Upper bound on Abelian lifts of complete graphs

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- Construction methods for lower bounds on $n(d,k)$.

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For other d,k we have $n(d,k) \leq M(d,k) - 2$ and $n(3,k) \leq M(3,k) - 4$.

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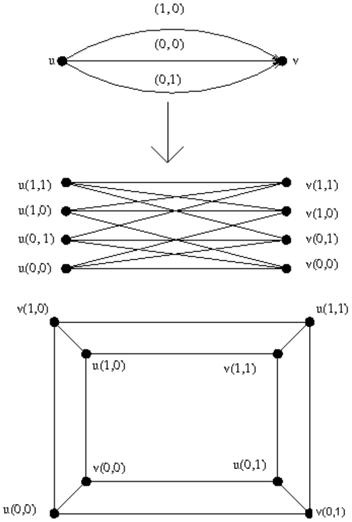
The set $\{(u, g), g \in \Gamma\}$ forms a fiber above u.

Example of a lift:



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Theorem

[J. Šiagiová '02] *For an arbitrary $d \geq 3$ let D be a dipole of degree d, and let α be a voltage assignment on D in an Abelian group such that the lift D^α is of diamter two. Then*

$$|V(D^\alpha)| \leq \frac{4(10+\sqrt{2})}{49}(d+0.34)^2 \approx 0.932d^2$$

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Lemma

Let α be an ordinary voltage assignment on a connected graph G in a group H and let k be a positive integer, and let u be a vertex of G. Then, the lift G^α has diameter at most k if and only if for every vertex v of G and for every element h of the group H there is a $u \to v$ path in G of length at most k such that $\alpha(W) = h$.

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Proposition

Let α be a voltage assignment on an n-pole $K_n(m,l)$ in an Abelian group A and suppose that the lift $K_n^\alpha(m,l)$ has diameter two. Then the number of vertices of $K_n^\alpha(m,l)$ is at most $\omega(m,l)$, where

$$\omega(m,l) = n \cdot \min\{1 + (n-1)m(m-1) + 2l(l+1), (n-2)m^2 + 4ml + m\} .$$

Theorem

Let $d \geq 3$ and $n \geq 2$ be integers and let α be a voltage assignment on an n-pole G of degree d in an Abelian group such that the lift G^α is of diameter 2. Then, for any fixed $n \geq 2$ the number of vertices of G^α is bounded above by

$$\frac{n^4 + 4n^3 + (2\sqrt{2} - 1)n^2 - (2\sqrt{2} + 2)n}{(n^2 + 2n - 1)^2} d^2 + O(d^{3/2})$$

as $d \to \infty$.

Sketch of proof:

We can express the polynomials from the Proposition in terms of degree d.

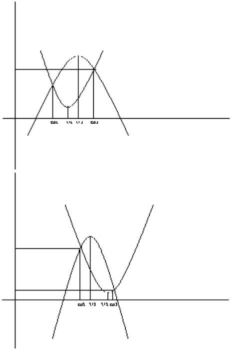
$$p_1(m) = (\frac{n^2}{2} - \frac{1}{2})m^2 + (-dn - 2n + d + 2)m + \frac{d^2}{2} + d + 1 \ , \quad \text{and}$$

$$p_2(m) = -nm^2 + (2d+1)m, \quad \text{where in both cases } \ 1 \leq m \leq d/(n-1) \ .$$

Since we need to find

$$M = \max_m \min \{p_1(m), p_2(m)\}$$

we have two possibilities of intersection of the parabolas.



Considering the x-coordinates of the vertices of parabolas $p_1(m), p_2(m)$ we get that

$$M = \max_m \ \min \{p_1(m), p_2(m)\}$$

is attained at x_2. Therefore

$$M = \max_m \ \min \{p_1(m), p_2(m)\} = p_1(x_2) = p_2(x_2)$$

If we substitute x_2 (expressed in terms of d, n) for m in $p_1(m)$ or $p_2(m)$ we gain the requesting result.

The upper bound for a diameter-two lift of $K_n(m, l)$ for some values of n:

n	**Order of lift of n-pole**
2	$0.932d^2 + O(d)$
3	$0.974d^2 + O(d)$
4	$0.987d^2 + O(d)$
5	$0.992d^2 + O(d)$
6	$0.995d^2 + O(d)$
7	$0.996d^2 + O(d)$

Conclusion

Our aim now is to investigate diameter-two lifts of other regular graphs on n vertices. We wish either to show that the upper bound obtained from the lifts of $K_n(m, l)$ is the best possible or to find a new regular graph on n vertices the lift of which has bigger order than presented.

THANK YOU FOR YOUR ATTENTION