T-Norms and Fuzzy Filters

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Motivation

- A great amount of papers about (special types of) fuzzy filters
- Idea to relax the 'min-condition' in a definition of fuzzy filter

Basic definition

Definition

Let **L** is an MTL-algebra. A fuzzy set μ of L is called a *fuzzy filter* if it satisfies the following conditions:

1. $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$,
2. $x \leq y \Rightarrow \mu(x) \leq \mu(y)$

for all $x, y \in L$.

Alternative Definition

Proposition

A fuzzy set μ of L is a fuzzy filter if and only if satisfies the following conditions:

1. $\mu(1) \geq \mu(x)$ *for all* $x \in L$,
2. $\mu(y) \geq min\{\mu(x \to y), \mu(x)\}$,

for all $x, y \in L$.

T-norms

Definition

A binary function $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if for all $a, b, c \in [0,1]$:

1. $T(a, T(b, c)) = T(T(a, b), c)$,
2. $T(a, b) = T(b, a)$,
3. $T(a, 1) = a$,
4. if $b \leq c$, then $T(a, b) \leq T(a, c)$.

Fuzzy Filters Revised

Definition

Let **L** is a MTL-algebra, T an arbitrary t-norm. A fuzzy set μ of L is called a *T-fuzzy filter* if it satisfies the following conditions:

1. $\mu(x * y) \geq T(\mu(x), \mu(y))$,
2. $x \leq y \Rightarrow \mu(x) \leq \mu(y)$

for all $x, y \in L$.

Every fuzzy filter on L is clearly a T-fuzzy filter, converse direction doesn't hold.

Alternative Definitions I.

Proposition

Let **L** *is a* MTL-*algebra,* T *an arbitrary t-norm. If* μ *is a* T*-fuzzy filter, then it satisfies the following conditions:*

1. $\mu(y) \geq T(\mu(x \to y), \mu(x))$,
2. $\mu(1) \geq \mu(x)$.

for all $x, y \in L$.

Proof.

$y \geq x * (x \to y)$, so from monotonicity of μ we get $\mu(y) \geq \mu(x * (x \to y))$. From the first condition in definition of T-fuzzy filter we obtain: $\mu(x * (x \to y)) \geq T(\mu(x), \mu(x \to y))$, thus $\mu(y) \geq T(\mu(x \to y), \mu(x))$. Second condition is obvious. □

Alternative Definitions II.

Proposition

Let **L** *is a* MTL*-algebra,* T *an arbitrary t-norm. If* μ *satisfies following:*

1. $\mu(y) \geq T(\mu(x \to y), \mu(x))$,
2. $\mu(1) = 1$,

for all $x, y \in L$*, then* μ *is a* T*-fuzzy filter on* L*.*

Alternative Definitions II. - cont.

Proof.

Let us assume $x \leq y$. Then $x \to y = 1$, so $\mu(x \to y) = \mu(1) = 1$. $\mu(y) \geq T(\mu(x \to y), \mu(x)) = T(1, \mu(x)) = \mu(x)$. So, μ is order preserving.
Next step: $x \leq y \to (x * y)$, so $\mu(x) \leq \mu(y \to (x * y))$, thus from monotonicity of t-norm we obtain
$T(\mu(x), \mu(y)) \leq T(y \to (x * y), \mu(y)) \leq \mu(x * y)$. □

Alternative Definitions III.

Proposition

Let **L** *is a* MTL*-algebra,* T *an arbitrary t-norm,* μ *a fuzzy set on* L *satisfying* $\mu(1) = 1$*. Then* μ *is a* T*-fuzzy filter if and only if implication* $a \leq b \to c \Rightarrow \mu(c) \geq T(\mu(a), \mu(b))$ *holds for all* $a, b, c \in L$*.*

Alternative Definitions III.

Proof.

Let us assume that the implication holds for all $a, b, c \in L$.
Let $a := x$, $b := y$ and $z := x * y$. We get
$x \leq y \rightarrow (x * y) \Rightarrow \mu(x * y) \geq T(\mu(x), \mu(y))$. In MTL-algebras $x \leq y \rightarrow (x * y)$ holds allways, so $\mu(x * y) \geq T(\mu(x), \mu(y))$.
Second condition: let $a := x$, $b := 1$, $c := y$. Thus
$x \leq 1 \rightarrow y = y \Rightarrow \mu(y) \geq T(\mu(x), \mu(1)) = \mu(x)$.
Conversely, let μ be a T-fuzzy filter and let $x \leq y \rightarrow z$, so (in MTL-algebra) $x * y \leq z$. Thus $\mu(x * y) \leq \mu(z)$. We know that $\mu(x * y) \geq T(\mu(x), \mu(y))$, hence $\mu(z) \geq T(\mu(x), \mu(y))$. □

Some Properties of T-fuzzy Filters

In 'classical' fuzzy filters we know that μ_t on L is an emptyset or a filter on L. This doesn't hold in T-fuzzy filters. But some properties of classical fuzzy filters hold also in this more general case, for example:

Proposition

F is a filter on L if and only if χ_F is a T-fuzzy filter.

Proof.

Assume F is a filter on L, let $x \leq y$. So obviously $\chi_F(x) \leq \chi_F(y)$. Also $\chi_F(x * y) \geq T(\chi_F(x), \chi_F(y))$ for all $x, y \in L$. Conversely, let χ_F be a T-fuzzy filter, $x \in F$ and $x \leq y$. Then $\chi_F(x) = 1$, $\chi_F(x) \leq \chi_F(y)$, so $y \in F$. If $x \in F$ and $y \in F$, then $T(\chi_F(x)) = 1$ and $\chi_F(y) = 1$, and, by T-fuzzy filter property $\chi_F(x * y) = 1$, thus $x * y \in F$.

Special Types of Fuzzy Filters and T-fuzzy Filters

Recall the definition of fuzzy boolean filter.

Definition

A fuzzy filter μ on L is a *fuzzy boolean filter* if $\mu(x \vee \neg x) = \mu(1)$.

Proposition

In any residuated lattice L/MTL-algebra the following assertions are equivalent:

1. *L is a boolean algebra.*
2. *Every fuzzy filter on L is a fuzzy boolean filter.*
3. *$\chi_{\{1\}}$ is a fuzzy boolean filter.*

We can rephrase this result in the environment of generalized fuzzy filters.

Rephrase of the Previous Result

Proposition

In any residuated lattice/MTL*-algebra* **L** *the following assertions are equivalent:*

1. **L** *is a boolean algebra,*
2. *Every T-fuzzy filter on* L *is a T-fuzzy boolean filter.*
3. $\chi_{\{1\}}$ *is a T-fuzzy boolean filter.*

Proof.

From 1 to 2: if **L** is a boolean algebra, then $x \vee \neg x = 1$ for all $x \in L$, thus $\mu(x \vee \neg x) = \mu(1)$. From 2 to 3: Obvious. From 3 to 1: Suppose $\chi_{\{1\}}$ is a T-fuzzy boolean filter on **L**. Then $\chi_{\{1\}}(x \vee \neg x) = \chi_{\{1\}}(1) = 1$, so $x \vee \neg x = 1$. □

Special Types of Generalized Fuzzy Filters

By the revision of the previous proof we see that:

1. the t-norm doesn't appear in the proof
2. the term $x \vee \neg x$ can be replaced by any term and we can generate dozens of similar results, for example: if we use term $\neg\neg x \rightarrow x$, we can get analogous result about IMTL-algebras etc.

Acknowledgement

Thank you for your attention!