OSTRAVSKÁ UNIVERZITA

DISERTAČNÍ PRÁCE

2023 VIEC BUI QUOC

OSTRAVSKÁ UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA
KATEDRA MATEMATIKY

Svazové Integrální Transformace A Jejich Aplikace

Disertační Práce

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2023

UNIVERSITY OF OSTRAVA
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

Lattice Integral Transforms and Their Applications

Ph.D. Thesis

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2023

OSTRAVSKÁ UNIVERZITA

Přírodovědecká fakulta

Akademický rok: 2022/2023

ZADÁNÍ DISERTAČNÍ PRÁCE

(projektu, uměleckého díla, uměleckého výkonu)

Jméno a příjmení:	**Viec BUI QUOC**
Osobní číslo:	**P18176**
Studijní program:	**P1103 Aplikovaná matematika**
Studijní obor:	**Aplikovaná matematika a Fuzzy modelování**
Téma práce:	**Svazové integrální transformace a jejich aplikace**
Téma práce anglicky:	**Lattice integral transforms and their applications**
Zadávající katedra:	**Katedra matematiky**

Zásady pro vypracování

Cílem práce je vybudovat teorii integrálních transformací pro funkce ohodnocené v úplných reziduovaných svazech, která zahrnuje svazové fuzzy transformace využívané pro horní a dolní aproximace funkcí. V rámci této teorie budou studovány základní vlastnosti tří typů integrálních transformací definovaných pomocí fuzzy integrálů Sugenova typu pro svazově ohodnocené funkce. Dále pak budou studovány aproximační vlastnosti kompozic vybraných integrálních transformací, které umožňují rekonstruovat původní signál. Nakonec budou zkoumány možnosti využití těchto integrálních transformací a jejich kompozic v řešení praktický úloh ve vícekriteriálním rozhodování a zpracování signálu a obrazu.

Rozsah pracovní zprávy:	
Rozsah grafických prací:	
Forma zpracování disertační práce:	**tištěná/elektronická**
Jazyk zpracování:	**Angličtina**

Seznam doporučené literatury:

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Školitel disertační práce: **Doc. RNDr. Michal Holčapek, Ph.D.**
Centrum excelence IT4Innovations

Datum zadání disertační práce: **1. října 2022**
Termín odevzdání disertační práce: **31. ledna 2023**

Doc. RNDr. Michal Holčapek, Ph.D.
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V Ostravě dne 25. ledna 2023

Abstrakt

Cílem práce je vybudovat teorii integrálních transformací pro funkce ohodnocené v úplných reziduovaných svazech, zkráceně svazových integrálních transformací, která zahrnuje svazové fuzzy transformace využívané pro horní a dolní aproximace funkcí. Integrální transformace jsou zavedeny podobně jako v klasickém případě reálných funkcí, kdy se integruje součin mezi funkcí a integrálním jádrem mezi vhodnými mezemi. V představené teorii se integruje pomocí fuzzy integrálů, které rozšiřují Sugenův integrál definovaný pro funkce s funkčními hodnotami ležící v úplném reziduovaném svazu, a integrální jádro je ve tvaru speciální binární fuzzy relace. Práce uvádí vybrané vlastnosti svazových integrálních transformací a mimo jiné ukazuje, že svazové fuzzy transformace jsou jejich speciálním případem. Dále jsou prezentovány základní aproximační vlastnosti kompozic svazových integrálních transformací, které rekonstruují původní funkce. Teoretické výsledky jsou ilustrovány a diskutovány na příkladu zpracování signálu včetně odstranění náhodného šumu. Vedle teoretických poznatků jsou v práci uvedeny dvě aplikace svazových integrálních transformací, konkrétně ve vícekriteriálním rozhodování a zpracování obrazu. Ve druhém případě jsou svazové integrální transformace použity k zavedení několika nových typů filtrů odtraňující šum typu sůl a pepř a rozšiřují tak mediánový filtr, dále pak je ukázána možnost komprese a dekomprese obrazu a v neposlední řadě je představeno zobecnění fuzzy morfologických operatorů eroze a dilatace.

Klíčová slova: Integrální transformace, Fuzzy transformace, Residuovaný svaz, Fuzzy integrál, Vícekriteriální rozhodování, Zpracování obrazu.

Abstract

The aim of the thesis is to develop a theory of integral transforms for complete residuated lattice-valued functions, lattice integral transforms for short, which includes lattice fuzzy transforms that are used for upper and lower approximations of functions. The integral transforms are introduced similarly as in the classical case of real functions, when the product of a function and an integral kernel is integrated between suitable limits. In the present theory, the integration is given by fuzzy integrals that extend the Sugeno integral for functions with function values in a complete residuated lattice and the integral kernel has the form of a binary fuzzy relation. The thesis presents selected properties of lattice integral transforms and shows, among other things, that lattice fuzzy transforms are special cases of them. The basic approximation properties of compositions of lattice integral transforms that reconstruct the original functions are also given. In addition to the theoretical findings, the thesis presents two applications of lattice integral transforms, namely, in multi-criteria decision making and image processing. In the second case, lattice integral transforms are used to introduce several new types of filters that filter out salt-and-pepper noise and thus extend the median filter, the possibility of compression and decompression of the image is shown, and finally a generalization of the fuzzy morphological operators of erosion and dilation is presented.

Keywords: Integral transforms, Fuzzy transforms, Residuated lattice, Fuzzy integral, Multicriteria decision making, Image processing.

Já, níže podepsaný student, tímto čestně prohlašuji, že text mnou odevzdané závěrečné práce v písemné podobě i na CD nosiči je totožný s textem závěrečné práce vloženým v databázi DIPL2.

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V Ostravě dne 30. 1. 2023

.....
podpis

First of all, I would like to express my deep gratitude to doc. RNDr. MICHAL HOLČAPEK, Ph.D., who is my supervisor at the university, who has guided me well and provided me with a lot of valuable support during my studies, and without his help this thesis could not have been completed. In addition, he is a good friend in my life who always listens to me, shares with me, understands and helps me in various ways. I look forward to learning a lot from him and contributing more in the future.

Secondly, I would like to sincerely thank my dear family: my wife and my daughter. Although we had many difficulties during my studies when we were separated, I always had their support. They are a strong motivation in my life.

Last but not least, I would like to thank all my colleagues at the Institute for Research and Applications of Fuzzy Modelling for all their help.

OSTRAVA, 30. 1. 2023 VIEC BUI QUOC

Contents

Foreword

Integral transforms are mathematical operators that produce a new function $g(y)$ by integrating the product of a given function $f(x)$ and an integral kernel function $K(x, y)$ between suitable limits. An integral kernel function forms a link between the domains of functions $f(x)$ and $g(y)$. The Fourier and Laplace transforms belong among the most popular integral transforms and are applied for real or complex valued functions. Integral transforms are very useful in solving practical problems from different areas of science and engineering as solving (partial) differential equations, signal and image processing, spectral analysis of stochastic processes (see, e.g., [8, 43, 48]).

In fuzzy set theory, we usually deal with functions whose function values belong to an appropriate algebra of truth values as a residuated lattice and its special variants as the BL-algebra, MV-algebra, IMTL-algebra (see, e.g., [6, 35, 16]). For this type of (residuated) lattice-valued functions, we can recognize a type of "integral" transforms which are hidden under the name lattice-valued upper and lower fuzzy transforms (lattice fuzzy transforms for short). These lattice fuzzy transforms were proposed by Perfilieva in [36] and further developed in several papers [44, 33, 30, 31, 34, 32, 37]. It is known that the key concept for lattice fuzzy transforms is the fuzzy partition of the domain of transformed functions, which is a system of fuzzy subsets defined on the domain that generalizes the classical (set) partition of the domain in a natural way (see, e.g., [36]). Using the fuzzy partition the direct and inverse lower and upper fuzzy transforms are introduced whose particular composition can be used to approximate the original functions from below and above. The quality of the approximation is then controlled by the setting of the fuzzy partition.

The analysis of the formulas introducing lattice fuzzy transforms leads to the interesting observation that both the direct and inverse transforms can be expressed as integral transforms using Sugeno-like fuzzy integrals introduced in [12] and fuzzy relations as integral kernels. More precisely, the fuzzy integrals used to interpret the direct and inverse lattice fuzzy transforms are defined using the least and highest fuzzy measure on the respective measurable space. This observation raises the very natural question of whether it is also possible, in the spirit of standard integral transforms, to develop a theory of integral transforms for functions valued in complete residuated lattices that works with different types of fuzzy integrals for these functions and with more general fuzzy measures.

The goal of this thesis is to provide an affirmative answer to this question and to introduce a theory of integral transforms for functions valued in complete residuated lattices, which we will call lattice integral transforms for short, and to show

that the provided theory can be used to solve practical problems. To achieve this goal we consider fuzzy and complementary fuzzy measure spaces and three types of Sugeno-like fuzzy integrals, namely, one that uses multiplication in its definition [12] and two others that are defined using the residuum [10, 11]. These fuzzy integrals are then used to introduce three types of lattice integral transforms whose properties are studied. Among the most important properties belong the preservation (reverseration) of constant functions which turns out to be essential for approximating functions. As noted above, compositions of direct and inverse lattice fuzzy transforms lead to upper and lower approximations of the original functions, and the natural question was whether a similar property would hold for compositions of the corresponding lattice integral transforms. To solve this problem, we introduce inverse kernels, which we use to show several approximation theorems for compositions of lattice integral transforms that, among other things, also estimate the quality of the approximation. Interestingly, in addition to constant functions, as a consequence of these theorems, it can be shown that compositions of lattice integral transforms preserve extensional functions with respect to specific fuzzy relations that slightly generalize the similarity relation. To show that the proposed theory can be applied in practice, we use lattice integral transforms to solve a multi-criteria decision problem and also in image processing, where we introduce new types of filters for noise reduction, a compression/decompression method, and generalize the fuzzy morphological operators of erosion and dilation and the related operators of opening and closing. The work is supplemented with examples that illustrate the newly introduced concepts and their properties.

The thesis is formally divided into seven chapters. The organization of the chapters may be briefly summarized as follows. We also add the author's published works related to the content of individual chapters.

Chapter 1 is a preliminary chapter devoted to the basic notions and properties of the truth values algebras and fuzzy set theory, which are used in the thesis.

In Chapter 2, we recall basic notions from the theory of fuzzy measure spaces and three types of Sugeno-like fuzzy integrals for functions with function values in a complete residuated lattice (lattice-valued functions for short), namely, one fuzzy integral based on the multiplication operation and two fuzzy integrals based on the residuum operation introduced in [10, 11, 12]. Furthermore, we analyze the measurability of lattice-valued functions and provide some other properties of fuzzy integrals.

In Chapter 3, we introduce the concept of integral kernel, which generalizes the fuzzy partition used in lattice fuzzy transforms, and three types of integral transforms based on the above mentioned fuzzy integrals. We present their basic properties, including the property of preservation (reversation) of constant functions, which is necessary for successful reconstruction of the original function. The theory is demonstrated on signal processing. The content of this chapter was partially published in [25, 24].

In Chapter 4, we analyze the approximation properties of compositions of respective lattice integral transforms. For this purpose, we introduce an inverse and dually inverse kernel, whose properties are studied. We show that the composition of lattice integral transforms that use the kernel and its (dual) inverse and preserve

constant functions gives an upper or lower approximation of the "smoothed" original function. This property generalizes the upper and lower approximation properties of the composition of the direct and inverse lattice fuzzy transform. Further, we introduce a modulus of continuity for lattice-valued functions which we use to estimate the quality of the approximation for lattice integral transforms and their compositions. The theory is demonstrated on signal reconstruction without and with present noise. Some of the presented results were published in [27].

Chapter 5 is devoted to the application of lattice integral transforms to multicriteria decision making, where we propose a new approach to the evaluation of alternatives with respect to global criteria and demonstrate it on the selection of a car from several alternatives. The content of this chapter is a part of [22].

In Chapter 6, we present the application of lattice integral transforms in image processing. We show that the lattice integral transforms can be used to filter out salt-and-pepper noise similarly to the median filter, which is a special case of them. We also provide a method for image compression and decompression and generalize the fuzzy morphological operators of dilation and erosion and the derived operators of opening and closing. The content of this chapter was partially published in [23].

The last chapter is a conclusion.

Chapter 1

Preliminaries

This chapter presents the basic concept used in the thesis. The first section is devoted to a brief overview of the algebraic structures in which the function values are interpreted. We chose the complete residuated lattice as the basic algebraic structure because it allows us to model a wide range of lattice integral transforms for the functions evaluated in the lattices. In particular, multiplication and residuum operations are powerful tools for combining lattice integral transforms that lead to a successful reconstruction of functions. This fact has been recognized in the seminal paper on fuzzy transforms [36]. The second section contains a review of concepts from fuzzy set theory. More details can be found in [6, 35] for residuated lattices and [29] for fuzzy sets.

1.1 Algebras of truth values

In this thesis, we deal with functions whose function values belong to a residuated lattice.

Definition 1.1. A *residuated lattice* is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \bot, \top \rangle \tag{1.1}$$

with four binary operations and two constants such that

(i) $\langle L, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice, where $\bot, \top$ denote the least and the greatest elements , respectively,

(ii) $\langle L, \otimes, \top \rangle$ is a commutative monoid, i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$,

(iii) the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{1.2}$$

for any $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering.

A pair $\langle \otimes, \rightarrow \rangle$ of operations is called the *adjoint pair*. The operations $\otimes$ and $\rightarrow$ are called the *multiplication* and *residuum*. A residuated lattice is said to be

complete (*linearly ordered*) if $\langle L, \wedge, \vee, \bot, \top\rangle$ is a complete (linearly ordered) lattice. A residuated lattice is *divisible* if $a \otimes (a \to b) = a \wedge b$ for any $a, b \in L$, and satisfies the *low of double negation* if $(a \to \bot) \to \bot = a$ for any $a \in L$. A residuated lattice is called the *Heyting algebra* if the multiplication is the meet operation of the corresponding lattice, i.e., $a \otimes b = a \wedge b$ for any $a, b \in L$. A divisible residuated lattice satisfying the law of double negation is called the MV–*algebra*.

Before we present examples of complete residuated lattices, we recall the definition of the t-norm (triangular norm) operation.

Definition 1.2. A binary function $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following properties hold for all $a, b, c \in [0,1]$:

(i) associativity, i.e., $T(a, T(b, c)) = T(T(a, b), c)$,

(ii) commutativity, i.e., $T(a, b) = T(b, a)$,

(iii) monotonicity, i.e., $b \leq c \quad \text{implies} \quad T(a, b) \leq T(a, c)$,

(iv) boundary condition, i.e., $T(a, 1) = a$.

Among the important residuated lattices, which are used in many valued logic, belong the residuate lattices defined by the left-continuous t-norm [18]. Recall that a t-norm is left-continuous provided that

$$\lim_{n\to\infty} T(a_n, b) = T(\lim_{n\to\infty} a_n, b)$$

for any non-decreasing sequence $a_1, a_2, \ldots$ in $[0,1]$ (see, [6, 28]).

Example 1.1. Let T be a left-continuous t-norm. Then the algebra

$$\mathbf{L}_T = \langle [0,1], \min, \max, T, \to_T, 0, 1\rangle,$$

where

$$a \to_T b = \bigvee \{c \in [0,1] \mid T(a, c) \leq b\} \tag{1.3}$$

is a complete linearly ordered residuated lattice. The most important examples of complete residuated lattices on $[0,1]$ are obtained from the minimum, product, and Łukasiewicz t-norms:

$$\begin{aligned} T_{\mathrm{G}}(a, b) &= \min(a, b), \\ T_{\mathrm{P}}(a, b) &= a \cdot b, \\ T_{\text{Ł}}(a, b) &= \max(a + b - 1, 0), \end{aligned}$$

respectively. Their residua are as follows

$$\begin{aligned} a \to_{\mathrm{G}} b &= \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise}, \end{cases} \\ a \to_{\mathrm{P}} b &= \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{otherwise}, \end{cases} \\ a \to_{\text{Ł}} b &= \min(1, 1 - a + b). \end{aligned}$$

The complete residuated lattices $\mathbf{L}_{T_{\mathrm{G}}}$, $\mathbf{L}_{T_{\mathrm{P}}}$ and $\mathbf{L}_{T_{\mathrm{Ł}}}$ on $[0,1]$ are called the *Gödel algebra*, *product algebra*, and *Łukasiewicz algebra*, respectively. It should be noted that the Łukasiewicz algebra is a canonical example of the MV–algebra.

The following example presents the Schweizer-Sklar class of t-norms from which complete residuated lattices can be introduced according to Example 1.1. Note that the Schweizer-Sklar class of t-norms provides a broad family of continuous t-norms except one (the drastic product), and therefore is appropriate for the application of lattice integral transforms on complete residuated lattices on $[0,1]$, i.e., in signal or image processing.

Example 1.2. The Schweizer-Sklar class of t-norms is defined for any $a, b \in [0,1]$ and $\lambda \in [-\infty, \infty]$ as

$$T_\lambda^{SS}(a,b) = \begin{cases} \min(a,b), & \lambda = -\infty, \\ (a^\lambda + b^\lambda - 1)^{\frac{1}{\lambda}}, & \lambda \in (-\infty, 0), \\ a \cdot b, & \lambda = 0, \\ (\max(0, (a^\lambda + b^\lambda - 1))^{\frac{1}{\lambda}}, & \lambda \in (0, \infty), \\ a \cdot_D b, & \lambda = \infty, \end{cases} \tag{1.4}$$

where

$$a \cdot_D b = \begin{cases} 0, & a, b \in [0,1), \\ \min(a,b), & 1 \in \{a, b\}. \end{cases}$$

Obviously, $T_{-\infty}^{SS}$, T_0^{SS} and T_1^{SS} are the minimum, product and Łukasiewicz t-norms introduced in Example 1.1, respectively. The t-norm T_∞^F is called the drastic product. For $\lambda \in [-\infty, \infty)$, the t-norm T_λ^{SS} is continuous, and the drastic product is only right-continuous, i.e., $\lim_{n\to\infty} T(a_n, b) = T(\lim_{n\to\infty} a_n, b)$ for any non-increasing sequence $\{a_n \in [0,1] \mid n = 1, 2, 3, ...\}$ (see, [28]). According to Example 1.1, we can determine the residuum for the Schweizer-Sklar t-norms (except $\lambda = \infty$) as follows. Let $a, b \in [0,1]$. For $a \leq b$, there is $a \to_{T_\lambda^{SS}} b = 1$, and for $b < a$, there is

$$a \to_{T_\lambda^{SS}} b = \begin{cases} b, & \lambda = -\infty, \\ \frac{b}{a}, & \lambda = 0, \\ (1 - a^\lambda + b^\lambda)^{\frac{1}{\lambda}}, & \lambda \in (-\infty, 0) \cup (0, \infty), \end{cases} \tag{1.5}$$

where we use $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, $\lambda + \infty = \infty + \lambda = \infty$ for any $\lambda \in (-\infty, \infty)$, and $\infty^\lambda = \infty$ for any $\lambda \in (0, \infty)$. Note that $a \to_{T_\lambda^{SS}} b = \min(1, (1 - a^\lambda + b^\lambda)^{\frac{1}{\lambda}})$ for any $\lambda \in (0, \infty)$ and $a, b \in [0,1]$.

Example 1.3. Let $a, b \in [0, \infty]$ be such that $a < b$. The algebra

$$\mathbf{L}_{[a,b]} = \langle [a,b], \min, \max, \min, \to, a, b \rangle,$$

where

$$c \to d = \begin{cases} b, & \text{if } c \leq d, \\ d, & \text{otherwise}, \end{cases} \tag{1.6}$$

is a complete residuated lattice. Note that $\mathbf{L}_{[a,b]}$ is a Heyting algebra.

On residuated lattices, we can introduce additional operations using the basic ones. An example of such operations are a unary operation of $\neg : L \to L$ called the *negation* and a binary operation $\leftrightarrow : L^2 \to L$ called the *biresiduum* given by

$$\neg a = a \to \bot, \tag{1.7}$$
$$a \leftrightarrow b = (a \to b) \wedge (b \to a). \tag{1.8}$$

for any $a, b \in L$. The following example presents the negation for residuated lattices from Example 1.1.

Example 1.4. The negation in the Łukasiewicz, Gödel, and product algebras are given as

$$\neg a = 1 - a, \qquad \text{(Łukasiewicz algebra)} \tag{1.9}$$
$$\neg a = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{if } a > 0. \end{cases} \qquad \text{(Gödel and product algebra)} \tag{1.10}$$

Note that the presented negations are well-established in fuzzy logic (see, [18, 35]).

A generalization of the concept of negation on L is as follows (see, [3]).

Definition 1.3. A unary operation $N : L \to L$ is called a *generalized negation* (*negation* for short) on L if N is a non-increasing map such that $N(\bot) = \top$ and $N(\top) = \bot$.

A canonical example of the generalized negation is the residuum based negation given as $N_{\text{res}}(a) = \neg a$ for $a \in L$. We say that a generalized negation N is *involutive* if $N(N(a)) = a$ for any $a \in L$.

Example 1.5. For the Schweizer-Sklar class of t-norms with $\lambda \in [-\infty, \infty)$, the canonical negation $N_\lambda^{SS} = N_{res,\lambda}^{SS}$ has the following form. For $a = 0$, there is $N_\lambda^{SS}(0) = 1$, for $a \neq 0$, there is

$$N_\lambda^{SS}(a) = \begin{cases} 0, & \lambda \in [-\infty, 0], \\ (1 - a^\lambda)^{\frac{1}{\lambda}}, & \lambda \in (0, \infty). \end{cases} \tag{1.11}$$

Note that for $\lambda \in (-\infty, 0)$, according to (1.5), we have $(1 - a^\lambda + 0^\lambda)^{\frac{1}{\lambda}} = (1 - (\frac{1}{a})^{-\lambda} + \infty^{-\lambda})^{\frac{1}{\lambda}} = \infty^{\frac{1}{\lambda}} = 0^{-\frac{1}{\lambda}} = 0$. Obviously, for $\lambda \in (0, \infty)$, it is sufficient to consider only one formula to express the negation, namely, $N_\lambda^{SS}(a) = (1 - a^\lambda)^{\frac{1}{\lambda}}$ for $a \in [0, 1]$. The negation N_λ^{SS} is continuous, and as a particular case, we obtain the negation N_1^{SS} in the Łukasiewicz algebra $L_{T_1^{SS}} = L_{T_Ł}$.

A generalized negation can be obtain, for example, as $N_\lambda^{gSS}(a) = N_\lambda^{SS}(a) \otimes N_\lambda^{SS}(a)$ for any $a \in L$.

The following example presents the biresiduum for residuated lattices from Example 1.1.

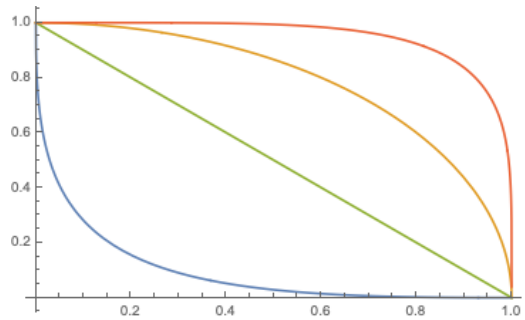


Figure 1.1: The negations N_λ^{SS} for $\lambda = 0.4$ (blue), $\lambda = 1$ (green), $\lambda = 2$ (yellow), and $\lambda = 5$ (orange) from Example 1.5.

Example 1.6. The biresiduum in the Łukasiewicz, Gödel, and product algebras are given as

$$a \leftrightarrow b = 1 - |a - b|, \qquad \text{(Łukasiewicz algebra)} \tag{1.12}$$

$$a \leftrightarrow b = \min(a, b), \qquad \text{(Gödel algebra)} \tag{1.13}$$

$$a \leftrightarrow b = \min\left(\frac{a}{b}, \frac{b}{a}\right), \qquad \text{(product algebra)} \tag{1.14}$$

where we put $\frac{a}{0} = 1$ for any $a \in [0, 1]$.

The following theorems summarize some basic properties of the residuated lattice and the distributivity of $\otimes$, $\to$ over $\wedge$, $\vee$ that are used in the thesis.

Theorem 1.1. *Let* $\mathbf{L}$ *be a residuated lattice. Then the following statements hold for every* $a, b, c, d \in L$*:*

(i) $a \otimes (a \to b) \leq b, \quad a \leq a \to (a \otimes b), \quad a \leq (a \to b) \to b,$

(ii) $a \leq b$ *iff* $a \to b = \top,$

(iii) $a \to a = \top, \quad a \to \top = \top, \bot \to a = \top,$

(iv) $a \otimes \bot = \bot, \quad \top \to a = a,$

(v) $a \otimes b \leq a \wedge b,$

(vi) $(a \otimes b) \to c = a \to (b \to c),$

(vii) $(a \to b) \otimes (b \to c) \leq a \to c,$

(viii) $b \otimes (a \to c) \leq a \to (b \otimes c),$

(ix) $a \to c \leq (a \otimes b) \to (c \otimes b),$

(x) $(a \to b) \otimes (c \to d) \leq (a \otimes c) \to (b \otimes d),$

(xi) $a \to (b \to c) = b \to (a \to c),$

(xii) $a \otimes \neg b \leq \neg(a \to b),$

(xiii) $a \to b \leq \neg b \to \neg a$,

(xiv) $a \leq \neg(\neg a)$.

Proof. The proof of statements (i)–(vii) can be found in [6]. For (viii), using (i) and the commutativity and associativity of $\otimes$, we get $a \otimes (b \otimes (a \to c)) = b \otimes (a \otimes (a \to c)) \leq b \otimes c$. The statement is a consequence of the adjointness property. For (ix), using (viii), the associativity of $\otimes$ and the adjointness property, we find that $(a \otimes b) \otimes (a \to c) \leq b \otimes c$, therefore, $a \to c \leq (a \otimes b) \to (b \otimes c)$. For (x), using (i), the commutativity and the associativity of $\otimes$ and the adjointness property, we have $(a \otimes c) \otimes (a \to b) \otimes (c \to d) = (a \otimes (a \to b)) \otimes (c \otimes (c \to d)) \leq b \otimes c$. The statement is a consequence of the adjointness property. The statement (xi) is a straightforward consequence of (vi) and the commutativity of $\otimes$. For (xii), using (i) and the adjointness property, we have $a \otimes \neg b \otimes (a \to b) \leq \neg b \otimes b = b \otimes (b \to \bot) \leq \bot$, therefore, $a \otimes \neg b \leq (a \to b) \to \bot = \neg(a \to b)$. The statement (xiii) follows from $\neg b \otimes (a \to b) = (a \to b) \otimes (b \to \bot) \leq a \to \bot$, where we used (vii). Hence, using the adjointness property, we get the desired inequality. The last statement follows from $a \otimes (a \to \bot) \leq \bot$ due to (i), therefore, $a \leq (a \to \bot) \to \bot = \neg(\neg a)$ by the adjointness property. □

Theorem 1.2. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $a \in L$ *and* $\{b_i \mid i \in I\}$ *is a set of elements from* L *over a non-empty index set* I*. Then it holds that*

(i) $a \otimes (\bigvee_{i \in I} b_i) = \bigvee_{i \in I}(a \otimes b_i)$,

(ii) $a \to \bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \to b_i)$,

(iii) $(\bigvee_{i \in I} b_i) \to a = \bigwedge_{i \in I}(b_i \to a)$,

(iv) $a \otimes \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I}(a \otimes b_i)$,

(v) $\bigvee_{i \in I}(a \to b_i) \leq a \to \bigvee_{i \in I} b_i$,

(vi) $\bigvee_{i \in I}(b_i \to a) \leq \bigwedge_{i \in I} b_i \to a$.

If $\mathbf{L}$ *is an* MV*–algebra, then the inequalities (iv)-(vi) may be replaced by the equalities and it holds that*

(vii) $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$,

(viii) $a \vee \bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \vee b_i)$.

Proof. See, [6]. □

The following theorem lists the basic properties of biresiduum used in our work.

Theorem 1.3. *Let* $\mathbf{L}$ *be a residuated lattice, and let* $a, b, c, d \in L$*. Then it holds that*

$$a \leftrightarrow a = \top, \tag{1.15}$$
$$a \leftrightarrow b = b \leftrightarrow a, \tag{1.16}$$
$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq a \leftrightarrow c, \tag{1.17}$$
$$(a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \otimes c) \leftrightarrow (b \otimes d), \tag{1.18}$$
$$(a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \to c) \leftrightarrow (b \to d), \tag{1.19}$$
$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d), \tag{1.20}$$
$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \vee c) \leftrightarrow (b \vee d). \tag{1.21}$$

Moreover, let $\mathbf{L}$ *be a complete residuated lattice. Then the following items hold for arbitrary sets* $\{a_i \mid i \in I\}$*,* $\{b_i \mid i \in I\}$ *of elements from* L *over an arbitrary set of indices* I*:*

$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq (\bigwedge_{i \in I} a_i) \leftrightarrow (\bigwedge_{i \in I} b_i), \tag{1.22}$$
$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq (\bigvee_{i \in I} a_i) \leftrightarrow (\bigvee_{i \in I} b_i). \tag{1.23}$$

Proof. See, [6]. □

1.2 Fuzzy sets

We assume that $\mathbf{L}$ is a fixed complete residuated lattice.

Definition 1.4. Let X be a non-empty set. A fuzzy set in X is a function

$$A : X \to L. \tag{1.24}$$

The set X is called a *universe of discourse* (universe for short). The function A is called the *membership function* of the fuzzy set A and the value $A(x)$ for $x \in X$ is called the *membership degree of* x *in* A . The set of all fuzzy sets in X is denoted by $\mathcal{F}(X)$. Let $A \in \mathcal{F}(X)$. The fuzzy set A is called the *singleton* if there is exactly one $x \in X$ whose membership degree is greater that $\perp$, i.e., $A(x) > \perp$ for some $x \in X$, and $A(x) = \perp$, otherwise, and the *constant fuzzy set* if there is $a \in L$ such that $A(x) = a$ for any $x \in X$. Such constant fuzzy set in X is also denote as $\underline{a}_X$. The fuzzy set A is said to be *empty* if $A(x) = \perp$ for any $x \in X$. The empty fuzzy set is denoted as $\emptyset$. The fuzzy set A is said to be *crisp* if $A(x) \in \{\perp, \top\}$. Obviously, the membership function of a crisp fuzzy set in X is nothing but the characteristic function of a subset of X. The characteristic function of a subset A of X is denoted as 1_A. The set of all crisp fuzzy sets (i.e., subsets) of X is denoted as $\mathcal{P}(X)$. The following definition presents three important sets determined from a fuzzy set.

Definition 1.5. Let $A \in \mathcal{F}(X)$.

(i) The *support* of A is a subset of X whose elements have the membership degree greater than $\perp$, i.e.,

$$\mathrm{Supp}(A) = \{x \in X \mid A(x) > \perp\}. \tag{1.25}$$

(ii) The *core* of A is a subset of X whose elements have the membership degree equal to $\top$, i.e.,

$$\mathrm{Core}(A) = \{x \in X \mid A(x) = \top\}. \tag{1.26}$$

(iii) The *a-cut* of A is a subset of X whose elements have the membership degree greater than a, i.e.,

$$A_a = \{x \in X \mid A(x) \geq a\}. \tag{1.27}$$

A fuzzy set A in X is said to be *normal* if $\mathrm{Core}(A) \neq \top$. Residuated lattice operations can be used to introduce operations between fuzzy sets. In the following definition, we recall the elementary operations for fuzzy sets.

Definition 1.6. Let $A, B \in \mathcal{F}(X)$, $\{A_i \mid i \in I\} \subseteq \mathcal{F}(X)$ and $x \in X$. Then

$$(A \cap B)(x) = A(x) \wedge B(x), \tag{1.28}$$
$$(A \cup B)(x) = A(x) \vee B(x), \tag{1.29}$$
$$(A \otimes B)(x) = A(x) \otimes B(x), \tag{1.30}$$
$$(A \to B)(x) = A(x) \to B(x), \tag{1.31}$$
$$(\bigcap_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x), \tag{1.32}$$
$$(\bigcup_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x), \tag{1.33}$$
$$(X \backslash A)(x) = \neg A(x). \tag{1.34}$$

Let $A, B \in \mathcal{F}(X)$. We say that a fuzzy set A is *less than or equal to* B and denote it as $A \leq B$ if $A(x) \leq B(x)$ for any $x \in X$.

Let X, Y be non-empty universes. A fuzzy set $K : X \times Y \to L$ is called a (binary) *fuzzy relation*. The *transpose* of a fuzzy relation K is a fuzzy relation $K^T : Y \times X \to L$ given by $K^T(y, x) = K(x, y)$ for any $(y, x) \in Y \times X$. For $X = Y$, we say that K is a fuzzy relation on X. A special fuzzy relation on X is a similarity, which is defined as follows.

Definition 1.7. A fuzzy relation on X is called the *similarity* if it satisfies the following properties for any $x, y, z \in X$:

1. $K(x, x) = \top$ (reflexivity),
2. $K(x, y) = K(y, x)$ (symmetry),
3. $K(x, y) \otimes K(y, z) \leq K(x, z)$ (transitivity).

For a fuzzy relation $K : X \times Y \to L$ and $x \in X$, a fuzzy set $K_x : Y \to L$ given as $K_x(y) = K(x, y)$ for $y \in Y$ is called the *x–projection* of K to Y. Similarly a *y–projection* of K to X for $y \in Y$ is given as $K_y(x) = K(x, y)$ for $x \in X$. A fuzzy relation K is said to be *normal*, whenever $\mathrm{Core}(K) \neq \emptyset$, *normal in the first coordinate*, whenever $\mathrm{Core}(K_x) \neq \emptyset$ for any $x \in X$, and similarly *normal in the second coordinate*, whenever $\mathrm{Core}(K_y) \neq \emptyset$ for any $y \in Y$.

Chapter 2

Fuzzy measure spaces and fuzzy integrals

In this chapter, we introduce basic concepts from fuzzy measure theory and three types of fuzzy integrals for functions whose function values belong to a complete residuated lattice, which extends in some way the well-known Sugeno fuzzy integral introduced in [41]. The first two sections are devoted to fuzzy measure spaces and measurable functions, where we study the conditions under which the operations of a residuated lattice preserve the measurability of functions. The third section presents Sugeno-like fuzzy integrals based on the multiplication operation, which were introduced by Dvořák and Holčapek in [11] and Dubois, Prade and Rico in [10], and two types of Sugeno-like fuzzy integrals based on the residuum operation in a complete residuated lattice. The first type of these residuum-based fuzzy integrals was introduced by Dvořák and Holčapek in [11] to model natural language quantifiers as "no", "little" or "few", and a second type, also called qualitative desintegral, was proposed by Dubois, Prade and Rico in [10] for reasoning in the case of a decreasing evaluation scale for data (i.e., the least value is the best evaluation) while the global evaluation has the standard ordering. A comparison of all three types of fuzzy integrals can be found in [24] and we will not present them here. In addition to basic definitions, we present several new results for these fuzzy integrals, which are used in the next part.

Throughout this chapter, we assume that the complete residuated lattice $\mathbf{L}$ is given, and we will not mention it explicitly except when we want to specify its form. To respect the notation used in measure and integral theory, we prefer $f, g, \ldots$ to denote the fuzzy sets $A, B, \ldots$ that will be integrated by a fuzzy integral.

2.1 Fuzzy measure spaces

In this work, we deal with fuzzy measures on algebras of sets. It should be noted that fuzzy measures can also be introduced for algebras of fuzzy subsets (see, [12]), but the computation of fuzzy integrals that use this type of fuzzy measure is rather difficult and impractical, especially if the computation is repeated many times and the result has to be obtained in real time (e.g. filtering or image compression). The complement of a subset A in X is denoted as $X \setminus A$, i.e., in the same way as the complement of a fuzzy set.

Definition 2.1. Let X be a non-empty set. A subset $\mathcal{F}$ of $\mathcal{P}(X)$ is an *algebra of sets on* X provided that

(i) $X \in \mathcal{F}$,

(ii) if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$,

(iii) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

A pair $\langle X, \mathcal{F} \rangle$ is called a *measurable space* (on X) if $\mathcal{F}$ is an algebra of sets on X. The sets from $\mathcal{F}$ are called *$\mathcal{F}$–measurable.* As a simple consequence of (i), (ii) of the previous definition and De Morgan's law, we find that the intersection of a finite number of $\mathcal{F}$–measurable sets is again $\mathcal{F}$–measurable.

Let $\langle X, \mathcal{F} \rangle$ be a measurable space and $A \in \mathcal{P}(X)$. We say that a set A is *$\mathcal{F}$–meaurable* if $A \in \mathcal{F}$. If there can be no confusion, we use *measurable* for short. It is easy to see that a finite intersection of measurable sets is a measurable set. A useful tool to introduce an algebra of sets on X is an algebra generated by a family of sets.

Definition 2.2. Let $\mathcal{H} \subseteq \mathcal{P}(X)$ be a non-empty family of sets. The smallest algebra on X containing $\mathcal{H}$ is denoted by $\mathrm{Alg}(\mathcal{H})$ and is called the *generated algebra by $\mathcal{H}$.*

Note that the intersection of algebras of sets is again an algebra of sets, and therefore the smallest algebra of sets on X containing $\mathcal{H}$ always exists and its unique. Moreover, the generated algebra $\mathrm{Alg}(\mathcal{H})$ can be easily constructed from the elements $\mathcal{H}$ as the set which consists of all finite unions applied on the set of all finite intersections over the elements of $\mathcal{H}$ and their complements. We now present some examples of algebras of sets.

Example 2.1. The sets $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are trivial algebras of fuzzy sets on X.

Example 2.2. Let τ_X be a topology on X. The algebra of sets $\mathrm{Alg}(\tau_X)$ is a generated algebra by a topology τ_X on X. Note that continuity and measurability (see the next section) are interrelated in algebras generated from topologies.

Example 2.3. Let $\langle L, \leq \rangle$ be a partially ordered set, and let $u : \mathcal{P}(L) \to \mathcal{P}(L)$ be given as

$$u(S) = \{x \in L \mid \exists\, a \in S,\, a \leq x\}, \tag{2.1}$$

for any $S \in \mathcal{P}(L)$. Obviously, $S \subseteq u(S)$. A set $S \in \mathcal{P}(L)$ such that $u(S) = S$ is called the *upper set* or *upset* for short. The set of all upsets in L is denoted by $\mathcal{U}(L)$. Trivially, we have $\emptyset, S \in \mathcal{U}(L)$. Moreover, it is easy to see that the intersection and the union of a non-empty family of upsets is an upset. Indeed, let us show that the claim is true for the intersection; analogously, the claim can be verified for the union. Let $\{S_i\}_{i \in I} \subseteq \mathcal{U}(L)$ and put $T = \bigcap_{i \in I} S_i$. If $T = \emptyset$, the claim is trivially true. In addition, we show that $u(T) \subseteq T$. Let $x \in u(T)$. Then, there exists $a \in T$ such that $a \leq x$. Since $a \in S_i$ for any $i \in I$ and S_i is an upset, we find that $x \in S_i$ for any $i \in I$, and therefore $x \in T$. Since the opposite inclusion is trivially true from the definition of u, we obtain $u(T) = T$ and $T \in \mathcal{U}(L)$. Obviously, $\mathcal{U}(L)$ is an example of Alexander topology. The algebra of sets generated by all upsets in L is denoted as $\mathcal{B}^u(L)$ or simply $\mathcal{B}^u$ if there can be no confusion, i.e., $\mathcal{B}^u(L) = \mathrm{Alg}(\mathcal{U}(L))$.

Example 2.4. Similarly to the previous example, let $\ell : \mathcal{P}(L) \to \mathcal{P}(L)$ be defined as

$$\ell(S) = \{x \in L \mid \exists a \in S,\, x \leq a\}, \tag{2.2}$$

for any $S \in \mathcal{P}(L)$. A set $S \in \mathcal{P}(L)$ for which $\ell(S) = S$ holds is called the *lower set* or *loset* for short. The set of all losets in L is denoted $\mathcal{L}(L)$ and the algebra of sets generated by all losets in L is denoted as $\mathcal{B}^\ell$ or simply $\mathcal{B}^\ell$ if there can be no confusion, i.e., $\mathcal{B}^\ell(L) = \mathrm{Alg}(\mathcal{L}(L))$.

Example 2.5. Let $\mathbf{L}$ be a residuated lattice on $[0,1]$, and $\mathcal{H} = \{[0,a] \mid a \in [0,1]\}$, where $[0,0] = \{0\}$ is a hybrid interval. Obviously, $\mathrm{Alg}(\mathcal{H})$ is a proper subset of the powerset of $[0,1]$. For example, there is $[a,1] \notin \mathrm{Alg}(\mathcal{H})$ for any $a \in (0,1]$.

In the following part, we introduce three types of set functions, namely, fuzzy measure, complementary fuzzy measure and conjugate fuzzy measure, with values in a complete residuated lattice. In addition to the well-known fuzzy measure, the notion of complementary fuzzy measure was introduced in [11] to define a residuum-based integral for modelling fuzzy quantifiers, and the conjugate fuzzy measure to a given fuzzy measure was proposed in [10] to define a residuum-based integral (qualitative desintegral) for decreasing local evaluation scales.

Definition 2.3. A function $\mu : \mathcal{F} \to L$ is called a *fuzzy measure* on a measurable space $\langle X, \mathcal{F}\rangle$ if

(i) $\mu(\emptyset) = \bot$ and $\mu(X) = \top$,

(ii) if $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet $\langle X, \mathcal{F}, \mu\rangle$ is called a *fuzzy measure space* whenever $\langle X, \mathcal{F}\rangle$ is a measurable space and μ is a fuzzy measure on $\langle X, \mathcal{F}\rangle$.

It should be noted that the term "fuzzy measure" was introduced by Sugeno in [41], but in the literature one can find the equivalent names for μ as a *capacity* or a *non-additive measure*. The following lemma shows an easy way to determine another fuzzy measure from a given fuzzy measure using a transformation function on $[0,1]$.

Lemma 2.1. *Let μ be a fuzzy measure on $\langle X, \mathcal{F}\rangle$, and let $\varphi : L \to L$ be a monotonically non-decreasing function with $\varphi(\bot) = \bot$ and $\varphi(\top) = \top$. Then the function $\mu_\varphi : \mathcal{F} \to L$ given by $\mu_\varphi(A) = \varphi(\mu(A))$ for any $A \in \mathcal{F}$ is a fuzzy measure on $\langle X, \mathcal{F}\rangle$.*

Proof. Obvious. □

Definition 2.4. A map $\nu : \mathcal{F} \to L$ is called a *complementary fuzzy measure* on a measurable space $\langle X, \mathcal{F}\rangle$ if

(i) $\nu(\emptyset) = \top$ and $\nu(X) = \bot$,

(ii) if $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\nu(A) \geq \nu(B)$.

A triplet $\langle X, \mathcal{F}, \nu \rangle$ is called the *complementary fuzzy measure space* whenever $\langle X, \mathcal{F} \rangle$ is a measurable space and ν is a complementary fuzzy measure on $\langle X, \mathcal{F} \rangle$.

The following lemma shows two ways in which a complementary fuzzy measure can be introduced from a fuzzy measure.

Lemma 2.2. *Let μ be a fuzzy measure on $\langle X, \mathcal{F} \rangle$, and let N be a generalized negation on L. Then a function $\nu : \mathcal{F} \to L$ given by $\nu(A) = \mu^c(A) = \mu(X \setminus A)$ or $\nu(A) = \mu^N(A) = N(\mu(A))$ for any $A \in \mathcal{F}$ is the complementary fuzzy measure.*

Proof. Obvious. □

A dual statement can be formulated for fuzzy measures determined from complementary fuzzy measures. Using Lemmas 2.1 and 2.2, we can introduce a broad class of fuzzy and complementary fuzzy measures. Finally, a conjugate fuzzy measure to a given fuzzy measure is introduced as follows.

Definition 2.5. Let μ be a fuzzy measure on $\langle X, \mathcal{F} \rangle$, and let N be a generalized negation on L. A function $\mu^{c,N} : \mathcal{F} \to L$ given by $\mu^{c,N}(A) = N \circ \mu^c(A) = \mu^N(X \setminus A) = N(\mu(X \setminus A))$ for any $A \in \mathcal{F}$ is called the *N–conjugate fuzzy measure to μ*.

Clearly, the N–conjugated fuzzy measure to μ is again a fuzzy measure, but constructed using two specific operations, namely generalized negation and set complement. For the negation N_1^{SS} in the Łukasiewicz algebra (see, Example 1.5), we obtain the definition of the conjugate fuzzy measure in [10]. An *N–conjugate complementary fuzzy measure* to a complementary fuzzy measure ν is defined analogously as $\nu^{c,N}(A) = N \circ \nu^c(A) = \nu^N(X \setminus A) = N(\nu(X \setminus A))$ for any $A \in \mathcal{F}$.

Since in our illustration of integral transforms we work with a finite number of data, we restrict our presentation of examples to (complementary, N–conjugate) fuzzy measures defined over measurable spaces with $X = \{x_1, \ldots, x_n\}$.

Example 2.6. Let $\langle X, \mathcal{F} \rangle$ be a finite measurable space. For two fuzzy measures μ_1, μ_2 on $\langle X, \mathcal{F} \rangle$, we say that μ_1 *is less than or equal to* μ_2 (denoted as $\mu_1 \preceq \mu_2$) if $\mu_1(A) \leq \mu_2(A)$ for any $A \in \mathcal{F}$. The *least* and the *highest* fuzzy measure on $\langle X, \mathcal{F} \rangle$ with respect to $\preceq$ is given by

$$\mu^{\perp}(A) = \begin{cases} \perp, & A \neq X, \\ \top, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \mu^{\top}(A) = \begin{cases} \perp, & A = \emptyset, \\ \top, & \text{otherwise}, \end{cases} \tag{2.3}$$

for any $A \in \mathcal{F}$, respectively.

Assume that $\mathbf{L}$ is a residuated lattice on $[0, 1]$. The following fuzzy measure belongs among the fundamental fuzzy measures.

Example 2.7. The *relative* fuzzy measure μ^r on $\langle X, \mathcal{F} \rangle$ is given as

$$\mu^r(A) = \frac{\#A}{\#X}, \tag{2.4}$$

for all $A \in \mathcal{F}$, where $\#A$ and $\#X$ denote the number of elements in A and X, respectively.

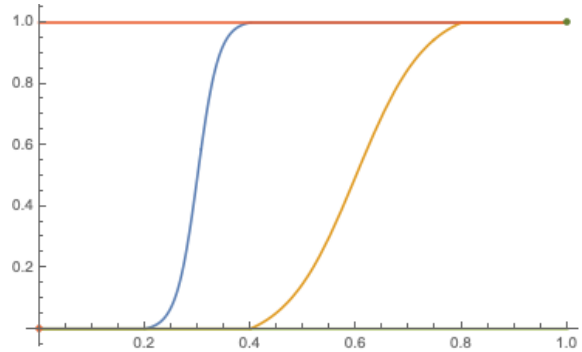


Figure 2.1: The functions $\varphi^1_{0,0}$ (orange), $\varphi^5_{0.2,0.4}$ (blue), $\varphi^3_{0.4,0.8}$ (yellow), and $\varphi^1_{1,1}$ (green) from Example 2.8 that are used to determine fuzzy measures.

By Lemma 2.1, the relative fuzzy measure μ^r can be modified by a non-decreasing function φ with $\varphi(0) = 0$ and $\varphi(1) = 1$ to get a fuzzy measure μ^r_φ. In the following example, we introduce a class of functions φ that determines a class of fuzzy measures on $L = [0, 1]$ from the relative fuzzy measure.

Example 2.8. Let $0 \leq \ell \leq u \leq 1$ and $0 < p$ be a natural number. Define $\psi^p_{\ell,u}, \varphi^p_{\ell,u} : [0, 1] \to [0, 1]$ as follows:

$$\psi^p_{\ell,u}(a) = \begin{cases} 0, & a = 0 \text{ or } a < \ell, \\ \frac{e^{p\left(2\frac{a-\ell}{u-\ell}-1\right)}}{e^{p\left(2\frac{a-\ell}{u-\ell}-1\right)}+1}, & \ell < a \leq u, \\ 1, & a = 1 \text{ or } u < a, \end{cases} \tag{2.5}$$

and

$$\varphi^p_{l,u}(a) = \begin{cases} \frac{\psi^p_{l,u}(a)(e^p+1)-1}{e^p-1}, & l < a \leq u, \\ \psi^p_{l,u}(a), & \text{otherwise.} \end{cases} \tag{2.6}$$

It could be simply verified that, for $\ell < u$, $\varphi^p_{\ell,u}(a)$ modifies $\psi^p_{\ell,u}(a)$ to obtain a continuous function on $[0, 1]$. For $\ell = u$, however, $\varphi^p_{\ell,u}(a)$ achieves only two values 0 and 1 with the jump at the point $a = \ell$. For example, if $\ell = u = 0$, then $\varphi^p_{0,0}(a) = \psi^p_{0,0}(0) = 0$ and $\varphi^p_{0,0}(a) = \psi^p_{0,0}(a) = 1$ for $a > 0$. Examples of the function $\varphi^p_{\ell,u}$ for the parameters $(\ell, u, p) \in \{(0, 0, 1), (0.2, 0.4, 5), (0.4, 0.8, 3), (1, 1, 1)\}$ are shown in Figure 2.1. The function $\varphi^p_{\ell,u}$ obviously satisfies the assumptions of Lemma 2.1, so it can be used to modify any fuzzy measure. Hence, we can introduce a class of fuzzy measures on $\langle X, \mathcal{F}\rangle$ derived from the relative fuzzy measure μ^r introduced in Example 2.7 as follows:

$$\mathcal{M}^r = \{\mu^r_{\varphi^p_{\ell,u}} \mid \ell, u \in [0, 1],\ \ell \leq u,\ p \in \mathbb{N},\ p > 0\}. \tag{2.7}$$

It is easy to see that $\mu^\perp = \mu^r_{\varphi^1_{1,1}}$, $\mu^\top = \mu^r_{\varphi^1_{0,0}}$ and $\mu^r = \mu^r_{\varphi^1_{0,1}}$.

Example 2.9. By $\varphi^p_{\ell,u}$ from the previous example, we can introduce two additional functions using which the complementary and conjugate fuzzy measures can be

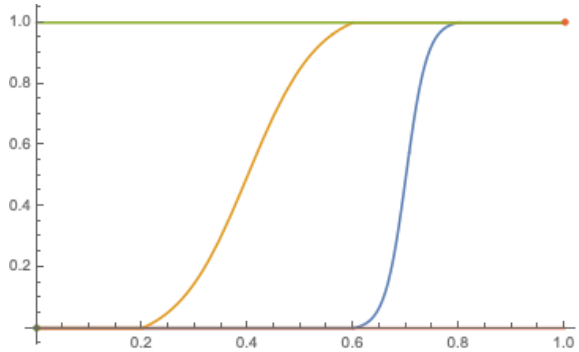


Figure 2.2: The functions $\varphi^{1,c,N}_{0,0}$ (orange), $\varphi^{5,c,N}_{0.2,0.4}$ (blue), $\varphi^{3,c,N}_{0.4,0.8}$ (yellow), and $\varphi^{1,c,N}_{1,1}$ (green) from Example 2.8 that are used to determine conjugate fuzzy measures.

determined from the relative fuzzy measure. Let N be a negation on $[0,1]$ and define $\varphi^{p,c}_{\ell,u}, \varphi^{p,N}_{\ell,u} : [0,1] \to [0,1]$ as follows

$$\varphi^{p,c}_{\ell,u}(a) = \varphi^{p}_{\ell,u}(1-a) \quad \text{and} \quad \varphi^{p,N}_{\ell,u}(a) = N(\varphi^{p}_{\ell,u}(a)), \tag{2.8}$$

for any $a \in [0,1]$. By Lemma 2.2, it is easy to check that $\mu^r_{\varphi^{p,c}_{\ell,u}}$ and $\mu^r_{\varphi^{p,N}_{\ell,u}}$ are complementary fuzzy measures on $\langle X, \mathcal{F}\rangle$. Define $\varphi^{p,c,N}_{\ell,u} = N \circ \varphi^{p,c}_{\ell,u}$, then $\mu^r_{\varphi^{p,c,N}_{\ell,u}}$ is the N–conjugate fuzzy measure to the fuzzy measure $\mu^r_{\varphi^{p}_{\ell,u}}$ on $\langle X, \mathcal{F}\rangle$. Indeed, put $\mu = \mu^r_{\varphi^{p}_{\ell,u}}$. Then, for any $A \in \mathcal{F}$, we have

$$\mu^{c,N}(A) = N(\mu(X \setminus A) = N(\varphi^{p}_{\ell,u}(\mu^r(X \setminus A))) = N(\varphi^{p}_{\ell,u}(1-\mu^r(A))) = \\ N(\varphi^{p,c}_{\ell,u}(\mu^r(A)) = \varphi^{p,c,N}_{\ell,u}(\mu^r(A)) = \mu^r_{\varphi^{p,c,N}_{\ell,u}}(A).$$

In Figure 2.2, we show the functions $\varphi^{p,c,N}_{\ell,u}$ for the same parameters as in Example 2.8 and $N(a) = 1-a$ for $a \in [0,1]$ (i.e., the negation in the Łukasiewicz algebra). It is easy to see that the conjugate fuzzy measure to $\mu^{\perp}$ ($\mu^{\top}$) is $\mu^{\top}$ ($\mu^{\perp}$); it is sufficient to compare green (orange) functions in Figure 2.1 and Figure 2.2. Fuzzy measures $\mu^r_{\varphi^{p}_{0,1}}$ are self-conjugate, which follows immediately from $\varphi^{p}_{0,1} = \varphi^{p,c,N}_{0,1}$.

Remark 2.1. One can see that all fuzzy measures (and similarly complementary and conjugate fuzzy measures) in the above examples are invariant with respect to the cardinality of sets, i.e., $\mu(A) = \mu(B)$, whenever A and B has the same number of elements, i.e., $\#A = \#B$. Such fuzzy measures are referred to *symmetric fuzzy measures*.

Remark 2.2 (Notation). Let $n > 0$ and $p > 0$ be natural numbers, $L, U \in [0,n]$ be real numbers such that $L \leq U$, and let $\langle X, \mathcal{F}\rangle$ be a finite measurable space with $X = \{x_1, \ldots, x_n\}$. A fuzzy measure on $\langle X, \mathcal{F}\rangle$ determined by the triplet $\langle L, U, p\rangle$ is denoted by $\mu^p_{L,U}$ and defined as

$$\mu^p_{L,U} = \mu^r_{\varphi^p_{L/n,U/n}}. \tag{2.9}$$

Obviously, $\mu^p_{L,U} \in \mathcal{M}^r$, where $\mathcal{M}^r$ is the class of fuzzy measure spaces introduced in Example 2.8. Similarly, we denote and define other fuzzy measures, namely,

$$\mu^{p,c}_{L,U} = \mu^r_{\varphi^{p,c}_{L/n,U/n}} \text{ and } \mu^{p,N}_{L,U} = \mu^r_{\varphi^{p,N}_{L/n,U/n}} \text{ and } \mu^{p,c,N}_{L,U} = \mu^r_{\varphi^{p,c,N}_{L/n,U/n}},$$

where $\mu^{p,c,N}_{L,U}$ is the N–conjugate fuzzy measure to $\mu^p_{L,U}$.

2.2 Measurable functions

An important concept in fuzzy measure theory is the measurability of functions. As we will see later, in general the definitions of Sugeno-like fuzzy integrals do not assume measurability of the functions being integrated, but under this assumption the original definition can be expressed in a more convenient way.

Definition 2.6. Let $\langle X, \mathcal{F}\rangle$ and $\langle Y, \mathcal{G}\rangle$ be measurable spaces, and let $f : X \to Y$ be a function. We say that f is $\mathcal{F}$-$\mathcal{G}$–measurable if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$.

In the case that $\mathcal{G}$ is a generated algebra, the verification of $\mathcal{F}$-$\mathcal{G}$-measurability of the function can be simplified as follows.

Lemma 2.3. *Let $\mathcal{H} \subseteq \mathcal{P}(Y)$ be a non-empty family of sets, and let $\langle X, \mathcal{F}\rangle$ be a measurable space. A function $f : X \to Y$ is $\mathcal{F}$-Alg$(\mathcal{G})$–measurable if and only if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$.*

Proof. ($\Rightarrow$) The implication is a simple consequence of $\mathcal{G} \subseteq \mathrm{Alg}(\mathcal{G})$.

($\Leftarrow$) Let $\mathcal{Q} = \{Z \mid f^{-1}(Z) \in \mathcal{F}\}$. Note that $\mathcal{Q}$ is called the preimage algebra on Y and $\mathcal{G} \subseteq \mathcal{Q}$. From the definition of the generated algebra Alg$(\mathcal{G})$ by the family $\mathcal{G}$, we find that $\mathrm{Alg}(\mathcal{G}) \subseteq \mathcal{Q}$. Hence, we obtain that $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathrm{Alg}(\mathcal{G})$, which means that f is $\mathcal{F}$-Alg$(\mathcal{G})$–measurable. □

In the following, we discuss the conditions under which the operations of a residuated lattice extended to fuzzy sets (see, Definition 1.6) preserve their measurability. More specifically, we show some conditions under which if f and g are $\mathcal{F}$-$\mathcal{B}^u$–measurable, then $f \star g$ is also $\mathcal{F}$-$\mathcal{B}^u$–measurable, where $\star = \{\vee, \wedge, \otimes, \to\}$ and $\mathcal{B}^u = \mathrm{Alg}(\mathcal{U}(L))$ is the generated algebra by all upsets in the support L of a residuated lattice $\mathbf{L}$ (see, Example 2.3).

Theorem 2.4. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space, and let $\mathcal{A} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-$\mathcal{B}^u$–measurable fuzzy sets. If $\mathbf{L}$ is linearly ordered, then*

$$f \wedge g, f \vee g \in \mathcal{A}, \quad f, g \in \mathcal{A}.$$

Proof. Since the proofs for both operations are analogous, here we verify only the case of $\wedge$. By Lemma 2.3, we have to prove that for any $f, g \in \mathcal{A}$ and $Y \in \mathcal{U}(L)$, we obtain $(f \wedge g)^{-1}(Y) \in \mathcal{F}$. Put $h = f \wedge g$. We show that $h^{-1}(Y) = f^{-1}(Y) \cap g^{-1}(Y)$. Let $x \in h^{-1}(Y)$. Then $h(x) \in Y$. Since $f(x) \geq h(x)$ and $g(x) \geq h(x)$ and $h(x) \in Y$, we find that $f(x), g(x) \in Y$. Hence, we obtain $x \in f^{-1}(Y)$ and simultaneously $x \in g^{-1}(Y)$; therefore, $x \in f^{-1}(Y) \cap g^{-1}(Y)$, and thus $h^{-1}(Y) \subseteq f^{-1}(Y) \cap g^{-1}(Y)$. Conversely, let $x \in f^{-1}(Y) \cap g^{-1}(Y)$. Then $f(x) \in Y$ and $g(x) \in Y$. Since $\mathbf{L}$ is

linearly ordered, we find that $h(x) = f(x)$ or $h(x) = g(x)$; therefore, $h(x) \in Y$. Hence, we obtain $f^{-1}(Y) \cap g^{-1}(Y) \subseteq h^{-1}(Y)$, and the equality is proved. Since $f^{-1}(Y), g^{-1}(Y) \in \mathcal{F}$, we find that $h^{-1}(Y) = f^{-1}(Y) \cap g^{-1}(Y) \in \mathcal{F}$. □

The previous result for non-linear residual lattices does not hold in general, but can be obtained under a different assumption. Moreover, this assumption is also sufficient for the multiplication operation.

Theorem 2.5. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space, and let $\mathcal{A} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-$\mathcal{B}^u$–measurable fuzzy sets. If $\mathcal{F}$ is closed over arbitrary unions, then*

$$f \wedge g, f \vee g, f \otimes g \in \mathcal{A}, \quad f, g \in \mathcal{A}.$$

Proof. Here we prove only the case of $\otimes$, since the proofs of $\wedge$ and $\vee$ are a complete analogy to the proof of $\otimes$.

Let $f, g \in \mathcal{A}$. We show that $f \otimes g \in \mathcal{A}$. Following the proof of Theorem 2.4, we have to prove that for any $Y \in \mathcal{U}(L)$, we obtain $(f \otimes g)^{-1}(Y) \in \mathcal{F}$. Put $h = f \otimes g$. We know that $x \in h^{-1}(Y)$ if and only if $h(x) \in Y$. Put $U_x = [f(x), \top]$ and $V_x = [g(x), \top]$. Obviously, $U_x, V_x \in \mathcal{U}(L)$ and $U_x \subseteq Y$ and $V_x \subseteq Y$. Moreover, $a \otimes b \in Y$ for any $a \in U_x$ and $b \in V_x$, that is, $U_x \otimes V_x = \{a \otimes b \mid a \in U_x, b \in V_x\} \subseteq Y$. Indeed, we have $a \otimes b \geq f(x) \otimes g(x) = h(x)$ and $h(x) \in Y$. Since f and g are $\mathcal{F}$-$\mathcal{B}^u$-measurable, we find that $f^{-1}(U_x), g^{-1}(V_x) \in \mathcal{F}$ and thus $f^{-1}(U_x) \cap g^{-1}(V_x) \in \mathcal{F}$ (recall that the intersection of a finite number of $\mathcal{F}$-measurable sets is again $\mathcal{F}$-measurable), where $x \in f^{-1}(U_x) \cap g^{-1}(V_x)$. Moreover, $f^{-1}(U_x) \cap g^{-1}(V_x) \subseteq h^{-1}(Y)$. Indeed, if $y \in f^{-1}(U_x) \cap g^{-1}(V_x)$, then $f(y) \in U_x$ and $g(y) \in V_x$, which implies $h(y) = f(y) \otimes g(y) \in U_x \otimes V_x \subseteq Y$, and thus $y \in h^{-1}(Y)$. Since $x \in f^{-1}(U_x) \cap g^{-1}(V_x)$, we trivially obtain

$$h^{-1}(Y) = \bigcup_{x \in h^{-1}(Y)} f^{-1}(U_x) \cap g^{-1}(V_x).$$

Since $\mathcal{F}$ is closed over arbitrary unions, we find that $h^{-1}(Y) \in \mathcal{F}$. □

The two previous assumptions together are sufficient to ensure that the residuum operation remains measurable, being monotonically non-increasing in its first argument and monotonically non-decreasing in its second argument.

Theorem 2.6. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space, and let $\mathcal{A} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-$\mathcal{B}^u$–measurable fuzzy sets. If $\mathbf{L}$ is linearly ordered and $\mathcal{F}$ is closed over arbitrary unions, then*

$$f \to g \in \mathcal{A}, \quad f, g \in \mathcal{A}.$$

Proof. Let $f, g \in \mathcal{A}$. We show that $f \to g \in \mathcal{A}$. Obviously, $(a, \top] \in \mathcal{U}(L)$ for any $a \in L$. Indeed, if $x \in u((a, \top])$, then there is $b \in (a, \top]$ such that $b \leq x$, which implies $a < b \leq x$ and $x \in (a, \top]$. Since L is linearly ordered, we find that $[\bot, a] = L \setminus (a, \top] \in \mathcal{B}^u$. Similarly to the previous cases, consider $Y \in \mathcal{U}(L)$, and we show that $(f \to g)^{-1}(Y) \in \mathcal{F}$. Put $h = f \to g$, and let $x \in h^{-1}(Y)$. Put $U_x = [\bot, f(x)]$ and $V_x = [g(x), \top]$. By the previous remark, we have $U_x, V_x \in \mathcal{B}^u$.

In addition, $a \to b \in Y$ for any $a \in U_x$ and $b \in V_x$, i.e., $U_x \to V_x = \{a \to b \mid a \in U_x, b \in V_x\} \subseteq Y$. Indeed, recall that $a \to b \leq a' \to b'$ for any $a, a', b, b' \in L$ such that $a' \leq a$ and $b \leq b'$. Hence, for any $a \in U_x$ and $b \in V_x$, we obtain that $h(x) = f(x) \to g(x) \leq a \to b$, which implies $a \to b \in Y$. Analogously to the proof of Theorem 2.5, we have $f^{-1}(V_x), g^{-1}(U_x) \in \mathcal{F}$ and $f^{-1}(V_x) \cap g^{-1}(U_x) \in \mathcal{F}$. Moreover, $f^{-1}(V_x) \cap g^{-1}(U_x) \subseteq f^{-1}(Y)$. Since $x \in f^{-1}(V_x) \cap g^{-1}(U_x)$, we trivially obtain

$$h^{-1}(Y) = \bigcup_{x \in h^{-1}(Y)} f^{-1}(U_x) \cap g^{-1}(V_x).$$

Since $\mathcal{F}$ is closed over arbitrary unions, we find that $h^{-1}(Y) \in \mathcal{F}$. □

Remark 2.3. The previous theorems remain true if the algebra of sets $\mathcal{B}^u$ is replaced by $\mathcal{B}^\ell$ and the $\mathcal{F}$-$\mathcal{B}^\ell$–measurability is considered. Note that $\mathcal{B}^u = \mathcal{B}^\ell$ for a linearly ordered residuated lattice, which is a simple consequence of the fact that $\mathcal{L}(L) = \{L \setminus S \mid S \in \mathcal{U}(L)\}$.

2.3 Multiplication-based fuzzy integral

The multiplication-based fuzzy integral is a direct generalization of Sugeno fuzzy integral introduced in [41] and further developed by many researchers (see, [45] for a review) for integrated functions evaluated in a residuated lattice, where the original meet (infimum) operation is replaced by a more general multiplication operation [10, 11]. The following definition of the fuzzy integral was proposed in [10] and coincides with the definition given in [11] (see also [12]) if the multiplication is distributive over the infimum in a given residual lattice (e.g. if $\mathbf{L}$ is an MV–algebra).

Definition 2.7. Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space, and let $f : X \to L$. The *$\otimes$–fuzzy integral of f on X* is given by

$$\int^{\otimes} f \, d\mu = \bigvee_{A \in \mathcal{F}} \left(\mu(A) \otimes \bigwedge_{x \in A} f(x) \right). \tag{2.10}$$

The next theorem presents the basic properties of the $\otimes$–fuzzy integral. Recall that the characteristic function of a subset Z of X is denoted by 1_Z.

Theorem 2.7. *Let $\underline{a}_X \in \mathcal{F}(X)$ be a constant function. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have*

(i) $\int^{\otimes} f \, d\mu \leq \int^{\otimes} g \, d\mu$ *if* $f \leq g$,

(ii) $\int^{\otimes} \underline{a}_X \, d\mu = a$,

(iii) $a \otimes \int^{\otimes} f \, d\mu \leq \int^{\otimes} \underline{a}_X \otimes f \, d\mu$,

(iv) $\int^{\otimes} \underline{a}_X \to f \, d\mu \leq a \to \int^{\otimes} f \, d\mu$,

(v) $\int^{\otimes} \underline{a}_X \otimes 1_Z \, d\mu = a \otimes \mu(Z)$ *for any* $Z \in \mathcal{F}$.

If $\mathbf{L}$ *is an MV–algebra, then inequality (iii) can be replaced by equality.*

Proof. See [11, 12]. □

One can see and may be surprised that we do not assume the $\mathcal{F}$-$\mathcal{B}^u$–measurability (or $\mathcal{F}$-$\mathcal{B}^\ell$–measurability) of the function f in the formula (2.10). Assuming the measurability of f, we obtain a very convenient formula for calculating the $\otimes$–fuzzy integral (see, [45] for the Sugeno fuzzy integral).

Theorem 2.8. *Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, and let $f : X \to L$ be $\mathcal{F}$-$\mathcal{B}^u$–measurable. Then*

$$\int^{\otimes} f \; d\mu = \bigvee_{a\in L} a \otimes \mu(\{x \in X \mid f(x) \geq a\}). \tag{2.11}$$

Proof. Let $a \in L$ and denote $L_a = \{x \in L \mid x \geq a\}$. Note that $u(\{a\}) = L_a$, where u is introduced in Example 2.3. By the assumption on the $\mathcal{F}$-$\mathcal{B}^u$–measurability of f, we have $f^{-1}(L_a) \in \mathcal{F}$, where $f^{-1}(L_a) = \{x \in X \mid f(x) \geq a\}$. Put $I = \bigvee_{A\in\mathcal{F}} \big(\mu(A) \otimes \bigwedge_{x\in A} f(x)\big)$ and $J = \bigvee_{a\in L} \big(a \otimes \mu(f^{-1}(L_a))\big)$. First, we show that $I \leq J$. Let $\lambda_f : \mathcal{F} \to L$ be a map given by $\lambda_f(A) = \bigwedge_{x\in A} f(x)$. Obviously, $A \subseteq f^{-1}(L_{\lambda_f(A)})$, and thus $\mu(A) \leq \mu(f^{-1}(L_{\lambda_f(A)}))$, where we used the fact that f is $\mathcal{F}$-measurable. Since $\lambda_f(\mathcal{F}) \subseteq L$, we obtain

$$I \leq \bigvee_{A\in\mathcal{F}} \lambda_f(A) \otimes \mu(f^{-1}(L_{\lambda_f(A)})) \leq J.$$

Further, let $\varrho_f : L \to \mathcal{F}$ be given by $\varrho_f(a) = f^{-1}(L_a)$. From the $\mathcal{F}$-$\mathcal{B}^u$–measurability of f, the map ϱ_f is well defined. Obviously, $\bigwedge_{x\in\varrho_f(a)} f(x) \geq a$ for any $a \in L$ and $\varrho_f(L) \subseteq \mathcal{F}$. Then, we obtain

$$J \leq \bigvee_{a\in L} \left(\mu(\varrho_f(a)) \otimes \bigwedge_{x\in\varrho_f(a)} f(x) \right) \leq I.$$

Hence, we obtain $I = J$ which concludes the proof. □

As a corollary, we get a simple computational formula for measurable functions defined on a finite set $X = \{x_1, \ldots, x_n\}$ assuming that the residuated lattice is linearly ordered. Denote $[n] = \{1, \ldots, n\}$ (see, [17, 28] for real functions).

Corollary 2.9. *Let* $\mathbf{L}$ *be linearly ordered, $\langle X, \mathcal{F}, \mu\rangle$ be a finite fuzzy measure space, i.e., $X = \{x_1, \ldots, x_n\}$, and let $f : X \to L$ be $\mathcal{F}$-$\mathcal{B}^u$–measurable. Then*

$$\int^{\otimes} f \; d\mu = \bigvee_{i\in[n]} (f_{\sigma(i)} \otimes \mu_i), \tag{2.12}$$

where σ is a permutation on $[n]$ such that $f_{\sigma(1)} \leq f_{\sigma(2)} \leq \cdots \leq f_{\sigma(n)}$, where $f_{\sigma(i)} = f(x_{\sigma(i)})$ for $i \in [n]$, and $\mu_i = \mu(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\})$.

Proof. It follows immediately from Theorem 2.8, where we restrict the calculation from $a \in L$ to $a \in \{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}$. Indeed, for $a = \bot$ or $f_{\sigma(n)} < a \leq \top$, we trivially

get $a \otimes \mu(\{x \in X \mid f(x) \geq a\}) = \bot$. If $f_{\sigma(i-1)} < a \leq f_{\sigma(i)}$ for $i \in [n]$ (we put $f_{\sigma(0)} = \bot$), then

$$a \otimes \mu(\{x \in X \mid f(x) \geq a\}) = a \otimes \mu_1 \leq f_{\sigma(i)} \otimes \mu_i,$$

where we used the fact that the multiplication is non-decreasing in both variables. Hence, we find that

$$\bigvee_{a \in [0,1]} (a \otimes \mu(\{x \in X \mid f(x) \geq a\})) \leq \bigvee_{i \in [n]} (f_{\sigma(i)} \otimes \mu_i).$$

Since the opposite inequality is trivially true, we obtain the desired equality. □

Remark 2.4. Note that for $\mathcal{F} = \mathcal{P}(X)$, and any function $f : X \to L$ is measurable, hence, formula (2.12) can be applied. Note that the calculation by formula (2.12) can even be simplified for fuzzy symmetric measures, because it is sufficient to define $\mu_i = \mu(\{x_i, \ldots, x_n\})$ for $i \in [n]$.

We now introduce the notion of *comonotonicity*, which is a sufficient condition ensuring the *maxitivity* and *minitivity* of the $\wedge$–fuzzy integral. Note that maxitivity (minitivity) means that the $\wedge$–fuzzy integral preserves the join (meet) of the lattice. Unfortunately, the comonotonicity is not a sufficient condition for the preservation of the join or meet for more general $\otimes$–fuzzy integrals.

Definition 2.8. Let $\langle X, \mathcal{F} \rangle$ be a measurable space. We say that $f, g \in \mathcal{F}$ are *comonotonic* if and only if there is no pair $x_1, x_2 \in X$ such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$.

Lemma 2.10. *Let* $\mathbf{L}$ *be linearly ordered, and let* $f, g \in \mathcal{F}(X)$*. Denote* $C_f = \{C_f(a) \mid a \in L\}$*, where* $C_f(a) = \{x \in X \mid f(x) \geq a\}$*. Then* C_f *is a chain with respect to* $\subseteq$*, and if* f *and* g *are comonotonic, then* $C_{f \odot g}(a) = C_f(a)$ *or* $C_{f \odot g}(a) = C_g(a)$ *for any* $a \in L$*, where* $\odot \in \{\wedge, \vee\}$*.*

Proof. The first statement is trivial. To proof the second statement, we restrict ourselves to the case $\otimes = \wedge$. The second case can be verified analogously.

First, let us show that $C_f(a) \cap C_g(a) = C_{f \wedge g}(a)$ holds for any $a \in L$. Let $x \in C_f(a) \cap C_g(a)$. Then $f(x) \geq a$ and $g(x) \geq a$. Hence, $f(x) \wedge g(x) \geq a$, which implies $x \in C_{f \wedge g}(a)$. Now, let $x \in C_{f \wedge g}(a)$. Since $f(x) \wedge g(x) \geq a$, we immediately get $x \in C_f(a)$ and $x \in C_g(a)$. Hence, $x \in C_f(a) \cap C_g(a)$. Further, we show that $C_{f \wedge g}(a) = C_f(a)$ or $C_{f \wedge g}(a) = C_g(a)$ for any $a \in L$, whenever f and g are comonotonic. Assume that $C_f(a) \not\subset C_g(a)$ and simultaneously $C_g(a) \not\subset C_f(a)$ for some $a \in L$. From $C_f(a) \not\subset C_g(a)$ there exists $x \in C_f(a)$ and $x \notin C_g(a)$, which implies $g(x) < a \leq f(x)$, and similarly, from $C_g(a) \not\subset C_f(a)$ there exists $y \in X$ such that $y \in C_g(a)$ and $y \notin C_f(a)$, which implies $f(y) < a \leq g(y)$, where we used the linearity of L. But this is a contradiction with the comonotonicity of f and g, since there exist $x, y \in X$ with $f(x) < f(y)$ and simultaneously $g(y) < g(x)$. □

The following theorem shows that $\wedge$–fuzzy integral is comonotonically minitive and comonotonically maxitive (see, [17, Theorem 4.44]).

Theorem 2.11. *Let* $\mathbf{L}$ *be linearly ordered complete Heyting algebra, and let* $f, g \in \mathcal{F}(X)$ *be comonotonic maps that are* $\mathcal{F}$-$\mathcal{B}^u$*–measurable. Then*

$$\int^{\wedge} (f \odot g)\, d\mu = \int^{\wedge} f\, d\mu \odot \int^{\wedge} g\, d\mu$$

for $\odot \in \{\wedge, \vee\}$.

Proof. We restrict ourselves to the proof of the case $\odot = \wedge$, the second case can be proved analogously. According to Theorem 2.5, the map $f \wedge g$ is $\mathcal{F}$-$\mathcal{B}^u$–measurable. Hence, we can use formula (2.11) to compute the $\wedge$–fuzzy integral, i.e.,

$$\int^{\wedge} (f \wedge g)\, d\mu = \bigvee_{a \in L} (a \wedge \mu(\{x \in X \mid f(x) \wedge g(x) \geq a\})) = \bigvee_{a \in L} (a \wedge \mu(C_{f \wedge g}(a))),$$

where we used the notation from Lemma 2.10. Since $C_{f \wedge g}(a) = C_f(a) \subseteq C_g(a)$ or $C_{f \wedge g}(a) = C_g(a) \subseteq C_f(a)$ for any $a \in L$, we obtain

$$\mu(C_{f \wedge g}(a)) = \mu(C_f(a)) \wedge \mu(C_g(a)),$$

where we used the monotonicity of μ. Hence, we obtain

$$\begin{aligned}\int^{\wedge} (f \wedge g)\, d\mu &= \bigvee_{a \in L} (a \wedge \mu(C_{f \wedge g}(a))) = \bigvee_{a \in L} (a \wedge (\mu(C_f(a)) \wedge \mu(C_g(a)))) \\ &\leq \bigvee_{a \in L} (a \wedge \mu(C_f(a))) \wedge \bigvee_{b \in L} (b \wedge \mu(C_g(b))) = \int^{\wedge} f\, d\mu \wedge \int^{\wedge} g\, d\mu.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int^{\wedge} f\, d\mu \wedge \int^{\wedge} g\, d\mu &= \bigvee_{a \in L} (a \wedge \mu(C_f(a))) \wedge \bigvee_{b \in L} (b \wedge \mu(C_g(b))) = \\ &\bigvee_{a \in L} \bigvee_{b \in L} (a \wedge \mu(C_f(a))) \wedge (b \wedge \mu(C_g(b)) \leq \\ &\bigvee_{a \in L} \bigvee_{b \in L} (a \wedge b) \wedge (\mu(C_f(a \wedge b)) \wedge \mu(C_g(a \wedge b))) \\ = \bigvee_{a \in L} \bigvee_{b \in L} (a \wedge b) \wedge \mu(C_{f \wedge g}(a \wedge b)) &= \bigvee_{a \in L} a \wedge \mu(C_{f \wedge g}(a) = \int^{\wedge} (f \wedge g)\, d\mu,\end{aligned}$$

where we used the distributivity of $\wedge$ over $\bigvee$, which holds in each Heyting algebra, and the fact that $C_f(a) \leq C_f(b)$ for any $a, b \in L$ such that $b \leq a$. $\square$

2.4 DH–residuum-based fuzzy integral

The residuum-based fuzzy integral was first proposed by Dvořák and Holčapek in [11] for modelling natural language quantifiers. Later, Dubois, Prade and Rico in [10] introduced another type of residuum-based fuzzy integral, which will be introduced in the next section.

Definition 2.9. Let $\langle X, \mathcal{F}, \nu \rangle$ be a complementary fuzzy measure space, and let $f : X \to L$. The $\to_{\mathrm{DH}}$*–fuzzy integral of* f *on* X is given by

$$\int_{\mathrm{DH}}^{\to} f \, d\nu = \bigwedge_{A \in \mathcal{F}} \left(\bigwedge_{x \in A} f(x) \to \nu(A) \right). \tag{2.13}$$

The basic properties of $\to_{\mathrm{DH}}$–fuzzy integral are summarized in the following theorem (for the proof, see, [11]).

Theorem 2.12. *Let* $\underline{a}_X \in \mathcal{F}(X)$ *be a constant function. For any* $f, g \in \mathcal{F}(X)$ *and* $a \in L$*, we have*

(i) $\int_{\mathrm{DH}}^{\to} f \, d\nu \geq \int_{\mathrm{DH}}^{\to} g \, d\nu$ *if* $f \leq g$*,*

(ii) $\int_{\mathrm{DH}}^{\to} \underline{a}_X \, d\nu = \neg a$*,*

(iii) $\int_{\mathrm{DH}}^{\to} \underline{a}_X \otimes f \, d\nu \leq a \to \int_{\mathrm{DH}}^{\to} f \, d\nu$*,*

(iv) $\int_{\mathrm{DH}}^{\to} \underline{a}_X \to f \, d\nu \geq a \otimes \int_{\mathrm{DH}}^{\to} f \, d\nu$*,*

(v) $\int_{\mathrm{DH}}^{\to} \underline{a}_X \otimes 1_Z \, d\nu = a \to \nu(Z)$ *for any* $Z \in \mathcal{F}$*.*

If $\mathbf{L}$ *is an MV-algebra, then inequality (iii) can be replaced by equality.*

Similarly to the multiplication-based fuzzy integral, we have an equivalent expression for $\to_{\mathrm{DH}}$–fuzzy integral assuming $\mathcal{F}$-$\mathcal{B}^u$–measurability of functions, which is very useful from a practical point of view.

Theorem 2.13. *Let* $\langle X, \mathcal{F}, \nu \rangle$ *be a complementary fuzzy measure space, and let* $f : X \to L$ *be* $\mathcal{F}$-$\mathcal{B}^u$*–measurable. Then*

$$\int_{\mathrm{DH}}^{\to} f \, d\nu = \bigwedge_{a \in L} (a \to \nu(\{x \in X \mid f(x) \geq a\})). \tag{2.14}$$

Proof. Similarly to the proof of Theorem 2.8, denote $L_a = \{x \in L \mid x \geq a\}$ for any $a \in L$, and recall that $f^{-1}(L_a) \in \mathcal{F}$ for any $a \in L$ as a consequence of $\mathcal{F} - \mathrm{Alg}(\mathcal{U}(L))$–measurability. Put $I = \bigwedge_{A \in \mathcal{F}^-} \left(\left(\bigwedge_{x \in A} f(x) \right) \to \nu(A) \right)$ and $J = \bigwedge_{a \in L} \left(a \to \nu\left(f^{-1}(L_a)\right)\right)$. First, we show that $I \geq J$. Let $\lambda_f : \mathcal{F} \to L$ be a map given by $\lambda_f(A) = \bigwedge_{x \in A} f(x)$. Obviously, we have $A \subseteq f^{-1}(L_{\lambda_f(A)})$, and thus $\nu(f^{-1}(L_{\lambda_f(A)})) \leq \nu(A)$. Since $\lambda_f(\mathcal{F}) \subseteq L$, we get

$$I = \bigwedge_{A \in \mathcal{F}} (\lambda_f(A) \to \nu(A)) \geq \bigwedge_{A \in \mathcal{F}} (\lambda_f(A) \to \nu(f^{-1}(L_{\lambda_f(A)}))) \geq J,$$

where we used the fact that the residuum is non-decreasing in its second component.

Further, we show that $I \leq J$. Let $\varrho_f : L \to \mathcal{F}$ be given by $\varrho_f(a) = f^{-1}(L_a)$. From the $\mathcal{F}$-$\mathrm{Alg}(\mathcal{U}(L))$–measurability of f, the map ϱ_f is well defined. Obviously, we have $\bigwedge_{x \in \varrho_f(a)} f(x) \geq a$ for any $a \in L$ and $\varrho_f(L) \subseteq \mathcal{F}$. Then, we obtain

$$I \leq \bigwedge_{a \in L} \left(\left(\bigwedge_{x \in \varrho_f(a)} f(x) \right) \to \nu(\varrho_f(a)) \right) \leq \bigwedge_{a \in L} (a \to \nu(f^{-1}(L_a)) = J,$$

where we used the fact that the residuum is non-increasing in its first component. Hence, we obtain $I = J$ and the proof is finished. □

As a corollary we get a simple computational formula for measurable functions defined on a finite set assuming that the residuated lattice is linearly ordered.

Corollary 2.14. *Let* $\mathbf{L}$ *be a linearly ordered, let* $\langle X, \mathcal{F}, \nu\rangle$ *be a finite complementary fuzzy measure space, i.e.,* $X = \{x_1, \dots, x_n\}$*, and let* $f : X \to L$ *be* $\mathcal{F}$*-*$\mathcal{B}^u$*–measurable. Then*

$$\int_{\mathrm{DH}}^{\to} f \; d\nu = \bigwedge_{i\in[n]} (f_{\sigma(i)} \to \nu_i), \tag{2.15}$$

where σ *is a permutation on* $[n]$ *such that* $f_{\sigma(1)} \le f_{\sigma(2)} \le \cdots \le f_{\sigma(n)}$*, where* $f_{\sigma(i)} = f(x_{\sigma(i)})$ *for* $i \in [n]$*, and* $\nu_i = \nu(\{x_{\sigma(i)}, \dots, x_{\sigma(n)}\})$*.*

Proof. Analogously to the proof of Corollary 2.9, for $a = \bot$, we trivially get $a \to \nu(\{x \in X \mid f(x) \ge a\}) = \top$, and for $f_{\sigma(n)} < a \le \top$, we find that $a \to \nu(\{x \in X \mid f(x) \ge a\}) = a \to \nu(\emptyset) = a \to \top = \top$. If $f_{\sigma(i-1)} < a \le f_{\sigma(i)}$ for $i \in [n]$ (we put $f_{\sigma(0)} = \bot$), then

$$a \to \nu(\{x \in X \mid f(x) \ge a\}) = a \to \nu_i \ge f_{\sigma(i)} \to \nu_i,$$

where we used the fact that the residuum is non-increasing in its first variable. Hence, we get

$$\bigwedge_{a\in[0,1]} (a \to \nu(\{x \in X \mid f(x) \ge a\})) \ge \bigwedge_{i\in[n]} (f_{\sigma(i)} \to \nu_i).$$

Since the opposite inequality is trivially true, we obtain the desired equality. □

Now we will present the comonotonic property of $\to_{\mathrm{DH}}$–fuzzy integral in the following theorem.

Theorem 2.15. *Let* $\mathbf{L}$ *be linearly ordered, and let* $f, g \in \mathcal{F}(X)$ *be comonotonic functions that are* $\mathcal{F}$*-*$\mathcal{B}^u$*–measurable. Then*

$$\int_{\mathrm{DH}}^{\to} (f \vee g) \; d\nu = \int_{\mathrm{DH}}^{\to} f \; d\nu \wedge \int_{\mathrm{DH}}^{\to} g \; d\nu.$$

Proof. According to Theorem 2.5, we have $f \vee g$ is $\mathcal{F}$-$\mathcal{B}^u$–measurable. Hence, we can use formula (2.14) to compute the $\to_{\mathrm{DH}}$–fuzzy integral, i.e., we have

$$\int_{\mathrm{DH}}^{\to} (f \vee g) \, d\nu = \bigwedge_{a\in L} (a \to \nu(\{x \in X \mid f(x) \vee g(x) \ge a\})) = \bigwedge_{a\in L} (a \to \nu(C_{f\vee g}(a))) \,,$$

where we used the notation from Lemma 2.10. Since $C_{f\vee g}(a) = C_f(a) \cup C_g(a)$, where $C_{f\vee g}(a) = C_f(a)$ or $C_{f\vee g}(a) = C_g(a)$ for any $a \in L$, we obtain

$$\nu(C_{f\vee g}(a)) = \nu(C_f(a)) \wedge \nu(C_g(a)),$$

where we apply the fact that ν is a non-increasing set mapping. Hence, we obtain

$$\int_{\mathrm{DH}}^{\rightarrow} (f \vee g)\ d\nu = \bigwedge_{a \in L} (a \rightarrow \nu(C_{f \vee g}(a)))$$
$$= \bigwedge_{a \in L} (a \rightarrow (\nu(C_f(a)) \wedge \nu(C_g(a)))) = \bigwedge_{a \in L} (a \rightarrow \nu(C_f(a))) \wedge (a \rightarrow \nu(C_g(a)))$$
$$= \bigwedge_{a \in L} (a \rightarrow \nu(C_f(a))) \wedge \bigwedge_{a \in L} (a \rightarrow \nu(C_g(a))) = \int_{\mathrm{DH}}^{\rightarrow} f\ d\nu \wedge \int_{\mathrm{DH}}^{\rightarrow} g\ d\nu,$$

where we used the distributivity of $\rightarrow$ over $\wedge$. □

2.5 DPR–residuum-based fuzzy integral

Another type of fuzzy integrals based on the operation of residuum was proposed by Dubois, Prade and Rico in [10] under the name desintegral for reasoning with a decreasing evaluation scale. The following definition slightly modifies the original definition, where the conjugate fuzzy measure is replaced by N–conjugate fuzzy measure introduced in Definition 2.5.

Definition 2.10. Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space, let $\mu^{c,N}$ denote the N–conjugate fuzzy measure to μ, and let $f : X \rightarrow L$. The $\rightarrow_{\mathrm{DPR}}$*–fuzzy integral of* f *on* X is given by

$$\int_{\mathrm{DPR}}^{\rightarrow} f\ d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \rightarrow \bigvee_{x \in A} f(x) \right). \tag{2.16}$$

Note that we use the original notation where the fuzzy measure μ is used in the integral, but its N–conjugate fuzzy measure is employed in the computational formula. The following theorem presents some basic properties of $\rightarrow_{\mathrm{DPR}}$–fuzzy integral.

Theorem 2.16. *Let* $a_X \in \mathcal{F}(X)$ *be a constant function. For any* $f, g \in \mathcal{F}(X)$ *and* $a \in L$*, we have*

(i) $\int_{\mathrm{DPR}}^{\rightarrow} f\, d\mu \leq \int_{\mathrm{DPR}}^{\rightarrow} g\, d\mu$ *if* $f \leq g$,

(ii) $\int_{\mathrm{DPR}}^{\rightarrow} \underline{a}_X\, d\mu = a$,

(iii) $a \otimes \int_{\mathrm{DPR}}^{\rightarrow} f\, d\mu \leq \int_{\mathrm{DPR}}^{\rightarrow} \underline{a}_X \otimes f\, d\mu$,

(iv) $\int_{\mathrm{DPR}}^{\rightarrow} \underline{a}_X \rightarrow f\, d\mu \leq a \rightarrow \int_{\mathrm{DPR}}^{\rightarrow} f\, d\mu$,

(v) If $N = N_{\mathrm{res}}$ *is the involutive negation, then* $\int_{\mathrm{DPR}}^{\rightarrow} \underline{a}_X \otimes 1_Z\ d\mu = a \wedge \mu(Z)$ *for any* $Z \in \mathcal{F}$.

Moreover, if $\mathbf{L}$ *is an MV–algebra, then inequality (iv) can be replaced by equality.*

Proof. It is easy to check that the N–conjugate fuzzy measure $\mu^{c,N}$ to a fuzzy measure μ is again a fuzzy measure.

(i) This is a straightforward consequence of the fact that the residuum is monotonically non-decreasing in the second argument.

(ii) Let $a \in L$. Then

$$\int_{\mathrm{DPR}}^{\to} \underline{a}_X \, d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} \underline{a}_X(x) \right) = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to a \right) = \mu^{c,N}(X) \to a = a,$$

where we used the fact that the residuum is monotonically non-increasing in the first component and $\top \to a = a$ (see, Theorem 1.1(iv)).

(iii) Let $a \in L$. Then

$$\int_{\mathrm{DPR}}^{\to} \underline{a}_X \otimes f \, d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} (a \otimes f(x)) \right) =$$

$$\bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to (a \otimes \bigvee_{x \in A} f(x)) \right) \geq \bigwedge_{A \in \mathcal{F}} a \otimes \left(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x) \right) \geq$$

$$a \otimes \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x) \right) = a \otimes \int_{\mathrm{DPR}}^{\to} f \, d\mu,$$

where we used $b \otimes (a \to c) \leq a \to (b \otimes c)$ (Theorem 1.1(viii)), and $\bigwedge_{i \in I} (a \otimes b_i) \geq a \otimes \bigwedge_{i \in I} b_i$ for any $a, b, c \in L$ (Theorem 1.2(iv)).

(iv) Let $a \in L$. Then

$$\int_{\mathrm{DPR}}^{\to} \underline{a}_X \to f \, d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} (a \to f(x)) \right) \leq$$

$$\bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to (a \to \bigvee_{x \in A} f(x)) \right) = \bigwedge_{A \in \mathcal{F}} a \to \left(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x) \right) =$$

$$a \to \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x) \right) = a \to \int_{\mathrm{DPR}}^{\to} f \, d\mu,$$

where we used $a \to (b \to c) = b \to (a \to c)$ (Theorem 1.1(ix)), and $\bigwedge_{i \in I} (a \to b_i) = a \to \bigwedge_{i \in I} b_i$ and $\bigvee_{i \in I} (a \to b_i) \leq a \to \bigvee_{i \in I} b_i$ for any $a, b, c \in L$ ((ii) and (v) of Theorem 1.2).

(v) Since the negation $N = N_{\mathrm{res}} = \neg$ is a residuum-based negation, see (1.7), and $\neg$ is involutive, i.e., $\neg(\neg a) = a$ for any $a \in L$, we get

$$\int_{\mathrm{DPR}}^{\to} \underline{a}_X \otimes 1_Z \, d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,\neg}(A) \to \bigvee_{x \in A} (\underline{a}_X(x) \otimes 1_Z(x)) \right) =$$

$$\bigwedge_{\substack{A \in \mathcal{F} \\ A \cap Z \neq \emptyset}} \left(\mu^{c,\neg}(A) \to \bigvee_{x \in A} (\underline{a}_X(x) \otimes 1_Z(x)) \right) \wedge \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus Z}} \left(\mu^{c,\neg}(A) \to \bigvee_{x \in A} (\underline{a}_X(x) \otimes 1_Z(x)) \right)$$

$$= (\mu^{c,\neg}(X) \to (a \otimes \top)) \wedge (\mu^{c,\neg}(X \backslash Z) \to (a \otimes \bot))$$
$$= (\top \to a) \wedge (\mu^{c,\neg}(X \backslash Z) \to \bot) = a \wedge (\mu^{c,\neg}(X \backslash Z) \to \bot)$$
$$= a \wedge (\neg(\mu(X \backslash (X \backslash Z)) \to \bot) = a \wedge \neg(\neg(\mu(Z))) = a \wedge \mu(Z),$$

where we used the fact that the residuum is monotonically non-increasing in the first component, $\top \to a = a$, and $a \otimes \bot = \bot$.

If $\mathbf{L}$ is an MV–algebra, then we have $\bigvee_{i \in I}(a \to b_i) = a \to \bigvee_{i \in I} b_i$ (see, Theorem 1.2 for an MV–algebra), which turns the inequality in the previous proof into equality. □

Again, the formula (2.16) can be simplified under the assumption of measurability of functions, where we now use $\mathcal{F}$-$\mathcal{B}^\ell$–measurability of functions, i.e., the algebra of sets on L is generated by losets (see, Example 2.3). Note that a similar result has been presented in [10] (Proposition 4) for Gödel and contrapositive Gödel implication.

Theorem 2.17. *Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space, $\mu^{c,N}$ be a N–conjugate fuzzy measure to μ and let $f : X \to L$ be $\mathcal{F}$-$\mathcal{B}^\ell$–measurable. Then*

$$\int_{\mathrm{DPR}}^{\to} f \; d\mu = \bigwedge_{a \in L} (\mu^{c,N}(\{x \in X \mid f(x) \leq a\}) \to a). \tag{2.17}$$

Proof. Let $a \in L$ and denote $L_a = \{x \in L \mid x \leq a\}$. Note that $\ell(\{a\}) = L_a$, where ℓ is given in formula (2.2). By the assumption on the $\mathcal{F}$-$\mathcal{B}^\ell$–measurability of f, we have $f^{-1}(L_a) \in \mathcal{F}$, where $f^{-1}(L_a) = \{x \in X \mid f(x) \leq a\}$. Put $I = \bigwedge_{A \in \mathcal{F}} \big(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x)\big)$ and $J = \bigwedge_{a \in L} \big(\mu^{c,N}(f^{-1}(L_a)) \to a\big)$. First, we show that $I \geq J$. Let $\lambda_f : \mathcal{F} \to L$ be a map given by $\lambda_f(A) = \bigvee_{x \in A} f(x)$. Obviously, $A \subseteq f^{-1}(L_{\lambda_f(A)})$, and thus $\mu^{c,N}(A) \leq \mu^{c,N}(f^{-1}(L_{\lambda_f(A)}))$, where we used the fact that f is $\mathcal{F}$-$\mathcal{B}^\ell$–measurable. Since $\lambda_f(\mathcal{F}) \subseteq L$, we obtain

$$I = \bigwedge_{A \in \mathcal{F}} \left(\mu^{c,N}(A) \to \bigvee_{x \in A} f(x) \right) \geq \bigwedge_{A \in \mathcal{F}} \big(\mu^{c,N}(f^{-1}(L_{\lambda_f(A)})) \to \lambda_f(A)\big) \geq J.$$

Furthermore, let $\varrho_f : L \to \mathcal{F}$ be given by $\varrho_f(a) = f^{-1}(L_a)$. From the $\mathcal{F}$-$\mathcal{B}^\ell$–measurability of f, the map ϱ_f is well defined. Obviously, we have $\bigvee_{x \in \varrho_f(a)} f(x) \leq a$ for any $a \in L$ and $\varrho_f(L) \subseteq \mathcal{F}$. Then, we obtain

$$I \leq \bigwedge_{a \in L} \left(\mu^{c,N}(\varrho_f(a)) \to \bigvee_{x \in \varrho_f(a)} f(x) \right) \leq \bigwedge_{a \in L} \big(\mu^{c,N}(f^{-1}(L_a)) \to a\big) = J.$$

Hence, we obtain $I = J$ which concludes the proof. □

As a corollary, we get a simple computational formula for measurable functions defined on a finite set assuming that the residuated lattice is linearly ordered.

Corollary 2.18. *Let* $\mathbf{L}$ *be a linearly ordered, let* $\langle X, \mathcal{F}, \mu\rangle$ *be a finite fuzzy measure space, i.e.,* $X = \{x_1, \dots, x_n\}$*,* $\mu^{c,N}$ *be the* N*–conjugate fuzzy measure to* μ*, and let* $f : X \to L$ *be* $\mathcal{F}$*-*$\mathcal{B}^\ell$*–measurable. Then*

$$\int_{\mathrm{DPR}}^{\to} f \; d\mu = \bigwedge_{i\in[n]} (\mu_i^{c,N} \to f_{\sigma(i)}), \tag{2.18}$$

where σ *is a permutation on* $[n]$ *such that* $f_{\sigma(1)} \geq f_{\sigma(2)} \geq \cdots \geq f_{\sigma(n)}$*, where* $f_{\sigma(i)} = f(x_{\sigma(i)})$ *for* $i \in [n]$*, and* $\mu_i^{c,N} = \mu^{c,N}(\{x_{\sigma(i)}, \dots, x_{\sigma(n)}\})$*.*

Proof. Analogously to the proof of Corollary 2.14, for $a = \top$, we trivially get $\mu^{c,N}(\{x \in X \mid f(x) \leq a\}) \to a = \top$, and for $\bot \leq a < f_{\sigma(n)}$, we find that $\mu^{c,N}(\{x \in X \mid f(x) \leq a\}) \to a = \mu^{c,N}(\emptyset) \to a = \bot \to a = \top$. If $f_{\sigma(i)} \leq a < f_{\sigma(i-1)}$ for $i \in [n]$ (we put $f_{\sigma(0)} = \top$), then

$$\mu^{c,N}(\{x \in X \mid f(x) \leq a\}) \to a = \mu_i^{c,N} \to a \geq \mu_i^{c,N} \to f_{\sigma(i)},$$

where we used the fact that the residuum is non-decreasing in its second variable. Hence, we get

$$\bigwedge_{a\in[0,1]} (\mu^{c,N}(\{x \in X \mid f(x) \leq a\}) \to a) \geq \bigwedge_{i\in[n]} (\mu_i^{c,N} \to f_{\sigma(i)}).$$

Since the opposite inequality is trivially true, we obtain the desired equality. □

In the following part, we show that $\to_{\mathrm{DPR}}$–fuzzy integral is commonotonic minitive.

Lemma 2.19. *Let* $\mathbf{L}$ *be linearly ordered, and let* $f, g \in \mathcal{F}(X)$*. Denote* $B_f = \{B_f(a) \mid a \in L\}$*, where* $B_f(a) = \{x \in X \mid f(x) \leq a\}$*. Then* B_f *is a chain with respect to* $\subseteq$*, and if* f *and* g *are comonotonic, then* $B_{f\odot g}(a) = B_f(a)$ *or* $B_{f\odot g}(a) = B_g(a)$ *for any* $a \in L$*, where* $\odot \in \{\wedge, \vee\}$*.*

Proof. It is easy to show that Lemma 2.10 remains true if $C_f(a) = \{x \in X \mid f(x) > a\}$. Put $C_f(a) = X \setminus B_f(a)$, $C_g(a) = X \setminus B_g(a)$ and $C_{f\odot g}(a) = X \setminus B_{f\odot g}(a)$. Since $C_{f\odot g}(a) = C_f(a)$ or $C_{f\odot g}(a) = C_g(a)$, we obtain $X \setminus B_{f\odot g}(a) = X \setminus B_f(a)$ or $X \setminus B_{f\odot g}(a) = X \setminus B_g(a)$, which implies the desired equalities. □

Theorem 2.20. *Let* $\mathbf{L}$ *be linearly ordered, and let* $f, g \in \mathcal{F}(X)$ *be comonotonic* $\mathcal{F}$*-*$\mathcal{B}^\ell$*–measurable functions. Then*

$$\int_{\mathrm{DPR}}^{\to} (f \wedge g) \; d\mu = \int_{\mathrm{DPR}}^{\to} f \; d\mu \wedge \int_{\mathrm{DPR}}^{\to} g \; d\mu.$$

Proof. According to Remark 2.3, we find that $f \wedge g$ is $\mathcal{F}$-$\mathcal{B}^\ell$–measurable. Considering the notation from Lemma 2.19, we obtain

$$\int_{\mathrm{DPR}}^{\to} (f \wedge g) \, d\mu = \bigwedge_{a\in L} \left(\mu^{c,N}(B_{f\wedge g}(a)) \to a\right).$$

Due to Lemma 2.19, we have $B_{f\wedge g}(a) = B_f(a) \cup B_g(a)$, where $B_{f\wedge g}(a) = B_f(a)$ or $B_{f\wedge g}(a) = B_g(a)$ for any $a \in L$. Hence, we obtain

$$\mu^{c,N}(B_{f\wedge g}(a)) = \mu^{c,N}(B_f(a)) \vee \mu^{c,N}(B_g(a)),$$

and using (iii) of Theorem 1.2, we find that

$$\begin{aligned}
\int_{\mathrm{DPR}}^{\rightarrow} (f \wedge g)\ d\mu &= \bigwedge_{a\in L} \big(\mu^{c,N}(B_{f\wedge g}(a)) \to a\big) \\
&= \bigwedge_{a\in L} \big(\mu^{c,N}(B_f(a)) \vee \mu^{c,N}(B_g(a)) \to a\big) = \\
&\bigwedge_{a\in L} (\mu^{c,N}(B_f(a)) \to a) \wedge (\mu^{c,N}(B_g(a)) \to a) \\
= \bigwedge_{a\in L} (\mu^{c,N}(B_f(a)) \to a) \wedge \bigwedge_{b\in L} (\mu^{c,N}(B_g(b)) \to b) &= \int_{\mathrm{DPR}}^{\rightarrow} f\ d\mu \wedge \int_{\mathrm{DPR}}^{\rightarrow} g\ d\mu,
\end{aligned}$$

and the proof is finished. □

Chapter 3

Lattice integral transforms

In this chapter, we introduce several types of integral transforms for residuated lattice-valued functions based on the Sugeno like fuzzy integrals presented in the previous chapter. We show some of their basic properties that will be used in the next part. The integral transforms were first proposed in papers [26, 25] and here we present slightly modified versions of them that are useful in practical tasks such as signal or image processing. Throughout this chapter, we assume that the complete residuated lattice $\mathbf{L}$ is given, and we will not mention it explicitly except when we want to specify its form.

3.1 Motivation

In [36], Perfilieva introduced, among others, upper and lower lattice fuzzy transforms to approximate functions whose function values belong to a complete residuated lattice. Lattice integral transforms are designed to generalize these two transforms naturally. To give a better idea, we briefly recall their definitions and show what the generalization consists of.

Let X, Y be non-empty sets, and let $\mathbf{A} = \{A_y \mid y \in Y\}$ be a family of fuzzy sets $A_y : X \to L$ such that $\bigcup_{y \in Y} A_y = X$ and $\mathrm{Core}(A_y) \cap \mathrm{Core}(A_z) = \emptyset$ for any $y, z \in Y$ with $y \neq z$. The family $\mathbf{A}$ is referred to as a *fuzzy partition* of X. The *direct upper lattice fuzzy transform* with respect to a fuzzy partition $\mathbf{A}$ is a map $F^{\uparrow}_{\mathbf{A}} : \mathcal{F}(X) \to \mathcal{F}(Y)$ given by

$$F^{\uparrow}_{\mathbf{A}}(f)(y) = \bigvee_{x \in X} f(x) \otimes A_y(x), \tag{3.1}$$

for any $f \in \mathcal{F}(X)$ and $y \in Y$, and the *direct lower lattice fuzzy transform* with respect to a fuzzy partition $\mathbf{A}$ is a map $F^{\downarrow}_{\mathbf{A}} : \mathcal{F}(X) \to \mathcal{F}(Y)$ given by

$$F^{\downarrow}_{\mathbf{A}}(f)(y) = \bigwedge_{x \in X} A_y(x) \to f(x), \tag{3.2}$$

for any $f \in \mathcal{F}(X)$ and $y \in Y$. Note that Perfilieva also proposed an inverse version for these two types of lattice fuzzy transforms, which have the same form and will be introduced in the next chapter. In Figure 3.1, we can see the results of the lower and upper lattice fuzzy transform applied on a signal given on $X = \{1, \ldots, 205\}$

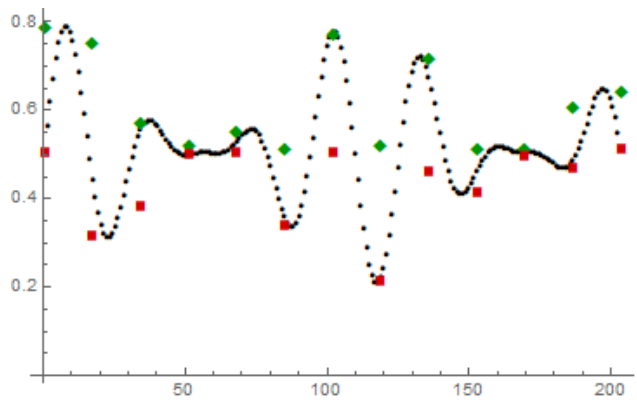


Figure 3.1: Upper (green diamonds) and lower (red squares) lattice fuzzy transforms.

and transformed to $Y = \{1, 18, 35, \ldots, 205\}$. We can see that the upper lattice fuzzy transform approximates the original signal from above and lower lattice fuzzy transform from below at the points of the set Y. Note that the composition of these lattice fuzzy transforms lead to the upper and lower approximation of the original signal, which will be discussed in the next chapter.

Now define a fuzzy relation $K : X \times Y \to L$ as $K(x, y) = A_y(x)$ for any $x \in X$ and $y \in Y$, and assume that $\mathcal{F} = \mathcal{P}(X)$ and $\mu = \mu^{\top}$ is the highest measure on $\langle X, \mathcal{F} \rangle$ (see, Example 2.6). It is easy to see that the upper lattice fuzzy transform $F^{\uparrow}_{\mathbf{A}}$ can be expressed as follows

$$F^{\uparrow}_{\mathbf{A}}(f)(y) = \int^{\otimes} f(x) \otimes K(x, y)\, d\mu, \tag{3.3}$$

where $\int^{\otimes}$ is the multiplication-based integral introduced in Section 2.3. Similarly, assuming $\mathcal{F} = \mathcal{P}(X)$ and $\mu = \mu^{\perp}$ is the least fuzzy measure on $\langle X, \mathcal{F} \rangle$, the lower lattice fuzzy transform $F^{\downarrow}_{\mathbf{A}}$ can be expressed as follows

$$F^{\downarrow}_{\mathbf{A}}(f)(y) = \int^{\otimes} K(x, y) \to f(x)\, d\mu. \tag{3.4}$$

We see that both lattice fuzzy transform can be introduced using a multiplication-based integral applied to the multiplication or residuum operation between the function f and the fuzzy relation (integral kernel) K, which is an identical scheme known for standard integral transforms for real and complex-valued functions. This motivates us to introduce a general framework for lattice fuzzy transforms, which we will call lattice integral transforms to keep the notation from the theory of integral transforms. The generalization consists in the use of more general fuzzy integrals and integral kernels that extends the concept of fuzzy partition. Since the lower and upper lattice fuzzy transforms have valuable approximation properties, a natural question is whether these approximation properties will also be achieved in the general framework of lattice integral transforms, which will be the subject of the next chapter.

3.2 Integral kernel

In this section, we generalize the fuzzy partition which is a key concept in lattice fuzzy transform (see, [36]). The following lemma characterizes a fuzzy partition of X in terms of an equivalence of sets.

Lemma 3.1. *A family $\mathbf{A} \subset \mathcal{F}(X)$ is a fuzzy partition of X if and only if*

$$R = \{(x, y) \in X \times X \mid \exists! A \in \mathbf{A} \,:\, A(x) = A(y) = \top\} \tag{3.5}$$

is an equivalence on X.

Proof. Let $\mathbf{A} \subset \mathcal{F}(X)$ be a family of fuzzy sets.

($\Rightarrow$) Denote $[x] = \{y \in X \mid (x, y) \in R\}$. Since $\mathrm{Core}(\mathbf{A}) = \{\mathrm{Core}(A) \mid A \in \mathbf{A}\}$ is a partition of X, we find that $x \in \mathrm{Core}(A)$ for some $A \in \mathbf{A}$. Moreover, if $y \in \mathrm{Core}(A)$, then $A(x) = A(y) = \top$, which implies $y \in [x]$, and thus $\mathrm{Core}(A) \subseteq [x]$. Conversely, if $y \in [x]$, then $A(x) = A(y) = \top$ by the definition of R, which implies $y \in \mathrm{Core}(A)$, and thus $[x] = \mathrm{Core}(A)$. Hence, $\{[x] \mid x \in X\}$ is a partition of X; therefore, R is an equivalence on X.

($\Leftarrow$) The reflexivity of R ensures that each fuzzy set $A \in \mathbf{A}$ is normal. We have shown above that $[x] = \mathrm{Core}(A)$, and moreover, $A \in \mathbf{A}$ is the unique fuzzy set for which this equality holds. Hence, there exists a bijection between $\mathrm{Core}(\mathbf{A})$ and $\{[x] \mid x \in X\}$, which implies that $\mathrm{Core}(\mathbf{A})$ is a partition of X and $\mathbf{A}$ a fuzzy partition of X. □

In [32], there is pointed out that a fuzzy partition of X can be equivalently expressed in terms of a fuzzy relation. In our notation, each fuzzy partition can be expressed as, for example, a fuzzy relation $K : X \times \mathbf{A} \to L$ given by $K(x, A) = A(x)$ for any $x \in X$ and $A \in \mathbf{A}$ such that for any $A \in \mathbf{A}$ there is $x \in X$ with $K(x, A) = \top$ and vice verse, i.e., the A–projection K_A and the x–projection K_x are normal fuzzy sets. This motivates us to introduce the integral kernel, which is the key concept for integral transforms as follows.

Definition 3.1. A fuzzy relation $K : X \times Y \to L$ is said to be an *integral kernel* provided that K is normal in both arguments, i.e., for any $x \in X$ there is $y \in Y$ such that $K(x, y) = \top$ and vice verse, for any $y \in Y$ there is $x \in X$ such that $K(x, y) = \top$.

Equivalently, we could say that a fuzzy relation K on $X \times Y$ is an integral kernel if and only if the x–projection and y–projection of K are normal fuzzy sets for any $x \in X$ and $y \in Y$. Obviously, the transpose of an integral kernel K is again an integral kernel $K^T : Y \times X \to L$. It should be noted that the original definition of integral kernel provided in [26] is weaker in the sense that $K(x, y) = \top$ is weakened by $K(x, y) > \bot$, reflecting a more general fuzzy partition introduced and discussed in [30]. However, the analysis of integral kernels related to the reconstruction of lattice-valued functions led us to assume the normality of integral kernels. We say that an integral kernel $K : X \times Y \to L$ *determines a fuzzy partition* $\mathbf{A}_K$ *of* X if $\mathbf{A}_K = \{K_y \mid y \in Y\}$ is a fuzzy partition. The following lemma shows a necessary and sufficient condition for an integral kernel to determine a fuzzy partition of X.

Lemma 3.2. *An integral kernel $K : X \times Y \to L$ determines a fuzzy partition of X if and only if the core of K_x is a singleton for any $x \in X$.*

Proof. First, assume that an integral kernel K is a fuzzy partition of X. Since $\mathrm{Core}(K_y) \cap \mathrm{Core}(K_z) = \emptyset$ for any $y, z \in Y$ such that $y \neq z$, we find that to each $x \in X$ there is exactly one $y \in Y$ such that $K(x, y) = \top$, i.e., $\mathrm{Core}(K_x) = \{y\}$.

Furthermore, let $\mathrm{Core}(K_x)$ be a singleton for any $x \in X$. Then $\mathrm{Core}(K_y) \cap \mathrm{Core}(K_z) = \emptyset$ for any $y, z \in X$ such that $y \neq z$, otherwise, there exists $x \in \mathrm{Core}(K_y) \cap \mathrm{Core}(K_z)$ for which $\{y, z\} \subseteq K_x$ and this is a contradiction to the assumption. If $x \in X$ such that $x \notin \bigcup_{y \in Y} \mathrm{Core}(K_y)$, then $K(x, y) < \top$ for any $y \in Y$, and thus $\mathrm{Core}(K_x) = \emptyset$, which is again a contradiction with the assumption. Hence, the family $\{K_y \mid y \in Y\}$ is a fuzzy partition of X. □

From the previous lemma, one can see that an integral kernel K determines a family $\mathbf{A} = \{K_y \mid y \in Y\}$ of fuzzy sets whose cores only cover X, i.e., $\bigcup \mathrm{Core}(\mathbf{A}) = X$, but generally $\mathrm{Core}(K_y) \cap \mathrm{Core}(K_z) \neq \emptyset$ for $y, z \in Y$ such that $y \neq z$. The family $\mathbf{A}$ can be called a *fuzzy covering* of X. Note that an integral kernel K also determines a family $\mathbf{B} = \{K_x \mid x \in X\}$ of fuzzy sets on Y, which is a fuzzy covering of Y. Obviously, both families $\mathbf{A}$ and $\mathbf{B}$ are fuzzy partitions only in a very specific case, particularly, the cardinalities of X and Y coincide and $K(x, y) = \top$ holds exactly for one pair $(x, y) \in X \times Y$ (i.e., obviously, a function $f : X \to Y$ defined as $f(x) = y$ if $K(x, y) = \top$ defines a bijection of sets X and Y).

In the following statement, we show that a similarity on X determines a fuzzy partition of X, and hence an integral kernel. We assume that the axiom of choice is true in our consideration.

Lemma 3.3. *Let R be a similarity on X. Then there exists a set $Y \subseteq X$ such $\mathbf{A} = \{A_y \mid y \in Y\}$, where $A_y(x) = R(x, y)$ for any $(x, y) \in X \times Y$, is a fuzzy partition of X and $K(x, y) = R(x, y)$ is an integral kernel.*

Proof. Denote $R^* = \{(x, y) \in X \times X \mid R(x, y) = \top\}$. Obviously, R^* is an equivalence on X. Further, consider a choice function $\lambda : X\backslash_{R^*} \to X$ such that $\lambda([x]) \in [x]$ holds for any $[x] \in X\backslash_{R^*}$, where $X\backslash_{R^*}$ is the set of all equivalence classes on X with respect to the equivalence R^* and $[x]$ denotes the class for x. Put $Y = \{\lambda([x]) \mid [x] \in X\backslash_{R^*}\}$. Obviously, we have $Y \subseteq X$. Now, consider $\mathbf{A} = \{A_y \mid y \in Y\}$, where $A_y(x) = R(x, y)$. Define a relation S on X by (3.5). To prove that $\mathbf{A}$ is a fuzzy partition of X, it is sufficient to show that S is an equivalence on X. We will demonstrate that $S = R^*$. If $(x, y) \in S$, then there exists a unique $z \in Y$ such that $A_z(x) = A_z(y) = \top$. Hence, $R(x, z) = R(y, z) = \top$, and by the symmetry and transitivity of R we find that $R(x, z) \otimes R(z, y) = \top \leq R(x, y)$; therefore, $(x, y) \in R^*$. Conversely, if $(x, y) \in R^*$, then $\{z\} = Y \cap [x]$ by the definition of Y, and thus $R(x, z) = A_z(x) = A_z(y) = R(y, z) = \top$. Since z is the unique element from Y that belongs to $[x]$, we obtain that $(x, y) \in S$, and thus S is an equivalence on X.

The second statement is a straightforward consequence of the fact that the x–projection and y–projection of K are normal fuzzy sets for any $x \in X$ and $y \in Y$. □

Remark 3.1. Note that a similarity on X is also an integral kernel on $X \times X$. As a consequence of Lemma 3.2, we find that a similarity R on X determines a fuzzy partition of X if and only if $R(x, y) = \top$ implies $x = y$ for any $x, y \in X$.

3.3 Multiplication-based lattice integral transforms

In this section, we introduce two types of lattice integral transforms that naturally generalize the lower and upper lattice fuzzy transforms for the residuated lattice-valued functions. The lattice integral transforms are constructed using the multiplication-based fuzzy integral, whose integrand is the transformed function multiplied by the integral kernel, where the multiplication $\star$ is one of the operations of $\otimes$ and $\rightarrow$. A lattice integral transform of fuzzy sets from $\mathcal{F}(X)$ to a fuzzy sets from $\mathcal{F}(Y)$ is defined as follows.

Definition 3.2. Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space, let $K : X \times Y \rightarrow L$ be an integral kernel, and let $\star = \{\otimes, \rightarrow\}$. A map $F^{\star}_{(K,\mu)} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ defined by

$$F^{\star}_{(K,\mu)}(f)(y) = \int^{\otimes} K(x, y) \star f(x)\, d\mu \tag{3.6}$$

is called a $(K, \mu, \star)$*–M–lattice integral transform.*

It should be noted that the original definition of the $(K, \mu, \star)$–M–lattice integral transform (see, [26]) considers only a semi-normality of the integral kernel in the second argument, which means that for any $y \in Y$ there is $x \in X$ such that $K(x, y) > \perp$ (i.e., it is not necessary that $K(x, y) = \top$ as assumed in this work). However, practical applications have shown that semi-normality is too weak to obtain good results, which motivated us to assume the normality of the integral kernel in both arguments. Moreover, we add the letter "M" to the notation to emphasize that the lattice integral transform is based on the multiplication. If the pair (K, μ) is known we simplify the notation of multiplication-based lattice integral transform to "M–lattice integral transform" or "M*–lattice integral transform", where $*$ is used to emphasize the operation used, or we express it using symbols such as "M–LIT" or "M*–LIT".

The following theorem shows that $(K, \mu, \star)$–M–lattice integral transforms indeed generalize the upper and lower lattice fuzzy transforms.

Theorem 3.4. *Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space such that $\mathcal{F} = \mathcal{P}(X)$, and let $K : X \times Y \rightarrow L$ be an integral kernel that determines a fuzzy partition $\mathbf{A}_K$ of X. Assume that $F^{\uparrow}_{\mathbf{A}_K}, F^{\downarrow}_{\mathbf{A}_K} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ are upper and lower lattice fuzzy transforms from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, respectively, and let $f \in \mathcal{F}(X)$.*

(i) If $\mu = \mu^{\top}$ is the highest fuzzy measure on $\langle X, \mathcal{F} \rangle$, then $F^{\uparrow}_{\mathbf{A}_K}(f) = F^{\otimes}_{(K,\mu^{\top})}(f)$.

(ii) If $\mu = \mu^{\perp}$ is the least fuzzy measure on $\langle X, \mathcal{F} \rangle$, then $F^{\downarrow}_{\mathbf{A}_K}(f) = F^{\rightarrow}_{(K,\mu^{\perp})}(f)$.

Proof. Let K be an integral kernel such that $\mathbf{A}_K = \{K_y \mid y \in Y\}$ is a fuzzy partition, and $y \in Y$.

(i) If $\mu = \mu^\top$, then

$$F^{\otimes}_{(K,\mu^\top)}(f)(y) = \bigvee_{A\in\mathcal{F}} (\mu^\top(A) \otimes \bigwedge_{x\in A} (K(x,y)\otimes f(x))) = \bigvee_{A\in\mathcal{F}} \bigwedge_{x\in A} (K(x,y)\otimes f(x))$$
$$= \bigvee_{x\in X} (K(x,y)\otimes f(x)) = \bigvee_{x\in X} (K_y(x)\otimes f(x)) = F^{\uparrow}_{\mathbf{A}_K}(f)(y).$$

(ii) If $\mu = \mu^\perp$, then

$$F^{\rightarrow}_{(K,\mu^\perp)}(f)(y) = \bigvee_{A\in\mathcal{F}} (\mu^\perp(A) \otimes \bigwedge_{x\in A} (K(x,y)\to f(x)))$$
$$= \bigwedge_{x\in X} (K(x,y)\to f(x)) = \bigwedge_{x\in X} (K_y(x)\to f(x)) = F^{\downarrow}_{\mathbf{A}_K}(f)(y),$$

and the proof is finished. □

Remark 3.2. Note that $\mathcal{F} = \mathcal{P}(X)$ in the previous theorem can be generalized by assuming that the algebra $\mathcal{F}$ contains all singletons in X, i.e., $\{\{x\} \mid x \in X\} \subset \mathcal{F}$. Obviously, the fuzzy measure is an additional parameter $(K,\mu,\star)$–M–lattice integral transforms in contrast to the upper or lower lattice fuzzy transforms.

The following theorem presents the basic properties of $(K,\mu,\star)$–M–lattice integral transforms (see, also [26]). We assume that a fuzzy measure space $\langle X,\mathcal{F},\mu\rangle$ is given and $K : X\times Y \to L$ is an integral kernel.

Theorem 3.5. *For any $f,g\in\mathcal{F}(X)$, $\star = \{\otimes,\to\}$ and for any $a\in L$, we have*

(i) $F^{\star}_{(K,\mu)}(f) \le F^{\star}_{(K,\mu)}(g)$ *if* $f\le g$,

(ii) $F^{\star}_{(K,\mu)}(f\cap g) \le F^{\star}_{(K,\mu)}(f) \wedge F^{\star}_{(K,\mu)}(g)$,

(iii) $F^{\star}_{(K,\mu)}(f) \vee F^{\star}_{(K,\mu)}(g) \le F^{\star}_{(K,\mu)}(f\cup g)$,

(iv) $a\otimes F^{\star}_{(K,\mu)}(f) \le F^{\star}_{(K,\mu)}(\underline{a}_X\otimes f)$,

(v) $F^{\star}_{(K,\mu)}(\underline{a}_X\to f) \le a\to F^{\star}_{(K,\mu)}(f)$.

If $\mathbf{L}$ *is an MV–algebra, then inequality (iv) can be replaced by equality.*

Proof. We prove only the first case $\star = \otimes$, the second case $\star = \to$ can be proved analogously.

(i)-(iii) This is a trivial consequence of the monotonicity of $\otimes$–fuzzy integral (Theorem 2.7(i)) and the fact that

$$K(x,y)\otimes f(x) \le K(x,y)\otimes g(x) \quad \text{for } f\le g,$$
$$K(x,y)\otimes (f\cap g)(x) \le (K(x,y)\otimes f(x)) \wedge (K(x,y)\otimes g(x)),$$
$$(K(x,y)\otimes f(x)) \vee (K(x,y)\otimes g(x) \le K(x,y)\otimes (f\cup g)(x),$$

for any $x\in X$ and $y\in Y$.

(iv) By Theorem 2.7(iii), we find that

$$F^{\otimes}_{(K,\mu)}(\underline{a}_X \otimes f)(y) = \int^{\otimes} K(x,y) \otimes (a \otimes f(x))\, d\mu = \int^{\otimes} a \otimes (K(x,y) \otimes f(x))\, d\mu \geq$$
$$= a \otimes \int^{\otimes} K(x,y) \otimes f(x)\, d\mu = a \otimes F^{\otimes}_{(K,\mu)}(f)(y).$$

Moreover, if $\mathbf{L}$ is an MV–algebra, then the previous inequality can be replaced by the equality according to Theorem 2.7(iii) for an MV–algebra.

(v) By Theorem 2.7(iv), we have

$$F^{\otimes}_{(K,\mu)}(\underline{a}_X \to f)(y) = \int^{\otimes} K(x,y) \otimes (a \to f(x))\, d\mu \leq$$
$$\int^{\otimes} a \to (K(x,y) \otimes f(x))\, d\mu \leq a \to \int^{\otimes} K(x,y) \otimes f(x)\, d\mu = a \to F^{\otimes}_{(K,\mu)}(f)(y),$$

where we used $b \otimes (a \to c) \leq a \to (b \otimes c)$ (see, the proof of Theorem 2.16 for details). □

As a straightforward consequence of the definition of multiplication-based lattice integral transform is the following statement.

Theorem 3.6. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space and μ, μ' be fuzzy measures on $\langle X, \mathcal{F}\rangle$, and let $K, K' : X \times Y \to L$ be integral kernels.*

(i) If $\mu \preceq \mu'$, then $F^{\star}_{(K,\mu)}(f) \leq F^{\star}_{(K,\mu')}(f)$ for any $f \in \mathcal{F}(X)$.

(ii) If $K \leq K'$, then $F^{\star}_{(K,\mu)}(f) \leq F^{\star}_{(K',\mu)}(f)$ for any $f \in \mathcal{F}(X)$.

The next theorem shows conditions under which a constant function (fuzzy set) $\underline{a}_X$ is transformed to a constant function $\underline{a}_Y$, i.e., $F^{\star}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$. Recall that K_y denotes the y-projection of K to X.

Theorem 3.7. *Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, let K be an integral kernel, and let $a \in L$.*

(i) If for any $y \in Y$ there exists $A_y \in \mathcal{F}$ such that $A_y \subseteq \mathrm{Core}(K_y)$ and $\mu(A_y) = \top$, then $F^{\otimes}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$.

(ii) If for any $y \in Y$ and for any $A \in \mathcal{F}$ with $A \subseteq X \setminus \mathrm{Core}(K_y)$ it holds that $\mu(A) \leq a$, then $F^{\to}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$.

Proof. (i) Let $a \in L$ and $y \in Y$. By the assumption of (i), we assume that there is $A_y \subseteq \mathrm{Core}(K_y)$ such that $\mu(A_y) = \top$. Then

$$F^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) = \int^{\otimes} K(x,y) \otimes \underline{a}_X(x)\, d\mu = \bigvee_{A \in \mathcal{F}} (\mu(A) \otimes \bigwedge_{x \in A} (K(x,y) \otimes \underline{a}_X(x))$$
$$= \bigvee_{A \in \mathcal{F}} (\mu(A) \otimes \bigwedge_{x \in A} (K(x,y) \otimes a) \geq \mu(A_y) \otimes \bigwedge_{x \in A_y} (K(x,y) \otimes a) = \top \otimes a = a.$$

On the other side, we trivially have $\mu(A) \otimes \bigwedge_{x\in A}(K(x,y) \otimes a) \leq a$ for any $A \in \mathcal{F}$. Hence, we find $F^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) \leq a$, which proves the desired equality.

(ii) Let $a \in L$ and $y \in Y$. By the assumption of (ii), for any $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$, it holds that $\mu(A) \leq a$. Since $\mathrm{Core}(K_y) \neq \emptyset$, we get $X \not\subseteq X \setminus \mathrm{Core}(K_y)$. Then

$$F^{\rightarrow}_{(K,\mu)}(\underline{a}_X)(y) = \int^{\otimes} K(x,y) \rightarrow \underline{a}_X(x)\, d\mu =$$

$$\bigvee_{\substack{A\in\mathcal{F}\\ A\not\subseteq X\setminus \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x\in A}(K(x,y) \rightarrow a)) \vee \bigvee_{\substack{A\in\mathcal{F}\\ A\subseteq X\setminus \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x\in A}(K(x,y) \rightarrow a))$$

$$\leq \bigvee_{\substack{A\in\mathcal{F}\\ A\not\subseteq X\setminus \mathrm{Core}(K_y)}} (a \otimes \mu(A)) \vee \bigvee_{\substack{A\in\mathcal{F}\\ A\subseteq X\setminus \mathrm{Core}(K_y)}} \mu(A)$$

$$\leq (a \otimes \bigvee_{\substack{A\in\mathcal{F}\\ A\not\subseteq X\setminus \mathrm{Core}(K_y)}} \mu(A)) \vee a = (a \otimes \mu(X)) \vee a = a,$$

where we used the distributivity of $\otimes$ over $\bigvee$, the equality

$$\bigwedge_{x\in A}(K(x,y) \rightarrow a) = a,$$

for any $A \not\subseteq X \setminus \mathrm{Core}(K_y)$, which follows from the fact that $K(x,y) = \top$ for some $x \in A$, $\top \rightarrow a = a$, and $b \rightarrow a \geq a$ for any $b \in L$, and the assumption stating that $\mu(A) \leq a$ for any $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$. Conversely, we have

$$F^{\rightarrow}_{(K,\mu)}(\underline{a}_X)(y) \geq \bigwedge_{x\in X} K(x,y) \rightarrow \underline{a}_X(x) = \bigwedge_{x\in X}(K(x,y) \rightarrow a) = a,$$

where we used the same arguments as above, which proves the desired equality. □

It is worth noting that the standard real-valued fuzzy transforms as well as lower and upper lattice fuzzy transforms preserve constant functions; therefore, it seems to be natural to assume that integral kernels and fuzzy measures as the parameters of the lattice integral transforms satisfy the conditions under which the constant functions are preserved. In addition, the preservation of constant functions proved to be an essential condition for the successful reconstruction of the original functions using lattice integral transforms, so we will discuss this property in more detail.

Definition 3.3. Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, let K be an integral kernel, and let $a \in L$. The sufficient condition in Theorem 3.7(i) is denoted by (C1) and we say that (K,μ) *satisfies (C1)*. The sufficient condition in Theorem 3.7(ii) is denoted by (C2) and we say that (K,μ) *satisfies (C2) for* a, where $a \in L$. If (K,μ) satisfies (C2) for any $a \in L$, then we say that (K,μ) *satisfies (C2)*.

It is easy to see that (K,μ) satisfies (C2) if and only if (K,μ) satisfies (C2) for $a = \bot$. The following example shows that if (C1) is not satisfied for a fuzzy measure and an integral kernel, the preservation of constant functions may fail.

Example 3.1. Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, and assume that $K : X \times Y \to L$ is given such that $K(x, y) \in \{\bot, \top\}$, and K determines a fuzzy partition of X. Let $a, a' \in L$, $y \in Y$, and assume that $\mu(A) < a' < \top$ for any $A \in \mathcal{F}$ such that $A \subseteq \mathrm{Core}(K_y)$, where $a \otimes a' < a$. Note that $K(x, y) = \bot$ for any $x \notin \mathrm{Core}(K_y)$. Then

$$F^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) = \bigvee_{A\in\mathcal{F}} (\mu(A) \otimes \bigwedge_{x\in A}(K(x, y) \otimes a)$$

$$= \bigvee_{\substack{A\in\mathcal{F}\\ A\subseteq \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x\in A}(K(x, y) \otimes a) \vee \bigvee_{\substack{A\in\mathcal{F}\\ A\not\subseteq \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x\in A}(K(x, y) \otimes a)$$

$$= \bigvee_{\substack{A\in\mathcal{F}\\ A\subseteq \mathrm{Core}(K_y)}} (\mu(A) \otimes a) \vee \bot \leq a' \otimes a < a.$$

Note that we assume $\mu(A) < a' < \top$ for any $\mathcal{F}$-measurable set $A \subseteq \mathrm{Core}(K_y)$, because if

$$\bigvee_{\substack{A\in\mathcal{F}\\ A\subseteq \mathrm{Core}(K_y)}} \mu(A) = \top,$$

the equality $F^{\otimes}_{(K,\mu)}(a_X)(y) = a$ would hold even when the assumption of the previous theorem is not satisfied.

It is rather difficult to establish a necessary and sufficient condition for the preservation of constant functions using $(K, \mu, \otimes)$–M–lattice integral transforms in a general case, but we can do it for finite sets, where we assume that $\mathbf{L}$ is linearly ordered and $\otimes$ satisfies the conditional cancellation law, i.e., $a \otimes b = a \otimes c > \bot$ implies $b = c$ for any $a, b, c \in L$.

Theorem 3.8. *Let* $\mathbf{L}$ *be linearly ordered and* $\otimes$ *satisfy the conditional cancellation law, and assume that* X *is a finite non-empty set. A* $(K, \mu, \otimes)$*–M–lattice integral transform preserves non-zero constant functions if and only if* (K, μ) *satisfies (C1).*

Proof. Let $\underline{a}_X \in \mathcal{F}(X)$ such that $a \neq \bot$, and let $y \in Y$. Due to Theorem 3.7(i), it is sufficient to prove the sufficient condition. As a consequence of the conditional cancellation law, we find that $K(x, y) \otimes a = a$ for some $(x, y) \in X \times Y$ if and only if $K(y, x) = \top$, i.e., $x \in K_y$. Since X is a finite set, we get

$$a = F^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) = \bigvee_{A\in\mathcal{F}} \mu(A) \otimes \bigwedge_{x\in A}(K(x, y) \otimes a)$$

if and only if there is $B \in \mathcal{F}$ such that $\mu(B) \otimes \bigwedge_{x\in B}(K(x, y) \otimes a) = a$. From the conditional cancellation law, we obtain that $\mu(B) = \top$ and $\bigwedge_{x\in B}(K(x, y) \otimes a) = a$. Indeed, we have $\bigwedge_{x\in B}(K(x, y) \otimes a) \leq a$ and to ensure the previous equality, it must be $\bigwedge_{x\in B}(K(x, y) \otimes a) \geq a$, which implies the desired equality. The equality $\mu(B) = \top$ immediately follows from the conditional cancellation law ($\mu(B) \otimes a = \top \otimes a$). In addition, we have $K(x, y) \otimes a = a$ for any $x \in B$, which implies $B \subseteq \mathrm{Core}(K_y)$. Hence, there exists $B \in \mathcal{F}$ such that $B \subseteq \mathrm{Core}(K_y)$ and $\mu(B) = \top$. $\square$

Note that $F^{\otimes}_{(K,\mu)}(\underline{0}_X) = \underline{0}_Y$ holds for arbitrary parameters of $(K, \mu, \otimes)$–M–lattice integral transforms, i.e., fuzzy measure spaces and integral kernels, which immediately follows from $\int^{\otimes} \underline{0}_X \, d\mu = 0$ (see, Theorem 2.7(ii)).

The following example shows that if (C2) is not satisfied, we can find a fuzzy measure and an integral kernel such that the constant function is not preserved.

Example 3.2. Let $\langle X, \mathcal{F}, \mu \rangle$ be a fuzzy measure space, and assume that $K : X \times Y \to L$ is given such that $K(x, y) \in \{\bot, \top\}$ and determines a fuzzy partition of X. Let $a \in L$, $y \in Y$, and assume that $\mu(B) > a$ for some $B \subseteq X \setminus \mathrm{Core}(K_y)$. Note that $K(x, y) = \bot$ for any $x \in B$, otherwise, $x \in \mathrm{Core}(K_y)$, which is a contradiction. Then

$$\begin{aligned} F^{\to}_{(K,\mu)}(\underline{a}_X)(y) &\geq \bigvee_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x \in A} (K(x, y) \to a)) \\ &\geq \mu(B) \otimes \bigwedge_{x \in B} (K(x, y) \to a) \\ &= \mu(B) \otimes (\bot \to a) = \mu(B) \otimes \top = \mu(B) > a, \end{aligned}$$

where we used that $\bot \to a = \top$ for any $a \in L$.

The next example shows that condition (C2) need not be satisfied to ensure the preservation of a constant function.

Example 3.3. Assume that $\mathbf{L}$ is the Łukasiewicz algebra on $[0, 1]$. It is easy to demonstrate that $b \otimes (b \to a) = a$ for any $a, b \in L$ such that $a \leq b$. Let $a, b \in L$ with $a < b$. Let $K : X \times Y \to L$ be an integral transform such that $K(x, y) \in \{b, 1\}$ for any $(x, y) \in X \times Y$ and $\mathrm{Core}(K_y) \neq X$ for any $y \in Y$. Consider a fuzzy measure μ on $\langle X, \mathcal{P}(X) \rangle$ for a non-empty set X, where $\mu(X \setminus \mathrm{Core}(K_y)) = b$ for any $y \in Y$. Then, for any $y \in Y$, we have

$$\begin{aligned} F^{\to}_{(K,\mu)}(\underline{a}_X)(y) = &\bigvee_{\substack{A \in \mathcal{F} \\ A \not\subseteq X \setminus \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x \in A} (K(x, y) \to a)) \vee \\ &\bigvee_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\mu(A) \otimes \bigwedge_{x \in A} (K(x, y) \to a)) = \bigwedge_{x \in X} (K(x, y) \to a) \vee \\ &\mu(X \setminus \mathrm{Core}(K_y)) \otimes \bigwedge_{x \in X \setminus \mathrm{Core}(K_y)} (K(x, y) \to a) \\ &= a \vee b \otimes (b \to a) = a, \end{aligned}$$

but for any $A \subseteq X \setminus \mathrm{Core}(K_y)$ we have $\mu(A) > a$.

Note that $F^{\to}_{(K,\mu)}(1_X) = 1_Y$ holds for arbitrary parameters of $(K, \mu, \to)$–M–lattice integral transforms, i.e., fuzzy measure spaces and integral kernels, which immediately follows from $\int^{\otimes} (K(x, y) \to 1_X(x)) \, d\mu = \int^{\otimes} 1_X \, d\mu = 1$ (see (ii) of Theorem 2.7). Recall that $\mu^{c,N}$ denotes the N–conjugate fuzzy measure to μ. The next theorem shows a useful relation between the satisfaction of conditions (C1) and (C2) for μ and its N–conjugate fuzzy measure.

Theorem 3.9. *Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, let K be an integral kernel.*

(i) If (K, μ) satisfies (C1), then $(K, \mu^{c,N})$ satisfies (C2).

(ii) If (K, μ) satisfies (C2) and $\mathrm{Core}(K_y) \in \mathcal{F}$ for any $y \in Y$, then $(K, \mu^{c,N})$ satisfies (C1).

Proof. (i) Assume that (K, μ) satisfies (C1), and let $y \in Y$ and $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$. By the assumption there exists $A_y \in \mathcal{F}$ such that $A_y \subseteq \mathrm{Core}(K_y)$ and $\mu(A_y) = \top$. Hence, we find that $A_y \subseteq \mathrm{Core}(K_y) \subseteq X \setminus A$, and thus $\top = \mu(A_y) \leq \mu(X \setminus A)$, which implies $\mu^{c,N}(A) = N(\mu(X \setminus A)) = N(\top) = \bot$. Therefore, (C2) is satisfied by $(K, \mu^{c,N})$.

(ii) Assume that (K, μ) satisfies (C2), let $y \in Y$ and put $A_y = \mathrm{Core}(K_y) \in \mathcal{F}$. Then trivially $X \setminus A_y \subseteq X \setminus \mathrm{Core}(K_y)$, and by the assumption, we have $\mu(X \setminus A_y) = \bot$ (recall that (C2) is satisfied for all $a \in L$, which is equivalent to (C2) is satisfied for $a = \bot$). Hence, we find that $\mu^{c,N}(A_y) = N(\mu(X \setminus A_y)) = N(\bot) = \top$. Therefore, $\mu^{c,N}$ satisfies (C1). □

In the next example, we give a class of fuzzy measures that, together with a given integral kernel, satisfy (C1). In addition, we introduce the N–conjugate fuzzy measures to them, which satisfy (C2).

Example 3.4. Let $\langle X, \mathcal{F}\rangle$ be a measurable space with $X = \{x_1, \ldots, x_n\}$, $L = [0, 1]$ and $\mathcal{F} = \mathcal{P}(X)$, and let $\mathcal{M}^r$ be the class of fuzzy measures on $\langle X, \mathcal{F}\rangle$ introduced in Example 2.8. Let $K : X \times Y \to [0, 1]$ be an integral kernel, where $Y = \{y_1, \ldots, y_m\}$ and put $u = \min\{\#\,\mathrm{Core}(K_{y_j}) \mid y_j \in Y\}/n$. Then the class

$$\mathcal{M}^r_u = \{\mu^r_{\varphi^p_{\ell,u}} \mid \ell \in [0, 1],\ \ell \leq u,\ p \in \mathbb{N},\ p > 0\} \tag{3.7}$$

consists of all fuzzy measures, for which the $(K, \mu, \otimes)$–M–lattice integral transform preserves constant functions. Indeed, we trivially have $\mathrm{Core}(K_{y_j}) \in \mathcal{F}$ for any $y_j \in Y$. By the definition of u, we find that for any $y_j \in Y$ it holds

$$\mu^r_{\varphi^p_{\ell,u}}(\mathrm{Core}(K_{y_j})) = \varphi^p_{\ell,u}(\#\,\mathrm{Core}(K_{y_j})/n) = 1$$

for any $\ell \leq u$ and $p > 0$, since $u \leq \#\,\mathrm{Core}(K_{y_j})/n$. Hence, condition (C1) is satisfied and thus $(K, \mu, \otimes)$–M–lattice integral transform preserves constant functions for any $\mu \in \mathcal{M}^r_u$. In addition, the class

$$\mathcal{M}^{r,c,N}_u = \{\mu^r_{\varphi^{p,c,N}_{\ell,u}} \mid \ell \in [0, 1],\ \ell \leq u,\ p \in \mathbb{N},\ p > 0\} \tag{3.8}$$

consists of all N–conjugate fuzzy measures to fuzzy measures from $\mathcal{M}^r_u$ (see, Example 2.9). By Theorem 3.9, we get that the $(K, \mu^r_{\varphi^{p,c,N}_{\ell,u}}, \to)$–M–lattice integral transform preserves constant functions.

In the following example, we demonstrate a multiplication-based lattice integral transform on a real function that imitates a part of a discrete signal.

Example 3.5. Assume that $\mathbf{L}$ is the Łukasiewicz algebra on $[0,1]$ (see, Example 1.1), and the discrete signal is given by the formula

$$f(x) = 0.3 \cdot \cos(x/36)^2 \cdot \sin(x/5) + 0.5 \tag{3.9}$$

on $X = \{1, 2, 3, \ldots, 204\}$. Let $Y = \{1, 8, 15, \ldots, 204\}$ be a subset of X such that the difference between two consecutive elements in Y is 7. In what follows, we present three cases to demonstrate the effect of

a) different fuzzy measures for the same integral kernel,

b) different integral kernels for the same fuzzy measure, and

c) "fuzziness", which means the fuzzy part of the integral kernel (i.e., the membership degrees different from 0 and 1),

to the output of the multiplication-based lattice integral transform. In all cases, the integral kernel K and the fuzzy measure μ are introduced in such a way that (K, μ) satisfies condition (C1) for $\star = \otimes$, and as a consequence of Theorem 3.9, we get that $(K, \mu^{c,N})$ satisfies condition (C2) for $\star = \rightarrow$. More specifically, we consider $\mu \in \mathcal{M}_u^r$ for $\star = \otimes$ and its N–conjugate fuzzy measure $\mu^{c,N} \in \mathcal{M}_u^{r,c,N}$ for $\star = \rightarrow$ (see, Example 3.4), where $N(a) = N_{\text{res}}(a) = 1 - a$ for any $a \in [0,1]$. This assumption ensures that the $(K, \mu, \otimes)$–M–lattice integral transform and the $(K, \mu^{c,N}, \rightarrow)$–M–lattice integral transform preserve an arbitrary constant function.

Case a) Consider the following integral kernel $K : X \times Y \rightarrow [0,1]$:

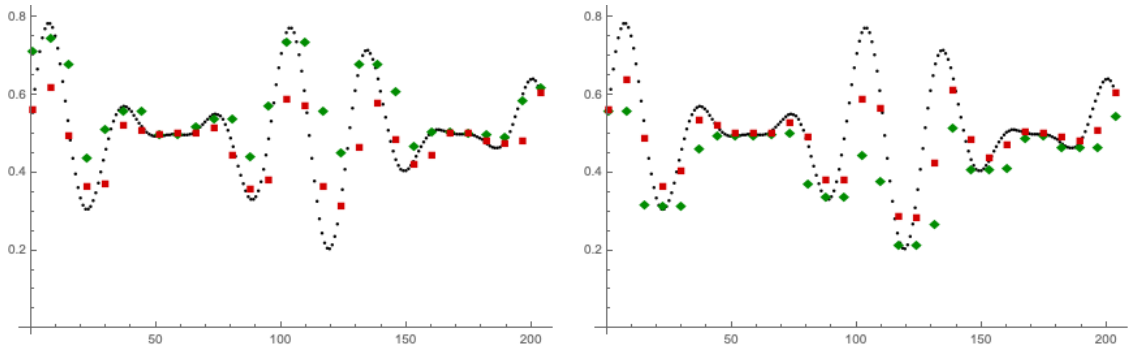
$$K(x,y) = \begin{cases} 1, & |x-y| \leq 8, \\ 0.8, & |x-y| \in \{9, 10\}, \\ 0.6, & |x-y| \in \{11, 12\}, \\ 0, & \text{otherwise}, \end{cases}$$

and define two fuzzy measures $\mu_1 = \mu_{2,6}^5$ and $\mu_2 = \mu_{7,12}^5$ on the measurable space $\langle X, \mathcal{P}(X) \rangle$.[1] Intuitively, the fuzzy measure μ_1 is set to choose a higher value from values $f(x) \otimes K(x,y)$, $x \in \text{Supp}(K_y)$, but not the highest or second highest value, which follows from the setting $U = 2$. The result of the $(K, \mu_1, \otimes)$–lattice integral transform is a (transformed) signal described by the green diamonds and shown in Figure 3.2(a) together with the original signal f. Obviously, the result of this transform is very similar to the upper lattice fuzzy transform ($\mu = \mu^\top = \mu_{0,0}^5$),[2] and the values of the original signal $f(x)$ at the points from Y are very similar or less than the values of the transformed signal. The fuzzy measure μ_2 is set to be smaller than μ_1 (i.e., $\mu_2 \preceq \mu_1$), which shifts the transformed signal down a bit (a consequence of Theorem 3.6(i)), as seen in Figure 3.2(a), where the transformed signal is described

[1] Recall that $\mu_{L,U}^p$ denotes the fuzzy measure $\mu^r_{\varphi^p_{L/n,U/n}}$ from Example 2.8. For details, see Remark 2.2.

[2] Note that the kernel K does not determined a fuzzy partition of X, but the interpretation of the direct upper and lower lattice fuzzy transforms remains the same even in this more general case.

by the red squares. Obviously, the latter and similar fuzzy measures are suitable for filtering out high frequencies, which will be demonstrated in the next chapter. On the other hand, the first and similar fuzzy measures can be suppressed high frequencies only in their "negative" parts, i.e., in the parts where the waves reach a local minimum.



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.2: M*–lattice integral transforms for a fixed integral kernel K and two different fuzzy measures $\mu^5_{2,6}$ (green diamonds) and $\mu^5_{7,12}$ (red squares).

The results of the $(K, \mu_i^{c,N}, \rightarrow)$–lattice integral transforms, $i = 1, 2$, are shown in Figure 3.2(b), where the transformed signal described by the green diamonds (red squares) is related to $\mu_1^{c,N}$ ($\mu_2^{c,N}$). The shape of both transformed signals is somehow dual to the shape of signals obtained by the previous integral transform ($\star = \otimes$), i.e., $\mu_1^{c,N}$ and similar fuzzy measures can be used to suppress high frequencies only in their "positive" parts, i.e., in the parts where the waves reach a local maximum. Note that the lower fuzzy transform also has this property.

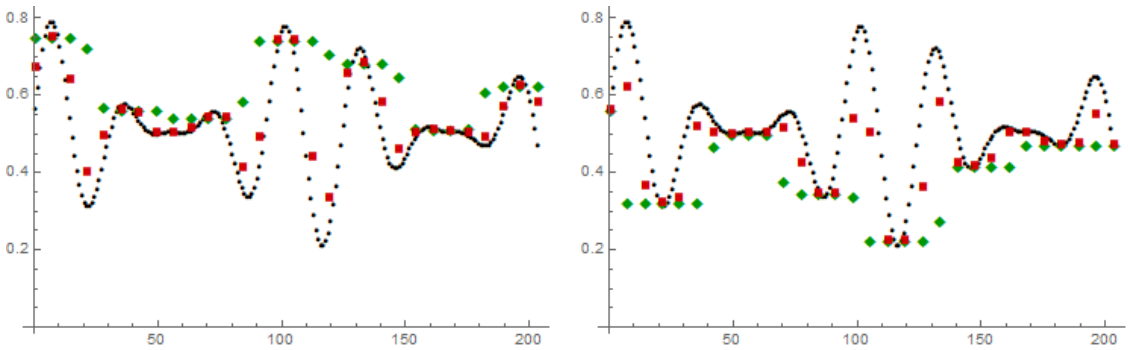
It should be noted that the near-constant parts of the original signal are transformed to near-constant parts of the transformed signal, with all lattice integral transforms resulting in nearly identical outputs. This is a consequence of the assumption that the fuzzy measure satisfies condition (C1) for $\star = \otimes$ and condition (C2) for $\star = \rightarrow$.

Case b) Now let us consider two integral kernels $K_1, K_2 : X \times Y \rightarrow [0, 1]$:

$$K_1(x, y) = \begin{cases} 1, & |x - y| \leq 16, \\ 0.8, & |x - y| = 17, \\ 0, & \text{otherwise}, \end{cases} \qquad K_2(x, y) = \begin{cases} 1, & |x - y| \leq 7, \\ 0.6, & |x - y| \in \{8, 9\}, \\ 0, & \text{otherwise}, \end{cases}$$

and the fuzzy measure $\mu = \mu^5_{3,6}$ on the measurable space $\langle X, \mathcal{P}(X)\rangle$. The fuzzy measure μ is set to satisfy condition (C1) for the integral kernel K_2 and take a value that is not far from the "middle" among the values $f(x) \otimes K_2(x, y)$, $x \in \mathrm{Supp}(K_{2,y})$. Unlike K_2, the fuzzy measure μ in the relationship to the integral kernel K_1 behaves as the fuzzy measure close to $\mu^\top$. The transformed signal obtained by the $(K_i, \mu, \otimes)$–lattice integral transform, $i = 1, 2$, is shown in Figure 3.3(a) together with the original signal, where the green diamonds (red squares) describe the transformed

signal related to K_1 (K_2). Since μ with respect to K_2 introduces a fuzzy integral that aggregates the values so they are not far from the "middle" value and the size of the support of K_2 is not high ($\#\mathrm{Supp}(K_{2,y})) = 19$), the transformed signal is close to the original signal at the points from Y. In contrast, the transformed signal for μ with respect to K_1 is similar to the transformed signal from case a) described by green diamonds in Figure 3.2(a), which suppresses or even removes the negative parts of higher frequencies.



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.3: M*–lattice integral transforms for a fixed fuzzy measure $\mu^5_{3,6}$ and two different integral kernels K_1 (green diamonds) and K_2 (red squares).

The transformed signal for the $(K_i, \mu^{c,N}, \rightarrow)$–M–lattice integral transform, $i = 1, 2$, is displayed in Figure 3.3(b). Similarly to case a), the results for this type of the lattice integral transform are somehow dual to the results for the previous type, leading to the suppression of the positive parts of higher frequencies. Again, the near-constant parts of the original signal are transformed to the near-constant parts of the transformed signal.

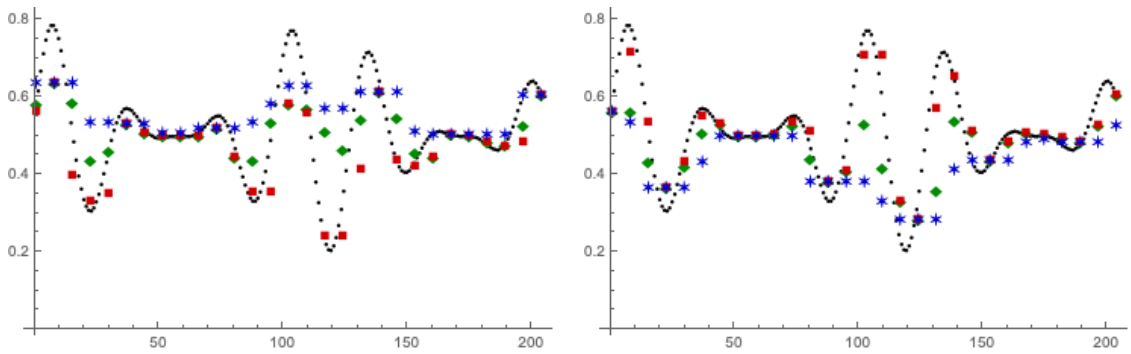
Case c) Finally, let us consider integral kernels $K, K^{\mathrm{Core}}, K^{\mathrm{Supp}} : X \times Y \rightarrow [0, 1]$:

$$K(x, y) = \begin{cases} 1, & |x - y| \leq 7, \\ 0.9, & |x - y| \in \{8, \ldots, 11\}, \\ 0.8, & |x - y| \in \{12, 13\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$K^{\mathrm{Core}}(x, y) = \begin{cases} 1, & |x - y| \leq 7, \\ 0, & \text{otherwise,} \end{cases} \qquad K^{\mathrm{Supp}}(x, y) = \begin{cases} 1, & |x - y| \leq 13, \\ 0, & \text{otherwise,} \end{cases}$$

and the fuzzy measure $\mu = \mu^5_{7,11}$ on the measurable space $\langle X, \mathcal{P}(X)\rangle$. The fuzzy measure μ is set to aggregate the values $f(x) \otimes K^{\mathrm{Core}}(x, y)$, $x \in \mathrm{Supp}(K^{\mathrm{Core}}_y)$ to a value that is not far from the "middle" value. In relation to K and K^{Supp}, the fuzzy measure μ is shifted to higher values. In Figure 3.4, we demonstrate that even if one integral kernel is the core or support of another, we obtain different transformed signals. More specifically, the transformed signal for the integral kernel

K with a fuzzy part is described by green diamonds. The crisp integral kernels K^{Core} determines the signal described by the red squares and K^{Supp} the signal described by the blue stars. All transformed signals differ from each other for both types of lattice integral transforms $\star = \otimes$ and $\star = \rightarrow$. We should note that the higher size of the



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.4: M*–lattice integral transforms for a fixed fuzzy measure $\mu^5_{7,11}$ and one integral kernel K with a fuzzy part (green diamonds) and two crisp integral kernels K^{Core} (red squares) and K^{Supp} (blue stars).

support of K^{Supp} ($\#\text{Supp}(K^{\text{Supp}}) = 27$) than in case of K^{Core} ($\#\text{Supp}(K^{\text{Core}}) = 15$) leads to a smoother output, which is mainly visible in Figure 3.4(a). However, appropriate setting of the fuzzy measure μ can lead to smoother output even in the case of $\star = \rightarrow$. Generally, integral kernels with a higher size of the support together with fuzzy measures using which the fuzzy integrals aggregate the values to a middle value are suitable for suppressing high frequencies in the signal, and thus we can obtain a smooth signal.

As a simple consequence of the comonotone property of the $\otimes$-fuzzy integral (see, Theorems 2.11), we can obtain an analogous result for the multiplication-based lattice integrals. Let K be an integral kernel, and let $\star = \{\otimes, \rightarrow\}$. We say that functions $f, g \in \mathcal{F}(X)$ are $(K, \star)$*–comonotonic* (or *comonotonically compatible with* K *and* $\star$) if $f \star K(\cdot, y)$ and $g \star K(\cdot, y)$ are comonotonic.

Theorem 3.10. *Let* $\mathbf{L}$ *be a linearly ordered complete Heyting algebra, let* $f, g, K(\cdot, y)$ *be* $\mathcal{F}$*-*$\mathcal{B}^u$*–measurable for any* $y \in Y$*, and let* $\odot \in \{\wedge, \vee\}$*.*

(i) If f *and* g *are* $(K, \wedge)$*–comonotonic, then*

$$F^{\wedge}_{(K,\mu)}(f \odot g) = F^{\wedge}_{(K,\mu)}(f) \odot F^{\wedge}_{(K,\mu)}(g). \tag{3.10}$$

(ii) If $\mathcal{F}$ *is closed under arbitrary union and* f *and* g *are* $(K, \rightarrow)$*–comonotonic, then*

$$F^{\rightarrow}_{(K,\mu)}(f \odot g) = F^{\rightarrow}_{(K,\mu)}(f) \odot F^{\rightarrow}_{(K,\mu)}(g). \tag{3.11}$$

Proof. (i) It is a simple consequence of Theorems 2.4 and 2.11 and the fact that $f \wedge K(\cdot, y)$ and $g \wedge K(\cdot, y)$ are comonotonic for any $y \in L$ and $\wedge$ is distributive over $\vee$.

(ii) From Theorems 2.5 and by the assumption, we have that the fuzzy sets $f \to K(\cdot, y)$ and $g \to K(\cdot, y)$ are $\mathcal{F}$-$\mathcal{B}^u$–measurable and $(K, \to)$–comonotonic for any $y \in Y$. Using Theorem 1.2(ii), the fact that $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$ holds in any linearly ordered residuated lattice and Theorem 2.11, we simply obtain the statement. □

3.4 DH–residuum-based lattice integral transforms

In this section, we introduce another type of lattice integral transforms that is based on the DH–residuum-based fuzzy integral. This type of integral transform was designed as a "negative" version of the multiplication-based lattice integral transform, whose output function reverses the values of the original function.[3] For example, the output of this lattice integral transform applied to an image is a negative (see, Subsection 6.3 on page 129). Nevertheless, its definition is completely analogous to the lattice integral transform based on multiplication, only the type of fuzzy integrals is changed.

Definition 3.4. Let $\langle X, \mathcal{F}, \nu \rangle$ be a complementary fuzzy measure space. Let $K : X \times Y \to L$ be an integral kernel, and $\star = \{\otimes, \to\}$. A map $G^{\star}_{(K,\nu)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by

$$G^{\star}_{(K,\nu)}(f)(y) = \int_{\mathrm{DH}}^{\to} K(x, y) \star f(x)\, d\nu \tag{3.12}$$

is called a $(K, \nu, \star)$–R_{DH}–*lattice integral transform.*

If the pair (K, ν) is known we simplify the notation of the DH–residuum based lattice integral transform to "$\mathrm{R}^*_{\mathrm{DH}}$–lattice integral transform" or express it using symbols "$\mathrm{R}^*_{\mathrm{DH}}$–LIT".

The following theorem summarizes basic properties of $(K, \nu, \star)$–R_{DH}–lattice integral transform (see, also [25]). We assume that a complementary fuzzy measure space $\langle X, \mathcal{F}, \nu \rangle$ is given and $K : X \times Y \to L$ is an integral kernel.

Theorem 3.11. *For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have*

(i) $G^{\star}_{(K,\nu)}(f) \geq G^{\star}_{(K,\nu)}(g)$ *if* $f \leq g$,

(ii) $G^{\star}_{(K,\nu)}(f \wedge g) \geq G^{\star}_{(K,\nu)}(f) \vee G^{\star}_{(K,\nu)}(g)$,

(iii) $G^{\star}_{(K,\nu)}(f) \wedge G^{\star}_{(K,\nu)}(g) \geq G^{\star}_{(K,\nu)}(f \vee g)$,

(iv) $G^{\star}_{(K,\nu)}(\underline{a}_X \otimes f) \leq a \to G^{\star}_{(K,\nu)}(f)$,

(v) $G^{\star}_{(K,\nu)}(\underline{a}_X \to f) \geq a \otimes G^{\star}_{(K,\nu)}(f)$.

If $\mathbf{L}$ *is an MV-algebra, then inequality (iv) can be replaced by equality.*

[3] Note that the negative output of the DH–residuum-based lattice integral transforms is a consequence of the relation between the multiplication and DH–residuum based fuzzy integrals where one is the (canonical) negation of the second one under certain conditions (e.g., in MV-algebra). For details, we refer to [11, 24].

Proof. We prove only the first case $\star = \otimes$, the second case $\star = \to$ can be proved analogously.

(i)-(iii) Similarly to the proof of Theorem 3.5(i)-(iii), it is a straightforward consequences of the monotonicity of the operation $\otimes$ (i.e., monotonically non-decreasing) and the $\to_{\mathrm{DH}}$–fuzzy integral, which reverses the ordering (Theorem 2.12(i)).

(iv) Using Theorem 2.12(iii) and the commutativity of $\otimes$, for any $y \in Y$, we have

$$G^{\otimes}_{(K,\nu)}(\underline{a}_X \otimes f)(y) = \int^{\to}_{\mathrm{DH}} K(x,y) \otimes (a \otimes f(x))\, d\nu$$
$$\leq a \to \int^{\to}_{\mathrm{DH}} K(x,y) \otimes f(x)\, d\nu = a \to G^{\otimes}_{(K,\nu)}(f)(y).$$

Moreover, if $\mathbf{L}$ is a complete MV–algebra, then the previous inequality can be replaced by the equality according to Theorem 2.12(iii) for an MV–algebra.

(v) Using (i) and (iv) of Theorem 2.12, we have

$$G^{\otimes}_{(K,\nu)}(\underline{a}_X \to f)(y) = \int^{\to}_{\mathrm{DH}} K(x,y) \otimes (a \to f(x))\, d\nu \geq$$
$$\int^{\to}_{\mathrm{DH}} (a \to (K(x,y) \otimes f(x))\, d\nu \geq a \otimes \int^{\to}_{\mathrm{DH}} K(x,y) \otimes f(x) = a \otimes G^{\otimes}_{(K,\nu)}(f)(y),$$

where we used $K(x,y) \otimes (a \to f(x)) \leq a \to (K(x,y) \otimes f(x))$ (see the proof of Theorem 2.16 for details). □

Again a straightforward consequence of the definition of the DH–residuum-based lattice integral transform is the following statement. We define the ordering of complementary fuzzy measures analogously to the ordering of fuzzy measures, i.e., $\nu_1 \preceq \nu_2$ if $\nu_1(A) \leq \nu_2(A)$ for any $A \in \mathcal{F}$.

Theorem 3.12. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space and ν, ν' be complementary fuzzy measures on $\langle X, \mathcal{F}\rangle$, and let $K, K' : X \times Y \to L$ be integral kernels.*

(i) If $\nu \preceq \nu'$, then $G^{\star}_{(K,\nu)}(f) \leq G^{\star}_{(K,\nu')}(f)$ for any $f \in \mathcal{F}(X)$.

(ii) If $K \leq K'$, then $G^{\star}_{(K,\nu)}(f) \geq G^{\star}_{(K',\nu)}(f)$ for any $f \in \mathcal{F}(X)$.

Similarly to Theorem 3.7, we are interested in sufficient conditions under which $(K, \nu, \star)$–R_{DH}–lattice integral transform ensure the reversation of constant functions, i.e., $G^{\star}_{(K,\nu)}(\underline{a}_X) = \underline{\neg a}_Y$.

Theorem 3.13. *Let $\langle X, \mathcal{F}, \nu\rangle$ be a complementary fuzzy measure space, let K be an integral kernel, and let $a \in L$.*

(i) If for any $y \in Y$ there exists $A_y \in \mathcal{F}$ such that $A_y \subseteq \mathrm{Core}(K_y)$ and $\nu(A_y) = \bot$, then $G^{\otimes}_{(K,\nu)}(\underline{a}_X) = \underline{\neg a}_Y$.

(ii) If for any $y \in Y$ and for any $A \in \mathcal{F}$ with $A \subseteq X \setminus \mathrm{Core}(K_y)$ it holds that $\nu(A) \geq \neg a$, then $G^{\to}_{(K,\nu)}(\underline{a}_X) = \underline{\neg a}_Y$.

Proof. (i) Let $y \in Y$, and $\underline{a}_X \in \mathcal{F}(X)$. Assume that there is $A_y \subseteq \mathrm{Core}(K_y)$ such that $\nu(A_y) = \bot$. Then

$$G^{\otimes}_{(K,\nu)}(\underline{a}_X)(y) = \int_{\mathrm{DH}}^{\rightarrow} K(x,y) \otimes \underline{a}_X(x)\, d\nu = \bigwedge_{A \in \mathcal{F}} (\bigwedge_{x \in A} (K(x,y) \otimes a) \rightarrow \nu(A))$$
$$\leq \bigwedge_{x \in A_y} (\top \otimes a) \rightarrow \bot = a \rightarrow \bot = \neg a = \underline{\neg a}_Y(y).$$

On the other side, for any $A \in \mathcal{F}$, we trivially have $\bigwedge_{x \in A}(K(x,y) \otimes a) \rightarrow \nu(A) \geq a \rightarrow \bot = \underline{\neg a}_Y(y)$, where we used the monotonically non-increasing in the first argument and the monotonically non-decreasing in the second argument of residuum. Hence, we find $G^{\otimes}_{(K,\nu)}(\underline{a}_X)(y) \geq \underline{\neg a}_Y(y)$, which proves the desired equality.

(ii) Let $y \in Y$. Since $\mathrm{Core}(K_y) \neq \emptyset$, so $X \not\subseteq X \setminus \mathrm{Core}(K_y)$. Then

$$G^{\rightarrow}_{(K,\nu)}(\underline{a}_X)(y) =$$
$$\int_{\mathrm{DH}}^{\rightarrow} K(x,y) \rightarrow \underline{a}_X(x)\, d\nu = \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\bigwedge_{x \in A} (K(x,y) \rightarrow a) \rightarrow \nu(A))$$
$$\wedge \bigwedge_{\substack{A \in \mathcal{F} \\ A \not\subseteq X \setminus \mathrm{Core}(K_y)}} (\bigwedge_{x \in A} (K(x,y) \rightarrow a) \rightarrow \nu(A)) \geq \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\top \rightarrow \nu(A))$$
$$\wedge \bigwedge_{\substack{A \in \mathcal{F} \\ A \not\subseteq X \setminus \mathrm{Core}(K_y)}} (a \rightarrow \nu(A)) \geq \neg a \wedge (a \rightarrow \bot) = \neg a = \underline{\neg a}_Y(y),$$

where we used that the residuum is monotonically non-increasing in the first component and the monotonically non-decreasing in the second component, i.e., trivially $K(x,y) \rightarrow a \leq \top$, therefore, $(K(x,y) \rightarrow a) \rightarrow \nu(A) \geq \top \rightarrow \nu(A) = \nu(A) \geq \neg a$ for $A \subseteq X \setminus \mathrm{Core}(K_y)$ by the assumption in (ii). Contrary, we have

$$G^{\rightarrow}_{(K,\nu)}(\underline{a}_X)(y) \leq \bigwedge_{x \in X} (K(x,y) \rightarrow a) \rightarrow \nu(X) = a \rightarrow \bot = \neg a = \underline{\neg a}_Y(y),$$

where we used the fact that $K(x,y) \rightarrow a \geq a$ for any $(x,y) \in X \times Y$ and $K(x,y) \rightarrow a = \top \rightarrow a = a$ for any $x \in \mathrm{Core}(K_y)$, which proves the desired equality. □

Definition 3.5. Let $\langle X, \mathcal{F}, \nu \rangle$ be a complementary fuzzy measure space, let K be an integral kernel. The sufficient condition in Theorem 3.13(i) is denoted by (C3) and we say that (K,ν) *satisfies (C3)*. The sufficient condition in Theorem 3.13(ii) is denoted by (C4) and we say that (K,ν) *satisfies (C4) for* a, where $a \in L$. If (K,ν) satisfies (C4) for any $a \in L$, then we say that (K,ν) *satisfies (C4)*.

Obviously, (K,ν) satisfies (C4) if and only if (K,ν) satisfies (C4) for $a = \top$. Recall that the N–conjugate complementary fuzzy measure to ν is denoted by $\nu^{c,N}$ and defined as $\nu^{c,N}(A) = N(\nu(X \setminus A))$ for any $A \in \mathcal{F}$. The next theorem provides an analogous statement as in Theorem 3.9 expressing the relation between the satisfaction of conditions (C3) and (C4) and the complementary fuzzy measures ν and $\nu^{c,N}$.

Theorem 3.14. *Let $\langle X, \mathcal{F}, \nu\rangle$ be a complementary fuzzy measure space, let K be an integral kernel.*

(i) If (K, ν) satisfies (C3), then $(K, \nu^{c,N})$ satisfies (C4).

(ii) If (K, ν) satisfies (C4) and $\mathrm{Core}(K_y) \in \mathcal{F}$ for any $y \in Y$, then $(K, \nu^{c,N})$ satisfies (C3).

Proof. (i) Let $y \in Y$ and $A_y \in \mathcal{F}$ be such that $A_y \subseteq \mathrm{Core}(K_y)$ and $\nu(A_y) = \bot$. Let $A \in \mathcal{F}$ be such that $A \subseteq X \setminus \mathrm{Core}(K_y)$. Since $A_y \subseteq \mathrm{Core}(K_y) \subseteq X \setminus A$ and $\nu(A_y) = \bot$, we get that $\nu(X \setminus A) = \bot$, since ν is a monotonically non-increasing set map, which implies $\nu^{c,N}(A) = N(\nu(X \setminus A)) = N(\bot) = \top$. Hence, we obtain that $\nu^{c,N}$ satisfies (C4).

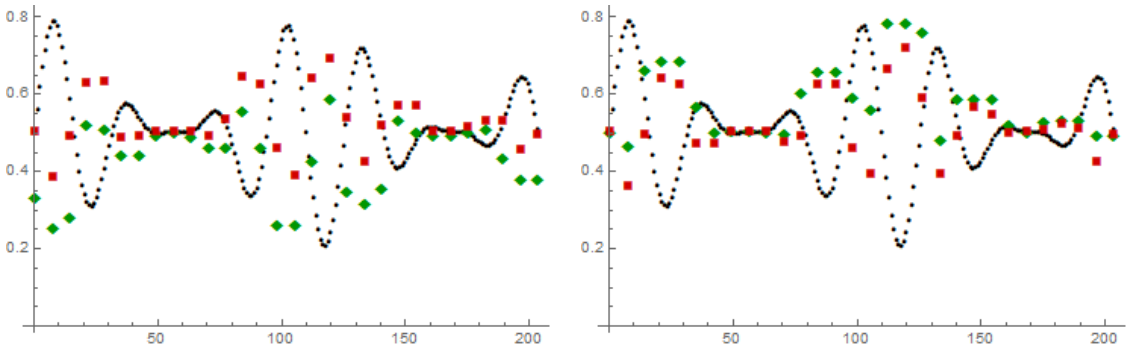
(ii) Let $y \in Y$ and assume that for any $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$ there is $\nu(A) = \top$. Since $\mathrm{Core}(K_y) \in \mathcal{F}$, put $A_y = \mathrm{Core}(K_y)$. Then trivially $X \setminus A_y \subseteq X \setminus \mathrm{Core}(K_y)$, and thus $\nu(X \setminus A_y) = \top$, which implies $\nu^{c,N}(A_y) = N(\nu(X \setminus A_y)) = N(\top) = \bot$. Hence, $(K, \nu^{c,N})$ satisfies (C3). □

Remark 3.3. From Lemma 2.2, we know that a complementary fuzzy measure can be introduced from a given fuzzy measure using a negation N on L as $\nu(A) = \mu^N(A) = N(\mu(A))$ for any $A \in \mathcal{F}$ (see, Lemma 2.2). It is easy to observe that if K is an integral kernel and μ a fuzzy measure such that condition (C1) (or (C2)) is satisfied, then $\nu = \mu^N$ is a complementary fuzzy measure that satisfies (C3) (or (C4)), where we assume that $N \geq N_{\mathrm{res}} = \neg$ (i.e., $N(a) \geq \neg a$ for any $a \in L$) for condition (C4). Indeed, if $\mu(A_y) = \top$ for some $A_y \subseteq \mathrm{Core}(K_y)$, then $\nu(A_y) = N(\mu(A_y)) = N(\top) = \bot$. Similarly, if for any $A \subset X \setminus \mathrm{Core}(K_y)$, we have $\mu(A) \leq a$, then $\nu(A) = N(\mu(A)) \geq N(a) \geq \neg a$. Obviously, if $G^{\rightarrow}_{(K,\nu)}$ has to reverse an arbitrary constant function, i.e., (C4) holds for any $a \in L$, which is equivalent to $\nu(A) = \top$ holds for any $A \in \mathcal{F}$ with $A \subseteq X \setminus \mathrm{Core}(K_y)$, then we can consider an arbitrary negation N on L to get the desired reversation. Using this observation and Theorem 3.9, we can can simply construct fuzzy measures and complementary fuzzy measures satisfying the corresponding sufficient conditions for a given integral kernel K from a fuzzy measure μ which, together with K, satisfies (C1).

The following example demonstrates the DH–residuum-based lattice integral transform on a real function that imitates a part of a discrete signal.

Example 3.6. We use the same residuated lattice, fuzzy measures (i.e., $\mu \in \mathcal{M}^r_u$) and integral kernels as in Example 3.5 and present the results of DH–residuum-based lattice integral transforms applied on the signal f given by formula (3.9) on $\{x_1, \ldots, x_n)$ for cases a) and b). We consider the complementary fuzzy measure defined as $\nu = \mu^N$, where $N = N_{\mathrm{res}}$ is the negation in the Łukasiewicz algebra, i.e., $N(a) = 1-a$ for any $a \in [0, 1]$. Since (K, μ) satisfies (C1) for all integral kernels used in Example 3.5, we get that (K, ν) satisfies (C3) by Remark 3.3. The N–conjugate complementary fuzzy measure to ν is given by $\nu^{c,N} = \mu^c$ (see, Lemma 2.2), i.e., $\nu^{c,N}(A) = \mu(X \setminus A)$.[4] Since the N–conjugate fuzzy measure $\mu^{c,N}$ together with all

[4]Note that the definition of $\nu^{c,N}$ is a consequence of the fact that N is involutive. Indeed, for $A \in \mathcal{F}$, we have $\nu^{c,N}(A) = N(\nu(X \setminus A)) = N(N(\mu(X \setminus A))) = \mu(X \setminus A) = \mu^c(A)$.



(a) $\star = \otimes$ (b) $\star = \rightarrow$

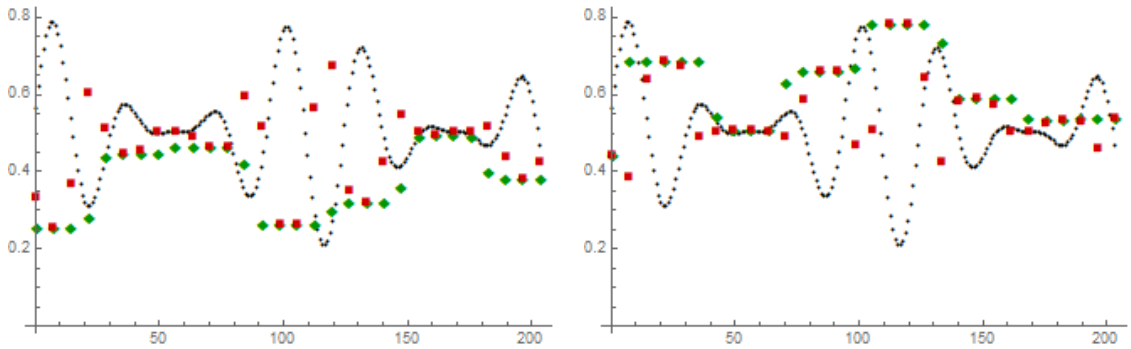
Figure 3.5: $\mathrm{R}^{\star}_{\mathrm{DH}}$–lattice integral transforms for a fixed integral kernel K and two different complementary fuzzy measures ν_1 (green diamonds) and ν_2 (red squares).

integral kernels used in Example 3.5 satisfies condition (C2) for any $a \in L$ and the negation of $\mu^{c,N}$ is the complementary fuzzy measure μ^c, we know from Remark 3.3 that μ^c together with all integral kernels used in Example 3.5 satisfies (C4) for any $a \in L$. Hence, the setting of ν and $\nu^{c,N}$ ensures that the $(K, \nu, \otimes)$–lattice integral transform and the $(K, \nu^{c,N}, \rightarrow)$–lattice integral transform reverse any constant function.

Case a) We consider one integral kernel $K : X \times Y \rightarrow [0, 1]$ and two complementary fuzzy measures ν_1 and ν_2 defined from the fuzzy measures $\mu_1 = \mu^5_{2,6}$ and $\mu_2 = \mu^5_{7,12}$. Figure 3.5(a) shows the resulting signals for the $(K, \nu_1, \otimes)$–lattice integral transform described by the green diamonds and for $(K, \nu_2, \otimes)$–lattice integral transform described by the red squares together with the original signal f. In contrast to the transformed signals in case a) of Example 3.5, these signals naturally reverse the ordering, i.e., higher values of the original signal are transformed to lower values and vice versa. This can also be understood that the DH–residuum based lattice integral transform is the multiplication-based lattice integral transform applied to the negation of the signal f, but it should be emphasized that the results of the two approaches coincide only in the special case. One could be surprised that the nearly-constant parts of the original signal are again transformed to the nearly-constant parts, but this is a consequence of the fact that the nearly constant parts are closed to the value 0.5 and $N(0.5) = 0.5$. However, the outputs here show similar behavior as for multiplication-based integral transform with respect to the fuzzy measures, only in the reverse ordering. For example, in case of ν_2 we obtain a signal where higher frequencies are suppressed (cf. the signals described by red squares in Figures 3.2(a) and 3.5(a)). The transformed signals for the $(K, \nu_i^{c,N}, \rightarrow)$–lattice integral transforms, $i = 1, 2$, are displayed in Figure 3.5(b). Their analysis is analogous to $* = \rightarrow$ in case a) of Example 3.5, only in the reverse ordering.

Case b) We consider two integral kernels $K_1, K_2 : X \times Y \rightarrow [0, 1]$ and one complementary fuzzy measure ν defined from the fuzzy measure $\mu = \mu^5_{3,6}$. Resulting signals of the $(K_i, \nu, \otimes)$–lattice integral transforms, $i = 1, 2$, are shown in Figure 3.6(a).

Again these signals are in the reverse ordering, but have similar behaviour as the transformed signals in case b) of Example 3.5 for $\star = \otimes$. For example, the signal described by the red squares is close (approximates) the negation of the original function at the points from Y, while the signal described by the green diamonds approximates only lower part of the negation of the the original function. Resulting signals of the $(K_i, \nu^{c,N}, \rightarrow)$–lattice integral transforms, $i = 1, 2$, are shown in Figure 3.6(b).



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.6: $\mathrm{R}^{\star}_{\mathrm{DH}}$–lattice integral transforms for a fixed complementary fuzzy measure ν and two different integral kernels K_1 (green diamonds) and K_2 (red squares).

We conclude this section with the statement similar to Theorem 3.10, which shows the linearity property of the DH–residuum-based lattice integral transform for comonotonic functions. In this case, the linearity of the transform means reversing the join operation into a meet operation.

Theorem 3.15. *Let* $\mathbf{L}$ *be a linearly ordered and assume that the algebra* $\mathcal{F}$ *is closed over arbitrary unions. Let* $f, g, K(\cdot, y)$ *be* $\mathcal{F}$-$\mathcal{B}^u$*–measurable for any* $y \in Y$*. If* $K(\cdot, y) \star f$ *and* $K(\cdot, y) \star g$ *are comonotonic for* $\star \in \{\otimes, \rightarrow\}$*, then*

$$G^{\star}_{(K,\nu)}(f \vee g) = G^{\star}_{(K,\nu)}(f) \wedge G^{\star}_{(K,\nu)}(g).$$

Proof. As a consequence of Theorem 2.5 and Theorem 2.6, we find that $K(\cdot, y) \star f$ and $K(\cdot, y) \star g$ are $\mathcal{F}$-$\mathcal{B}^u$–measurable. Since $K(x, y) \star (f(x) \vee g(x) = (K(x, y) \star f(x)) \vee (K(x, y) \star g(x))$ for any $x \in X$ and $y \in Y$ holds in any linearly ordered residuated lattice, we get by Theorem 2.15 the statement. □

3.5 DPR–residuum-based lattice integral transforms

The last type of lattice integral transforms presented in this chapter is defined using the DPR–residuum based fuzzy integral. The motivation for introducing this type of transform was to create an alternative version of the multiplication-based lattice integral transformation, where the input values are processed in the reverse order, i.e., the smallest value gives the best evaluation in decision making, but the final evaluation has a standard order.

Definition 3.6. let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, let $K : X \times Y \to L$ be an integral kernel, and let $\star = \{\otimes, \to\}$. A map $H^{\star}_{(K,\mu)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by

$$H^{\star}_{(K,\mu)}(f)(y) = \int_{\mathrm{DPR}}^{\to} K(x,y) \star f(x)\, d\mu \tag{3.13}$$

is called a $(K, \mu, \star)$–R_{DPR}–*lattice integral transform.*

If the pair (K, μ) is known we simplify the notation of the DPR–residuum based lattice integral transform to "$\mathrm{R}^{*}_{\mathrm{DPR}}$–lattice integral transform" or express it using symbols "$\mathrm{R}^{*}_{\mathrm{DPR}}$–LIT". The following theorem presents the basic properties of $(K, \mu, \star)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms (see, also [25]).

Theorem 3.16. *For any $f, g \in \mathcal{F}(X)$, $\star = \{\otimes, \to\}$ and $a \in L$, we have*

(i) $H^{\star}_{(K,\mu)}(f) \leq H^{\star}_{(K,\mu)}(g)$ *if* $f \leq g$,

(ii) $H^{\star}_{(K,\mu)}(f \wedge g) \leq H^{\star}_{(K,\mu)}(f) \wedge H^{\star}_{(K,\mu)}(g)$,

(iii) $H^{\star}_{(K,\mu)}(f \vee g) \geq H^{\star}_{(K,\mu)}(f) \vee H^{\star}_{(K,\mu)}(g)$,

(iv) $H^{\star}_{(K,\mu)}(\underline{a}_X \otimes f) \geq a \otimes H^{\star}_{(K,\mu)}(f)$,

(v) $H^{\star}_{(K,\mu)}(\underline{a}_X \to f) \leq a \to H^{\star}_{(K,\mu)}(f)$.

If $\mathbf{L}$ *is an MV–algebra, then inequality (v) can be replaced by equality.*

Proof. The theorem can be proved completely analogously to Theorem 3.5 using the properties of $\to_{\mathrm{DPR}}$–fuzzy integral (Theorem 2.16), so we omit its proof here. □

The next theorem is analogous to Theorem 3.6 and is a straightforward consequence of the definition of the DPR–residuum-based lattice integral transform and the fact that $\mu_2^{c,N} \preceq \mu_1^{c,N}$ whenever $\mu_1 \preceq \mu_2$.

Theorem 3.17. *Let $\langle X, \mathcal{F}\rangle$ be a measurable space and μ, μ' be fuzzy measures on $\langle X, \mathcal{F}\rangle$, and let $K, K' : X \times Y \to L$ be integral kernels.*

(i) If $\mu \preceq \mu'$, then $H^{\star}_{(K,\mu)}(f) \leq H^{\star}_{(K,\mu')}(f)$ for any $f \in \mathcal{F}(X)$.

(ii) If $K \leq K'$, then $H^{\star}_{(K,\mu)}(f) \leq H^{\star}_{(K',\mu)}(f)$ for any $f \in \mathcal{F}(X)$.

Similarly to Theorem 3.7 and Theorem 3.13, we are interested in sufficient conditions under which $(K, \mu, \star)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms ensure the preservation of constant functions, i.e., $H^{\star}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$ for any constant function $\underline{a}_X \in \mathcal{F}(X)$. The following theorem shows that conditions (C1) and (C2) introduced in Subsection 3.3 are the sufficient conditions for the preservation of constant functions.

Theorem 3.18. *Let $\langle X, \mathcal{F}, \mu\rangle$ be a fuzzy measure space, let $\mu^{c,N}$ be the N–conjugate fuzzy measure to μ, let K be an integral kernel, and let $a \in L$.*

(i) If $(K, \mu^{c,N})$ satisfies (C2) for $\neg a$, then $H^{\otimes}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$.

(ii) If $(K, \mu^{c,N})$ satisfies (C1), then $H^{\to}_{(K,\mu)}(\underline{a}_X) = \underline{a}_Y$.

Proof. (i) Let $y \in Y$. Assume that for any $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$, we have $\mu^{c,N}(A) \leq \neg a$. Hence, we get

$$H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) = \int^{\to}_{\mathrm{DPR}} K(x,y) \otimes \underline{a}_X(x)\, d\mu = \bigwedge_{A \in \mathcal{F}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \otimes \underline{a}_X(x)))$$

$$= \bigwedge_{\substack{A \in \mathcal{F} \\ A \not\subseteq X \setminus \mathrm{Core}(K_y)}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \otimes a)) \wedge \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \otimes a))$$

$$\geq \bigwedge_{\substack{A \in \mathcal{F} \\ A \not\subseteq X \setminus \mathrm{Core}(K_y)}} (\top \to \bigvee_{x \in A} (K(x,y) \otimes a)) \wedge \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq X \setminus \mathrm{Core}(K_y)}} (\neg a \to \bigvee_{x \in A} (K(x,y) \otimes a)) \geq$$

$$= (\top \to a) \wedge (\neg a \to \bot) = a \wedge \neg\neg(a) \geq a = a_Y(y),$$

where we used the fact that the residuum is non-increasing in the first argument and $\mu^{c,N}(A) \leq \top$ for $A \not\subseteq X \setminus \mathrm{Core}(K_y)$ and $\mu^{c,N}(A) \leq \neg a$ for $A \subseteq X \setminus \mathrm{Core}(K_y)$, the residuum is non-decreasing in the second argument and $\bigvee_{x \in A}(K(x,y) \otimes a)) = a \otimes \bigvee_{x \in A} K(x,y) = a \otimes \top = a$ for $A \not\subseteq X \setminus \mathrm{Core}(K_y)$, i.e., $A \cap \mathrm{Core}(K_y) \neq \emptyset$, and $\bigvee_{x \in A}(K(x,y) \otimes a)) \geq \bot$ for $A \subseteq X \setminus \mathrm{Core}(K_y)$, and $\top \to a = a$ (Theorem 1.1(iv)) and $a \leq \neg(\neg a)$ (Theorem 1.1(xiv)). Hence, we get $H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) \geq \underline{a}_Y(y)$. On the other side, we have

$$H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) = \int^{\to}_{\mathrm{DPR}} K(x,y) \otimes \underline{a}_X(x)\, d\mu = \bigwedge_{A \in \mathcal{F}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \otimes \underline{a}_X(x)))$$

$$\leq \mu^{c,N}(X) \to \bigvee_{x \in X} (K(x,y) \otimes a) = \top \to (a \otimes \bigvee_{x \in X} K(x,y)) = \top \to a = a = \underline{a}_Y(y),$$

where we used $\bigvee_{x \in X} K(x,y) = \top$. Thus, $H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) \leq \underline{a}_Y(y)$, which implies the desired equality.

(ii) Let $y \in Y$. Assume that there is $A_y \subseteq \mathrm{Core}(K_y)$ such that $\mu^{c,N}(A_y) = \top$. Then

$$H^{\to}_{(K,\mu)}(\underline{a}_X)(y) = \int^{\to}_{\mathrm{DPR}} K(x,y) \to \underline{a}_X(x)\, d\mu = \bigwedge_{A \in \mathcal{F}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \to a))$$

$$\leq \mu^{c,N}(A_y) \to \bigvee_{x \in A_y} (K(x,y) \to a) = \top \to a = a = \underline{a}_Y(y),$$

where we used $K(x,y) \to a = \top \to a = a$ for any $x \in A_y$. Hence, we get $H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) \leq \underline{a}_Y(y)$. On the other hand, for any $A \in \mathcal{F}$, we have

$$\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \to a) \geq \top \to \bigvee_{x \in A} (K(x,y) \to a) =$$

$$\bigvee_{x \in A} (K(x,y) \to a) \geq a = \underline{a}_Y(y),$$

where we used the fact that the residuum is non-increasing in the first argument, $K(x,y) \to a \geq \top \to a = a$ for any $x \in A$ and Theorem 1.1(iv). Thus, we get $H^{\otimes}_{(K,\mu)}(\underline{a}_X)(y) \geq \underline{a}_Y(y)$, which implies the desired equality. $\square$

Obviously, $H^{\otimes}_{(K,\mu)}$ preserves any constant function in $\mathcal{F}(X)$ if $(K, \mu^{c,N})$ satisfies (C2) for $\neg a$ for any $a \in L$, which is equivalent to $\mu^{c,N}(A) = \bot$ for any $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$ for any $y \in Y$. As a consequence of Theorem 3.9, we find that if (K, μ) satisfies (C1), then $(K, \mu^{c,N})$ satisfies (C2) and thus $H^{\otimes}_{(K,\mu)}$ preserves any constant function in $\mathcal{F}(X)$. Assuming that $\mathrm{Core}(K_y) \in \mathcal{F}$ for any $y \in Y$, if (K, μ) satisfies (C2), then $(K, \mu^{c,N})$ satisfies (C1) and $H^{\to}_{(K,\mu)}$ preserves any constant function in $\mathcal{F}(X)$. In addition, if N is involutive generalized negation, we get that the N–conjugate fuzzy measure to $\mu^{c,N}$ is just the fuzzy measure μ. [5] Hence, if (K, μ) satisfies (C1), we simply obtain that $H^{\otimes}_{(K,\mu)}$ and $H^{\to}_{(K,\mu^{c,N})}$ preserve constant functions, which is a very simple way to construct the DPR–residuum-based lattice integral transforms in practice.

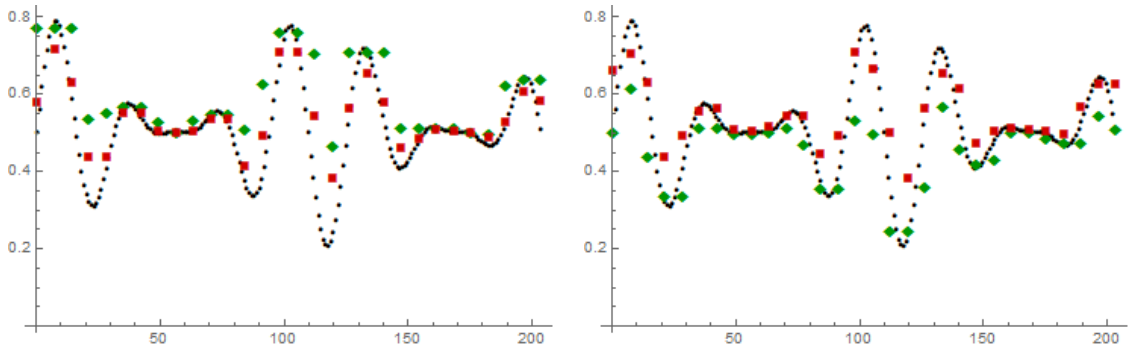
The following example demonstrates the DPR–residuum-based lattice integral transform on a real function.

Example 3.7. We use the same residuated lattice, fuzzy measures (i.e., $\mu \in \mathcal{M}^r_u$) and integral kernels as in Example 3.5 and present the results of DPR–residuum-based lattice integral transforms applied on the signal f given by formula (3.9) on $\{x_1, \ldots, x_n)$ for cases a) and b). From Example 3.5 we know that (K, μ) satisfies (C1) and $(K, \mu^{c,N})$ satisfies (C2) for all considered integral kernels, where $N = N_{\mathrm{res}}$ is the negation in the Łukasiewicz algebra. By Theorem 3.18 and the discussion in the above paragraph, we get that the $(K, \mu, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and the $(K, \mu^{c,N}, \to)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform preserve any constant function in $\mathcal{F}(X)$.

Case a) We consider the integral kernel $K : X \times Y \to [0,1]$ and the fuzzy measures $\mu_1 = \mu^5_{2,6}$ and $\mu_2 = \mu^5_{7,12}$ given in case a) of Example 3.5. Figure 3.7(a) shows the output signals of the $(K, \mu_1, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the green diamonds and the $(K, \mu_2, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the red squares together with the original signal f. The output signals of $(K, \mu_i^{c,N}, \to)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform, $i = 1, 2$, are displayed in Figure 3.7(b). Since the analysis of results is very similar to the analysis of case a) in Example 3.5, we omit it here.

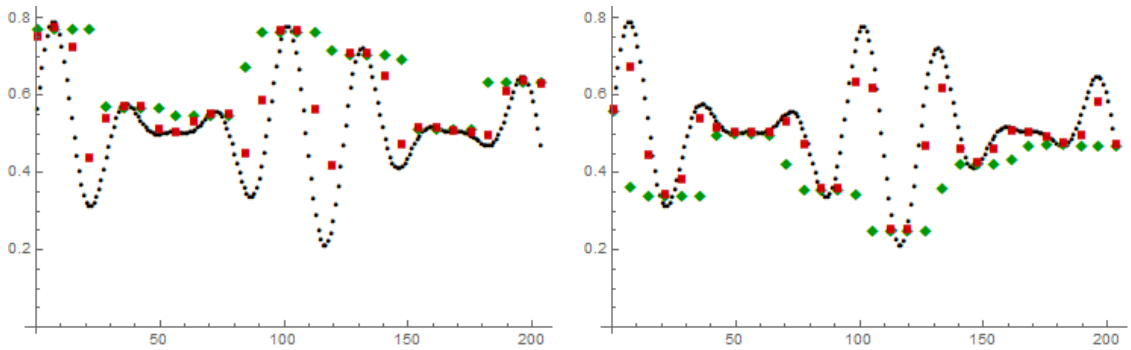
Case b) We consider two integral kernels $K_1, K_2 : X \times Y \to [0,1]$ and the fuzzy measure $\mu = \mu^5_{3,6}$ in case b) of Example 3.5. Figure 3.8(a) shows the output signals of $(K_1, \mu, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the green diamonds and $(K_2, \mu, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the red squares together with the original function f. Figure 3.8(b) shows output signals of the $(K_1, \mu^{c,N}, \to)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the green diamonds and

[5] Indeed, we have $N(\mu^{c,N}(X \setminus A)) = N(N(\mu(X \setminus (X \setminus A)))) = \mu(A)$ for any $A \in \mathcal{F}$.



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.7: $\mathrm{R}^{\star}_{\mathrm{DPR}}$–lattice integral transforms for a fixed integral kernel K and two different fuzzy measures μ_1 (green diamonds) and μ_2 (red squares).



(a) $\star = \otimes$ (b) $\star = \rightarrow$

Figure 3.8: $\mathrm{R}^{\star}_{\mathrm{DPR}}$–lattice integral transforms for a fixed fuzzy measure μ and two different integral kernels K_1 (green diamonds) and K_2 (red squares).

the $(K_2, \mu^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by the red squares. Again the analysis of results is very similar to the analysis of case b) in Example 3.5, therefore, we omit it here.

We conclude this chapter with the statement on the linearity property of the DPR–residuum-based lattice integral transform for comonotonic functions. More precisely, we show that this type of lattice integral transform is comonotonic minitive.

Theorem 3.19. *Let* $\mathbf{L}$ *be a linearly ordered and assume that the algebra* $\mathcal{F}$ *is closed over arbitrary unions. Let* $f, g, K(\cdot, y)$ *be* $\mathcal{F}$-$\mathcal{B}^{\ell}$*–measurable for any* $y \in Y$*. If* $K(\cdot, y) \star f$ *and* $K(\cdot, y) \star g$ *are comonotonic for* $\star \in \{\otimes, \rightarrow\}$*, then*

$$H^{\star}_{(K,\mu)}(f \wedge g) = H^{\star}_{(K,\mu)}(f) \wedge H^{\star}_{(K,\mu)}(g).$$

Proof. As a consequence of Theorem 2.5 and Theorem 2.6, which are adopted for $\mathcal{F}$-$\mathcal{B}^{\ell}$–measurability (see, Remark 2.3), we find that $K(\cdot, y) \star f$ and $K(\cdot, y) \star g$ are $\mathcal{F}$-$\mathcal{B}^{\ell}$–measurable. Since $K(x, y) \star (f(x) \wedge g(x) = (K(x, y) \star f(x)) \wedge (K(x, y) \star g(x))$ for any $x \in X$ and $y \in Y$ holds in any linearly ordered residuated lattice, we get by Theorem 2.20 the statement. □

Chapter 4

Approximation of functions based on lattice integral transforms

In this chapter we show how lattice integral transforms can be used to approximate lattice-valued functions. More specifically, we show that the composition of appropriate lattice integral transforms introduced in Chapter 3 leads to an approximation of the original functions and develop a basic approximation theory for lattice integral transforms, where some results generalize well-known results on function approximation by lower and upper fuzzy transforms [36]. The results are illustrated with examples of signal approximation, including signals with the presence of noise.

4.1 Motivation

As we have mentioned in the motivation part of Chapter 3, the lower and upper fuzzy transforms were designed to approximate lattice-valued functions (functions for short). More precisely, Perfilieva in [36] shows that a suitable composition of a direct upper (lower) lattice fuzzy transform and an inverse upper (lower) lattice fuzzy transform approximates the original function from above (below). For a better understanding, recall that the *inverse upper lattice fuzzy transform* with respect to a fuzzy partition $\mathbf{A} = \{A_y \mid y \in Y\}$ of X is a map $G^{\uparrow}_{\mathbf{A}} : \mathcal{F}(Y) \to \mathcal{F}(X)$ given by

$$G^{\uparrow}_{\mathbf{A}}(g)(x) = \bigwedge_{y \in Y} A_y(x) \to g(y) \tag{4.1}$$

for any $g \in \mathcal{F}(Y)$ and $x \in X$, and the *inverse lower lattice fuzzy transform* with respect to a fuzzy partition $\mathbf{A}$ of X is a map $G^{\downarrow}_{\mathbf{A}} : \mathcal{F}(Y) \to \mathcal{F}(X)$ given by

$$G^{\downarrow}_{\mathbf{A}}(g)(x) = \bigvee_{y \in Y} g(y) \otimes A_y(x) \tag{4.2}$$

for any $g \in \mathcal{F}(Y)$ and $x \in X$. Formally, the following approximation theorem was proved:

$$G^{\downarrow}_{\mathbf{A}} \circ F^{\downarrow}_{\mathbf{A}}(f)(x) \leq f(x) \leq G^{\uparrow}_{\mathbf{A}} \circ F^{\uparrow}_{\mathbf{A}}(f)(x) \tag{4.3}$$

for any $f \in \mathcal{F}(X)$ and $x \in X$. An example of the upper and lower approximation of a function using lattice fuzzy transforms is presented in Figure 4.1(a). It is easy to see

that the inverse lattice fuzzy transforms have the same formula as the direct lattice fuzzy transforms only in the opposite direction. In fact, we can introduce a family $\mathbf{B} = \{B_x \mid x \in X\}$ of fuzzy sets on Y from the fuzzy partition $\mathbf{A}$ of X such that $B_x(y) = A_y(x)$ for any $x \in X$ and $y \in Y$. If we admit that $\mathrm{Core}(B_x) = \mathrm{Core}(B_z)$ also for some $x, z \in X$ such that $x \neq z$,[1] the family $\mathbf{B}$ is a fuzzy partition of Y, where $\mathrm{Core}(B_x)$ is a singleton for any $x \in X$, which is a simple consequence of the definition of the fuzzy partition $\mathbf{A}$. Then we find that

$$G^{\uparrow}_{\mathbf{A}} = F^{\downarrow}_{\mathbf{B}} \quad \text{and} \quad G^{\downarrow}_{\mathbf{A}} = F^{\uparrow}_{\mathbf{B}},$$

where $F^{\uparrow}_{\mathbf{B}}$ and $F^{\downarrow}_{\mathbf{B}}$ are the direct upper and lower lattice fuzzy transforms with respect to $\mathbf{B}$ from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$, respectively. Hence, the inverse lattice fuzzy transforms are nothing else than special multiplication-based lattice integral transforms for some inverse integral kernel K^{-1} to the integral kernel $K : X \times Y \to L$, which is determined from the fuzzy partition $\mathbf{A}$ as $K(x, y) = A_y(x)$ for any $x \in X$ and $y \in Y$. From the definition of the family $\mathbf{B}$, we can simply deduce that $K^{-1} = K^T$. Now, the approximation theorem expressed as (4.3) can be rewritten in the terminology of lattice integral transforms as

$$F^{\otimes}_{(K^{-1},\mu^{\top}_Y)} \circ F^{\rightarrow}_{(K,\mu^{\perp}_X)}(f)(x) \leq f(x) \leq F^{\rightarrow}_{(K^{-1},\mu^{\perp}_Y)} \circ F^{\otimes}_{(K,\mu^{\top}_X)}(f)(x) \tag{4.4}$$

for any $f \in \mathcal{F}(X)$ and $x \in X$, where the fuzzy measures $\mu^{\top}_X$ and $\mu^{\perp}_X$ denote the highest and the least fuzzy measure on the measurable space $\langle X, \mathcal{P}(X)\rangle$ and similarly for the fuzzy measures $\mu^{\top}_Y$ and $\mu^{\perp}_Y$ (see, Theorem 3.4). Naturally, these relations between the original function and its approximations can be studied in a more general setting for various integral kernels, their inverses and fuzzy measures. This is the first goal of this chapter.

In addition, an interesting and challenging question arises whether we can express the approximation quality of the composition of lattice integral transforms, which means estimating the closeness of the original function and its approximation, i.e.,

$$F^{\rightarrow}_{(K^{-1},\mu')} \circ F^{\otimes}_{(K,\mu)}(f)(x) \approx f(x), \quad x \in X,$$

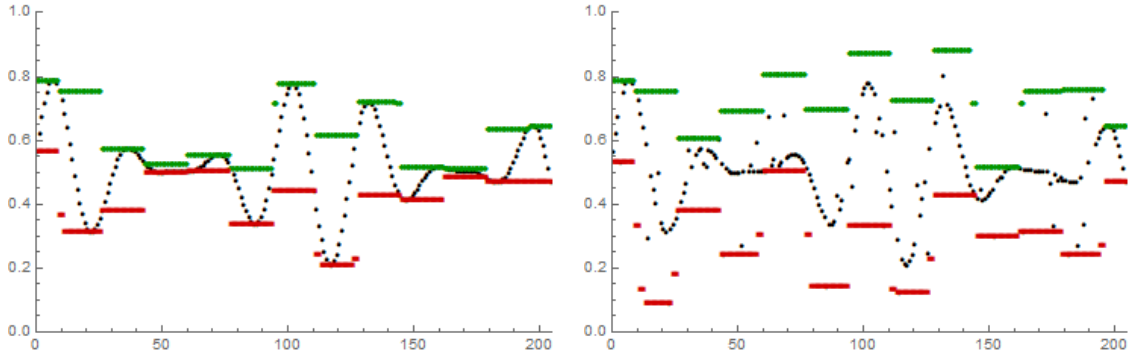
for a suitable setting of the fuzzy measure μ', and similarly for the reverse composition. The second goal of this chapter is to focus on this problem.

The disadvantage of upper and lower approximation of a function using lattice fuzzy transforms is that these transforms cannot be used as a filter similarly to the real-valued fuzzy transform or Fourier and some other integral transforms, as can be seen in Figure 4.1(b), where the original function is a bit noisy (30% function values are biased). The final goal of this chapter is to show that more general lattice integral transforms can suppress high frequencies and can serve as a random noise filter.

4.2 Inverse and dually inverse integral kernels

In the first part of this subsection, we introduce the inverse of an integral kernel $K : X \times Y \to L$, which is an integral kernel $K' : Y \times X \to L$ that will be used in

[1] That is, we also consider the family of sets $\{X_i \mid i \in I\}$ such that $\bigcup_{i\in I} X_i = X$ and $X_i \cap X_j = \emptyset$ or $X_i = X_j$ for any $i, j \in I$ to be a partition of X.



(a) Upper (green) and lower (red) approximations of smooth function.

(b) Upper (green) and lower (red) approximations of 30% noisy function values.

Figure 4.1: Original function f (black) and its approximations using lattice fuzzy transforms.

the reconstruction of functions by the composition of M*–lattice integral transforms or $\mathrm{R}^*_{\mathrm{DPR}}$–lattice integral transforms introduced in Chapter 3.

Let $K : X \times Y \to L$ and $K' : Y \times X \to L$ be two integral kernels. First, we introduce a new type of integral kernel on X to express the relationship between K and K'.

Definition 4.1. An integral kernel $Q : X \times X \to L$ is said to be *compatible with* K and K' or also (K, K')*–compatible* provided that

$$Q(x, z) \otimes K'(y, z) \leq K(x, y) \tag{4.5}$$

holds for any $x, z \in X$ and $y \in Y$.

The following lemma shows that if an integral kernel Q on X is simultaneously (K, K_i)–compatible for $i \in I$, then Q is also compatible with K and $\bigcup_{i\in I} K_i$.

Lemma 4.1. *Let $Q : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels, and let $\{K_i : Y \times X \to L \mid i \in I\}$ be a system of integral kernels such Q is (K, K_i)–compatible for any $i \in I$. Then $\bigcup_{i\in I} K_i$ is an integral kernel and Q is compatible with K and $\bigcup_{i\in I} K_i$.*

Proof. For any $y \in Y$, there is $x \in X$ such that $K_i(y, x) = \top$ holds at least for one $i \in I$. Hence, we o ain $\bigvee_{i\in I} K_i(y, x) = \top$. Similarly, for any $x \in X$ there is $y \in Y$ such that $K_i(y, x) = \top$ for at least one $i \in I$, which implies $\bigvee_{i\in I} K_i(y, x) = \top$. Hence, we find that $\bigcup_{i\in I} K_i$ is an integral kernel. From the (K, K_i)–compatibility of Q, the following inequality holds for any $i \in I$:

$$Q(x, y) \otimes K_i(y, z) \leq K(x, y), \quad x, z \in X,\ y \in Y.$$

Using Theorem 1.2(i), we find that

$$\bigvee_{i\in I}(Q(x, y) \otimes K_i(z, y)) = Q(x, y) \otimes \bigvee_{i\in I} K_i(z, y) \leq K(x, y),$$

which implies that Q is $(K, \bigcup_{i\in I} K_i)$–compatible. □

The notion of inverse of K is related to the integral kernel of Q on X, which allows to introduce a wider class of inverses that can be taken into account when approximating the original functions.

Definition 4.2. Let $Q : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels. An integral kernel $K' : Y \times X \to L$ is said to be an *Q–inverse of K* if

$$Q(x, z) = \bigwedge_{y \in Y} K'(y, z) \to K(x, y), \quad x, z \in X. \tag{4.6}$$

From the previous definition, we can see that Q is uniquely determined from K and K' by (4.6)formula. In other words, if K' is a Q–inverse of K, then K' cannot be also a Q'–inverse of K for an integral kernel Q' different from Q. So, when we write that K' is an Q–inverse of K, we mean that Q is an integral kernel determined from K and K' by formula (4.6). As a simple consequence of the adjointness property, Q is compatible with K and K'. Obviously, not every integral kernel K has its Q–inverse for every integral kernel Q, and also not all pairs of integral kernels K and K' lead to an integral kernel Q given by (4.6), i.e., Q is a fuzzy relation which is not normal in both arguments. As we have indicated, Q–inverses of K are different with respect to different Q. On the other hand, each integral kernel K has its transpose as an Q-inverse, where Q is determined by (4.6), as will be seen in Theorem 4.3 on page 66. Interestingly, we can have different Q–inverses of K for the same Q as the following example shows, so, the Q–inverse is not defined uniquely.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, and consider the Łukasiewicz algebra on $[0, 1]$. Assume the integral kernels expressed by matrices as follows:

$$\boldsymbol{K} = \begin{pmatrix} 1 & 0.8 \\ 0.9 & 1 \\ 0.7 & 1 \end{pmatrix} \quad \boldsymbol{K}^T = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.8 & 1 & 1 \end{pmatrix} \quad \boldsymbol{K}' = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.7 & 1 & 1 \end{pmatrix}.$$

We see that $K^T \neq K'$, since $K^T(y_2, x_1) = \boldsymbol{K}^T_{21} = 0.8 \neq 0.7 = \boldsymbol{K}'_{21} = K'(y_2, x_1)$. More specifically, there is $K^T > K'$. Introducing the matrix operation for a $p \times q$–matrix K and a $q \times p$–matrix L as

$$(\boldsymbol{K} \to \boldsymbol{L})_{ik} = \min\{\boldsymbol{K}_{jk} \to \boldsymbol{L}_{ij} \mid j = 1, \ldots, q\} \tag{4.7}$$

for any $i, k = 1, \ldots, p$, we simply find that the integral kernel Q defined in (4.6) expressed by the matrix form is

$$\boldsymbol{Q} = \boldsymbol{K} \to \boldsymbol{K}^T = \boldsymbol{K} \to \boldsymbol{K}' = \begin{pmatrix} 1 & 0.8 & 0.8 \\ 0.9 & 1 & 1 \\ 0.7 & 0.8 & 1 \end{pmatrix}.$$

Hence, there are two different Q–inverses of K, namely, the transpose of K and the integral kernel K'.

By Lemma 4.1, if there is a Q–inverse of K, then there is also the *maximal Q–inverse of K*, where maximality is characterized with respect to the ordering of

fuzzy sets. Indeed, if the set $\{K_i \mid i \in I\}$ of all Q–inverses of K is non-empty, then from (4.6) and the adjointness property, we have

$$K_i(y,z) \otimes Q(x,z) \leq K(x,y)$$

for any $x, z \in X$, $y \in Y$ and $i \in I$. Using Lemma 4.1, we get that $\bigcup_{i\in I} K_i$ is an integral kernel for which

$$\left(\bigcup_{i\in I} K_i\right)(y,z) \otimes Q(x,z) \leq K(x,y)$$

holds for any $x, z \in X$ and $y \in Y$. Due to the adjointness property and the fact that the residuum is non-increasing in the first argument, we obtain

$$Q(x,z) \leq \bigwedge_{y\in Y} \left(\bigcup_{i\in I} K_i\right)(y,z) \to K(x,y) \leq \bigwedge_{y\in Y} K_i(y,z) \to K(x,y) = Q(x,z)$$

for any $x, z \in X$, which implies that $\bigcup_{i\in I} K_i$ is the Q–inverse of K.

In what follows, we use K^{-1} to denote an arbitrary Q–inverse of K. More precisely, if we use K^{-1}, then we mean one of the Q–inverses of K, including the maximal one.

Remark 4.1. A reason why the Q–inverse of K is not defined as the maximal integral kernel satisfying (4.6) is twofold. First, we have the same estimate of the approximation of the original function for any Q–inverse of K with the same integral kernel Q. Second, we want to establish a unified theory for all lattice integral transforms, and the notion of the dual inverse of K used in reconstructing functions using R_{DH}–lattice integral transforms has a slightly different standing for the transpose of K as in the case of the Q–inverse of K, as will be seen later. But the transpose of K is the simplest and most natural way to express a (dual) inverse of K in practice, which motivates us to introduce the Q–inverse of K that admits more than one inverse.

A natural question is whether we are able to determine an inverse of K from a given integral kernel Q. The following theorem shows that from a suitable integral kernel Q, we can determine a maximal inverse of K.

Theorem 4.2. *Let $Q : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels. If the fuzzy relation $K' : Y \times X \to L$ given by*

$$K'(y,x) = \bigwedge_{u\in X} Q(u,x) \to K(u,y), \quad x \in X,\ y \in Y, \tag{4.8}$$

is an integral kernel, then K' is the maximal Q^*–inverse of K.

Proof. From the definition of K', we simply find that

$$K'(y,z) \otimes Q(x,z) \leq K(x,y) \tag{4.9}$$

holds for any $x, z \in X$ and $y \in Y$. Hence, Q is (K, K')–compatible. Let Q be (K, K'')–compatible for some integral kernel $K'' : Y \times X \to L$. Since

$$K''(y,z) \otimes Q(x,z) \leq K(x,y)$$

for any $x, z \in X$ and $y \in Y$, we find that

$$K''(y,z) \leq \bigwedge_{x \in X} Q(x,z) \to K(x,y) = K'(y,z)$$

for any $z \in X$ and $y \in Y$, and thus $K'' \leq K'$. Hence, K' is the maximal integral kernel such that Q is (K, K')–compatible. From (4.9), for any $x, z \in X$, we obtain

$$Q(x,z) \leq K'(y,z) \to K(x,y)$$

for any $y \in Y$, which implies

$$Q(x,z) \leq \bigwedge_{y \in Y} K'(y,z) \to K(x,y) = Q^*(x,z),$$

and thus $Q \leq Q^*$. Since Q is an integral kernel, the same holds for Q^*. By the definition, K' is the Q^*–inverse of K, and it remains to prove that K' is maximal. Let K'' be another Q^*–inverse of K. Since Q^* is (K, K'')–compatible and $Q \leq Q^*$, we have

$$K''(y,z) \otimes Q(x,z) \leq K''(y,z) \otimes Q^*(x,z) \leq K(x,y)$$

for any $x, z \in X$ and $y \in Y$, where we used the monotonicity of $\otimes$. Hence, Q is also (K, K'')–compatible, which implies that $K'' \leq K'$, since K' is the maximal integral kernel such that Q is (K, K')–compatible, which completes the proof. □

It is know that the inverse lattice fuzzy transforms use the transpose of the integral kernel K as the inverse integral kernel (see formula (4.4) in the motivation part). The following theorem shows that K^T is the maximal Q–inverse of K for the integral kernel Q determined by (4.6).

Theorem 4.3. *If K is an integral kernel, then K^T is the maximal Q–inverse of K.*

Proof. For any $x \in X$, we have

$$Q(x,x) = \bigwedge_{y \in Y} K^T(y,x) \to K(x,y) = \top,$$

which implies that Q is an integral kernel on X, and thus K^T is the Q–inverse of K. In addition, we find that

$$K^T(y,x) \leq \bigwedge_{u \in Y} Q(u,x) \to K(u,y) = K'(y,x),$$

for any $x \in X$ and $y \in Y$, and thus $K^T \leq K'$, where K' is the maximal Q^*–inverse of K according to Theorem 4.2. On the other side, we have

$$K'(y,x) = \bigwedge_{u \in Y} Q(u,x) \to K(u,y) \leq Q(x,x) \to K(x,y) =$$
$$\top \to K(x,y) = K(x,y) = K^T(y,x)$$

for any $x \in X$ and $y \in Y$. Hence, $K' \leq K^T$, which implies $K' = K^T$ and $Q = Q^*$. □

The natural question is what is the position of the inverse defined as the transpose of an integral kernel among other inverses. The following theorem provides an answer to this question. Recall that a fuzzy relation Q on X is reflexive if $Q(x,x) = \top$ for any $x \in X$.

Theorem 4.4. *Let K be an integral kernel, and let K^{-1} be a Q–inverse of K. Assume that K is the P–inverse of K^{-1} for an integral kernel P on Y. If Q and P are reflexive fuzzy relations, then $K^{-1} = K^T$.*

Proof. Since Q is (K, K^{-1})–compatible and reflexive, we simply find that

$$\top = Q(x,x) \leq \bigwedge_{y \in Y} K^{-1}(y,x) \to K(x,y)$$

for any $x \in X$, which implies $K^{-1}(y,x) \leq K(x,y)$ for any $x \in X$ and $y \in Y$, i.e. $K^{-1} \leq K^T$. By similar arguments, we find that

$$\top = P(y,y) \leq \bigwedge_{x \in X} K(x,y) \to K^{-1}(y,x)$$

for any $y \in Y$, which implies $K(x,y) \leq K^{-1}(y,x)$ for any $x \in X$ and $y \in Y$, i.e., $K^T \leq K^{-1}$. Hence, we get $K^{-1} = K^T$. □

The previous theorem shows that $K^{-1} = K^T$ if K and K^{-1} are mutually inverse for reflexive integral kernels Q and P, which is a natural requirement that holds for the integral kernels in the classical theory of integral transforms.

In the rest of this subsection we introduce dual notions to the notions of compatible and inverse integral kernel. The dual inverse of an integral kernel is the key concept in the reconstruction of the functions using the compositions of R_{DH}–lattice integral kernels.

Definition 4.3. An integral kernel $Q^d : X \times X \to L$ is said to be *dually compatible with* K and K' or also (K, K')*–dually compatible* provided that

$$K(x,y) \leq K'(y,z) \to Q^d(x,z) \tag{4.10}$$

holds for any $x, z \in X$ and $y \in Y$.

The following lemma shows that the same result presented in Lemma 4.1 holds also for the dually compatible integral kernels.

Lemma 4.5. *Let $Q^d : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels, and let $\{K_i : Y \times X \to L \mid i \in I\}$ be a system of integral kernels such Q^d is (K, K_i)–dually compatible for any $i \in I$. Then Q^d is dually compatible with K and $\bigcup_{i \in I} K_i$.*

Proof. From the (K', K)–dual compatibility of Q^d and using the adjointness property, we have for any $i \in I$:

$$K_i(y,z) \otimes K(x,y) \leq Q^d(x,z), \quad x, z \in X,\ y \in Y.$$

Using Theorem 1.2(i), we find that

$$\bigvee_{i\in I}(K_i(z,y)\otimes K(x,y)) = \left(\bigvee_{i\in I} K_i(z,y)\right)\otimes K(x,y) \le Q^d(x,z),$$

and by the adjointness property, we get

$$K(x,y) \le \left(\bigvee_{i\in I} K_i(z,y)\right) \to Q^d(x,z),$$

which implies that Q^d is $(K, \bigcup_{i\in I} K_i)$-dually compatible. □

The dual inverse of an integral kernel is defined as follows.

Definition 4.4. Let $Q^d : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels. An integral kernel $K' : Y \times X \to L$ is said to be a Q^d*–dual inverse* of K if

$$Q^d(x,z) = \bigvee_{y\in Y} K(x,y)\otimes K'(y,z), \quad x,z \in X. \tag{4.11}$$

Similarly to Q–inverses of K, we can have more than one Q^d-dual inverse of K, as the following example shows.

Example 4.2. Similarly to Example 4.1, we assume $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, and consider the Łukasiewicz algebra on $[0,1]$. The integral kernels are expressed by matrices as follows:

$$\boldsymbol{K} = \begin{pmatrix} 1 & 0.8 \\ 0.9 & 1 \\ 0.7 & 1 \end{pmatrix} \quad \boldsymbol{K}^T = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.8 & 1 & 1 \end{pmatrix} \quad \boldsymbol{K}' = \begin{pmatrix} 1 & 0.9 & 0.8 \\ 0.7 & 1 & 1 \end{pmatrix},$$

where the integral kernel K is the same as in the mentioned example. We see that $K^T \neq K$, since $K^T(y_1, x_3) = \boldsymbol{K}^T_{13} = 0.7 \neq 0.8 = \boldsymbol{K}'_{13} = K'(y_1, x_3)$. More specifically, there is $K^T < K'$. Introducing the matrix operation for a $p \times q$–matrix K and a $q \times p$–matrix L as

$$(\boldsymbol{K}\otimes\boldsymbol{L})_{ik} = \max\{\boldsymbol{K}_{ij}\otimes\boldsymbol{L}_{jk} \mid j = 1,\dots,q\} \tag{4.12}$$

for any $i,k = 1,\dots,p$, we simply find that the integral kernel Q^d defined in (4.11) expressed by the matrix form is

$$\boldsymbol{Q}^d = \boldsymbol{K}\otimes\boldsymbol{K}^T = \boldsymbol{K}\otimes\boldsymbol{K}' = \begin{pmatrix} 1 & 0.9 & 0.8 \\ 0.9 & 1 & 1 \\ 0.8 & 1 & 1 \end{pmatrix}.$$

Hence, there are two different Q^d–dual inverses of K, namely, the transpose of K and the integral kernel K'.

Similarly to Theorem 4.2, we are able to determine the Q^d–dually inverse integral kernel whenever it exists.

Theorem 4.6. *Let $Q^d : X \times X \to L$ and $K : X \times Y \to L$ be integral kernels. If the fuzzy relation $K' : Y \times X \to L$ given by*

$$K'(y,x) = \bigwedge_{u \in X} K(u,y) \to Q^d(u,x), \quad x \in X,\ y \in Y \tag{4.13}$$

is an integral kernel, then K' is the maximal Q^{d*}–dual inverse of K.

Proof. From the definition of K', we simply find that

$$K(x,y) \leq K'(y,z) \to Q^d(x,z) \tag{4.14}$$

holds for any $x, z \in X$ and $y \in Y$. Hence, Q^d is (K, K')–dually compatible. Let Q^d be (K, K'')–dually compatible for some integral kernel $K'' : Y \times X \to L$. Since

$$K(x,y) \leq K''(y,z) \to Q^d(x,z)$$

holds for any $x, z \in X$ and $y \in Y$, using the adjointness property we find that

$$K''(y,z) \leq \bigwedge_{x \in X} K(x,y) \to Q^d(x,z) = K'(y,z)$$

for any $z \in X$ and $y \in Y$, and thus $K'' \leq K'$. Hence, K' is the maximal integral kernel such that Q^d is (K, K')–dually compatible. Since K and K' are the integral kernels, for any $x \in X$, there is $u \in Y$ such that $K(x,u) = \top$, and also for u there is $z \in X$ such that $K'(u,z) = \top$, which implies

$$Q^{d*}(x,z) = \bigvee_{y \in Y} K(x,y) \otimes K'(y,z) \geq K(x,u) \otimes K'(u,z) = \top.$$

Therefore, for any $x \in X$ there is $z \in X$ such that $Q^{d*}(x,z) = \top$. Similarly, we can prove that for any $z \in X$ there is $x \in X$ such that $Q^{d*}(x,z) = \top$. Hence, we find that Q^{d*} is an integral kernel. By the definition, K' is an Q^{d*}–inverse of K and remains to show that K' is maximal. Assume that K'' is an Q^{d*}–inverse of K. Then

$$K(x,y) \leq K''(z,y) \to Q^{d*}(x,z) \leq K''(z,y) \to Q^d(x,z)$$

holds for any $x, z \in X$ and $y \in Y$, where we used the fact that the residuum is non-decreasing in the second component and $Q^{d*} \leq Q^d$. Hence, K'' is also an integral kernel such that Q^d is (K, K'')–dually compatible. Since K' is the maximal kernel such that Q^d is (K, K')–dually compatible, we find that $K'' \leq K'$, which implies that K' is also the maximal Q^{d*}–dual inverse of K and the proof is finished. □

The following theorem shows that, as in the case of the Q–inversion, the transpose of K is the Q^d–dual inverse of K. Nevertheless, the difference is that K^T is not the maximal Q^d–dual inverse of K, as was demonstrated in Example 4.2.

Theorem 4.7. *If K is an integral kernel, then K^T is a Q^d–dual inverse of K.*

Proof. For any $x \in X$, we have

$$Q^d(x,x) = \bigvee_{y \in Y} K^T(y,x) \otimes K(x,y) = \top,$$

since for any $x \in X$ there is $y \in Y$ such that $K(x,y) = K^T(y,x) = \top$. Hence, Q^d is an integral kernel on X, and K^T is an Q^d–dual inverse of K. □

4.3 Approximation of functions based on M–lattice integral transforms

This section aims to investigate the approximation of an original lattice-valued function using a combination of two types of M–lattice integral transforms that are introduced in Subsection 3.3. Throughout this section, we assume that $\langle X, \mathcal{F}, \mu_X \rangle$ and $\langle Y, \mathcal{G}, \mu_Y \rangle$ are fuzzy measure spaces, $K : X \times Y \to L$ is an integral kernel and $K^{-1} : Y \times X \to L$ is a Q–inverse integral kernel, where Q is the integral kernel on X satisfying (4.6).

4.3.1 Upper and lower approximation of functions

We start with a theorem that gives a generalization of the right inequality in (4.4) and in a sense shows the approximation from above of the original function using the composition of M–lattice integral transforms. Recall that the $\otimes$–fuzzy integral is a monotonically non-decreasing map (see, Theorem 2.7).

Theorem 4.8. *Let $F^{\otimes}_{(K,\mu_X)}$ be an M–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, and let $F^{\to}_{(K^{-1},\mu_Y)}$ be an M–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)} \geq F^{\otimes}_{(Q,\mu_X)}. \tag{4.15}$$

Proof. From Definition 4.1 and using (ii) and (iv) of Theorem 2.7, we simply find for any $f \in \mathcal{F}$ and $z \in X$ that

$$\begin{gathered}
F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(z) = \int_Y^{\otimes} (K^{-1}(y,z) \to F^{\otimes}_{(K,\mu_X)}(f)(y))\, d\mu_Y = \\
\int_Y^{\otimes} \left(K^{-1}(y,z) \to \int_X^{\otimes} K(x,y) \otimes f(x)\, d\mu_X \right) d\mu_Y \geq \\
\int_Y^{\otimes} \left(\int_X^{\otimes} (\underline{K^{-1}(y,z)}_X(x) \to (K(x,y) \otimes f(x)))\, d\mu_X \right) d\mu_Y \geq \\
\int_Y^{\otimes} \left(\int_X^{\otimes} (\underline{K^{-1}(y,z)}_X(x) \to K(x,y)) \otimes f(x))\, d\mu_X \right) d\mu_Y \geq \\
\int_Y^{\otimes} \underline{\left(\int_X^{\otimes} Q(x,z) \otimes f(x)\, d\mu_X \right)}_Y (y)\, d\mu_Y = \int_X^{\otimes} Q(x,z) \otimes f(x)\, d\mu_X = F^{\otimes}_{(Q,\mu_X)}(f)(z),
\end{gathered}$$

where we used $a \to (b \otimes c) \geq b \otimes (a \to c)$ (Theorem 1.1(viii)), $K^{-1}(y,z)$ is a constant for the fuzzy integral $\int_X^{\otimes}$ (therefore, we may apply Theorem 2.7(iv)), the fact that Q is a (K, K^{-1})–compatible integral kernel, i.e., $\underline{K^{-1}(y,z)}_X(x) \to K(x,y) = K^{-1}(y,z) \to K(x,y) \geq Q(x,z)$ for any $x, z \in X$, and

$$\underline{\left(\int_X^{\otimes} Q(x,z) \otimes f(x)\, d\mu_X \right)}_Y$$

is a constant function on Y integrated by $\int_Y^{\otimes}$. $\square$

In contrast to the original approximation of the original functions from above by a composition of lattice fuzzy transforms, the composition of M–lattice integral transforms approximates the original function such that the reconstructed function is over the $\mathrm{M}^{\otimes}$–lattice integral transform of the original functions with respect to the integral kernel Q on X, which is determined from the integral kernel K and its inverse K^{-1}. The $\mathrm{M}^{\otimes}$–lattice integral transform on X with respect to Q can be viewed as a smoothing filter that filters out the high frequencies presented in the functions. Thus, the composition of the M–lattice integral transform does not in general approximate the original function from above, but its smoothing given by the $\mathrm{M}^{\otimes}$–lattice integral transform. This property also shows that the M–lattice integral transforms can be applied as filters for high frequencies or random noise (see, Subsection 4.3.4).

Similarly, a generalization of the approximation from below of the original function by a composition of M–lattice integral transforms is given in the following theorem.

Theorem 4.9. *Let $F^{\rightarrow}_{(K,\mu_X)}$ be an M–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, and let $F^{\otimes}_{(K^{-1},\mu_Y)}$ be an M–lattice integral transform $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)} \leq F^{\rightarrow}_{(Q,\mu_X)}. \tag{4.16}$$

Proof. From Definition 4.1 and using (i) and (iii) of Theorem 2.7, we have

$$\begin{aligned}
F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(z) &= \int^{\otimes}_{Y} K^{-1}(y,z) \otimes F^{\rightarrow}_{(K,\mu_X)}(f)(y)\, d\mu_Y = \\
&\int^{\otimes}_{Y} K^{-1}(y,z) \otimes \left(\int^{\otimes}_{X} (K(x,y) \to f(x))\, d\mu_X \right) d\mu_Y \leq \\
&\int^{\otimes}_{Y} \left(\int^{\otimes}_{X} \underline{K^{-1}(y,z)}_X(x) \otimes (K(x,y) \to f(x))\, d\mu_X \right) d\mu_Y \leq \\
&\int^{\otimes}_{Y} \left(\int^{\otimes}_{X} ((\underline{K^{-1}(y,z)}_X(x) \to K(x,y)) \to K(x,y)) \otimes (K(x,y) \to f(x))\, d\mu_X \right) d\mu_Y \\
&\leq \int^{\otimes}_{Y} \left(\int^{\otimes}_{X} ((\underline{K^{-1}(y,z)}_X(x) \to K(x,y)) \to f(x))\, d\mu_X \right) d\mu_Y \leq \\
&\int^{\otimes}_{Y} \underline{\left(\int^{\otimes}_{X} Q(x,z) \to f(x)\, d\mu_X \right)}_Y (y)\, d\mu_Y \\
&= \int^{\otimes}_{X} Q(x,z) \to f(x)\, d\mu_X = F^{\rightarrow}_{(Q,\mu_X)}(f)(z),
\end{aligned}$$

where we used $a \leq (a \to b) \to b$ and $(a \to b) \otimes (b \to c) \leq a \to c$, ((i) and (vii) of Theorem 1.1), the fact that the residuum is non-increasing in the first component, Q is a (K, K^{-1})–compatible integral kernel (see the proof of Theorem 4.8 for details), and again

$$\underline{\left(\int^{\otimes}_{X} Q(x,z) \to f(x)\, d\mu_X \right)}_Y$$

is a constant function integrated on Y by $\int^{\otimes}_{Y}$. □

Note that $\mathrm{M}^{\to}$–lattice integral transform on X with respect to the integral kernel Q can be viewed as another smoothing filter that filters out the high frequencies presented in the functions. Thus, the reverse composition of M–lattice integral transforms approximates from below the smoothed original function given by the $\mathrm{M}^{\to}$–lattice integral transform on X with respect to Q.

As a consequence of the previous theorems, we can derive another approximation of the original function using M–lattice integral transforms in the case that Q is a reflexive fuzzy relation on X. Recall that Q_y denotes the y-projection, i.e., $Q_y(x) = Q(x, y)$ for any $x \in X$

Corollary 4.10. *For any $f \in \mathcal{F}(X)$ and $y \in X$, it holds that*

(i) $F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(y) \geq \int_X^{\otimes} f \otimes 1_{\mathrm{Core}(Q_y)}\, d\mu_X$,

(ii) $F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f)(y) \leq \int_X^{\otimes} 1_{\mathrm{Core}(Q_y)} \to f\, d\mu_X$.

Proof. Let $\Delta : L \to L$ be given as follows

$$\Delta(a) = \begin{cases} \top, & a = \top, \\ \bot, & \text{otherwise.} \end{cases} \tag{4.17}$$

Further, define $\Delta Q : X \times X \to L$ as $\Delta Q(x, y) = \Delta(Q(x, y))$ for any $x, y \in X$. Obviously, it holds that $\Delta Q \leq Q$.

(i) By Theorem 2.7(i), for any $y \in Y$, we simply get

$$\begin{aligned} F^{\otimes}_{(Q,\mu)}(f)(y) &= \int_X^{\otimes} Q(x, y) \otimes f(x)\, d\mu_X \\ \geq \int_X^{\otimes} \Delta Q(x, y) \otimes f(x)\, d\mu_X &= \int_X^{\otimes} f(x) \otimes 1_{\mathrm{Core}(Q_y)}\, d\mu_X, \end{aligned}$$

where we used that $\otimes$ is monotonically non-decreasing in both argument. The statement is the straightforward consequence of (4.15).

(ii) Again by Theorem 2.7(i), for any $y \in Y$, we simply get

$$\begin{aligned} F^{\to}_{(Q,\mu_X)}(f)(y) &= \int_X^{\otimes} Q(x, y) \to f(x)\, d\mu_X \\ \leq \int_X^{\otimes} \Delta Q(x, y) \to f(x)\, d\mu_X &= \int_X^{\otimes} 1_{\mathrm{Core}(Q_y)} \to f(x)\, d\mu_X, \end{aligned}$$

where we used that $\to$ is monotonically non-increasing in the first component. The statement is the straightforward consequence of (4.16). □

The last corollary in this subsection gives a generalization of the upper and lower approximation of the original function expressed in (4.4) for more general integral kernels under the assumption the least and the highest fuzzy measures are considered.

Corollary 4.11. *Let K be an integral kernel, and let K^{-1} be the Q–inverse of K such that Q is a reflexive integral kernel. Then, for any $f \in \mathcal{F}(X)$, it holds that*

$$F^{\otimes}_{(K^{-1},\mu_Y^{\top})} \circ F^{\to}_{(K,\mu_X^{\bot})}(f) \leq f \leq F^{\to}_{(K^{-1},\mu_Y^{\bot})} \circ F^{\otimes}_{(K,\mu_X^{\top})}(f). \tag{4.18}$$

Proof. From Corollary 4.10(i), we get

$$F^{\to}_{(K^{-1},\mu^{\perp}_{Y})} \circ F^{\otimes}_{(K,\mu^{\top}_{X})}(f)(y) \geq \int_X^{\otimes} f \otimes 1_{\mathrm{Core}(Q_y)}\, d\mu^{\top}_X =$$
$$\bigvee_{A\in\mathcal{F}} (\bigwedge_{x\in A} f(x) \otimes 1_{\mathrm{Core}(Q_y)}(x)) \otimes \mu^{\top}_X(A) = \bigvee_{x\in \mathrm{Core}(Q_y)} f(x) \geq f(y),$$

where we used the fact that Q is reflexive, and thus $y \in \mathrm{Core}(Q_y)$. Hence, the right inequality in (4.18) is proved.

From Corollary 4.10(ii), we get

$$F^{\otimes}_{(K^{-1},\mu^{\top}_{Y})} \circ F^{\to}_{(K,\mu^{\perp}_{X})}(f)(y) \leq \int_X^{\otimes} 1_{\mathrm{Core}(Q_y)} \to f\, d\mu^{\perp}_X =$$
$$\bigvee_{A\in\mathcal{F}} (\bigwedge_{x\in A} 1_{\mathrm{Core}(Q_y)}(x) \to f(x)) \otimes \mu^{\perp}_X(A) = \bigwedge_{x\in \mathrm{Core}(Q_y)} f(x) \leq f(y),$$

where again we used the fact that $y \in \mathrm{Core}(Q_y)$, and the property $\top \to a = a$ (Theorem 1.1(iv)). Hence, the left inequality in (4.18) is proved. □

4.3.2 Estimation of approximation quality

In this part, we focus on the quality estimation of the approximation of the original function using the M–lattice integral transforms. One tool to measure the quality of the approximation is to determine the proximity of the original function and the reconstructed function with respect to the modulus of continuity. For the classical (higher degree) fuzzy transform, we can find several approximation theorems based on the modulus of continuity, see, for example, [4, 21, 40]. In the case of a lattice-valued function, it is difficult to apply the arithmetic of real numbers; therefore, we propose the following, rather abstract, definition of modulus of continuity suitable for our purpose. Let $\mathcal{E}(X)$ denote the set of all equivalences on X.

Definition 4.5. The map $\omega : \mathcal{F}(X) \times \mathcal{E}(X) \to L$ given by

$$\omega(f, E) = \bigwedge_{(x,y)\in E} f(x) \leftrightarrow f(y) \tag{4.19}$$

is called the *modulus of continuity*.

The modulus of continuity provides a degree of proximity of function values at points that are equivalent with respect to a chosen equivalence E. For E expressing the equality on X, we trivially obtain $\omega(f, E) = \top$. In contrast to the classical definition, we specify a structure on X directly using the equivalence relation.

We first show how the functional values of the result of an M–lattice integral transform are close to the function values of the original function for a certain equivalence relation. Define $\nabla : L \to L$ as

$$\nabla(a) = \begin{cases} \perp, & a = \perp, \\ \top, & \text{otherwise}, \end{cases} \tag{4.20}$$

for any $a \in L$. The operator ∇ on L is dual to the operator Δ on L introduced in (4.17). Recall that a constant function on X is denoted by $\underline{a}_X$. The following theorem shows an estimate of the approximation quality for the $M^{\otimes}$–lattice integral transform.

Theorem 4.12. *Let (K, μ) satisfy (C1), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Define an equivalence E_y on X as $(x, z) \in E_y$ if $x = z$ or $\nabla K_y(x) \otimes \nabla K_y(z) = \top$ for any $x, z \in X$. Then*

$$F^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow f(x) \geq \omega(f, E_y), \tag{4.21}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. Let $y \in Y$. First, let us show that E_y is an equivalence on X, so E_y is well-defined. The reflexivity of E_y is trivially true, and the symmetry immediately follows from the commutativity of $\otimes$. If $(u, v), (v, w) \in E_y$, then from $\nabla K_y(u) \otimes \nabla K_y(v) = \top$ and $K_y(v) \otimes \nabla K_y(w) = \top$, we get that $\nabla K_y(u) = \top = \nabla K_y(w)$, and thus $\nabla K_y(u) \otimes \nabla K_y(w) = \top$ and $(u, w) \in E_y$. Further, let us show that

$$F^{\otimes}_{(K,\mu)}(f)(y) = \bigvee_{\substack{A \in \mathcal{F} \\ A \subseteq \mathrm{Supp}(K_y)}} (\bigwedge_{x \in A} (K(x, y) \otimes f(x)) \otimes \mu(A).$$

Note that the satisfaction of (C1) by (K, μ) ensures that there is $A \in \mathcal{F}$ such that $A \subseteq \mathrm{Supp}(K_y)$ with $\mu(A) = \top$, i.e. $A \neq \emptyset$. Moreover, if $A \not\subseteq \mathrm{Supp}(K_y)$, then we have $\bigwedge_{x \in A} K(x, y) \otimes f(x) = \bot$; therefore, it is sufficient to restrict the supremum to $\mathcal{F}$–measurable sets that are subsets of $\mathrm{Supp}(A)$. Denote by $\mathcal{F}_y$ the set of all $\mathcal{F}$-measurable sets such that $A \subseteq \mathrm{Supp}(K_y)$. According to (C1), we know that there is $A \in \mathcal{F}$ such $A \subseteq \mathrm{Supp}(K_y)$ for which $\mu(A) = \top$. Finally, since $F^{\otimes}_{(K,\mu)}$ preserves constant functions due to (C1), using (18), (22) and (23) of Theorem 1.3, we have

$$F^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow f(x) = F^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow F^{\otimes}_{(K,\mu)}(\underline{f(x)}_X)(y) =$$

$$\bigvee_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z, y) \otimes f(z)) \otimes \mu(A) \leftrightarrow \bigvee_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z, y) \otimes \underline{f(x)}_X(z)) \otimes \mu(A) \geq$$

$$\bigwedge_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z, y) \otimes f(z)) \leftrightarrow \bigwedge_{z \in A} (K(z, y) \otimes \underline{f(x)}_X(z))) \otimes (\mu(A) \leftrightarrow \mu(A)) \geq$$

$$\bigwedge_{A \in \mathcal{F}_y} \bigwedge_{z \in A} (K(z, y) \leftrightarrow K(z, y)) \otimes (f(z) \leftrightarrow \underline{f(x)}_X(z)) \geq$$

$$\bigwedge_{z \in \mathrm{Supp}(K_y)} (f(z) \leftrightarrow f(x)) \geq \bigwedge_{(u,v) \in E_y} (f(u) \leftrightarrow f(v)) = \omega(f, E_y),$$

where we used the fact that $z \in \mathrm{Supp}(K_y)$ implies $(x, z) \in E_y$. □

The next theorem shows that the same estimate of the approximation quality holds also for the $M^{\rightarrow}$–lattice integral transform.

Theorem 4.13. *Let (K, μ) satisfy (C2), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Define the equivalence E_y on X as in Theorem 4.12. Then*

$$F^{\rightarrow}_{(K,\mu)}(f)(y) \leftrightarrow f(x) \geq \omega(f, E_y), \tag{4.22}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. From Theorem 4.12 we know that E_y is well-defined. First, let us show that

$$F^{\rightarrow}_{(K,\mu)}(f)(y) = \bigvee_{\substack{A\in\mathcal{F}\\ A\cap \mathrm{Core}(K_y)\neq\emptyset}} (\bigwedge_{x\in A\cap \mathrm{Supp}(K_y)} (K(x,y)\to f(x))\otimes \mu(A).$$

Since (K,μ) satisfy (C2), i.e., $\mu(A)=\bot$ for any $A\in\mathcal{F}$ such that $A\subseteq X\setminus \mathrm{Core}(K_y)$, we can write

$$F^{\rightarrow}_{(K,\mu)}(f)(y) = \bigvee_{\substack{A\in\mathcal{F}\\ A\cap \mathrm{Core}(K_y)\neq\emptyset}} (\bigwedge_{x\in A} (K(x,y)\to f(x))\otimes \mu(A), \tag{4.23}$$

and from

$$\bigwedge_{x\in A} (K(x,y)\to f(x)) = \bigwedge_{x\in A\cap \mathrm{Supp}(K_y)} (K(x,y)\to f(x)),$$

where we used that $K(x,y)\to f(x)=\top$ for $x\in A$ such that $x\notin \mathrm{Supp}(K_y)$, we find the desired modification of the definition of $F^{\rightarrow}_{(K,\mu)}(f)$. Denote by $\mathcal{F}_y$ the set of all $\mathcal{F}$–measurable sets such that $A\cap \mathrm{Core}(K_y)\neq\emptyset$. Since $F^{\rightarrow}_{(K,\mu)}$ preserves constant functions due to (C2), using (15),(19), (22) and (23) of Theorem 1.3, we have

$$F^{\rightarrow}_{(K,\mu)}(f)(y)\leftrightarrow f(x) = F^{\rightarrow}_{(K,\mu)}(f)(y)\leftrightarrow F^{\rightarrow}_{(K,\mu)}(\underline{f(x)}_X)(y) =$$

$$\bigvee_{A\in\mathcal{F}_y} (\bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (K(z,y)\to f(z))\otimes \mu(A) \leftrightarrow$$

$$\bigvee_{A\in\mathcal{F}_y} (\bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (K(z,y)\to \underline{f(x)}_X(z))\otimes \mu(A) \geq$$

$$\bigwedge_{A\in\mathcal{F}_y} (\bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (K(z,y)\to f(z)) \leftrightarrow \bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (K(z,y)\to \underline{f(x)}_X(z)))$$

$$\otimes(\mu(A)\leftrightarrow \mu(A)) \geq \bigwedge_{A\in\mathcal{F}_y}\bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (K(z,y)\leftrightarrow K(z,y))\otimes (f(z)\leftrightarrow \underline{f(x)}_X(z)) \geq$$

$$\bigwedge_{A\in\mathcal{F}_y}\bigwedge_{z\in A\cap \mathrm{Supp}(K_y)} (f(z)\leftrightarrow f(x)) \geq \bigwedge_{z\in X\cap \mathrm{Supp}(K_y)} (f(z)\leftrightarrow f(x)) =$$

$$\bigwedge_{z\in \mathrm{Supp}(K_y)} (f(z)\leftrightarrow f(x)) \geq \bigwedge_{(u,v)\in E_y} (f(u)\leftrightarrow f(v)) = \omega(f,E_y),$$

where we used the same notation and arguments as in the proof of Theorem 4.12. □

The following statements present the estimation of the approximation quality of the reconstructed function.

Theorem 4.14. *Let K be an integral kernel, K^{-1} be an Q–inverse of K for a reflexive integral kernel Q, and let $f\in\mathcal{F}(X)$. Assume that (K,μ_X) satisfies (C1) and (K^{-1},μ_Y) satisfies (C2) and define $\omega(f)=\bigwedge_{y\in Y}\omega(f,E_y)$, where E_y is the equivalence defined in Theorem 4.12. Then*

$$F^{\rightarrow}_{(K^{-1},\mu_Y)}\circ F^{\otimes}_{(K,\mu_X)}(f)(x)\leftrightarrow f(x)\geq \omega(f) \tag{4.24}$$

for any $x\in X$.

Proof. Since Q is reflexive, we get $\top = Q(x,x) = \bigwedge_{y\in Y} K^{-1}(y,x) \to K(x,y)$ for any $x \in X$, which implies $K^{-1}(y,x) \leq K(x,y)$ for any $x \in X$ and $y \in Y$. Hence, we get that if $y \in \mathrm{Supp}(K_x^{-1})$, then $x \in \mathrm{Supp}(K_y)$, i.e., $\nabla K_y(x) = \top$. Similarly to the proof of Theorem 4.13, for any $x \in X$, denote $\mathcal{G}_x$ the set of all $\mathcal{G}$–measurable sets A such that $A \cap \mathrm{Core}(K_x^{-1}) \neq \emptyset$. Let $x \in X$. Since both M–lattice integral transforms preserve constant functions, according to Theorem 4.12, using (4.23) and the fact that $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1})$, we have

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) =$$

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(\underline{f(x)}_X)(x) =$$

$$F^{\to}_{(K^{-1},\mu_Y)}(F^{\otimes}_{(K,\mu_X)}(f))(x) \leftrightarrow F^{\to}_{(K^{-1},\mu_Y)}(\underline{f(x)}_Y)(x) \geq$$

$$\bigvee_{A\in\mathcal{G}_x} \Big(\bigwedge_{y\in A\cap \mathrm{Supp}(K_x^{-1})} (K^{-1}(y,x) \to F^{\otimes}_{(K,\mu_X)}(f)(y)) \otimes \mu_Y(A) \leftrightarrow$$

$$\bigvee_{A\in\mathcal{G}_x} \Big(\bigwedge_{y\in A\cap \mathrm{Supp}(K_x^{-1})} (K^{-1}(y,x) \to \underline{f(x)}_Y(y)) \otimes \mu_Y(A) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A\cap \mathrm{Supp}(K_x^{-1})} (F^{\otimes}_{(K,\mu_X)}(f)(y) \leftrightarrow \underline{f(x)}_Y(y)) =$$

$$\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A\cap \mathrm{Supp}(K_x^{-1})} (F^{\otimes}_{(K,\mu_X)}(f)(y) \leftrightarrow f(x)) \geq \bigwedge_{y\in \mathrm{Supp}(K_x^{-1})} \omega(f,E_y) \geq \omega(f),$$

where the verification skips the analogous steps as in the proof of Theorem 4.12. □

Theorem 4.15. *Let K be an integral kernel, K^{-1} be an Q–inverse of K for a reflexive integral kernel Q, and let $f \in \mathcal{F}(X)$. Assume that (K,μ_X) satisfies (C2) and (K^{-1},μ_Y) satisfies (C1), and let $\omega(f)$ be defined as in Theorem 4.14. Then*

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \omega(f) \tag{4.25}$$

for any $x \in X$.

Proof. Similarly to the proof of Theorem 4.12, denote $\mathcal{G}_x$ the set of all $\mathcal{G}$-measurable sets A such that $A \subseteq \mathrm{Supp}(K_x^{-1})$. Let $x \in X$. Since both M–lattice integral transforms preserve constant functions, according to Theorem 4.13, using (4.21) and the fact that $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1})$ (see the proof of Theorem 4.14), we have

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) =$$

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f)(x) \leftrightarrow F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(\underline{f(x)}_X)(x) =$$

$$F^{\otimes}_{(K^{-1},\mu_Y)}(F^{\to}_{(K,\mu_X)}(f))(x) \leftrightarrow F^{\otimes}_{(K^{-1},\mu_Y)}(\underline{f(x)}_Y)(x) \geq$$

$$\bigvee_{A\in\mathcal{G}_x} \Big(\bigwedge_{y\in A} (K^{-1}(y,x) \otimes F^{\to}_{(K,\mu_X)}(f)(y)) \otimes \mu_Y(A) \leftrightarrow$$

$$\bigvee_{A\in\mathcal{G}_x} \Big(\bigwedge_{y\in A} (K^{-1}(y,x) \to \underline{f(x)}_Y(y)) \otimes \mu_Y(A) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x}\bigwedge_{y\in A}(F^{\rightarrow}_{(K,\mu_X)}(f)(y)\leftrightarrow \underline{f(x)}_Y(y)) =$$

$$\bigwedge_{A\in\mathcal{G}_x}\bigwedge_{y\in A}(F^{\rightarrow}_{(K,\mu_X)}(f)(y)\leftrightarrow f(x)) \geq \bigwedge_{y\in\mathrm{Supp}(K_x^{-1})}\omega(f,E_y)\geq\omega(f),$$

where the verification skips the analogous steps as in the proof of Theorem 4.13. □

In the end of this subsection, we provide an estimation of the approximation quality of M–lattice integral transforms for very special functions which are extensional with respect to a fuzzy relation on the space X. We know that $K^{-1}=K^T$ is the Q–inverse of K, where Q is the fuzzy relation on X given by formula (4.6). Let $Y\subseteq X$. We say that a fuzzy relation Q on X is Y*–transitive* if

$$Q(x,y)\otimes Q(y,z)\leq Q(x,z) \tag{4.26}$$

holds for any $x,z\in X$ and $y\in Y$. Obviously, Q is transitive if $X=Y$. The following lemma shows the properties of Q, when the integral kernel K is determined by a similarity relation on X.

Lemma 4.16. *Let $Y\subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K:X\times Y\to L$ given as $K(x,y)=P(x,y)$ for any $x\in X$ and $y\in Y$ is an integral kernel, and let $K^{-1}=K^T$ be a Q–inverse of K, i.e., Q is give by formula (4.6), Then Q is a reflexive and Y–transitive integral kernel on X such that $P\leq Q$. In addition, $P(x,y)=Q(x,y)$ holds for any $x\in X$ and $y\in Y$.*

Proof. In the proof of Theorem 4.3, we have verify that Q is reflexive. Let $x,z\in X$ and $y\in Y$. Then

$$\begin{aligned}
Q(x,y)\otimes Q(y,z) &= \left(\bigwedge_{u\in Y}K^{-1}(u,y)\to K(x,u)\right)\otimes\left(\bigwedge_{v\in Y}K^{-1}(v,z)\to K(y,v)\right)\leq\\
&\bigwedge_{u\in Y}\bigwedge_{v\in Y}\big(K^{-1}(u,y)\to K(x,u)\big)\otimes\big(K^{-1}(v,z)\to K(y,v)\big)\leq\\
&\bigwedge_{u\in Y}\bigwedge_{v\in Y}\big((K^{-1}(u,y)\otimes K^{-1}(v,z))\to(K(x,u)\otimes K(y,v))\big)\leq\\
&\bigwedge_{v\in Y}\big((K^{-1}(y,y)\otimes K^{-1}(v,z))\to(K(x,y)\otimes K(y,v))\big)\leq\\
&\bigwedge_{v\in Y}(K^{-1}(v,z)\to K(x,v))=Q(x,z),
\end{aligned}$$

where we used Theorem 1.1(x), the monotonicity of the residuum in the second argument, and the transitivity of P, i.e., $K(x,y)\otimes K(y,v)=P(x,y)\otimes P(y,v)\leq P(x,v)=K(x,v)$. Since P is a similarity, we have

$$P(x,y)=\bigwedge_{z\in X}P(y,z)\to P(x,z)$$

for any $x, z \in X$. Indeed, $P(x,y) \leq P(y,z) \to P(x,z)$, which implies $P(x,y) \leq \bigwedge_{z\in X} P(y,z) \to P(x,z)$. On the other hand, $\bigwedge_{z\in X} P(y,z) \to P(x,z) \leq P(y,y) \to P(x,y) = P(x,y)$. Hence, we obtain

$$P(x,y) = \bigwedge_{z\in X} P(y,z) \to P(x,z) = \bigwedge_{z\in X} P^T(z,y) \to P(x,z) \leq$$
$$\bigwedge_{z\in Y} P^T(z,y) \to P(x,z) = \bigwedge_{z\in Y} K^{-1}(z,y) \to K(x,z) = Q(x,y)$$

for any $x \in X$ and $y \in Y$, and thus $P \leq Q$. Finally, assume that $y \in Y$. Then

$$Q(x,y) = \bigwedge_{u\in Y} K^{-1}(u,y) \to K(x,u) = \bigwedge_{u\in Y} P^T(u,y) \to P(x,u) \leq$$
$$P^T(y,y) \to P(x,y) = \top \to P(x,y) = P(x,y).$$

Since $P \leq Q$ as shown, we get $Q(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$. □

Let $f : X \to L$ be a function, and let Q be a reflexive and Y–transitive fuzzy relation on X. We say that f *is extensional with respect to* Q if

$$f(x) \otimes Q(x,y) \leq f(y) \text{ and } Q(x,y) \otimes f(y) \leq f(x) \tag{4.27}$$

holds for any $x, y \in X$. Note the standard concept of the extensionality of f is related to the similarity relation on X, where it is sufficient to consider only one of the above inequalities. In our case, the fuzzy relation Q is not symmetric, therefore, we need to consider both inequalities to introduce the extensionality of f with respect to Q. The following theorems provide an estimation of the approximation of the extensional functions.

Theorem 4.17. *Let $Y \subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K : X \times Y \to L$ given as $K(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$ is an integral kernel, and let $K^{-1} = K^T$ be a Q–inverse of K. If f is extensional with respect to Q and (K^{-1}, μ_Y) satisfies (C2), then*

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int_X^{\otimes} Q^2(y,x)\, d\mu_X \tag{4.28}$$

for any $x \in X$.

Proof. Let f be an extensional function with respect to Q. By Lemma 4.16, we have $K(x,y) = P(x,y) = Q(x,y)$ and so $K^{-1}(y,x) = Q^T(y,x)$ for any $x \in X$ and $y \in Y$. Then, for any $y \in Y$, we have

$$F^{\otimes}_{(K,\mu_X)}(f)(y) = \int_X^{\otimes} K(x,y) \otimes f(x)\, d\mu_X =$$
$$\int_X^{\otimes} Q(x,y) \otimes f(x)\, d\mu_X \leq \int_X^{\otimes} \underline{f(y)}_X(x)\, d\mu_X = f(y),$$

where we used Theorem 2.7(ii), and the extensionality of f with respect to Q. It is easy to see that $K^2(x,y) = Q^2(x,y) = Q(x,y) \otimes Q(x,y)$ for $x \in X$ and $y \in Y$ is

again an integral kernel, since $\top = \top \otimes \top$. Note that $(K^2)^T = (K^T)^2 = (K^{-1})^2$. In addition, it holds that $\mathrm{Core}(K_x^{-1}) = \mathrm{Core}(K_x^T) = \mathrm{Core}((K^2)_x^T)$ for any $x \in X$. Indeed, if $y \in \mathrm{Core}(K_x^T)$, then $K^T(y,x) = \top = (K^2)^T(y,x)$ and therefore $y \in \mathrm{Core}((K^2)_x^T)$. If $y \in \mathrm{Core}((K^2)_x^T)$, then $(K^2)^T(y,x) = (K^T)^2 = \top$. Since $\top$ is the only solution of the equation $a \otimes a = \top$, we get $K^T(y,x) = \top$ and therefore $y \in \mathrm{Core}(K_x^T)$. Since (K^T, μ_Y) satisfies (C2), we find that $((K^2)^T, \mu_Y)$ also satisfies (C2). Hence, for any $x \in X$, we have

$$F^{\rightarrow}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) = \int_Y^{\otimes} K^{-1}(y,x) \rightarrow F^{\otimes}_{(K,\mu_X)}(f)(y)\, d\mu_Y =$$
$$\int_Y^{\otimes} Q^T(y,x) \rightarrow F^{\otimes}_{(K,\mu_X)}(f)(y)\, d\mu_Y \leq$$
$$\int_Y^{\otimes} Q^T(y,x) \rightarrow f(y)\, d\mu_Y \leq \int_Y^{\otimes} (Q^T(y,x) \otimes Q^T(y,x)) \rightarrow (Q^T(y,x) \otimes f(y))\, d\mu_Y =$$
$$\int_Y^{\otimes} (Q^T(y,x) \otimes Q^T(y,x)) \rightarrow (Q(x,y) \otimes f(y))\, d\mu_Y \leq$$
$$\int_Y^{\otimes} (Q^2)^T(y,x) \rightarrow \underline{f(x)}_Y(y)\, d\mu_Y = F^{\rightarrow}_{((K^2)^T,\mu_Y)}(\underline{f(x)}_Y)(x) = f(x),$$

where we used Theorem 1.1(x), the inequality $F^{\otimes}_{(K,\mu_X)}(f)(y) \leq f(y)$ that holds for any $y \in Y$, and the fact that $((K^2)^T, \mu_Y)$ satisfies (C2), and therefore, $F^{\rightarrow}_{((K^2)^{-1},\mu_Y)}$ preserves constant functions. As a consequence of the previous inequality, we obtain

$$F^{\rightarrow}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) \rightarrow f(x) = \top \tag{4.29}$$

for any $x \in X$. Further, for any $x \in X$, we have

$$(F^{\rightarrow}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)})(f)(x) \geq F^{\otimes}_{(Q,\mu_X)}(f)(x) = \int_X^{\otimes} Q(y,x) \otimes f(y)\, d\mu_X \geq$$
$$\int_X^{\otimes} Q(y,x) \otimes (Q(y,x) \otimes \underline{f(x)}_X(y))\, d\mu_X = \int_X^{\otimes} Q^2(y,x) \otimes \underline{f(x)}_X(y)\, d\mu_X \geq$$
$$f(x) \otimes \int_X^{\otimes} Q^2(y,x)\, d\mu_X,$$

where we used Theorem 2.7(iii) and Theorem 4.8. By the adjointness property, we get

$$f(x) \rightarrow (F^{\rightarrow}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)})(f)(x) \geq \int_X^{\otimes} Q^2(y,x)\, d\mu_X \tag{4.30}$$

for any $x \in X$. By combining (4.29) and (4.30), we get the desired inequality in (4.28). □

Theorem 4.18. *Let $Y \subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K : X \times Y \rightarrow L$ given as $K(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$ is an integral kernel, and let $K^{-1} = K^T$ be a Q–inverse of K. If f is extensional with respect to Q and (K, μ_X) satisfies (C2), then*

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int_Y^{\otimes} Q^2(x,y)\, d\mu_Y \tag{4.31}$$

for any $x \in X$.

Proof. Let f be an extensional function with respect to Q. By Lemma 4.16, we have $K(x,y) = P(x,y) = Q(x,y)$ and so $K^{-1}(y,x) = Q^T(y,x)$ for any $x \in X$ and $y \in Y$. First, we show that under the assumptions of the theorem, it holds that

$$F^{\rightarrow}_{(K,\mu_X)}(f)(y) = f(y), \quad y \in Y. \tag{4.32}$$

It is easy to see that $\mathrm{Core}(K_y) \subseteq \mathrm{Core}(Q^2_y)$ for any $y \in Y$, where $Q^2_y(x) = Q_y(x) \otimes Q_y(x)$, $x \in X$. Indeed, due to Lemma 4.16, for any $y \in Y$, we have $K_y(x) = K(x,y) = P(x,y) \leq Q(x,y) = Q_y(x)$ for any $x \in X$. Hence, if $x \in \mathrm{Core}(K_y)$, then $K_y(x) = \top = Q_y(x)$, therefore, $Q^2_y(x) = \top$ and $x \in \mathrm{Core}(Q^2_y)$. Now, if $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(Q^2_y)$, then $A \subseteq X \setminus \mathrm{Core}(K_y)$, which implies $\mu_X(A) = \bot$, since (K, μ_X) satisfies (C2). As a consequence, we obtain that (Q^2, μ_X) also satisfies (C2), and $F^{\rightarrow}_{(Q^2,\mu_X)}$ preserves constant functions. Then, for any $y \in Y$, we have

$$F^{\rightarrow}_{(K,\mu_X)}(f)(y) = \int_X^{\otimes} K(x,y) \rightarrow f(x)\, d\mu_X = \int_X^{\otimes} Q(x,y) \rightarrow f(x)\, d\mu_X \leq$$
$$\int_X^{\otimes} Q^2(x,y) \rightarrow (f(x) \otimes Q(x,y))\, d\mu_X \leq \int_X^{\otimes} Q^2(x,y) \rightarrow \underline{f(y)}_X(x)\, d\mu_X =$$
$$F^{\rightarrow}_{(Q^2,\mu_X)}(\underline{f(y)}_X)(y) = f(y),$$

where we used Theorem 1.1(x) and the extensionality of f with respect to Q. On the other hand, for any $x, y \in X$, $Q(x,y) \otimes f(y) \leq f(x)$ implies $f(y) \leq Q(x,y) \rightarrow f(x)$ due to the adjointness property. Using Theorem 2.7(ii), we simply find that

$$F^{\rightarrow}_{(K,\mu_X)}(f)(y) = \int_X^{\otimes} K(x,y) \rightarrow f(x)\, d\mu_X = \int_X^{\otimes} Q(x,y) \rightarrow f(x)\, d\mu_X \geq$$
$$\int_X^{\otimes} \underline{f(y)}_X(x)\, d\mu_X = f(y),$$

which implies (4.32). Using this equality, for any $x \in X$, we get

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(x) = \int_Y^{\otimes} K^{-1}(y,x) \otimes F^{\rightarrow}_{(K,\mu_X)}(f)(y)\, d\mu_Y =$$
$$\int_Y^{\otimes} Q^T(y,x) \otimes f(y)\, d\mu_Y = \int_Y^{\otimes} Q(x,y) \otimes f(y)\, d\mu_Y \leq \int_Y^{\otimes} \underline{f(x)}_Y(y)\, d\mu_Y = f(x),$$

where we used Theorem 2.7(ii) and the extensionality of f with respect to Q. As a consequence of this inequality, we get

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(x) \rightarrow f(x) = \top. \tag{4.33}$$

Since $f(x) \otimes Q(x,y) = f(x) \otimes Q^T(y,x) \leq f(y)$ for any $x \in X$ and $y \in Y$, we get

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(x) = \int_Y^{\otimes} Q^T(y,x) \otimes F^{\rightarrow}_{(K,\mu_X)}(f)(y)\, d\mu_Y =$$
$$\int_Y^{\otimes} Q^T(y,x) \otimes f(y)\, d\mu_Y \geq \int_Y^{\otimes} Q^T(y,x) \otimes (Q^T(y,x) \otimes \underline{f(x)}_Y(y))\, d\mu_Y =$$
$$\int_Y^{\otimes} (Q^T)^2(y,x) \otimes \underline{f(x)}_Y(y))\, d\mu_Y \geq f(x) \otimes \int_Y^{\otimes} (Q^T)^2(y,x)\, d\mu_Y,$$

where we used Theorem 2.7(iii) and (4.32). Due to the adjointness property, we find that

$$f(x) \to F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f)(x) \geq \int_Y^{\otimes} Q^2(x,y)\, d\mu_Y, \tag{4.34}$$

where we used the fact that $(Q^T)^2(y,x) = Q^2(x,y)$, which follows from $(Q^T)^2(y,x) = Q^T(y,x) \otimes Q^T(y,x) = Q(x,y) \otimes Q(x,y) = Q^2(x,y)$ for any $x, y \in X$. By combining (4.33) and (4.34), we obtain the desired inequality in (4.31). □

We have shown that the composition of M–lattice integral transforms preserves constant functions under the satisfaction of conditions (C1) and (C2). The following corollaries show that the same conditions ensure the preservation of extensional functions with respect to Q, where Q is determined from a similarity relation on X.

Corollary 4.19. *Let the assumption of Theorem 4.17 be satisfied, and let (K, μ_X) satisfy (C1). Then*

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f) = f \tag{4.35}$$

for any extensional function f on X with respect to Q.

Proof. From Lemma 4.16, we know that Q is Y–transitive and $K(y,x) = P(y,x) \leq Q(y,x)$ for any $x, y \in X$. Let $x \in X$. Then there is $z \in Y$ such that $K(x,z) = P(x,z) = \top$, since K is an integral kernel. By the Y–transitivity of Q, for any $y \in X$, we get $Q(y,x) \geq Q(y,z) \otimes Q(z,x) \geq P(y,z) \otimes P(z,x) = P(y,z) \otimes P(x,z) = P(y,z) = K(y,z)$, where we used the fact that P is symmetric. Hence, we get $\mathrm{Core}(K_z) \subseteq \mathrm{Core}(Q_x)$. Since (K, μ_X) satisfies (C1), we get that (Q, μ_X) also satisfies (C1). Hence, for any $x \in X$ there is $A_x \in \mathcal{F}$ such that $A_x \subseteq \mathrm{Core}(Q_x)$ and $\mu(A_x) = \top$. Obviously, $Q^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$. Indeed, for $y \in A_x$, we have $\top = 1_{A_x}(y) = Q(y,x) = Q^2(y,x)$. If $y \notin A_x$, then trivially $\bot = 1_{A_x}(y) \leq Q^2(y,x)$. By Theorem 4.17, we simply find

$$F^{\to}_{(K^{-1},\mu_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int_X^{\otimes} Q^2(y,x)\, d\mu_X \geq \int_X^{\otimes} 1_{A_x}(y)\, d\mu_X = \mu_X(A_x) = \top,$$

where we used Theorem 2.7(v). Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

Corollary 4.20. *Let the assumption of Theorem 4.18 be satisfied, and let (K^{-1}, μ_Y) satisfy (C1). Then*

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu_X)}(f) = f \tag{4.36}$$

for any extensional function f on X with respect to Q.

Proof. Using Lemma 4.16, we have $Q^T(y,x) = P^T(y,x) = K^{-1}(y,x)$ for any $x \in X$ and $y \in Y$. Since (K^{-1}, μ_Y) satisfies (C1), for any $x \in X$, there is $A_x \in \mathcal{G}$ such that $A_x \subseteq \mathrm{Core}(K_x^{-1})$ and $\mu_Y(A_x) = \top$. Obviously, we have $(K^{-1})^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$, which can be verified analogously as in the proof of Corollary 4.19. In addition, it holds $(K^{-1})^2 = (K^2)^T$. By Theorem 4.18, we find that

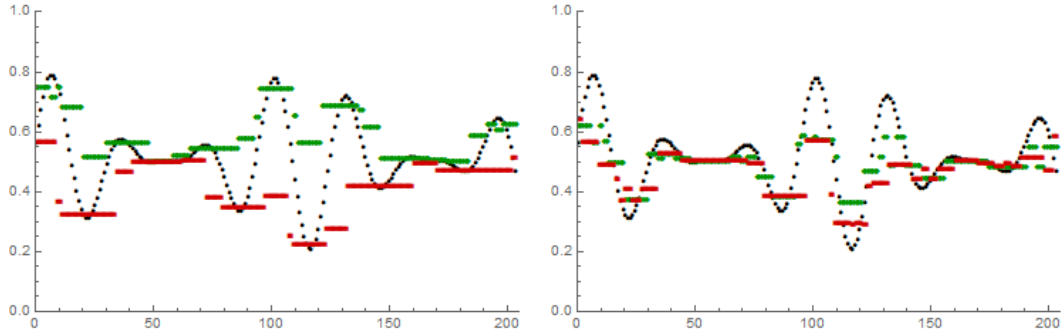
$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int_Y^{\otimes} Q^2(x,y)\, d\mu_Y =$$
$$\int_Y^{\otimes} (K^2)^T(y,x)\, d\mu_Y = \int_Y^{\otimes} (K^{-1})^2(y,x)\, d\mu_Y \geq \int_Y^{\otimes} 1_{A_x}(y)\, d\mu_Y = \mu(A_x) = \top,$$

where we used Theorem 2.7(v). Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

4.3.3 Illustration on signal reconstruction

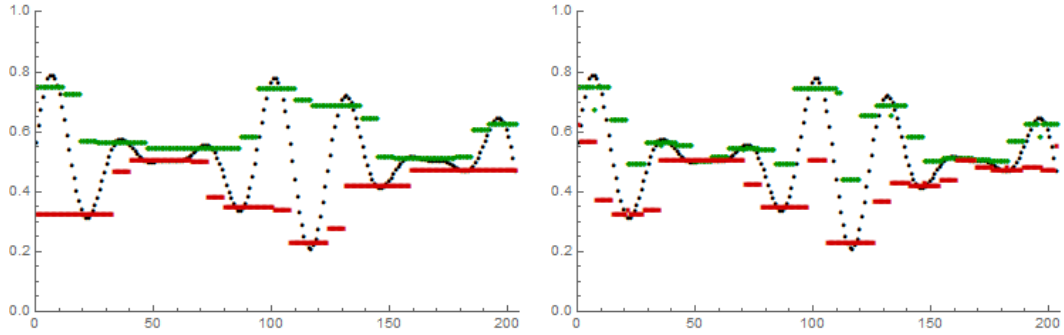
In this part, we demonstrate the reconstruction of discrete signals using compositions of M–lattice integral transforms. We will follow the setting of Example 3.5 and reconstruct the original signal given by formula (3.9) from the transformed functions (i.e., the outputs of M–lattice integral transforms) presented there. Recall that all the integral kernels $K : X \times Y \to L$ together with the fuzzy measures μ_X (originally denoted as μ) on the measurable space $\langle X, \mathcal{P}(X)\rangle$ are introduced in that example in such a way that (K, μ_X) satisfies condition (C1) for $\star = \otimes$ and $(K, \mu_X^{c,N})$ satisfies condition (C2) for $\star = \rightarrow$ due to Theorem 3.9. Therefore, the M*–lattice integral transforms from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ preserve constant functions. To illustrate, we similarly introduce M–lattice integral transforms from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$, namely, an Q–inverse integral kernel $K^{-1} : Y \times X \to L$ is given by $K^{-1} = K^T$ and the related fuzzy measure μ_Y on the measurable space $\langle Y, \mathcal{P}(Y)\rangle$ is defined such that (K^{-1}, μ_Y) satisfies (C1) for $\star = \otimes$ and consequently $(K^{-1}, \mu_Y^{c,N})$ satisfies (C2) for $\star = \rightarrow$ due to Theorem 3.9. Thus, both M–lattice integral transforms in a particular composition for reconstructing the original signal always preserve constant signals (cf., Theorems 4.14 and 4.15). In the following we will consider cases a) and b) studied in Example 3.5.

Case a) We assume the integral kernel $K : X \times Y \to [0,1]$ and two associated fuzzy measures $\mu_{X1} = \mu^5_{2,6}$ and $\mu_{X2} = \mu^5_{7,12}$ on $\langle X, \mathcal{P}(X)\rangle$ established in case a) of Example 3.5 that specify the $(K, \mu_{Xi}, \otimes)$–M–lattice integral transform and the $(K, \mu_{Xi}^{c,N}, \rightarrow)$–M–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ for $i = 1, 2$. Further, we define the fuzzy measure $\mu_Y = \mu^5_{1,2}$ on $\langle Y, \mathcal{P}(Y)\rangle$ to introduce the $(K^{-1}, \mu_Y, \otimes)$–M–lattice integral transform and the $(K^{-1}, \mu_Y^{c,N}, \rightarrow)$–M–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. It is easy to see that $\#\mathrm{Core}(K_x^{-1}) \in \{2, 3\}$ for any $x \in X$, therefore, μ_Y satisfies (C1) and $\mu_Y^{c,N}$ satisfies (C2) due to Theorem 3.9. Note that the space for setting fuzzy measures on $\langle Y, \mathcal{P}(Y)\rangle$ to ensure that condition (C1) is satisfied is very small, so we use only one fuzzy measure for both reconstructions. Since this fuzzy measure is very close to the highest measure $\mu^{\top}$ on $\langle Y, \mathcal{P}(Y)\rangle$, we get that the M–lattice integral transforms from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ are very similar to the direct lattice fuzzy transforms. In Figure 4.2(a), we show the reconstruction



(a) μ_{X1} (b) μ_{X2}

Figure 4.2: Original function f (black) and its approximation using $\mathrm{M}^{\rightarrow} \circ \mathrm{M}^{\otimes}$ (green diamonds) and $\mathrm{M}^{\otimes} \circ \mathrm{M}^{\rightarrow}$ (red squares) for a fixed integral kernel K and two different fuzzy measures μ_{X1} and μ_{X2}.



(a) K_1 (b) K_2

Figure 4.3: Original function f (black) and its approximation using $\mathrm{M}^{\rightarrow} \circ \mathrm{M}^{\otimes}$ (green diamonds) and $\mathrm{M}^{\otimes} \circ \mathrm{M}^{\rightarrow}$ (red squares) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.

of the original signal using the composition of the $(K, \mu_{X1}, \otimes)$–M–lattice integral transform and the $(K^{-1}, \mu_Y^{c,N}, \rightarrow)$–M–lattice integral transform described by green diamonds ($\mathrm{M}^{\rightarrow} \circ \mathrm{M}^{\otimes}$ for short) and the same reconstruction using the composition of the $(K, \mu_{X1}^{c,N}, \rightarrow)$–M–lattice integral transform and the $(K^{-1}, \mu_Y, \otimes)$–M–lattice integral transform described by red squares ($\mathrm{M}^{\otimes} \circ \mathrm{M}^{\rightarrow}$ for short) together with the original function. Since the output signals of the M–lattice integral transforms from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ are very similar to the signals obtained by the direct lattice fuzzy transforms (a consequence of the setting of $\mu_{X1} = \mu_{2,6}^5$, see Figure 3.2), it is not surprising that the reconstructed signals provide near lower and upper approximation of the original function. Analogous reconstructions of the original signal, but for the fuzzy measure μ_{X2}, are shown in Figure 4.2(b). As we stated in Example 3.5, the fuzzy measure μ_{X2} and similar can be used to filter out high frequencies in a signal using the M–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, and the output of the corresponding M–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ is the smoothed original signal. A suitable composition of M–lattice integral transforms can thus serve as a high-pass filter in signal processing.

Case b) We assume two integral kernels $K_1, K_2 : X \times Y \to [0,1]$ and the fuzzy measure $\mu_X = \mu^5_{3,6}$ on $\langle X, \mathcal{P}(X)\rangle$ established in case b) of Example 3.5 that specify the $(K_i, \mu_X, \otimes)$–M–lattice integral transform and the $(K_i, \mu_X^{c,N}, \to)$–M–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ for $i = 1, 2$. Further, we consider the same fuzzy measure μ_Y on $\langle Y, \mathcal{P}(Y)\rangle$ as above in case a) to introduce the $(K_i^{-1}, \mu_Y, \otimes)$–M–lattice integral transform and the $(K_i^{-1}, \mu_Y^{c,N}, \to)$–M–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ for $i = 1, 2$. Note that $\#\mathrm{Core}(K_{1x}^{-1}) \in \{4, 5\}$ and $\#\mathrm{Core}(K_{2x}^{-1}) \in \{2, 3\}$ for any $x \in X$, therefore, (K_i^{-1}, μ_Y) satisfies (C1) and $(K_i^{-1}, \mu_Y^{c,N})$ satisfies (C2) due to Theorem 3.9. In Figure 4.3, similarly to case a), we show reconstructed signals for the compositions of M–lattice integral transforms with different integral kernels K_1 and K_2. Since $K_{2y} \subseteq K_{1y}$ for any $y \in Y$, more function values are aggregated inside the M*–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ with regard to K_1 than K_2, which leads to a better approximation of the original function for K_2, shown in Figure 4.3(b) than for K_1, shown in Figure 4.3(a). This influence of the setting of the integral kernel on the approximation quality is not surprising and its a consequence of the theorems presented in Subsection 4.3.2. Obviously, if $K_{2y} \subseteq K_{1,y}$ for any $y \in Y$, then $\mathrm{Supp}(K_{2y}) \subseteq \mathrm{Supp}(K_{1y})$, which leads to $\omega(f, E_{1y}) \leq \omega(f, E_{2y})$, where $\omega(f, E_{iy})$, $i = 1, 2$, is defined in Theorem 4.12. Then

$$\omega_1(f) = \bigwedge_{y \in Y} \omega(f, E_{1y}) \leq \bigwedge_{y \in Y} \omega(f, E_{2y}) = \omega_2(f),$$

and by Theorems 4.14 and 4.15 we find that the reconstructed signal by the compositions of M–lattice integral transforms with regard to K_2 should provide a better approximation of the original function f than in the case of K_1.

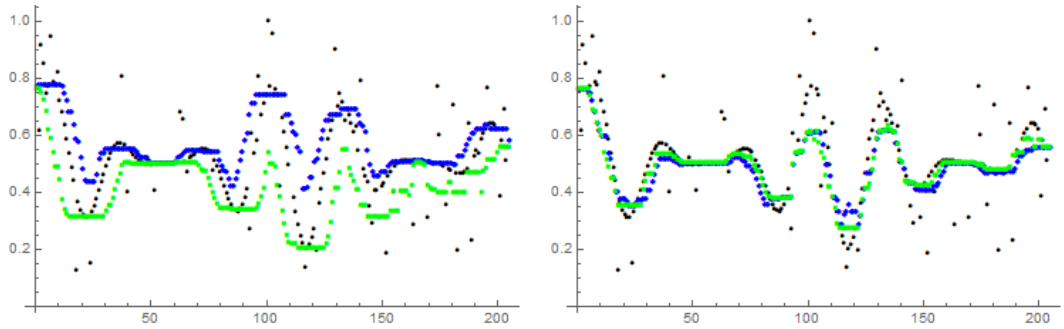
Finally, we should note that the composition $\mathrm{M}^{\to} \circ \mathrm{M}^{\otimes}$ is over the composition $\mathrm{M}^{\otimes} \circ \mathrm{M}^{\to}$ in all of the above cases except one, which is shown in Figure 4.2(b). This observation is very interesting and leads to a natural question under which general conditions this claim is true, i.e.,

$$F^{\otimes}_{(K^{-1},\mu_Y)} \circ F^{\to}_{(K,\mu'_X)}(f) \leq F^{\to}_{(K^{-1},\mu'_Y)} \circ F^{\otimes}_{(K,\mu_X)}(f) \tag{4.37}$$

holds for any $f \in \mathcal{F}(X)$. Specifically, we can assume that $\mu'_X = \mu_X^{c,N}$ and $\mu'_Y = \mu_Y^{c,N}$ and (K, μ_X) and (K^{-1}, μ_Y) satisfy (C1), but this is not sufficient, as shown in Figure 4.2(b). The answer to this question is the subject of our future research.

4.3.4 Filtering of random noise

As we mentioned in the motivation in Subsection 4.1 of this chapter, lattice fuzzy transforms cannot in principle filter out random noise present in the signal (see, Figure 4.1(b)). The goal of this subsection is to show that M–lattice integral transforms can be used to filter out random noise in signal processing. A further illustration will be given in Chapter 6, where M–lattice integral transforms are used to filter out salt-and-pepper noise in images. For the demonstration, we again use the function f given by the formula in (3.9), to which we add 30% random noise determined by a uniform distribution. In what follows, we will present two applications of M–lattice integral transforms in random noise filtering:



(a) μ_{X1} (b) μ_{X2}

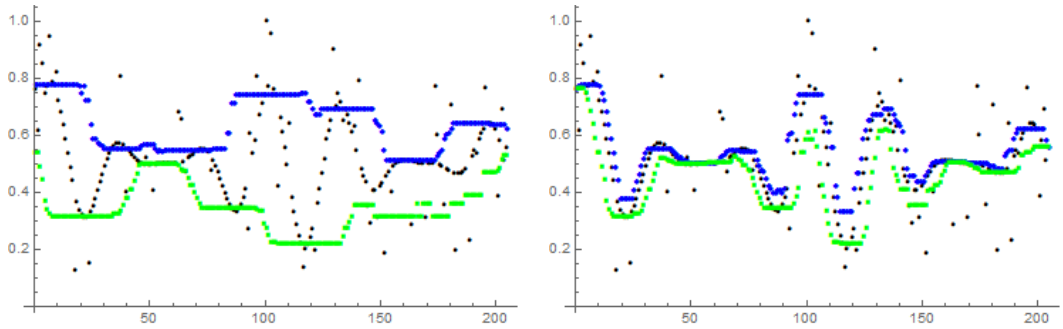
Figure 4.4: Filtering random noise using M$^{\otimes}$–LIT (light blue) and M$^{\rightarrow}$–LIT (light green) for a fixed integral kernel K and two different fuzzy measures μ_{X1} and μ_{X2}.

- Filter based on a single M–lattice integral transform with an integral kernel on X.
- Filter based on a composition of M–lattice integral transforms as mentioned above.

In both applications, we consider the same fuzzy measures μ_X, μ_{X1} and μ_{X2} on $\langle X, \mathcal{P}(X)\rangle$ and μ_Y on $\langle Y, \mathcal{P}(Y)\rangle$ specified in the previous illustration subsection.

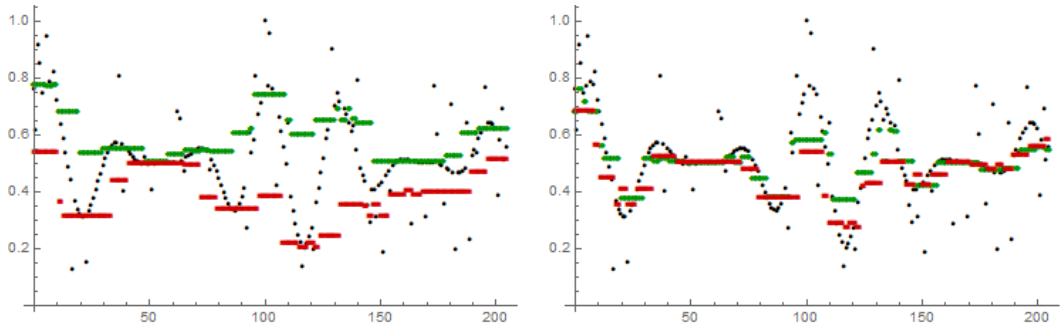
Filter based on single M–lattice integral transform The idea of filtering out random noise from a signal using a single M–lattice integral transform is inspired by data analysis methods such as moving average or median. It consists in using an integral kernel defined only on X (i.e., $X = Y$), which we use to aggregate function values over each point in X. By adjusting the integral kernel and in particular the fuzzy measure, we can control how random noise is removed. For illustration, we assume that $X = \{1, 2, \ldots, 204\}$ and the integral kernels $K, K_1, K_2 : X \times X \to L$ are are defined by the same formulas as in Example 3.5 only Y is replaced by X. Again, we distinguish two cases, namely, case a) the fixed integral kernel K and two fuzzy measures μ_{X1} and μ_{X2}, case b) the fixed fuzzy measure μ_X and two integral kernels K_1 and K_2.

The results of filtering random noise for case a) are displayed in Figure 4.4. Recall that the size of the core and support of K is quite high, so in addition to noise filtering, there is also suppression of higher frequencies, which is particularly noticeable in the case of μ_{X2}, shown in Figure 4.4(b), which is set to aggregate function values similar to the median. It seems that the fuzzy measure μ_{X1} preserves higher frequencies and filters noise better than μ_{X2}, namely the M$^{\otimes}$–LIT reconstructs the upper part of the signal well, while the M$^{\rightarrow}$–LIT reconstructs the lower part well. The results for case b) are shown in Figure 4.5. Since the fuzzy measure μ_X with respect to K_1 is very similar to the highest fuzzy measure $\mu_X^{\top}$, as discussed in Example 3.5), so it is not surprising that the noise is not well filtered, as shown in Figure 4.5(a), similar to the lattice fuzzy transforms (cf. figure 4.1(b)). Visually, the best result is obtained by the M$^{\otimes}$–LIT for K_2, which, together with the fuzzy measure μ_X, behaves almost like a median filter. Again, the M$^{\otimes}$–LIT reconstructs



(a) K_1 (b) K_2

Figure 4.5: Filtering random noise using $\mathrm{M}^{\otimes}$–LIT (light blue) and $\mathrm{M}^{\rightarrow}$–LIT (light green) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.



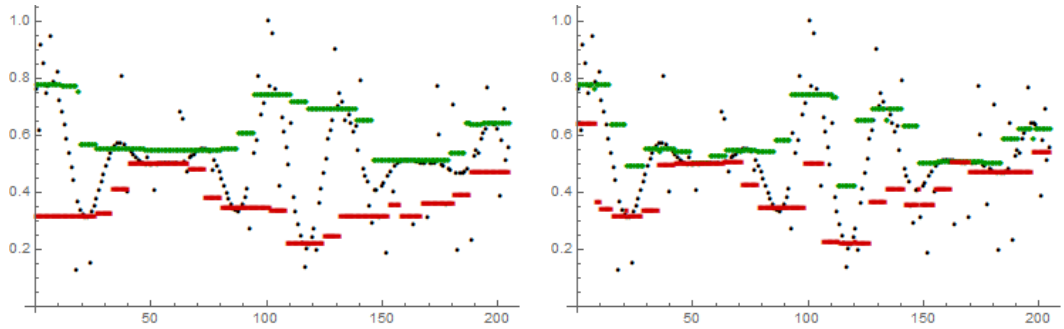
(a) μ_{X1} (b) μ_{X2}

Figure 4.6: Filtering random noise using composition of $\mathrm{M}^{\rightarrow} \circ \mathrm{M}^{\otimes}$ (green diamonds) and $\mathrm{M}^{\otimes} \circ \mathrm{M}^{\rightarrow}$ (red squares) for a fixed integral kernel and two different fuzzy measures μ_{X1} and μ_{X2}.

the upper part of the signal better, while the $\mathrm{M}^{\rightarrow}$–LIT reconstructs the lower part better.

Filter based on composition of M–lattice integral transforms For noise filtering, we can generally use more than one filter, with filters applied in sequence. Thus, we consider the composition of the same or different filters. We can even transform the noisy signal from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ for its compression, assuming that $Y \subset X$, and back again to recover it with noise suppression. In this part, we show that the compositions of M–lattice integral transforms presented in the previous subsection suppress noise in the reconstructed signals.

We consider the same setting of the integral kernels K, K_1 and K_2 as in Example 3.5. The results are shown in Figures 4.6 and 4.7. We can see that all the resulting signals are noise-free and are very similar to the reconstructed signals in the previous subsection, which are derived from the original (noise-free) signal.



(a) K_1 (b) K_2

Figure 4.7: Filtering random noise using composition of $\mathrm{M}^{\to} \circ \mathrm{M}^{\otimes}$(green diamonds) and $\mathrm{M}^{\otimes} \circ \mathrm{M}^{\to}$ (red squares) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.

4.4 Approximation of functions based on R_{DH}–lattice integral transforms

In this section, we continue our investigation of the estimation of an original lattice-valued function using a combination of two types of R_{DH}–lattice integral transforms that are introduced in Subsection 3.4. Throughout this section, we assume that $\langle X, \mathcal{F}, \nu_X \rangle$ and $\langle Y, \mathcal{G}, \nu_Y \rangle$ are complementary fuzzy measure spaces, $K : X \times Y \to L$ is an integral kernel and $K^{-1,d} : Y \times X \to L$ is a Q^d–dual inverse integral kernel of K, where Q^d is the integral kernel on X satisfying (4.11)

4.4.1 Upper and lower approximation of functions

Similarly to the previous section, we start with a theorem showing in a sense a generalization of the approximation from above of the original function using the composition of R_{DH}–lattice integral transforms. Recall that the fuzzy integral $\int^{\to}_{\mathrm{DH}Y}$ is monotonically non-increasing map (see, Theorem 2.12).

Theorem 4.21. *Let $G^{\otimes}_{(K,\nu_X)}$ be an R_{DH}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and $G^{\to}_{(K^{-1,d},\nu_Y)}$ be an R_{DH}–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)} \geq \neg G^{\otimes}_{(Q^d,\nu_X)}. \tag{4.38}$$

Proof. From Definition 4.4 and using (ii) and (iii) of Theorem 2.12, we have for any $f \in \mathcal{F}$ and $z \in X$

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(z) = \int^{\to}_{\mathrm{DH}\,Y} (K^{-1,d}(y,z) \to G^{\otimes}_{(K,\nu_X)}(f)(y))\, d\nu_Y =$$
$$\int^{\to}_{\mathrm{DH}\,Y} \left(K^{-1,d}(y,z) \to \int^{\to}_{\mathrm{DH}\,X} K(x,y) \otimes f(x)\, d\nu_X \right) d\nu_Y \geq$$
$$\int^{\to}_{\mathrm{DH}\,Y} \left(\int^{\to}_{\mathrm{DH}\,X} \underline{(K^{-1,d}(y,z)}_X(x) \otimes (K(x,y) \otimes f(x)))\, d\nu_X \right) d\nu_Y =$$

$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\left(\int_{\mathrm{DH}\,X}^{\rightarrow}\left(\underline{K^{-1,d}(y,z)}_X(x)\otimes K(x,y)\right)\otimes f(x)\,d\nu_X\right)d\nu_Y\geq$$
$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\underline{\left(\int_{\mathrm{DH}\,X}^{\rightarrow}Q^d(x,z)\otimes f(x)\,d\nu_X\right)}_Y(y)\,d\nu_Y=$$
$$\neg\int_{\mathrm{DH}\,X}^{\rightarrow}Q^d(x,z)\otimes f(x)\,d\nu_X=\neg G^{\otimes}_{(Q^d,\nu_X)}(f)(z),$$

where we used the associativity of $\otimes$, $K^{-1,d}(y,z)$ is a constant for $\int_{\mathrm{DH}\,X}^{\rightarrow}$ (therefore, we may apply Theorem 2.12(iii)), the fact that Q^d is a $(K,K^{-1,d})$–dually compatible integral kernel, i.e., $\underline{K^{-1,d}(y,z)}_X(x)\otimes K(x,y)=K^{-1,d}(y,z)\otimes K(x,y)\leq Q^d(x,z)$ for any $x,z\in X$, and

$$\underline{\left(\int_{\mathrm{DH}\,X}^{\rightarrow}Q^d(x,z)\otimes f(x)\,d\nu_X\right)}_Y$$

is a constant function integrated by $\int_{\mathrm{DH}\,Y}^{\rightarrow}$. □

Note that the use of negation on the right-hand side of the inequality in (4.38) is very natural because R_{DH}–lattice integral transforms give a negative output, so the negation provides a comparison of two positive outputs. The approximation from below of the original function using the second composition of R_{DH}–lattice integral transforms is presented in the following theorem.

Theorem 4.22. *Let $G^{\rightarrow}_{(K,\nu_X)}$ be a R_{DH}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and $G^{\otimes}_{(K^{-1,d},\nu_Y)}$ be a R_{DH}–lattice integral transform $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$G^{\otimes}_{(K^{-1,d},\nu_Y)}\circ G^{\rightarrow}_{(K,\nu_X)}\leq\neg G^{\rightarrow}_{(Q^d,\nu_X)}. \tag{4.39}$$

Proof. From Definition 4.4, using (ii) and (iv) of Theorem 2.12, and for any $f\in\mathcal{F}$ and $z\in X$, we have

$$G^{\otimes}_{(K^{-1,d},\nu_Y)}\circ G^{\rightarrow}_{(K,\nu_X)}(f)(z)=\int_{\mathrm{DH}\,Y}^{\rightarrow}(K^{-1,d}(y,z)\otimes G^{\rightarrow}_{(K,\nu_X)}(f)(y))\,d\nu_Y=$$
$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\left(K^{-1,d}(y,z)\otimes\int_{\mathrm{DH}\,X}^{\rightarrow}(K(x,y)\rightarrow f(x))\,d\nu_X\right)d\nu_Y\leq$$
$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\left(\int_{\mathrm{DH}\,X}^{\rightarrow}(\underline{K^{-1,d}(y,z)}_X(x)\rightarrow(K(x,y)\rightarrow f(x)))\,d\nu_X\right)d\nu_Y=$$
$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\left(\int_{\mathrm{DH}\,X}^{\rightarrow}((\underline{K^{-1,d}(y,z)}_X(x)\otimes(K(x,y))\rightarrow f(x))\,d\nu_X\right)d\nu_Y\leq$$
$$\int_{\mathrm{DH}\,Y}^{\rightarrow}\underline{\left(\int_{\mathrm{DH}\,X}^{\rightarrow}Q^d(x,z)\rightarrow f(x)\,d\nu_X\right)}_Y(y)\,d\nu_Y=$$
$$\neg\left(\int_{\mathrm{DH}\,X}^{\rightarrow}Q^d(x,z)\rightarrow f(x)\,d\nu_X\right)=\neg G^{\rightarrow}_{(Q^d,\nu_X)}(f)(z),$$

where we used $a\rightarrow(b\rightarrow c)=(a\otimes b)\rightarrow c$ (Theorem 1.1(vi)), $K^{-1,d}(y,z)$ is a constant for $\int_{\mathrm{DH}\,X}^{\rightarrow}$ (therefore, we may apply Theorem 2.12(iv)), the fact that the residuum is

non-increasing in the first argument, Q^d is a $(K, K^{-1,d})$–dually compatible integral kernel (see the proof of Theorem 4.21 for details), and

$$\underline{\left(\int_{\mathrm{DH}\,X}^{\rightarrow} Q^d(x,z) \rightarrow f(x)\, d\nu_X\right)}_Y$$

is a constant function integrated by $\int_{\mathrm{DH}_Y}^{\rightarrow}$. □

Similarly to Corollary 4.10, we show another estimation of the approximation of the original function using the composition of R_{DH}–lattice integral transforms in the case that Q^d is a reflexive fuzzy relation on X. Denote Q^d_y–projection, i.e., $Q^d_y(x) = Q^d(x,y)$ for any $x \in X$.

Corollary 4.23. *For any $f \in \mathcal{F}(X)$ and $y \in X$, it holds that*

(i) $G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(y) \geq \neg \int_{\mathrm{DH}\,X}^{\rightarrow} 1_{\mathrm{Core}(Q^d_y)} \otimes f(x)\, d\nu_X$,

(ii) $G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(y) \leq \neg \int_{\mathrm{DH}\,X}^{\rightarrow} 1_{\mathrm{Core}(Q^d_y)} \rightarrow f\, d\nu_X$.

Proof. Let $\Delta : L \rightarrow L$ be the operation defined by (4.17) on page 72, and $f \in \mathcal{F}(X)$.

(i) Using Theorem 2.12(i), for any $y \in X$, we have

$$G^{\otimes}_{(Q^d,\nu_X)}(f)(y) = \int_{\mathrm{DH}\,X}^{\rightarrow} Q^d(x,y) \otimes f(x)\, d\nu_X$$
$$\leq \int_{\mathrm{DH}\,X}^{\rightarrow} \Delta Q^d(x,y) \otimes f(x)\, d\nu_X = \int_{\mathrm{DH}\,X}^{\rightarrow} 1_{\mathrm{Core}(Q^d_y)} \otimes f(x)\, d\nu_X,$$

Applying the negation, the previous inequality is reversed, and the desired inequality in (a) is the straightforward consequence of (4.38).

(ii) Again using Theorem 2.12, for any $y \in X$, we have

$$G^{\rightarrow}_{(Q^d,\nu_X)}(f)(y) = \int_{\mathrm{DH}\,X}^{\rightarrow} Q^d(x,y) \rightarrow f(x)\, d\nu_X$$
$$\geq \int_{\mathrm{DH}\,X}^{\rightarrow} \Delta Q^d(x,y) \rightarrow f(x)\, d\nu_X = \int_{\mathrm{DH}\,X}^{\rightarrow} 1_{\mathrm{Core}(Q^d_y)} \rightarrow f(x)\, d\nu_X,$$

where we used the fact that the residuum is monotonically non-increasing in the first component. Using the negation, the previous inequality is reversed and the desired inequality in (b) is the straightforward consequence of (4.38). □

4.4.2 Estimation of approximation quality

This part is devoted to the quality estimation of the approximation of the original function using the R_{DH}–lattice integral transforms. For the estimation, we use the modulus of continuity introduced in Definition 4.5. Since the R_{DH}–lattice integral transform transforms a positive input to a negative output, we compare the transformed values with the negation of the original values. Recall that ∇ is the operator assigning $\perp$, if $a = \perp$, and $\top$, otherwise (see, (4.20) on page 73 for the definition). The following theorem shows an estimate of the approximation quality for the $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–lattice integral transform transform.

Theorem 4.24. *Let (K,ν) satisfy (C3), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Let E_y be the equivalence on X defined in Theorem 4.12. Then*

$$G^{\otimes}_{(K,\nu)}(f)(y) \leftrightarrow \neg f(x) \geq \omega(f, E_y), \tag{4.40}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. From Theorem 4.12 we know that E_y is well-defined. Similarly to the reformulation of the definition of $F^{\otimes}_{(K,\mu)}$, we have

$$G^{\otimes}_{(K,\nu)}(f)(y) = \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq \mathrm{Supp}(K_y)}} (\bigwedge_{x \in A} (K(x,y) \otimes f(x)) \to \nu(A)). \tag{4.41}$$

Indeed, if $A \in \mathcal{F}$ such that $A \not\subseteq \mathrm{Supp}(K_y)$, then we get $\bigwedge_{x \in A}(K(x,y) \otimes f(x)) \to \nu(A) = \bot \to \nu(A) = \top$. Hence, the infimum over all $\mathcal{F}$–measurable sets can be reduced to all $\mathcal{F}$–measurable sets that are subsets of $\mathrm{Supp}(K_y)$. We use $\mathcal{F}_y$ to denote the set of all $\mathcal{F}$–measurable sets $A \subseteq \mathrm{Supp}(K_y)$. Since $G^{\otimes}_{(K,\nu)}$ preserves constant functions due to (C3), and using (15), (18), (19) and (22) of Theorem 1.3, we have

$$\begin{aligned}
&G^{\otimes}_{(K,\nu)}(f)(y) \leftrightarrow \neg f(x) = G^{\otimes}_{(K,\nu)}(f)(y) \leftrightarrow G^{\otimes}_{(K,\nu)}(\underline{f(x)}_X)(y) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z,y) \otimes f(z)) \to \nu(A)) \leftrightarrow \bigwedge_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z,y) \otimes \underline{f(x)}_X(z)) \to \nu(A)) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\bigwedge_{z \in A} (K(z,y) \otimes f(z)) \to \nu(A)) \leftrightarrow (\bigwedge_{z \in A} (K(z,y) \otimes \underline{f(x)}_X(z)) \to \nu(A))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\bigwedge_{z \in A} (K(z,y) \otimes f(z)) \leftrightarrow \bigwedge_{z \in A} (K(z,y) \otimes \underline{f(x)}_X(z)) \otimes (\nu(A) \leftrightarrow \nu(A))) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\bigwedge_{z \in A} (K(z,y) \otimes f(z)) \leftrightarrow \bigwedge_{z \in A} (K(z,y) \otimes \underline{f(x)}_X(z))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} \bigwedge_{z \in A} (K(z,y) \leftrightarrow K(z,y)) \otimes (f(z) \leftrightarrow \underline{f(x)}_X(z)) \geq \\
&\bigwedge_{z \in \mathrm{Supp}(K_y)} (f(z) \leftrightarrow f(x)) \geq \bigwedge_{(u,v) \in E_y} (f(u) \leftrightarrow f(v)) = \omega(f, E_y),
\end{aligned}$$

where we used the fact that if $z \in \mathrm{Supp}(K_y)$, then $(x,z) \in E_y$. □

The next theorem shows that the same approximation quality can be achieved also for the $\mathrm{R}^{\to}_{\mathrm{DH}}$–lattice integral transform transform.

Theorem 4.25. *Let (K,ν) satisfy (C4), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Let E_y be the equivalence on X defined in Theorem 4.12. Then*

$$G^{\to}_{(K,\nu)}(f)(y) \leftrightarrow \neg f(x) \geq \omega(f, E_y), \tag{4.42}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. From Theorem 4.12 we know that E_y is well-defined. Similarly to the reformulation of the definition of $F^{\to}_{(K,\mu)}$, we have

$$G^{\to}_{(K,\nu)}(f)(y) = \bigwedge_{\substack{A\in\mathcal{F}\\ A\cap\mathrm{Core}(K_y)\neq\emptyset}} \Big(\bigwedge_{x\in A\cap\mathrm{Supp}(K_y)} (K(x,y)\to f(x)) \to \nu(A)\Big). \tag{4.43}$$

Indeed, since (K,ν) satisfy (C4), i.e., $\nu(A)=\top$ for any $A\subseteq X\setminus\mathrm{Core}(K_y)$, we get

$$\bigwedge_{\substack{A\in\mathcal{F}\\ A\cap\mathrm{Core}(K_y)=\emptyset}} \Big(\bigwedge_{x\in A\cap\mathrm{Supp}(K_y)} (K(x,y)\to f(x)) \to \nu(A)\Big) = \top,$$

which follows from $a\to\top=\top$ for any $a\in L$ (Theorem 1.1(iii)). Hence, it is sufficient to consider only $\mathcal{F}$–measurable sets such that $A\cap\mathrm{Core}(K_y)\neq\emptyset$. In addition, we find that for such an $\mathcal{F}$-measurable set A, it holds that

$$\bigwedge_{x\in A}(K(x,y)\to f(x)) = \bigwedge_{x\in A\cap\mathrm{Supp}(K_y)} (K(x,y)\to f(x)),$$

where we used that $x\in A\setminus\mathrm{Supp}(K_y)$ leads to $K(x,y)\to f(x)=\bot\to f(x)=\top$, therefore, we can restrict the calculation of the infimum to the elements from $A\cap\mathrm{Supp}(K_y)$. Denote by $\mathcal{F}_y$ the set of all $\mathcal{F}$–measurable sets A such that $A\cap\mathrm{Core}(K_y)\neq\emptyset$. Since $G^{\to}_{(K,\nu)}$ preserves constant functions due to (C4), and using (15), (18), (19), and (22) of Theorem 1.3, we have

$$G^{\to}_{(K,\nu)}(f)(y)\leftrightarrow\neg f(x) = G^{\to}_{(K,\nu)}(f)(y)\leftrightarrow G^{\to}_{(K,\nu)}(\underline{f(x)}_X)(y) =$$

$$\bigwedge_{A\in\mathcal{F}_y}\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to f(z))\to\nu(A)\Big)\leftrightarrow$$

$$\bigwedge_{A\in\mathcal{F}_y}\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to \underline{f(x)}_X(z))\to\nu(A)\Big)\geq$$

$$\bigwedge_{A\in\mathcal{F}_y}\Big(\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to f(z))\to\nu(A)\Big)\leftrightarrow$$

$$\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to \underline{f(x)}_X(z))\Big)\to\nu(A)\Big)\geq$$

$$\bigwedge_{A\in\mathcal{F}_y}\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to f(z))\leftrightarrow\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to \underline{f(x)}_X(z))$$

$$\otimes(\nu(A)\leftrightarrow\nu(A))\Big) = \bigwedge_{A\in\mathcal{F}_y}\Big(\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to f(z))\leftrightarrow$$

$$\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\to \underline{f(x)}_X(z))\Big)\geq$$

$$\bigwedge_{A\in\mathcal{F}_y}\bigwedge_{z\in A\cap\mathrm{Supp}(K_y)}(K(z,y)\leftrightarrow K(z,y))\otimes(f(z)\leftrightarrow \underline{f(x)}_X(z))\geq$$

$$\bigwedge_{z\in X\cap\mathrm{Supp}(K_y)}(f(z)\leftrightarrow f(x))\geq\bigwedge_{z\in\mathrm{Supp}(K_y)}(f(z)\leftrightarrow f(x))\geq$$

$$\bigwedge_{(u,v)\in E_y}(f(u)\leftrightarrow f(v)) = \omega(f,E_y),$$

where we used the fact that if $z \in \mathrm{Supp}(K_y)$, then $(x, z) \in E_y$. $\square$

Similarly to the previous section, the next two statements characterize the approximation quality of the reconstructed function obtained by the respective composition of R_{DH}–lattice integral transforms. Recall that the negation $\neg$ is involutive, if $\neg(\neg a) = a$ for any $a \in L$.

Theorem 4.26. *Let K be an integral kernel, $K^{-1,d}$ be a Q^d–dual inverse of K such that $K^{-1,d}(y, x) > \bot$ if and only if $K(x, y) > \bot$ holds for any $x \in X$ and $y \in Y$, and let $f \in \mathcal{F}(X)$. Assume that (K, ν_X) satisfies (C3) and $(K^{-1,d}, \nu_Y)$ satisfies (C4), let $\omega(f)$ be defined as in Theorem 4.14, and $\neg$ is involutive. Then*

$$G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \omega(f) \tag{4.44}$$

for any $x \in X$.

Proof. As a simple consequence of the assumption on K and $K^{-1,d}$, we get $\nabla K_y(x) = \top$ for any $x \in X$ and $y \in \mathrm{Supp}(K_x^{-1,d})$. Similarly to Theorem 4.25, we use $\mathcal{G}_x$ to denote the set of all $\mathcal{G}$–measurable sets A such that $A \cap \mathrm{Core}(K_x^{-1,d}) \neq \emptyset$. Since both R_{DH}–lattice integral transforms reverse constant functions (Theorem 3.13) and the negation is involutive, according to Theorem 4.24, using (4.43) and $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1,d})$, we have

$$\begin{gathered}
G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) = \\
G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \leftrightarrow G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(\underline{f(x)}_X)(x) = \\
G^{\rightarrow}_{(K^{-1,d},\nu_Y)}(G^{\otimes}_{(K,\nu_X)}(f))(x) \leftrightarrow G^{\rightarrow}_{(K^{-1,d},\nu_Y)}(\neg \underline{f(x)}_Y)(x) \geq \\
\bigwedge_{A \in \mathcal{G}_x} \Big(\bigwedge_{y \in A \cap \mathrm{Supp}(K_x^{-1,d})} (K^{-1,d}(y, x) \to G^{\otimes}_{(K,\nu_X)}(f)(y)) \to \nu_Y(A) \Big) \leftrightarrow \\
\bigwedge_{A \in \mathcal{G}_x} \Big(\bigwedge_{y \in A \cap \mathrm{Supp}(K_x^{-1,d})} (K^{-1,d}(y, x) \to \neg \underline{f(x)}_Y(y)) \to \nu_Y(A) \Big) \geq \\
\bigwedge_{A \in \mathcal{G}_x} \bigwedge_{y \in A \cap \mathrm{Supp}(K_x^{-1,d})} (G^{\otimes}_{(K,\nu_X)}(f)(y) \leftrightarrow \neg \underline{f(x)}_Y(y)) = \\
\bigwedge_{A \in \mathcal{G}_x} \bigwedge_{y \in A \cap \mathrm{Supp}(K_x^{-1,d})} (G^{\otimes}_{(K,\nu_X)}(f)(y) \leftrightarrow \neg f(x)) \geq \bigwedge_{y \in \mathrm{Supp}(K_x^{-1,d})} \omega(f, E_y) \geq \omega(f),
\end{gathered}$$

where the verification skips the analogous steps as in the proof of Theorem 4.24 and we used

$$\begin{gathered}
G^{\rightarrow}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(\underline{f(x)}_X)(x) = G^{\rightarrow}_{(K^{-1,d},\nu_Y)}(\underline{\neg f(x)}_Y)(x) = \\
\underline{\neg(\neg f(x))}_X(x) = \neg(\neg f(x)) = f(x)
\end{gathered}$$

as the consequence of the fact that the negation is involutive. $\square$

Note that we cannot use the same assumption on the integral kernels K and $K^{-1,d}$ as in Theorem 4.14, where we assume that Q is a reflexive integral kernel. Unfortunately, the reflexivity of Q^d does not ensure that $\nabla K_y(x) = \top$ holds for all $y \in \mathrm{Supp}(K_x^{-1,d})$.

Theorem 4.27. *Let K be an integral kernel, $K^{-1,d}$ be a Q^d–dual inverse of K such that $K^{-1,d}(y,x) > \bot$ if and only if $K(x,y) > \bot$ holds for any $x \in X$ and $y \in Y$, and let $f \in \mathcal{F}(X)$. Assume that (K, ν_X) satisfies (C4) and $(K^{-1,d}, \nu_Y)$ satisfies (C3), let $\omega(f)$ be defined as in Theorem 4.14, and $\neg$ is involutive. Then*

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \omega(f) \tag{4.45}$$

for any $x \in X$.

Proof. Similarly to Theorem 4.24, we use $\mathcal{G}_x$ to denote the set of all $\mathcal{G}$–measurable sets A such that $A \subseteq \mathrm{Supp}(K_x^{-1,d})$. Let $x \in X$. Since both R_{DH}–lattice integral transforms reverse constant functions, according to Theorem 4.25, using (4.41) and the fact that $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1,d})$ (see the proof of Theorem 4.26), we have

$$\begin{gathered}
G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) = \\
G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \leftrightarrow G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(\underline{f(x)}_X)(x) = \\
G^{\otimes}_{(K^{-1,d},\nu_Y)}(G^{\rightarrow}_{(K,\nu_X)}(f))(x) \leftrightarrow G^{\otimes}_{(K^{-1,d},\nu_Y)}(\neg\underline{f(x)}_Y)(x) \geq \\
\bigwedge_{A\in\mathcal{G}_x} (\bigwedge_{y\in A} (K^{-1,d}(y,x) \otimes G^{\rightarrow}_{(K,\nu_X)}(f)(y)) \rightarrow \nu_Y(A)) \leftrightarrow \\
\bigwedge_{A\in\mathcal{G}_x} (\bigwedge_{y\in A} (K^{-1,d}(y,x) \otimes \neg\underline{f(x)}_Y(y)) \rightarrow \nu_Y(A)) \geq \\
\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A} (G^{\rightarrow}_{(K,\nu_X)}(f)(y) \leftrightarrow \neg\underline{f(x)}_Y(y)) = \\
\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A} (G^{\rightarrow}_{(K,\nu_X)}(f)(y) \leftrightarrow \neg f(x)) \geq \bigwedge_{y\in\mathrm{Supp}(K_x^{-1,d})} \omega(f, E_y) \geq \omega(f),
\end{gathered}$$

where the verification skips the analogous steps as in the proof of Theorem 4.25 and we used the fact that $G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(\underline{f(x)}_X)(x) = f(x)$, which can be shown similarly as in the proof of Theorem 4.26. □

The end of this section will be devoted to an estimation of the approximation quality of R_{DH}–lattice integral transforms for extensional functions with respect to a reflexive and Y-transitive fuzzy relation on X. We show that the results achieved in the previous section for M–lattice integral transforms also hold in a reformulated version for R_{DH}–lattice integral transforms.

First, we provide a similar result in Lemma 4.16 showing that Q^d is a reflexive and Y-transitive fuzzy relation on X.

Lemma 4.28. *Let $Y \subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K : X \times Y \to L$ given as $K(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$ is an integral kernel, and let $K^{-1,d} = K^T$ be a Q^d–dual inverse of K, i.e., Q^d is given by formula (4.11), Then Q^d is a reflexive and Y–transitive integral kernel on X such that $Q^d \leq P$. In addition, $Q^d(x,y) = P(x,y) = Q^d(y,x)$ for any $x \in X$ and $y \in Y$.*

Proof. The reflexivity of Q^d has been verified in the proof of Theorem 4.7. As a consequence of the symmetry of P, we get $P = P^T$. Then, for any $x, z \in X$ and $y \in Y$, we have

$$Q^d(x,y) \otimes Q^d(y,z) = \left(\bigvee_{u \in Y} K^{-1,d}(u,y) \otimes K(x,u) \right) \otimes \left(\bigvee_{v \in Y} K^{-1,d}(v,z) \otimes K(y,u) \right)$$
$$= \left(\bigvee_{u \in Y} P^T(u,y) \otimes P(x,u) \right) \otimes \left(\bigvee_{v \in Y} P^T(v,z) \otimes P(y,u) \right) \leq$$

$$P(x,y) \otimes P(y,z) \leq \bigvee_{y \in Y} P^T(y,z) \otimes P(x,y) = \bigvee_{w \in Y} K^{-1,d}(w,z) \otimes K(x,w) = Q^d(x,z),$$

where we used the transitivity of P. Hence, we obtain that Q^d is Y–transitive. Further, it holds that

$$P(x,y) = \bigvee_{u \in X} P(x,u) \otimes P(u,y), \quad x, y \in X.$$

Indeed, as a consequence of the transitivity of P, we have $P(x,y) \geq \bigvee_{u \in X} P(x,u) \otimes P(u,y)$ for any $x, y \in X$. Moreover,

$$\bigvee_{u \in X} P(x,u) \otimes P(u,y) \geq P(x,x) \otimes P(x,y) = \bot \otimes P(x,y) = P(x,y)$$

for any $x, y \in X$. Hence, we get

$$Q^d(x,y) = \bigvee_{u \in Y} K^{-1,d}(u,y) \otimes K(x,u) = \bigvee_{u \in Y} P(x,u) \otimes P(u,y) \leq P(x,y),$$

where we used that $P = P^T$ and $Y \subseteq X$; therefore, $Q^d \leq P$. Finally, for $x \in X$ and $y \in Y$, we find that

$$Q^d(x,y) = \bigvee_{u \in Y} K^{-1,d}(u,y) \otimes K(x,u) \geq P^T(y,y) \otimes P(x,y) = \top \otimes P(x,y) = P(x,y)$$

and

$$Q^d(y,x) = \bigvee_{u \in Y} K^{-1,d}(u,x) \otimes K(y,u) \geq P^T(y,x) \otimes P(y,y) = P(x,y) \otimes \top = P(x,y)$$

which together with $Q^d \leq P$ leads $Q^d(x,y) = P(x,y) = Q^d(y,x)$ for any $x \in X$ and $y \in Y$. □

The following lemma shows that the extensionality of f is preserved even for its negation $\neg f$.

Lemma 4.29. *Let Q^d be a reflexive and Y–transitive fuzzy relation on X, and let f be extensional with respect to Q^d. Then $\neg f$ is also extensional with respect to Q^d.*

Proof. Let $x, y \in X$. Since f is extensional with respect to Q^d, we have $f(x) \otimes Q^d(x,y) \leq f(y)$ and also $Q^d(x,y) \otimes f(y) \leq f(x)$. Hence, and using Theorem 1.1(xiii), we get

$$Q^d(x,y) \leq f(x) \to f(y) \leq \neg f(y) \to \neg f(x),$$

which implies $Q^d(x,y) \otimes \neg f(y) \leq \neg f(x)$. Similarly, we get $\neg f(x) \otimes Q^d(x,y) \leq \neg f(y)$. □

Theorem 4.30. *Let $Y \subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K : X \times Y \to L$ given as $K(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$ is an integral kernel, let $K^{-1,d} = K^T$ be a Q^d–dual inverse of K, and let $\neg$ be involutive. If f is extensional with respect to Q^d and $(K^{-1,d}, \nu_Y)$ satisfies (C4), then*

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \neg \int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X \tag{4.46}$$

for any $x \in X$.

Proof. Let f be extensional function with respect to Q^d, and let $x \in X$. Note that if $a \leq b$, then $\neg b \leq \neg a$, which is a consequence of Theorem 1.1(xiii). Using the extensionality of f and (i) and (iii) of Theorem 2.12, we have

$$\int^{\to}_{\mathrm{DH}\, X} Q^d(y,x) \otimes f(y)\, d\nu_X \leq \int^{\to}_{\mathrm{DH}\, X} Q^d(y,x) \otimes (Q^d(y,x) \otimes \underline{f(x)}_X(y))\, d\nu_X \leq$$
$$\int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x) \otimes \underline{f(x)}_X(y)\, d\nu_X \leq f(x) \to \int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X$$

By Theorem 4.21 and using Theorem 1.1(xii), we get

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \geq \neg \int^{\to}_{\mathrm{DH}\, X} Q^d(y,x) \otimes f(y)\, d\nu_X \geq$$
$$\neg(f(x) \to \int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X) \geq f(x) \otimes \neg \int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X.$$

Due to the adjointness property, we get

$$f(x) \to G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \geq \neg \int^{\to}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X. \tag{4.47}$$

For $y \in Y$, we have

$$G^{\otimes}_{(K,\nu_X)}(f)(y) = \int^{\to}_{\mathrm{DH}\, X} K(x,y) \otimes f(x)\, d\nu_X = \int^{\to}_{\mathrm{DH}\, X} P(x,y) \otimes f(x)\, d\nu_X \geq$$
$$\int^{\to}_{\mathrm{DH}\, X} \underline{f(y)}_X = \neg f(y),$$

where we used that $P(x,y) \otimes f(x) = Q^d(x,y) \otimes f(x) \leq f(y)$. From Lemma 4.29, we have $\neg f(x) \otimes Q^d(x,y) \leq \neg f(y)$ then we obtain $\neg f(x) \leq Q^d(x,y) \to \neg f(y)$ using

adjunction property. Hence,

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) = \int^{\to}_{\mathrm{DH}\,Y} \left(K^{-1,d}(y,x) \to G^{\otimes}_{(K,\nu_X)}(f)(y)\right) d\nu_Y \leq$$
$$\int^{\to}_{\mathrm{DH}\,Y} \left(K^{-1,d}(y,x) \to \neg f(y)\right) d\nu_Y = \int^{\to}_{\mathrm{DH}\,Y} \left(Q^T(y,x) \to \neg f(y)\right) d\nu_Y \leq$$
$$\int^{\to}_{\mathrm{DH}\,Y} \underline{\neg f(x)}_Y \, d\nu_Y = \neg(\neg f(x)) = f(x),$$

where we used the involutive property of $\neg$, the monotonically non-increasing of the $\int^{\to}_{\mathrm{DH}\,Y}$ and $Q^d(x,y) = (Q^d)^T(y,z) = K^{-1,d}(y,x)$ from Lemma 4.28. Therefore, using the adjunction property we have

$$G^{\to}_{(K^{-1,d},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \to f(x) = \top. \tag{4.48}$$

By combining (4.47) and (4.48), we get the desired inequality in (4.46). □

Theorem 4.31. *Let $Y \subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K : X \times Y \to L$ given as $K(x,y) = P(x,y)$ for any $x \in X$ and $y \in Y$ is an integral kernel, let $K^{-1,d} = K^T$ be a Q^d–dual inverse of K, and let $\neg$ be involutive. If f is extensional with respect to Q^d and (K,ν_X) satisfies (C4), then*

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\to}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \neg \int^{\to}_{\mathrm{DH}\,Y} (Q^d)^2(x,y)\, d\nu_Y \tag{4.49}$$

for any $x \in X$.

Proof. Let f be an extensional function with respect to Q^d. By Lemma 4.28, we have $K(x,y) = P(x,y) = Q^d(x,y)$ and so $K^{-1,d}(y,x) = (Q^d)^T(y,x)$ for any $x \in X$ and $y \in Y$. First, we show that under the assumptions of the theorem, it holds that

$$G^{\to}_{(K,\nu_X)}(f)(y) = \neg f(y). \tag{4.50}$$

For any $x, y \in X$, $Q^d(x,y) \otimes f(y) \leq f(x)$ implies $f(y) \leq Q^d(x,y) \to f(x)$ using the adjointness property. Hence, using Theorem 2.12(ii), we have

$$G^{\to}_{(K,\nu_X)}(f)(y) = \int^{\to}_{\mathrm{DH}\,X} (Q^d(y,x) \to f(x))\, d\nu_X \leq \int^{\to}_{\mathrm{DH}\,X} \underline{f(y)}_X \, d\nu_X = \neg f(y).$$

On the other hand, it is easy to see that $\mathrm{Core}((Q^d_y)^2) = \mathrm{Core}(K_y)$ for any $y \in Y$, where $(Q^d_y)^2(x) = Q^d_y(x) \otimes Q^d_y(x)$, $x \in X$. Indeed, due to Lemma 4.28, for any $y \in Y$, we have $K_y(x) = K(x,y) = P(x,y) = Q^d(x,y) = Q^d_y(x)$ for any $x \in X$. Hence, if $x \in \mathrm{Core}((Q^d_y)^2)$, then $(Q^d_y)^2(x) = \top$, therefore, $Q^d_y(x) = \top$, which implies $K_y(x) = \top$ and $x \in \mathrm{Core}(K_y)$. If $x \in \mathrm{Core}(K_y)$, then $K_y(x) = Q^d_y(x) = (Q^d_y)^2(x)\top$, which implies $x \in \mathrm{Core}((Q^d_y)^2)$. Now, if $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$, then $A \subseteq X \setminus \mathrm{Core}((Q^d_y)^2)$, which implies $\nu_X(A) = \top$, since (K,ν_X) satisfies (C4). As a consequence, we obtain that $((Q^d)^2,\nu_X)$ also satisfies (C4), and $G^{\to}_{((Q^d)^2,\nu_X)}$ preserves constant functions. Then, for any $y \in Y$, we have

$$G^{\to}_{(K,\nu_X)}(f)(y) = \int^{\to}_{\mathrm{DH}\,X} K(x,y) \to f(x)\, d\nu_X = \int^{\to}_{\mathrm{DH}\,X} Q^d(x,y) \to f(x)\, d\nu_X \geq$$
$$\int^{\to}_{\mathrm{DH}\,X} (Q^d)^2(x,y) \to (f(x) \otimes Q^d(x,y))\, d\nu_X \geq \int^{\to}_{\mathrm{DH}\,X} (Q^d)^2(x,y) \to \underline{f(y)}_X(x)\, d\nu_X =$$
$$G^{\to}_{((Q^d)^2,\nu_X)}(\underline{f(y)}_X)(y) = \neg f(y),$$

where we used Theorem 1.1(x) and the extensionality of f with respect to Q^d. Using the equality in (4.50), for any $x \in X$, we get

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) = \int^{\rightarrow}_{\mathrm{DH}\,Y} K^{-1,d}(y,x) \otimes G^{\rightarrow}_{(K,\nu_X)}(f)(y)\, d\nu_Y =$$
$$\int^{\rightarrow}_{\mathrm{DH}\,Y} (Q^d)^T(y,x) \otimes \neg f(y)\, d\nu_Y = \int^{\rightarrow}_{\mathrm{DH}\,Y} Q^d(x,y) \otimes \neg f(y)\, d\nu_Y \geq$$
$$\int^{\rightarrow}_{\mathrm{DH}\,Y} \underline{\neg f(x)}_Y(y)\, d\nu_Y = \neg(\neg f(x)) = f(x),$$

where we used Theorem 2.12(ii), and the extensionality of $\neg f$ with respect to Q^d according to Lemma 4.29. As a consequence of this inequality, we get

$$f(x) \rightarrow G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) = \top. \tag{4.51}$$

Since $\neg f(x) \otimes Q^d(x,y) = \neg f(x) \otimes (Q^d)^T(y,x) \leq \neg f(y)$ for any $x \in X$ and $y \in Y$, we get

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) = \int^{\rightarrow}_{\mathrm{DH}\,Y} (Q^d)^T(y,x) \otimes G^{\rightarrow}_{(K,\nu_X)}(f)(y)\, d\nu_Y =$$
$$\int^{\rightarrow}_{\mathrm{DH}\,Y} (Q^d)^T(y,x) \otimes \neg f(y)\, d\nu_Y \leq \int^{\rightarrow}_{\mathrm{DH}\,Y} (Q^d)^T(y,x) \otimes ((Q^d)^T(y,x) \otimes \underline{\neg f(x)}_Y(y))\, d\nu_Y$$
$$= \int^{\rightarrow}_{\mathrm{DH}\,Y} ((Q^d)^T)^2(y,x) \otimes \underline{\neg f(x)}_Y(y))\, d\nu_Y \leq \neg f(x) \rightarrow \int^{\rightarrow}_{\mathrm{DH}\,Y} ((Q^d)^T)^2(y,x)\, d\nu_Y,$$

where we used Theorem 2.12(iii) and (4.50). Using the adjointness property, we get

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \otimes \neg f(x) \leq \int^{\rightarrow}_{\mathrm{DH}\,Y} ((Q^d)^T)^2(y,x)\, d\nu_Y.$$

Due to Theorem 1.1(xii) and $a \leq b$ implies $\neg a \geq \neg b$, we find that

$$\neg\left(G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \otimes \neg f(x)\right) = G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \rightarrow f(x),$$

which implies

$$G^{\otimes}_{(K^{-1,d},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \rightarrow f(x) \geq \neg \int^{\rightarrow}_{\mathrm{DH}\,Y} (Q^d)^2(x,y)\, d\nu_Y, \tag{4.52}$$

where we used the fact that $((Q^d)^T)^2(y,x) = (Q^d)^2(x,y)$. By combining (4.51) and (4.52), we get the desired inequality in (4.49). □

We know that the composition of R_{DH}–lattice integral transforms preserves constant functions under the satisfaction of conditions (C3) and (C4). The following corollaries show that these conditions ensure the preservation of extensional functions with respect to Q^d, where Q^d is determined from a similarity relation on X.

Corollary 4.32. *Let the assumption of Theorem 4.30 be satisfied, and let (K,ν_X) satisfy (C3). Then*

$$G^{\rightarrow}_{(K^{-1},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f) = f \tag{4.53}$$

for any extensional function f on X with respect to Q^d.

Proof. From Lemma 4.28, we know that Q^d is Y–transitive and $K(x,y) = P(x,y) = Q^d(x,y) = Q^d(y,x)$ for any $x \in X$ and $y \in Y$. Let $x \in X$. Then there is $z \in Y$ such that $K(x,z) = P(x,z) = \top$, since K is an integral kernel. By the Y-transitivity of Q^d, for any $y \in X$, we get $Q^d(y,x) \geq Q^d(y,z) \otimes Q^d(z,x) = P(y,z) \otimes P(z,x) = P(y,z) \otimes P(x,z) = P(y,z) = K(y,z)$, where we used the fact that P is symmetric. Hence, we get $\mathrm{Core}(K_z) \subseteq \mathrm{Core}(Q^d_x)$. Since (K, ν_X) satisfies (C3), we get that (Q^d, ν_X) also satisfies (C3). Hence, for any $x \in X$ there is $A_x \in \mathcal{F}$ such that $A_x \subseteq \mathrm{Core}(Q^d_x)$ and $\nu(A_x) = \bot$. Obviously, $(Q^d)^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$. Indeed, for $y \in A_x$, we have $\top = 1_{A_x}(y) = Q^d(y,x) = (Q^d)^2(y,x)$. If $y \notin A_x$, then trivially $\bot = 1_{A_x}(y) \leq (Q^d)^2(y,x)$. By Theorem 4.30, we simply find

$$G^{\rightarrow}_{(K^{-1},\nu_Y)} \circ G^{\otimes}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \neg \int^{\rightarrow}_{\mathrm{DH}\, X} (Q^d)^2(y,x)\, d\nu_X \geq$$
$$\neg \int^{\rightarrow}_{\mathrm{DH}\, X} 1_{A_x}(y)\, d\nu_X = \neg \nu_X(A_x) = \top,$$

where we used Theorem 2.12(v). Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

Corollary 4.33. *Let the assumption of Theorem 4.31 be satisfied, and let (K^{-1}, ν_Y) satisfy (C3). Then*

$$G^{\otimes}_{(K^{-1},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f) = f \tag{4.54}$$

for any extensional function f on X with respect to Q^d.

Proof. Using Lemma 4.28, we have $(Q^d)^T(y,x) = P^T(y,x) = K^{-1}(y,x)$ for any $x \in X$ and $y \in Y$. Since (K^{-1}, ν_Y) satisfies (C3), for any $x \in X$, there is $A_x \in \mathcal{G}$ such that $A_x \subseteq \mathrm{Core}(K^{-1}_x)$ and $\nu_Y(A_x) = \bot$. Obviously, we have $(K^{-1})^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$, which can be verified analogously as in the proof of Corollary 4.32. In addition, it holds $(K^{-1})^2 = (K^2)^T$. By Theorem 4.31, we find that

$$G^{\otimes}_{(K^{-1},\nu_Y)} \circ G^{\rightarrow}_{(K,\nu_X)}(f)(x) \leftrightarrow f(x) \geq \neg \int^{\rightarrow}_{\mathrm{DH}\, Y} (Q^d)^2(x,y)\, d\nu_Y =$$
$$\neg \int^{\rightarrow}_{\mathrm{DH}\, Y} (K^{-1})^2(y,x)\, d\nu_Y \geq \neg \int^{\rightarrow}_{\mathrm{DH}\, Y} 1_{A_x}(y)\, d\nu_Y = \neg \nu_X(A_x) = \top,$$

where we used Theorem 2.12(v). Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

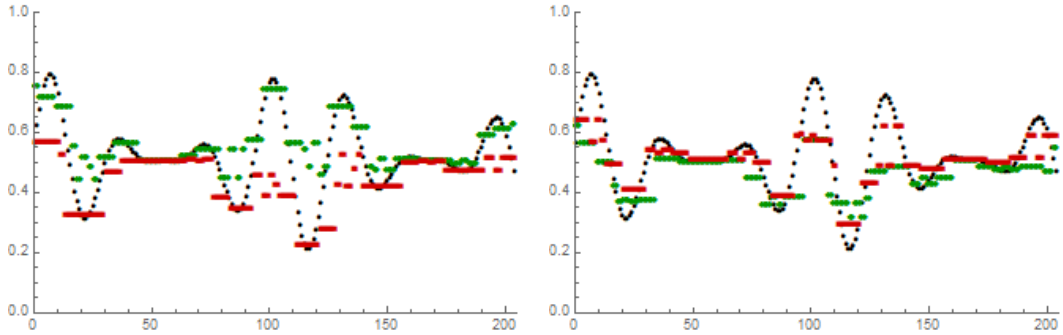
4.4.3 Illustration on signal reconstruction

In this part, we present the reconstruction of signals using the composition of R_{DH}–lattice integral transforms. We use the setting of Example 3.6 and reconstruct the original signal given by formula (3.9) from the transformed functions presented in the mentioned example. Recall that all the integral kernels $K : X \times Y \to L$ together with the complementary fuzzy measures ν_X (originally denoted as ν) on the measurable space $\langle X, \mathcal{P}(X) \rangle$ are introduced in such a way that (K, ν_X) satisfies condition (C3) for $\star = \otimes$ and $(K, \nu^{c,N}_X)$ satisfies condition (C4) for $\star = \rightarrow$

due to Theorem 3.14. Therefore, the R_{DH}–lattice integral transforms from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ reverse constant functions. For the reconstruction, we similarly introduce R_{DH}–lattice integral transforms from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$, namely, an Q^d–dual inverse integral kernel $K^{-1,d} : Y \times X \to L$ is given by $K^{-1,d} = K^T$ and the related complementary fuzzy measure ν_Y on the measurable space $\langle Y, \mathcal{P}(Y)\rangle$ is defined such that $(K^{-1,d}, \nu_Y)$ satisfies (C3) for $\star = \otimes$ and consequently $(K^{-1,d}, \nu_Y^{c,N})$ satisfies (C4) for $\star = \to$ due to Theorem 3.14. Thus, both R_{DH}–lattice integral transforms in a particular composition always reverse constant functions. Since $N = N_{res}$ is the negation in the Łukasiewicz algebra, which is involutive, the composition of R_{DH}–lattice integral transforms preserves constant functions as can be also seen from Theorems 4.26 and 4.27. In the following, we will consider cases a) and b) studied in Example 3.6.

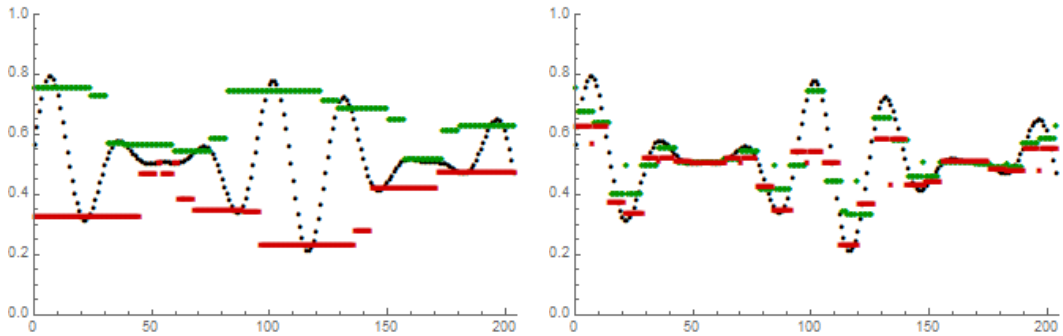
Case a) We consider the integral kernel $K : X \times Y \to [0,1]$ and two associated complementary fuzzy measures $\nu_{X1} = \mu_{2,6}^{5,N}$ and $\nu_{X2} = \mu_{7,12}^{5,N}$ on $\langle X, \mathcal{P}(X)\rangle$ introduced in case a) of Example 3.6 that specify the $(K, \nu_{Xi}, \otimes)$–R_{DH}–lattice integral transform and the $(K, \nu_{Xi}^{c,N}, \to)$–R_{DH}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ for $i = 1, 2$. Further, we consider the complementary fuzzy measure $\nu_Y = \mu_{1,2}^{5,N}$ on $\langle Y, \mathcal{P}(Y)\rangle$, where $\mu_{1,2}^{5}$ is given in Subsection 4.3.3, to introduce the $(K^{-1,d}, \nu_Y, \otimes)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform and $(K^{-1,d}, \nu_Y^{c,N}, \to)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. In Figure 4.8(a), we show the reconstruction of the original signal using the composition of the $(K, \nu_{X1}, \otimes)$–R_{DH}–lattice integral transform and the $(K^{-1,d}, \nu_Y^{c,N}, \to)$ –R_{DH}–lattice integral transform described by green diamonds ($\mathrm{R}_{\mathrm{DH}}^{\to} \circ \mathrm{R}_{\mathrm{DH}}^{\otimes}$ for short) and the analogous reconstruction using the composition of the $(K, \nu_{X1}^{c,N}, \to)$–R_{DH}–lattice integral transform and the $(K^{-1,d}, \nu_Y, \otimes)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform described by red squares ($\mathrm{R}_{\mathrm{DH}}^{\otimes} \circ \mathrm{R}_{\mathrm{DH}}^{\to}$ for short) together with the original function. The reconstructed signals for the complementary fuzzy measure ν_{X2} are displayed in Figure 4.8(b). Comparing the signals reconstructed here with the reconstructed signals in Figure 4.2, we see the similar effect of the setting of complementary fuzzy measures, namely, using ν_{X2} we get a suppression of higher frequencies as using the fuzzy measure μ_{X2}, whereas, ν_{X1} leads to a near lower and upper approximation of the original signal depending on the type of composition. In Figure 4.8(b), we can also see that the output signal of $\mathrm{R}_{\mathrm{DH}}^{\to} \circ \mathrm{R}_{\mathrm{DH}}^{\otimes}$ (green diamonds) is below the output signal of $\mathrm{R}_{\mathrm{DH}}^{\otimes} \circ \mathrm{R}_{\mathrm{DH}}^{\to}$ (red squares), confirming that the inequality similar to (4.37) between the types of the composition of R_{DH}–lattice integral transforms does not hold in general.

Case b) We consider two integral kernels $K_1, K_2 : X \times Y \to [0,1]$ and the complementary fuzzy measures $\nu_X = \mu_{3,6}^{5,N}$ on $\langle X, \mathcal{P}(X)\rangle$ introduced in case case b) of Example 3.6 that specify the $(K_i, \nu_X, \otimes)$–R_{DH}–lattice integral transform and the $(K_i, \nu_X^{c,N}, \to)$–R_{DH}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ for $i = 1, 2$. Further, we consider the same complementary fuzzy measure $\nu_Y = \mu_{1,2}^{5,N}$ on $\langle Y, \mathcal{P}(Y)\rangle$ as above in case a) to introduce the $(K_i^{-1,d}, \nu_Y, \otimes)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform and the $(K_i^{-1,d}, \nu_Y^{c,N}, \to)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ for $i = 1, 2$. In Figure 4.9, similarly to case a), we illustrate reconstructed signals for the compositions of R_{DH}–lattice integral transforms with different integral kernels K_1 and



(a) ν_{X1} (b) ν_{X2}

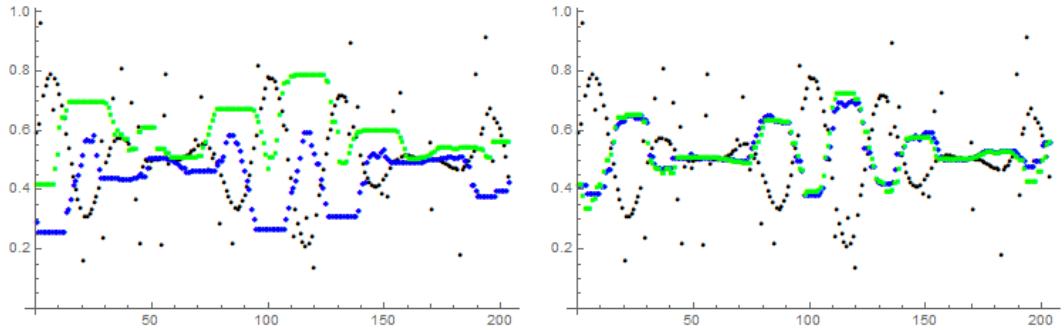
Figure 4.8: Original function f (black) and its approximation using $\mathrm{R}^{\rightarrow}_{\mathrm{DH}} \circ \mathrm{R}^{\otimes}_{\mathrm{DH}}$ (green diamonds) and $\mathrm{R}^{\otimes}_{\mathrm{DH}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DH}}$ (red squares) for a fixed integral kernel K and two different complementary fuzzy measures ν_{X1} and ν_{X2}.



(a) K_1 (b) K_2

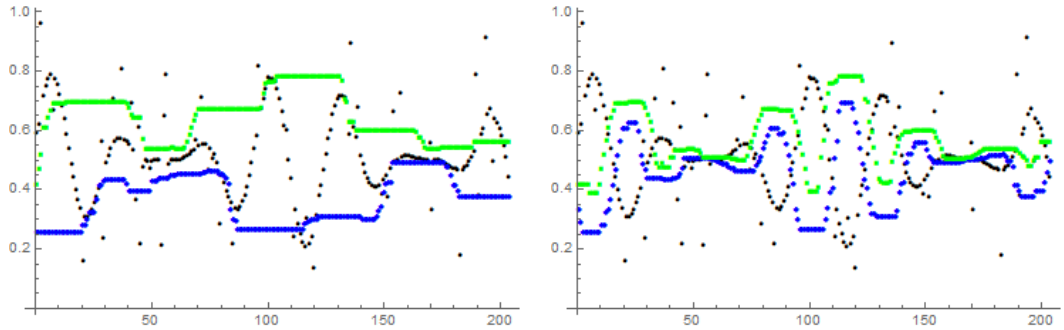
Figure 4.9: Original function f (black) and its approximation using $\mathrm{R}^{\rightarrow}_{\mathrm{DH}} \circ \mathrm{R}^{\otimes}_{\mathrm{DH}}$ (green diamonds) and $\mathrm{R}^{\otimes}_{\mathrm{DH}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DH}}$ (red squares) for a fixed complementary fuzzy measure ν_X and two different integral kernels K_1 and K_2.

K_2. As we stated in case b) of Example 3.6, $K_{2y} \subseteq K_{1y}$ for any $y \in Y$, which again leads to a better approximation of the original signal for K_2, shown in Figure 4.9(b) than for K_1, shown in Figure 4.9(a). Thus, the use of a smaller kernel (with respect to the ordering of fuzzy sets) results in a better approximation of the original function in both types of lattice integral transforms, which is the common property with the standard (real-valued) fuzzy transform (see, [36]). Note that the results of $\mathrm{R}^{\star}_{\mathrm{DH}}$–lattice integral transforms from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ given in Example 3.6 are negative, i.e., in the reverse ordering (it can be also seen as a negation of the results for the M–lattice integral transform, as discussed in Example 3.6), but the results of the reconstruction of function from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ given by the $\mathrm{R}^{\star}_{\mathrm{DH}}$–lattice integral transform are transformed back to positive. Thus, to get a reasonable reconstruction, we must of course assume that the negation is involutive, which is also an important assumption in the approximation theorems presented in the previous section.



(a) ν_{X1} (b) ν_{X2}

Figure 4.10: Filtering random noise using $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT (light blue), $\mathrm{R}^{\rightarrow}_{\mathrm{DH}}$–LIT (light green) for a fixed integral kernel K and two different complementary fuzzy measures ν_{X1} and ν_{X2}.



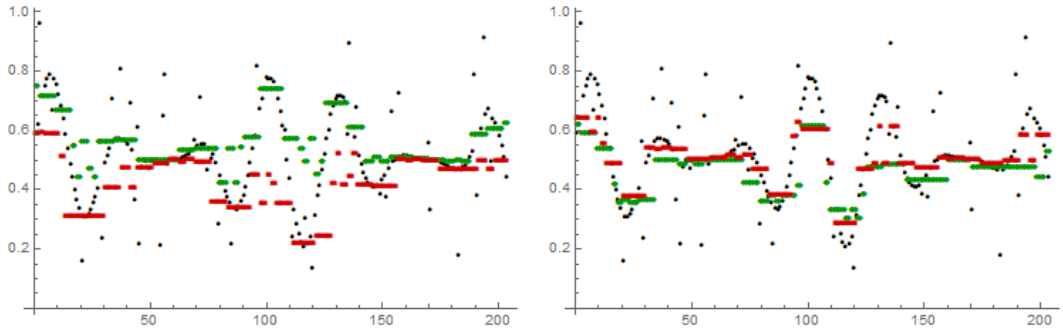
(a) K_1 (b) K_2

Figure 4.11: Filtering random noise using $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT (light blue), $\mathrm{R}^{\rightarrow}_{\mathrm{DH}}$–LIT (light green) for a fixed complementary fuzzy measure ν_X and two different integral kernels K_1 and K_2.

4.4.4 Filtering of random noise

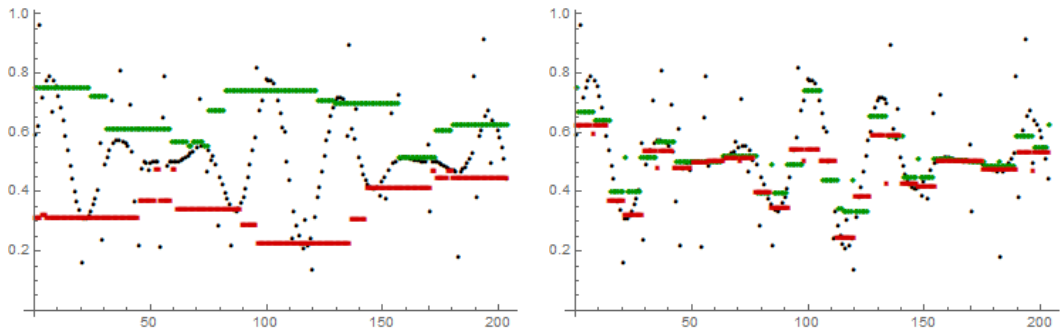
In this part, we will show that R_{DH}–lattice integral transforms can be used to filter out random noise in signal processing, but with a negative output in case of the filter based on a single transform. A further demonstration will be shown in Chapter 6, where R_{DH}–lattice integral transforms are used to filter out salt-and-pepper noise in images. For the illustration, we again use the function f given by the formula in (3.9), to which we add 30% random noise determined by a uniform distribution. Similarly to Subsection 4.3.4, we present filters based on a single R_{DH}–lattice integral transform and their compositions. In both applications, we consider the same complementary fuzzy measures ν_X, ν_{X1} and ν_{X2} on $\langle X, \mathcal{P}(X)\rangle$ and ν_Y on $\langle Y, \mathcal{P}(Y)\rangle$ specified in the previous subsection.

Filter based on single R_{DH}–lattice integral transform Recall that we can control how random noise is removed by adjusting the integral kernel and in particular the complementary fuzzy measure. For demonstration, we assume that



(a) ν_{X1} (b) ν_{X2}

Figure 4.12: Filtering random noise using composition of $\mathrm{R}^{\rightarrow}_{\mathrm{DH}} \circ \mathrm{R}^{\otimes}_{\mathrm{DH}}$ (green diamonds), and $\mathrm{R}^{\otimes}_{\mathrm{DH}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DH}}$ (red squares) for a fixed integral kernel K and two different complementary fuzzy measures ν_{X1} and ν_{X2}.



(a) K_1 (b) K_2

Figure 4.13: Filtering random noise using composition of $\mathrm{R}^{\rightarrow}_{\mathrm{DH}} \circ \mathrm{R}^{\otimes}_{\mathrm{DH}}$ (green diamonds), and $\mathrm{R}^{\otimes}_{\mathrm{DH}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DH}}$ (red squares) for a fixed complementary fuzzy measure ν_X and two different integral kernels K_1 and K_2.

$X = Y = \{1, 2, \ldots, 204\}$ and the integral kernels $K, K_1, K_2 : X \times X \to L$ are defined by the same formulas as in Example 3.6 only Y is replaced by X. Again, we distinguish two cases, namely, case a) the fixed integral kernel K and two complementary fuzzy measures ν_{X1} and ν_{X2}, case b) the fixed complementary fuzzy measure ν_X and two integral kernels K_1 and K_2.

The results of filtering random noise for case a) are shown in Figure 4.10. Although, the output signals are negative, we can see that the random noise is filtered out similarly as in the case of the M–lattice integral transform. The results for case b) are presented in Figure 4.11. Note that this type of filter does not appear to be useful in practice, only when we filter out the noise and simultaneously convert the signal or better image from positive to negative.

Filter based on composition of R_{DH}–lattice integral transforms In this part, we consider the same setting of the integral kernels K, K_1 and K_2 and the complementary fuzzy measures ν_X, ν_{X1} and ν_{X2} as in Subsection 4.4.3. The reconstructed signals are shown in Figures 4.12 and 4.13. Since the R_{DH}–lattice integral

transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ transforms negative signals to positive ones, the output signals approximate the original signal. We can see that all the reconstructed signals are noise-free and are very similar to the reconstructed signals in the previous subsection, which are derived from the original (noise-free) signal. However, to obtain a reasonable approximation, we must assume that the negation used is involutive, as discussed in case (b) of the previous subsection.

4.5 Approximation of functions based on $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms

In this section, we complete our investigation of the estimation of an original lattice-valued function using a combination of two types of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms that are introduced in Subsection 3.5. Throughout this section, we assume that $\langle X, \mathcal{F}, \mu_X \rangle$ is a fuzzy measure space and $\mu_X^{c,N}$ denotes the N–conjugate fuzzy measure to μ_X and $\langle Y, \mathcal{G}, \mu_Y \rangle$ is a fuzzy measure space and $\mu_Y^{c,N}$ denotes the N–conjugate fuzzy measure to μ_Y. Further, we assume that $K : X \times Y \to L$ is an integral kernel and $K^{-1} : Y \times X \to L$ is an Q–inverse integral kernel of K, where Q is the integral kernel that satisfies (4.6).

4.5.1 Upper and lower approximation of functions

In this part, we show similar results presented in the previous two sections, which in a sense generalize the approximation of the original functions from below and above by a composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms. Comparing the properties of the $\otimes$–fuzzy integral and the $\to_{\mathrm{DPR}}$–fuzzy integral, we see that they are identical except for (v) in Theorem 2.7 and Theorem 2.16, where the latter assumes an involutive negation based on the residuum. Since the proofs of all the statements on the upper and lower estimation of functions using M–lattice integral transforms in Subsection 4.3.1 are based only on the properties (i)-(iv) of Theorem 2.7, the same statements for the $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms can be proved quite analogously. We therefore present only analogous statements and omit their proofs.

The following theorem shows a generalization of the approximation from above of the original function using the composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms.

Theorem 4.34. *Let $H^{\otimes}_{(K,\mu_X)}$ be an R_{DPR}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and $H^{\to}_{(K^{-1},\mu_Y)}$ be an R_{DPR}–lattice integral transform from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$H^{\to}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)} \geq H^{\otimes}_{(Q,\mu_X)}. \tag{4.55}$$

Similarly to the upper approximation of the original functions by the composition of M–lattice integral transforms, the composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms provides the upper approximation of the smoothed original function given by the $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–lattice integral transform on X with respect to the integral kernel Q derived from the kernel K and its inverse K^{-1}.

The approximation from below of the original function using the composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms is presented in the following theorem.

Theorem 4.35. *Let $H^{\rightarrow}_{(K,\mu_X)}$ be an R_{DPR}–lattice integral transform from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and $H^{\otimes}_{(K^{-1},\mu_Y)}$ be a R_{DPR}–lattice integral transform $\mathcal{F}(Y)$ to $\mathcal{F}(X)$. Then*

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)} \leq H^{\rightarrow}_{(Q,\mu_X)}. \tag{4.56}$$

Analogously to Corollary 4.10, we show another approximation of the original function using the composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms in case that Q is a reflexive fuzzy relation on X.

Corollary 4.36. *For any $f \in \mathcal{F}(X)$ and $y \in X$, it holds that*

(i) $H^{\rightarrow}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)} \geq \int^{\rightarrow}_{\mathrm{DPR}\,X} f \otimes 1_{\mathrm{Core}(Q_y)}\, d\mu_X$,

(ii) $H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)} \leq \int^{\rightarrow}_{\mathrm{DPR}\,X} 1_{\mathrm{Core}(Q_y)} \rightarrow f\, d\mu_X$.

The last corollary is the analogous statement given in Corollary 4.11 and shows that under certain assumptions, compositions of integral transformations of the $\mathrm{R}_{\mathrm{DPR}}$–lattice can approximate the original function from below and above.

Corollary 4.37. *Let K be an integral kernel, and let K^{-1} be the Q–inverse of K such that Q is a reflexive integral kernel. Then, for any $f \in \mathcal{F}(X)$, it holds that*

$$H^{\otimes}_{(K^{-1},\mu^{\top}_Y)} \circ H^{\rightarrow}_{(K,\mu^{\perp}_X)}(f) \leq f \leq H^{\rightarrow}_{(K^{-1},\mu^{\perp}_Y)} \circ H^{\otimes}_{(K,\mu^{\top}_X)}(f). \tag{4.57}$$

4.5.2 Estimation of approximation quality

In this section we show that all the statements about the quality of the approximation that hold for M–lattice integral transforms also hold for the $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms. The first theorem shows an estimate of the approximation quality for the $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–lattice integral transform.

Theorem 4.38. *Let $(K, \mu^{c,N})$ satisfy (C2), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Define the equivalence E_y on X as in Theorem 4.12. Then*

$$H^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow f(x) \geq \omega(f, E_y), \tag{4.58}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. From Theorem 4.12 we know that E_y is well-defined. Since $(K, \mu^{c,N})$ satisfies (C2), we can write

$$H^{\otimes}_{(K,\mu)}(f)(y) = \bigwedge_{\substack{A \in \mathcal{F} \\ A \cap \mathrm{Core}(K_y) \neq \emptyset}} (\mu^{c,N}(A) \rightarrow \bigvee_{x \in A \cap \mathrm{Supp}(K_y)} (K(x,y) \otimes f(x))). \tag{4.59}$$

Indeed, due to (C2), we know that for any $A \in \mathcal{F}$ such $A \subseteq X \setminus \mathrm{Core}(K_y)$ it holds that $\mu^{c,N}(A) = \perp$. Hence, we find that

$$H^{\otimes}_{(K,\mu)}(f)(y) = \bigwedge_{\substack{A \in \mathcal{F} \\ A \cap \mathrm{Core}(K_y) \neq \emptyset}} (\mu^{c,N}(A) \rightarrow \bigvee_{x \in A} (K(x,y) \otimes f(x))).$$

Moreover, if $x \in A \setminus \mathrm{Supp}(K_y)$, then $K(x,y) \otimes f(x) = \bot \otimes f(x) = \bot$, therefore, we obtain

$$\bigvee_{x \in A} (K(x,y) \otimes f(x)) = \bigvee_{x \in A \cap \mathrm{Supp}(K_y)} (K(x,y) \otimes f(x)),$$

which implies the desired modification of the definition of $H^{\otimes}_{(K,\mu)}(f)$. Denote $\mathcal{F}_y$ the set of all $\mathcal{F}$–measurable sets A such that $A \cap \mathrm{Core}(K_y) \neq \emptyset$. Since $H^{\otimes}_{(K,\mu)}$ preserves constant functions due to (C2), using (18), (19), (22) and (23) of Theorem 1.3, we have

$$\begin{aligned}
H^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow f(x) &= H^{\otimes}_{(K,\mu)}(f)(y) \leftrightarrow H^{\otimes}_{(K,\mu)}(\underline{f(x)}_X)(y) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\mu^{c,N}(A) \to \bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes f(z))) \\
&\leftrightarrow \bigwedge_{A \in \mathcal{F}_y} (\mu^{c,N}(A) \to \bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes \underline{f(x)}_X(z))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\mu^{c,N}(A) \to \bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes f(z))) \\
&\leftrightarrow (\mu^{c,N}(A) \to \bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes \underline{f(x)}_X(z)))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\mu^{c,N}(A) \leftrightarrow \mu^{c,N}(A)) \otimes (\bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes f(z)) \leftrightarrow \\
&\bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes \underline{f(x)}_X(z))) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\bigvee_{z \in A \cap \mathrm{Supp} K_y} (K(z,y) \otimes f(z)) \leftrightarrow \bigvee_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \otimes \underline{f(x)}_X(z))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} \bigwedge_{z \in A \cap \mathrm{Supp}(K_y)} (K(z,y) \leftrightarrow K(z,y)) \otimes (f(z) \leftrightarrow \underline{f(x)}_X(z)) \geq \\
&\bigwedge_{z \in \mathrm{Supp}(K_y)} (f(z) \leftrightarrow f(x)) \geq \bigwedge_{(u,v) \in E_y} (f(u) \leftrightarrow f(v)) = \omega(f, E_y),
\end{aligned}$$

where we used the fact that if $z \in \mathrm{Supp}(K_y)$, then $(x,z) \in E_y$. □

The next theorem provides an estimate of the approximation quality for the $\mathrm{R}^{\to}_{\mathrm{DPR}}$–lattice integral transform.

Theorem 4.39. *Let $(K, \mu^{c,N})$ satisfy (C1), and let $f \in \mathcal{F}(X)$ and $y \in Y$. Define the equivalence E_y on X as in Theorem 4.12. Then*

$$H^{\to}_{(K,\mu)}(f)(y) \leftrightarrow f(x) \geq \omega(f, E_y), \tag{4.60}$$

for any $x \in X$ such that $\nabla K_y(x) = \top$.

Proof. From Theorem 4.12 we know that E_y is well-defined. Further, we can write

$$H^{\to}_{(K,\mu)}(f)(y) = \bigwedge_{\substack{A \in \mathcal{F} \\ A \subseteq \mathrm{Supp}(K_y)}} (\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \to f(x))). \tag{4.61}$$

Note that due to (C1), there is $A \in \mathcal{F}$ such that $A \subseteq \mathrm{Supp}(K_y)$ and $\mu^{c,N}(A) = \top$, i.e., $A \neq \emptyset$. Moreover, if $A \in \mathcal{F}$ is such that $A \not\subseteq \mathrm{Supp}(K_y)$, then

$$\mu^{c,N}(A) \to \bigvee_{x \in A} (K(x,y) \to f(x)) = \mu^{c,N}(A) \to \top = \top,$$

since $K(x,y) \to f(x) = \bot \to f(x) = \top$ for any $x \in A \setminus \mathrm{Supp}(K_y)$. Hence, we can restrict the infimum in (4.61) to all $A \in \mathcal{F}$ such that $A \subseteq \mathrm{Supp}(K_y)$. Denote $\mathcal{F}_y$ the set of all $\mathcal{F}$–measurable sets such that $A \subseteq \mathrm{Supp}(K_y)$. Since $H^{\to}_{(K,\mu)}$ preserves constant functions due to (C1), using (18), (19), (22) and (23) of Theorem 1.3, we have

$$\begin{aligned}
H^{\to}_{(K,\mu)}(f)(y) \leftrightarrow f(x) &= H^{\to}_{(K,\mu)}(f)(y) \leftrightarrow H^{\to}_{(K,\mu)}(\underline{f(x)}_X)(y) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\mu^{c,N}(A) \to \bigvee_{z \in A} (K(z,y) \to f(z))) \\
&\qquad \leftrightarrow \bigwedge_{A \in \mathcal{F}_y} (\mu^{c,N}(A) \to \bigvee_{z \in A} (K(z,y) \to \underline{f(x)}_X(z))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\mu^{c,N}(A) \to \bigvee_{z \in A} (K(z,y) \to f(z))) \\
&\qquad \leftrightarrow (\mu^{c,N}(A) \to \bigvee_{z \in A} (K(z,y) \to \underline{f(x)}_X(z)))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} ((\mu^{c,N}(A) \leftrightarrow \mu^{c,N}(A)) \otimes (\bigvee_{z \in A} (K(z,y) \to f(z)) \\
&\qquad \leftrightarrow \bigvee_{z \in A} (K(z,y) \to \underline{f(x)}_X(z))) = \\
&\bigwedge_{A \in \mathcal{F}_y} (\bigvee_{z \in A} (K(z,y) \to f(z)) \leftrightarrow \bigvee_{z \in A} (K(z,y) \to \underline{f(x)}_X(z))) \geq \\
&\bigwedge_{A \in \mathcal{F}_y} \bigwedge_{z \in A} (K(z,y) \leftrightarrow K(z,y)) \otimes (f(z) \leftrightarrow \underline{f(x)}_X(z)) \geq \\
&\bigwedge_{z \in \mathrm{Supp}(K_y)} (f(z) \leftrightarrow f(x)) \geq \bigwedge_{(u,v) \in E_y} (f(u) \leftrightarrow f(v)) = \omega(f, E_y),
\end{aligned}$$

where we used the fact that if $z \in \mathrm{Supp}(K_y)$, then $(x,z) \in E_y$. □

The following two statements present the estimation of the approximation quality of the reconstructed function.

Theorem 4.40. *Let K be an integral kernel, K^{-1} be an Q–inverse of K for a reflexive integral kernel Q, and let $f \in \mathcal{F}(X)$. Assume that $(K, \mu^{c,N}_X)$ satisfies (C2) and $(K^{-1}, \mu^{c,N}_Y)$ satisfies (C1), and let $\omega(f)$ be defined as in Theorem 4.12. Then*

$$H^{\to}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \omega(f) \tag{4.62}$$

for any $x \in X$.

Proof. From the proof of Theorem 4.12, we know that $\nabla K_y(x) = \top$ for any $x \in X$ and $y \in \mathrm{Supp}(K_x^{-1})$. Similarly to Theorem 4.39, denote $\mathcal{G}_x$, the set of all $\mathcal{G}$–measurable sets A such that $A \subseteq \mathrm{Supp}(K_x^{-1})$. Let $x \in X$. Since both $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms preserve constant functions, according to Theorem 4.38, using (4.61) and the fact that $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1})$, we have

$$H^{\rightarrow}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) =$$

$$H^{\rightarrow}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow H^{\rightarrow}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)}(\underline{f(x)}_X)(x) =$$

$$H^{\rightarrow}_{(K^{-1},\mu_Y)}(H^{\otimes}_{(K,\mu_X)}(f))(x) \leftrightarrow H^{\rightarrow}_{(K^{-1},\mu_Y)}(\underline{f(x)}_Y)(x) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x} (\mu_Y^{c,N}(A) \rightarrow \bigvee_{y\in A} (K^{-1}(y,x) \rightarrow H^{\otimes}_{(K,\mu_X)}(f)(y))) \leftrightarrow$$

$$\bigwedge_{A\in\mathcal{G}_x} (\mu_Y^{c,N}(A) \rightarrow \bigvee_{y\in A} (K^{-1}(y,x) \rightarrow \underline{f(x)}_Y(y))) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A} (H^{\otimes}_{(K,\mu_X)}(f)(y) \leftrightarrow \underline{f(x)}_Y(y)) =$$

$$\bigwedge_{A\in\mathcal{G}_x} \bigwedge_{y\in A} (H^{\otimes}_{(K,\mu_X)}(f)(y) \leftrightarrow f(x)) \geq \bigwedge_{y\in\mathrm{Supp}(K_x^{-1})} \omega(f,E_y) \geq \omega(f),$$

where we omit the same steps in the verification used in the proof of Theorem 4.38. □

Theorem 4.41. *Let K be an integral kernel, K^{-1} be an Q–inverse of K for a reflexive integral kernel Q, and let $f \in \mathcal{F}(X)$. Assume that $(K, \mu_X^{c,N})$ satisfies (C1) and $(K^{-1}, \mu_Y^{c,N})$ satisfies (C2), and let $\omega(f)$ be defined as in Theorem 4.14. Then*

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \omega(f) \tag{4.63}$$

for any $x \in X$.

Proof. Similarly to the proof of Theorem 4.38, we use $\mathcal{G}_x$ to denote the set of all $\mathcal{G}$-measurable sets A such that $A \cap \mathrm{Core}(K_x^{-1}) \neq \emptyset$. Let $x \in X$. Since both $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms preserve constant functions, according to Theorem 4.39, using (4.59) and the fact that $\nabla K_y(x) = \top$ for any $y \in \mathrm{Supp}(K_x^{-1})$, we have

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) =$$

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(\underline{f(x)}_X)(x) =$$

$$H^{\otimes}_{(K^{-1},\mu_Y)}(H^{\rightarrow}_{(K,\mu_X)}(f))(x) \leftrightarrow H^{\otimes}_{(K^{-1},\mu_Y)}(\underline{f(x)}_Y)(x) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x} (\mu_Y^{c,N}(A) \rightarrow \bigvee_{y\in A\cap\mathrm{Supp}(K_x^{-1})} (K^{-1}(y,x) \otimes H^{\rightarrow}_{(K,\mu_X)}(f)(y))) \leftrightarrow$$

$$\bigwedge_{A\in\mathcal{G}_x} (\mu_Y^{c,N}(A) \rightarrow \bigvee_{y\in A\cap\mathrm{Supp}(K_x^{-1})} (K^{-1}(y,x) \otimes \underline{f(x)}_Y(y))) \geq$$

$$\bigwedge_{A\in\mathcal{G}_x}\bigwedge_{y\in A\cap\mathrm{Supp}(K_x^{-1})}(H^{\rightarrow}_{(K,\mu_X)}(f)(y)\leftrightarrow \underline{f(x)}_Y(y)) =$$

$$\bigwedge_{A\in\mathcal{G}_x}\bigwedge_{y\in A\cap\mathrm{Supp}(K_x^{-1})}(H^{\rightarrow}_{(K,\mu_X)}(f)(y)\leftrightarrow f(x)) \geq \bigwedge_{y\in\mathrm{Supp}(K_x^{-1})}\omega(f,E_y)\geq\omega(f),$$

where we omit the same steps in the verification used in the proof of Theorem 4.39. □

In the next part we give an estimate of the quality of the approximation for the extensional functions. Since the proofs of the following two theorems are complete analogies of the proofs of Theorems 4.17 and 4.18, we omit them.

Theorem 4.42. *Let $Y\subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K: X\times Y\to L$ given as $K(x,y)=P(x,y)$ for any $x\in X$ and $y\in Y$ is an integral kernel, and let $K^{-1}=K^T$ be a Q–inverse of K. If f is extensional with respect to Q and $(K^{-1},\mu_Y^{c,N})$ satisfies (C1), then*

$$H^{\rightarrow}_{(K^{-1},\mu_Y)}\circ H^{\otimes}_{(K,\mu_X)}(f)(x)\leftrightarrow f(x)\geq\int^{\rightarrow}_{\mathrm{DPR}\,X}Q^2(y,x)\,d\mu_X \tag{4.64}$$

for any $x\in X$.

Theorem 4.43. *Let $Y\subseteq X$ be a non-empty set, let P be a similarity relation on X such that $K: X\times Y\to L$ given as $K(x,y)=P(x,y)$ for any $x\in X$ and $y\in Y$ is an integral kernel, and let $K^{-1}=K^T$ be a Q–inverse of K. If f is extensional with respect to Q and $(K,\mu_X^{c,N})$ satisfies (C1), then*

$$H^{\otimes}_{(K^{-1},\mu_Y)}\circ H^{\rightarrow}_{(K,\mu_X)}(f)(x)\leftrightarrow f(x)\geq\int^{\rightarrow}_{\mathrm{DPR}\,Y}Q^2(x,y)\,d\mu_Y \tag{4.65}$$

for any $x\in X$.

We have shown that the M–lattice integral transforms preserve extensional functions with respect to Q, where Q is determined from a similarity relation on X. The following corollaries show that $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms have the same property, provided that the negation N is determined by the residuum and is involutive.

Corollary 4.44. *Let the assumption of Theorem 4.42 be satisfied, let (K,μ_X) satisfy (C1), and let $N=N_{res}$ be involutive. Then*

$$H^{\rightarrow}_{(K^{-1},\mu_Y)}\circ H^{\otimes}_{(K,\mu_X)}(f)=f \tag{4.66}$$

for any extensional function f on X with respect to Q.

Proof. First, observe that the definition of the integral kernel K (derived from the similarity relation P on X) in this corollary coincides with the definition of K in Corollary 4.19. Therefore, from the proof of Corollary 4.19, we know that for any $x\in X$ there is $z\in Y$ such that $\mathrm{Core}(K_z)\subseteq\mathrm{Core}(Q_x)$. Since (K,μ_X) satisfies (C1), we get that there is $A_z\in\mathcal{F}$ such that $A_z\subseteq\mathrm{Core}(K_z)$ and $\mu_X(A_z)=\top$. Since $A_z\subseteq\mathrm{Core}(Q_x)$, we find that (Q,μ_X) also satisfies (C1).

Let $x \in X$. Then there is $A_x \subseteq \mathrm{Core}(Q_x)$ such that $\mu_X(A_x) = \top$. Obviously, $Q^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$ (see the proof of Corollary 4.19). By Theorem 4.42, Theorem 2.16(v) and the fact that $N = N_{res}$ is nilpotent, we find

$$H^{\rightarrow}_{(K^{-1},\mu_Y)} \circ H^{\otimes}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int^{\rightarrow}_{\mathrm{DPR}\,X} Q^2(y,x)\, d\mu_X \geq \int^{\rightarrow}_{\mathrm{DPR}\,X} 1_{A_x}(y)\, d\mu_X = \top \wedge \mu_X(A_x) = \mu_X(A_x) = \top.$$

Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

Corollary 4.45. *Let the assumption of Theorem 4.43 be satisfied, let (K^{-1}, μ_Y) satisfy (C1), and let $N = N_{res}$ be involutive. Then*

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(f) = f \tag{4.67}$$

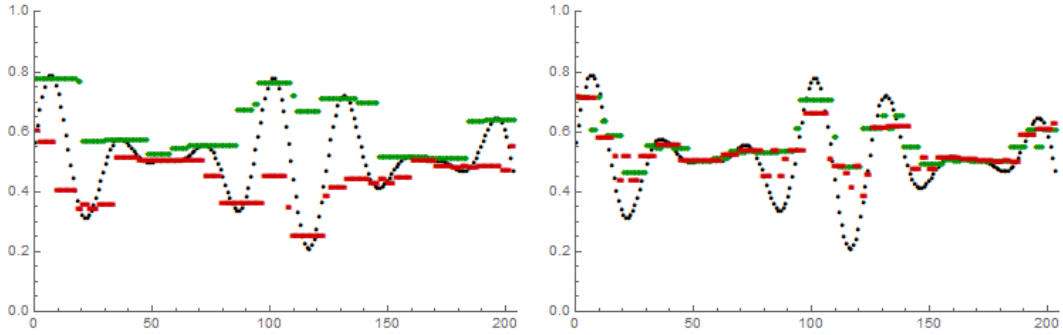
for any extensional function f on X with respect to Q.

Proof. First, observe that the definition of the integral kernel K (derived from the similarity relation P on X) in this corollary coincides with the definition of K in Corollary 4.20, from the proof of which we know that $Q^T(y,x) = K^{-1}(y,x)$ for any $x \in X$ and $y \in Y$. Let $x \in X$. Since (K^{-1}, μ_Y) satisfies (C1), there is $A_x \in \mathcal{G}$ such that $A_x \subseteq \mathrm{Core}(K^{-1}_x)$ and $\mu_Y(A_x) = \top$. Further, we have $(K^{-1})^2(y,x) \geq 1_{A_x}(y)$ for any $y \in Y$ and $(K^{-1})^2 = (K^2)^T$ (see, Corollary 4.20). By Theorem 4.44, Theorem 2.16(v) and the fact that $N = N_{res}$ is nilpotent, we find that

$$H^{\otimes}_{(K^{-1},\mu_Y)} \circ H^{\rightarrow}_{(K,\mu_X)}(f)(x) \leftrightarrow f(x) \geq \int^{\rightarrow}_{\mathrm{DPR}\,Y} Q^2(x,y)\, d\mu_Y = \int^{\rightarrow}_{\mathrm{DPR}\,Y} (K^2)^T(y,x)\, d\mu_Y = \int^{\rightarrow}_{\mathrm{DPR}\,Y} (K^{-1})^2(y,x)\, d\mu_Y \geq \int^{\rightarrow}_{\mathrm{DPR}\,Y} 1_{A_x}(y)\, d\mu_Y = \top \wedge \mu(A_x) = \top.$$

Hence, we get the desired equality as the consequence of the fact that $a \leftrightarrow b = \top$ if and only if $a = b$. □

Remark 4.2. We assume that (K, μ_X) satisfies (C1) in the first corollary to find a suitable set $A_x \in \mathcal{F}$ such that $A_x \subseteq \mathrm{Core}(Q_x)$ and $\mu_X(A_x)$, which allows us to construct a function 1_{A_x} whose DPR–residuum based fuzzy integral is equal to $\top$. In general, we cannot replace this assumption by $(K, \mu_X^{c,N})$ satisfy (C2), which would be expected in this case, since condition (C2) does not guarantee the existence of the desired set A_x. Nevertheless, if we assume that $\mathrm{Core}(K_y) \in \mathcal{F}$ for any $y \in Y$, then the N–conjugate fuzzy measure to $\mu_X^{c,N}$ is the fuzzy measure μ_X, since $N = N_{res}$ is an involutive negation, and μ_X satisfies (C1) by Theorem 3.9. Thus, assuming that each set $\mathrm{Core}(K_y)$ is $\mathcal{F}$–measurable for $y \in Y$, then we can substitute the assumption "(K, μ_X) satisfy (C1)" by the assumption "$(K, \mu_X^{c,N})$ satisfy (C2)" in Corollary 4.44. The same observation also holds for the second corollary.



(a) μ_{X1} (b) μ_{X2}

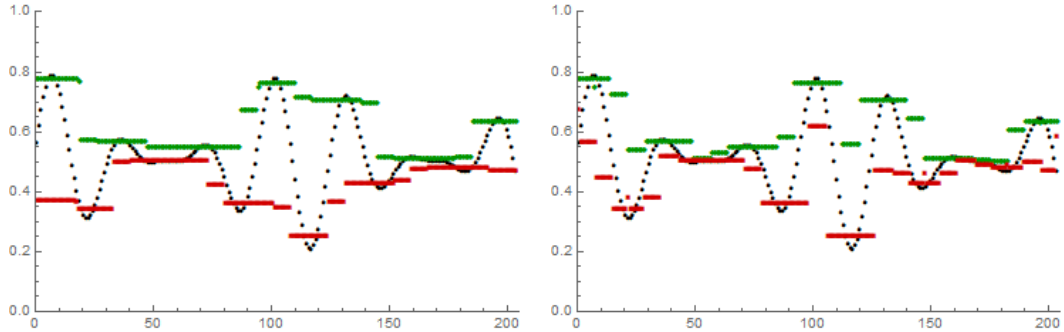
Figure 4.14: Original function f (black) and its approximation $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$ (green diamonds) and $\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$ (red squares) for a fixed integral kernel K and two different fuzzy measures μ_{X1} and μ_{X2}.

4.5.3 Illustration on signal reconstruction

In this part, we demonstrate the reconstruction of signals using the composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms. We use the setting of Example 3.7 and reconstruct the original signal given by formula (3.9) from the transformed functions presented in the mentioned example. Recall that all the integral kernels $K : X \times Y \to L$ together with the fuzzy measures μ_X (originally denoted as μ) on the measurable space $\langle X, \mathcal{P}(X)\rangle$ are introduced in such way that (K, μ_X) satisfies condition (C1) and $(K, \mu_X^{c,N})$ satisfies condition (C2), and due to Theorem 3.18, we get that the $(K, \mu_X, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and the $(K, \mu_X^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform preserve any constant function in $\mathcal{F}(X)$. In the following, we will consider cases a) and b) studied in Example 3.7.

Case a) We use the same setting of the kernel K and the fuzzy measures μ_{X1} and μ_{X2} as in case a) of Subsection 4.3.3. In Figure 4.14(a), we show the reconstruction of the original signal using composition of $(K, \mu_{X1}, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and $(K^{-1}, \mu_Y^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by green diamonds ($\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$ for short) and the same reconstruction using the composition of the $(K, \mu_{X1}^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and $(K^{-1}, \mu_Y, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform described by red squares ($\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$ for short) together with the original function. We see that the constructed signals are similar to those obtained by the previous two types of lattice integral transforms, so we can fully adopt their analysis. The same applies to the reconstruction of the original signal with the fuzzy measure μ_{X2}, which is shown in Figure 4.14(b).

Case b) Again we use the same setting of the kernels K_1 and K_2 and the fuzzy measure μ_X as in case b) of Subsection 4.3.3. In Figure 4.15, similarly to case a), we illustrate reconstructed signals for compositions of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms with different integral kernels K_1 and K_2. As shown in Subsection 4.3.3, since $K_{2y} \subseteq K_{1y}$ for any y in Y, we get that the approximation of the original signal for K_2, shown in Figure 4.15(b), is better than that for K_1, shown in Figure 4.15(a),



(a) K_1 (b) K_2

Figure 4.15: Original function f (black) and its approximation using $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$ (green diamonds) and $\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$ (red squares) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.

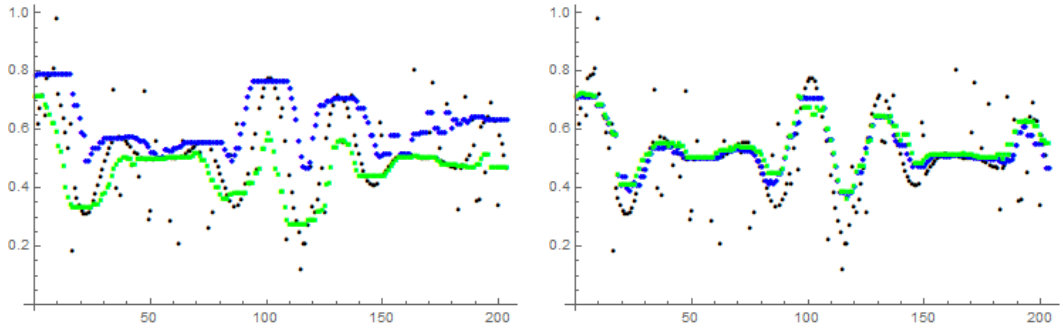
which is a consequence that appears in the previous two types of lattice integral transforms.

4.5.4 Filtering of random noise

In this part, we will show that $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms can be used to filter out random noise in signal processing. For the illustration, we again use the function f given by the formula (3.9), to which we add 30% random noise determined by a uniform distribution. We present filters based on a single $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and their compositions. In both applications, we consider the same fuzzy measures μ_X, μ_{X1} and μ_{X2} on $\langle X, \mathcal{P}(X)\rangle$ and μ_Y on $\langle Y, \mathcal{P}(Y)\rangle$ specified in Subsection 4.3.3.

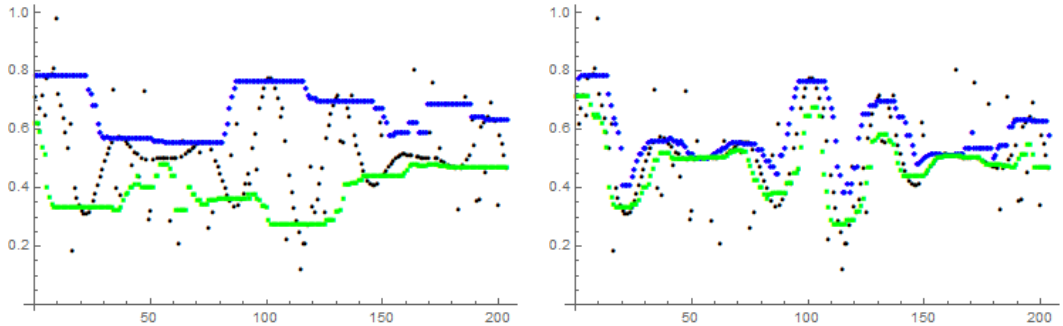
Filter based on single $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform Again, for demonstration, we assume that $X = Y = \{1, 2, \ldots, 204\}$ and integral kernels $K, K_1, K_2 : X \times X \to L$ are defined by the same formulas as in Example 3.7 only Y is replaced by X. Again, we distinguish two cases, namely, case a) the fixed integral kernel K and two fuzzy measures μ_{X1} and μ_{X2}, case b) the fixed fuzzy measure μ_X and two integral kernels K_1 and K_2. The results of filtering random noise for both cases are shown in Figures 4.16 and 4.17. Comparing these results with those obtained using a filter based on a single M–lattice integral transform, we see little difference, so that the behavior of the two filters is almost identical. An interesting question is whether we can determine the effectiveness of one filter over another with respect to the signal and the noise presented inside.

Filter based on composition of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms This part presents noise suppression using the compositions of $\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms shown in previous subsection on signal reconstruction. From Figures 4.18 and 4.19 we can see that all the resulting signals significantly filter out the noise and reconstruct the original signal without noise similarly to the previous subsection.



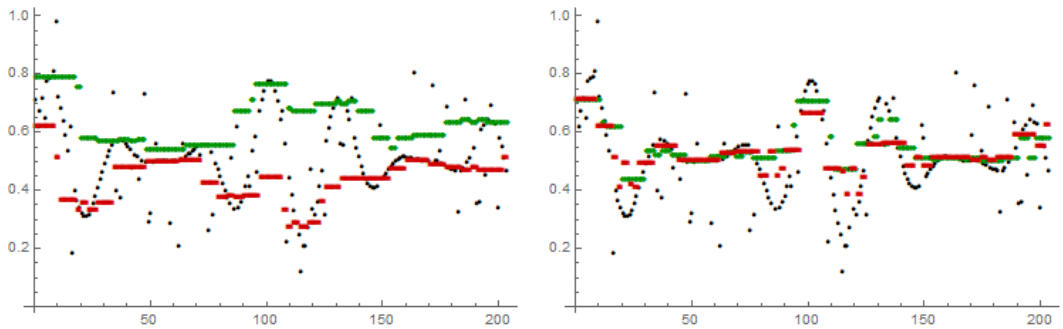
(a) μ_{X1} (b) μ_{X2}

Figure 4.16: Filtering random noise using $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT (light blue), $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$–LIT (light green) for a fixed integral kernel K and two different fuzzy measures μ_{X1} and μ_{X2}.



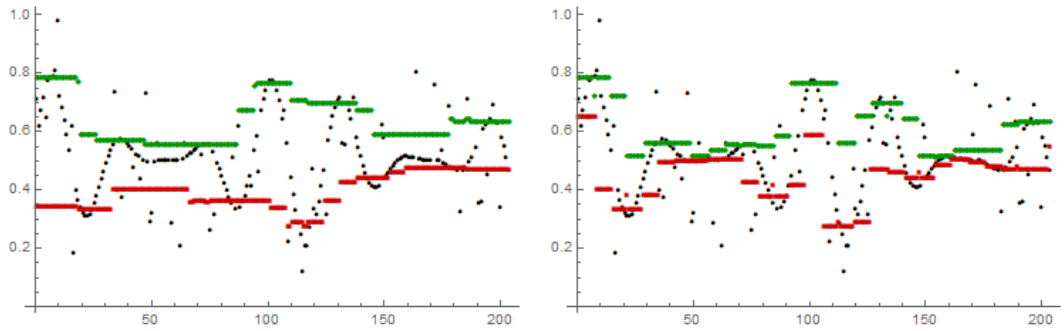
(a) K_1 (b) K_2

Figure 4.17: Filtering random noise using $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT (light blue), $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$–LIT (light green) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.



(a) μ_{X1} (b) μ_{X2}

Figure 4.18: Filtering random noise using $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$ (green diamonds), and $\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$ (red squares) for a fixed integral kernel K and two different fuzzy measures μ_{X1} and μ_{X2}.



(a) K_1 (b) K_2

Figure 4.19: Filtering random noise using $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$ (green diamonds), and $\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$ (red squares) for a fixed fuzzy measure μ_X and two different integral kernels K_1 and K_2.

Chapter 5

Application of M–lattice integral transforms to multicriteria decision making

Multicriteria decision making (MCDM) is used in screening, prioritising, ranking, or selecting a set of alternatives under usually independent, incommensurate or conflicting criteria. An MCDM problem is usually characterized by the ratings of each alternative with respect to criteria and weights determining their significance (see, [2, 7, 5, 14]). The evaluation of alternatives is provided by an aggregation of values expressing the degrees to which criteria are satisfied, taking into account the weights of their importance in a decision making. The most popular aggregation function in practice is the weighted average (generally OWA operators can be applied in [46]) when we assume that sum of all weights of the importance of criteria equal to 1. However, the linearly ordered scale L may not satisfy generally all the requirements for the application of the weighted average. This can occur when the standard arithmetic operations cannot be, in principle, used for the values of the scale L (e.g., the values of a scale are only linearly ordered labels like *low*, *medium*, *high* or *bad*, *good*, *excellent*) or even can but the weighted average provides wrong results [1]. In this case, it is reasonable to use aggregation operators on bounded linearly ordered sets (or even bounded partial ordered sets or lattices) such as the weighted minimum or maximum proposed in [9], weighted median in [47] or linguistic OWA operator in [20]. The theory of such aggregation operators "often referred to as qualitative", with other examples can be found in [15].

In this chapter, we are interested in MCDM, where the alternatives are evaluated in a linearly ordered set endowed by additional operations (precisely, in a residuated lattice) and, moreover, the evaluation is not a single value for each alternative but a vector whose values determine the satisfaction of alternatives with respect to global criteria describing suitable features. This seems to be advantageous in a situation

[1] For example, let $r_1 = r(a_1)(\text{price}) = 0.2$, $r_2 = r(a_2)(\text{price}) = 0.5$ and $r_3 = r(a_3)(\text{price}) = 0.8$ denote the satisfactions of the criterion "price" by three alternatives (cars). Although we have $r_2 - r_1 = r_3 - r_2$, the real prices of alternatives may not capture the same differences, because of different considerations when real prices are lower and when higher. This type of heterogeneity is quite common, especially when quantifying something which is not well measurable (e.g. car design), and in this case, the weighted average can lead to an incorrect evaluation of alternatives.

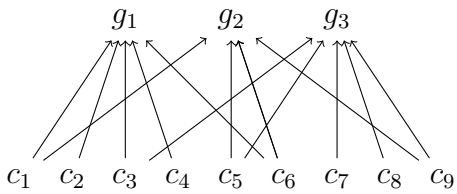


Figure 5.1: Relationship between criteria from $C = \{c_j \mid j = 1, \ldots, 9\}$ and global criteria from $G = \{g_k \mid k = 1, 2, 3\}$, where the displayed arrows indicate an existing importance that can be described by degrees of importance and missing arrows indicate non importance.

when it is difficult to specify the importance of criteria with respect to one global criterion formally expressing "to be the best alternative", whose satisfaction by alternatives corresponds to the evaluation of alternatives [2], and it is easier to select global criteria related only to criteria from certain subgroups of all criteria, which allows us to simply determine the importance of criteria with respect to the related global criteria. Subgroups of criteria may overlap which means that one criterion has an influence in the evaluation of more than one global criterion as it is demonstrated in Figure 5.1. The evaluated alternatives can be used directly for a decision (by a comparison of vectors), or can serve as input values for another MCDM (e.g., a hierarchical model is considered).

The aim of this chapter is to introduce the evaluation of alternatives with respect to global criteria by a novel approach, which is based on the M–lattice integral transform (M–LIT) of lattice-valued functions presented in Subsection 3.3 of Chapter 3 (also see [26]). The proposed approach will be demonstrated and compared with a common approach on a car selection problem.

5.1 MCDM based on the M–lattice integral transform

As an example, let us consider a problem of selecting a car, the aim of which is to buy a new car from a set of cars of different brands. This set is called the set of alternatives. To select the best car, it is necessary to determine suitable criteria (e.g., price, brand, design, safety, performance) together with their degrees of importance according to which it will be decided. More formally, let $A = \{a_1, \ldots, a_n\}$ denote a set of alternatives, let $C = \{c_1, \ldots, c_m\}$ denote a set of criteria, and let $\mathbf{L}$ be a complete linearly ordered residuated lattice as the scale for the evaluation of alternatives. A satisfaction of criteria by alternatives can be described as a function $r : A \to L^C$, where $r(a_i)(c_j)$ expresses the degree to which the j-th criterion c_j is satisfied by the i-th alternative a_i (i.e., L^C denotes the set of all functions from C to

[2] It is easy to see that the evaluation of alternatives described as a function $u : A \to L$ can be identically expressed by a function $u' : A \to L^{\{g\}}$, where g represents a global criterion "to be the best alternative", and the evaluation of alternatives can be equivalently described by the degrees to which the alternatives satisfy the global criterion g, i.e., $u(a_i) = u'(a_i)(g)$ for $i = 1, \ldots, n$. The importance of criteria can be equivalently expressed as a function $w' : C \times \{g\} \to L$, where $w'(c_j, g)$ determines the degree to which the criterion c_j is important in the evaluation of the global criterion g.

L). The importance of criteria can be described as a function $w : C \to L$, where the higher value of $w(c_j)$ means the higher importance of criterion c_j. The evaluation of alternatives is then a function $u : A \to L$ given as

$$u(a_i) = h_w(r(a_i)(c_1), \ldots, r(a_i)(c_m)), \tag{5.1}$$

where $h_w : L^m \to L$ is an aggregation function respecting the importance of criteria expressed by the function w. In addition, let G denote the set of global criteria, then the evaluation of alternatives with respect to global criteria can be described as a function $u : A \to L^G$ which is defined through the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "u"'] & L^C \arrow[d, "h"] \\ & L^G, \end{tikzcd} \tag{5.2}$$

where the function r expresses a satisfaction of criteria from C by alternatives from A, and h is an "extended" aggregation function, which will be introduced as the lattice integral transform of the space L^C to the space L^G with an integral kernel $w : C \times G \to L$, where $w(c_j, g_k)$ determines the degree to which the criterion c_j is important in the evaluation of alternatives by the criterion g_k.[3] If a criterion c_j is not important at all for a global criterion g_k, then $w(c_j, g_k)$ is equal to the bottom element of L. Note that, for $G = \{g\}$, the extended aggregation function h defined by the lattice integral transform involves the weighted maximum mentioned above (i.e., similarly the weighted minimum can be obtained as a special case of the residuum base lattice integral transform proposed in [25]). Of course, the extended aggregation function can be obtained also in other ways, but the lattice integral transform provides a consistent way for the evaluation of alternatives with a possibility of changing parameters.

Now, for a more detailed expression, let $\langle C, \mathcal{F}, \mu \rangle$ be a fuzzy measure space over the set of criteria C. According to (5.2), the evaluation of alternatives $u : A \to L^G$ is determined by a $(w, \mu, \otimes)$–M–lattice integral transform as follows

$$u(a_i)(g_k) := F^{\otimes}_{(w,\mu)}(r(a_i))(g_k) = \int^{\otimes} w(c_j, g_k) \otimes r(a_i)(c_j)\, d\mu, \quad g_k \in G, \tag{5.3}$$

where the kernel function $w : C \times G \to L$ determines the importance of the criteria from C in the evaluation of alternatives with respect to the global criteria from G assuming that w is semi-normal in the second component, i.e., for any $g_k \in G$, there exists at least one $c_j \in C$ such that $w(c_j, g_k) > 0$.

We should note that the setting of kernel function w is hard work for an expert with experience because its values significantly influence the decision. Following the assumptions on the weighted maximum proposed by Dubois and Prade in [9], one could even assume that w is normal in the second component, which means that each function w_{g_k} is a possibility distribution (i.e., $\max_{c_j \in C} w_{g_k}(c_j) = 1$), and in the

[3] We use w instead of K for the denotation of the integral kernel to keep the notation in the paper [22].

case of lower lattice fuzzy transform proposed by Perfilieva in [36], one could be even stronger and assume that the sets $\mathrm{Core}(w_{g_1}), \ldots, \mathrm{Core}(w_{g_\ell})$ form a partition of C. But w does not provide the only parameter of our approach. Other parameters are the fuzzy measure space and the selection of residuated lattice, especially, the multiplication operation. For example, if the measurable space $\mathcal{F} = \mathcal{P}(C)$, $L = [0,1]$ and the fuzzy measure is defined as $\mu(X) = 1$ for any $X \in \mathcal{F} \setminus \{\emptyset\}$, and $\mu(\emptyset) = 0$, the evaluation of alternatives can be expressed as

$$u_{\otimes}^{\mathrm{WM}}(a_i)(g_k) := \bigvee_{c_j \in C} w(c_j, g_k) \otimes r(a_i)(c_j), \quad a_i \in A,\ g_k \in G, \tag{5.4}$$

which can be seen as a $\otimes$–weighted maximum generalizing the weighted maximum with $\otimes = \wedge$, i.e., $u_{\wedge}^{\mathrm{WM}}$. It is easy to see that for any fuzzy measure μ on a measurable space $\langle C, \mathcal{F} \rangle$, the evaluation of alternatives u given by (5.3) cannot be higher than the evaluation $u_{\otimes}^{\mathrm{WM}}$ given by the $\otimes$–weighted maximum, i.e., $u(a_i)(g_k) \leq u_{\otimes}^{\mathrm{WM}}(a_i)(g_k)$ for any $a_i \in A$ and $g_k \in G$.

5.2 Illustrative example

We consider a car selection problem, that is, we would like to buy a new car from some famous car brands. The MCDM problem is to select an appropriate car from the following four alternatives: *Toyota Wigo, Hyundai Grand i10, Honda City* and *Nissan Terra*, i.e., we consider the set $A = \{\mathrm{Wigo}, \mathrm{Grand\ i10}, \mathrm{City}, \mathrm{Terra}\}$ as the set of alternatives. Our global criteria for the evaluation of alternatives that form the set G are *looks, safety* and *performance*. To determine the evaluation of alternatives with respect to the global criteria we consider ten criteria, namely, *Price, Logo, Year of Manufacture, Top Speed, Fuel consumption, Style, Insurance quote, Boot space, Warranty* and *Equipment*, which form the set C.

For a comparison of the evaluation of alternatives based on the M–LIT with other evaluations based on quantitative and qualitative aggregations, we consider the residuated lattices L defined by the left-continuous t-norms on $[0,1]$ (see, Example 1.1). The satisfaction of criteria by alternatives (i.e., $r(a_i, c_j)$) is displayed in Table 5.1.

One can see that, example, $r(\mathrm{Wigo, Price}) = 0.2 < 0.5 = r(\mathrm{Grand\ i10, Price})$, which corresponds to the higher price of Toyota Wigo than the price of Hyundai Grand i10, and a lower price naturally increases the satisfaction of the criterion Price.

For the purpose of the evaluation of alternatives, we consider the integral kernel $w : C \times G \to [0,1]$ whose values are displayed in Table 5.2. It can be seen that the set $\mathrm{Supp}(w_{\mathrm{Looks}})$ consists of seven criteria from C, namely, Price, Logo, Year, Style, Insurance quote, Warranty and Equipment, that are important in a certain non-zero degree for the evaluation of alternatives with respect to the global criterion (a car feature) Looks. Similarly the sets $\mathrm{Supp}(w_{\mathrm{Safety}})$ and $\mathrm{Supp}(w_{\mathrm{Performance}})$ consist of four and five criteria from C, respectively. One could see that the functions w_{g_k}, $g_k \in G$, have the non-empty cores, hence, these functions are possibility distributions (see, [9]), which seems to be a reasonable requirement reflecting the fact that there is at least one fully important criterion for each global criterion.

Criteria	**Cars**			
	Wigo	Grand i10	City	Terra
Price	0.2	0.5	0.7	0.4
Logo	0.9	0.7	0.6	0.8
Year	0.6	0.2	0.8	0.4
Top.sp/mph	0.6	0.8	0.2	0.4
Fuel.co/mpg	0.9	0.5	0.4	0.7
Style	0.7	0.9	0.6	0.8
Insurance.qu	0.1	0.7	0.6	0.9
Boot.sp/litres	0.2	0.5	0.7	0.3
Warranty	0.3	0.5	0.2	0.8
Equipment	0.8	0.7	0.9	0.6

Table 5.1: Satisfactions of the criteria by the alternatives.

Criteria	**Global criteria**		
	Looks	Safety	Performance
Price	1	0	0
Logo	0.2	0.3	0.5
Year	1	0	0
Top.sp/mph	0	0	1
Fuel.co/mpg	0	0	1
Style	0.6	0.4	0
Insurance.qu	0.5	0.5	0
Boot.sp/litres	0	0	1
Warranty	0.6	0	0.4
Equipment	0.7	1	0

Table 5.2: Integral kernel determining the importance of criteria for the evaluation of alternatives with respect to global criteria.

To ensure that the evaluation of alternatives is "fair" and respects only the important criteria, we define a fuzzy measure μ on $\langle C, \mathcal{F}(C)\rangle$ as follows

$$\mu(X) = \begin{cases} 1, & \#X \geq 4, \\ \frac{\#X}{4}, & \text{otherwise}, \end{cases}$$

for any $X \in \mathcal{F}(C)$. The "fair" evaluation is reflected in the fact that the fuzzy measure μ is symmetric. Moreover, we set $\mu(X) = 1$ for $\#X \geq 4$, which is motivated by the numbers of criteria in the support of functions w_{g_k}, $g_k \in G$. More specifically, we use the minimum number 4 to allow the maximum evaluation of all alternatives equal to 1, ideally when the satisfaction of criteria is equal to 1 for all alternatives and the non-zero degrees of importance in Table 5.2 would be changed to 1. Our setting of fuzzy measure does not influence the evaluation of alternatives in the above-mentioned ideal case, although we consider arbitrary left-continuous t-norm as the multiplication on the residuated lattice, which seems to be a reasonable requirement. A stronger requirement defined analogously could be introduced using

the number of elements in the cores of functions w_{g_k}, $g_k \in G$, which will guarantee the preservation of constant satisfactions of criteria by alternatives.

In Table 5.3, we present the evaluations of alternatives u^T with respect to global criteria for the fundamental continuous t-norms, namely, the minimum, product and Łukasiewicz t-norms. To compare the proposed approach based on the M–LIT with some, say, representatives of the standard (quantitative and qualitative) approaches, we show in the same table the evaluation of alternatives using the weighted average given as

$$u^{\mathrm{WA}}(a_i)(g_k) := \frac{\sum_{j=1}^{10} w(c_j, g_k) \cdot r(a_i)(c_j)}{\sum_{j=1}^{10} w(c_j, g_k)}, \quad a_i \in A,\ g_k \in G, \tag{5.5}$$

representing the quantitative approach, and the weighted maximum $u_{\wedge}^{\mathrm{WM}}$ representing the qualitative approach, although, it can be obtained as a special case of the lattice integral transform.

Evaluations	**Global criteria**	**Cars**			
		Wigo	Grand i10	City	Terra
$u^{T_{\mathrm{M}}}$	Looks	0.6	0.5	0.7	0.6
	Safety	0.4	0.5	0.5	0.5
	Perfor	0.5	0.5	0.5	0.5
$u^{T_{\mathrm{P}}}$	Looks	0.315	0.3675	0.4725	0.42
	Safety	0.2025	0.2625	0.225	0.24
	Perfor	0.3375	0.375	0.225	0.32
$u^{T_{\mathrm{Ł}}}$	Looks	0.2	0.2	0.35	0.4
	Safety	0.05	0	0.15	0.1
	Perfor	0.2	0.25	0.1	0.3
u^{WA}	Looks	0.476	0.547	0.658	0.606
	Safety	0.636	0.736	0.736	0.731
	Perfor	0.582	0.602	0.43	0.543
$u_{\wedge}^{\mathrm{WM}}$	Looks	0.7	0.7	0.8	0.6
	Safety	0.8	0.7	0.9	0.6
	Perfor	0.9	0.8	0.7	0.7

Table 5.3: Evaluation of alternatives with respect to global criteria determined by the M–LITs, the weighted average and the weighted maximum.

To select the best car we aggregate the values of vectors evaluating alternatives related to global criteria in Table 5.3 to one value using the weighted average with respect to the weights: $w(\text{Looks}) = 0.35, w(\text{Safety}) = 0.4$ and $w(\text{Performance}) = 0.25$, with a total sum equal to 1, expressing their importance for our selection of the best car. The results are displayed in Table 5.4. To compare the resulting evaluations of cars we determine the orders of cars that correspond to the orders of their evaluations presented in Table 5.3, e.g., we get Honda City, Nissan Terra, Hyundai Grand i10, Toyota Wigo for the evaluation $u^{T_{\mathrm{M}}}$, where Honda City has the highest evaluation 0.57, while Toyota Wigo the lowest evaluation 0.495. Surprisingly,

Evaluations	**Cars**			
	Wigo	Grand i10	City	Terra
$u^{T_{\mathrm{M}}}$	0.495	0.5	0.57	0.535
$u^{T_{\mathrm{P}}}$	0.275	0.327	0.311	0.323
$u^{T_{\mathrm{Ł}}}$	0.14	0.132	0.207	0.255
u^{WA}	0.566	0.636	0.632	0.64
$u_{\wedge}^{\mathrm{WM}}$	0.79	0.725	0.815	0.625

Table 5.4: Aggregation of various evaluations of alternatives to order the alternatives.

there are no two evaluations resulting in the same order of cars, but three of all evaluations indicate Toyota Wigo as the car with the worst evaluation. Clearly the candidates for the best car are Honda City and Nissan Terra with the two highest evaluations. It is probably impossible to say, what evaluation of alternatives is right or even the best in this illustrative example, since each uses a different type of aggregation, but summing the ranking numbers of cars (i.e., a car gets the ranking number n if it stands on the n-th position in an order of cars. The ranking number 1 (4) indicates the best (worst) car with respect to considered evaluation of cars) over all evaluations to get an overall ranking number we can conclude that Honda City and Nissan Terra occupy the first and second place with the overall ranking number 10 (e.g., $10 = 1+3+2+3+1$ for Honda City). The third place gets Hyundai Grand i10 with the overall ranking number 13, and Toyota Wigo gets the last place with the overall ranking number 17. If we remove the weighted maximum evaluation in the summation of ranking numbers, which depends on only one maximum value, the best car is Nissan Terra with the overall ranking number $6 = 2 + 2 + 1 + 1$.

Chapter 6

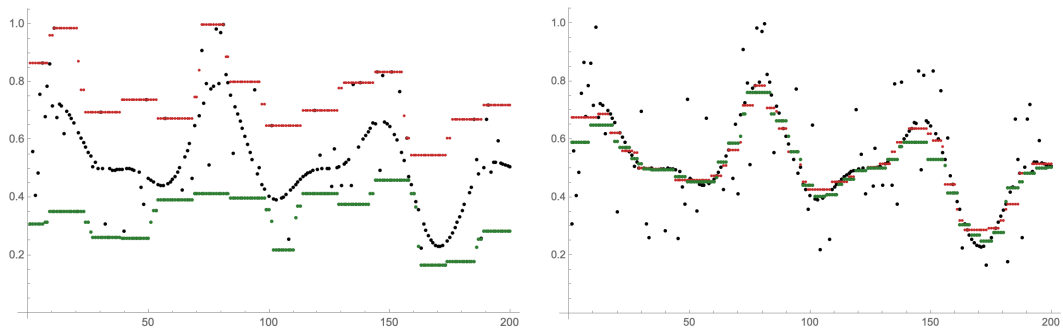
Application of lattice integral transforms in image processing

Lattice integral transforms for lattice-valued functions were introduced to provide a theoretical framework for transformations of functions whose functional values cannot in principle be handled by standard arithmetic of real or complex numbers or application of standard arithmetic have certain disadvantages. For example, non-additive noise in signal or image processing is filtered out by the methods that do not use the standard arithmetic, but order statistic functions like median are applied (see, [1]). Another example can be mathematical morphology on complete lattices, which provides morphological operators whose mathematically coherent application to gray-scale images has already been justified (see, [39, 38, 19]).

Similarly to the lattice fuzzy transforms, in Chapter 4, we have shown the composition of two types of lattice integral transforms, for example,

$$F^{\otimes}_{(K,\mu_X)} : \mathcal{F}(X) \to \mathcal{F}(Y) \text{ and } F^{\rightarrow}_{(K^{-1},\mu_Y)} : \mathcal{F}(Y) \to \mathcal{F}(X),$$

where K is an integral kernel, $K^{-1} = K^T$ is an Q-inverse of K, μ_X is an appropriate fuzzy measure on $\langle X, \mathcal{F}\rangle$ and μ_Y is an appropriate fuzzy measure on $\langle Y, \mathcal{G}\rangle$, approximates original functions. In addition, we have also shown that the random noise present can be filtered out, unlike lattice fuzzy transforms, as shown in Figure 6.1.



(a) lattice fuzzy transforms (b) M–lattice integral transforms

Figure 6.1: A comparison of noisy signal reconstructions based on lower and upper approximations using lattice fuzzy transforms and M–lattice integral transforms.

The aim of this chapter is to present the use of all types of lattice integral transforms introduced in Chapter 3 in image processing, specifically non-linear filtering, compression/decompression and opening/closing of images. We show that filters based on lattice integral transforms can be seen as a generalization of the known median filter as well as minimum and maximum filters. Note that these filters are popular for removing of salt-and-pepper noise, namely, the minimum (maximum) filter removes the salt (pepper) noise because it has very high (low) values of intensities. The median filter removes both types of noise. The minimum and maximum filters are also associated with the most common morphological operations of erosion and dilation, because the minimum filter erodes shapes on the image, whereas the maximum filter extends object boundaries (see, [13]). The opening and closing filters are achieved by combining the morphological operations of erosion and dilation, in our case, we will consider their definitions in fuzzy mathematical morphology (see, [42]). We illustrate the proposed methods in various selected images.

6.1 Introduction

In this chapter, we apply the $(K, \mu, \star)$–M–lattice integral transforms ($\mathrm{M}^\star$–LIT), the $(K, \nu, \star)$–R_{DH}–lattice integral transforms ($\mathrm{R}^\star_{\mathrm{DH}}$–LIT) and the $(K, \mu, \star)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms ($\mathrm{R}^\star_{\mathrm{DPR}}$–LIT) introduced in Chapter 3 and their compositions investigated in Chapter 4 to the following image processing tasks: non-linear filtering, compression/decompression and closing/opening of images. For their application, we restrict ourselves to grayscale images. Note that the color image is divided into color channels and independently processed one by one. The standard RGB color model is suitable for noise filtering. For compression, it is preferable to use the YUV color model, where U and V are compressed more strongly than the Y component that contains information important to human perception.

In what follows, we assume that an image I of the size $N \times M$ (the number of pixels in rows and columns) is a function $I : D \to [0, 1]$, where

$$D = \{(i, j) \mid 1 \leq i \leq N, 1 \leq j \leq M\}$$

and the value $I(i, j)$ expresses the intensity of shades of gray from black to white for the pixel at the position $(i, j) \in D$. In our terminology, the image I is nothing but a fuzzy relation on D. For simplicity, we assume that the shade of gray is determined for any number from $[0, 1]$. Since an image is a two-dimensional function, we consider lattice integral transforms for fuzzy relations from $\mathcal{F}(D_1)$ to $\mathcal{F}(D_2)$, where D_1 is the domain of original (input) images and D_2 is the domain of transformed (output) images (e.g., compressed images). In the following sections, we first describe in details the way of how the lattice integral transforms are applied to the above tasks, and then demonstrate it in various images.

6.2 Method description

Let N, M, ϱ be natural numbers such that ϱ divides N and M. Denote $n = N/\varrho$ and $m = M/\varrho$. The number ϱ will be called the *shift* and $\varrho^2 : 1$ expresses the

compression ratio. Let D be the domain of input images and denoted $D_\varrho = \{(i,j) \mid 1 \le i \le n, 1 \le j \le m\}$ the domain for compressed images ($\varrho > 1$) or filtered images ($\varrho = 1$).

Let r, s be natural numbers such that $\varrho \le r \ll N$ and $\varrho \le s \ll M$, and denote $[-r, r] = \{-r, \dots, 0, \dots, r\}$ and similarly for $[-s, s]$. Let $W = \{w_{i,j} \mid i \in [-r, r], j \in [-s, s]\}$ be a $2r+1 \times 2s+1$ matrix of values from $[0,1]$, which will be referred as the *window* of size $R \times S$, where $R = 2r+1$ and $S = 2s+1$. A window W specifies the weights that are assigned to pixels in the neighborhood of a corresponding pixel in the input image. In our application, we assume that $w_{00} = 1$. Note that W can be viewed as a normal fuzzy relation on $[-r, r] \times [-s, s]$.

To properly process the pixels at the edges of images, we extend D to a broader domain given as

$$D_{r,s} = \{(i,j) \mid -r+1 \le i \le N+r, -s+1 \le j \le M+s\},$$

and consider an operator $\widehat{\ }\colon \mathcal{F}(D) \to \mathcal{F}(D_{r,s})$ that each image $I \in \mathcal{F}(D)$ extends to an image $\widehat{I} \in \mathcal{F}(D_{r,s})$ such that $\widehat{I}(i,j) = I(i,j)$ for any $(i,j) \in D$. The extension of I for pixels from $D_{r,s} \setminus D$ can be adjusted in different ways according to the given task. For example, we can consider the following extending operators:

(i) $\widehat{I}(i,j) = I(i,j)$, for $(i,j) \in D$, and $\widehat{I}(i,j) = \top$, otherwise, which is used in dilation,

(ii) $\widehat{I}(i,j) = I(i,j)$, for $(i,j) \in D$, and $\widehat{I}(i,j) = \bot$, otherwise, which is used in erosion,

(iii) $\widehat{I}(i,j) = I(i',j')$, where

$$i' = 2 \cdot \max(1, \min(N, i)) - i \quad \text{and} \quad j' = 2 \cdot \max(1, \min(M, j)) - j.$$

It is easy to see that $i' = i$ and $j' = j$, whenever $1 \le i \le N$ and $1 \le j \le M$, therefore, so the extension in case (iii) is well defined, where the grayscale intensity in the new pixels is mirrored across the edges.

Image filtering Filtering is a technique for adjusting or enhancing an image. For example, we can filter an image to emphasize certain elements or remove other elements. Filtering is a neighborhood operation, in which the value of any given pixel in the output image is determined by applying some algorithm to the values of the pixels in the neighborhood of the corresponding pixel in the input image. Our approach based on the lattice integral transforms provides a class of non-linear filters which includes some of the known filters as median filter, or minimum and maximum filter.

The *M⋆–LIT–filter* for images in $\mathcal{F}(D)$ is defined as an M⋆–lattice integral transform $F^\star_{(K_W,\mu)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D)$, where $K_W : D_{r,s} \times D \to [0,1]$ is an integral kernel determined by a widow W of size $R \times S$ given by

$$K_W((i,j),(i',j')) = \begin{cases} W(i'-i, j'-j), & |i'-i| \le r \text{ and } |j'-j| \le s, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

and μ is a fuzzy measure defined on $\langle D_{r,s}, \mathcal{P}(D_{r,s})\rangle$. In addition, we assume that K_W and μ are adjusted in such a way that $F^{\star}_{(K_W,\mu)}$ preserves constant functions, see Theorem 3.7. By Example 3.4, we select $\mu \in \mathcal{M}^r_u$ for $\star = \otimes$ and $\mu^{c,N} \in \mathcal{M}^{r,c,N}_u$ for $\star = \rightarrow$.

One can see that image filtering is provided by aggregation based on a Sugeno-like integral applied on the values in specific neighborhoods that are adjusted by the weights in the window W. More precisely, for any $(i', j') \in D$, a neighborhood in $D_{r,s}$ is determined as follows:

$$N(i', j') = \{(i' + k, j' + \ell) \mid -r \leq k \leq r, -s \leq \ell \leq s\}$$

collecting positions of pixels that are actually processed. The calculation of the output pixel value at position $(i', j') \in D$ is given by Corollary 2.9 as follows:

$$F^{\star}_{(K_W,\mu)}(\widehat{I})(i', j') = \bigvee_{k\in[p]} \left(\widehat{I}_{\sigma(k)} \star K_W(\sigma(k), (i', j'))\right) \otimes \mu_k,$$

where $p = \#N(i', j')$, $[p] = \{1, \ldots, p\}$ and $\sigma : [p] \rightarrow N(i', j')$ is a bijection such that

$$\widehat{I}_{\sigma(1)} \star K_W(\sigma(1), (i', j')) \leq \widehat{I}_{\sigma(2)} \star K_W(\sigma(2), (i', j')) \leq \cdots \leq \widehat{I}_{\sigma(n)} \star K_W(\sigma(n), (i', j'))$$

with $\widehat{I}_{\sigma(k)} = \widehat{I}(\sigma(k))$ and $\mu_k = \mu(\{\sigma(k), \ldots, \sigma(n)\})$. Hence, the procedure of calculation of image filtering is very simple and fast.

By setting of the window (integral kernel), an operation $\star \in \{\otimes, \rightarrow\}$, and a fuzzy measure μ on $\langle D_{r,s}, \mathcal{P}(D_{r,s})\rangle$, we can determine various types of non-linear filters. Assume that the window W consists of weights $w_{i,j} \in \{0, 1\}$ for any $i \in [-r, r]$ and $j \in [-s, s]$. In Table 6.1, we display the operation $\star$ and the fuzzy measure μ in Example 2.8 specifying the integral transform $F^{\star}_{(K_W,\mu)}$ that determine the classical non-linear filters. Note that the weighted median could also be introduced within the

Filters	**Type of M–LIT** ($\star$)	**Fuzzy measures** ($\mu^p_{L,U} = \mu^r_{\varphi^p_{L/n,U/n}}$)
Standard median	$\otimes$	$\mu^1_{(R\times S)/2,(R\times S)/2}$
Minimum	$\rightarrow$	$\mu^1_{R\times S,R\times S} = \mu^{\perp}$
Maximum	$\otimes$	$\mu^1_{0,0} = \mu^{\top}$

Table 6.1: The classical types of non-linear filters.

framework of integral transform, but the definition is not straightforward, because the window used for the weighted median contains natural numbers that determine the repetition of pixels in the window from which the median is calculated (see, [1]). A solution of this task is to extend the input image domain in a suitable way to respect the repetition of pixels according to the weights in the window and define the weighted median as a lattice integral transform of the images with the extended domain to the original domain with the integral kernel that connects the positions of the pixels according to the repetitions in the window. It should be noted that the specific choice of operation $\star$ has no influence on the result, because the weights

are only 0 and 1, and $\star$ for 0 and 1 always give the same results regardless of the specific operation.

Similarly, the *$R^\star_{DH}$–LIT–filter* is defined as an $\mathrm{R}^\star_{\mathrm{DH}}$–lattice integral transform $G^\star_{(K_W,\nu)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D)$, where K_W is the same integral kernel as in the previous case, and ν is a complementary fuzzy measure on $\langle D_{r,s}, \mathcal{P}(D_{r,s})\rangle$ such that constant functions are reversed, see Theorem 3.13 and Remark 3.3. In contrast to the M*–LIT–filter, the $\mathrm{R}^\star_{\mathrm{DH}}$–LIT–filter provides a negative output image (see, images (c) and (d) in Figure 6.4). The calculation of the output pixel values is again simple and fast and is given by Corollary 2.14 as follows:

$$G^\star_{(K_W,\nu)}(\widehat{I})(i',j') = \bigwedge_{k\in[p]} \left(\widehat{I}_{\sigma(k)} \star K_W(\sigma(k),(i',j'))\right) \to \nu_k,$$

where $p = \#N(i',j')$, $[p] = \{1,\dots,p\}$ and $\sigma : [p] \to N(i',j')$ is a bijection such that

$$\widehat{I}_{\sigma(1)} \star K_W(\sigma(1),(i',j')) \le \widehat{I}_{\sigma(2)} \star K_W(\sigma(2),(i',j')) \le \dots \le \widehat{I}_{\sigma(n)} \star K_W(\sigma(n),(i',j'))$$

with $\widehat{I}_{\sigma(k)} = \widehat{I}(\sigma(k))$ and $\nu_k = \nu(\{\sigma(k),\dots,\sigma(n)\})$.

Finally, the *$R^\star_{DPR}$–LIT–filter* is defined as an $\mathrm{R}^\star_{\mathrm{DPR}}$–lattice integral transform $H^\star_{(K_W,\mu)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D)$, where K_W is the same integral kernel as in the previous case and μ is a fuzzy measure on $\langle D_{r,s}, \mathcal{P}(D_{r,s})\rangle$ such that constant functions are preserved, see Theorem 3.18 and subsequent discussion. Similarly to the M*–LIT–filter and $\mathrm{R}^\star_{\mathrm{DH}}$–LIT–filter, due to Corollary 2.18 the simple and fast calculation of the output pixel values is given by

$$H^\star_{(K_W,\mu)}(\widehat{I})(i',j') = \bigwedge_{k\in[p]} \left(\mu^{c,N}_k \to \widehat{I}_{\sigma(k)} \star K_W(\sigma(k),(i',j'))\right),$$

where $p = \#N(i',j')$, $[p] = \{1,\dots,p\}$ and $\sigma : [p] \to N(i',j')$ is a bijection such that

$$\widehat{I}_{\sigma(1)} \star K_W(\sigma(1),(i',j')) \ge \widehat{I}_{\sigma(2)} \star K_W(\sigma(2),(i',j')) \ge \dots \ge \widehat{I}_{\sigma(n)} \star K_W(\sigma(n),(i',j'))$$

with $\widehat{I}_{\sigma(k)} = \widehat{I}(\sigma(k))$ and $\mu^{c,N}_k = \mu^{c,N}(\{\sigma(k),\dots,\sigma(n)\})$.

Image compression Image compression is a technique for reducing image size. Similarly to image filtering, we introduce the *M*–LIT–image compression* as an M*–lattice integral transform $F^\star_{(K_W,\mu_1)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D_\varrho)$, where $\varrho > 1$ is the shift, $K_W : D_{r,s} \times D_\varrho \to [0,1]$ is an integral kernel determined by a widow W of size $R \times S$ given by

$$K_W((i,j),(i',j')) = \begin{cases} W((i'-1)\varrho - i, (j'-1)\varrho - j), & |(i'-1)\varrho - i| \le r \\ & \text{and } |(j'-1)\varrho - j| \le s, \\ 0, & \text{otherwise}, \end{cases} \tag{6.2}$$

and μ_1 is a fuzzy measure defined on $\langle D_{r,s}, \mathcal{P}(D_{r,s})\rangle$. Again, we assume that K_W and μ_1 are adjusted in such a way that $F^\star_{(K_W,\mu_1)}$ preserves constant functions. The

procedure of calculation of image compression is performed in the same way as for image filtering, only the neighborhood in $D_{r,s}$ for $(i',j') \in D_\varrho$ is determined as

$$N_\varrho(i',j') = \{((i'-1)\varrho + 1 + k, (j'-1)\varrho + 1 + \ell) \mid -r \le k \le r,\ -s \le \ell \le s\}.$$

Obviously, $N = N_\varrho$ for $\varrho = 1$.

The $R^\star_{DH}$*–LIT–image compression* is defined analogously as an $\mathrm{R}^\star_{\mathrm{DH}}$–lattice integral transform $G^\star_{(K_W,\nu_1)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D_\varrho)$, where K_W is the same integral kernel as in the previous case, and ν_1 is a complementary fuzzy measure on $\langle D_{r,s}, \mathcal{F}(D_{r,s})\rangle$ such that constant functions are reversed.

Finally, the $R^\star_{DPR}$*–LIT–image compression* is defined as an $\mathrm{R}^\star_{\mathrm{DPR}}$–lattice integral transform $H^\star_{(K_W,\mu_1)} : \mathcal{F}(D_{r,s}) \to \mathcal{F}(D_\varrho)$, where K_W is the same integral kernel as in the previous case, and μ_1 is a fuzzy measure on $\langle D_{r,s}, \mathcal{F}(D_{r,s})\rangle$ such that constant functions are preserved due to Theorem 3.18.

Image decompression Conversely to image compression, the image decompression is used to reconstruct the original image from its compression. To introduce image decompression, in the first step, we extend the domain D_ϱ to the domain $D_{\varrho,u,v}$, where u denotes the integer part of r/ϱ and similarly v denotes the integer part of s/ϱ. To better understand our motivation for the definition of extension, let us consider the situation $\varrho = r = s$, i.e., $u = 1 = v$. For any pixel position $(i',j') \in D_\varrho$ of output images, the neighborhood $N_\varrho(i',j')$ of the pixel at the position $((i'-1)\varrho + 1, (j'-1)\varrho + 1)$ in $D_{r,s}$, over which the calculation is provided, contains positions $((i'+a-1)\varrho+1, (j'+b-1)\varrho+1)$ for $-u \le a \le u$ and $-v \le b \le v$, where $(i'+a, j'+b) \notin D_\varrho$ can occur in general. So, once we use the pixel values at positions $((i'+a-1)\varrho + 1, (j'+b-1)\varrho + 1)$ to calculate image compression, it seems reasonable to use pixel values at positions $(i'+a, j'+b)$ into account for reconstructions of compressed images.

Assume that the $\mathrm{M}^\star$–LIT–image compression with the ratio $\varrho^2 : 1$ is realized by $F^\star_{(K_W,\mu_1)}$ for $\star \in \{\otimes, \to\}$, and denote $\bar\star$ the adjoined operation to $\star$, e.g., if $\star = \otimes$, then $\bar\star = \to$. The $M^\star$*–LIT–image decompression* is introduced as an $\mathrm{M}^\star$–lattice integral transform $F^{\bar\star}_{(K_W^{-1},\mu_2)} : \mathcal{F}(D_{\varrho,u,v}) \to \mathcal{F}(D)$, where $K_W^{-1} : D_{\varrho,u,v} \times D \to [0,1]$ is the integral kernel determined by a widow W of size $R \times S$ given by

$$K_W^{-1}((i',j'),(i,j)) = \begin{cases} W((i'-1)\varrho - i, (j'-1)\varrho - j), & |(i'-1)\varrho - i| \le r \\ & \text{and } |(j'-1)\varrho - j| \le s, \\ 0, & \text{otherwise}, \end{cases} \tag{6.3}$$

μ_2 is a fuzzy measure defined on $\langle D_{\varrho,u,v}, \mathcal{P}(D_{\varrho,u,v})\rangle$. In addition, we assume that K_W^{-1} and μ_2 are adjusted in such a way that $F^{\bar\star}_{(K_W^{-1},\mu_2)}$ preserves constant functions.

It is easy to see that the integral kernel K_W^{-1} is the inverse to K_W if we restrict ourselves to original domains D and D_ϱ, i.e., $K_W^{-1}((i',j'),(i,j)) = K_W^T((i',j'),(i,j))$ for any $(i,j) \in D$ and $(i',j') \in D_\varrho$.

The $R^\star_{DH}$*–LIT–image decompression* is defined analogously as an $\mathrm{R}^\star_{\mathrm{DH}}$–lattice integral transform $G^{\bar\star}_{(K_W^{-1},\nu_2)} : \mathcal{F}(D_{\varrho,u,v}) \to \mathcal{F}(D)$, where K_W^{-1} is the same integral kernel

as in the previous case, and ν_2 is complementary fuzzy measure on $\langle D_{\varrho,u,v}, \mathcal{F}(D_{\varrho,u,v})\rangle$ such that constant functions are reversed. It should be noted that the decompression of the negative image, which is a result of the compression procedure, we get again a positive image, as demonstrated in Figure 6.5.

Finally, the $R^{\star}_{DPR}$*–LIT–image decompression* is defined as an $\mathrm{R}^{\star}_{\mathrm{DPR}}$–lattice integral transform $H^{\bar{\star}}_{(K_W^{-1},\mu_2)} : \mathcal{F}(D_{\varrho,u,v}) \to \mathcal{F}(D)$, where K_W^{-1} is the same integral kernel as in the previous case, and μ_2 is a fuzzy measure on $\langle D_{\varrho,u,v}, \mathcal{F}(D_{\varrho,u,v})\rangle$ such that constant functions are preserved due to Theorem 3.18.

Opening and closing Opening and closing are two important morphological operators. They are both derived from the fundamental operations of erosion and dilation, namely, the opening is defined as an erosion followed by a dilation, and vice versa for closing. Opening is generally used to restore the original image to the maximum possible extent. It eliminates the thin protrusions of the obtained image and is also used to remove internal noise. Closing is generally used to smooth the contour of the distorted image and fuse back the narrow breaks and long thin gulfs. It is also used to remove the small holes in the obtained image.

In our case, we consider the opening and closing defined by fuzzy morphological erosion and dilation, which correspond to the direct lower and upper lattice fuzzy transforms, respectively, as shown in [42]. The fuzzy morphological erosion (dilation) is defined in a similar way as the minimum (maximum) filter introduced in Table 6.1. More precisely, the fuzzy morphological erosion is the $(K_W, \mu^{\perp}, \rightarrow)$–M–lattice integral transform from $\mathcal{F}(D_{r,s})$ to $\mathcal{F}(D)$, where $\mu^{\perp}$ denotes the least fuzzy measure on the powerset $\mathcal{P}(D_{r,s})$ and the window W consists of arbitrary weights from $[0,1]$, as described in Subsection 6.2. The fuzzy morphological dilation is the $(K_W, \mu^{\top}, \otimes)$–M–lattice integral transform from $\mathcal{F}(D_{r,s})$ to $\mathcal{F}(D)$, where $\mu^{\top}$ denotes the highest fuzzy measure on the powerset $\mathcal{P}(D_{r,s})$ and again the window W consists of arbitrary weights from $[0,1]$. However, the lattice integral transforms provide an opportunity to generalize the fuzzy morphological erosion (dilation) so that instead of the least (highest) fuzzy measure, we can consider fuzzy measures that are close (but not equal) to the least (highest) fuzzy measure. The combination of more general fuzzy morphological operations introduces a generalization of opening and closing. More precisely, a generalized opening (M–LIT–opening) operation is obtained as the composition of $(K_W, \mu, \rightarrow)$–M–lattice integral transform (M–LIT–erosion) and $(K_W, \mu', \otimes)$–M–lattice integral transform (M–LIT–dilation) which are set to preserve constant functions and μ is close to $\mu^{\perp}$ and μ' to $\mu^{\top}$. The reverse composition of the previous lattice integral transforms leads to a generalized closing (M–LIT–closing) operation. Other alternatives of opening and closing can be obtained by applying R_{DH}–LIT (also $\mathrm{R}_{\mathrm{DPR}}$–LIT).

6.3 Filtering, compression/decompression, opening/closing

In this part, we illustrate our method based on M–LIT, R_{DH}–LIT, $\mathrm{R}_{\mathrm{DPR}}$–LIT for lattice-valued functions. We do not have the ambition to present results that surpass current approaches, but we want to show that integral transforms provide an extension of selected methods with a wide space for setting parameters that can be

used to solve various tasks in image processing. We believe that certain parameter settings could provide interesting alternatives to popular techniques such as the median, minimum or maximum filter, opening and closing, and the M–LIT (R_{DH}–LIT and $\mathrm{R}_{\mathrm{DPR}}$–LIT) can be used to introduce other useful types of filters and morphological operators. However, a detailed analysis is beyond the scope of this thesis, and we leave it for our future work.

To illustrate, we assume the complete residuated lattices determined by continuous t-norms from the Schweizer-Sklar class of t-norms T_λ^{SS} in Example 1.2 and negations $N = N_\lambda^{SS}$ determined by the residuum $\rightarrow_{T_\lambda^{SS}}$ for $\lambda > 0$ in Example 1.5. Note that N_λ^{SS} are involutive for $\lambda > 0$, i.e., $N_\lambda^{SS} \circ N_\lambda^{SS} = id_{[0,1]}$, where $id_{[0,1]}$ denotes the identity function on $[0,1]$. In addition, we consider a fuzzy measure $\mu = \mu_{L,U}^{p} = \mu^{r}_{\varphi^{p}_{L/n,U/n}}$ for the $(K_W, \mu, \otimes)$–M–lattice integral transform and the N–conjugate fuzzy measure $\mu^{c,N} = \mu_{L,U}^{p,c,N} = \mu^{r}_{\varphi^{p,c,N}_{L/n,U/n}}$ to μ for the $(K, \mu^{c,N}, \rightarrow)$ –M–lattice integral transform (see, Examples 2.8 and 2.9 and Remark 2.2). Further, we consider the complementary fuzzy measure $\nu = \mu_{L,U}^{p,N} = \mu^{r}_{\varphi^{p,N}_{L/n,U/n}}$ for the $(K, \nu, \otimes)$–R_{DH}–lattice integral transform and the N–conjugate complementary fuzzy measure $\nu^{c,N} = \mu_{L,U}^{p,c} = \mu^{r}_{\varphi^{p,c}_{L/n,U/n}}$ for the $(K, \nu^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DH}}$–lattice integral transform (see, Remark 3.3). Thus, all types of fuzzy measures are determined from the fuzzy measure μ. Finally, we consider the fuzzy measure μ and the N–conjugate fuzzy measure $\mu^{c,N}$ for the $(K_W, \mu, \otimes)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform and the $(K_W, \mu^{c,N}, \rightarrow)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transform, respectively.

Image filtering For illustration, we consider the Cameraman image (256x256) with 30% and 40% of salt-and-pepper noise, see Figure 6.2(b) and Figure 6.3(b), where we assume that the salt-and-pepper noise is in the ratio 2:1 and 3:1, respectively. The reason for the non-uniform distribution of salt and pepper noise is to show that M–LIT–filters (R_{DH}–LIT–filters, $\mathrm{R}_{\mathrm{DPR}}$–LIT–filters) provide a more efficient way to reduce noise due to greater parameter flexibility than the standard median filter and its combination with minimum filter (maximum filter). Note that the median filter provided the best solution in case of the uniform distribution of salt and pepper noise in our experiment. To compare the results of M–LIT–filters (R_{DH}–LIT–filters, $\mathrm{R}_{\mathrm{DPR}}$–LIT–filters) with the median filter approach, we use the same window of size 3x3 with all weights equal to 1. Furthermore, we consider $\lambda = 1$, which specifies the t norm, residuum, and negation used. The fuzzy measure is the crucial parameter in our experiment, and its setting will be specified for each result of M–LIT–filters (R_{DH}–LIT–filters, $\mathrm{R}_{\mathrm{DPR}}$–LIT–filters).

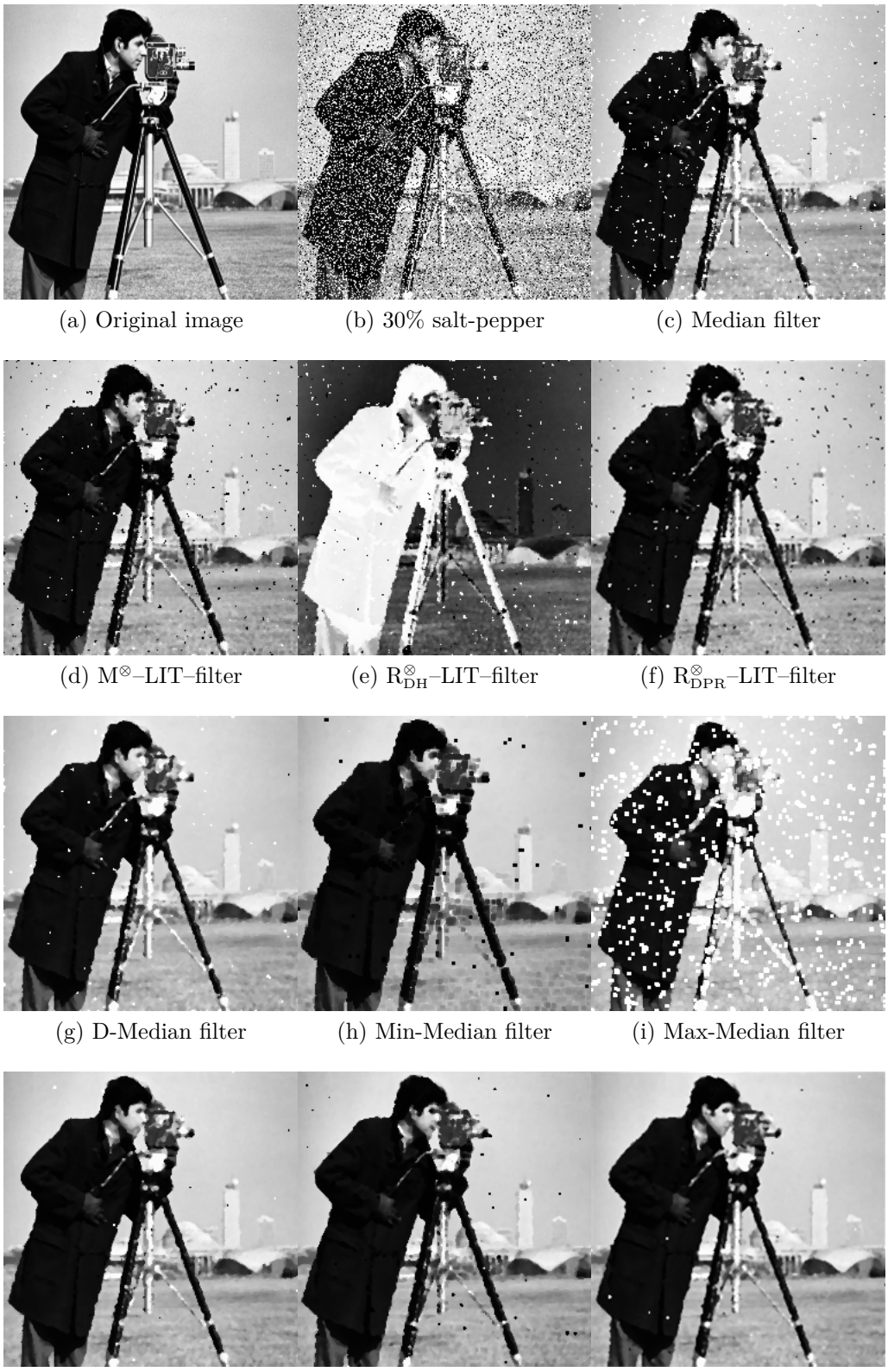
The filtering results of 30% salt-and-pepper noise (2:1 ratio) for different types of filters are shown in Figure 6.2. In our demonstration, we consider the application of all three filters in succession to demonstrate the effect of the composition of M–LIT–filters (R_{DH}–LIT–filters, $\mathrm{R}_{\mathrm{DPR}}$–LIT–filters) determined by the multiplication and residuum. In Figure 6.2(c-f), we can see the results of the median filter, $\mathrm{M}^{\otimes}$–LIT–filter, $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter and $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter, where the respective (complementary, N–conjugate) fuzzy measures are determined from the fuzzy measure $\mu = \mu^{5}_{5,6}$. By adjusting the fuzzy measure μ, we can remove more salt noise, see

Figure 6.2(d,f), compared to the median filter, see Figure 6.2(c), with the presence of a higher proportion of pepper noise. The negative image as the result of the $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter seems unnecessary at first glance, especially if we want to work with it immediately without further processing. In Figure 6.2(g-l), we can see the results of combinations of two filters. Particularly, we use the double application of the median filter (D-Median filter) and the application of the median filter and then the minimum filter (Min-Median filter) and the maximum filter (Max-Median filter). Further, the composition of the M–LIT–filters with the multiplication and residuum ($\mathrm{M}^{\rightarrow}\circ\ \mathrm{M}^{\otimes}$–LIT–filter), where the $\mathrm{M}^{\rightarrow}$–LIT–filter with the N–conjugate fuzzy measure $\mu^{c,N}$ derived from $\mu^{1}_{4.5,4.5}$ (the same fuzzy measure as for the median filter, see, Table 6.1) is applied on the result of the $\mathrm{M}^{\otimes}$–LIT–filter shown in Figure 6.2(d). The composition of the R_{DH}–LIT–filters with the multiplication and residuum ($\mathrm{R}^{\rightarrow}_{\mathrm{DH}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter), where the $\mathrm{R}^{\rightarrow}_{\mathrm{DH}}$–LIT-filter with the N–conjugate complementary fuzzy measure $\nu^{c,N}$ derived from $\mu = \mu^{5}_{5,6}$ is applied on the result of the $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT-filter shown in Figure 6.2(e). Finally, the composition of the $\mathrm{R}_{\mathrm{DPR}}$–LIT–filters with the multiplication and residuum ($\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter), where the $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$–LIT–filter with the N–conjugate fuzzy measure to μ is applied on the result of the $\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter shown in Figure 6.2(f). Visual comparison of the results shows that the best filtering is provided by the $\mathrm{M}^{\rightarrow}\circ\ \mathrm{M}^{\otimes}$–LIT–filter in Figure 6.2(j). This claim is also underlined by the highest PSNR among others in Table 6.2.

The filtering results of 40% salt-and-pepper noise (3:1 ratio) for different filters are displayed in Figure 6.3. We consider the same filters as in the previous case. The $\mathrm{M}^{\otimes}$–LIT–filter has the same fuzzy measure as above. For the $\mathrm{M}^{\rightarrow}\circ\ \mathrm{M}^{\otimes}$–LIT–filter, we consider the conjugate fuzzy measure $\mu^{c,N}$ derived from $\mu = \mu^{5}_{5,6}$ in the setting of $\mathrm{M}^{\rightarrow}$–LIT–filter, which is applied on the result of the $\mathrm{M}^{\otimes}$–LIT–filter in Figure 6.3(d). Again, the $\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter ($\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter) and $\mathrm{R}^{\rightarrow}_{\mathrm{DH}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter ($\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter) have the same setting as in the previous case. The $\mathrm{R}^{\rightarrow}_{\mathrm{DH}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter provides the best result both visually and supported by the highest PSNR, as seen in Table 6.2.

Filters	**PSNR for 30% noise (dB)**	**PSNR for 40% noise (dB)**
Median filter	18.0779	13.5977
$\mathrm{M}^{\otimes}$–LIT filter	18.8865	16.7818
$\mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter	3.2306	3.2385
$\mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter	18.6335	16.579
D-Median filter	20.4318	15.734
Min-Median filter	16.4346	16.9958
Max-Median filter	11.9095	8.2982
$\mathrm{M}^{\rightarrow}\circ\ \mathrm{M}^{\otimes}$–LIT–filter	**21.365**	18.9119
$\mathrm{R}^{\rightarrow}_{\mathrm{DH}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DH}}$–LIT–filter	19.1213	**19.4209**
$\mathrm{R}^{\rightarrow}_{\mathrm{DPR}}\circ\ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT–filter	20.9759	18.5289

Table 6.2: PSNR for different methods of filtering of salt-and-pepper noise.



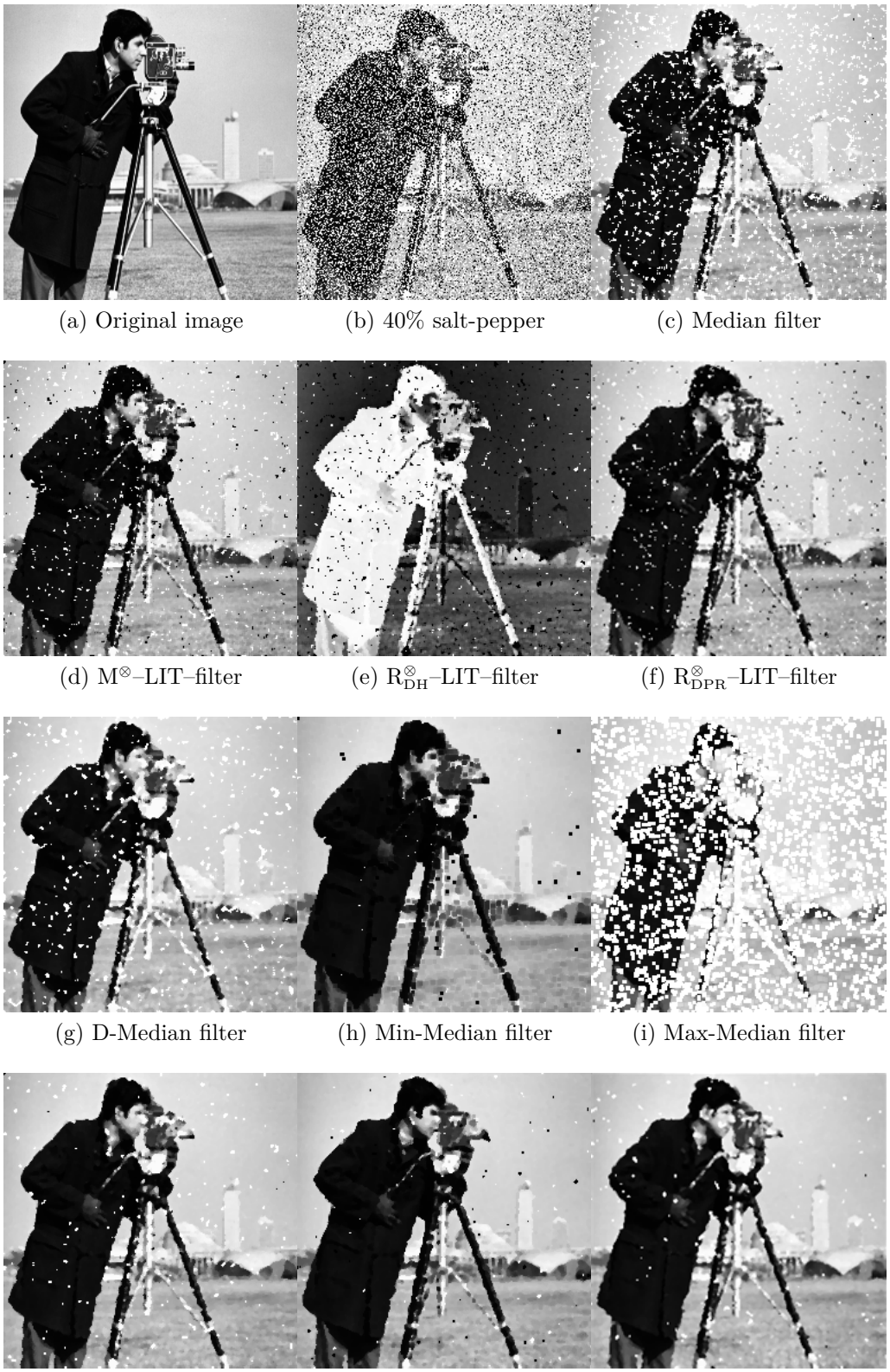
(a) Original image (b) 30% salt-pepper (c) Median filter

(d) $M^{\otimes}$–LIT–filter (e) $R^{\otimes}_{DH}$–LIT–filter (f) $R^{\otimes}_{DPR}$–LIT–filter

(g) D-Median filter (h) Min-Median filter (i) Max-Median filter

(j) $M^{\rightarrow} \circ M^{\otimes}$–LIT filter (k) $R^{\rightarrow}_{DH} \circ R^{\otimes}_{DH}$–LIT filter (l) $R^{\rightarrow}_{DPR} \circ R^{\otimes}_{DPR}$–LIT filter

Figure 6.2: Filtering Cameraman image with 30% salt-and-pepper noise (2:1 ratio) using standard filters and new filters based on $M^{\star}$–LIT filter, $R^{\star}_{DH}$–LIT filter, and $R^{\star}_{DPR}$–LIT filter with the window 3x3.



(a) Original image (b) 40% salt-pepper (c) Median filter

(d) $M^{\otimes}$–LIT–filter (e) $R^{\otimes}_{DH}$–LIT–filter (f) $R^{\otimes}_{DPR}$–LIT–filter

(g) D-Median filter (h) Min-Median filter (i) Max-Median filter

(j) $M^{\rightarrow} \circ M^{\otimes}$–LIT filter (k) $R^{\rightarrow}_{DH} \circ R^{\otimes}_{DH}$–LIT filter (l) $R^{\rightarrow}_{DPR} \circ R^{\otimes}_{DPR}$–LIT filter

Figure 6.3: Filtering Cameraman image with 40% salt-and-pepper noise (3:1 ratio) using standard filters and new filters based on the $M^{\star}$–LIT filter, $R^{\star}_{DH}$–LIT filter, and $R^{\star}_{DPR}$–LIT filter with window 3x3.

To summarize results, the filters based on (R_{DH}–LIT, R_{DPR}–LIT) M–LIT seem to be useful in filtering non-uniform salt-and-pepper noise from images. For the sake of comparison, the only parameter here was the fuzzy measure whose setting improves the results for the median filter. The further development of more sophisticated filters based on (R_{DH}–LIT, R_{DPR}–LIT) M–LIT is the subject of future research.

Image compression and decompression For illustration of this part, we use the Lena image (512x512) and the compression ratio 4:1, i.e., the shift is $\varrho = 2$. We consider the window of size 7x7 with the weights equal to 1 around the center and other less than 1, more specifically, $w_{ij} = 1$ for $i, j \in [-2, 2]$ and $w_{ij} < 1$, otherwise. Here, we use different values less than 1 for different λ. The respective integral kernel is denoted by K. Further, we consider $\lambda \in \{0.5, 1, 2\}$, which specify the used operations.

For compression, we apply the fuzzy measure $\mu_1 = \mu^6_{18,24}$, and similarly to image filtering, the remaining fuzzy measures are derived from μ_1 as follows: $\mu_1^{c,N}$ is the N–conjugate fuzzy measure to μ_1, $\nu_1 = \mu_1^N$ is the complementary fuzzy measure of μ_1, and $\nu_1^{c,N}$ is the N–conjugate complementary fuzzy measure to ν_1. The results of M–LIT–image compression of the Lena image for different settings of λ are shown in Figures 6.4(a-b). The negative images as the results of R_{DH}–LIT–image compression of the Lena image are then shown in Figures 6.4(c-d). Finally, the results of R_{DPR}–LIT–image compression of Lena image are shown in Figure 6.4(f-g). On compressed images we can observe that a higher value of the parameter λ makes the $M^{\otimes}$–LIT–image compression darker, while the opposite effect appears for the $M^{\rightarrow}$–LIT–image compression. Similar observations can be found for other types of compression.

For decompression, we consider the fuzzy measure $\mu_2 = \mu^1_{1,2}$, which is close to the highest fuzzy measure on $\mathcal{P}(D_{\varrho,u,v})$. The remaining fuzzy measures are derived from μ_2 in the same way as for μ_1. We chose μ_2 because it experimentally provided the best decompression. The results of decompression of compressed Lena image in Figure 6.4 for different settings are shown in Figure 6.5. A comparison of all image decompression results with the original Lena image using PSNR is shown in Table 6.3. We can see that a good result of LIT–image decompression can be provided by the R_{DH}–LIT–image decompression in case of $R^{\otimes}_{DH} \circ R^{\rightarrow}_{DH}$ with $\lambda = 2$.

Type of LIT-decompression	**PSNR**($\lambda = 0.5$)	**PSNR**($\lambda = 1$)	**PSNR**($\lambda = 2$)
$M^{\rightarrow} \circ M^{\otimes}$	26.3666	23.7756	22.3109
$M^{\otimes} \circ M^{\rightarrow}$	24.069	25.9264	23.7151
$R^{\rightarrow}_{DH} \circ R^{\otimes}_{DH}$	26.1738	26.6169	26.5364
$R^{\otimes}_{DH} \circ R^{\rightarrow}_{DH}$	24.3746	25.9614	**28.6796**
$R^{\rightarrow}_{DPR} \circ R^{\otimes}_{DPR}$	25.6844	25.6013	24.4589
$R^{\otimes}_{DPR} \circ R^{\rightarrow}_{DPR}$	24.7239	24.0233	21.4154

Table 6.3: PSNR for the decompression of Lena image using M–LIT decompression, R_{DH}–LIT decompression, and R_{DPR}–LIT decompression with a window 7x7.

(a) $M^{\otimes}$–LIT compression.

(b) $M^{\rightarrow}$–LIT compression.

(c) $R^{\otimes}_{DH}$–LIT compression.

(d) $R^{\rightarrow}_{DH}$–LIT compression.

continued figure

(e) $R^{\otimes}_{DPR}$–LIT compression

(f) $R^{\rightarrow}_{DPR}$–LIT compression

Figure 6.4: Compression of Lena image with ratio of 4:1 using M–LIT compression, R_{DH}–LIT compression, and R_{DPR}–LIT compression with a window 7x7 for various operations determined by $\lambda \in \{0.5, 1, 2\}$ where $\lambda = 0.5$(left), $\lambda = 1$(middle), and $\lambda = 2$(right).

(a) Original Lena

(b) $M^{\rightarrow} \circ M^{\otimes}$–LIT decompression.

(c) $M^{\otimes} \circ M^{\rightarrow}$–LIT decompression.

(d) $R^{\rightarrow}_{DH} \circ R^{\otimes}_{DH}$–LIT decompression.

continued figure

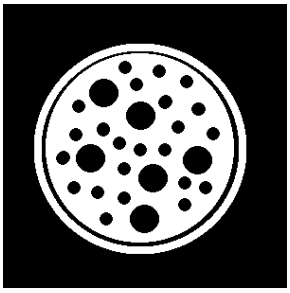
(e) $\mathrm{R}^{\otimes}_{\mathrm{DH}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DH}}$–LIT decompression.

(f) $\mathrm{R}^{\rightarrow}_{\mathrm{DPR}} \circ \mathrm{R}^{\otimes}_{\mathrm{DPR}}$–LIT decompression.

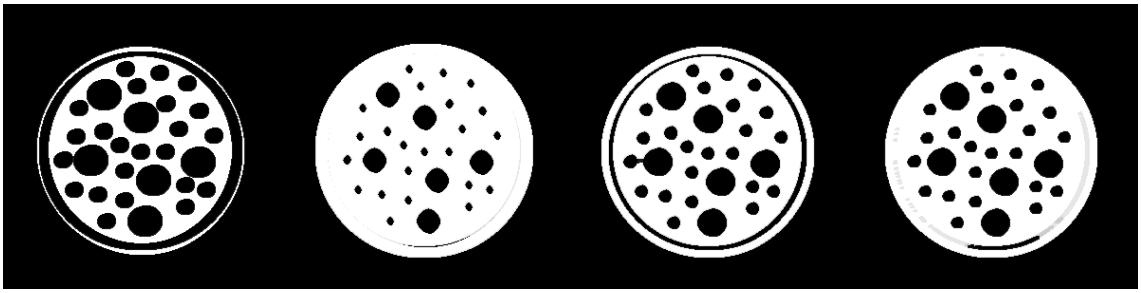
(g) $\mathrm{R}^{\otimes}_{\mathrm{DPR}} \circ \mathrm{R}^{\rightarrow}_{\mathrm{DPR}}$–LIT decompression.

Figure 6.5: Decompression of Lena image, which was previously compressed at a ratio of 4: 1 using M–LIT decompression, R_{DH}–LIT decompression and $\mathrm{R}_{\mathrm{DPR}}$–LIT decompression with a 7x7 window for various operations specified by $\lambda \in \{0, 5, 1, 2\}$, where $\lambda = 0.5$ (left), $\lambda = 1$ (middle) and $\lambda = 2$ (right).

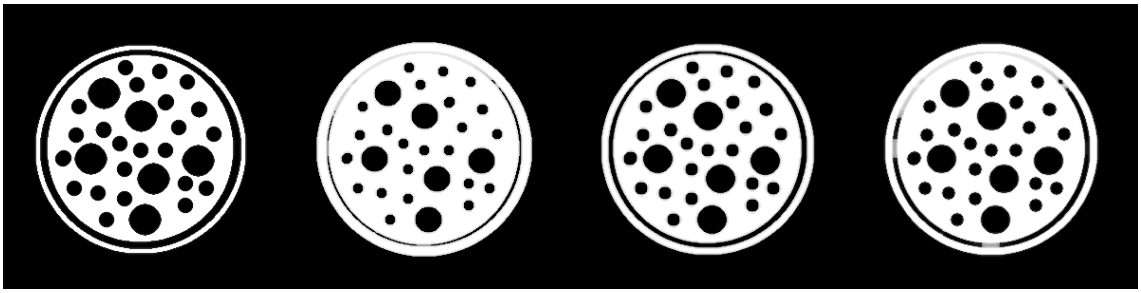
In principle, lattice integral transforms lead to lossy (irreversible) compression, and the question is whether the quality of reconstructed images can be improved by appropriate parameter settings, which is the subject of our future research. However, we see the primary purpose of lattice integral transforms in image filtering demonstrated in the previous paragraph and introducing new types of (morphological) operations, which is the topic of the next paragraph.



(a) 300x300 image



(b) Fuzzy erosion (c) Fuzzy dilation (d) Fuzzy opening (e) Fuzzy closing



(f) M–LIT–erosion (g) M–LIT–dilation (h) M–LIT–opening (i) M–LIT–closing

Figure 6.6: Comparison of fuzzy and M–LIT–morphological operations of erosion, dilation, opening and closing with the use of window of size 5x5.

Opening and closing For the last illustration in this chapter, we use a 300x300 binary image with black balls inside a white circle, which can be seen in Figure 6.6(a). Similarly to compression, we consider the window (structuring element) W of size 5x5 with weights equal to 1 around the center and others small than 1. By setting the window W, the effect of the morphological operations can be seen on the white pixels. As we stated above, in our case, opening and closing operations are fuzzy morphological operations that can be expressed in terms of M–LIT as compositions of fuzzy morphological erosion and dilation.

The results of fuzzy morphological erosion, dilation, opening and closing for the considered image with respect to the given window are shown in Figure 6.6(b-e). For example, we can see that the white space erodes in Figure 6.6(b) and is extended in Figure 6.6(c). For comparison, we consider the M–LIT–dilation defined as the $(K_W, \mu, \otimes)$–M–lattice integral transform with $\mu = \mu^5_{3,7}$, which is close to the highest fuzzy measure $\mu^\top$, and the M–LIT–erosion as the $(K_W, \mu', \rightarrow)$–M–lattice integral transform with $\mu' = \mu^{c,N}$, which is close to $\mu^\perp$. Further, we use the t-norm T^{SS}_λ

and its residuum $\rightarrow_{T_\lambda^{SS}}$ with $\lambda = 1$ in the definitions of M–lattice integral transforms. The M–LIT–opening is defined as the composition of M–LIT–erosion and M–LIT–dilation, and vice verse for the M–LIT–closing. The results of all modified fuzzy morphological operations are displayed in Figure 6.6(f-i). The effect of the newly defined conjugate fuzzy measure μ' in the M–LIT–erosion is obvious and consists in a smaller erosion of white part in the image in contrast to the fuzzy erosion. The opposite effect can be recognized for the M–LIT–dilation defined by the fuzzy measure μ. The M–LIT–opening provides a better restoration of the original image than fuzzy opening, and the M–LIT–closing leaves more of the black circle than the fuzzy closing. Interestingly, morphological operations based on lattice integral transforms better preserves the shape of black balls in image.

Chapter 7

Conclusion

In the thesis, we introduced a theory of integral transforms for functions valued in complete residuated lattice (lattice integral transforms) and showed their usefulness in solving practical problem. The motivation for this theory was the discovery that the lattice fuzzy transforms proposed in [36] and used for the lower and upper approximation of functions can be expressed in the same form as the standard integral transforms, except that a Sugeno-type fuzzy integral is used for integration and the binary fuzzy relation represents the integral kernel. To develop the theory of lattice integral transforms, we employed the three types of Sugeno-like fuzzy integrals, which were established to integrate functions with function values in complete residuated lattices. More specifically, we used a multiplication-based fuzzy integral ($\otimes$–fuzzy integral) and two residuum-based fuzzy integrals ($\rightarrow_{\mathrm{DH}}$–fuzzy integral and $\rightarrow_{\mathrm{DPR}}$–fuzzy integral) proposed in [10, 11]. The basic properties of these fuzzy integrals and computational methods useful for solving practical problems were studied in Chapter 2.

Following the standard scheme, we introduced three types of lattice integral transforms using the fuzzy integrals mentioned above and investigated their basic properties in Chapter 3. It is well-known that the key element of lattice fuzzy transforms is the fuzzy partition of the function domain. Therefore, we provided a representation of the fuzzy partition using the integral kernel which enabled us to prove that the lattice fuzzy transforms are particular cases of lattice integral transforms with respect to the top and bottom fuzzy measures. We also analyzed the sufficient conditions ensuring the preservation (reversation) of constant functions by lattice integral transforms. This property proved to be essential for the successful approximation of the original functions. The theoretical results were demonstrated on signal processing.

The approximation properties of integral lattice transforms were investigated in Chapter 4. In particular, we were interested in the quality of the approximation achieved by the composition of two suitable lattice integral transforms, where we were inspired by the composition of the direct and inverse lattice fuzzy transforms leading to a lower and upper approximation of the original function. For this purpose, we introduced a modulus of continuity for functions valued in a complete residuated lattice. Further, we designed two inverse kernels according to the type of integral transform and thoroughly investigated their properties. The results were used to prove several approximation theorems for each type of integral transform.

An interesting and surprising result is that the composition of lattice integral transforms can even preserve extensional-like functions with respect to a similarity relation. The approximation abilities of the composition of lattice integral transforms were illustrated on a signal without and with noise.

An application of lattice integral transforms to multicriteria decision making was developed in Chapter 5. In particular, we proposed an approach based on the multiplication-based lattice integral transform to evaluate the alternatives in a linearly ordered set endowed by additional operations when the evaluation need not be a single value for each alternative but a vector whose values determine the satisfaction of alternatives with respect to global criteria describing suitable features. We demonstrated our approach on the problem of car selection, where the obtained results are compared with the results of the evaluation of alternatives using the standard weighted average and weighted maximum. Since the lattice integral transform can be seen as an "extended" qualitative aggregation function, our approach provide a tool for qualitative evaluations as opposed to quantitative evaluations based on the weighted average or more general OWA operators.

Other applications of lattice integral transforms in image were shown in Chapter 6. We proposed non-linear filters based on lattice integral transforms that generalize the popular median filter as well as minimum and maximum filters. We also designed a method for image compression/decompression and generalized fuzzy morphological operators of erosion and dilation and derived operators of opening and closing of images. All the new approaches to image processing were illustrated on various selected images for different types and settings of lattice integral transforms and the quality of the results were compared with each other using PSNR.

The proposed theory of lattice integral transforms provides a theoretical background for data processing, especially for data whose values cannot in principle be processed within the arithmetic of real or complex numbers but has a lattice structure. In addition, the proposed tools can be used to develop new "qualitative" methods that are as oppose to standard "quantitative" methods. Although the theory provides some interesting results, it can only be considered as an introduction to lattice integral transforms and motivation for further research. In the future, we plan to focus our attention on studying different types of kernels and their inverses together with their effect on function approximation. For particular complete residuated lattices, we want to investigate the quality of the approximation in more detail and show that some functions defined over a real interval (or more general compact set) can be approximated with arbitrary accuracy. Motivated by examples, we see as a challenge the theoretical analysis of lattice integral transforms applied to signals with noise. Finally, we plan to develop methods based on lattice integral transforms in signal and image processing to provide alternative approaches to solve various tasks such as non-additive noise filtering, image compression and decompression, or image processing using novel and interesting fuzzy morphological operators.

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Nomenclature

$\mathbf{A}_K$ fuzzy partition determined by an integral kernel K, page 37

$\mathrm{Alg}(\mathcal{H})$ the generated algebra by $\mathcal{H}$, page 16

$\mathcal{B}^\ell(L)$ algebra of sets generated by all losets in L, page 17

$\mathcal{B}^u(L)$ algebra of sets generated by all upsets in L, page 16

$\mathcal{F}(X)$ set of all fuzzy sets in X, page 13

$\int^\otimes$ $\otimes$–fuzzy integral, page 23

$\int^\rightarrow_{\mathrm{DH}}$ $\rightarrow_{\mathrm{DH}}$–fuzzy integral, page 27

$\int^\rightarrow_{\mathrm{DPR}}$ $\rightarrow_{\mathrm{DPR}}$–fuzzy integral, page 29

$\mathbf{L}$ residuated lattice, page 7

$\mathbf{L}_{T_{\text{Ł}}}$ Łukasiewicz algebra, page 9

$\mathbf{L}_{T_{\mathrm{G}}}$ Gödel algebra, page 9

$\mathbf{L}_{T_{\mathrm{P}}}$ product algebra, page 9

$\mathcal{L}(L)$ set of all losets in L, page 17

$\mathcal{P}(X)$ set of all crisp fuzzy sets of X, page 13

$\mu^\top$ the highest fuzzy measure, page 18

$\mu^\perp$ the least fuzzy measure, page 18

$\mu^{c,N}$ N–conjugate fuzzy measure, page 18

$\mu^p_{L,U}$ fuzzy measure determined by a the triplet $\langle L, U, p\rangle$ on a finite measurable space, page 20

μ fuzzy measure, page 17

ν complementary fuzzy measure, page 17

$\nu^{c,N}$ N–conjugate complementary fuzzy measure to ν, page 18

$\otimes$ multiplication, page 7

$\preceq$ the ordering of fuzzy measures, page 18

Q^d a dually compatible integral, page 67

$\mathrm{Core}(A)$ core of A, page 14

$\mathrm{Supp}(A)$ support of A, page 13

$\mathcal{U}(L)$ set of all upsets in L, page 16

$\vee$ maximum, page 11

$\wedge$ minimum, page 11

$\perp$ the least element, page 7

A_a a-cut of A, page 14

$F^{\star}_{(K,\mu)}$ $(K,\mu,\star)$–M–lattice integral transform, page 39

$G^{\star}_{(K,\nu)}$ $(K,\nu,\star)$–R_{DH}–lattice integral transform, page 50

$H^{\star}_{(K,\mu)}$ $(K,\mu,\star)$–$\mathrm{R}_{\mathrm{DPR}}$–lattice integral transforms, page 56

K an integral kernel, page 37

K^T a transpose of fuzzy relation K, page 14

K^{-1} a Q–inverse of K, page 65

K_x x–projection of a fuzzy relation K to Y, page 14

K_y y–projection of a fuzzy relation K to X, page 14

N generalized negation, page 10

Q a compatible integral, page 63

T^{SS} Schweizer-Sklar class of t-norms, page 9

X universe of discourse, page 13

$\top$ the greatest element, page 7

$\rightarrow$ residuum, page 7

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