Moments of Markov-switching models

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Introduction

- Overview to MSW models
- Moments of MSW models
  - State-independent MSW model
  - State-dependent MSW model

Overview to MSW models

- If we come out from AR(p), then
  $X_t = \phi_{0,q_t} + \phi_{1,q_t} X_{t-1} + \cdots + \phi_{p,q_t} X_{t-p} + \varepsilon_t \quad \text{for } q_t = 1,2$
  where $\{\varepsilon_t\}$ - process i.i.d. ≈ N(0, 1).

- special case of discrete first order Markov process (important is only the actual and the previous state) => regime q_t depends only on the regime q_{t-1}
  $$P\left(q_t = S_j \mid q_{t-1} = S_i\right) = p_{ij}$$

- state transition probabilities from one state to another:
  $$\sum_{j=1}^{N} p_{ij} = 1 \qquad p_{ij} \geq 0$$

Overview to MSW models

- **B** – the (N x N) matrix of transition probabilities for the 'time-reversed' Markov chain that moves back in time

$$P(q_t = S_j \mid q_{t+1} = S_i) = b_{ij} \qquad 0 \le b_{ij} \le 1 \qquad \sum_{j=1}^{N} b_{ij} = 1$$

Since

$$P(q_t = S_j \cap q_{i+1} = S_i) = P(q_{i+1} = S_i \mid q_t = S_j)P(q_t = S_j) =$$
$$= P(q_t = S_j \mid q_{i+1} = S_i)P(q_{i+1} = S_i)$$

the 'backward' transition probability matrix **B** is related to the 'forward' transition probabilities as follows:

$$b_{ij} = p_{ji}\left(\frac{\pi_j}{\pi_i}\right)$$

First order MSW model

a) Simple autoregressive MSW model

$$X_t = \phi_{q_t} + \phi_1 X_{t-1} + \varepsilon_t$$

b) State-dependent autoregressive MSW

$$X_t = \phi_{q_t} + \phi_{q_t,1} X_{t-1} + \varepsilon_t$$

First order MSW model

Let's have simple autoregressive MSW model

$$X_t = \phi_{q_t} + \phi_1 X_{t-1} + \varepsilon_t$$

then for state-dependent mean value holds

$$\mu_{q_t} = \phi_{q_t} + \phi_1 \mu_{q_{t-1}}$$

After substituting into the basic model we get

$$X_t = \mu_{q_t} + \phi_1 \left(X_{t-1} - \mu_{q_{t-1}} \right) + \varepsilon_t \qquad (1)$$

and after substituting backwards we obtain

$$X_t - \mu_{q_t} = \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}$$

Derivation of the second central moment

$$E\left[(X_t-\mu)^2\right] = E\left[(\mu_{q_t}-\mu)^2 + \phi_1^2\left(X_{t-1}-\mu_{q_{t-1}}\right)^2 + \varepsilon_t^2\right] +$$
$$+ 2\phi_1 Cov\left(\mu_{q_t}-\mu, X_{t-1}-\mu_{q_{t-1}}\right) + 2Cov\left(\mu_{q_t}-\mu, \varepsilon_t\right) +$$
$$+ 2\phi_1 Cov\left(X_{t-1}-\mu_{q_{t-1}}, \varepsilon_t\right)$$

- From the assumption $\{\varepsilon_t\}$ - process i.i.d. ≈ N(0, 1) and from independence between ε_{t-1} and q_t for i = 0, ±1, ±2,... => all covariances = 0 =>

$$E\left[(X_t-\mu)^2\right] = E\left[E\left[(X_t-\mu)^2\right] \mid q_t\right] =$$

$$= E\left[E\left[(\mu_{q_t}-\mu)^2 + \phi_1^2\left(X_{t-1}-\mu_{q_{t-1}}\right)^2 + \varepsilon_t^2\right] \mid q_t\right] =$$

Derivation of the second central moment

$$= \boldsymbol{\pi}'\left(\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\otimes\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\right)+\boldsymbol{\pi}'\phi_1^2\left(\sum_{i=0}^{\infty}\phi_1^{2i}\mathbf{P}^{\mathbf{i}}\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\end{pmatrix}\right)+\boldsymbol{\pi}'\begin{pmatrix}\sigma_\varepsilon^2\\ \sigma_\varepsilon^2\\ \sigma_\varepsilon^2\end{pmatrix}$$

$$= \boldsymbol{\pi}'\left(\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\otimes\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\right)+\phi_1^2\left(\sum_{i=0}^{\infty}\phi_1^{2i}\boldsymbol{\pi}'\mathbf{P}^{\mathbf{i}}\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\end{pmatrix}\right)+\boldsymbol{\pi}'\begin{pmatrix}\sigma_\varepsilon^2\\ \sigma_\varepsilon^2\\ \sigma_\varepsilon^2\end{pmatrix}=$$

$$= \boldsymbol{\pi}'\left(\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\otimes\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\right)+\phi_1^2\left(\sum_{i=0}^{\infty}\phi_1^{2i}\boldsymbol{\pi}'\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\end{pmatrix}\right)+\boldsymbol{\pi}'\begin{pmatrix}\sigma_\varepsilon^2\\ \sigma_\varepsilon^2\\ \sigma_\varepsilon^2\end{pmatrix}=$$

Derivation of the second central moment

$$= \pi' \left[\begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} \otimes \begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} + \phi_1^2 \left(1 - \phi_1^2\right)^{-1} \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \end{pmatrix} + \begin{pmatrix} \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \end{pmatrix} \right] =$$

$$= \pi' \left[\begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} \otimes \begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} + \begin{pmatrix} \dfrac{\phi_1^2 \sigma_1^2}{\left(1 - \phi_1^2\right)} \\ \dfrac{\phi_1^2 \sigma_2^2}{\left(1 - \phi_1^2\right)} \\ \dfrac{\phi_1^2 \sigma_3^2}{\left(1 - \phi_1^2\right)} \end{pmatrix} + \begin{pmatrix} \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \end{pmatrix} \right]$$

- In the proof we use that $\sum_{i=0}^{\infty} \phi^{2i} = \frac{1}{1-\phi^2}$ and π' is vector of steady-state probabilities, where $\pi' \mathbf{P}^{\mathbf{i}} = \pi'$.

The simple state-independent first-order autoregressive MSW model

2. central moment

$$E\left[(y_t-\mu)^2\right]=\pi'\left[\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\otimes\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}+\begin{pmatrix}\dfrac{\phi_1^2\sigma_1^2}{\left(1-\phi_1^2\right)}\\ \dfrac{\phi_1^2\sigma_2^2}{\left(1-\phi_1^2\right)}\\ \dfrac{\phi_1^2\sigma_3^2}{\left(1-\phi_1^2\right)}\end{pmatrix}+\begin{pmatrix}\sigma_\varepsilon^2\\ \sigma_\varepsilon^2\\ \sigma_\varepsilon^2\end{pmatrix}\right]$$

The second central moment of MSW model for 3 states (case ***b***)

From

$$X_t = \mu_{q_t} + \phi_{1,q_{t-1}}\left(X_{t-1} - \mu_{q_{t-1}}\right) + \varepsilon_t$$

where $\phi_{1,q_{t-1}}$ is first-order autoregressive coefficient for state q_{t-1}. After substituting backwards we get

$$X_t - \mu_{q_t} = \sum_{l=1}^{\infty}\left(\prod_{j=1}^{l}\phi_{1,q_{t-j}}\right)\varepsilon_{t-j} + \varepsilon_t$$

Derivation of the second central moment

$$E\left[(X_t-\mu)^2\right]=E\left[\left(\mu_{q_t}-\mu\right)^2+\phi^2_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right)^2+\varepsilon_t^2\right]+$$
$$+2Cov\left(\mu_{q_t}-\mu,\varepsilon_t\right)+2Cov\left(\mu_{q_t}-\mu,\phi_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right)\right)+$$
$$+2Cov\left(\phi_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right),\varepsilon_t\right)$$

From the assumption $\{\varepsilon_t\}$ - process i.i.d. ≈ N(0, 1) and from independence between ε_{t-1} and q_t for i = 0, ±1, ±2,... => all covariances = 0 =>

$$E\left[(X_t-\mu)^2\right]=E\left[\left(\mu_{q_t}-\mu\right)^2+\phi^2_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right)^2+\varepsilon_t^2\right]$$

$$E\left[\left(\mathbf{X_t}-\boldsymbol{\mu}_{\mathbf{q_t}}\right)^{\mathbf{2}}\middle|\mathbf{q_t}\right]=E\left[\left(\phi^2_{1,q_{t-1}}\left(\mathbf{X_{t-1}}-\boldsymbol{\mu}_{\mathbf{q_{t-1}}}\right)^{\mathbf{2}}+2\phi_{1,q_{t-1}}\left(\mathbf{X_{t-1}}-\boldsymbol{\mu}_{\mathbf{q_{t-1}}}\right)\varepsilon_t+\varepsilon_t^2\right)\middle|\mathbf{q_t}\right]$$

Derivation of the second central moment

$$E\left[\left(X_t-\mu_{q_t}\right)^2 \middle| q_t=i\right]=\sum_{j=1}^{3} E\left[\phi^2_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right)^2 \middle| q_{t-1}=j \cap q_t=i\right]\cdot P\left(q_{t-1}=j \middle| q_t=i\right)+\sigma^2_{\varepsilon}=$$

$$=\sum_{j=1}^{3} E\left[\phi^2_{1,q_{t-1}}\left(X_{t-1}-\mu_{q_{t-1}}\right)^2 \middle| q_{t-1}=j\right]\cdot b_{ij}+\sigma^2_{\varepsilon}$$

Label $\boldsymbol{\sigma}^2_{\varepsilon}=\begin{pmatrix}\sigma^2_{\varepsilon}\\ \sigma^2_{\varepsilon}\\ \sigma^2_{\varepsilon}\end{pmatrix}$, Φ is 3 x 3 diagonal matrix of autoregressive coefficients $(\phi)_{i,\ r=1,2,3}$

Thus

$$E\left[\left(X_t-\mu_{q_t}\right)^2 \mid \mathbf{q}_t\right]=\mathbf{B}\cdot\mathbf{\Phi}^2 E\left[\left(X_{t-1}-\mu_{q_{t-1}}\right)^2 \middle| \mathbf{q}_{t-1}\right]+\boldsymbol{\sigma}^2_{\varepsilon}$$

Derivation of the second central moment

We assume stationary process $\{X_t\}$ =>

$$E\left[X_t \middle| \begin{matrix} q_t = 1 \\ q_t = 2 \\ q_t = 3 \end{matrix}\right] = E\left[X_{t-1} \middle| \begin{matrix} q_{t-1} = 1 \\ q_{t-1} = 2 \\ q_{t-1} = 3 \end{matrix}\right] = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

$$E\left[\left(X_{t-1} - \boldsymbol{\mu}_{q_{t-1}}\right)^{\mathbf{2}} \middle| q_{t-1}\right] = \left(\mathbf{I}_3 - \mathbf{B} \cdot \boldsymbol{\Phi}^{\mathbf{2}}\right)^{-1} \boldsymbol{\sigma}_{\varepsilon}^{\mathbf{2}}$$

and thus

$$E\left[(X_t - \mu)^2\right] = E\left[E\left[(X_t - \mu)^2\right] \mid q_t\right] =$$

$$= E\left[E\left[\left(\left(\mu_{q_t} - \mu\right)^2 + \phi_{1,q_{t-1}}^2 \left(X_{t-1} - \mu_{q_{t-1}}\right)^2 + \varepsilon_t^2\right)\right] \mid q_t\right] =$$

$$= \boldsymbol{\pi}' \left(\begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} \otimes \begin{pmatrix} \mu_1 - \mu \\ \mu_2 - \mu \\ \mu_3 - \mu \end{pmatrix} \right) + \boldsymbol{\pi}' \boldsymbol{\Phi}^{\mathbf{2}} \left(\mathbf{I_3} - \mathbf{B}\boldsymbol{\Phi}^{\mathbf{2}}\right)^{-1} \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \end{pmatrix} + \boldsymbol{\pi}' \begin{pmatrix} \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \\ \sigma_\varepsilon^2 \end{pmatrix}$$

State-dependent first-order autoregressive MSW model

2. central moment

$$E\left[(X_t-\mu)^2\right]=\boldsymbol{\pi}'\left[\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}\otimes\begin{pmatrix}\mu_1-\mu\\ \mu_2-\mu\\ \mu_3-\mu\end{pmatrix}+\boldsymbol{\Phi}^2\left(\mathbf{I}_3-\mathbf{B}\boldsymbol{\Phi}^2\right)^{-1}\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\end{pmatrix}+\begin{pmatrix}\sigma_\varepsilon^2\\ \sigma_\varepsilon^2\\ \sigma_\varepsilon^2\end{pmatrix}\right]$$

Thank you for your attention