Department of Mathematics, University of Latvia

# On F-transforms with spline based generalized fuzzy partitions

Irina Vaviļčenkova

13TH INTERNATIONAL STUDENT CONFERENCE ON APPLIED MATHEMATICS AND INFORMATICS

Malenovice, Czech Republic, May 10-13, 2012

This talk is devoted to F-transform (fuzzy transform). The core idea of fuzzy transform is inwrought with an interval fuzzy partitioning into fuzzy subsets, determined by their membership functions. In this work we consider polynomial splines of degree $m$ and defect 1 with respect to the mesh of an interval with some additional nodes. The idea of the direct F-transform is transformation from a function space to a finite dimensional vector space. The inverse F-transform is transformation back to the function space.

# Fuzzy partitions

Let $\Delta_n = \{x_1, x_2, \dots, x_n | x_1 < x_2 < \cdots < x_n\}$ be a fixed mesh of $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$.

We consider two types of fuzzy partitions:

- usual (fuzzy partition)
  which consists of membership functions $A_{k,}$ with support $[x_{k-1}, x_{k+1}] \cap [a, b]$
  for $k = 1, \dots, n$;

- generalized (fuzzy m-partition)
  which consists of membership functions $A_{k,}^{(m)}$ with support
  $[x_{k-m}, x_{k+m}] \cap [a, b]$ for $k = -m + 2, \dots, n + m - 1$;

**The fuzzy transform was proposed by I. Perfilieva in 2003 and studied in several papers**

[1] **I. Perfilieva,** Fuzzy transforms: Theory and applications, *Fuzzy Sets and Systems,* 157 (2006) 993–1023.

[2] **I. Perfilieva**, Fuzzy transforms, in: J.F. Peters, et al. (Eds.), Transactions on Rough Sets II, Lecture Notes in Computer Science, vol. 3135, 2004, pp. 63–81.

[3] **L. Stefanini,** F- transform with parametric generalized fuzzy partitions, Fuzzy Sets and Systems 180 (2011) 98–120.

[4] **I. Perfilieva, V. Kreinovich,** Fuzzy transforms of higher order approximate derivatives: A theorem , Fuzzy Sets and Systems 180 (2011) 55–68.

# Classical fuzzy partition

We say that fuzzy sets $A_1, A_2, \dots, A_n$, identified with their membership functions $A_1(x), A_2(x), \dots, A_n(x)$ defined on $[a, b]$, form fuzzy partition of $[a, b]$ if they fulfill the following conditions for $k = 1, \; 2, \dots, n$ :

**1)** $A_k$: $[a, b] \to [0, 1], \; A_k(x_k) = 1$;

**2)** $A_k(x) = 0, \;$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotion, we put $x_0 = a$ un $x_{n+1} = b$;

**3)** $A_k$ is continuous;

**4)** $A_k$, if $k = 2, \dots, n$, strictly increases on $[x_{k-1}, x_k]$ and $A_k$, $k = 1, \dots, n-1$, strictly decreases on $[x_k, x_{k+1}]$;

**5)** for all $x \in [a, b]$ $\sum_{k=1}^{n} A_k(x) = 1$;

**6)** $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0, h]$, $k = 2, \dots, n$;

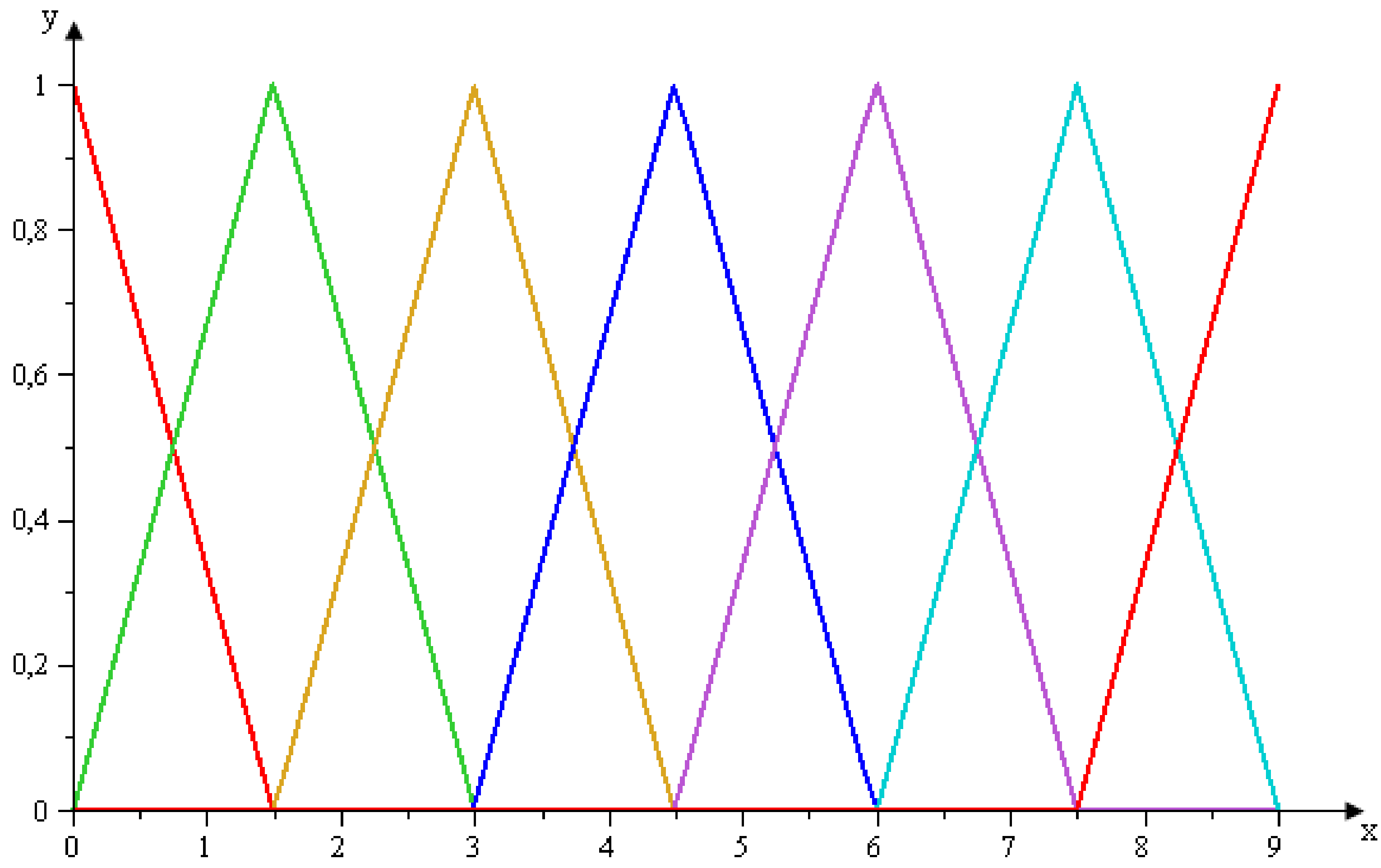
**7)** $A_k(x) = A_{k-1}(x - h)$, for all $k = 2, \dots, n-1$ and $x \in [x_k, x_{k+1}]$ and $A_{k+1}(x) = A_k(x - h)$, for all $k = 2, \dots, n-1$ and $x \in [x_{k-1}, x_k]$.

The membership functions $A_1, A_2, \dots, A_n$ are called basic functions.

In this talk we consider splines of degree $m$ and defect 1 with respect to $\Delta_{m,n}$ ($\Delta_{m,n}$ is an extension of $\Delta_n$) as basic functions of the fuzzy partition

$$A_k^m \epsilon\, S_{m,1}(\Delta_{m,n}), k = 1,2, \dots, n.$$

The first degree splines ($m = 1$) are triangular shaped basic functions. Next we will also consider the second and the third degree splines. And through this talk we are going to slightly modify notations by adding the degree of a spline used for constructing basic functions in superscript.




**An uniform fuzzy partition of the interval *[0, 9]* by fuzzy sets with triangular shaped membership funcions *n=7***

# Polynomial splines

Let $\Delta_{m,n} = \{t_1, t_2, \ldots, t_N | a = t_1 < t_2 < \ldots < t_N = b\}$ be a given mesh for the interval $[a, b]$.

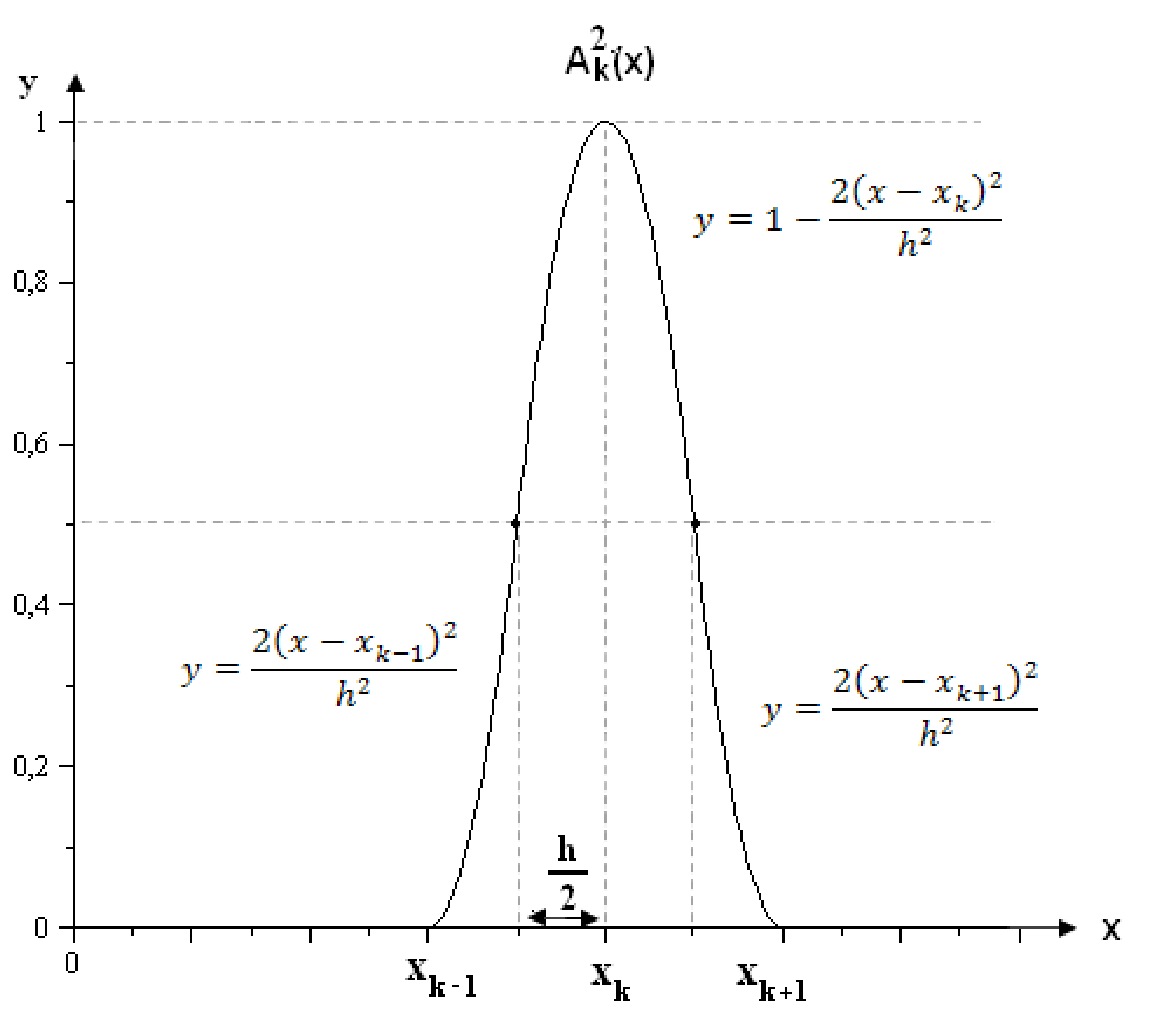
$S_{m,1}(\Delta_{m,n})$ is the space of polynomial splines of degree $m$ and defect 1 with respect to $\Delta_{m,n}$.

$s \in S_{m,1}(\Delta_{m,n})$:

- $s$ is a polynomial of degree $m$ on each interval $[t_{i-1}, t_i]$, $i = 2, \ldots, N$;
- $s \in C^{m-1}[a, b]$

$$s(x) = \sum_{j=0}^{m} a_j x^j + \sum_{i=2}^{N-1} \beta_i (x - t_i)_+^m$$

# Quadratic spline construction

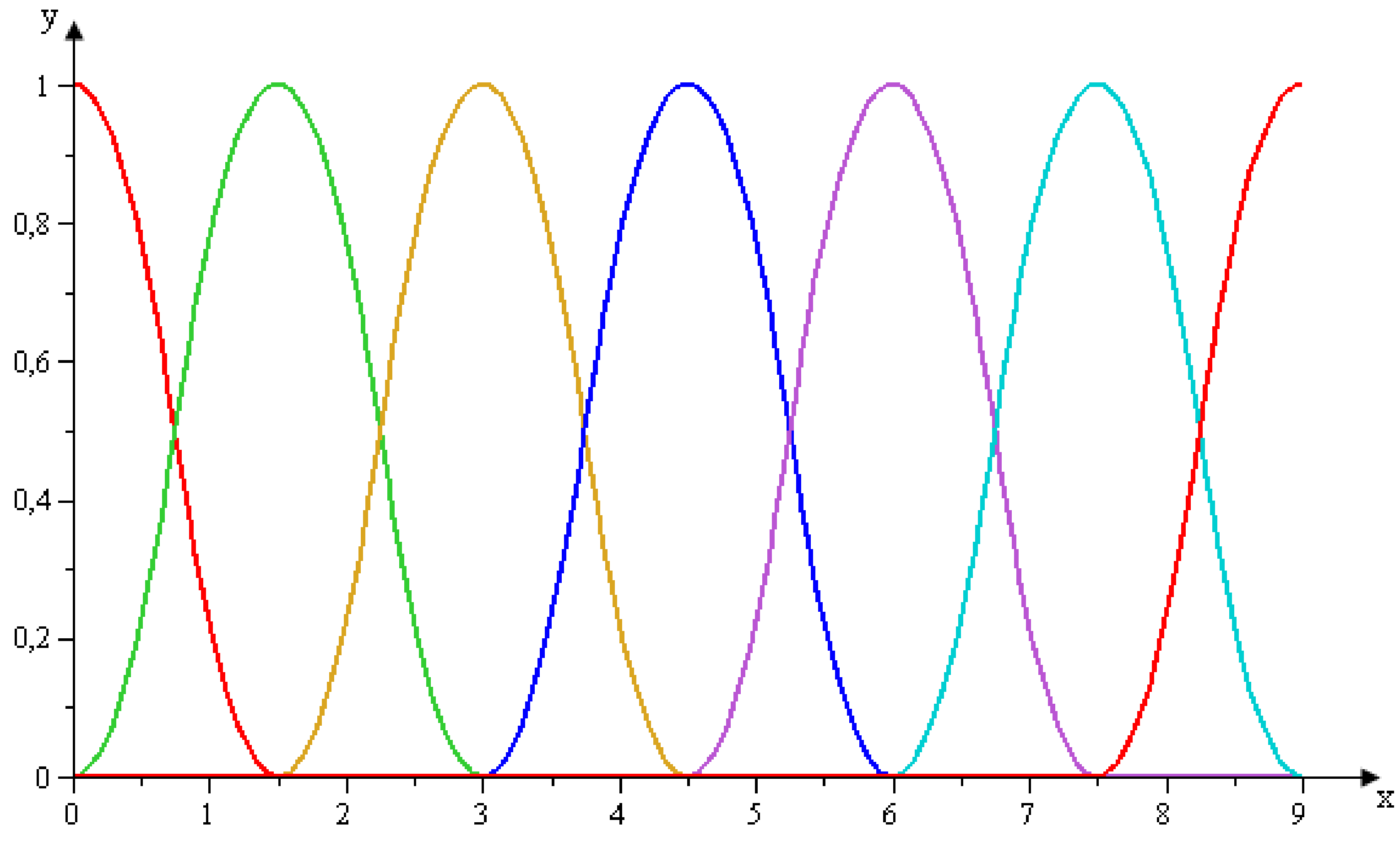


# Quadratic spline membership functions

$$A_1^2(x) = \begin{cases} 1 - \dfrac{2(x-x_1)^2}{h^2}, & x \in \left[x_1, \quad x_1 + \dfrac{h}{2}\right], \\ \dfrac{2(x-x_2)^2}{h^2}, & x \in \left[x_1 + \dfrac{h}{2}, \quad x_2\right], \\ 0, & x \in [x_2, \ x_n], \end{cases}$$

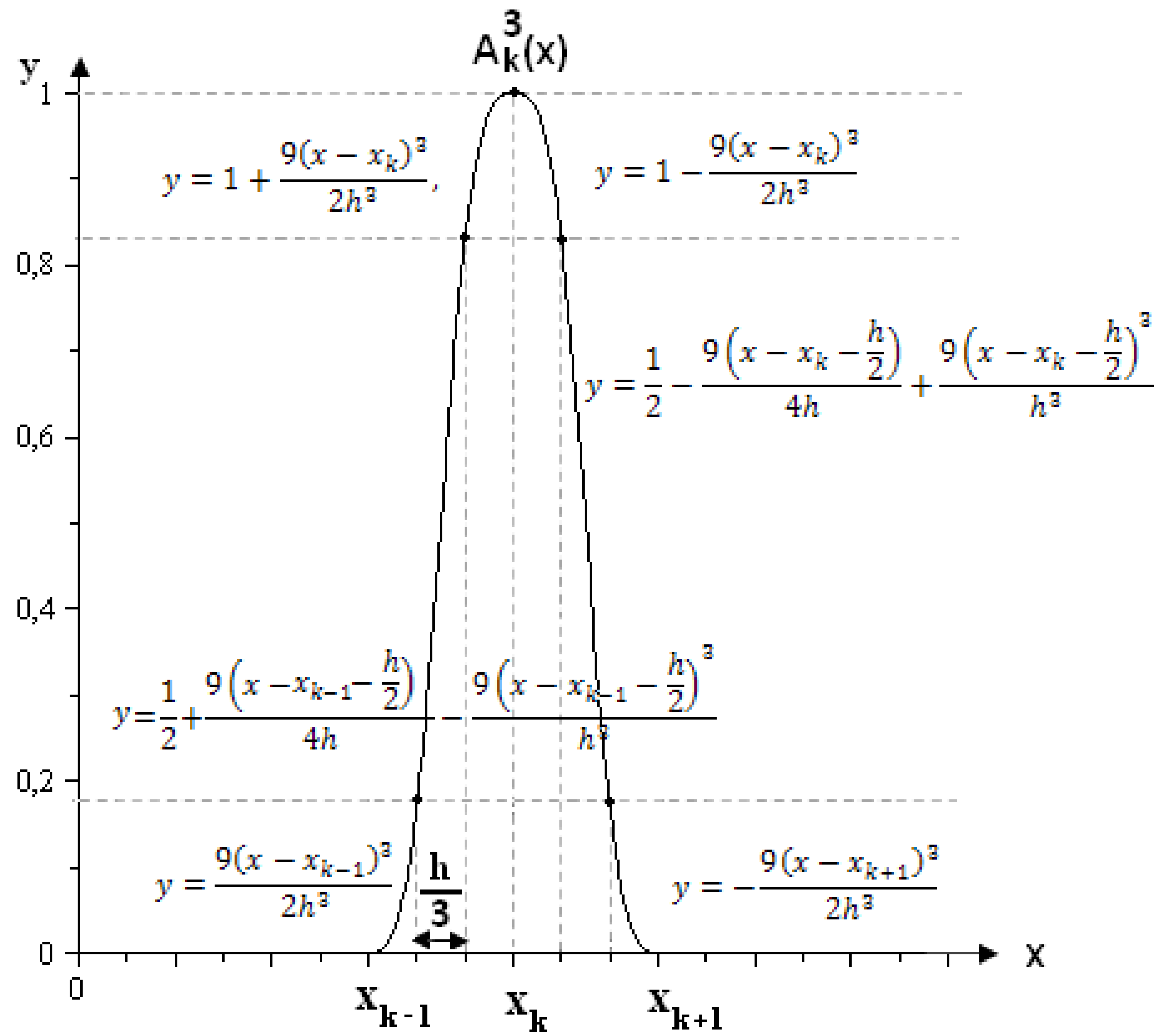
$$A_k^2(x) = \begin{cases} \dfrac{2(x-x_{k-1})^2}{h^2}, & x \in \left[x_{k-1}, \ \ x_{k-1} + \dfrac{h}{2}\right], \\ 1 - \dfrac{2(x-x_k)^2}{h^2}, & x \in \left[x_{k-1} + \dfrac{h}{2}, \ x_k + \dfrac{h}{2}\right], \\ \dfrac{2(x-x_{k+1})^2}{h^2}, & x \in \left[x_k + \dfrac{h}{2}, \quad x_{k+1}\right], \\ 0, & x \in [x_1, x_{k-1}] \cup [x_{k+1}, x_n], \end{cases}$$

$$A_n^2(x) = \begin{cases} \dfrac{2(x-x_{n-1})^2}{h^2}, & x \in \left[x_{n-1}, \ x_{n-1} + \dfrac{h}{2}\right], \\ 1 - \dfrac{2(x-x_n)^2}{h^2}, & x \in \left[x_{n-1} + \dfrac{h}{2}, \quad x_n\right], \\ 0, & x \in [x_1, x_{n-1}], \end{cases}$$




**An uniform fuzzy partition of the interval *[0, 9]* by fuzzy sets with quadratic spline membership funcions *n=7***

# Cubic spline construction

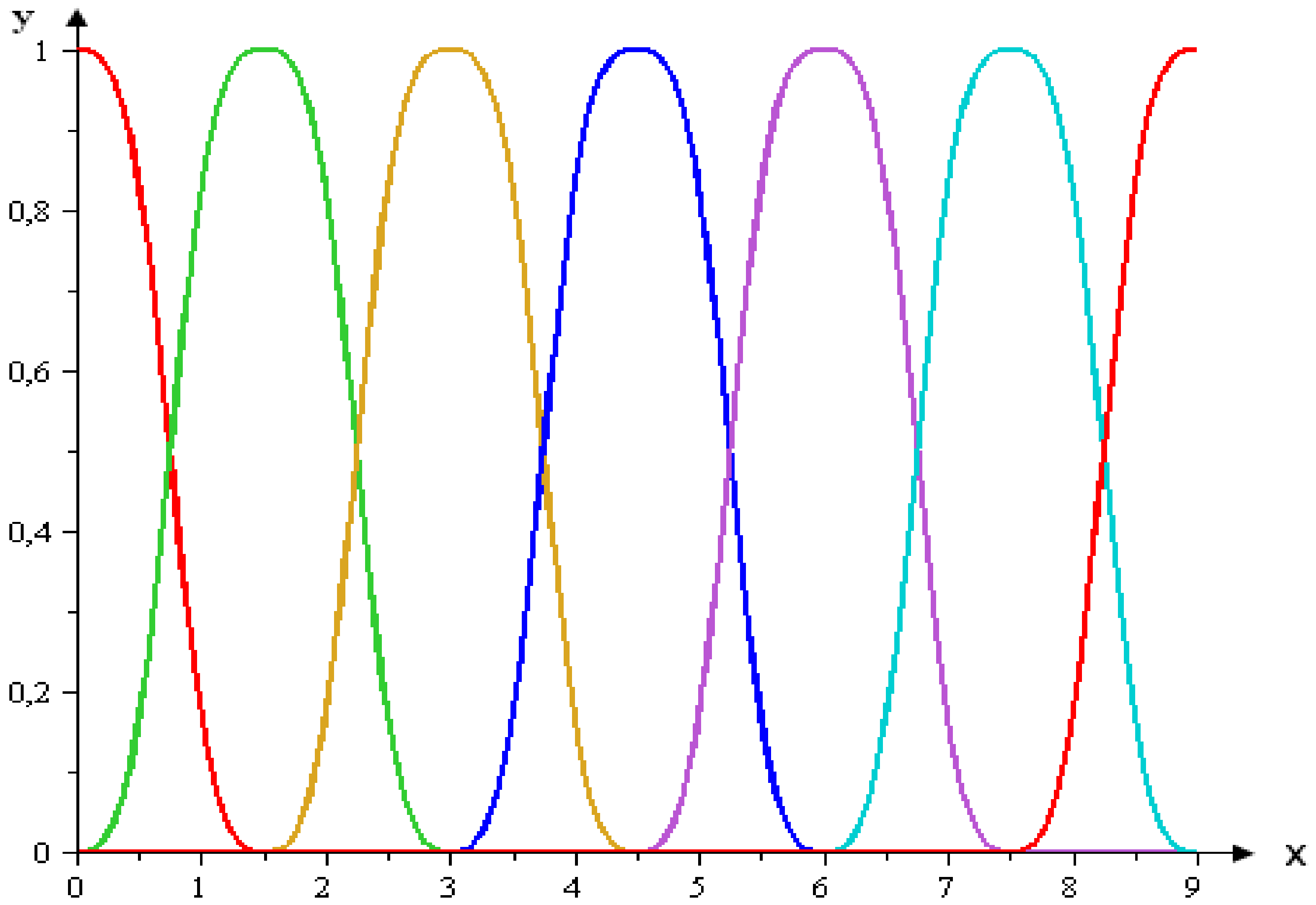


# Cubic spline membership functions

$$A_k^3(x) = \begin{cases} \dfrac{9(x - x_{k-1})^3}{2h^3}, & x \in \left[x_{k-1},\ x_{k-1} + \dfrac{h}{3}\right], \\ \dfrac{1}{2} + \dfrac{9\left(x - x_{k-1} - \dfrac{h}{2}\right)}{4h} - \dfrac{9\left(x - x_{k-1} - \dfrac{h}{2}\right)^3}{h^3}, & x \in \left[x_{k-1} + \dfrac{h}{3},\ x_{k-1} + \dfrac{2h}{3}\right], \\ 1 + \dfrac{9(x - x_k)^3}{2h^3}, & x \in \left[x_{k-1} + \dfrac{2h}{3},\ x_k\right], \\ 1 - \dfrac{9(x - x_k)^3}{2h^3}, & x \in \left[x_k,\ x_k + \dfrac{h}{3}\right], \\ \dfrac{1}{2} - \dfrac{9\left(x - x_k - \dfrac{h}{2}\right)}{4h} + \dfrac{9\left(x - x_{k-1} - \dfrac{h}{2}\right)^3}{h^3}, & x \in \left[x_k + \dfrac{h}{3},\ x_k + \dfrac{2h}{3}\right], \\ \dfrac{9(x - x_{k+1})^3}{2h^3}, & x \in \left[x_k + \dfrac{2h}{3},\ x_{k+1}\right], \\ 0, & x \in [x_1, x_{k-1}] \cup [x_{k+1}, x_n], \end{cases}$$

$$A_1^3(x) = \begin{cases} 1 - \dfrac{9(x-x_1)^3}{2h^3}, & x \in \left[x_1, \quad x_1 + \dfrac{h}{3}\right], \\ \dfrac{1}{2} - \dfrac{9\left(x - x_1 - \dfrac{h}{2}\right)}{4h} - \dfrac{9\left(x - x_1 - \dfrac{h}{2}\right)^3}{h^3}, & x \in \left[x_1 + \dfrac{h}{3}, x_1 + \dfrac{2h}{3}\right], \\ -\dfrac{9(x-x_2)^3}{2h^3}, & x \in \left[x_1 + \dfrac{2h}{3}, x_2\right], \\ 0, & x \in [x_2, \quad x_n], \end{cases}$$

$$A_n^3(x) = \begin{cases} \dfrac{9(x - \ x_{n-1})^3}{2h^3}, & x \in \left[\ x_{n-1}, \ x_{n-1} + \dfrac{h}{3}\right], \\ \dfrac{1}{2} + \dfrac{9\left(x - \ x_{n-1} - \dfrac{h}{2}\right)}{4h} + \dfrac{9\left(x - \ x_{n-1} - \dfrac{h}{2}\right)^3}{h^3}, & x \in \left[\ x_{n-1} \ + \dfrac{h}{3}, \ x_{n-1} \ + \dfrac{2h}{3}\right], \\ 1 + \dfrac{9(x-x_n)^3}{2h^3}, & x \in \left[\ x_{n-1} + \dfrac{2h}{3}, \quad x_n\right], \\ 0, & x \in [x_1, x_{n-1}], \end{cases}$$




**An uniform fuzzy partition of the interval *[0, 9]* by fuzzy sets with cubic spline membership funcions *n=7***

## On additional nodes

Generally basic functions are specified by the mesh:

$$\Delta_n = \{x_1, x_2, \ldots, x_n | a = x_1 < x_2 < \cdots < x_n = b\},$$

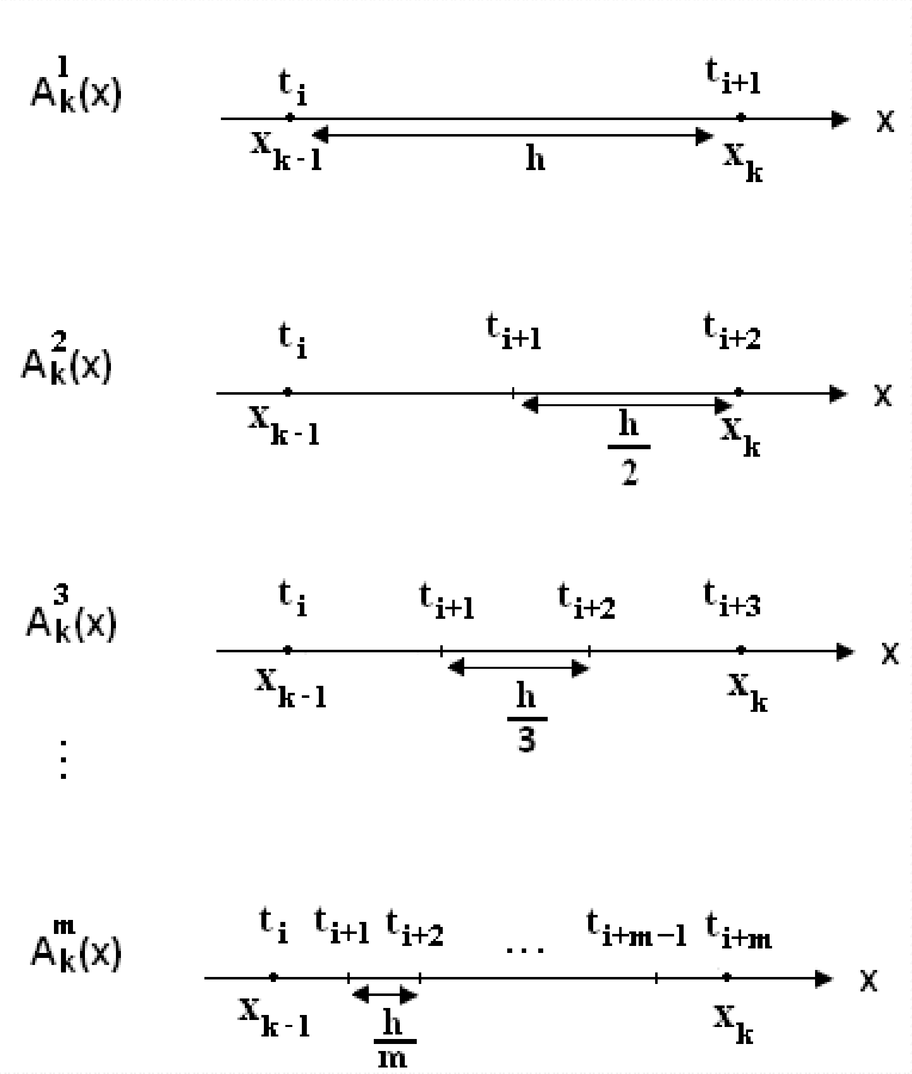
but when we construct those functions with the help of polynomial splines we use additional nodes, this extended mesh:

$$\Delta_{m,n} = \{t_1, t_2, \ldots, t_N | a = t_1 < t_2 < \ldots < t_N = b\},$$

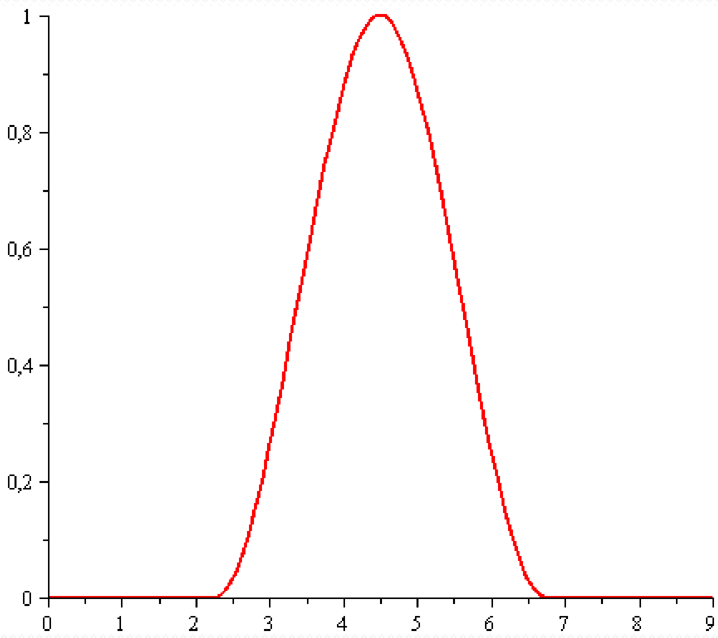
where *N* directly depends on the degree of a spline used for constructing basic functions and the number of nodes in the original mesh

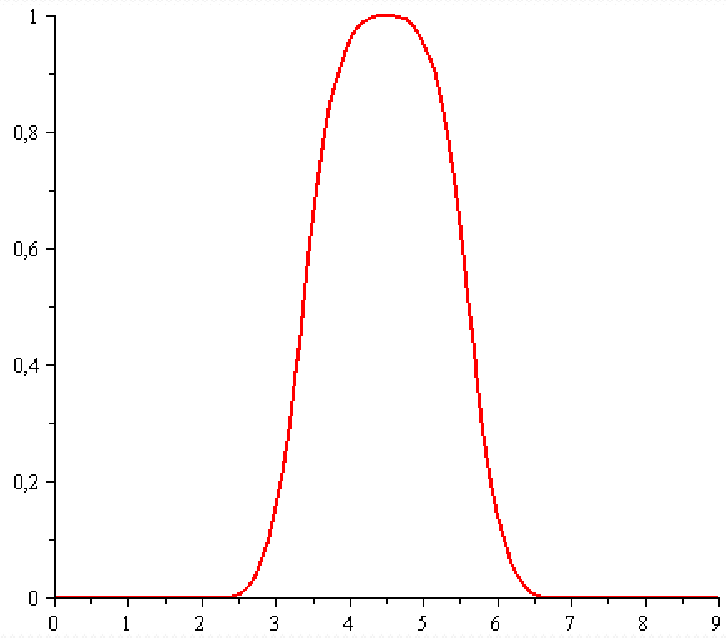
$$N = m(n-1).$$

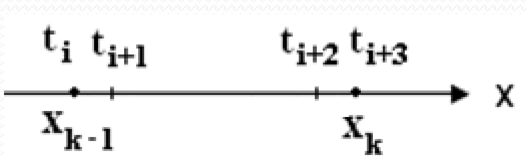
# Additional nodes for splines with degree 1,2,3, ..., m (uniform partition)

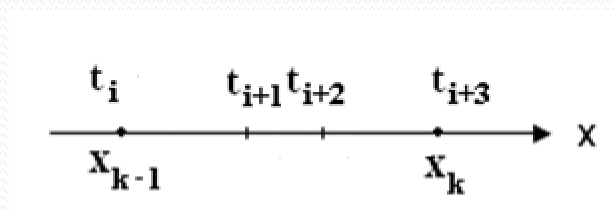


## We consider additional nodes as parameters of fuzzy partition: the shape of basic functions depends of the choise of additional nodes



# Direct F – transform

Let $A_1^m, A_2^m, \dots, A_n^m$ be basic functions which form a fuzzy partition of $[a, b]$ and $f$ be any function from $C([a,b])$. We say $n$-dimensional vector $\boldsymbol{F}^m[f] = (F_1^m, F_2^m, \dots, F_n^m)$ given by

$$F_k^m = \frac{\int_a^b f(x) A_k^m(x) dx}{\int_a^b A_k^m(x) dx}, \qquad k = 1, 2, \dots, n,$$

is the F-transform of $f$ with respect to $A_1^m, A_2^m, \dots, A_n^m$.

If partition $A_1^m, A_2^m, \dots, A_n^m$ of $[a, b]$ is uniform then components of the F – transform $F_k^m$ may be simplified

$$F_1^m = \frac{2}{h} \int_{x_1}^{x_2} f(x) A_1^m(x) dx,$$

$$F_k^m = \frac{1}{h} \int_{x_{k-1}}^{x_{k+1}} f(x) A_k^m(x) dx, \qquad k = 2, \dots, n-1,$$

$$F_n^m = \frac{2}{h} \int_{x_{n-1}}^{x_n} f(x) A_n^m(x) dx.$$

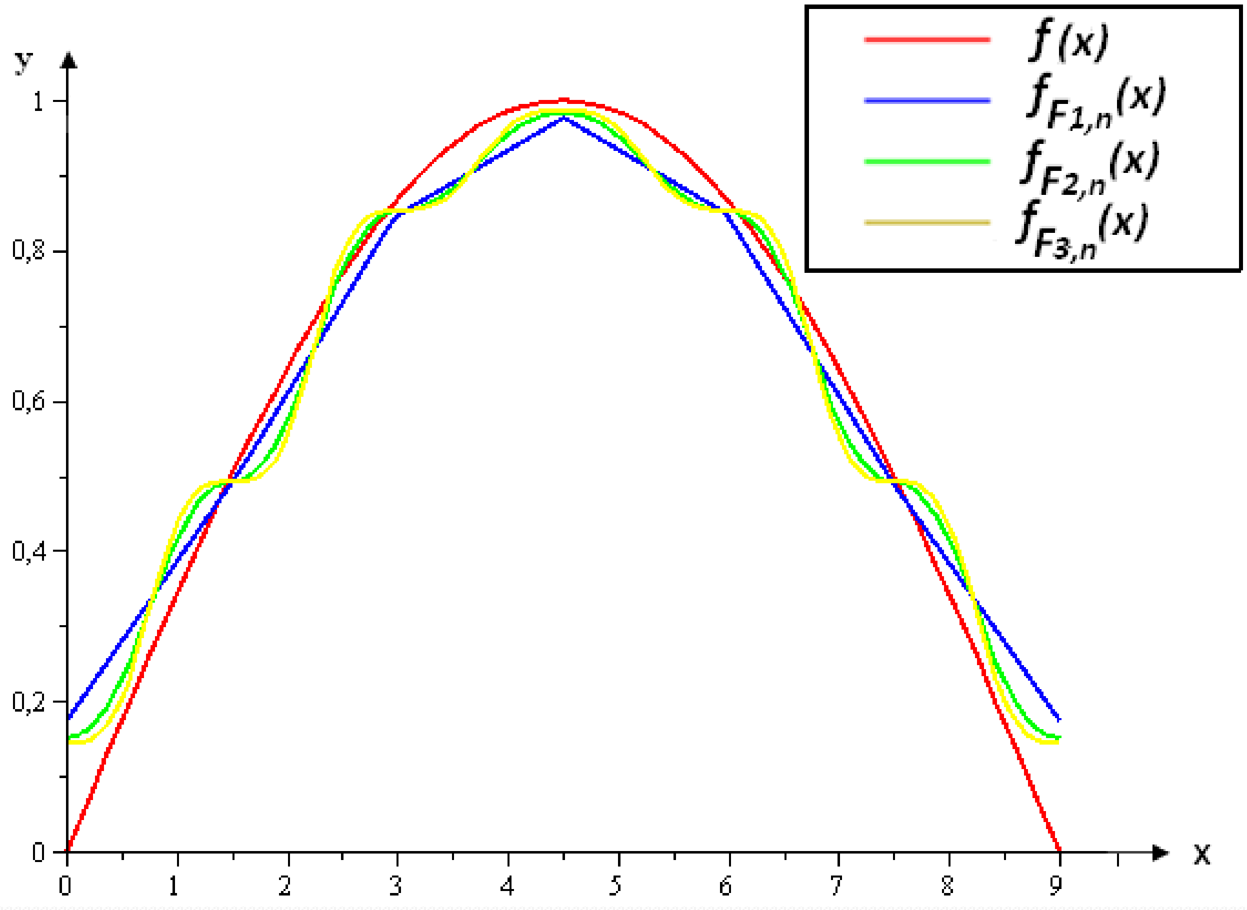
# Inverse F-transform

Let $A_1^m, A_2^m, \dots, A_n^m$ be basic functions which form a fuzzy partition of $[a, b]$ and $f$ be any function from $C([a, b])$. Let $\boldsymbol{F}^m[f] = (F_1^m, F_2^m, \dots, F_n^m)$ be the F – transform of $f$ with respect to $A_1^m, A_2^m, \dots, A_n^m$. Then the function

$$f_{F,m,n}(x) = \sum_{k=1}^{n} F_k^m A_k^m(x)$$

is called the inverse F – transform.

# Inverse F-transforms of the test function *f(x)=sin(xπ/9)*



# Extension of the original mesh

To extend $\Delta_n$ let us take $2(2m-1)$ additional nodes outside the interval $[a,b]$ and denote $\bar{a} = x_{-2m+2}$ and $\bar{b} = x_{n+2m-1}$. Our new mesh:

$$\Delta_n^{(m)} = \{x_{-2m+2}, x_{-2m+3}, \dots, x_{n+2m-1} \,|\, \bar{a} = x_{-2m+2} < x_{-2m+3} < \dots < x_{n+2m-1} = \bar{b}\}.$$

## Generalized fuzzy *m*-partition

We say that fuzzy sets $A_{-m+2}^{(m)}, A_{-m+3}^{(m)}, \dots, A_{n+m-1}^{(m)}$, identified with their membership functions $A_{-m+2}^{(m)}(x)$, $A_{-m+3}^{(m)}(x), \dots$, $A_{n+m-1}^{(m)}(x)$ $(n+2m-2$ continuous functions) defined on $[\bar{a}, \bar{b}]$, form uniform fuzzy $m$ –partition of $[a, b]$ if they fulfill the following conditions for $k = -m+2, \dots, n+m-1$:

1) $A_k^{(m)}: [\bar{a}, \bar{b}] \to [0,1]$, $A_k^{(m)}(x_k) = 1$;
2) $A_k^{(m)}(x) = 0$ for $x \notin [x_{k-m}, x_{k+m}]$;
3) $A_k^{(m)}$ is a continuous function;

4) $A_k^{(m)}$ is decreasing on $[x_k, x_{k+m}]$;

5) $A_k^{(m)}$ is increasing on $[x_{k-m}, x_k]$;

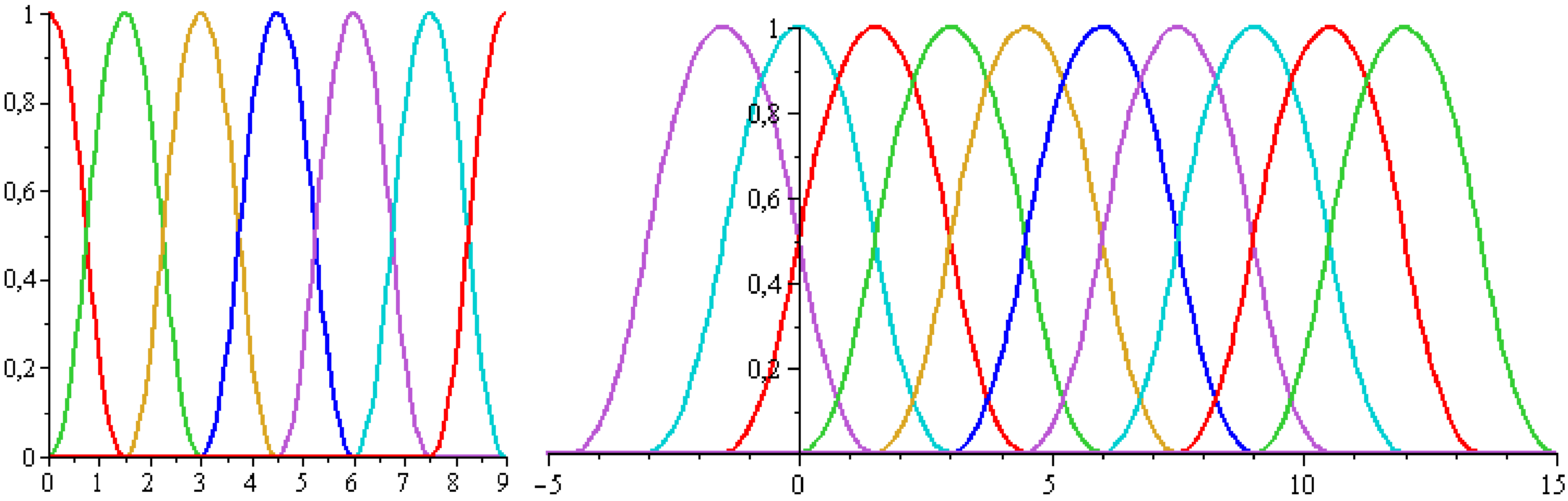
6) $A_k^{(m)}(x_k - x) = A_k^{(m)}(x_k + x)$, for all $x \in [0, mh]$;

7) $A_k^{(m)}(x) = A_{k-1}^{(m)}(x - h)$, for all $x \in [x_{k-m}, x_{k+m}]$, if $k = -m + 3, \ldots, n + m - 1$;

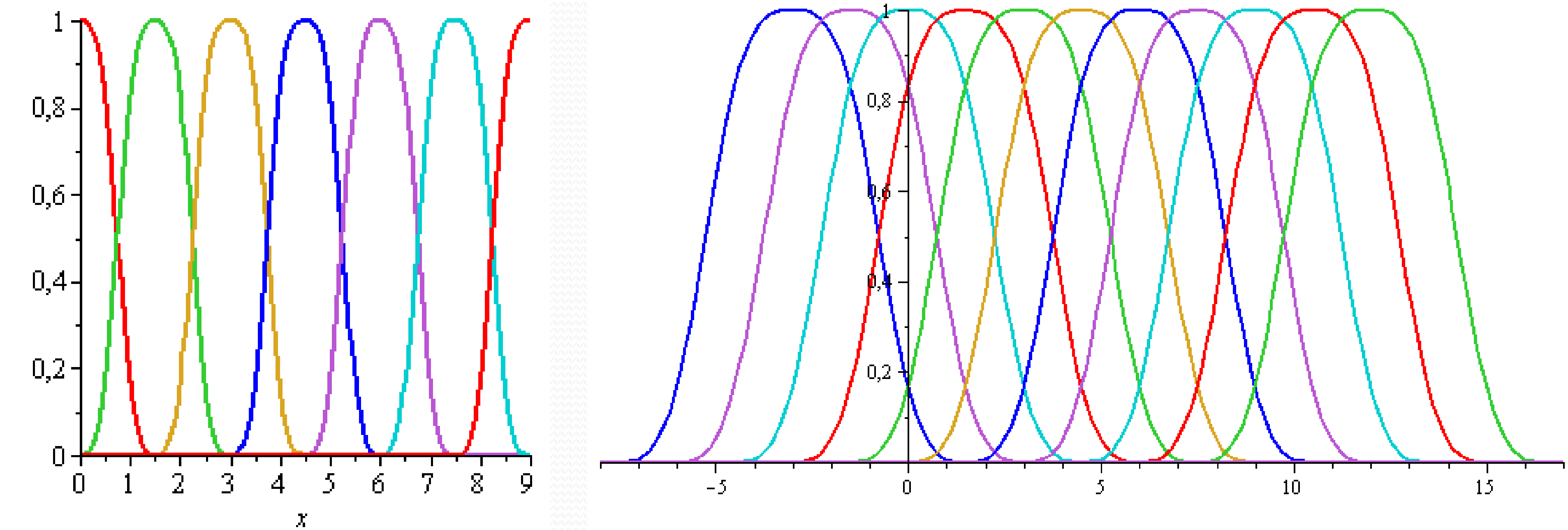
8) for all $x \in [a, b]$ $\sum_{k=-m+2}^{n+m-1} A_k^{(m)}(x) = m$.

In this talk we consider fuzzy m-partition described by splines of degree $m$ and defect 1 with respect to $\Delta_n^{(m)}$ ($\Delta_n^{(m)}$ - an extension of $\Delta_n$).

$$A_k^{(m)} \in S_{m,1}\left(\Delta_n^{(m)}\right), k = -m + 2, \dots, n + m - 1.$$




**An uniform fuzzy partition and a fuzzy m-partition of the interval *[0, 9]* based on quadratic spline membership funcions, *n=7, m=2***




**An uniform fuzzy partition and a fuzzy m-partition of the interval *[0, 9]* based on cubic spline membership funcions, *n=7, m=3***

## Direct and inverse *F*–transform with respect to the fuzzy m-partition

The direct F-transform (of integer bandwidth $m \geq 1$) based on the given generalized fuzzy $m$-partition is defined by the vector $\boldsymbol{F}^{(m)}[f] = (F_{-m+2}^{(m)}, F_2^{(m)}, \ldots, F_{n+m-1}^{(m)})$, where

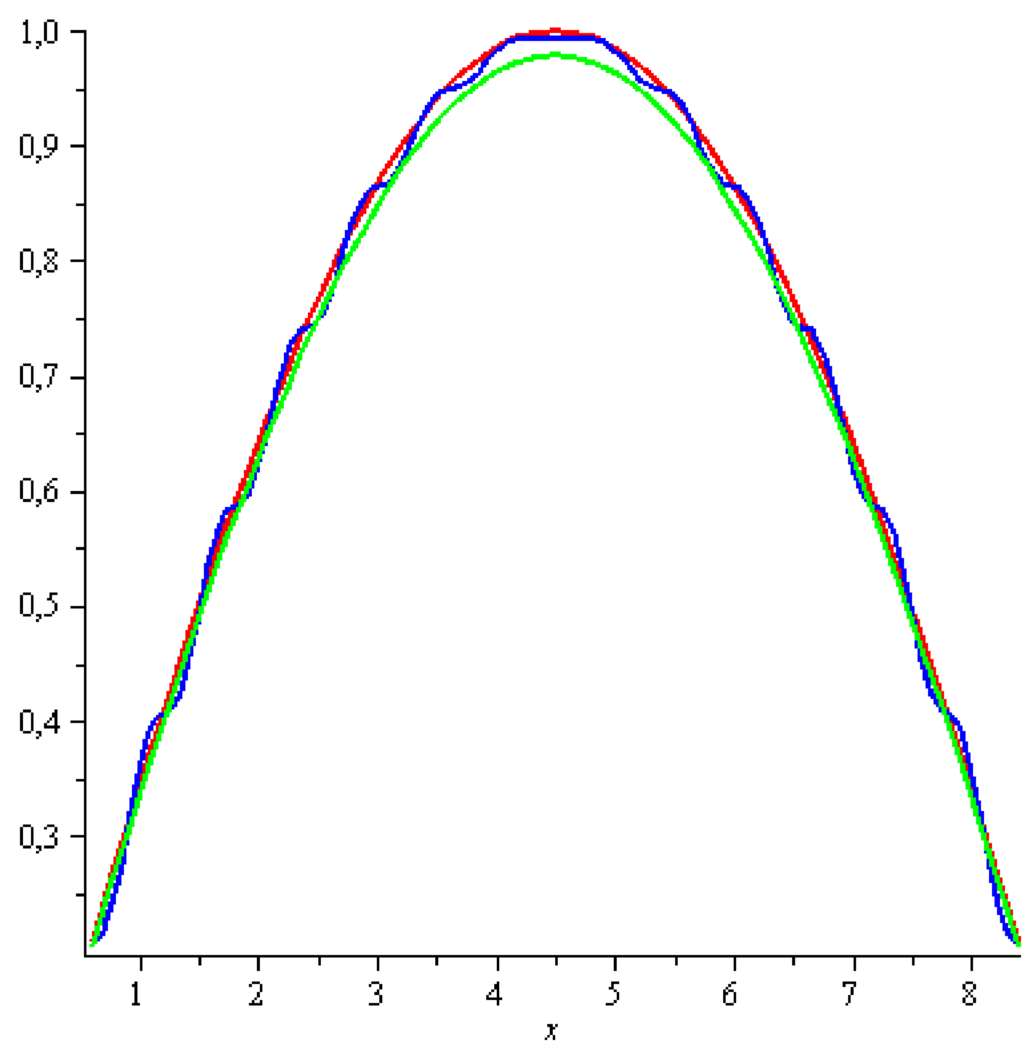
$$F_k^{(m)} = \frac{\int_{x_{k-m}}^{x_{k+m}} f(x) A_k^{(m)}(x)dx}{\int_{x_{k-m}}^{x_{k+m}} A_k^{(m)}(x)dx}$$

for $k = -m + 2, \ldots, n + m - 1$.

Correspondingly, the inverse F-transform function (of bandwidth m) is

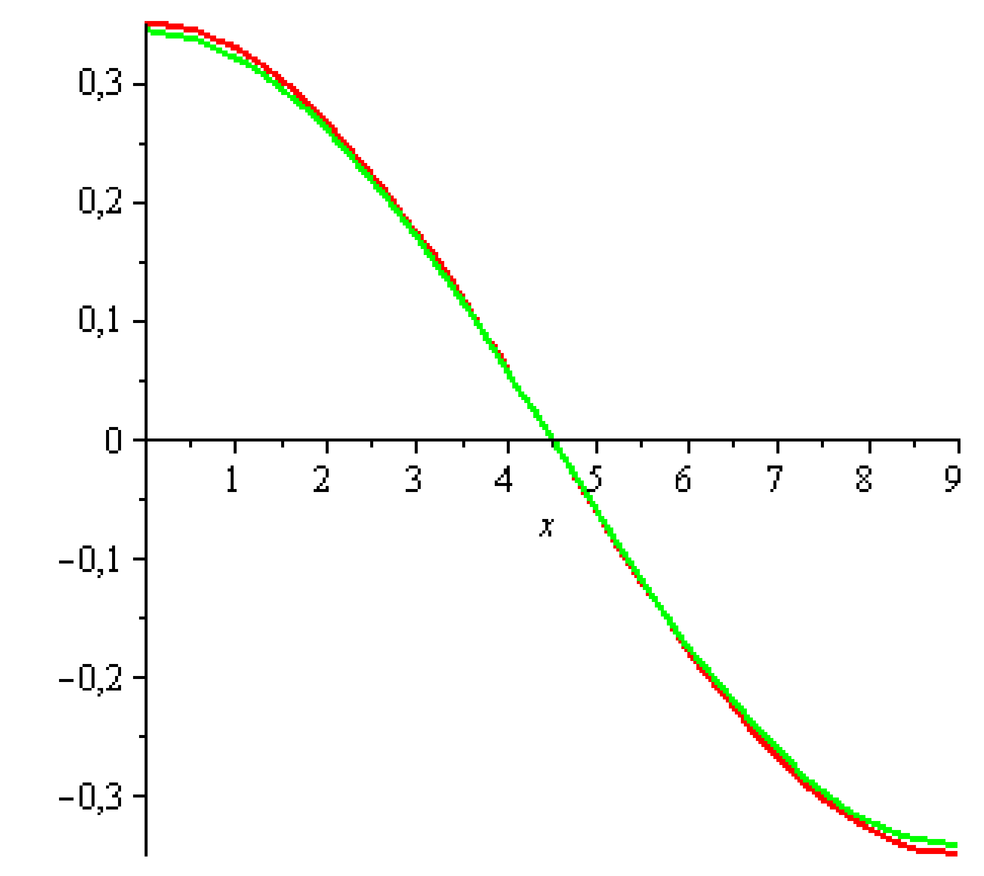
$$f_{F,n}^{(m)}(x) = \frac{1}{m} \sum_{k=-m+2}^{n+m-1} F_k^{(m)} A_k^{(m)}(x).$$

# Numerical examples

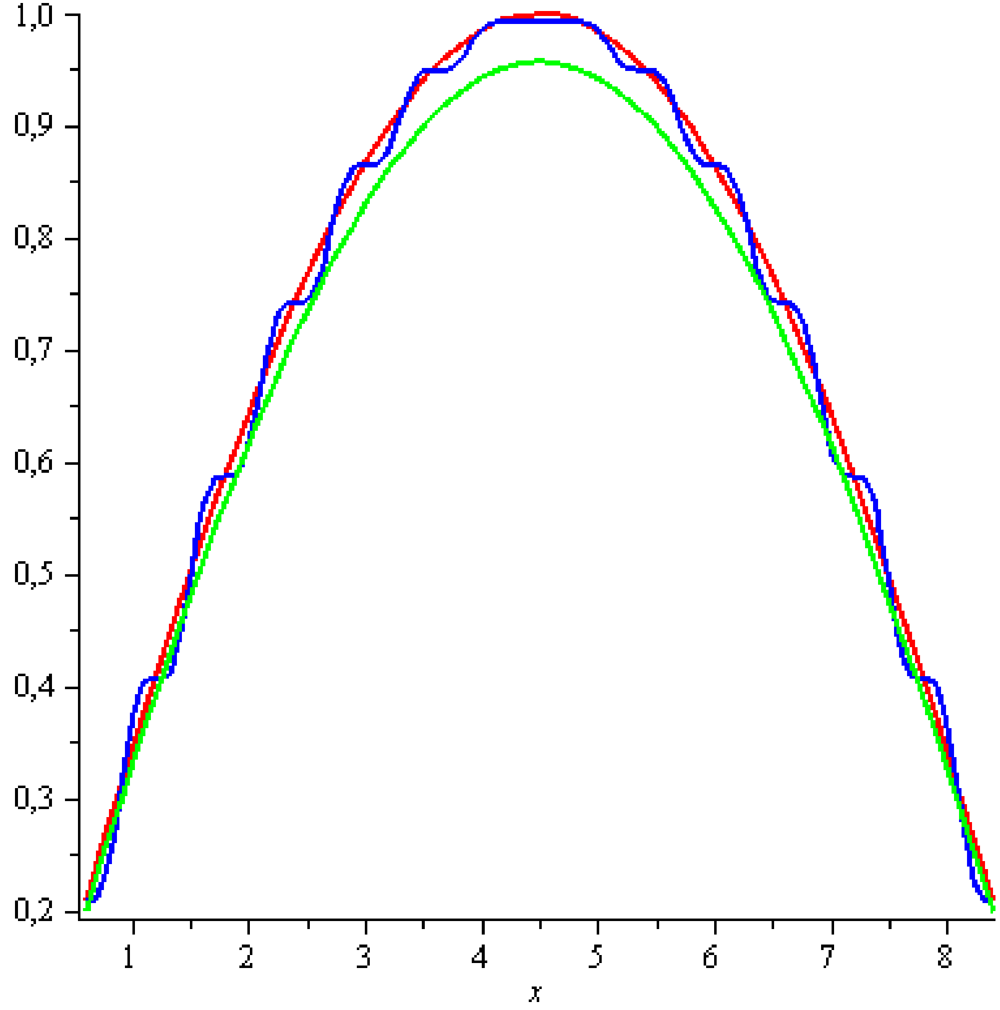



**Inverse *F*–transforms of the test function *f(x)=sin(xπ/9)* in case of quadratic spline basic funcions, *n=20, m=2***

# Approximation of derivatives

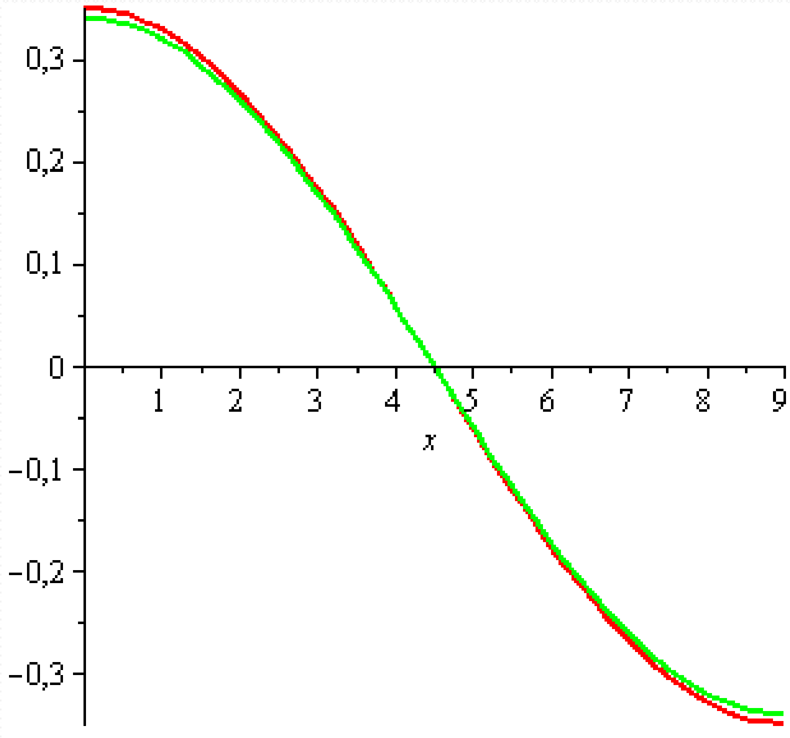


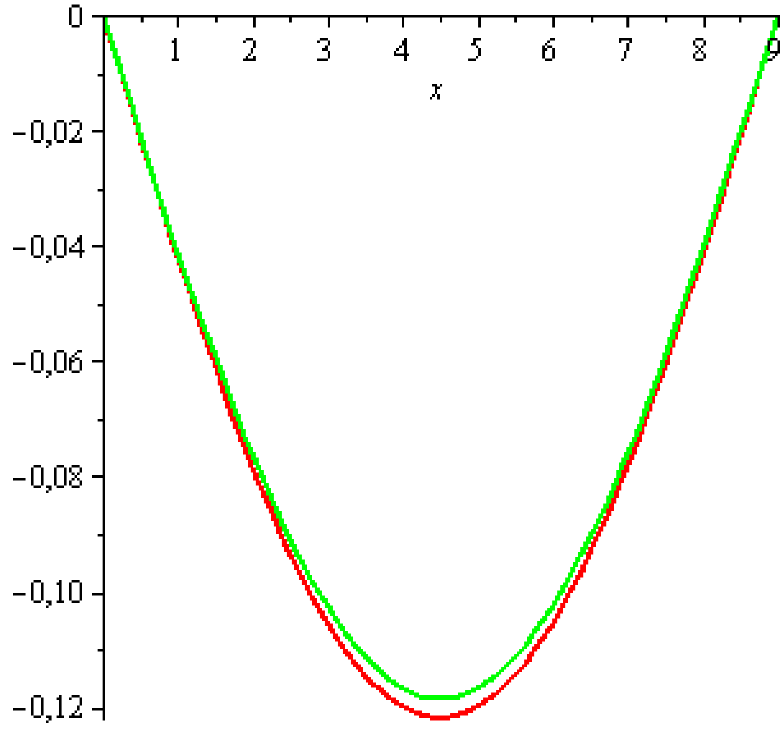

**Inverse *F*–transforms of the test function's *f(x)=sin(xπ/9)* first derivative in case of quadratic spline basic funcions, *n=20, m=2***




**Inverse *F*–transforms of the test function *f(x)=sin(xπ/9)* in case of cubic spline basic funcions, *n=20, m=3***

# Approximation of derivatives




**Inverse *F*–transforms of the test function's *f(x)=sin(xπ/9)* first and second derivative in case of cubic spline basic funcions, *n=20, m=3***

# Estimations of approximation of derrivatives

- $F$ −transform based on quadratic splines, generalized $m$-partition;

Let $f$ be a continuously differentiable function on $[\bar{a}, \bar{b}]$. Then for each $x \in [a, b]$ the following estimation hold

$$\left| f'(x) - \left( f_{F,n}^{(2)} \right)'(x) \right| \leq \omega(4h, f') .$$

- $F$ −transform based on cubic splines, generalized $m$-partition;

Let $f$ be two times continuously differentiable function on $[\bar{a}, \bar{b}]$. Then for each $x \in [a, b]$ the following estimation hold

$$\left| f''(x) - \left( f_{F,n}^{(3)} \right)''(x) \right| \leq \omega(6h, f''),$$

where $\omega(\delta, g) = \max_{|x-y| \leq \delta} |g(x) - g(y)|$, is the moduls of continuity of $g$ on $[a, b]$.

# Thank you for your attention!