Internal Operators: Application to Image Processing

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May 12, 2012

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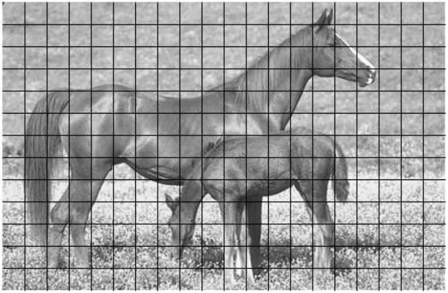
Example: Image Reduction



Preliminaries

Definition

A function $M : [0,1]^n \to [0,1]$ is called an n-ary aggregation function if it is monotone non-decreasing in each variable and such that $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$.

Definition

We say that an aggregation function M is

- idempotent if $M(x, \ldots, x) = x$ for all $x \in [0, 1]$.
- averaging if $min(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n) \leq max(x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in [0, 1]^n$.
- conjuctive if $M(x_1, \ldots, x_n) \leq min(x_1, \ldots, x_n)$.
- disjunctive if $M(x_1, \ldots, x_n) \geq max(x_1, \ldots, x_n)$.
- mixed if it is not conjunctive, disjunctive nor averaging, that is, it has different behaviours on different parts of the domain.
- simmetric if $M(x_1, \ldots, x_n) = M(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every $(x_1, \ldots, x_n) \in [0, 1]^n$ and every permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$.

Some families:

Definition

Let $g : [0,1] \to [-\infty, \infty]$ be a continuous and strictly monotone function. We call quasi-arithmetic mean the mapping $M_g : [0,1]^n \to [0,1]$ defined as

$$M_g(x_1, \ldots, x_n) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right)$$

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Definition

A mapping $F : [0,1]^n \to [0,1]$ is called an OWA operator of dimension n if there exists a weighting vector $W = (w_1, \ldots, w_n) \in [0,1]^n$ with $\sum_i w_i = 1$ and such that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{(i)}$$

Internal operators. Definition and properties

Definition

An internal operator is a mapping $F : [0,1]^n \to [0,1]$ such that

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}.$$

1. An internal operator needs not to be homogeneous. Consider the following *IF* operator:

$$IF(x_1,\ldots,x_n) = \begin{cases} min(x_1,\ldots,x_n) & \text{if } max(x_1,\ldots,x_n) \leq \frac{1}{4} \\ x_1 & \text{otherwise} \end{cases}$$

Then, if we take $n = 3$ and $\lambda = 0.5$, we have that $IF\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right) = \frac{1}{4}$ and $IF\left(\lambda\frac{1}{2},\lambda\frac{1}{3},\lambda\frac{1}{4}\right) = \frac{1}{4} \neq \lambda IF\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right)$.

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2. An internal operator needs not to be shift-invariant. Consider the same example as below. Then for $n=3$ and $r=\frac{1}{2}$ we have that $IF\left(\frac{1}{4},\frac{1}{6},\frac{1}{8}\right)=\frac{1}{8}$ and $IF\left(\frac{1}{2}+r,\frac{1}{3}+r,\frac{1}{4}+r\right)=\frac{3}{4}\neq IF\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right)+r$.

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3. An internal operator needs not to be monotone. When there are at least one input repeated, the mode is an example of a non-monotone internal operator.

Proposition

Let F be an internal operator. The following items hold:

i) *F is idempotent;*

ii) *F is compensative, that is,*

$$min(x_1, \ldots, x_n) \leq F(x_1, \ldots, x_n) \leq max(x_1, \ldots, x_n).$$

Internal Aggregation Functions

Theorem

Let $F : [0,1]^n \to [0,1]$ be an internal operator. If F is monotone non-decreasing in each variable, then F is an averaging aggregation function.

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Corollary

Under conditions of Theorem 6, F is jointly strict monotone, that is

$$x_i < y_i \text{ for all } i \in \{1,\ldots,n\} \text{ implies } F(x_1,\ldots,x_n) < F(y_1,\ldots,y_n).$$

Theorem

The following items hold:

i) *There is no conjunctive internal operators other than the minimum.*

ii) *There is no disjunctive internal operators other than the maximum.*

Theorem

Let $M_g : [0,1]^n \to [0,1]$ be a quasi-arithmetic mean. The following items are equivalent:

i) *M_g es internal operator;*

ii) *there exist $x_j \in \{x_1, \ldots, x_n\}$ such that*

$$x_j = M_g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = g^{-1}\left(\frac{1}{n-1}\sum_{\substack{i=1 \\ i \neq j}}^{n} g(x_i)\right).$$

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Corollary

There is not any internal quasi-arithmetic mean for all $(x_1, \ldots, x_n) \in [0,1]^n$.

Theorem

Let $F : [0,1]^n \to [0,1]$ be an OWA operator of dimension n such that $w_j \in [0,1[$ for all $j \in \{1,\ldots,n\}$. F is internal if and only if there exists $x_{(j)} \in \{x_{(1)},\ldots,x_{(n)}\}$ such that

$$x_{(j)} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{w_i}{1-w_j} x_{(i)}.$$

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Corollary

Let $F : [0,1]^n \to [0,1]$ be an OWA operator of dimension n associated with the weighting vector $w = (w_1,\dots,w_n)$. F is internal if and only if there exists $i \in \{1,\dots,n\}$ such that $w_i = 1$ and $w_j = 0$ for every $j \neq i$.

Internal operators and penalty functions

In a recent paper, we have studied the aggregation of color pixels by means of Penalty Functions.
How can we define aggregation functions by means of penalty functions?

- The arithmetic mean and the median are the solution to optimization problems;

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- The arithmetic mean and the median are the solution to optimization problems;
- Suppose we have the input $\mathbf{x} = (x_1, \ldots, x_n)$ and a possible output y;
- We want to measure the disagreement between input and output;
- If some input $x_i \neq y$, we impose a penalty for the disagreement.

Definition

Let $P : [a, b]^{n+1} \to \mathcal{R}$ be a penalty function with the properties

i) $P(\mathbf{x}, y) \geq 0$ for all $\mathbf{x}, y$;

ii) $P(\mathbf{x}, y) = 0$ if all $x_i = y$;

iii) $P(x, y)$ is quasi-convex in y for any $\mathbf{x}$.

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The penalty based function is

$$f(\mathbf{x}) = \arg\min_{y} P(\mathbf{x}, y)$$

if y is the unique minimizer, and $y = \frac{a+b}{2}$ if the set of minimizers is the interval $[a, b]$.

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Theorem

Any averaging aggregation function can be expressed as a penalty based function.

Examples:

- The arithmetic mean is the value that minimizes

$$f(\mathbf{x}) = \arg\min_{y} \sum_{i=1}^{n} (x_i - y)^2$$

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- The median is the value that minimizes

$$f(\mathbf{x}) = \arg\min_y \sum_{i=1}^{n} |x_i - y|$$

Definition

Let $P : [0,1]^{n+1} \to [0,1]$ be a penalty function. We call internal penalty based function F to

$$F(x_1, \ldots, x_n) = \arg\min_{x_i} P(\mathbf{x}, x_i)$$

Theorem

Let $F : [0,1]^n \to [0,1]$ *be an internal penalty based function given by*

$$F(x_1, \ldots, x_n) = \arg\min_{x_i} P(\mathbf{x}, x_i).$$

Then, F *is an internal operator.*

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- The internal version of the median

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- The internal version of the geometric mean

$$f(\mathbf{x}) = \arg\min_{x_j} \sum_{i=1}^{n} (-\log(x_i) + \log(x_j))^2$$

Conclusions

- We have seen the need of using internal aggregation functions in some processes of image processing
- We have studied the conditions under which well known aggregation functions are internal or not
- This study has led us to study a way of constructiong new internal aggregation functions
- We have given a way of constructing these functions by minimizing special penalty functions