Dynamic Stochastic Accumulation Model

with Application to Risk Management in Slovak Pension System

Darina Graczová, Daniel Ševčovič

Comenius University
Faculty of Mathematics, Physics and Informatics

13th of May 2012

Outline

- Saving problem
- Modeling of returns with fat-tailed distributions
- Accumulation model for II. pillar of Slovak pension system
- Numerical computation
- Results with the NIG distributed returns
- Conclusions

Pension Problem - II. Pillar

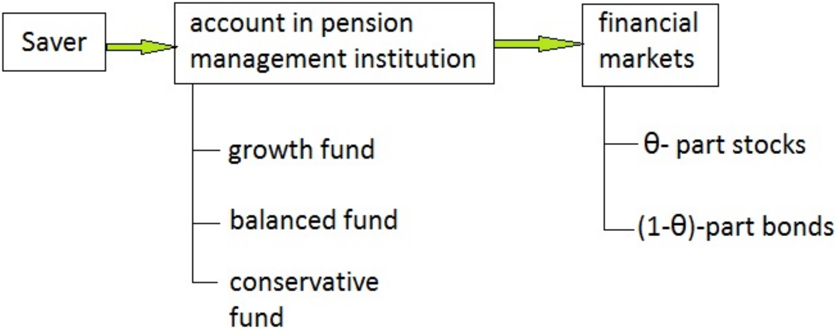
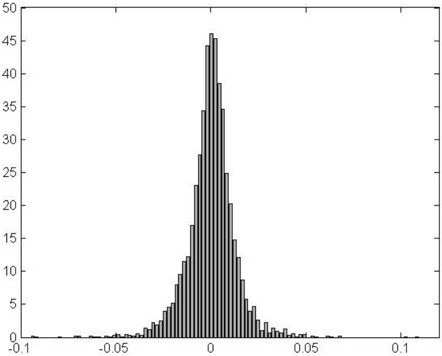
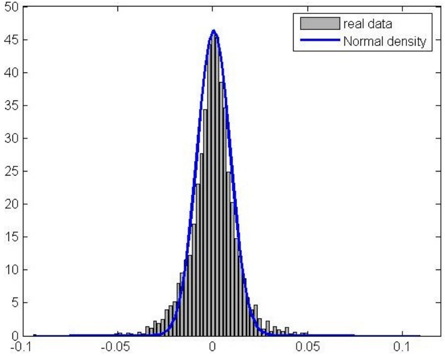


Figure: Histogram of log returns for daily data for S&P500 Index (Jan-1996 - Feb-2012).



Modeling Returns - Normal distribution

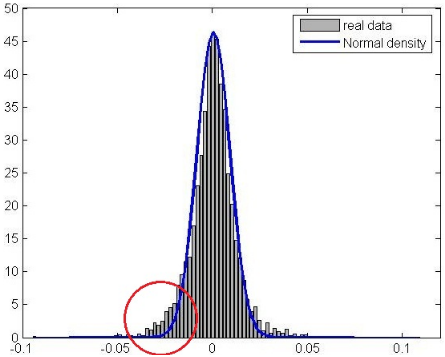
Figure: Histogram of log returns for daily data for S&P500 Index (Jan-1996 - Feb-2012).



Parameters:

mean value: 0.02%; standard deviation: 1.31%;

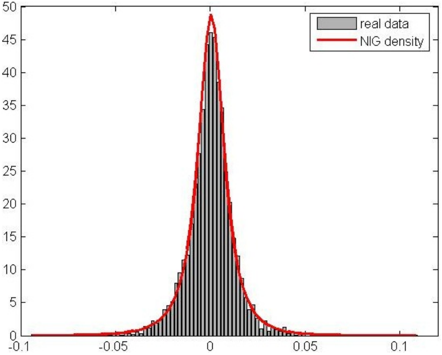
Figure: Histogram of log returns for daily data for S&P500 Index (Jan-1996 - Feb-2012).



Parameters:

mean value: 0.02%; standard deviation: 1.31%;

Figure: Histogram of log returns for daily data for S&P500 Index (Jan-1996 - Feb-2012).



Parameters:

mean value: 0.02%; standard deviation: 1.31%;
skewness: -0.22; kurtosis: 10.08;

Stochastic Dynamic Problem

$$\max_{\theta} \mathbb{E}(U(d_T)),$$

subject to

$$\begin{aligned} d_{t+1} &= F_t(d, \theta(d,t), r_t^{\theta}) \\ d_1 &= \tau \end{aligned}$$

where T is the retirement time, d_t is a measure of saved amount of money and θ is the selected stock-to-bond proportion with saver's utility function U.

Bellman Equation

Using the tower law for conditional expectation, we obtain the Bellman equation

$$\begin{aligned} V_T(d) &= U(d), \\ V_t(d) &= \max_{\theta} \mathbb{E}(V_{t+1}(d_{t+1})) = \max_{\theta} \mathbb{E}(V_{t+1}(F_t(d, \theta(d,t), r_t^{\theta}))) \\ &= \max_{\theta} \int_{\mathbb{R}} V_{t+1}(d_{t+1}) f^{\theta}(r) \mathrm{d}r. \end{aligned}$$

The problem we solve in each time step t is to find the maximum over θ of the integral

$$\int_{\mathbb{R}} V_{t+1}(d_{t+1}) f^{\theta}(r) \mathrm{d}r.$$

Definition

The random variable X is Normal Inverse Gaussian distributed $NIG(\alpha, \beta, \mu, \delta)$ if its probability density function is given by

$$f(x) = \frac{\alpha}{\pi} \exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\} \frac{K_1(\alpha\delta\sqrt{1 + (\frac{x-\mu}{\delta})^2})}{\sqrt{1 + (\frac{x-\mu}{\delta})^2}}$$

where K_1 denotes the modified Bessel function of the third kind, $K_1(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(y+y^{-1})}\mathrm{d}y$, and the conditions for the parameters are $\alpha > 0, \delta > 0, \mu \in \mathbb{R}, 0 \leq |\beta| \leq \alpha$.

The problem: What will be the distribution of the convolution of two NIG distributed random variables if their parameters will be different?

Properties of NIG Distribution

The problem: What will be the distribution of the convolution of two NIG distributed random variables if their parameters will be different?

The NIG class of densities has the following properties:

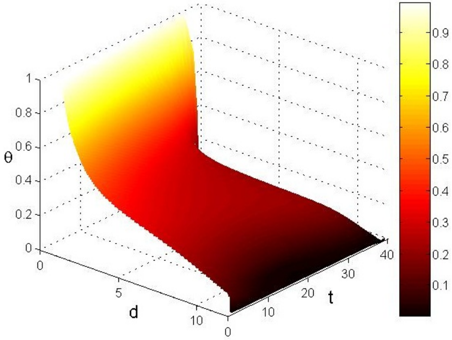
1. Scaling property: If $X \sim NIG(\alpha, \beta, \mu, \delta)$, then $Y = cX \sim NIG(\alpha/c, \beta/c, c\mu, c\delta)$.
2. Convolution property: If $X_1 \sim NIG(\alpha, \beta, \mu_1, \delta_1)$ and $X_2 \sim NIG(\alpha, \beta, \mu_2, \delta_2)$ are independent, then the sum $Y = X_1 + X_2 \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$.
3. Standardization: If $X \sim NIG(\alpha, \beta, \mu, \delta)$, then variable $Y = (X - \mu)/\delta$ has the *Standard Normal Inverse Gaussian Distribution* $NIG(\alpha\delta, \beta\delta, 0, 1)$.

The procedure:

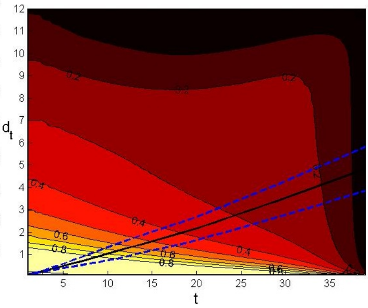
1. generation of the random vectors:
$$Z_B \sim NIG(\alpha_B\delta_B, \beta_B\delta_B, 0, 1)$$
$$Z_S \sim NIG(\alpha_S\delta_S, \beta_S\delta_S, 0, 1)$$
2. construction of the correlated random vectors so that
$$X_B = \delta_B Z_B + \mu_B$$
$$X_S = \delta_S(\rho_{B,S} Z_B + \sqrt{1-\rho_{B,S}^2} Z_S) + \mu_S$$
where the vectors X_B and X_S represent the bond and stock log-returns, respectively.
3. computing the log-return of the portfolio with stock to bond proportion θ in the way: $X^\theta = \theta X_S + (1-\theta) X_B$
4. computing the values of the first four moments

$\rightarrow$ NIG parameters

Optimal Proportion in Portfolio



(a) 3D plot of optimal choice of θ



(b) contour

Figure: 3D plot of the of the optimal choice of θ as a function of time t and current accumulated sum d_t for $\mathcal{S} = -0.2, \mathcal{K} = 10$.

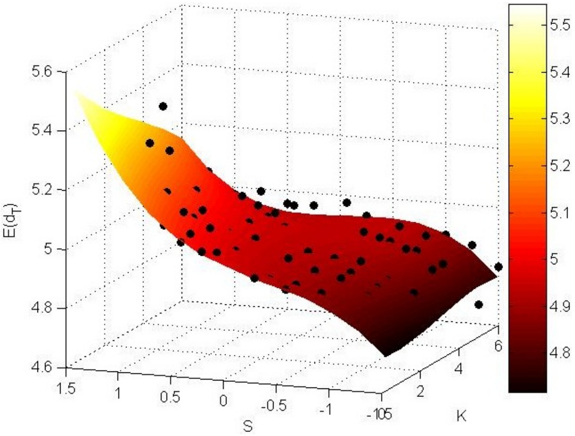


Figure: 3D plot of the final mean value of the accumulated sum as a function of skewness $\mathcal{S}$ and excess kurtosis $\mathcal{K}$. Figures shows different views.

Conclusion

- We have introduced a saving problem.
- We have analyzed S&P500 Index in more detail and showed that the Index is characterized by higher kurtosis and is skewed to the left.
- We have presented the dynamic stochastic model for determining the optimal value of the stock to bond proportion in pension funds.
- We have formulated the problem in term of Bellman equation.
- We have presented the numerical procedure and results.
- We have shown different levels of skewness and kurtosis of the underlying assets influence the optimal choice and the level of accumulated sum.

Thank you for your attention.