Generating fuzzy implications given strictly decreasing functions

Dana Hliněná and Vladislav Biba

Brno University of Technology, Brno

Outline

1, Basic definitions

2, Classes of generated implications

3, Generated implications I_f^N

4, Implications related to I_f^N

Basic definitions

Definition

A decreasing function $N : [0,1] \to [0,1]$ is called a *fuzzy negation* if for each $a, b \in [0,1]$ it satisfies the following conditions

- (i) $a < b \Rightarrow N(b) \leq N(a)$,
- (ii) $N(0) = 1, N(1) = 0$.

Basic definitions

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- (ii) $N(0) = 1, N(1) = 0$.

Remark

A fuzzy negation N is called strict *if N is strictly decreasing and continuous for arbitrary $x, y \in [0,1]$. Fuzzy negations such that $N(N(x)) = x$ for all $x \in [0,1]$ this equality are called* involutive negations. *The strict fuzzy negation is* strong *if and only if it is involutive.*

Definition (Klement, Mesiar, Pap)

A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$, the following four axioms are satisfied:

(T1) *Commutativity* $T(x, y) = T(y, x)$,

(T2) *Associativity* $T(x, T(y, z)) = T(T(x, y), z)$,

(T3) *Monotonicity* $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

(T4) *Boundary Condition* $T(x, 1) = x$.

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Remark

Commonly used fuzzy disjunctions in fuzzy logic are the triangular conorms*. A triangular conorm (also called a $t-$conorm) is a binary operation S on the unit interval $[0,1]$ which, for all $x, y, z \in [0,1]$, satisfies $(T1)-(T3)$ and $(S4)$ $S(x,0) = x$.*

Definition

A function $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

(I1) I is decreasing in its first variable,

(I2) I is increasing in its second variable,

(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

Definition

A fuzzy implication $I : [0,1]^2 \to [0,1]$ satisfies:

(NP) the left neutrality property if

$$I(1, y) = y; \quad y \in [0,1],$$

(IP) the identity principle if

$$I(x, x) = 1; \quad x \in [0,1],$$

(OP) the ordering property if

$$x \le y \iff I(x, y) = 1; \quad x, y \in [0,1],$$

(CP) the contrapositive symmetry with respect to a given fuzzy negation N if

$$I(x, y) = I(N(y), N(x)); \quad x, y \in [0,1],$$

Definition

A fuzzy implication $I : [0,1]^2 \to [0,1]$ satisfies:

(EP) the exchange principle if

$$I(x, I(y,z)) = I(y, I(x,z)) \text{ for all } x, y, z \in [0,1],$$

(LI) the law of importation with a t-norm T if

$$I(\,T(x,y), z) = I(x, I(y,z)); \quad x, y \in [0,1],$$

Definition

Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the natural negation of I.

Definition

A function $I : [0,1]^2 \to [0,1]$ is called an (S,N)–implication if there exists a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x),y), \quad x,y \in [0,1].$$

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Theorem (Baczyński and Jayaram)

For a function $I : [0,1]^2 \to [0,1]$, the following statements are equivalent:

- *I is an (S,N)–implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation N.*
- *I satisfies (I2), (EP) and N_I is a continuous (strict, strong) fuzzy negation.*

Definition

Another way of extending the classical binary implication to the unit interval $[0, 1]$ is based on the residuation operator with respect to a left-continuous triangular norm T:

$$I_T(x, y) = \max\{z \in [0, 1]; T(x, z) \leq y\}.$$

Elements of this class are known as R–implications.

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Elements of this class are known as R–implications.

Theorem (Fodor and Roubens)

For a function $I : [0, 1]^2 \to [0, 1]$, the following statements are equivalent:

- *I is an R–implication based on some left-continuous t-norm T.*
- *I satisfies (I2), (OP), (EP), and $I(x, .)$ is a right-continuous for any $x \in [0, 1]$.*

Classes of generated implications

Definition

Let $\varphi : [0,1] \to [0,\infty]$ be an increasing and non-constant function. The function $\varphi^{(-1)}$ defined by

$$\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\}$$

is called the *pseudo-inverse* of φ, with the convention $\sup \emptyset = 0$.

Definition

Let $f : [0,1] \to [0,\infty]$ be a decreasing and non-constant function. The function $f^{(-1)}$ defined by

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\}$$

is called the pseudo-inverse of f, with the convention $\sup \emptyset = 0$.

Theorem (Smutná)

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing function such that* $f(1) = 0$. *Then the function* $I_f(x,y) : [0,1]^2 \to [0,1]$ *which is given by*

$$I_f(x,y) = \begin{cases} 1 & \textit{if } x \leq y, \\ f^{(-1)}(f(y^+) - f(x)) & \textit{otherwise,} \end{cases}$$

where $f(y^+) = \lim\limits_{y \to y^+} f(y)$ *and* $f(1^+) = f(1)$, *is a fuzzy implication.*

Theorem (Smutná)

Let $g : [0,1] \to [0,\infty]$ *be a strictly increasing function such that* $g(0) = 0$. *Then the function* $I^g(x,y) : [0,1]^2 \to [0,1]$ *which is given by*

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)),$$

is a fuzzy implication.

Generated implications I_f^N

Theorem (Biba, Hliněná, Kalina and Kráľ)

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing function with* $f(1) = 0$ *and* $N : [0,1] \to [0,1]$ *be a fuzzy negation. Then the function* $I_f^N : [0,1]^2 \to [0,1]$ *defined by*

$$I_f^N(x,y) = N\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right),$$

is a fuzzy implication.

Example

Let $f_1(x) = 1 - x,\ f_2(x) = -\ln x$, and $N_1(x) = 1 - x,\ N_2(x) = \sqrt{1 - x^2}$. Then the functions $f_1^{(-1)}$ and $f_2^{(-1)}$ are given by $f_1^{(-1)}(x) = \max(1 - x, 0)$ and $f_2^{(-1)}(x) = e^{-x}$. The fuzzy implications I_f^N are given by

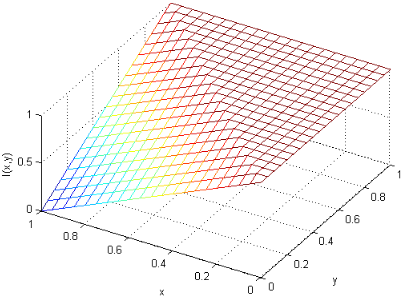
$$I_{f_1}^{N_1}(x, y) = \min(1 - x + y, 1),$$

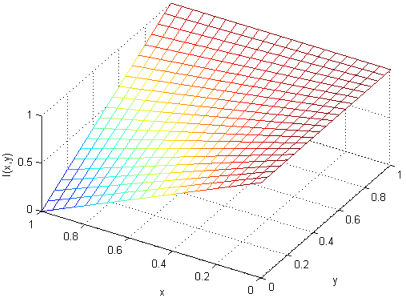
$$I_{f_2}^{N_1}(x, y) = 1 - x + x \cdot y,$$

$$I_{f_2}^{N_2}(x, y) = \sqrt{1 - x^2 + x^2 \cdot y^2}.$$

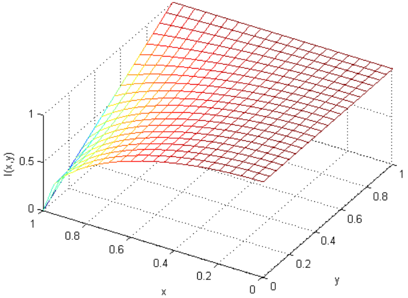
Note, that $I_{f_1}^{N_1}$ and $I_{f_2}^{N_1}$ are the well-known Łukasiewicz and Reichenbach implication, respectively.

Generated implications $I_{f_1}^{N_1}$ and $I_{f_2}^{N_1}$





Generated implication $I_{f_2}^{N_2}$



Proposition

Let c be a positive constant and $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the implications I_f^N and $I_{(c \cdot f)}^N$ which are based on functions f and $(c \cdot f)$, respectively, are identical.

Proposition

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Proposition

Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function such that $f(1) = 0$ and N be an arbitrary negation. Then the natural negation N_I given by I_f^N is $N_{I_f^N}(x) = N(x)$.

Proposition

Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the fuzzy implication I_f^N satisfies (NP) if and only if N is an involutive negation.

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Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the fuzzy implication I_f^N satisfies (CP) with respect to N if and only if N is an involutive negation.

Proposition

Let $f : [0,1] \to [0,\infty]$ *be a continuous and bounded strictly decreasing function such that* $f(1) = 0$ *and* $N(x) = f^{-1}(f(0) - f(x))$. *Then the fuzzy implication* I_f^N *satisfies (OP).*

Remark

Let $f : [0,1] \to [0,\infty]$ *be a not bounded function. Then the fuzzy implication* I_f^N *does not hold (OP). This is due from the fact that for all* $x, y \in]0,1[$ *we get* $f(x) + f(N(y)) < f(0)$ *and consequently* $I_f^N(x,y) < 1$.

Proposition

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing and continuous function such that* $f(1) = 0$. *Let* $N : [0,1] \to [0,1]$ *be a strong negation. Then the fuzzy implication* I_f^N *satisfies (EP).*

Proposition

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Proposition

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing and continuous function such that* $f(1) = 0$. *Let* $N : [0,1] \to [0,1]$ *be a strong negation. Then the fuzzy implication* I_f^N *satisfies (LI) with a t-norm* $T(x,y) = f^{(-1)}(f(x) + f(y))$.

Theorem

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing and continuous function such that* $f(1) = 0$. *Let* $N : [0,1] \to [0,1]$ *be a strong negation. Then* I_f^N *is* (S,N)*–implication. Moreover, if* f *is bounded function and* $N(x) = f^{-1}(f(0) - f(x))$, *then* I_f^N *is an* R*–implication as well.*

Implications related to I_f^N

Lemma (Biba, Hliněná, Kalina and Král')

Let $N : [0,1] \to [0,1]$ *be a fuzzy negation. Then* $N^{(-1)}$ *is a fuzzy negation if and only if*

$$N(x) = 0 \quad \Leftrightarrow \quad x = 1. \tag{1}$$

Theorem (Biba, Hliněná, Kalina and Král')

Let $f : [0,1] \to [0,\infty]$ *be a strictly decreasing function with* $f(1) = 0$*, and* $N : [0,1] \to [0,1]$ *be a fuzzy negation such that Formula* (1) *is fulfilled for* N*. Then the function* $I_f^{(N,N^{(-1)})} : [0,1]^2 \to [0,1]$ *defined by*

$$I_f^{(N,N^{(-1)})}(x,y) = N^{(-1)}\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right)$$

is a fuzzy implication.

Remark

A dual fuzzy negation based on a fuzzy negation N *is given by* $N^d(x) = 1 - N(1-x)$.

Theorem (Biba, Hliněná, Kalina and Kráľ)

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$$I_f^{(N,N^d)}(x,y) = N^d \left(f^{(-1)} \left(f(x) + f(N(y)) \right) \right),$$

is fuzzy implication.

Example

Let functions f be $f_1(x) = 1 - x$ and $f_2(x) = -\ln(x)$. Let negation N be given by $N(x) = 1 - x^2$. Then the implications $I_{f_1}^{N,N^{(-1)}}$ and $I_{f_2}^{N,N^{(-1)}}$ are given by

$$I_{f_1}^{N,N^{(-1)}}(x, y) = \sqrt{\min\left(1 - x + y^2, 1\right)},$$

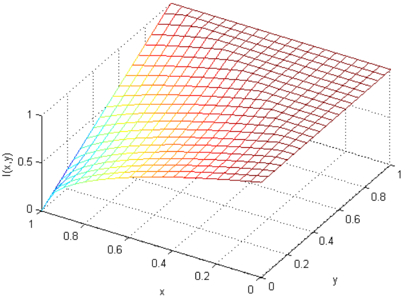
$$I_{f_2}^{N,N^{(-1)}}(x, y) = \sqrt{1 - x + x \cdot y^2}.$$

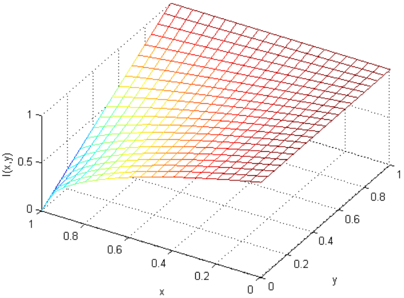
The dual negation to the negation N is given by $N^d(x) = (1 - x)^2$, therefore the implications $I_{f_1}^{N,N^d}$ and $I_{f_2}^{N,N^d}$ are given by

$$I_{f_1}^{N,N^d}(x, y) = \min\left(\left(1 - x + y^2\right)^2, 1\right),$$

$$I_{f_2}^{N,N^d}(x, y) = \left(1 - x + x \cdot y^2\right)^2.$$

Generated implications $I_{f_1}^{N,N^{(-1)}}$ and $I_{f_2}^{N,N^{(-1)}}$





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