An analysis of the parameters of Bilevel Linear Programming Problems

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Mathematical model of Optimal Location Problem

$$\sum_{i \in I} d_i U_i \longrightarrow min$$

$$T \longrightarrow min$$

$$\sum_{r \in R} \sum_{j \in J} \sum_{l \in L} tr_{rjl} Z_{rjl} + \sum_{j \in J} op_j V_j \longrightarrow min$$

$$\begin{cases} \sum_{j \in N_i} V_j + U_i \geq 1 & \forall i \in I \\ \sum_{j \in J} V_j = w & \\ pt_j V_j \leq T & \forall j \in J \\ \sum_{r \in R} Z_{rjl} = a_{jl} V_j & \forall j \in J, \ \forall l \in L \\ V_j \in \{0, 1\} & \forall j \in J \\ U_i \in \{0, 1\} & \forall i \in I \\ T \geq 0 & \\ Z_{rjl} \geq 0 & \forall r \in R, \ \forall j \in J, \quad \forall l \in L \end{cases}$$

Multi-objective Linear Programming Problem (MOLP)

$$y_0(x) = c_{01}x_1 + c_{02}x_2 + ... + c_{0k}x_k \longrightarrow \min$$

$$y_1(x) = c_{11}x_1 + c_{12}x_2 + ... + c_{1k}x_k \longrightarrow \min$$

$$\dots$$

$$y_n(x) = c_{n1}x_1 + c_{n2}x_2 + ... + c_{nk}x_k \longrightarrow \min$$

$$D : \begin{cases} a_{j1}x_1 + a_{j2}x_2 + ... + a_{jk}x_k \leq b_j, \ j = 1, ..., m, \\ x_l \geq 0, \ l = 1, ..., k. \end{cases}$$

We suppose that D is non-empty bounded set.

Bilevel Linear Programming Problem (BLPP)

P^U – upper level problem,
$P^L = (P_1^L, P_2^L, ..., P_n^L)$ – lower level problems:

$$P^U : \quad y_0(x) = c_{01}x_1 + c_{02}x_2 + ... + c_{0k}x_k \longrightarrow \min$$

$$P_1^L : \quad y_1(x) = c_{11}x_1 + c_{12}x_2 + ... + c_{1k}x_k \longrightarrow \min$$

$$\dots$$

$$P_n^L : \quad y_n(x) = c_{n1}x_1 + c_{n2}x_2 + ... + c_{nk}x_k \longrightarrow \min$$

$$D : \begin{cases} a_{j1}x_1 + a_{j2}x_2 + ... + a_{jk}x_k \le b_j, \ j = 1, ..., m, \\ x_l \ge 0, \ l = 1, ..., k. \end{cases}$$

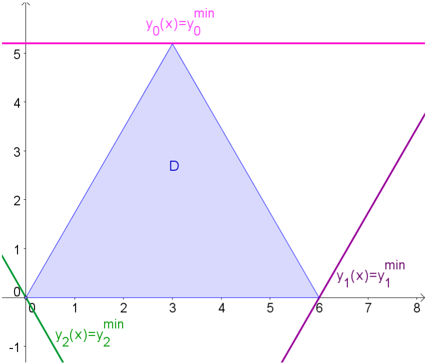
Example (Part I)

$$y_0(x) = -x_2 \longrightarrow \min$$

$$y_1(x) = -3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

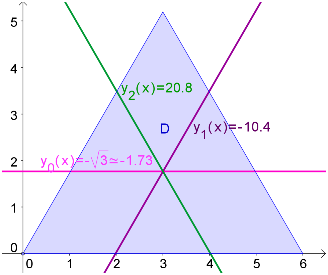
$$y_2(x) = 3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$D: \begin{cases} 3\sqrt{3}x_1 + 3x_2 \leq 18\sqrt{3} \\ -3\sqrt{3}x_1 + 3x_2 \leq 0 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

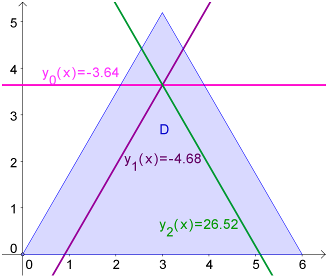


Individual solutions

Example (Part II)



MOLP



BLPP

Fuzzy numbers

Definition

A fuzzy subset of a set X is a map $\mu : X \to [0, 1]$.

Definition

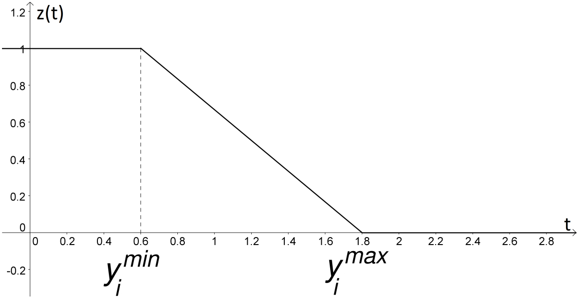
A fuzzy set $z : \Re \to [0, 1]$ is called B. Hutton's fuzzy number, if it satisfies:

- z is monotone;
- $\lim_{t \to \infty} z(t) = 0$;
- $\lim_{t \to -\infty} z(t) = 1$;
- z is a left semi continuous map: $\lim_{t \to t_0 - 0} z(t) = z(t_0)$ for all $t_0 \in \Re$.

Membership functions of objectives

$$y_i^{min} = \min_{x \in D} y_i(x), \quad y_i^{max} = \max_{x \in D} y_i(x), \quad i = 0, \ldots, n.$$

$$z_i(t) = \begin{cases} 1, & t < y_i^{min}, \\ \dfrac{t - y_i^{max}}{y_i^{min} - y_i^{max}}, & y_i^{min} \le t \le y_i^{max}, \\ 0, & t > y_i^{max}. \end{cases}$$



Method for solving MOLP

We denote:

$$\mu_i(x) = z_i(y_i(x)), i = \overline{0, n}.$$

Using the notation above it is reasonable to rewrite the problem:

$$\min_{i \in \{0,\dots,n\}} \mu_i(x) \longrightarrow \max_{x \in D},$$

which can be reduced to the linear programming problem (H.J. Zimmermann, 1978):

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_0(x) \geq \sigma \\ \dots \\ \mu_n(x) \geq \sigma \\ x \in D \end{cases}$$

Let us denote by (x^*, σ^*) the optimal solution.

Pareto optimal solution

Definition

x^* is said to be a **Pareto optimal solution** if and only if there does not exist another $x \in D$ such that $y_i(x) \leq y_i(x^*)$ for all i and $y_j(x) \neq y_j(x^*)$ for at least one j.

It means that any improvement of one objective function by changing decisions can be achieved only at the expense of at least one of the other objective functions.

Proposition

An optimal solution x^* of the problem is a Pareto optimal solution if it is an unique optimal solution.

Parameters of BLPP solving algorithm

Parameters $\delta, \Delta_L, \Delta_U$ are introduced to describe criteria of the solution (x^{**}, σ^{**}) for BLPP (M. Sakawa, I. Nishizaki, 2002):

1. $\mu_0(x^{**}) \geq \delta$;
2. $\Delta_L \leq \Delta = \dfrac{\min\{\mu_1(x^{**}), ..., \mu_n(x^{**})\}}{\mu_0(x^{**})} \leq \Delta_U$.

$$\min_{i=\overline{1,n}} \mu_i(x) \longrightarrow \max_{x \in D,\ \mu_0(x) \geq \delta}$$

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_0(x) \geq \delta \\ \mu_1(x) \geq \sigma \\ \dots \\ \mu_n(x) \geq \sigma \\ x \in D \end{cases}$$

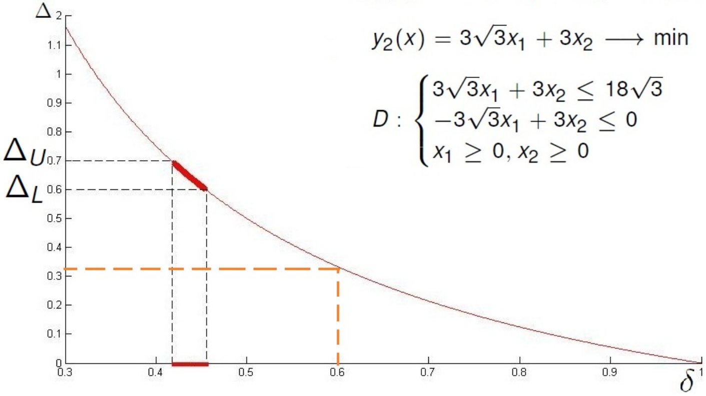
Analysis of parameters of BLPP solving algorithm

$$y_0(x) = -x_2 \longrightarrow \min$$

$$y_1(x) = -3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$y_2(x) = 3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$D : \begin{cases} 3\sqrt{3}x_1 + 3x_2 \leq 18\sqrt{3} \\ -3\sqrt{3}x_1 + 3x_2 \leq 0 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$



Aggregation operator designed for solving BLPP

To study in details the parameters of BLPP solving algorithm, a special aggregation has been constructed. The aggregation observes objective functions on the lower level considering the classes of equivalence generated by a function on the upper level.

$$A_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \max_{\mu_0(x)=\mu_0(u)} \min(\mu_1(u), \mu_2(u), ..., \mu_n(u)),$$

where

$$x, u \in D,$$

$$\mu_0, \mu_1, ..., \mu_n : D \to [0, 1] \text{ are fuzzy subsets of } D.$$

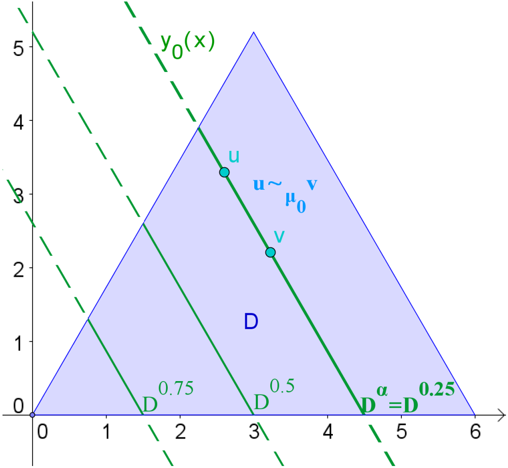
Factoraggregation

Equivalence relation generated by μ_0:

$$u \sim_{\mu_0} v \iff \mu_0(u) = \mu_0(v).$$

Relation $\sim_{\mu_0}$ factorizes D into the classes of equivalence:

$$D^{\alpha} = \{x \in D | \mu_0(x) = \alpha\}.$$



General aggregation operator

General aggregation operator is denoted: $\tilde{A}(\mu_1, \mu_2, ..., \mu_n)$, where $\mu_1, \mu_2, ..., \mu_n \in [0, 1]^D$ are fuzzy subsets: $\mu_1, ..., \mu_n : D \to [0, 1]$.

Definition

(A. Takaci, 2003) A mapping $\tilde{A}([0, 1]^D)^n \to [0, 1]^D$ is called a general aggregation operator if the following conditions hold:

($\tilde{A}$1) $\tilde{A}(\tilde{0}, \ldots, \tilde{0}) = \tilde{0}$;

($\tilde{A}$2) $\tilde{A}(\tilde{1}, \ldots, \tilde{1}) = \tilde{1}$;

($\tilde{A}$3) $\forall \mu_1, \mu_2, ..., \mu_n, \eta_1, \eta_2, ..., \eta_n \in [0, 1]^D$:
$\{\mu_i \preceq \eta_i, i = \overline{1, n}\} \Longrightarrow \{\tilde{A}(\mu_1, \ldots, \mu_n) \preceq \tilde{A}(\eta_1, \ldots, \eta_n)\}$.

Here $\tilde{0}$, $\tilde{1}$ are indicators of $\varnothing$ and X respectively.

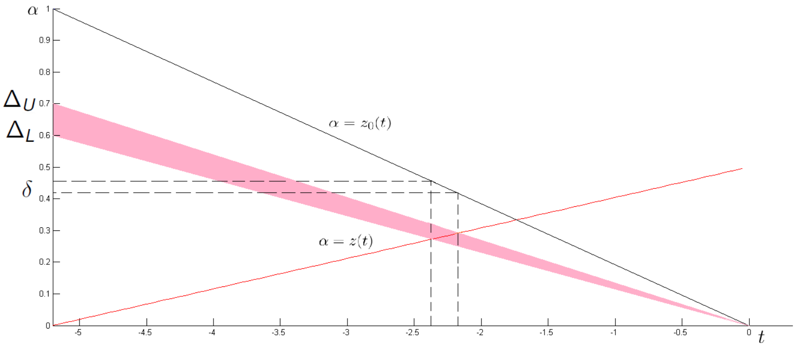
Denoting

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \mu(x),$$

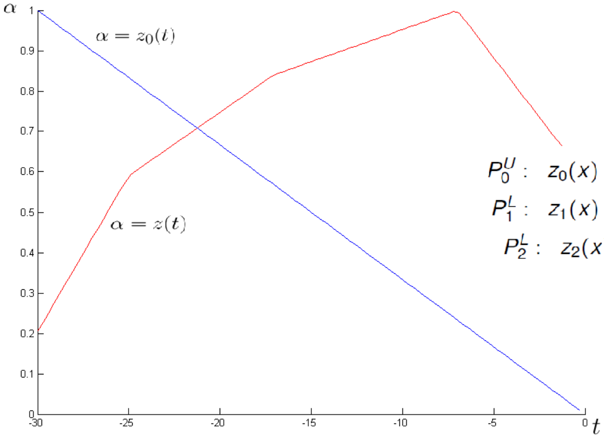
$$z(y_0(x)) = \mu(x),$$

$$t = y_0(x),$$

we consider the figures $\alpha = z_0(t)$ *and* $\alpha = z(t)$:



Analysis of parameters of BLPP



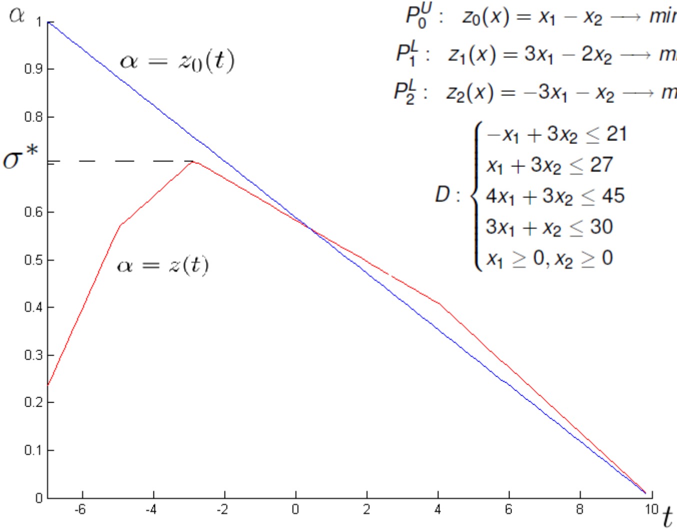
$$P_0^U: \quad z_0(x) = -3x_1 - x_2 \longrightarrow min$$

$$P_1^L: \quad z_1(x) = 3x_1 - 2x_2 \longrightarrow min$$

$$P_2^L: \quad z_2(x) = x_1 - x_2 \longrightarrow min$$

$$D: \begin{cases} -x_1 + 3x_2 \le 21 \\ x_1 + 3x_2 \le 27 \\ 4x_1 + 3x_2 \le 45 \\ 3x_1 + x_2 \le 30 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

Analysis of parameters of BLPP



$P_0^U: \quad z_0(x) = x_1 - x_2 \longrightarrow min$

$P_1^L: \quad z_1(x) = 3x_1 - 2x_2 \longrightarrow min$

$P_2^L: \quad z_2(x) = -3x_1 - x_2 \longrightarrow min$

$$D: \begin{cases} -x_1 + 3x_2 \leq 21 \\ x_1 + 3x_2 \leq 27 \\ 4x_1 + 3x_2 \leq 45 \\ 3x_1 + x_2 \leq 30 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

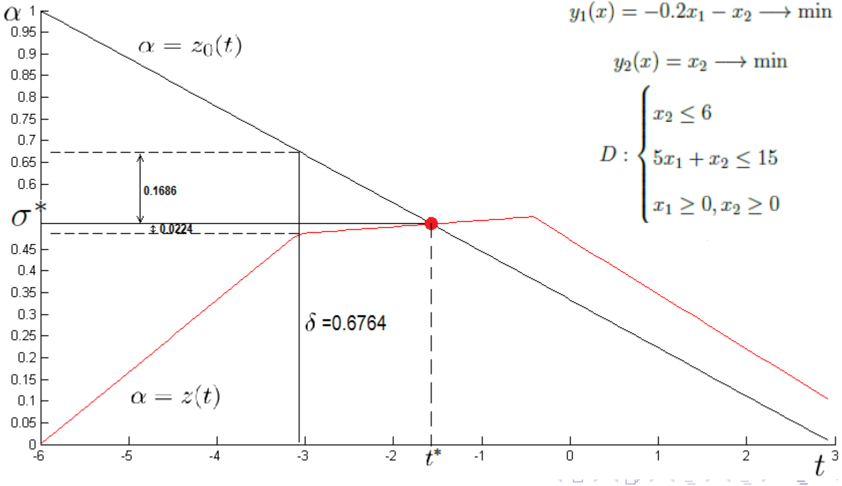
Properties of function $z(t)$ obtained by the factoraggregation

- $\sigma^* = min\{z(t^*),\ z_0(t^*)\},\ where\ t^* = y_0(x^*).$
- $\max\limits_{t \in [y_0^{min},\ t^*]} z(t) = z(t^*).$
- Function z is monotone on interval $[y_0^{min},\ t^*]$:

$$\forall t^1,\ t^2 \in [y_0^{min},\ t^*] : t^1 < t^2 \Longrightarrow z(t^1) \leq z(t^2).$$

- Function z is convex on interval $[y_0^{min}, t^*]$.

Analysis of parameters of BLPP



$$y_0(x) = x_1 - x_2 \longrightarrow \min$$

$$y_1(x) = -0.2x_1 - x_2 \longrightarrow \min$$

$$y_2(x) = x_2 \longrightarrow \min$$

$$D: \begin{cases} x_2 \leq 6 \\ 5x_1 + x_2 \leq 15 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Mathematical model of Optimal Location Problem

$$P^U: \quad \sum_{i\in I} d_i U_i \longrightarrow min$$

$$P_1^L: \quad T \longrightarrow min$$

$$P_2^L: \quad \sum_{r\in R}\sum_{j\in J}\sum_{l\in L} tr_{rjl} Z_{rjl} + \sum_{j\in J} op_j V_j \longrightarrow min$$

$$\begin{cases} \sum_{j\in N_i} V_j + U_i \geq 1 & \forall i \in I \\ \sum_{j\in J} V_j = w & \\ pt_j V_j \leq T & \forall j \in J \\ \sum_{r\in R} Z_{rjl} = a_{jl} V_j & \forall j \in J,\ \forall l \in L \\ V_j \in \{0,1\} & \forall j \in J \\ U_i \in \{0,1\} & \forall i \in I \\ T \geq 0 & \\ Z_{rjl} \geq 0 & \forall r \in R,\ \forall j \in J,\quad \forall l \in L \end{cases}$$

Numerical example data

I=J=	1	2	3	4	5	6	7	8	9	10
R=	1	2	3	4	5	6				
w=	2									
D=	550	950	750	580	882	695	550	500	800	700
A=	150	150	200	250	100	150	150	100	150	250
PT=	14	36	28	3	34	32	32	26	18	27
OP=	550	500	546	851	550	559	850	565	800	900
S=	1.751772	1.470759	1.553435	2.145118	1.69329	2.476794	1.378829	1.905565	1.384255	1.608837

Numerical example data

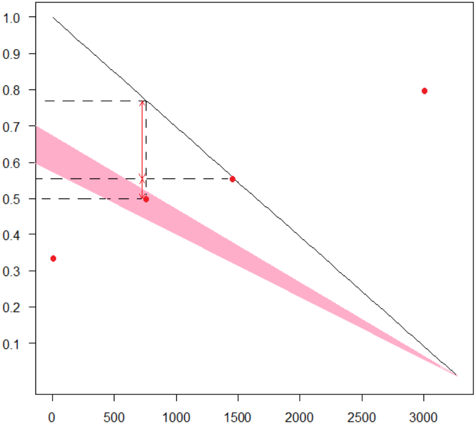
DT=

0	1.2944348	1.3465438	0.7079721	2.065761	0.5749347	2.038723	0.7757845	2.072512	2.1349263
2.1568812	0	0.8114536	0.6235684	1.630472	1.3049564	1.6685582	1.388409	2.020143	1.8865061
1.8233544	1.431371	0	1.3942606	1.472506	1.4226399	0.9525999	2.4531135	1.597618	2.0918023
2.110518	0.9763891	2.462184	0	2.422386	1.5915896	0.5176596	1.1948526	1.484914	1.4315101
1.0411216	2.2498542	0.9209467	2.2831905	0	0.6268405	1.9892725	1.3227986	1.324131	2.4132678
0.7322807	1.4504811	2.3340591	1.3803333	1.981185	0	1.9266907	0.6291684	1.219984	2.4775985
0.6719005	2.3651323	0.6466632	1.7435931	2.351354	1.4030028	0	1.4847138	1.610082	2.4547413
1.6957376	2.110361	2.2529788	2.3173106	2.136109	1.9152288	1.808521	0	2.048829	2.1360651
1.7812104	0.8904434	0.691735	1.4291347	1.234546	0.605377	1.2717766	0.8658483	0	0.8131684
1.9768812	1.2667223	2.3130064	1.4875493	1.133137	0.7390524	1.8422769	2.3184107	1.388832	0

TR=

23.97388	37.76728	33.25608	18.65846	11.13965	36.53304	25.96655	36.37172	16.43025	31.01209
28.00979	29.4132	33.28828	25.00592	24.61955	23.11904	34.31485	29.28181	27.29868	28.26903
17.39327	18.75648	19.91189	31.52574	18.18374	34.96356	25.46274	31.65748	49.17829	27.06982
25.36817	30.97285	20.48736	19.32844	25.79524	17.45482	25.98798	29.29086	45.76908	24.08064
24.2461	20.1284	29.02453	11.3684	18.36425	28.00202	17.96694	13.19316	27.58653	32.42413
31.67597	10.08487	20.48307	14.50817	19.23274	15.279	38.761	10.06928	34.5559	35.88458

Analysis of numerical example



$$\Delta_L = 0.6,\ \Delta_U = 0.7$$

$$\mu_0(x) = 0.7727,\ \mu_1(x) = 0.5,\ \mu_2(x) = 0.7491,\ \Delta = \frac{0.5}{0.7727} = 0.6471$$

Thank you for your attention!