone w.r.t. $\leq$

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Definition (continued)

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity)

(E8) $a \otimes b \leq a \sim b$ (boundedness)

Implication: $a \to b = (a \wedge b) \sim a$

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Implication: $a \rightarrow b = (a \wedge b) \sim a$

Special EQ-algebras

EQ-algebra is

(a) good if $a \sim \mathbf{1} = a$

(b) residuated if
$(a \otimes b) \wedge c = a \otimes b \quad \text{iff} \quad a \wedge ((b \wedge c) \sim b) = a$

(c) involutive if $\neg\neg a = a$ (**IEQ-algebra**)

(d) prelinear if for all $a, b \in E$ $\sup\{a \to b, b \to a\} = \mathbf{1}$.

(e) lattice EQ-algebra if it is a lattice-ordered and for all $a, b, c, d \in E$ $((a \vee b) \sim c) \otimes (d \sim a) \leq (d \vee b) \sim c$ (ℓ**EQ-algebra**)

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Special EQ-algebras

A lattice EQ$_\Delta$-algebra (ℓEQ$_\Delta$-algebra)

$\mathcal{E}_\Delta = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$

- $\langle E, \wedge, \vee, \otimes, \sim, \mathbf{0}, \mathbf{1} \rangle$ is a good non-commutative and bounded ℓEQ-algebra.
- $\Delta \mathbf{1} = \mathbf{1}$
- $\Delta a \leq \Delta\Delta a$
- $\Delta(a \sim b) \leq \Delta a \sim \Delta b$
- $\Delta(a \wedge b) = \Delta a \wedge \Delta b$
- $\Delta a = \Delta a \otimes \Delta a$
- $\Delta(a \vee b) \leq \Delta a \vee \Delta b$
- $\Delta a \vee \neg\Delta a = 1$
- $\Delta(a \sim b) \leq (a \otimes c) \sim (b \otimes c)$
- $\Delta(a \sim b) \leq (c \otimes a) \sim (c \otimes b)$

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Representation of ℓEQ_Δ-algebras

Lemma

If a good EQ-algebra $\mathcal{E}$ satisfies

$$(a \to b) \vee (d \to (d \otimes (c \to ((b \to a) \otimes c)))) = \mathbf{1} \qquad (1)$$

for all $a, b, c, d \in \mathcal{E}(\mathcal{E}_\Delta)$ then it is prelinear.

Theorem (M. El Zekey)

Let $\mathcal{E}_\Delta$ be ℓEQ_Δ-algebra. The following are equivalent:

(a) *$\mathcal{E}_\Delta$ is subdirectly embeddable into a product of linearly ordered good ℓEQ_Δ-algebras.*

(b) *$\mathcal{E}_\Delta$ satisfies (1).*

Representation of ℓEQ_{Δ}-algebras

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Why EQ-logics

EQ-logics — special class of many-valued logics
truth values form an EQ-algebra

- Equivalence as the basic connective instead of implication
- Proofs in equational style
- Even more general than MTL-logics

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Language

- Propositional variables $p_1, p_2, \ldots$
- Connectives: $\wedge$ (conjunction), $\&$ (fusion), $\equiv$ (equivalence),
- Logical constant $\top$ (true)

Implication:

$$A \Rightarrow B := (A \wedge B) \equiv A$$

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Logical axioms

(EQ1) $(A \equiv \top) \equiv A$

(EQ2) $A \wedge B \equiv B \wedge A$

(EQ3) $(A \bigcirc B) \bigcirc C \equiv A \bigcirc (B \bigcirc C), \quad \bigcirc \in \{\wedge, \&\}$

(EQ4) $A \wedge A \equiv A$

(EQ5) $A \wedge \top \equiv A$

(EQ6) $A \& \top \equiv A$

(EQ7) $\top \& A \equiv A$

(EQ8a) $((A \wedge B) \& C) \Rightarrow (B \& C)$

(EQ8b) $(C \& (A \wedge B)) \Rightarrow (C \& B)$

(EQ9) $((A \wedge B) \equiv C) \& (D \equiv A) \Rightarrow (C \equiv (D \wedge B))$ (substitution)

(EQ10) $(A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D)$ (congruence)

(EQ11) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$ (monotonicity)

Inference rules

Equanimity rule

From A *and* $A \equiv B$ **infer** B

Leibniz rule

From $A \equiv B$ **infer** $C[p := A] \equiv C[p := B]$

$C[p := A]$ denotes a formula resulting from C by replacing all occurrences of a variable p in C by the formula A.

Semantics

Truth values

The set of truth values is a good non-commutative EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$

Truth evaluation $e : F_J \longrightarrow E$

$e(p) \in E$
$e(A \equiv B) = e(A) \sim e(B)$
$e(A \wedge B) = e(A) \wedge e(B)$
$e(A \,\&\, B) = e(A) \otimes e(B)$ (non-commutative)

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Completeness

Theorem (Completeness)

For every formula $A \in F_J$ the following is equivalent:

(a) $\vdash A$

(b) $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every good non-commutative EQ-algebra* $\mathcal{E}$.

Other EQ-logics

- Involutive EQ-logic (with double negation)
- Prelinear EQ-logic (stronger variant of the completeness theorem)
- EQ(MTL)-logic (equivalent with MTL-logic)

Not strong enough for development of the predicate EQ-logic!

- Basic EQ_Δ-logic (weaker variant of the completeness theorem)
- Prelinear EQ_Δ-logic

Theorem (Deduction)

For each theory T and formulas $A, B, C \in F_J$:
$T \cup \{A \equiv B\} \vdash C$ *iff* $T \vdash \Delta(A \equiv B) \Rightarrow C$

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Prelinear EQ$_\Delta$-logic

Language

The language of basic EQ-logic extended by unary connective Δ, binary connective $\vee$ and logical constant $\bot$.

Negation $\neg A := A \equiv \bot$

Axioms (EQ1)–(EQ11) and

$((((A \wedge B) \vee C) \equiv D) \& (F \equiv C)) \& (E \equiv A) \Rightarrow (D \equiv (F \vee (B \wedge E)))$

(EQ12) $(A \vee B) \vee C \equiv A \vee (B \vee C)$

(EQ13) $A \vee (A \wedge B) \equiv A$

(EQ14) $(A \wedge \bot) \equiv \bot$

(EQ15) $(A \Rightarrow B) \vee (D \Rightarrow (D \& (C \Rightarrow ((B \Rightarrow A) \& C))))$

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(EQ12) $(A \vee B) \vee C \equiv A \vee (B \vee C)$

(EQ13) $A \vee (A \wedge B) \equiv A$

(EQ14) $(A \wedge \bot) \equiv \bot$

(EQ15) $(A \Rightarrow B) \vee (D \Rightarrow (D \,\&\, (C \Rightarrow ((B \Rightarrow A) \,\&\, C))))$

Prelinear EQ$_\Delta$-logic

Axioms (continued)

(EQΔ1) $\Delta A \Rightarrow \Delta\Delta A$

(EQΔ2) $\Delta(A \equiv B) \Rightarrow (\Delta A \equiv \Delta B)$

(EQΔ3) $\Delta(A \wedge B) \equiv (\Delta A \wedge \Delta B)$

(EQΔ4) $\Delta A \equiv (\Delta A \,\&\, \Delta A)$

(EQΔ5) $\Delta(A \vee B) \Rightarrow (\Delta A \vee \Delta B)$

(EQΔ6) $\Delta A \vee \neg\Delta A$

(EQΔ7) $\Delta(A \equiv B) \Rightarrow ((A \,\&\, C) \equiv (B \,\&\, C))$

(EQΔ8) $\Delta(A \equiv B) \Rightarrow ((C \,\&\, A) \equiv (C \,\&\, B))$

Prelinear EQ_Δ-logic

Inference rules

- Equanimity rule
- Leibniz rule
- Necessitation rule

$$\textit{From } A \ \textbf{infer} \ \Delta A$$

Semantics

An ℓEQ_Δ-algebras in which (1) is satisfied.

Prelinear EQ$_\Delta$-logic

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$$\textit{From } A \ \textbf{infer} \ \Delta A$$

Semantics

An ℓEQ$_\Delta$-algebras in which (1) is satisfied.

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Prelinear EQ$_\Delta$-logic

Theorem (Completeness)

For every formula $A \in F_J$ and every theory T the following is equivalent:

(a) $T \vdash A$

(b) $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every linearly ordered,* ℓEQ_Δ*-algebra* $\mathcal{E}$.

(c) $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every* ℓEQ_Δ*-algebra* $\mathcal{E}$ *satisfying (1).*

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Syntax

Language

- Object variables $x, y, \ldots$.
- Set of object constants $\mathrm{Const} = \{\mathbf{u}, \mathbf{v}, \ldots\}$.
- Non-empty set of n-ary predicate symbols $\mathrm{Pred} = \{P, Q, \ldots\}$.
- Binary connectives $\wedge, \vee, \boldsymbol{\&}, \equiv$ and unary connective $\boldsymbol{\Delta}$.
- Logical (truth) constants $\top$ (true) and $\bot$ (false).
- Quantifiers $\forall, \exists$.
- Auxiliary symbols: brackets.

Syntax

Terms

Object variables and object constants are terms.

Formulas

- If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms then $P(t_1, \ldots, t_n)$ is atomic formula.
- Logical constants $\top$ and $\perp$ are formulas.
- If A, B are formulas then $A \wedge B$, $A \vee B$, $A \,\&\, B$, $A \equiv B$, ΔA are formulas.
- If A is formula and x is an object variable then $(\forall x)A$, $(\exists x)A$ are formulas.

Semantics

Structure for language J

$$\mathcal{M}^{\mathcal{E}} = \langle M, \mathcal{E}, \{r_P\}_{P \in Pred}, \{m_u\}_{u \in Const} \rangle$$

$\mathcal{E} = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$ is a non-commutative linearly ordered ℓEQ$_\Delta$-algebra,
$r_P : M^n \longrightarrow E$ is n-ary relation, $m_u \in M$.

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Interpretation of terms and formulas

v — assignment of elements from M to variables
$\mathcal{M}^{\mathcal{E}}(x) = v(x), \mathcal{M}^{\mathcal{E}}(\mathbf{u}) = m_u,$

$$
\begin{gathered}
\mathcal{M}_v^{\mathcal{E}}(P(t_1,\dots,t_n)) = r_P(\mathcal{M}^{\mathcal{E}}(t_1),\dots,\mathcal{M}^{\mathcal{E}}(t_n)),\\
\mathcal{M}_v^{\mathcal{E}}(A \wedge B) = \mathcal{M}_v^{\mathcal{E}}(A) \wedge \mathcal{M}_v^{\mathcal{E}}(B),\\
\mathcal{M}_v^{\mathcal{E}}(A \,\&\, B) = \mathcal{M}_v^{\mathcal{E}}(A) \otimes \mathcal{M}_v^{\mathcal{E}}(B),\\
\mathcal{M}_v^{\mathcal{E}}(A \equiv B) = \mathcal{M}_v^{\mathcal{E}}(A) \sim \mathcal{M}_v^{\mathcal{E}}(B),\\
\mathcal{M}_v^{\mathcal{E}}(\top) = \mathbf{1}, \mathcal{M}_v^{\mathcal{E}}(\bot) = \mathbf{0},\\
\mathcal{M}_v^{\mathcal{E}}((\forall x)A) = \inf\{\mathcal{M}_v^{\mathcal{E}}(A_x[\mathbf{m}]) \mid m \in M\},\\
\mathcal{M}_v^{\mathcal{E}}((\exists x)A) = \sup\{\mathcal{M}_v^{\mathcal{E}}(A_x[\mathbf{m}]) \mid m \in M\}
\end{gathered}
$$

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Logical Axioms

(EQ1)–(EQ15), (EQΔ1-EQΔ8) plus

(EQ$\forall$1) $(\forall x)A(x) \Rightarrow A(t)$ (t substituable for x in $A(x)$),
(EQ$\exists$1) $A(t) \Rightarrow (\exists x)A(x)$ (t substituable for x in $A(x)$),
(EQ$\forall$2) $(\forall x)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x)B)$ (x not free in A),
(EQ$\exists$2) $(\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow B)$ (x not free in B),
(EQ$\forall$3) $(\forall x)(A \vee B) \Rightarrow ((\forall x)A \vee B)$ (x not free in B),

Inference Rules

- Equanimity rule
- Leibniz rule
- Necessitation rule
- Rule of Generalization

$$\textit{From } A \;\; \textbf{infer} \;\; (\forall x)A$$

Model

Definition

Structure $\mathcal{M}^{\mathcal{E}}$ is a model of a theory T if $\mathcal{M}^{\mathcal{E}}_{v}(A) = \mathbf{1}$ holds for all axioms of T.

Theorem (Soundness)

If $T \vdash A$ then $\mathcal{M}^{\mathcal{E}}_{v}(A) = \mathbf{1}$ holds for every assignment v and every model $\mathcal{M}^{\mathcal{E}}$ of T.

Main properties

Lemma

(i) $\vdash (\forall x)(A \Rightarrow B) \equiv (A \Rightarrow (\forall x)B)$, *x not free in A*

(ii) $\vdash (\forall x)(B \Rightarrow A) \equiv ((\exists x)B \Rightarrow A)$, *x not free in A*

(iii) $\vdash (\forall x)(A \Rightarrow B) \Rightarrow ((\forall x)A \Rightarrow (\forall x)B)$

(iv) $\vdash (\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow (\exists x)B)$

(v) $\vdash (\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow (\exists x)B)$

Theorem (Deduction theorem)

$T \cup \{A \equiv B\} \vdash C \quad \textit{iff} \quad T \vdash \Delta(A \equiv B) \Rightarrow C.$

Completeness

Definition

(i) Theory T is consistent if there is a formula A unprovable in T.

(ii) T is linear (complete) if for every two formulas A, B, $T \vdash A \Rightarrow B$ or $T \vdash B \Rightarrow A$.

(iii) T is extensionally complete if for every closed formula $(\forall x)(A(x) \equiv B(x))$, $T \nvdash (\forall x)(A(x) \equiv B(x))$ there is a constant $\mathbf{u}$ such that $T \nvdash (A_x[\mathbf{u}] \equiv B_x[\mathbf{u}])$

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Completeness

Theorem

Every consistent theory T can be extended to a maximally consistent linear theory.

Theorem

Every consistent theory T can be extended to an extensionally complete consistent theory T.

Completeness

Theorem (Completeness)

(a) *A theory T is consistent iff it has a safe model $\mathcal{M}$.*

(b) $T \vdash A \quad \textit{iff} \quad T \models A$

Outline

Conclusion

- Δ-connective (deduction theorem) is indispensable for development of the first-order EQ-logic.
- Formal system of predicate EQ-logic with equivalence as main connective.
- Possible direction in the development of mathematical fuzzy logics in which axioms are formed as identities and proofs naturally have equational form.

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Thank you for your attention.