Models of interest rates
Three factor convergence model

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ISCAMI 11. 5. 2012

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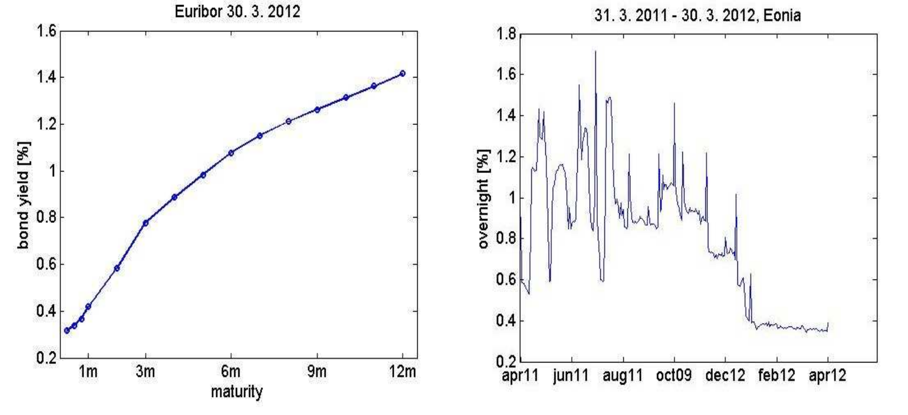
General terms

- Bond
  - Let $P(t, T)$ be a price of a discount bond, which is given by formula
  $$P(t, T) = e^{-R(t,T)(T-t)},$$
  from where we can express $R(t, T)$, which is
- Term structure of interest rates
  - given by formula
  $$R(t, T) = -\frac{\ln(P(t, T))}{(T - t)},$$
  - explains the relation between the time to maturity of a discount bond and its present price.
- Short rate
  - is instantenous interest rate for short time, it represents the beginning of the bond yield $r(t) = \lim_{T \to t^+} R(t, T)$.

- Term structure and short rate

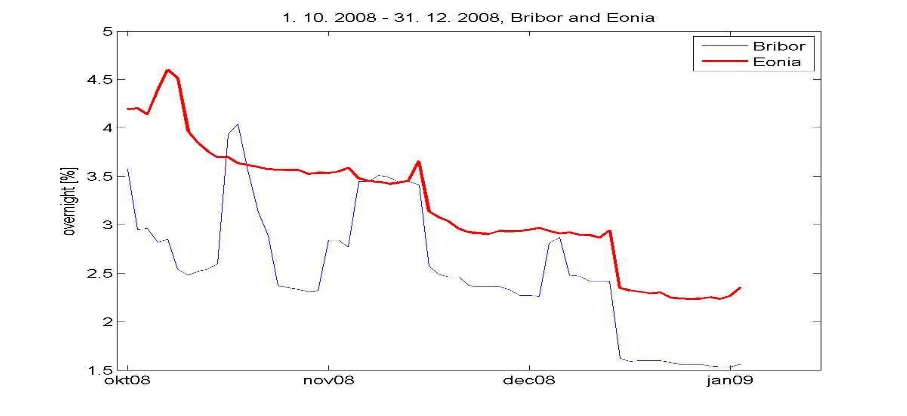


Models of interest rates

- Define term structure of interest rates.
- Are formulated by stochastic differential equation for instantaneous short rate:
$$dX = \mu(X,t)dt + \sigma(X,t)dW,$$
where W is Wiener process, function $\mu(X,t)$ is trend or drift part of this equation and $\sigma(X,t)$ presents fluctuations around the drift.
- Choosing different drift part $\mu(X,t)$ and volatility part $\sigma(X,t)$ we obtain various one-factor (where X is scalar) and multi-factor models (where X is vector).

Convergence models

- Special case of two-factor models.
- Explain the evolution of interest rate in connection with the adoption of the Euro currency.



Slovakia interest rate before enterning the monetary union

Two-factor convergence model of Vasicek type in real measure

$$\begin{aligned} dr_d &= [a + b(r_u - r_d)]dt + \sigma_d dW_d, \\ dr_u &= c(d - r_u)dt + \sigma_u dW_u, \\ Cov[dW_d, dW_u] &= \rho dt. \end{aligned}$$

Two-factor convergence model of CIR type in real measure

$$\begin{aligned} dr_d &= [a + b(r_u - r_d)]dt + \sigma_d \sqrt{r_d} dW_d, \\ dr_u &= c(d - r_u)dt + \sigma_u \sqrt{r_u} dW_u, \\ Cov[dW_d, dW_u] &= \rho dt. \end{aligned}$$

Two-factor convergence model of CKLS type in risk-neutral measure

Two-factor model for the domestic short rate r_d and European short rate r_e

$$\begin{aligned} dr_d &= (a_1 + a_2 r_d + a_3 r_e)dt + \sigma_d r_d^{\gamma_d} dW_d, \\ dr_e &= (b_1 + b_2 r_e)dt + \sigma_e r_e^{\gamma_e} dW_e, \\ Cov[dW_d, dW_e] &= \rho dt. \end{aligned}$$

- The bond price $P(r_d, r_e, \tau)$ with maturity τ satisfies the partial differential equation

$$-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e}$$
$$+\frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\frac{\partial^2 P}{\partial r_e^2} + \rho\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0$$

  which holds for $r_d, r_e > 0, \tau \in (0, T)$, with initial condition $P(r_d, r_e, 0) = 1$ for $r_d, r_e > 0$.
- For Vasicek ($\gamma_d = \gamma_e = 0$) and Cox-Ingersoll-Ross (CIR) model ($\gamma_d = \gamma_e = \frac{1}{2}$) with zero correlation $\rho = 0$ closed form solutions in a separate form are known:

$$P(r_d, r_e, \tau) = e^{A(\tau) - D(\tau) r_d - U(\tau) r_e},$$

  whereby
  - in a case of Vasicek model is given explicit formula for A, D, U,
  - in a case of CIR model it is system of ordinary differential equations (ODE), which can be solved by numerical methods.

Approximation of the solution for the CKLS model on the basis of the solution of the Vasicek model

- Approximation of the one-factor CKLS model used on approximation European bonds, where for constant volatility we substitute instantaneous volatility σr^{γ}.
- For domestic bonds - in the solution of Vasicek model we substitute terms σ_d, σ_e by the terms $\sigma_d r_d^{\gamma_d}$ and $\sigma_e r_e^{\gamma_e}$.
- We have obtained an approximation of the solution for the CKLS model, in a case $a_2 \neq b_2$: $P(r_d, r_e, \tau) = e^{A(\tau) - D(\tau) r_d - U(\tau) r_e}$

$$
\begin{aligned}
D(\tau) &= \frac{-1 + e^{a_2 \tau}}{a_2}, \\
U(\tau) &= \frac{a_3 \left(a_2 - a_2 e^{b_2 \tau} + b_2 \left(-1 + e^{a_2 \tau}\right)\right)}{a_2 \left(a_2 - b_2\right) b_2}, \\
A(\tau) &= \int_0^{\tau} -a_1 D(s) - b_1 U(s) + \frac{\sigma_d^2 r_d^{2\gamma_d} D^2(s)}{2} + \frac{\sigma_e^2 r_e^{2\gamma_e} U^2(s)}{2} \\
&+ \rho \sigma_d r_d^{\gamma_d} \sigma_e r_e^{\gamma_e} D(s) U(s) ds.
\end{aligned}
$$

Accuracy of the approximation for CKLS model

We have obtained accuracy of the approximation for CIR model with $\rho = 0$:

$$\ln P^{CIR,\rho=0,ap} - \ln P^{CIR,\rho=0} = \frac{1}{24}\left(-a_2\sigma_d^2 r_d - a_1\sigma_d^2 - a_3\sigma_d^2 r_e\right)\tau^4 + O(\tau^4).$$

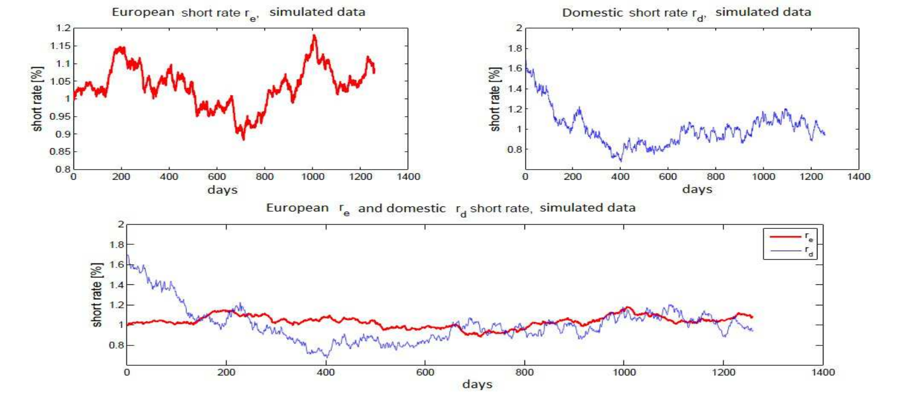
The accuracy of the approximation in general case for CKLS model is τ^4

$$\ln P^{CKLS,ap} - \ln P^{CKLS} = c_4(r_d, r_e)\tau^4 + O(\tau^4),$$

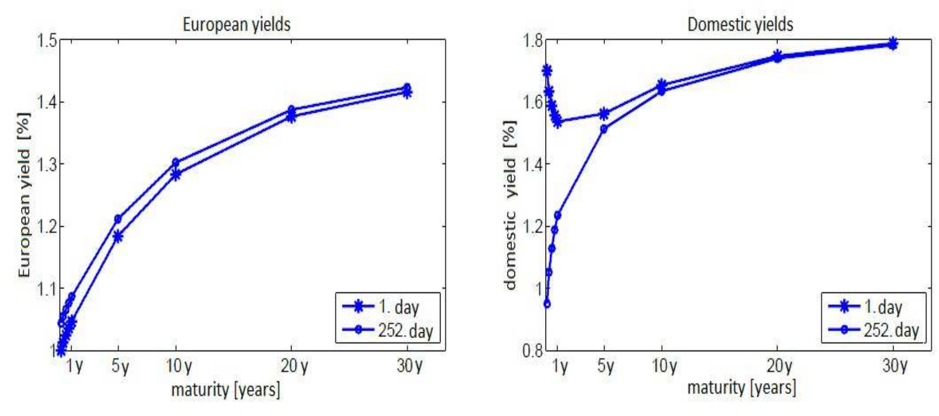
$$\text{where} \quad c_4(r_d, r_e) = \ldots$$

Numerical results for CIR model and its approximation

- We use simulated data with parameters: $a_1 = 0.0075$, $a_2 = -2$, $a_3 = 2$, $b_1 = 0.003$, $b_2 = -0.2$, $\sigma_d = 0.03$, $\sigma_e = 0.01$.
- We generate European and domestic short rate with initial values $r_e = 1\%$, $r_d = 1.7\%$.



- European and domestic yield curve



Obr.: Yield curves for 1. a 252. day in simulated data

Exact and approximative interest rate (acquired through proposed approximation for CIR model) and its difference.

- 252nd day of observation, where $r_d = 1.75\%$, $r_e = 1.06\%$

Maturity [in years]	Exact interest r. [%]	Approximate interest r. [%]	Difference [%]
0.25	1.08249	1.08250	-8.2e-006
0.5	1.15994	1.15996	-1.7e-005
0.75	1.21963	1.21964	-7.0e-006
1	1.26669	1.26671	-1.6e-005
5	1.53685	1.53691	-6.2e-005
10	1.65113	1.65127	-1.4e-004
20	1.74855	1.74884	-2.9e-004
30	1.78879	1.78918	-3.9e-004

- We have observed very small differences between the exact and theoretical term structure.
- Real market data Euribor are known with the accuracy 10^{-3}.
- Accuracy of our approximation is 10^{-5} till 10^{-6} for domestic bonds with maturity till one year and 10^{-4} till 10^{-5} for bonds with the maturity till 30 years.

Proposal of the calibration algorithm

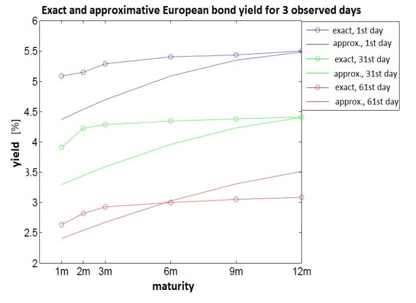
- Mutual frame - formulation of the optimization tasks.
- Algorithm:
  - Estimation of the European parameters.
  - Estimation of the domestic parameters:
    - Estimation of the risk neutral drift.
    - Estimation of the volatility.
    - Final modification of the parameters.
- Testing of the proposed algorithm on:
  - simulated data,
  - real market data.

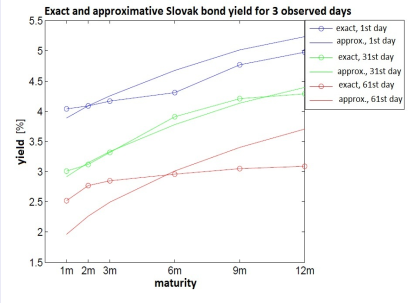
Illustration of one obtained estimation for CIR model

parameter	b_1	b_2	σ_e	a_1	a_2	a_3	σ_d
exact values	0.003	-0.2	0.01	0.0075	-2	2	0.03
estimated values	0.00299	-0.19999	0.010004	0.007502	-2.00028	2.000017	0.015566

Real market data

Real market data - Bribor





differences in European bond yields

Mat. [year]	Exact yield [%]	Approx. yield [%]	Difference [%]
$\frac{1}{12}$	3.9140	3.2953	0.6187
$\frac{2}{12}$	4.2260	3.4461	0.7799
$\frac{3}{12}$	4.2860	3.5891	0.6969
$\frac{6}{12}$	4.3450	3.9604	0.3846
$\frac{9}{12}$	4.3810	4.2327	0.1483
$\frac{12}{12}$	4.4120	4.4065	0.0055

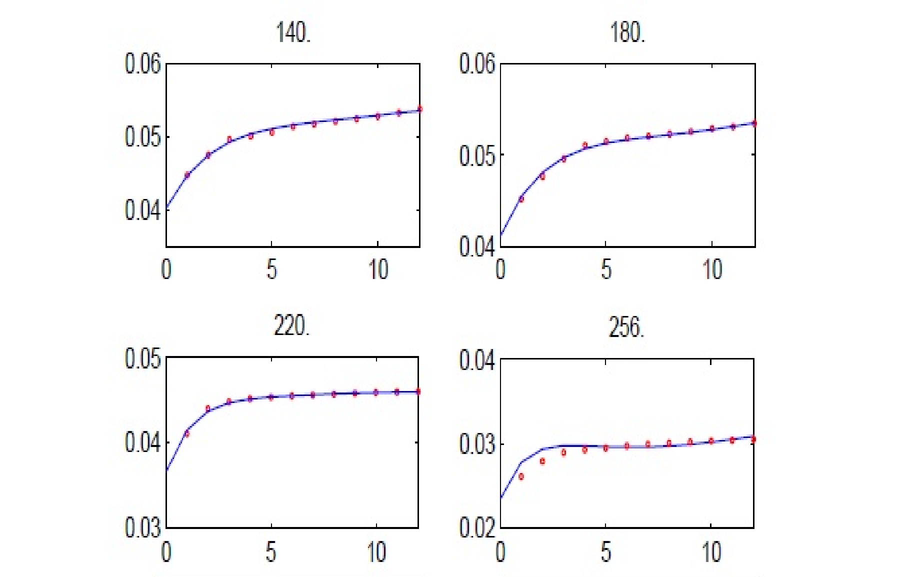
differences in domestic bond yields

Mat. [year]	Exact yield [%]	Approx. yield [%]	Difference [%]
$\frac{1}{12}$	3.0100	2.9174	0.0926
$\frac{2}{12}$	3.1200	3.1490	-0.0290
$\frac{3}{12}$	3.3200	3.3336	-0.0136
$\frac{6}{12}$	3.9100	3.7805	0.1295
$\frac{9}{12}$	4.2100	4.1358	0.0742
$\frac{12}{12}$	4.2900	4.3957	-0.1057

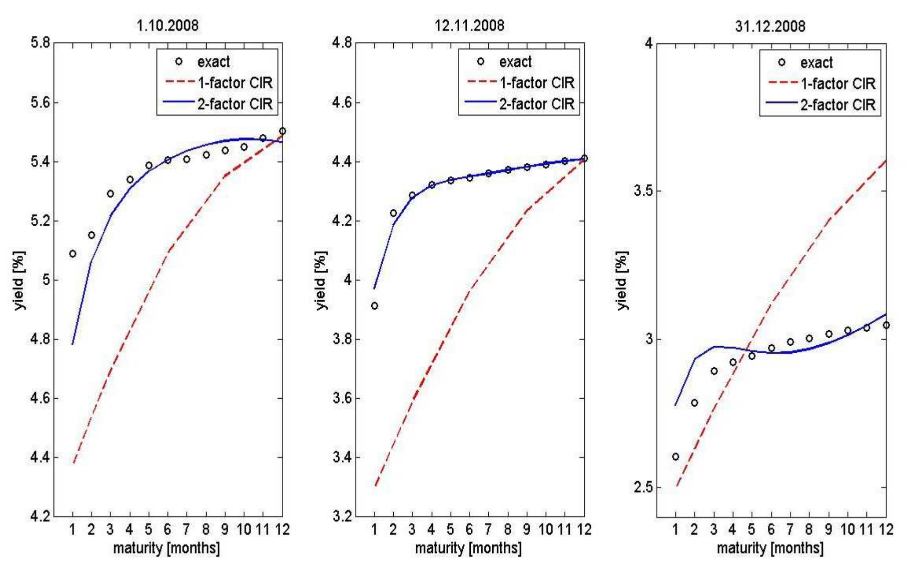
Problems

- The problem is already in the estimation of the European yield curves. Therefore we propose a different model to estimate the European parameters.
- Inspiration:
  Halgašová, J.: 2011, The approximation of bond prices in the two-factor interest rate models, Master Thesis
- In this thesis was reached a very good match yield curves.

Real data, CIR model



Comparison of these two models



- Two-factor model (shortrate is defined as the sum of two factors.)

$$\begin{aligned} r &= r_1 + r_2, \\ dr_1 &= (a_1 + b_1 r_1)dt + \sigma_1 r_1^{\gamma_1} dW_1, \\ dr_2 &= (a_1 + b_2 r_2)dt + \sigma_2 r_2^{\gamma_2} dW_2, \\ Cov[dW_1, dW_2] &= \rho dt. \end{aligned}$$

- PDE for the bond price $P(r_1, r_2, t)$ does not have explicit solution.
- We are searching for suitable approximation.
- It was proposed the same approximation as in previous case.

The accuracy of the approximation is τ^4

$$f^{ap}(r_d, r_e, \tau) - f^{ex}(r_d, r_e, \tau) = c_4(r_d, r_e)\tau^4 + O(\tau^4).$$

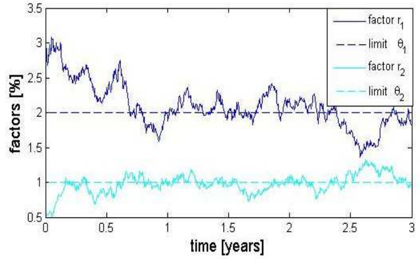
Formulation of the new model

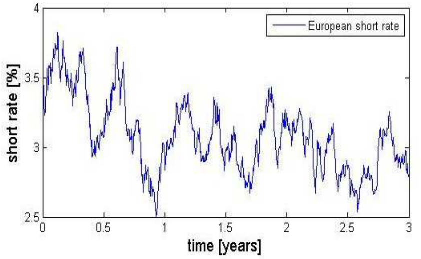
- The convergence model for the evolution of domestic interest rates, while the European rate is defined as the sum of two other unobservable factors.
- Dynamics of the system is described by three stochastic differential equations. First one for development of the domestic short rate r_d, the second one for development of the factor r_1 and third one for development of the factor r_2; where r_1, r_2 are individual components of the short rate $r_e = r_1 + r_2$.

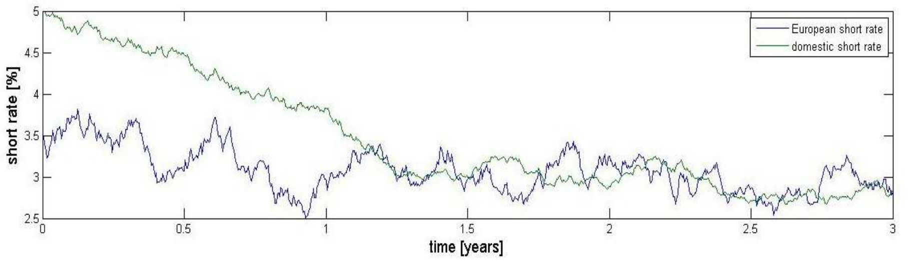
$$
\begin{aligned}
\mathrm{d}r_d &= (a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\mathrm{d}t + \sigma_d r_d^{\gamma_d} \mathrm{d}W_d, \\
\mathrm{d}r_1 &= (b_1 + b_2 r_1)\mathrm{d}t + \sigma_1 r_1^{\gamma_1} \mathrm{d}W_1, \\
\mathrm{d}r_2 &= (c_1 + c_2 r_2)\mathrm{d}t + \sigma_2 r_2^{\gamma_2} \mathrm{d}W_2, \\
Cov[\mathrm{d}W_d, \mathrm{d}W_1] &= \rho_1 \mathrm{d}t, \\
Cov[\mathrm{d}W_d, \mathrm{d}W_2] &= \rho_2 \mathrm{d}t, \\
Cov[\mathrm{d}W_1, \mathrm{d}W_2] &= \rho_3 \mathrm{d}t.
\end{aligned}
$$

Evolution of the factors and the interest rates

- We used following set of parameters:
  $\kappa_1 = 3, \theta_1 = 0.02, \sigma_1 = 0.05, \gamma_1 = 0.5, \kappa_2 = 10, \theta_2 = 0.01; \sigma_2 = 0.05, \gamma_2 = 0.5, \kappa_d = 1, \sigma_d = 0.02, \gamma_d = 0.5, \rho_{ij} = 0$ for all i, j.







Simulation of the European short rate r_e and the domestic short rate r_d.

Derivation of partial differential equations for the bond price

Using Ito lema and by construction risk-free portfolio of partial differential equations (PDEs) for the bondprice $P(r_d, r_1, r_2, \tau)$ with maturity τ.

$$
\begin{gathered}
-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_1 + a_4 r_2 - \lambda_1 \sigma_d r_d^{\gamma_d}) \frac{\partial P}{\partial r_d} + (b_1 + b_2 r_1 - \lambda_2 \sigma_1 r_1^{\gamma_1}) \frac{\partial P}{\partial r_1} + \\
(c_1 + c_2 r_2 - \lambda_3 \sigma_2 r_2^{\gamma_2}) \frac{\partial P}{\partial r_2} + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2} \frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_1^2 r_1^{2\gamma_1}}{2} \frac{\partial^2 P}{\partial r_1^2} + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2} \frac{\partial^2 P}{\partial r_2^2} + \\
\rho_1 \sigma_d r_d^{\gamma_d} \sigma_1 r_1^{\gamma_1} \frac{\partial^2 P}{\partial r_d \partial r_1} + \rho_2 \sigma_d r_d^{\gamma_d} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_d \partial r_2} + \rho_3 \sigma_1 r_1^{\gamma_1} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_1 \partial r_2} - r_d P = 0,
\end{gathered}
$$

where $r_d, r_1, r_2 > 0, \tau \in (0, T)$
with the initial condition $P(r_d, r_1, r_2, 0) = 1$ for $r_d, r_1, r_2 > 0$.

Theorem

Let $P^{CIR,\rho=0}$ be the bond price in the CIR-type convergence model with zero correlations and let $P^{CIR,\rho=0,ap}$ be its approximation. Then

$$\ln P^{CIR,\rho=0,ap} - \ln P^{CIR,\rho=0} = -\frac{1}{24}\sigma_d^2 \left(a_1 + a_2 r_d + a_3 r_1 + a_4 r_2\right) \tau^4 + o(\tau^4)$$

for $\tau \to 0^+$.

maturity	exact	approx.	exact	approx.	exact	approx.
0	4.00000	4.00000	4.00000	4.00000	4.00000	4.00000
0.25	4.06607	4.06607	4.01638	4.01638	3.96668	3.96668
0.5	4.05591	4.05591	3.95219	3.95219	3.84847	3.84847
0.75	4.00932	4.00931	3.87493	3.87493	3.74055	3.74054
1	3.94734	3.94733	3.7995	3.79949	3.65166	3.65165
2	3.69802	3.69796	3.56221	3.56217	3.4264	3.42638
3	3.52184	3.52171	3.41487	3.41479	3.30791	3.30788
4	3.40688	3.40669	3.32208	3.32196	3.23728	3.23724
5	3.32995	3.32972	3.26077	3.26062	3.19158	3.19153

Exact interest rates and their approximations obtained by the proposed formula. The domestic short rate is 4%, the European short rate is 5%, the columns correspond to the different values of the factors: $r_1 = 4\%, r_2 = 1\%$ (left), $r_1 = 2.5\%, r_2 = 2.5\%$ (middle), $r_1 = 1\%, r_2 = 4\%$ (right).

Proposal of the calibration method

. . . just in progress . . .

- Two ideas:
  - Similar idea as in the first convergence model - domestic short rate r_d is known.
  - Domestic short rate is unknown and it is neccesarry to estimate it.

Conslusion

- We studied 2 two-factor models and their properties:
  - Two-factor convergence model
    (We are not satisified with estimation for European parameters.)
  - Two-factor nonconvergence model
    (Very good fit - exact and approximative yield curves.)
- Proposed model - three-factor convergence model:
  - Combination of previous two models.
  - We proposed approximation of solution.
  - We found accury of its approximation.
  - We proposed calibration method.
- Next aim:
  - Test new proposed model using simulated and real market data.
  - Study different types od models: LIBOR market models.

Thank you for your attention.