UNIVERSITY OF OSTRAVA
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

A FORMAL THEORY OF INTERMEDIATE GENERALIZED QUANTIFIERS

Ph.D. THESIS

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2011

OSTRAVSKÁ UNIVERZITA V OSTRAVĚ
PŘÍRODOVĚDECKÁ FAKULTA
KATEDRA MATEMATIKY

FORMÁLNÍ TEORIE ZOBECNĚNÝCH INTERMEDIÁLNÍCH KVANTIFIKÁTORŮ

DOKTORSKÁ DISERTAČNÍ PRÁCE

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2011

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Summary

Fuzzy logic is a mathematical discipline which was started to be applied at the end of the 80's and at the beginning of the 90's of the last century. Nowadays in the rush world almost everybody has a chance to meet "fuzzy" or "sixth sense" in a washing machine, camera, motor vehicle, etc. The most successful applications were carried out in the area of decision-making and fuzzy regulation. What is the key for such success of fuzzy logic? The main source is the fact that fuzzy logic enables to include vagueness and uncertainty. That is why fuzzy logic is important for many areas of human activities and new possibilities of its use still arise. One of the activities is also the use of IF - THEN rules for decision-making and regulation, which are established on the mentioned expressions of a natural language.

Another area which applies the terms of a natural language are generalized syllogisms. The goal of this thesis is primarily to suggest a formal mathematical apparatus and to find syntactic proofs of all 105 generalized Aristotle's syllogisms. It looks like the fuzzy logic established on Łukasiewicz algebra is sufficiently powerful to be applied in modeling the vagueness phenomenon, or in modeling of the commonsense reasoning. That is why the proposed mathematical apparatus, which will be used by us, is Łukasiewicz fuzzy type theory.

For the purpose of a proof related to all the 105 forms of generalized syllogisms, we will apply definitions of generalized intermediate quantifiers. At the end of the chapter relating to the syllogisms we will demonstrate an example of the generalized syllogisms, which include expressions of the natural language.

In the next part we will perform an analysis of generalized Aristotle's square, which is called as complete square of opposition in the classical logic. We will show that the square may be constructed for generalized intermediate quantifiers in Łukasiewicz fuzzy type theory.

Keywords: Fuzzy type theory; Generalized quantifiers; Intermediate quantifiers; Aristotle's syllogisms; Aristotle's square of opposition;

Anotace

Fuzzy logika je matematická disciplína, která si našla velké uplatnění již na konci osmdesátých a na začátku devadesátých let. Však skoro každý se dnes v moderním uspěchaném světě setkal s nápisem "fuzzy" nebo "šestý smysl" v pračkách, myčkách, fotoaparátech, autech atd. Nejúspěšnější aplikace však byly uskutečněny v oblasti řízení a fuzzy regulaci. V čem je schovaná taková úspěšnost fuzzy logiky? Hlavním zdrojem je to, že fuzzy logika umožňuje zahrnout nepřesnost a umí dobře pracovat s významy slov přirozeného jazyka. Proto fuzzy logika zasahuje do mnoha oblastí lidské činnosti a stále se objevují její možnosti využití. Jednou s takových oblastí je také využití fuzzy pravidel typu JESTLIŽE-PAK k rozhodování a řízení, která jsou založena na již zmíněných výrazech přirozeného jazyka.

Další významnou oblastí, která využívá výrazy přirozeného jazyka jsou zobecněné sylogismy. Cílem disertační práce je nejprve navrhnout formální matematický aparát a najít syntaktické důkazy všech 105 zobecněných Aristotelových sylogismů. Pro práci s jazykovými výrazy v praxi je nejlepší fuzzy logika založena na Łukasiewiczově algebře. Proto navrhovaným matematický aparátem, se kterým budeme pracovat, je Łukasiewiczova fuzzy teorie typů.

K účelu dokázat všech 105 forem zobecněných sylogismů, nejprve zavedeme definice zobecněných intermediálních kvantifikátorů. V závěru kapitoly týkající se sylogismů uvedeme příklady zobecněných sylogismů, které obsahují právě výrazy přirozeného jazyka.

V další části provedeme analýzu zobecněného Aristotelova čtverce, který je v klasické logice nazýván úplný čtverec opaků. Ukážeme, že tento čtverec můžeme zkonstruovat pro zobecněné intermediální kvantifikátory v Łukasiewiczově fuzzy teorii typů.

Klíčová slova: Fuzzy teorie typů; Zobecněné kvantifikátory; Intermediální kvantifikátory; Aristotelovy sylogismy; Aristotelův čtverec opaků;

Preface

The many valued logics were initiated already during the first half of the 20^{th} century by J. Łukasiewicz in [23] and also by K. Gődel in [20]. The study of vagueness by means of the many valued approach began only after L.A. Zadeh published his seminar paper [51] in 1965, where the concept of fuzzy set together with the basic principles of fuzzy set theory were introduced. Since then, the notation of fuzziness has been extended into many mathematical disciplines: fuzzy arithmetic, fuzzy logic, fuzzy topology, etc. The development of mathematical fuzzy logic was followed in 1969 by J.A. Goguen in [12]. His inventions were first followed by J. Pavelka in [42] and V. Novák in [29] who presented the fuzzy logic with evaluated syntax. This means that axioms can be only partially true. This area is thoroughly studied in the monograph of V. Novák et al. [40].

Another point of view was developed by Hájek and Gottwald in [14, 13] with traditional syntax. Many special formal systems of propositional and predicate first-order fuzzy logics are proposed in [7], where 57 systems are studied as extensions of the MTL-logic, a logic based on the MTL-algebra of truth values [9]. These logics are examples of *core* fuzzy logics [15].

Naturally, the development of fuzzy logic continued in fuzzy logic of higher order which was proposed by Novák in [31, 32] and in [35] in more detail. This higher order fuzzy logic is called *fuzzy type theory* denoted by FTT. This is generalization of the classical type theory introduced by B. Russell in [46]. This work was further elaborated by A. Church, L. Henkin and P. Andrews in [2, 5, 16, 17].

There is another branch proposed by L. Běhounek and P. Cintula in [3] where the special formal system called *fuzzy class theory* is studied. All their results may be also shown in FTT.

We continue developing the formal theory of the so-called *intermediate quantifiers* introduced in [38]. Remember that intermediate quantifiers are linguistic expressions such as *most, few, almost all, a lot of, many, a great deal of, a large part of, a small part of*, etc. This class of quantifiers was deeply studied by Peterson in the book [45] from the point of view of their semantics and general logical properties. However, Peterson did not introduce any special formal logical system for them. Moreover, despite a typically vague character of intermediate quantifiers, the proposed semantics is basically classical.

The main idea consists in the assumption that intermediate quantifiers are just classical quantifiers ∀ or ∃ whose universe of quantification is modified using an evaluative linguistic expression (an expression such as "very small", "roughly big", "more or less medium", etc.). The meaning of the latter, however, is imprecise and so, the meaning of intermediate quantifiers is imprecise as well. Therefore, we propose their model in the frame of fuzzy logic. Namely, intermediate quantifiers are represented in our theory by special formulas consisting of two parts:

(i) Characterization of the size of a given fuzzy set using specific measure and some evaluative linguistic expression,

(ii) ordinary quantification (general or existential) of the resulting formula.

The formulas are constructed in a certain extension of a special formal theory T^{Ev} of FTT which describes semantics of *trichotomous evaluative linguistic expressions* (see [37]).

The following are the main merits of our theory: it is relatively simple (intermediate quantifiers are taken as special formulas of the already established theory); it is sufficiently general so that a wide class of generalized quantifiers is encompassed within it; the definition of all of them is unified; their properties can be studied in syntax only in a way that we are free to consider a variety of possible interpretations.

The FTT should be based on the same structure of truth values as the above introduced core fuzzy logic, i.e. the structure of truth valued should be an extension of the MTL-algebra. IMTL-FTT proposed by V. Novák based on $\mathrm{IMTL}_{\boldsymbol{\Delta}}$-algebra (see [35]), BL-FTT based on $\mathrm{BL}_{\boldsymbol{\Delta}}$-algebra (see [34]). We introduce Łukasiewicz fuzzy type theory (denoted by Ł-FTT) based on a linearly ordered $\mathrm{MV}_{\boldsymbol{\Delta}}$-algebra. The development of Ł-FTT is one from the main results of this work.

One of the essential contributions of the Peterson's analysis is a list of generalized Aristotle's syllogisms. Namely, he introduced and informally demonstrated validity of 105 of them. In [38], it was proved that 24 of them are valid as well in our theory. In this work, we continue in proving the remaining ones so that, finally, we *formally* prove the validity of all the mentioned 105 generalized syllogisms. At the same time, we will also prove that various syllogisms listed in [45] as invalid are also invalid in our theory. Therefore, we believe that our theory provides a reasonable mathematical model of the generalized syllogistics. Let us also emphasize that all our proofs are syntactical, thus, our theory is very general. That is the second important result of this work.

In the view of the classical theory of generalized quantifiers, our quantifiers are of type $\langle 1, 1\rangle$ (cf. [19, 43, 49]), which are *isomorphism-invariant* (cf. [18, 8]).

How to find the relationships between the intermediate generalized quantifiers? We can see a more important meaning in the generalized Aristotle's square which was introduced by Peterson in [44] where he proposed new intermediate quantifiers "Few" and "Many". This work was formally elaborated by Thompson in [47] where the intermediate quantifier "Most" was introduced and a complete square of opposition as generalization of Aristotle's square was studied. Our main idea is to find generalization of a complete square of opposition in Ł-FTT for intermediate generalized quantifiers. Many results are presented in the last section of this work.

Outline of this thesis

This thesis can be divided into three main parts. We start with Chapter 1 where we introduce the main definitions used later. Chapter 2 proposes Łukasiewicz fuzzy type theory (Ł-FTT). In this chapter we prove main properties of Ł-FTT which are used for the proof of deduction theorem and for the construction of the canonical model of Ł-FTT. The main goal of the author was to provide formal proofs of the mentioned properties. We did not repeat construction of the canonical model because this is the same as in [39]. In Chapter 3 we briefly overview the formal theory of trichotomous evaluative linguistic expressions which are used in the definitions of intermediate quantifiers. This chapter is based on [37].

In Chapter 4 we first review the formal theory of intermediate quantifiers and list some of their basic properties used later. Then we give formal proofs of validity of 105 generalized Aristotle's syllogism which were informally studied in [45]. This chapter is closed with discussion of our theory and demonstration of its behavior on a simple model. The final Chapter 5 contains generalization of the complete square of opposition in Ł-FTT. Let us emphasize that the main results of the author are given in Chapter 4 and Chapter 5.

I would like to express my gratitude to my supervisor Prof. Vilém Novák for his support, valuable comments and permanent encouragement which made it possible to finish this thesis. Warm thanks goes to all my present and former colleagues from the institute for making a friendly and creative atmosphere. I would especially highlight Viktor Pavliska for his co-operation with LaTeX.

Table of Contents

List of Symbols

T – "Most B are A", 76
D – "Most B are not A", 76
K – "Many B are A", 76
G – "Many B are not A", 76

Contr(P_1, P_2) – contraries in Ł-FTT, 121
Sub-contr(P_1, P_2) – sub-contraries in Ł-FTT, 121
Contrad(P_1, P_2) – contradictories in Ł-FTT, 121
Subaltern(P_1, P_2) – subalterns in Ł-FTT, 122

$\mathcal{M}$ – safe general model, 34

Types– set of types, 25
$Form_\alpha$ – set of formulas of type α, 26

$*$ – triangular norm (t-norm), 24
$\otimes$ – Łukasiewicz t-norm, 18
$\oplus$ – Łukasiewicz disjunction, 20
$\odot$ – Product t-norm, 18
$\wedge$ – minimum t-norm, 18

Chapter 1

Preliminaries

1.1 Structure of truth values

1.1.1 Residuated lattice

Residuated lattices were introduced by M. Ward and R.P. Dilworth in [48]. Later the residuated lattices were studied by the group of authors in [10]. We will work with the definition of residuated lattices which is described below.

Definition 1
A residuated lattice is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle, \tag{1.1.1}$$

with four binary operations and two constants such that $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice with ordering defined using the operations* $\wedge$, $\vee$ *and* $\mathbf{0}$,$\mathbf{1}$ *are its least and greatest elements, respectively.* $L = \langle L, \otimes, \mathbf{1} \rangle$ *is a commutative monoid (i.e.* $\otimes$ *is associative, commutative and the identity* $a \otimes \mathbf{1} = a$ *holds for any* $a \in L$*) and the property of* adjunction *is satisfied, i.e.*

$$a \otimes b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c$$

holds for all $a, b, c \in L$ *(*$\leq$ *denotes the corresponding lattice ordering). The operations* $\otimes$ *and* $\rightarrow$ *are called* multiplication *and* residuum, *respectively.*

The additional operations are introduced as follows:

(i) $\neg a = a \rightarrow \mathbf{0}$ $\quad$ (*negation*),

(ii) $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ (*biresiduum*)

Example 1
The Boolean algebra *for classical logic is an algebra*

$$\mathcal{L}_B = \langle \{\mathbf{0}, \mathbf{1}\}, \vee, \wedge, \to, \mathbf{0}, \mathbf{1} \rangle,$$

where $\to$ is the classical implication (the multiplication $\otimes = \wedge$) is the residuated lattice.

We say that a residuated lattice is a *complete* or *linearly ordered* if the corresponding lattice $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a complete or linearly ordered, respectively.

We introduce a few examples of complete residuated lattices on a unit interval which are determined by the left-continuous t-norms (this notion will be introduced at the end of this chapter). These residuated lattices will be denoted by $\mathcal{L}_t$ where t denotes a certain left continuous t-norms.

Example 2
Let $\otimes$ be a Łukasiewicz t-norm and $\to_{Ł}$ be defined as follows:

$$a \to_{Ł} = min(1 - a + b, 1).$$

Then $\mathcal{L}_{Ł} = \langle [0, 1], \vee, \wedge, \otimes, \to_{Ł}, \mathbf{0}, \mathbf{1} \rangle$, is the complete residuated lattice called the Łukasiewicz MV- algebra *which is used for construction of Łukasiewicz logic (see e.g. [14, 40]).*

Example 3
Let $\odot$ be a product t-norm and $\to_P$ be defined as follows:

$$a \to_P b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

Then $\mathcal{L}_P = \langle [0, 1], \vee, \wedge, \odot, \to_P, \mathbf{0}, \mathbf{1} \rangle$, is a complete residuated lattice called product algebra *or also* Goguen algebra *which is used in the product logic (see [14]).*

Example 4
Let $\wedge$ be minimum t-norm and $\to_G$ be defined as follows:

$$a \to_G b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Then $\mathcal{L}_G = \langle [0, 1], \vee, \wedge, \to_G, \mathbf{0}, \mathbf{1} \rangle$, is a complete residuated lattice called the Gödel algebra*(see [14]).*

1.1.2 BL-algebra

This algebra has been introduced by P. Hájek as a basic structure for many-valued logic (see [14]).

Definition 2
Let $\mathcal{L}$ be a residuated lattice. It is called the BL-algebra if the following holds for every $a, b \in L$.

(a) $a \otimes (a \to b) = a \wedge b$ *(divisibility property),*

(b) $(a \to b) \vee (b \to a) = \mathbf{1}$ *(preliniearity).*

1.1.3 MV-algebra

The notion of MV-algebra which means "many valued", has been introduced by C.C. Chang (see [4]) as an algebraic system corresponding to the $\aleph_0$-valued propositional calculus. More details can be found in [6, 40]).

From the algebraic point of view, the MV-algebra differs from the Boolean one by absence of the idempotency law for their algebraic operations (addition and multiplication) and also by the missing law of exclude middle for the lattice operations.

Definition 3 (C.C. Chang)
An MV-algebra is an algebra

$$\mathcal{L}_{MV} = \langle L, \oplus, \otimes, \neg, \mathbf{0}, \mathbf{1} \rangle, \tag{1.1.2}$$

where the following identities are valid.

$$\begin{aligned}
a \oplus b &= b \oplus a, & a \otimes b &= b \otimes a,\\
a \oplus (b \oplus c) &= (a \oplus b) \oplus c, & a \otimes (b \otimes c) &= (a \otimes b) \otimes c,\\
a \oplus \mathbf{0} &= a, & a \otimes \mathbf{1} &= a,\\
a \oplus \mathbf{1} &= \mathbf{1}, & a \otimes \mathbf{0} &= \mathbf{0},\\
a \oplus \neg a &= \mathbf{1}, & a \otimes \neg a &= \mathbf{0},\\
\neg(a \oplus b) &= \neg a \otimes \neg b, & \neg(a \otimes b) &= \neg a \oplus \neg b,\\
a &= \neg\neg a, & \neg \mathbf{0} &= \mathbf{1},\\
\neg(\neg a \oplus b) \oplus b &= \neg(\neg b \oplus a) \oplus a.
\end{aligned}$$

The lattice operations can be introduced by

$$
\begin{aligned}
a \vee b &= \neg(\neg a \oplus b) \oplus b = (a \otimes \neg b) \oplus b,\\
a \wedge b &= \neg(\neg a \vee \neg b) = (a \oplus \neg b) \otimes b,\\
a \to b &= \neg a \oplus b.
\end{aligned}
$$

Example 5
The Łukasiewicz algebra from Example 2 is an MV-algebra. It can be written as follows:

$$\mathcal{L}_{Ł} = \langle [0,1], \oplus, \otimes, \neg, 0, 1 \rangle$$

where $\otimes$ *is the* Łukasiewicz conjunction *defined above,* $\oplus$ *is called the* Łukasiewicz disjunction *defined by*

$$a \oplus b = 1 \wedge (a + b)$$

and its negation operation defined by $\neg a = 1 - a$.

Further, we will work with an MV-algebra which is based on the residuated lattice.

Definition 4 (MV-algebra based on residuated lattices)
Let $\mathcal{L}$ *be a residuated lattice. It is called the MV-algebra if the following holds for every* $a, b \in L$.

(a) $a \otimes (a \to b) = a \wedge b$ *(divisibility property),*

(b) $(a \to b) \vee (b \to a) = \mathbf{1}$ *(preliniearity),*

(c) $\neg\neg a = a$ *(involutive negation).*

We can see that the MV-algebra is the BL-algebra extended by the law of double negation.

Furthermore, we define the following operations for all $a, b \in L$:

(i) $a \oplus b = \neg(\neg a \otimes \neg b)$ *(strong sum),*

(ii) $a^n = \underbrace{a \otimes \cdots \otimes a}_{n-\text{times}}$ *(strong power),*

(iii) $na = \underbrace{a \oplus \cdots \oplus a}_{n-\text{times}}$ *(n− fold strong sum),*

(iv) $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ *(biresiduation).*

1.1.4 MV$_\Delta$-algebra

We continue with the definition of MV$_\Delta$-algebra which is an MV-algebra defined in Definition 4 extended by *Bazz delta* (cf. [14, 35]) operation which sends all truth values smaller than 1 to 0.

Definition 5 (MV$_\Delta$-algebra)
The MV$_\Delta$-algebra is an algebra

$$\mathcal{L}_\Delta = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta \rangle, \tag{1.1.3}$$

where $\langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$ *is an MV-algebra defined as residuated lattice and* Δ *is an unary operation defined on* $[0, 1]$ *which fulfils the following additional conditions:*

(i) $\Delta a \vee \neg \Delta a = 1$,

(ii) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,

(iii) $\Delta a \leq a$,

(iv) $\Delta a \leq \Delta\Delta a$,

(v) $\Delta(a \rightarrow b) \leq \Delta a \rightarrow \Delta b$,

(vi) $\Delta \mathbf{1} = \mathbf{1}$.

Example 6
A special case of MV$_\Delta$ algebra is the standard Łukasiewicz MV$_\Delta$-algebra

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \rightarrow_{\L}, 0, 1, \Delta \rangle \tag{1.1.4}$$

where

$$\begin{aligned} \wedge &= \textit{minimum}, & \vee &= \textit{maximum}, \\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow_{\L} b &= 1 \wedge (1 - a + b), \\ \neg a &= a \rightarrow_{\L} 0 = 1 - a, & \Delta(a) &= \begin{cases} 1 & \textit{if } a = 1, \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

1.1.5 Basic properties

The following properties are continued in $\mathrm{MV}_{\boldsymbol{\Delta}}$ algebras. The proofs can be found in [35] and in [40].

Lemma 1
Let $\mathcal{L}_{\Delta}$ be an $MV_{\boldsymbol{\Delta}}$ algebra. Then for every $a, b, c \in L$ the following is true.

(i) $a \otimes b \leq a, \quad a \otimes b \leq b, \quad a \otimes b \leq a \vee b$

(ii) $b \leq a \rightarrow b,$

(iii) $a \otimes (a \rightarrow b) \leq b, \quad b \leq a \rightarrow (a \otimes b),$

(iv) *if* $a \leq b$ *then* $c \rightarrow a \leq c \rightarrow b,$

(v) *if* $a \leq b$ *then* $a \rightarrow c \geq b \rightarrow c,$

(vi) $a \otimes (a \rightarrow \mathbf{0}) = \mathbf{0},$

(vii) $a \rightarrow (b \rightarrow c) = (a \otimes b) \rightarrow c,$

(viii) $(a \wedge b) \otimes c = (a \otimes c) \wedge (b \otimes c),$

(ix) $a \wedge b \leq ((a \rightarrow b) \rightarrow b) \vee ((b \rightarrow a) \rightarrow a),$

(x) $a \rightarrow b = \neg b \rightarrow \neg a,$

(xi) $a \otimes b = \neg(a \rightarrow \neg b),$

(xii) $a \rightarrow b = (a \wedge b) \leftrightarrow a,$

(xiii) $(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d),$

(xiv) $\Delta(a \leftrightarrow b) \leq \Delta a \leftrightarrow \Delta b,$

(xv) $\Delta(a \wedge b) \leq \Delta a \wedge \Delta b.$

1.2 Theory of Fuzzy Sets

1.2.1 Basic definitions

The basic elements of the fuzzy sets and fuzzy logic were introduced by L.A. Zadeh in [51]. His idea was to generalize the concept of an ordinary set. Namely, every classical set A may be identified with a function $\chi_A\colon U \to \{0,1\}$ where U is a domain, such that

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{1.2.1}$$

This leads to the following definition.

Definition 6
Let L be an algebra of truth values. Let U be a non-empty set. Then the L-fuzzy set A be identified by the function

$$A\colon U \longrightarrow L$$

which to each element $x \in U$ assigns a value $A(x) \in L$ representing the degree of membership of x in the fuzzy set A. We will write $A \underset{\sim}{\subset} U$.

The theory of fuzzy sets is elaborated in [28]. Generally, the fuzzy sets are applied for the modeling of the linguistic evaluative expressions which will be studied in Chapter 3. Among these expressions belong for example *small, very big, extremally slim*, etc.

Definition 7
Let $A \underset{\sim}{\subset} U$. Then the support *of A is the following set:*

$$Supp(A) = \{x \mid x \in U, A(x) > 0\}.$$

Definition 8
Let $A \underset{\sim}{\subset} U$. Then the kernel *of U is the following set:*

$$Ker(A) = \{x \mid x \in U, A(x) = 1\}.$$

Definition 9
Let $A \underset{\sim}{\subset} U$. We say that A is a normal *fuzzy set if $Ker(A) \neq \emptyset$.*

1.2.2 T-norms and fuzzy relation

The original motivation of K. Menger (see [25]) to introduce a class of generalized multiplications known as *triangular norms* (t-norms) was not logical. The main idea was to generalize the concept of the triangular inequality.

Definition 10
A binary operation $* : [0,1]^2 \to [0,1]$ *is called* triangular norm *(t-norms) if it fulfills commutativity, associativity, monotonicity and the boundary condition. This means that for all* $a, b, c \in [0,1]$, *the following is true:*

$$a * b = b * a, \qquad (commutativity) \tag{1.2.2}$$

$$a * (b * c) = (a * b) * c, \qquad (associativity) \tag{1.2.3}$$

$$a \leq b \Longrightarrow a * c \leq b * c, \qquad (monotonicity) \tag{1.2.4}$$

$$a * 1 = a. \qquad (boundary\ condition) \tag{1.2.5}$$

Example 7
Below, we introduce the most known examples of continuous t-norms which serve as natural interpretations of a general conjunction:

(a) Minimum t-norm $a * b = a \wedge b$,

(b) Product t-norm $a \odot b = a \cdot b$,

(c) Łukasiewicz t-norm (*Łukasiewicz conjunction*) $a \otimes b = \max(0, a + b - 1)$.

More details can be found in [21].

Definition 11 (Fuzzy equality)
Let U *be a non-empty set. A fuzzy relation* R *is a fuzzy set* $R \underset{\sim}{\subset} U \times U$. *We say that it is a* fuzzy equality *if the following conditions are true:*

(i) reflexivity $R(m, m') = \mathbf{1}$, $m, m' \in U$,

(ii) symmetry $R(m, m') = R(m', m)$, $m, m' \in U$,

(iii) $\otimes$-transitivity $R(m, m') \otimes R(m', m'') \leq R(m, m'')$, $m, m', m'' \in U$.

Chapter 2

Łukasiewicz fuzzy type theory

The first of the main goals of this thesis is to propose a mathematical theory. Using it we may prove validity of the all the generalized intermediate syllogisms introduced in Chapter 4.

Łukasiewicz fuzzy type theory (denoted by Ł-FTT) is a higher order fuzzy logic [35]. It is a special case of FTT presented in [35] and it is based on a linearly ordered MV_{Δ} -algebra presented in the previous section and has the form (1.1.3). We except that the reader is familiar with the classical type theory.

2.1 Syntax of Ł-FTT

2.1.1 Basic syntactical elements

Definition 12 (Types)
Let ϵ, o be distinct objects. The set of types is the smallest set Types *satisfying the following:*

(i) $\epsilon, o \in$ *Types,*

(ii) If $\alpha, \beta \in$ *Types then* $(\alpha\beta) \in$ *Types.*

The type ϵ represents elements and o truth values.

The following symbols define the language and are called *basic syntactical elements*. Using them we construct formulas in Ł-FTT.

Definition 13 (Language)
The language *J of Ł-FTT consists of:*

(i) variables $x_\alpha, \ldots$ where $\alpha \in$ Types,

(ii) special constants $c_\alpha, \ldots$ where $\alpha \in$ Types. We will consider the following concrete special constants: $\mathbf{E}_{(o\alpha)\alpha}$, for every $\alpha \in$ Types, $\mathbf{C}_{(oo)o}$, $\mathbf{D}_{oo}$ and descriptions operators $\iota_{\epsilon(o\epsilon)}$ and $\iota_{o(oo)}$.

(iii) Auxiliary symbols: λ, brackets

Definition 14 (Formulas)
A set of formulas over the language J is the smallest set such that for each $\alpha, \beta \in$ Types the following is specified:

(i) If $x_\alpha \in J$ is a variable, $\alpha \in$ Types, then x_α is a formula of type α.

(ii) If $c_\alpha \in J$ is a constant, $\alpha \in$ Types, then c_α is a formula of type α.

(iii) If $B_{\beta\alpha}$ is a formula of type $\beta\alpha$ and A_α is a formula of type α, then $(B_{\beta\alpha} A_\alpha)$ is a formula of type β.

(iv) If A_β is a formula of type β and $x_\alpha \in J$ is a variable of type α, then $\lambda x_\alpha A_\beta$ is a formula of type $\beta\alpha$.

A set of formulas*) of type $\alpha, \beta \in$ *Types*, is denoted by $Form_\alpha$. The set of all formulas is $Form = \bigcup_{\alpha \in Types} Form_\alpha$. If $A \in Form_\alpha$ is a formula of the type $\alpha \in$ *Types* then we will write A_α.

Remark 1
An occurrence of x_α is free *in B_β iff it is not in a part of B_β of the form $\lambda x_\alpha C_\delta$. We say that occurrence of x_α is* bound *in B_β iff it is in part of B_β of the form $\lambda x_\alpha C_\delta$.*

2.1.2 Basic definitions

Before introducing basic definitions, let's stress that we will quite often use the (meta-)symbol ":=" which means "is defined by".

The following special formulas are defined:

*) In the up-to-date type theory, "formulas" are quite often called "lambda-terms". We prefer the former in this thesis because FTT is logic and so the term "formula" is more natural.

(i) *Equivalence:* $\equiv \; := \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha, \qquad \alpha \in \mathit{Types}.$

As usual, we will write $x_o \equiv y_o$ instead of $(\equiv y_o)x_o$ and similarly for other formulas defined below. Note that if A_α, B_α are formulas then $(A_\alpha \equiv B_\alpha)$ is a formula of type o; if $\alpha = o$ then $\equiv$ is a logical equivalence.

(ii) *Conjunction:* $\boldsymbol{\wedge} := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o}\, y_o) x_o$.

We will write $x_o \wedge y_o$ instead of $(\wedge x_o) y_o$.

(iii) *Delta connective:* $\boldsymbol{\Delta} := \lambda x_o \mathbf{D}_{oo} x_o$.

2.1.3 Derived connectives

(i) *Representation of truth:* $\top := \lambda x_o x_o \equiv \lambda x_o x_o$.

(ii) *Representation of falsity:* $\bot := \lambda x_o x_o \equiv \lambda x_o \top$.

(iii) *Negation:* $\boldsymbol{\neg} := \lambda x_o (x_o \equiv \bot)$.

(iv) *Implication:* $\boldsymbol{\Rightarrow} \; := \lambda x_o \lambda y_o \, (x_o \boldsymbol{\wedge} y_o) \equiv x_o$.

(v) *Strong conjunction:* $\boldsymbol{\&} := \lambda x_o (\lambda y_o (\boldsymbol{\neg}(x_o \boldsymbol{\Rightarrow} \boldsymbol{\neg} y_o)))$.

(vi) *Strong disjunction:* $\boldsymbol{\nabla} := \lambda x_o (\lambda y_o (\boldsymbol{\neg}(\boldsymbol{\neg} x_o \,\boldsymbol{\&}\, \boldsymbol{\neg} y_o)))$.

(vii) *Disjunction:* $\boldsymbol{\vee} := \lambda x_o (\lambda y_o (x_o \boldsymbol{\Rightarrow} y_o) \boldsymbol{\Rightarrow} y_o)$.

(viii) *General quantifier:* Let $A_o \in \mathit{Form}_o$ and x_α be a variable of type α. Then we put

$$(\forall x_\alpha) A_o := (\lambda x_\alpha A_o) \equiv (\lambda x_\alpha \top).$$

(ix) *Existential quantifier:*

$$(\exists x_\alpha) A_o := \boldsymbol{\neg}(\forall x_\alpha) \boldsymbol{\neg} A_o.$$

Furthermore, the n-times strong conjunction of A_o is denoted by A_o^n and n-times strong disjunction (denoted by nA_o). From the definition above it is obvious that $\top, \bot, (\forall x_\alpha)A_o, (\exists x_\alpha)A_o \in \mathit{Form}_o$, $\boldsymbol{\neg} \in \mathit{Form}_{oo}$ and $\boldsymbol{\Rightarrow}, \boldsymbol{\wedge}, \boldsymbol{\vee}, \boldsymbol{\&}, \boldsymbol{\nabla} \in \mathit{Form}_{(oo)o}$.

2.2 Semantics of Ł-FTT

2.2.1 Basic frame

Definition 15

Let D be a set of objects and L be a set of truth values. The basic frame *based on D, L is a family of sets $(M_\alpha)_{\alpha \in Types}$ where*

(a) $M_\epsilon = D$ is a set of objects,

(b) $M_o = L$ is a set of truth values,

(c) for each type $\gamma = \beta\alpha$, the corresponding set $M_\gamma \subseteq M_\beta^{M_\alpha}$.

The elements from M_α will be denoted by m_α. Let $m_{\beta\alpha} \in M_\beta^{M_\alpha}$ be an element of the type $\beta\alpha$. This means that it is a function $m_{\beta\alpha} : M_\alpha \longrightarrow M_\beta$ assigning to each element $m_\alpha \in M_\alpha$ some element $m_\beta \in M_\beta$. So, if M_α is a set from the basic frame then a fuzzy equality (was defined in Definition 11) on M_α is denoted by $=_\alpha$. We closed this subsection with the lemma below which can be found with a proof in [35].

Lemma 2

Let $=_\beta$ be a fuzzy equality. Then the function

$$=_{\beta\alpha} : M_\beta^{M_\alpha} \times M_\beta^{M_\alpha} \longrightarrow L$$

defined for every $m_{\beta\alpha}, m'_{\beta\alpha} \in M_\beta^{M_\alpha}$ by

$$[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] = \bigwedge_{m_{\alpha \in M_\alpha}} [m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] \tag{2.2.1}$$

is a fuzzy equality.

2.2.2 Extensional function

Extensionality is the well-known notion from the classical set theory. The first generalization of this description was introduced by F. Klawon and R. Kruse in [26]. There, the existential fuzzy relations are defined w.r.t. a similarity relation on their domain, where the notation of similarity stands for a generalized relation of equality between objects.

We will define extensional functions using the strong power defined in Chapter 1.

Definition 16
We say that F *is* extensional w.r.t fuzzy equalities $=_{\alpha_1}, \dots =_{\alpha_n}, =_\beta$ *if there are exponents* $q_1, \dots, q_n \geq 1$ *such that the inequality*

$$[m_{\alpha_1} =_{\alpha_1} m'_{\alpha_1}]^{q_1} \otimes \cdots \otimes [m_{\alpha_n} =_{\alpha_n} m'_{\alpha_n}]^{q_n} \leq [F(m_{\alpha_1}, \dots, m_{\alpha_n}) =_\beta F(m'_{\alpha_1}, \dots, m'_{\alpha_n})] \quad (2.2.2)$$

If $q_1 = \cdots = q_n = 1$ holds in (2.2.2) then we say that F is *strongly extensional* otherwise it is simply *extensional*.

Definition 17
We say that F *is* weakly extensional *if*

$$[m_{\alpha_1} =_{\alpha_1} m'_{\alpha_1}] = \cdots = [m_{\alpha_n} =_{\alpha_n} m'_{\alpha_n}] = \mathbf{1}$$

implies that

$$[F(m_{\alpha_1}, \dots, m_{\alpha_n}) =_\beta F(m'_{\alpha_1}, \dots, m'_{\alpha_n})] = \mathbf{1}.$$

2.2.3 General frame

Definition 18 (General frame)
Let J *be a language of Ł-FTT and* $(M_\alpha)_{\alpha \in Types}$ *be a basic frame. The* general frame *is a tuple*

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \quad (2.2.3)$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ *is a structure of truth values (i.e., an* MV_Δ*-algebra). We put* $M_o = L$ *and assume that the set* $M_{oo} \cup M_{(oo)o}$ *contains all the operations from* $\mathcal{L}_\Delta$.

(ii) $=_\alpha$ *is a fuzzy equality on* M_α *and* $=_\alpha \in M_{(o\alpha)\alpha}$ *for every* $\alpha \in Types$*. Moreover,*

(a) if $\alpha = o$ *then* $=_o$ *is* $\leftrightarrow$*,*

(b) If $\alpha = \epsilon$ *then* $=_\epsilon \subsetapprox M_\epsilon \times M_\epsilon$ *is a fuzzy equality on the set* M_ϵ *(remember that* M_ϵ *is a set of objects).*

(c) If $\alpha \neq \emptyset, \epsilon$ *then* $=_\alpha$ *is the fuzzy equality given in (2.2.1).*

(iii) If $\alpha = \gamma\beta$ *then each function* $F \in M_{\gamma\beta}$ *is weakly extensional w.r.t* $=_\beta$ *and* $=_\gamma$*.*

2.2.4 Interpretation of formulas

Let us fix the general frame $\mathcal{M}$. Before defining of an interpretation over $\mathcal{M}$ of all formulas, we will define an assignment p for the variables over $\mathcal{M}$.

A function p such that $p(x_\alpha) \in M_\alpha$, $\alpha \in \mathit{Types}$ is a weakly extensional function is an assignment of elements from $\mathcal{M}$ to variables. The set of all assignments over $\mathcal{M}$ will be denoted by $Asg(\mathcal{M})$.

An interpretation $\mathcal{M}_p$ is a function that assigns every formula A_α, $\alpha \in \mathit{Types}$ and every assignment p a corresponding element of type α.

Let x_α be a variable and $p, p' \in Asg(\mathcal{M})$ be two assignments such that $p'(y_\gamma) \neq p(y_\gamma)$ for all $y_\gamma \neq x_\alpha$ (i.e. p' differs from p only in the variable x_α). Then, in this case, we will write $p' = p\backslash x_\alpha$.

Definition 19
Let $\mathcal{M}$ be a general frame and $p \in Asg(\mathcal{M})$ an assignment. Then we define:

(i) If x_α is a variable then $\mathcal{M}_p(x_\alpha) = p(x_\alpha)$.

(ii) If c_α is a constant then $\mathcal{M}_p(c_\alpha)$ is an element from M_α. If $\alpha \neq o, \epsilon$ then $p(c_\alpha)$ is a weakly extensional function . As a special case:

(a) $\mathcal{M}_p(\mathbf{E}_{(o\alpha)\alpha})\colon M_\alpha \longrightarrow L^{M_\alpha}$ is a fuzzy equality $=_\alpha$. Precisely, it is a function such that for all $m, m' \in M_\alpha$

$$\mathcal{M}_p(\mathbf{E}_{(o\alpha)\alpha})(m')(m) = [m =_\alpha m'] \in L$$

holds true.

(b) $\mathcal{M}_p(\mathbf{C}_{(oo)o})\colon L \longrightarrow L^L$ is the meet operation $\wedge$. Thus,

$$\mathcal{M}_p(\mathbf{C}_{(oo)o})(a)(b) = a \wedge b$$

for all $a, b \in L$.

(c) $\mathcal{M}_p(\mathbf{D}_{oo})\colon L \longrightarrow L$ is the Baaz delta operation $\mathbf{\Delta}$. Thus,

$$\mathcal{M}_p(\mathbf{D}_{oo})(a) = \mathbf{\Delta} a$$

for all $a \in L$.

(d) Interpretation of the description operator $\mathcal{M}_p(\iota_{\epsilon(o\epsilon)})$ (or $\mathcal{M}_p(\iota_{o(oo)})$) is a function assigning to each non-empty fuzzy set in M_ϵ (or in M_o) an element from its kernel provided that the latter is non-empty; otherwise it is not determined.

(iii) *Interpretation of a formula* $B_{\beta\alpha}A_\alpha$ *of type* β *is*

$$\mathcal{M}_p(B_{\beta\alpha}A_\alpha) = \mathcal{M}_p(B_{\beta\alpha})(\mathcal{M}_p(A_\alpha)).$$

(iv) *Interpretation of a formula* $\lambda x_\alpha A_\beta$ *of type* $\beta\alpha$ *is a function*

$$\mathcal{M}_p(\lambda x_\alpha A_\beta) = F\colon M_\alpha \longrightarrow M_\beta,$$

which is weakly extensional w.r.t "$=_\alpha$*" and "*$=_\beta$*" and such that for each* $m_\alpha \in M_\alpha$, $F(m_\alpha) = \mathcal{M}_{p'}(A_\beta)$ *for some assignment* $p' = p\backslash x_\alpha$.

From the previous definition it is obvious that

$$\mathcal{M}_p(A_\alpha \equiv B_\alpha) = [\mathcal{M}_p(A_\alpha) =_\alpha \mathcal{M}_p(B_\alpha)],$$

where $=_\alpha$ is $\leftrightarrow$ if $\alpha = o$, $=_\alpha$ is $=_\epsilon$ if $\alpha = \epsilon$, and

$$[\mathcal{M}_p(A_{\gamma\beta}) =_{\gamma\beta} \mathcal{M}_p(B_{\gamma\beta})] = \bigwedge_{m_\beta \in M_\beta} [\mathcal{M}_p(A_{\gamma\beta})(m_\beta) =_\gamma \mathcal{M}_p(B_{\gamma\beta})(m_\beta)]$$

if $\alpha = \gamma\beta$.

The proof of the following lemma can be found in [35].

Lemma 3
Let $A_o, B_o \in Form_o$. *Then the following holds true for every assignment* $p \in Asg(\mathcal{M})$*:*

(a) $\mathcal{M}_p(\top) = \mathbf{1}$,

(b) $\mathcal{M}_p(\bot) = \mathbf{0}$,

(c) $\mathcal{M}_p(\neg A_o) = \mathcal{M}_p \rightarrow \mathbf{0}$,

(d) $\mathcal{M}_p(A_o \mathbin{\pmb{\wedge}} B_o) = \mathcal{M}_p(A_o) \wedge \mathcal{M}_p(B_o)$,

(e) $\mathcal{M}_p(A_o \Rightarrow B_o) = \mathcal{M}_p(A_o) \rightarrow \mathcal{M}_p(B_o)$,

(f) $\mathcal{M}_p(A_o \,\&\, B_o) = \mathcal{M}_p(A_o) \otimes \mathcal{M}_p(B_o)$,

(g) $\mathcal{M}_p(A_o \,\nabla\, B_o) = \mathcal{M}_p(A_o) \oplus \mathcal{M}_p(B_o)$,

(h) $\mathcal{M}_p((\forall x_\alpha)A_o) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha, \\ p' = p \setminus x_\alpha}} \mathcal{M}_{p'}(A_o)$,

(i) $\mathcal{M}_p((\exists x_\alpha)A_o) = \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha, \\ p' = p \setminus x_\alpha}} \mathcal{M}_{p'}(A_o)$,

2.3 Axioms and inference rules.

2.3.1 Axioms

The Ł-FTT has the following logical axioms:

(ŁFT1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$,

(ŁFT2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$,

(ŁFT2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$,

(ŁFT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$, where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*),

(ŁFT4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon)$,

(ŁFT5) $(A_o \equiv B_o) \equiv ((A_o \Rightarrow B_o) \wedge (B_o \Rightarrow A_o))$,

(ŁFT6) $(A_o \equiv \top) \equiv A_o$,

(ŁFT7) $A_o \Rightarrow (B_o \Rightarrow A_o)$,

(ŁFT8) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$,

(ŁFT9) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$,

(ŁFT10) $(A_o \vee B_o) \equiv (B_o \vee A_o)$, (ŁFT11) $(A_o \wedge B_o) \equiv A_o \,\&\, (A_o \Rightarrow B_o)$,

(ŁFT12) $(g_{oo}(\boldsymbol{\Delta} x_o) \wedge g_{oo}(\neg \boldsymbol{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\boldsymbol{\Delta} y_o)$,

(ŁFT13) $\boldsymbol{\Delta}(A_o \wedge B_o) \equiv \boldsymbol{\Delta} A_o \wedge \boldsymbol{\Delta} B_o$,

(ŁFT14) $\boldsymbol{\Delta}(A_o \vee B_o) \Rightarrow \boldsymbol{\Delta} A_o \vee \boldsymbol{\Delta} B_o$,

(ŁFT15) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$, where x_α is not free in A_o.

(ŁFT16) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon.$

The axioms which are introduced above can be divided into two groups: axioms for general types and axioms characterizing only truth values. Axiom (ŁFT1) states that all functions must be weakly extensional. Axiom (ŁFT2_1) states that if two functions are equal for all arguments, they are equal. Axiom (ŁFT2_2) has the opposite meaning that if two functions are equal then they must be equal for their arguments. Axioms (ŁFT4) states that fuzzy equality on objects is transitive. Axiom (ŁFT5) characterize logical equivalence as biresiduation. Axioms (ŁFT7-11) are for logical implication and conjunction. Axioms (ŁFT13-14) characterize the structure of truth values. Axiom (ŁFT15) is the classical axiom of predicate logic. Finally, the axioms of descriptions contain *description operators* $\iota_{\alpha(o\alpha)}$.

Lemma 4
For every general frame $\mathcal{M}$, interpretation and assignment p, $\mathcal{M}_p$(ŁFTi)=$\mathbf{1}$ where i= $1, \ldots, 16$.

PROOF: The equality for the axioms (ŁFT1)-(ŁFT3), axioms of descriptions, and (ŁFT12) were proved in [35]. Other axioms of delta and truth values follow from the MV$_\Delta$ axioms and the assumed properties. □

2.3.2 Inference rules

Definition 20 (Inference rules)

(R) Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.

(N) Let $A_o \in Form_o$. Then, from A_o infer $\boldsymbol{\Delta} A_o$.

The proof of following lemma is analogous to the proof of Lemma 12 from [35].

Lemma 5
The inference rules (R) and (N) are sound, i.e. the following holds for every general frame $\mathcal{M}$ and an assignment $p \in Asg(\mathcal{M})$:

(R) if $\mathcal{M}_p(A_\alpha \equiv A'_\alpha) = 1$, then $\mathcal{M}_p(B_o) = \mathcal{M}_p(B'_o)$.

(N) if $\mathcal{M}_p(A_o) = 1$, then also $\mathcal{M}_p(\mathbf{\Delta} A_o) = 1$.

Remark 2
Let us stress that the inference rules of modus ponens (denoted by MP) and generalization (denoted by G) are derived rules in Ł-FTT.

The concept of *provability* and *proof* are defined in the same way as in classical logic. A *theory* T over Ł-FTT is a set of formulas of type o ($T \subseteq Form_o$). By $J(T)$ we denote the language of the theory T. If $A_o \in Form_o$ is a formula then it is provable in T if there is a proof of A_o. Then we will write $T \vdash A_o$.

Let T be a theory, $J' \supseteq J(T)$ an extension of the language of T and $K \subseteq Form_{J',o}$ a set of formulas in the language J'. Then $T' = T \cup K$ is an *extension* of T where K is added to the special axioms of T. The extension T' is *conservative* if $T' \vdash A_o$ implies $T \vdash A_o$ for every formula $A_o \in Form_{J(T),o}$.

2.4 Models of Ł-FTT

We start with the definition of the *safe*[†)] *general model* and *model* in Ł-FTT. Then we will continue with the lemma which demonstrates that the Łukasiewicz fuzzy type theory does not collapse into classical type theory.

Definition 21 (Safe general model)
A safe general model is a general frame $\mathcal{M}$ such that for every A_α, $\alpha \in Types$ and every assignment $p \in Asg(\mathcal{M})$, the interpretation $\mathcal{M}_p$ gives

$$\mathcal{M}_p(A_\alpha) \in M_\alpha.$$

This means that each set M_α from the frame $\mathcal{M}$ has enough elements so that the interpretation $\mathcal{M}_p(A_\alpha)$ is always defined. As a special case (analogously as in the concept of the safe model in [14]), if the formula A_α contains quantifiers then all the necessary suprema and infima are included in $\mathcal{L}$.

[†)] *The notation of the safe structure has been introduced by P.Hájek in his book and it means that all the needed infima and suprema exist.*

Definition 22 (Model)
A safe general model $\mathcal{M}$ is a $safe$ $model$ of a $theory$ T (further "model of T" denoted by $\mathcal{M} \models T$) if

$$\mathcal{M}(A_o) = \mathbf{1}$$

holds for all axioms of T.

If A_o is true in the degree $\mathbf{1}$ in all safe models of T then we write $T \models A_o$.

Let T be a theory. A formula A_o is *true* in the degree $a \in L$ in T, if

$$a = \bigwedge\{\mathcal{M}_p(A_o) \mid \mathcal{M} \models T, p \in Asg(\mathcal{M})\}. \tag{2.4.1}$$

We will write (2.4.1) as $T \models_a A_o$. If $a = 1$ then we omit the subscript .

We close this subsection with two theorems which are proved analogously as in [35].

Theorem 1 (Soundness)
Let T be a theory. Then the following holds for every theory T: If $T \vdash A_o$ then $\mathcal{M}_p(A_o) = \mathbf{1}$ holds for every assignment $p \in Asg(\mathcal{M})$ and every safe general model $\mathcal{M}$ of T.

The following theorem demonstrates that the law of excluded middle $A_o \vee \neg A_o$ is not true. Thus we can find atomic formulas of type o whose interpretation in a general model is different from $\mathbf{1}$ and $\mathbf{0}$.

Theorem 2
There is a safe general model $\mathcal{M}$ in Łukasiewicz fuzzy type theory which is not classical; there is formulas A_o and assignment p such that

$$\mathcal{M}_p(A_o) \neq \mathbf{1}, \mathbf{0} \quad \textit{and} \quad \mathcal{M}_p(A_o \vee \neg A_o) \neq \mathbf{1}.$$

Corollary 1
The exclude middle formula $A_o \vee \neg A_o$ is not provable in Ł-FTT.

2.4.1 Fuzzy sets in Ł-FTT

In this thesis, we identify fuzzy sets with their membership functions. Interpretation of a formula $A_{o\alpha}$ is a function from M_α to truth values. Let B_o be a formula of type o whose interpretation is a truth value. Let the variable x_α occur freely in B_o. Then we can write a fuzzy set explicitly as a formula

$$A_{o\alpha} := \lambda u_\alpha B_{o,x_\alpha}[u_\alpha],$$

where $B_{o,x_\alpha}[u_\alpha]$ denotes instance of B_o in which all free occurrences of x_α are replaced by u_α.

We continue with the definition of operations on fuzzy sets in Ł-FTT which will be used in Chapter 4.

Definition 23
Let $u \in Form_\alpha$. Operations[‡)] *on fuzzy sets can be introduced as special formulas in the following way:*

$$\emptyset_{o\alpha} \equiv \lambda u\alpha \bot, \tag{2.4.2}$$

$$V_{o\alpha} \equiv \lambda u\alpha \top, \tag{2.4.3}$$

$$\subseteq_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha}\lambda y_{o\alpha}(\forall u\alpha)(x_{o\alpha}u\alpha \Rightarrow y_{o\alpha}u\alpha), \tag{2.4.4}$$

$$\subset_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha}\lambda y_{o\alpha}(\forall u\alpha)(x_{o\alpha}u\alpha \Rightarrow y_{o\alpha}u\alpha) \wedge (x_{o\alpha} \not\equiv y_{o\alpha}), \tag{2.4.5}$$

$$\cap_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha}\lambda y_{o\alpha}\lambda u_\alpha(x_{o\alpha}u_\alpha \wedge y_{o\alpha}u_\alpha), \tag{2.4.6}$$

$$\cup_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha}\lambda y_{o\alpha}\lambda u_\alpha(x_{o\alpha}u_\alpha \vee y_{o\alpha}u_\alpha), \tag{2.4.7}$$

$$-_{o(o\alpha)(o\alpha)} \equiv \lambda x_{o\alpha}\lambda y_{o\alpha}\lambda u_\alpha(x_{o\alpha}u_\alpha \,\&\, \neg y_{o\alpha}u_\alpha). \tag{2.4.8}$$

2.5 Main properties of Ł-FTT

We will present the important properties of Ł-FTT which will be used in the sequel.

2.5.1 Basic logical properties

We start with the basic properties. The proofs of the properties which are summarized in the following theorem can be found in [35].

Lemma 6
The following is provable in Ł-FTT.

(a) *If* $\vdash A_o$ *and* $\vdash A_o \equiv B_o$ *then* $\vdash B_o$.

(b) $\vdash A_\alpha \equiv A_\alpha, \alpha \in$ *Types.*

(c) $\vdash \top$.

(d) $\vdash \boldsymbol{\Delta}\top \equiv \top$.

‡) *We introduce the main operations which will be used later.*

(e) If $\vdash A_\alpha \equiv B_\alpha$, *then* $\vdash B_\alpha \equiv A_\alpha$.

(f) $\vdash A_o$ *iff* $\vdash A_o \equiv \top$.

The following lemma introduces the properties with $\neg$.

Lemma 7

(a) $\vdash (A_o \equiv B_o) \equiv (\neg B_o \equiv \neg A_o)$.

(b) $\vdash A_o \equiv \neg\neg A_o$.

(c) $\vdash \neg\bot \equiv \top$ *and* $\vdash \neg\top \equiv \bot$.

Remark 3

In the next subsections of this chapter we will show the connection with axioms of IMTL-FTT (denoted by (FT1)-(FT16)) which have been introduced in [35].

2.5.2 Łukasiewicz properties

Lemma 8

The following properties are provable in Ł-FTT.

(a) $\vdash (A_o \Rightarrow B_o) \equiv \neg(A \mathbin{\&} \neg B_o)$.

(b) $\vdash A_o \nabla B_o \equiv (\neg A_o \Rightarrow B_o)$.

PROOF: (a)

(L1) $\vdash (A_o \mathbin{\&} \neg B_o) \equiv \neg(A_o \Rightarrow B_o)$ from def.of $\&$, by Lemma 7(b) and by Rule (R),

(L2) $\vdash (A_o \Rightarrow B_o) \equiv \neg(A \mathbin{\&} \neg B_o)$ from (L1) by Lemma 7(a,b) and Rule (R).

(b)

(L1) $\vdash A_o \nabla B_o \equiv \neg(\neg A_o \mathbin{\&} \neg B_o)$ from the def. of ∇,

(L2) $\vdash \neg(\neg A_o \mathbin{\&} \neg B_o) \equiv (\neg A_o \Rightarrow B_o)$ from def. of $\&$, by Lemma 7(b) and Rule (R),

(L3) $\vdash A_o \nabla B_o \equiv (\neg A_o \Rightarrow B_o)$ from (L1),(L2) by rule (R).

□

Lemma 9 (de Morgan rules)
Ł-FTTproves the following properties of $\wedge, \&, \vee, \nabla$*:*

(a) $\vdash \neg(A_o \wedge B_o) \equiv (\neg A_o \vee \neg B_o)$,

(b) $\vdash \neg(A_o \vee B_o) \equiv (\neg A_o \wedge \neg B_o)$,

(c) $\vdash \neg(A_o \& B_o) \equiv (\neg A_o \nabla \neg B_o)$

(d) $\vdash \neg(A_o \nabla B_o) \equiv (\neg A_o \& \neg B_o)$,

PROOF: (a)

(L1) $\vdash \neg(A_o \wedge B_o) \equiv \neg(A_o \& (A_o \Rightarrow B_o))$ by (ŁFT11) then using Lemma 7(a) and Lemma 6(a),

(L2) $\vdash \neg(A_o \wedge B_o) \equiv \neg(A_o \& \neg(A_o \& \neg B_o))$ from (L1) by Lemma 8(a) and by Rule (R),

(L3) $\vdash \neg(A_o \wedge B_o) \equiv (A_o \Rightarrow (A_o \& \neg B_o))$ from (L2) one more by Lemma 8(a) and by (R),

(L4) $\vdash \neg(A_o \wedge B_o) \equiv (\neg(A_o \& \neg B_o) \Rightarrow \neg A_o)$ from (L3) by (ŁFT9),

(L5) $\vdash \neg(A_o \wedge B_o) \equiv ((A_o \Rightarrow B_o) \Rightarrow \neg A_o)$ from (L4) by Lemma 8(a) and by (R),

(L6) $\vdash \neg(A_o \wedge B_o) \equiv ((\neg B_o \Rightarrow \neg A_o) \Rightarrow \neg A_o)$ from (L5) by (ŁFT9) and Rule (R),

(L7) $\vdash \neg(A_o \wedge B_o) \equiv \neg B_o \vee \neg A_o$ from (L6) by def. of $\vee$ and by (R).

(b) is analogous to (a).

(c)

(L1) $\vdash \neg(A_o \& B_o) \equiv (A_o \Rightarrow \neg B_o)$, from def. of $\&$, by Lemma 7(a,b) and by Lemma 6(a),

(L2) $\vdash (A_o \Rightarrow \neg B_o) \equiv (\neg A_o \nabla \neg B_o)$ by Lemma 8(b),

(L3) $\vdash \neg(A_o \& B_o) \equiv (\neg A_o \nabla \neg B_o)$ from (L1), (L2) by Rule (R).

(d)

(L1) $\vdash \neg(A_o \nabla B_o) \equiv \neg(\neg A_o \Rightarrow B_o)$ by Lemma 8(b), by Lemma 7(a) and by Lemma 6(a),

(L2) $\vdash \neg(\neg A_o \Rightarrow B_o) \equiv (\neg A_o \,\&\, \neg B_o)$ by Lemma 8(a) then by Lemma 7(a,b) and by Lemma 6(a),

(L3) $\vdash \neg(A_o \nabla B_o) \equiv (\neg A_o \,\&\, \neg B_o)$ from (L1) and (L2) by Rule (R).

□

The following lemma proves the commutativity rule for $\wedge, \&, \nabla$. The associativity of $\wedge, \&, \nabla$ will be introduced later, because their proofs need the properties which are derived using the inference rule of modus ponens which will be proved later.

Lemma 10 (commutativity of ∧, &, ∇)
The following properties are provable in Ł-FTT:

(a) $\vdash (A_o \wedge B_o) \equiv (B_o \wedge A_o)$,

(b) $\vdash (A_o \,\&\, B_o) \equiv (B_o \,\&\, A_o)$,

(c) $\vdash A_o \,\&\, B_o \Rightarrow A_o$,

(d) $\vdash A_o \wedge B_o \Rightarrow A_o$,

(e) $\vdash B_o \Rightarrow (A_o \nabla B_o)$,

(f) $\vdash A_o \nabla B_o \equiv B_o \nabla A_o$.

PROOF: (a)

(L1) $\vdash A_o \wedge B_o \equiv \neg(\neg A_o \vee \neg B_o)$ from Lemma 9(a), by Lemma 7(a,b), by (R) and by Lemma 6(a),

(L2) $\vdash A_o \wedge B_o \equiv \neg(\neg B_o \vee \neg A_o)$,from (L1) by (ŁFT10) and by (R),

(L3) $\vdash A_o \wedge B_o \equiv B_o \wedge A_o$ from (L2) by Lemma 9(a), using Lemma 7(b) and by (R).

(b)

(L1) $\vdash (A_o \Rightarrow \neg B_o) \equiv (B_o \Rightarrow \neg A_o)$ by (ŁFT9), using Lemma 7(b) and buy (R), (L2) $\vdash \neg(A_o \Rightarrow \neg B_o) \equiv \neg(B_o \Rightarrow \neg A_o)$, from (L1) by Lemma 7(a) and by Lemma 6(a),

(L3) $\vdash A_o \,\&\, B_o \equiv B_o \,\&\, A_o$ from (L2) using the def. of **&**.

(c)

(L1) $\vdash \neg A_o \Rightarrow (B_o \Rightarrow \neg A_o)$ is an instance of the axiom (ŁFT7),

(L2) $\vdash \neg(B_o \Rightarrow \neg A_o) \Rightarrow A_o$ from (L1) using (ŁFT9), by Lemma 7(b) and Rule (R),

(L3) $\vdash A_o \,\&\, B_o \Rightarrow A_o$ from (L2) using def.of **&**, property (b) and Rule (R).

(d) immediately results from (c) using axiom (ŁFTT11).

(e)

(L1) $\vdash B_o \Rightarrow (\neg A_o \Rightarrow B_o)$, it is instance of (ŁFT7),

(L2) $\vdash B_o \Rightarrow A_o \,\nabla\, B_o$ from (L1) by Lemma 8(b) using Rule (R).

(f)

(L1) $\vdash A_o \,\nabla\, B_o \equiv \neg A_o \Rightarrow B_o$ by Lemma 8(b),

(L2) $\vdash A_o \,\nabla\, B_o \equiv \neg B_o \Rightarrow A_o$ from (L1) by (ŁFT9) and by (R),

(L3) $\vdash A_o \,\nabla\, B_o \equiv B_o \,\nabla\, A_o$ from (L2) using Lemma 8(b) and by (R).

□

We can see that the property (d) is just the axiom (FT12) and the property (a) is the axiom (FT11) of IMTL-FTT.

2.5.3 First-order properties

The following theorems show that substitutable axioms and the rule of generalization are provable in Ł-FTT. The proofs are analogous as the proofs in [35].

Theorem 3
(a) $\vdash (\forall x_\alpha) B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$

(b) $\vdash B_{o,x_\alpha}[A_\alpha] \Rightarrow (\exists x_\alpha)B_o$

provided that A_α is substitutable to B_o for all free occurrences of x_α.

Theorem 4
If $T \vdash A_o$ then $T \vdash (\forall x_\alpha)A_o$.

Using the axioms of Ł-FTT and by the properties above, we can prove the following lemma (can be found in [35]).

Lemma 11

(a) $\vdash \bot \Rightarrow A_o$,

(b) $\vdash A_o \Rightarrow \top$ *and* $\vdash A_o \wedge \top \equiv A_o$,

(c) $\vdash \top \Rightarrow A_o) \equiv A_o$,

(d) $\vdash (A_o \equiv B_o) \Rightarrow (A_o \Rightarrow B_o)$ *and* $\vdash (A_o \equiv B_o) \Rightarrow (B_o \Rightarrow A_o)$,

(e) if $\vdash A_o$ *and* $\vdash B_o$ *then* $A_o \wedge B_o$.

PROOF: The proofs of (a),(b),(c),(d) are constructed analogously as in [35] but instead of the axiom (FT12) we used the property of Lemma 10(d). Let us stres that the proof of (a) is constructed using Theorem 3(a).

(d)

(L1) $\vdash A_o$ it is assumption,

(L2) $\vdash A_o \equiv \top$ from (L1) by Lemma 6(f),

(L3) $\vdash B_o$ it is assumption,

(L4) $\vdash B_o \equiv \top$ in the same way as (L2),

(L5) $\vdash \top \wedge \top \equiv \top$ using (b),

(L6) $\vdash \top \wedge \top$ from (L5) by Lemma 6(c) and Lemma 6(a),

(L7) $A_o \wedge B_o$ from (L2), (L4) and (L6) using Rule (R).

□

Theorem 5 (Rule of modus ponens)
If $T \vdash A_o$ and $T \vdash A_o \Rightarrow B_o$ then $T \vdash B_o$.

PROOF:

(L1) $T \vdash A_o \equiv \top$ assumption and by Lemma 6(f),

(L2) $T \vdash \top \Rightarrow B_o$ assumption and from (L1) by Rule (R),

(L3) $T \vdash \top \wedge B_o \equiv \top$ from (L2) and by def. of $\Rightarrow$,

(L4) $T \vdash \top \wedge B_o$ from (L3) by Lemma 6(a),(c),

(L5) $T \vdash \top \wedge B_o \equiv B_o$ from Lemma 11(b) and by Lemma 10(a),

(L6) $T \vdash B_o$ from (L4),(L5) and by Lemma 6(a).

$\square$

Definition 24
We say that A'_o is the variant of the formula A_o if A'_o gradually develops from A_o replaced by the subformulas in the following:

$$(Qx_\alpha)B_o$$

replaced by

$$(Qy_\alpha)B_{o,x_\alpha}[y_\alpha]$$

where Q is a general or existential quantifier and y_α is not a free variable in B_o.

Theorem 6
Let A'_o be a variant of A_o. Then $\vdash A_o \equiv A'_o$.

PROOF: Let x_α, y_α be different variables. Using Theorem 3(a) we get

$$\vdash (\forall x_\alpha)B_o \Rightarrow B_{o,x_\alpha}[y_\alpha].$$

By Theorem 4 we obtain

$$\vdash (\forall x_\alpha)B_o \Rightarrow (\forall y_\alpha)B_{o,x_\alpha}[y_\alpha].$$

Let us denote by B'_o the formula $B_{o,x_\alpha}[y_\alpha]$. Because x_α is not free in B'_o then once more by Theorem 3(a) we get

$$\vdash (\forall y_\alpha)B'_o \Rightarrow (\forall x_\alpha)B'_{o,y_\alpha}[x_\alpha].$$

But $B'_{o,y_\alpha}[x_\alpha]$ is the formula B_o. Thus we also have the opposite implication. Finally, by Lemma 11(e), using (ŁFT5) and by Lemma 6(a) we obtain that $\vdash (\forall x_\alpha)B_o \equiv (\forall y_\alpha)B_{o,x_\alpha}[y_\alpha]$. The proof for the existential quantifier is constructed analogously.

$\square$

Lemma 12

(a) $\vdash (A_o \& B_o) \Rightarrow (B_o \& A_o)$,

(b) $\vdash A_o \&(A_o \Rightarrow B_o) \Rightarrow (B_o \&(B_o \Rightarrow A_o))$,

(c) $\vdash A_o \Rightarrow ((A_o \Rightarrow B_o) \Rightarrow B_o)$,

(d) $\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o))$,

(d') $\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \equiv (B_o \Rightarrow (A_o \Rightarrow C_o))$,

(g) $\vdash A_o \Rightarrow (B_o \Rightarrow C_o) \equiv (A_o \& B_o) \Rightarrow C_o$,

(h) $\vdash A_o \Rightarrow A_o$,

(i) $\vdash \neg\neg A_o \Rightarrow A_o$,

(j) $\vdash (A_o \Rightarrow B_o) \Rightarrow (\neg B_o \Rightarrow \neg A_o)$.

PROOF: (a) It results from Lemma 10(b) and using Lemma 11(d) and MP.

(b) Immediately from Lemma 10(a) then using the axiom (ŁFT11), by Lemma 11(d) and by MP.

(c)

(L1) $\vdash A_o \Rightarrow ((B_o \Rightarrow A_o) \Rightarrow A_o)$ is an instance of (ŁFT7),

(L2) $\vdash A_o \Rightarrow ((A_o \Rightarrow B_o) \Rightarrow B_o)$ by (ŁFT10), using def. of $\vee$ and by (R).

(d)

(L1) $\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (((B_o \Rightarrow C_o) \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$ by (ŁFT8),

(L2) $\vdash (B_o \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow C_o)) \Rightarrow [(((B_o \Rightarrow C_o) \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o))]$ one more by (ŁFT8),

(L3) $\vdash (((B_o \Rightarrow C_o) \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o))$ from (L2) by MP using (c),

(L4) $\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o))$ from (L1), (L3) using transitivity.

(d')

(L1) $\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o))$ is an instance of (d), (L2) $\vdash (B_o \Rightarrow (A_o \Rightarrow C_o)) \Rightarrow (A_o \Rightarrow (B_o \Rightarrow C_o))$ is an instance of (d),

From Lemma 11(e) we obtain

$$\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o)) \wedge (B_o \Rightarrow (A_o \Rightarrow C_o)) \Rightarrow (A_o \Rightarrow (B_o \Rightarrow C_o)). \quad (2.5.1)$$

Then using (ŁFT5) and by Lemma 6(a) we get

$$\vdash (A_o \Rightarrow (B_o \Rightarrow C_o)) \equiv (B_o \Rightarrow (A_o \Rightarrow C_o)).$$

(g)

(L1) $\vdash (((A_o \,\&\, B_o) \wedge C_o) \equiv C_o) \equiv (((A_o \,\&\, B_o) \wedge C_o) \equiv C_o)$ is an instance of Lemma 6(b),

(L2) $\vdash ((A_o \,\&\, B_o) \Rightarrow C_o) \equiv (\neg(A_o \Rightarrow \neg B_o) \Rightarrow C_o)$ from (L1) by definition of $\Rightarrow$ and $\&$,

(L3) $\vdash ((A_o \,\&\, B_o) \Rightarrow C_o) \equiv A_o \Rightarrow (\neg C_o \Rightarrow \neg B_o)$ from (L2) by (ŁFT9), using Lemma 7(b), by (d') and (R),

(L4) $\vdash (A_o \,\&\, B_o) \Rightarrow C_o \equiv (A_o \Rightarrow (B_o \Rightarrow C_o))$ from (L3) by (ŁFT9), Rule (R).

(h)

(L1) $\vdash A_o \Rightarrow (\top \Rightarrow A_o)$ is an instance of (ŁFT7),

(L2) $\vdash \top \Rightarrow (A_o \Rightarrow A_o)$ from (L1) using Lemma 12(d') and Lemma 6(a),

(L3) $\vdash (A_o \Rightarrow A_o)$ from (L2) by Lemma 6(e) and MP.

(i)

(L1) $\vdash A_o \equiv \neg\neg A_o$ from Lemma 7(b),

(L2) $\vdash \neg\neg A_o \Rightarrow$ from (L1) by (ŁFT5), using Lemma 10(d) and by MP.

(j) It results from the axiom (ŁFT9), (ŁFT5) then by Lemma 10(d) and by MP. □

Lemma 13 (associativity of &, ∇)

(a) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv (A_o \mathbin{\&} B_o) \mathbin{\&} C_o$,

(b) $\vdash A_o \nabla (B_o \nabla C_o) \equiv (A_o \nabla B_o) \nabla C_o$,

PROOF: (a)

(L1) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg(A_o \Rightarrow \neg(B_o \mathbin{\&} C_o))$, using def. of $\&$,

(L2) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg((B_o \mathbin{\&} C_o) \Rightarrow \neg A_o)$ from (L1) by (ŁFT9) and (R),

(L3) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg(B_o \Rightarrow (C_o \Rightarrow \neg A_o))$, from (L2) using Lemma 12(g) and by (R),

(L4) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg(B_o \Rightarrow (A_o \Rightarrow \neg C_o))$ from (L3) using (ŁFT9) and by (R),

(L5) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg((B_o \mathbin{\&} A_o) \Rightarrow \neg C_o)$ from (L4) using Lemma 12(g) and by (R),

(L6) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv \neg((A_o \mathbin{\&} B_o) \Rightarrow \neg C_o)$ from (L5) by Lemma 10(b) and using (R),

(L7) $\vdash A_o \mathbin{\&} (B_o \mathbin{\&} C_o) \equiv (A_o \mathbin{\&} B_o) \mathbin{\&} C_o$ from (L6) using def of $\&$ and by (R).

(a)

(L1) $\vdash A_o \nabla (B_o \nabla C_o) \equiv \neg A_o \equiv \neg A_o \Rightarrow (\neg B_o \Rightarrow C_o)$

(L2) $\vdash A_o \nabla (B_o \nabla C_o) \equiv (\neg A_o \mathbin{\&} \neg B_o) \Rightarrow C_o$ from (L1) using Lemma 12(g) and by (R),

(L3) $\vdash A_o \nabla (B_o \nabla C_o) \equiv \neg(A_o \nabla B_o) \Rightarrow C_o$ from (L2) using Lemma 9(d) and by (R),

(L4) $\vdash A_o \nabla (B_o \nabla C_o) \equiv (A_o \nabla B_o) \nabla C_o$ from (L3) by Lemma 8(b) and using (R).

□

Lemma 14

(a) $\vdash (A_o \wedge B_o) \equiv (A_o \nabla \neg B_o) \mathbin{\&} B_o$,

(b) $\vdash (A_o \vee B_o) \equiv (A_o \mathbin{\&} \neg B_o) \nabla B_o$,

(c) $\vdash (A_o \mathbin{\&} \neg B_o) \nabla B_o \equiv A_o \nabla (B_o \mathbin{\&} \neg A_o)$,

(d) $\vdash (A_o \nabla \neg B_o) \mathbin{\&} B_o \equiv A_o \mathbin{\&} (B_o \nabla \neg A_o)$,

(e) $\vdash A_o \nabla \neg A_o$.

PROOF: (a)

(L1) $\vdash A_o \wedge B_o \equiv B_o \wedge A_o$ it follows from Lemma 10(a),

(L2) $\vdash A_o \wedge B_o \equiv B_o \mathbin{\&} (B_o \Rightarrow A_o)$ from (L1) using (ŁFT11) and by (R),

(L3) $\vdash A_o \wedge B_o \equiv (\neg A_o \Rightarrow \neg B_o) \mathbin{\&} B_o$, from (L2) using (ŁFT9) and by (R),

(L4) $\vdash A_o \wedge B_o \equiv (A_o \nabla \neg B_o) \mathbin{\&} B_o$ from (L3) using Lemma 8(b) and by (R).

(b)

(L1) $\vdash A_o \vee B_o \equiv (A_o \Rightarrow B_o) \Rightarrow B_o$ from def. of $\vee$,

(L2) $\vdash A_o \vee B_o \equiv \neg(A_o \mathbin{\&} \neg B_o) \Rightarrow B_o$ from (L1) using Lemma 8(a) and by (R),

(L3) $\vdash A_o \vee B_o \equiv (A_o \mathbin{\&} \neg B_o) \nabla B_o$ from (L2) by Lemma 8(b) and by (R).

The properties (c) and (d) result from commutativity of $\wedge$ and $\vee$ (where proved above) and from (a,b).

(e)

(L1) $\vdash \neg A_o \Rightarrow \neg A_o$ is an instance of Lemma 12(h),

(L2) $\vdash A_o \nabla \neg A_o$ from (L1) using Lemma 8(b), by Lemma 7(b), by (R) and finally by Lemma 6(a).

□

Lemma 15 (prelinearity)

$$\vdash (A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o). \tag{2.5.2}$$

PROOF: The proof of this property will be constructed in the same way as in [14]. We start with

(L1) $\vdash ((A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o) \equiv (A_o \Rightarrow B_o) \,\&\, \neg(B_o \Rightarrow A_o)) \nabla (B_o \Rightarrow A_o)$ it follows from Lemma 14(b),

Then we continue with commutativity and associativity of $\&, \nabla$, de-Morgan rules and by property from Lemma 14(b). Finally, we derive that $\vdash A_o \nabla \neg A_o \nabla \ldots$ which is a provable formula in Ł-FTT. □

Lemma 16

(a) $\vdash (A_o \vee A_o) \Rightarrow A_o$,

(b) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow ((A_o \vee B_o) \Rightarrow C_o))$.

PROOF:

(L1) $\vdash ((A_o \vee A_o) \Rightarrow A_o) \equiv (((A_o \Rightarrow A_o) \Rightarrow A_o) \Rightarrow A_o)$ from def of $\vee$ and by (R),

(L2) $\vdash ((A_o \vee A_o) \Rightarrow A_o) \equiv ((A_o \Rightarrow (A_o \Rightarrow A_o)) \Rightarrow (A_o \Rightarrow A_o))$ from (L1) using (ŁFT10) and by def. of $\vee$.

Due to the fact that $\vdash A_o \Rightarrow A_o$ is provable (it was shown in Lemma 12(h)) then property (a) is also provable using Lemma 6(a).

(b)

(L1) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((C_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow B_o))$ (ŁFT8),

(L2) $\vdash ((C_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow B_o)) \Rightarrow (((A_o \Rightarrow B_o) \Rightarrow B_o) \Rightarrow ((C_o \Rightarrow B_o) \Rightarrow B_o))$ one more by (ŁFT8),

(L3) $\vdash (A_o \Rightarrow C_o) \Rightarrow (((A_o \Rightarrow B_o) \Rightarrow B_o) \Rightarrow ((C_o \Rightarrow B_o) \Rightarrow B_o))$ from (L1) and (L2) by (ŁFT8),

(L4) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((A_o \vee B_o) \Rightarrow (C_o \vee B_o))$ from (L3) using def. of $\vee$ and by (R),

(L5) $\vdash (B_o \Rightarrow C_o) \Rightarrow ((B_o \vee C_o) \Rightarrow (C_o \vee C_o))$ analogously as in (L4),

(L6) $\vdash (B_o \Rightarrow C_o) \Rightarrow ((B_o \vee C_o) \Rightarrow C_o)$ from (L5) using (a) and by (ŁFT8),

(L7) $\vdash ((A_o \Rightarrow C_o) \,\&\, (A_o \vee B_o)) \Rightarrow (C_o \vee B_o)$ from (L4) by Lemma 12(g) and using Lemma 6(a), (L8) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((A_o \vee B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow C_o))$ from (L6), (L7) by Lemma 12(d) and MP, by transitivity, using the commutativity of $\vee$ and finally by Lemma 12(g),

(L9) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \vee B_o) \Rightarrow C_o)$ from (L8) by Lemma 12(d) and using transitivity.

□

Lemma 17

$$\vdash ((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o).$$

PROOF:

(L1) $\vdash ((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow (((A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o)) \Rightarrow C_o))$ is an instance of Lemma 16(b),

(L2) $\vdash ((A_o \Rightarrow B_o) \Rightarrow C_o) \,\&\, ((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow (((A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o)) \Rightarrow C_o)$ from (L1) by Lemma 12(g) and using Lemma 6(a),

(L3) $\vdash ((A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o)) \Rightarrow (((A_o \Rightarrow B_o) \Rightarrow C_o) \,\&\, ((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o)$ from (L2) using Lemma 12(d') and using Lemma 6(a),

(L4) $\vdash ((A_o \Rightarrow B_o) \Rightarrow C_o) \,\&\, ((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o$ from (L3) by prelinearity and MP,

(L5) $\vdash ((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o)$ from (L4) by Lemma 12(g) using Lemma 6(a).

□

Remark 4
Remember that (A1)–(A7) are understood as the axioms of basic fuzzy logic introduced by P. Hájek in [14].

We have shown that using the axioms of Ł-FTT we can prove all the axioms of BL-logic. Namely, (ŁFT8) is (A1), (A2) is the property in Lemma 10(a), (A3) and (A4) are properties from Lemma 12(a) and (b), Lemma 12(g) is just the axiom (A5), (A6) is provable in Lemma 17 and finally, Lemma 11(a) proves the last axiom (A7).

Together with the rule of modus ponens (Theorem 5), we obtain a formal system of propositional BL- logic. Thus all the other theorems of propositional BL-logic (which are also theorems of Łukasiewicz propositional logic) are also provable in Ł-FTT. Using propositional BL-logic together with the property of double negation (proved in Lemma 12(i)) we may prove other main properties of Ł-FTT. From this it results that all the axioms of IMTL-FTT and all the main properties of IMTL-FTT are also provable in Ł-FTT. We summarized in the following theorem that some of these properties needed in the sequel.

2.5.4 List of the main properties of Ł-FTT

Theorem 7
Let $A_o, B_o, C_o \in Form_o$ be formulas. Then the following properties are provable in Ł-FTT.

(P1) $\vdash (A_o \,\&\, (A_o \Rightarrow B_o)) \Rightarrow B_o$,

(P2) $\vdash A_o \Rightarrow (B_o \Rightarrow (A_o \,\&\, B_o))$,

(P3) $\vdash (A_o \Rightarrow B_o) \Rightarrow ((A_o \,\&\, C_o) \Rightarrow (B_o \,\&\, C_o))$,

(P4) $\vdash (A_o \Rightarrow B_o) \Rightarrow ((A_o \wedge C_o) \Rightarrow (B_o \wedge C_o))$,

(P5) $\vdash (A_o \,\&\, B_o) \Rightarrow (A_o \wedge B_o)$,

(P6) $\vdash ((A_o \Rightarrow B_o) \wedge (A_o \Rightarrow C_o)) \Rightarrow (A_o \Rightarrow (B_o \wedge C_o))$,

(P7) $\vdash ((A_o \Rightarrow B_o) \,\&\, (A_o \Rightarrow C_o)) \Rightarrow (A_o \Rightarrow (B_o \wedge C_o))$,

(P8) $\vdash (C_o \Rightarrow A_o) \Rightarrow ((C_o \Rightarrow B_o) \Rightarrow (C_o \Rightarrow (B_o \wedge A_o))$,

(P9) $\vdash A_o \wedge (B_o \wedge C_o) \equiv (A_o \wedge B_o) \wedge C_o$,

(P10) $\vdash (A_o \vee B_o) \vee C_o \equiv A_o \vee (B_o \vee C_o)$,

(P11) $\vdash (A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (A_o \wedge B_o))$,

(P12) $\vdash ((A_o \Rightarrow B_o) \Rightarrow B_o) \equiv ((B_o \Rightarrow A_o) \Rightarrow A_o)$

(P13) $\vdash (A_o \Rightarrow (A_o \vee B_o))$,

(P14) $\vdash (A_o \Rightarrow B_o) \Rightarrow ((A_o \vee B_o) \Rightarrow B_o)$,

(P15) $\vdash (A_o \vee B_o) \wedge A_o \equiv A_o$ *and* $\vdash (A_o \wedge B_o) \vee A_o \equiv A_o$,

(P16) $\vdash (A_o \,\&\, \top) \equiv A_o$,

(P17) $\vdash (A_o \wedge B_o) \,\&\, (A_o \wedge C_o) \Rightarrow (A_o \wedge (B_o \,\&\, C_o))$,

(P18) $\vdash (A_o \wedge (B_o \Rightarrow C_o)) \Rightarrow ((A_o \wedge B_o) \Rightarrow (A_o \wedge C_o))$,

(P19) $\vdash (A_o \,\&\, (B_o \Rightarrow C_o)) \Rightarrow ((A_o \,\&\, B_o) \Rightarrow (A_o \,\&\, C_o))$,

(P20) $\vdash A_o \,\&\, (B_o \wedge C_o) \equiv (A_o \,\&\, B_o) \wedge (A_o \,\&\, C_o)$,

(P21) $\vdash (A_o \,\&\, \neg A_o) \Rightarrow B_o$,

(P22) $\vdash (A_o \,\&\, \neg A_o) \equiv \bot$

The following lemma will be used in Chapter 4. Using its and others properties we will prove that all 105 generalized syllogism are strongly valid.

Lemma 18
Let $A_o, B_o, C_o, D_o \in Form_o$. Then the following formulas are provable in Ł-FTT.

(a) $A_o \Rightarrow (B_o \Rightarrow C_o) \vdash A_o \Rightarrow ((B_o \,\&\, D_o) \Rightarrow (C_o \,\&\, D_o))$,

(b) $A_o \Rightarrow (B_o \Rightarrow C_o) \vdash A_o \Rightarrow ((B_o \wedge D_o) \Rightarrow (C_o \wedge D_o))$,

(c) $A_o \Rightarrow B_o, C_o \Rightarrow D_o \vdash (A_o \wedge C_o) \Rightarrow (B_o \wedge D_o)$,

(d) $A_o \Rightarrow B_o, C_o \Rightarrow D_o \vdash (A_o \,\&\, C_o) \Rightarrow (B_o \,\&\, D_o)$,

(e) $A_o \Rightarrow B_o, C_o \Rightarrow D_o \vdash (A_o \nabla C_o) \Rightarrow (B_o \nabla D_o)$,

(f) $A_o, B_o \vdash A_o \,\&\, B_o$.

PROOF: Let $\vdash A_o \Rightarrow (B_o \Rightarrow C_o)$.

(a) Then by Lemma 12(g) and MP we have $\vdash (A_o \,\&\, B_o) \Rightarrow C_o$ and hence by (P3) we have $\vdash (A_o \,\&\, B_o) \,\&\, D_o \Rightarrow (C_o \,\&\, D_o)$. Thus by Lemma 13(a) and by (R) we obtain $\vdash A_o \,\&\, (B_o \,\&\, D_o) \Rightarrow (C_o \,\&\, D_o)$ and hence by Lemma 12(g) we get $\vdash A_o \Rightarrow ((B_o \,\&\, D_o) \Rightarrow (C_o \,\&\, D_o))$.

(b)

(L1) $\vdash A_o \Rightarrow (B_o \Rightarrow C_o)$ assumption, (L2) $\vdash B_o \Rightarrow (A_o \Rightarrow C_o)$ from (L1) by Lemma 12(d') and by Lemma 6(a),

(L3) $\vdash B_o \wedge D_o \Rightarrow ((A_o \Rightarrow C_o) \wedge D_o)$ from (L2) by (P4) and using MP,

(L4) $\vdash ((A_o \,\&\, D_o) \wedge C_o) \Rightarrow (D_o \wedge C_o)$ from Lemma 10(d) and by (P4), ,

(L5) $\vdash (A_o \,\&\, D_o) \wedge (A_o \,\&(A_o \Rightarrow C_o)) \Rightarrow ((A_o \,\&\, D_o) \wedge C_o)$ from (L4) by (P1) using (P4),

(L6) $\vdash A_o \,\&(D_o \wedge (A_o \Rightarrow C_o)) \Rightarrow ((A_o \,\&\, D_o) \wedge C_o)$ from (L5) by (P20) using Rule (R),

(L7) $\vdash A_o \,\&(D_o \wedge (A_o \Rightarrow C_o)) \Rightarrow (D_o \wedge C_o)$ from (L4), (L6) using (ŁFT8),

(L8) $\vdash ((A_o \Rightarrow C_o) \wedge D_o) \Rightarrow (A_o \Rightarrow (D_o \wedge C_o))$ from (L7) by Lemma 12(g),

(L9) $\vdash (B_o \wedge D_o) \Rightarrow (A_o \Rightarrow (D_o \wedge C_o))$ from (L3), (L8) using (ŁFT8),

(L10) $\vdash A_o \Rightarrow ((B_o \wedge D_o) \Rightarrow (D_o \wedge C_o))$ from (L9) by Lemma 12(d) and by MP.

(c) and (d) are provable analogously as in [35].

(e)

(L1) $\vdash A_o \Rightarrow B_o$ assumption,

(L2) $\vdash C_o \Rightarrow D_o$ assumption,

(L3) $\vdash \neg B_o \Rightarrow \neg A_o$ from (L1) by (ŁFT9) and using (R),

(L4) $\vdash \neg D_o \Rightarrow \neg C_o$ from (L2) by (ŁFT9) and using (R),

(L5) $\vdash (\neg B_o \,\&\, \neg D_o) \Rightarrow (\neg A_o \,\&\, \neg C_o)$ from (L3) and (L4) by the property (d),

(L6) $\vdash \neg(\neg A_o \,\&\, \neg C_o) \Rightarrow \neg(\neg B_o \,\&\, \neg D_o)$ from (L5) by (ŁFT9) and using Lemma 6(a),

(L7) $\vdash (A_o \,\nabla\, C_o) \Rightarrow (B_o \,\nabla\, D_o)$ from (L6) by def. of ∇ and by (R).

(f)

(L1) $\vdash A_o$ assumption,

(L2) $\vdash (A_o \equiv \top) \equiv A_o$ it is instance of (ŁFT6),

(L3) $\vdash (A_o \equiv \top)$ from (L1),(L2) by Rule (R),

(L4) $\vdash B_o$ assumption,

(L5) $\vdash (B_o \equiv \top)$ analogous to (L3),

(L6) $\vdash \top \mathbin{\&} \top$ it is an instance of (P16),

(L7) $\vdash A_o \mathbin{\&} B_o$ from (L3), (L5), (L6) by Rule (R).

□

2.5.5 Predicate Łukasiewicz properties

The axioms (ŁFT7-ŁFT10) are axioms of propositional Łukasiewicz logic and together with the rule of modus ponens (Theorem 5), they constitute its formal system. Together with Łukasiewicz-fuzzy propositional logic and rule of generalization (Theorem 4), we obtain a formal system of predicate Łukasiewicz fuzzy logic and hence, all its theorems are also provable in Ł-FTT. We introduce a lemma which summarizes the predicate properties of Łukasiewicz fuzzy logic.

Lemma 19
Let $A_o, B_o \in Form_o$ be formulas. Then the following is true.

(PP1) $(\forall x_\alpha)(A_o \Rightarrow B_o) \equiv (A_o \Rightarrow (\forall x_\alpha)B_o)$, *where x_α is not free in A_o,*

(PP2) $(\forall x_\alpha)(A_o \Rightarrow B_o) \equiv ((\exists x_\alpha)A_o \Rightarrow B_o)$, *where x_α is not free in B_o,*

(PP3) $(\exists x_\alpha)(A_o \Rightarrow B_o) \equiv (A_o \Rightarrow (\exists x_\alpha)B_o)$, *where x_α is not free in A_o,*

(PP4) $(\exists x_\alpha)(A_o \Rightarrow B_o) \equiv ((\forall x_\alpha)A_o \Rightarrow B_o)$, *where x_α is not free in B_o,*

(PP5) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\forall x_\alpha)A_o \Rightarrow (\forall x_\alpha)B_o)$,

(PP6) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha)A_o \Rightarrow (\exists x_\alpha)B_o)$

(PP7) $((\forall x_\alpha)A_o \mathbin{\&} (\exists x_\alpha)B_o) \Rightarrow (\exists x_o)(A_o \mathbin{\&} B_o)$,

(PP8) $(\forall x_\alpha)A_o(x_\alpha) \equiv (\forall y_\alpha)A_o(y_\alpha), (\exists x_\alpha)A_o(x_\alpha) \equiv (\exists y_\alpha)A_o(y_\alpha)$ *if* y_α *substitutable for* x_α *in* $A_o(x_\alpha)$,

(PP9) $(\exists x_\alpha)(A_o \,\&\, B_o) \equiv ((\exists x_\alpha)A_o \,\&\, B_o)$, *where* x_α *is not free in* B_o,

(PP10) $(\exists x_\alpha)(A_o \,\&\, A_o) \equiv ((\exists x_\alpha)A_o \,\&(\exists x_o)A_o)$,

(PP11) $(\exists x_\alpha)A_o \equiv \neg(\forall x_\alpha)\neg A_o$,

(PP12) $\neg(\exists x_\alpha)A_o \equiv (\forall x_\alpha)\neg A_o$,

(PP13) $(\exists x_\alpha)(A_o \wedge B_o) \equiv (A_o \wedge (\exists x_o)B_o)$, *where* x_α *is not free in* A_o,

(PP14) $(\exists x_\alpha)(A_o \vee B_o) \equiv (A_o \vee (\exists x_\alpha)B_o)$, *where* x_α *is not free in* A_o,

(PP15) $(\forall x_\alpha)(A_o \wedge B_o) \equiv (A_o \wedge (\forall x_\alpha)B_o)$, *where* x_α *is not free in* A_o,

(PP16) $(\forall x_\alpha)(A_o \vee B_o) \equiv (A_o \vee (\forall x_\alpha)B_o)$, *where* x_α *is not free in* A_o,

(PP17) $(\exists x_\alpha)(A_o \vee B_o) \equiv ((\exists x_\alpha)A_o \vee (\exists x_\alpha)B_o)$,

(PP18) $(\forall x_\alpha)(A_\alpha \wedge B_\alpha) \equiv ((\forall x_\alpha)A_o \wedge (\forall x_\alpha)B_o)$.

2.5.6 Properties of $\mathbf{\Delta}$

We continue with a theorem which presented the propositional properties of $\mathbf{\Delta}$. All proofs can be found in [35].

Theorem 8
Let $A_o, B_o, C_o \in Form_o$ *be formulas. Then the following properties are provable in Ł-FTT.*

(P$_\Delta$1) $\vdash \mathbf{\Delta}A_o \Rightarrow A_o$,

(P$_\Delta$2) $\vdash \mathbf{\Delta}\bot \equiv \bot$,

(P$_\Delta$3) $\vdash \bot \equiv (\forall y_o)\mathbf{\Delta}y_o$,

(P$_\Delta$4) $\vdash \mathbf{\Delta}A_o \vee \neg\mathbf{\Delta}A_o$,

(P$_\Delta$5) $\vdash \mathbf{\Delta}A_o \Rightarrow \mathbf{\Delta}\mathbf{\Delta}A_o$.

(P$_\Delta$6) $\vdash \mathbf{\Delta}(A_o \Rightarrow B_o) \Rightarrow (\mathbf{\Delta}A_o \Rightarrow \mathbf{\Delta}B_o)$,

($P_\Delta 7$) $\vdash \mathbf{\Delta}(\neg A_o) \Rightarrow \neg\mathbf{\Delta} A_o$,

($P_\Delta 8$) $\vdash \mathbf{\Delta}(A_o \Rightarrow B_o) \vee \mathbf{\Delta}(B_o \Rightarrow A_o)$,

($P_\Delta 9$) $\vdash \mathbf{\Delta} A_o \equiv \mathbf{\Delta} A_o \,\&\, \mathbf{\Delta} A_o$,

($P_\Delta 10$) $\vdash (\mathbf{\Delta} A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow ((\mathbf{\Delta} A_o \Rightarrow B_o) \Rightarrow (\mathbf{\Delta} A_o \Rightarrow C_o))$,

($P_\Delta 11$) $\vdash (\mathbf{\Delta} A_o \,\&\, \mathbf{\Delta}(A_o \Rightarrow B_o)) \Rightarrow \mathbf{\Delta} B_o$.

One may verify that properties ($P_{\mathbf{\Delta}}1$), ($P_{\mathbf{\Delta}}4$)-($P_{\mathbf{\Delta}}6$) are syntactic counterparts of properties (i), (iii)-(iv) of the MV_Δ-algebra. The axiom (ŁFT14) represents the property (ii) of the MV_Δ-algebra .

2.5.7 Deduction theorem

In the sections above we proved the main properties of Ł-FTT and also all axioms of IMTL-FTT. Namely, (ŁFT1)-(ŁFT6) are the same axioms as in IMTL-FTT axioms (FT1)-(FT6). Axiom (ŁFT8) is (FT7) of IMTL-FTT, the property from Lemma 12(d') is just axiom (FT8) of IMTL-FTT and finally, Lemma 17 is the axiom (FT9). Axiom (ŁFT9) is just axiom (FT10). Axioms (FT11) and (FT12) are provable in Lemma 10(a),(d). Axiom (FT13) is the property (P9). Axioms (FT14)-(FT16) are included among the axioms of Ł-FTT. Thus all others properties of IMTL-FTT are also provable in Ł-FTT.

We continue with a deduction theorem. Its proof is analogous to [35].(All properties which are used in the proof are also provable in our theory.) We start with a lemma which is used in the proof of deduction theorem and which may also be proved in our theory.

Lemma 20
Let $A_o \in Form_o$ be a closed formula and T be a theory such that $T \vdash \mathbf{\Delta} A_o \Rightarrow (D_\alpha \equiv E_\alpha)$. Then $T \vdash \mathbf{\Delta} A_o \Rightarrow (B_\beta \equiv C_\beta)$ where C_β is a formula resulting from B_β by replacing one occurrence of D_α in B_β by E_α under the same restrictions as in Rule (R).

Theorem 9 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ be a closed formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

Remark 5

Generally, the deduction theorem holds for every formula $A_o \in Form_o$ if all free variables included in A_o are also free in B_o. In other words, the rule of generalization is not used in the proof of B_o from $T \cup \{A_o\}$ for any variables which is free in A_o, then we may formulate the deduction theorem within the assumption that A_o is closed.

2.5.8 Predicate properties with Δ operation

We introduce a Lemma which summarized the properties of $\boldsymbol{\Delta}$ and an existential quantifier. We prove property (PP$_\Delta$1) because other properties are proved in [35].

Lemma 21

(PP$_\Delta$1) $T \vdash (\exists x_{\alpha_1}), \ldots, (\exists x_{\alpha_n})\boldsymbol{\Delta} B[x_{\alpha_1}, \ldots, x_{\alpha_n}]$ *iff*
$T \cup \{B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_\mathbf{n}}]\}$ *is a conservative extension of* T
where $\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_\mathbf{n}} \notin J(T)$ *Rule (C),*

(PP$_\Delta$2) $\vdash (\exists x_o)\boldsymbol{\Delta} A_o \Rightarrow \boldsymbol{\Delta}(\exists x_o)A_o$,

(PP$_\Delta$3) $\vdash (\exists x_o)\boldsymbol{\Delta} A_o \Rightarrow (\exists x_o)A_o$,

(PP$_\Delta$4) $\vdash (\exists x_o)(\exists y_o)\boldsymbol{\Delta} A_o \equiv (\exists x_o)\boldsymbol{\Delta}(\exists y_o)\boldsymbol{\Delta} A_o$.

PROOF: (PP$_\Delta$1): Let $T \vdash (\exists x_{\alpha_1}), \ldots, (\exists x_{\alpha_n})\boldsymbol{\Delta} B[x_{\alpha_1}, \ldots, x_{\alpha_n}]$. Let $T \cup \{B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_n}]\} \vdash A$ where A does not contain $\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_n}$. By deduction theorem we obtain that

$$T \vdash \boldsymbol{\Delta} B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_\mathbf{n}}] \Rightarrow A.$$

We replace all occurrences of $\mathbf{u}_{\alpha_1}, \ldots, \mathbf{u}_{\alpha_n}$ in the proof of A by variables $y_{\alpha_1}, \ldots, y_{\alpha_n}$ not occurring in it. Thus

$$T \vdash \boldsymbol{\Delta} B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[y_{\alpha_1}, \ldots, y_{\alpha_n}] \Rightarrow A.$$

If we use n-times the rule of generalization and using (PP2) n-times, we get

$$T \vdash (\exists x_{\alpha_1}), \ldots, (\exists x_{\alpha_n})\boldsymbol{\Delta} B \Rightarrow A.$$

From this we obtain $T \vdash A$ using the assumption and by (MP) which proves conservativeness.

The opposite implication: let $T' = T \cup \{B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u_{\alpha_1}},\ldots,\mathbf{u_{\alpha_n}}]\}$ is a conservative extension of T where $\mathbf{u_{\alpha_1}},\ldots,\mathbf{u_{\alpha_n}} \notin J(T)$. Then

$$T' \vdash B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u_{\alpha_1}},\ldots,\mathbf{u_{\alpha_n}}].$$

By Rule (N) we infer

$$T' \vdash \mathbf{\Delta} B_{x_{\alpha_1},\ldots,x_{\alpha_n}}[\mathbf{u_{\alpha_1}},\ldots,\mathbf{u_{\alpha_n}}].$$

From this and using Theorem 3(b) n-times, we obtain

$$T' \vdash (\exists x_{\alpha_1}),\ldots,(\exists x_{\alpha_n})\mathbf{\Delta} B.$$

From the assumption that T' is a conservative extension of T we obtain

$$T \vdash (\exists x_{\alpha_1}),\ldots,(\exists x_{\alpha_n})\mathbf{\Delta} B.$$

$\square$

2.5.9 Properties of equality

In this section, we introduce properties demonstrating syntactical properties of the fuzzy equality. All proofs can be found in [35].

Theorem 10

(a) $\vdash (\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha) \equiv (f_{\beta\alpha} \equiv g_{\beta\alpha})$,

(b) $\vdash ((x_\gamma \equiv y_\gamma)\,\&\,(y_\gamma \equiv z_\gamma)) \Rightarrow (x_\gamma \equiv z_\gamma)$, *for all* $\gamma \in$ *Types,*

(c) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)$,

(d) $\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta) \Rightarrow (\mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta))$,

(d') $\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta)\,\&\,\mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta)$.

Property (b) demonstrates that the fuzzy equality is transitive for all types. Properties (c),(d) characterize weak extensionality of all functions with respect to the fuzzy equality.

2.6 Contradictory, consistent and extensionally complete theory

Definition 25
Let T be a theory. We say that:

(i) T is contradictory if

$$T \vdash \bot.$$

Otherwise it is consistent.

(ii) T is maximal consistent if each its extension T',$T' \supset T$ is inconsistent.

(iii) T is linear if for every two formulas A_o, B_o, the following is true:

$$T \vdash A_o \Rightarrow B_o \quad or \quad T \vdash B_o \Rightarrow A_o.$$

(iv) T is extensionally complete if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ it follows that there is a closed formula C_α such that $T \not\vdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha$.

It can be shown that this definition generalizes the definition of Henkin theory in the sense of Hájek. Indeed, if we put $A_{o\alpha} := \lambda x_\alpha A_o$ and $B_{o\alpha} := \lambda x_\alpha \top$ then the formula $A_{\beta\alpha} \equiv B_{\beta\alpha}$ is equivalent to $(\forall x_\alpha)A_o$. Thus for this case the definition above is just the definition of Henkin theory presented by Hájek in [14].

Lemma 22
A theory T is contradictory iff $T \vdash A_o$ for each A_o.

PROOF: If T proves each formula, then it proves $\top$. Conversely, if $T \vdash \top$, then $T \vdash A_o$ from Lemma 11(a) and by MP. □

Lemma 23
Let T be a theory and $T \vdash A_o$. Then $T \vdash \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}A_o$.

PROOF:

(L1) $T \vdash A_o$ assumption,

(L2) $T \vdash \boldsymbol{\Delta}A_o$ (L1), Rule (N), (L3) $T \vdash \mathbf{\Delta} A_o \equiv \neg\neg\mathbf{\Delta} A_o$ instance of Lemma 7(b),Lemma 6(a)

(L4) $T \vdash \mathbf{\Delta}\neg\neg\mathbf{\Delta} A_o$ from (L2), (L3) by Lemma 6(a) and by Rule (N),

(L5) $T \vdash \neg\mathbf{\Delta}\neg\mathbf{\Delta} A_o$ by ($P_\Delta 7$).

□

Theorem 11
Every consistent theory T can be extended to a maximal consistent linear theory $\bar{T}$.

PROOF: Analogously as Theorem 19 in [35] we apply Zorn's lemma. Then we continue with the same steps as in [35]. We used the property from Lemma 16(b) and then ($P_\Delta 8$). □

Theorem 12
To every consistent theory T, there is an extensionally complete theory $\bar{T}$ which is an extension of T.

PROOF: Analogously as the proof of Theorem 20 in [35] by deduction theorem, using the property from Lemma 16(b), by MP and using ($P_\Delta 8$). Then we continue with the same steps with two cases:

- case(a): we used (P13), MP, Lemma 23, axiom (ŁFT$_2$) and rule of substitution.
- case(b): by (PP16), by Theorem 10(a) and Rule (R)

□

2.7 Canonical model of Ł-FTT

In this section we will construct a canonical model of a consistent theory of Ł-FTT in the same way as in the classical type theory. We start with constructing of the set M_o of truth values and its appropriate algebraic structure.

Let us define an equivalence on the set of closed formulas from $Form_o$ by

$$A_o \approx B_o \quad \text{iff} \quad T \vdash A_o \equiv B_o. \tag{2.7.1}$$

Using Lemma 6(b),(e) and by Theorem 10(b), we can verify that $\approx$ is the equivalence. The equivalence class of a formula A_o is denoted by $|A_o|$ and we put $\bar{M}_o = Form_o\,|\!\approx$.

Definition 26
We will define the operations on the set $\bar{M}_o$ as follows:

(i) $|A_o| \wedge_T |B_o| = |A_o \wedge B_o|$,

(ii) $|A_o| \vee_T |B_o| = |A_o \vee B_o|$,

(iii) $|A_o| \otimes_T |B_o| = |A_o \,\&\, B_o|$,

(iv) $|A_o| \rightarrow_T |B_o| = |A_o \Rightarrow B_o|$,

(v) $\boldsymbol{\Delta}_T(|A_o|) = |\boldsymbol{\Delta} A_o|$,

(vi) *put* $\mathbf{1}_T = |\top|, \mathbf{0}_T = |\bot|$.

Theorem 13
Let T be a linear theory. Then the algebra

$$\mathcal{L}_T = \langle \bar{M}_o, \wedge_T, \vee_T, \otimes_T, \rightarrow_T, \Delta_T, \mathbf{1}_T, \mathbf{0}_T \rangle \tag{2.7.2}$$

is a linearly ordered MV$_\Delta$-algebra.

PROOF: The proof proceeds in the same way as the proof of analogous theorem in IMTL-FTT. We introduce properties from which it follows that $\mathcal{L}_T$ is linearly ordered and so MV$_\Delta$-algebra. The property from Lemma 10(a) and axiom (ŁFT10) demonstrate that $\wedge, \vee$ are commutative. Properties (P9) and (P10) show that $\wedge, \vee$ are associative. (P15) is an absorbtion and the property from Lemma 13(a) verifies that **&** is associative. Finally, by (P16), using the property from Lemma 12(g)(it is adjunction) and by the property from Lemma 10(b)(commutativity of **&**), we may show that $\mathcal{L}_T$ is a residuated lattice. Lemma 15 proves prelinearity and Lemma 7(b) proves involution. Finally, axiom (ŁFT11) guarantees divisibility. Thus $\mathcal{L}_T$ is an MV-algebra. From linearity of T it is obvious that $\mathcal{L}_T$ is linearly ordered. Properties for $\boldsymbol{\Delta}$ can be proved using $(P_\Delta 1)$,$(P_\Delta 2)$, $(P_\Delta 4)-(P_\Delta 6)$ and by axiom (ŁFT14). Thus we conclude that $\mathcal{L}_T$ is a linearly ordered MV$_\Delta$-algebra. □

2.7.1 Construction of the canonical model

Let T be a linear and extensionally complete theory. We will extend the equivalence (2.7.1) to closed formulas of all types as follows:

$$A_\alpha \sim B_\alpha \quad \text{iff} \quad T \vdash A_\alpha \equiv B_\alpha. \tag{2.7.3}$$

By the same way as in (2.7.1), we may verify that (2.7.3) is also an equivalence. The class of a formula A_α of type α is denoted by $|A_\alpha|$. Now we continue with the definition of a special function $\mathcal{V}$, whose domain and range are formulas or their equivalence classes.

Basic canonical frame

Definition 27

Let $\mathcal{V}$ be a special function, whose domain and range are formulas of their equivalence classes of formulas. Then the basic canonical frame *is as follows:*

$$M_\alpha = \{\mathcal{V}(A_\alpha) \mid A_\alpha \in \mathit{Form}_\alpha\}, \quad \alpha \in \mathit{Types}. \tag{2.7.4}$$

where

(i) if $\alpha = o$ then $\mathcal{V}(A_o) = |A_o|$, i.e $M_o = \mathit{Form}_o| \approx$,

(ii) if $\alpha = \epsilon$ then $\mathcal{V}(A_\epsilon) = |A_\epsilon|$, i.e $M_\epsilon = \mathit{Form}_\epsilon| \approx$,

(iii) if $\alpha = \gamma\beta$ then we put $\mathcal{V}(A_{\gamma\beta}) \subseteq M_\beta \times M_\gamma$ which is a relation consisting of couples

$$\langle \mathcal{V}(B_\beta), \mathcal{V}(A_{\gamma\beta}B_\beta)\rangle$$

for all closed $B_\beta \in \mathit{Form}_\beta$ and $A_{\gamma\beta}\mathit{Form}_{\gamma\beta}$.

Definition 28

Let M_α be a family sets from (2.7.4). Then the fuzzy equality in each set M_α, $\alpha \neq o$, is defined by the following:

$$=_\alpha (\mathcal{V}(A_\alpha), \mathcal{V}(B_\alpha)) := |A_\alpha \equiv B_\alpha|. \tag{2.7.5}$$

Lemma 24

The relation (2.7.5) is a fuzzy equality on M_α, $\alpha \in \mathit{Types}$.

PROOF: The reflexivity and symmetry of (2.7.5) follow from Lemma 6(b),(e) and transitivity follows from Theorem 10(b). □

The proof of the following lemma is analogous to the proof of [39] Lemma 21.

Lemma 25

(a) If T is an extensionally complete theory, then for all types $\alpha = \gamma\beta$

$$[\mathcal{V}(A_{\gamma\beta}) =_{\gamma\beta} \mathcal{V}(B_{\gamma\beta})] = \bigwedge_{C_\beta \in \mathit{Form}_\beta} [\mathcal{V}(A_{\gamma\beta}C_\beta) =_\gamma \mathcal{V}(B_{\gamma\beta}C_\beta)] \tag{2.7.6}$$

where formulas C_β in (2.7.6) are closed.

(b) *For all* $\alpha \in$ *Types*

$$[\mathcal{V}(A_\alpha) =_\alpha \mathcal{V}(B_\alpha)] = \mathbf{1} \quad \textit{iff} \quad T \vdash A_\alpha \equiv B_\alpha.$$

The proof of the following lemma is analogous to the proof of [39] Lemma 22.

Lemma 26
Let T be an extensionally complete theory. Then $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function.

General canonical frame

Definition 29
Let M_α be a basic canonical frame from (2.7.4) and fuzzy equalities $=_\alpha$ be given in (2.7.5). Let $\mathcal{L}_T$ be a linearly ordered MV$_{\mathbf{\Delta}}$ algebra from (2.7.2). Then the general canonical frame *is a tuple*

$$\mathcal{M}^T = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_T \rangle. \tag{2.7.7}$$

2.7.2 Canonical model of Ł-FTT

We start with a definition of interpretation of formulas in the general canonical frame.

Definition 30
Let p be an assignment of elements to variables, i.e. $p(x_\alpha) = \mathcal{V}(A_\alpha) \in M_\alpha$ for all $\alpha \in$ Types. Then we put:

(i) *If x_α is a variable then $\mathcal{M}^T_p(x_\alpha) = p(x_\alpha)$.*

(ii) *If c_α, $\alpha \neq o$ is a constant then $\mathcal{M}^T_p(c_\alpha) = \mathcal{V}(c_\alpha) \in M_\alpha$. Furthermore,*

  (a) *$\mathcal{M}^T_p(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality (2.7.5), $\alpha \in$ Types.*

  (b) *Interpretation of the conjunction $\mathbf{C}_{(oo)o}$ is given by (i) in Definition 26. Interpretation of the delta operation $\mathbf{D}_{(oo)o}$ is given by (v) Definition 26.*

(iii) *Interpretation of the formula $\lambda x_\alpha A_\beta$ of type $\beta\alpha$ is the function*

$$\mathcal{M}^T_p(\lambda x_\alpha A_\beta)\colon \mathcal{V}(B_\beta) \mapsto \mathcal{V}((\lambda x_\alpha A_\beta) B_\alpha) \tag{2.7.8}$$

for each assignment $p' = p \backslash x_\alpha$, where $p'(x_\alpha) = \mathcal{V}(B_\alpha)$.

(iv) Interpretation of the description operator is the function

$$\mathcal{M}_p^T(\iota_{\alpha(o\alpha)})\colon \mathcal{V}(A_{o\alpha}) \mapsto \mathcal{V}(\iota_{\alpha(o\alpha)}A_{o\alpha}), \quad \alpha = o, \epsilon. \tag{2.7.9}$$

To verify that (2.7.8) and (2.7.9) are weakly extensional, see subsection 5.3 in [39].

We conclude that (2.7.7) is a general *safe canonical model* of Ł-FTT.

2.7.3 Completeness theorems in Ł-FTT

The proofs of the following two theorems are analogous to the proofs of the same theorems presented in subsection 5.4 in [39]).

Theorem 14
A theory T is consistent iff it has a safe general model $\mathcal{M}$.

PROOF: If T is inconsistent then $T \vdash \bot$. Thus, if $\mathcal{M} \models T$ then $\mathcal{M}_p(\bot) = \mathbf{1}$, which is impossible. Conversely: first, we extend T to a linear extensionally complete theory T' and construct its canonical model. Then from the construction of the canonical model it is obvious that $\mathcal{M}_p^T(A_\alpha) \in M_\alpha$ holds for every formula A_α and for every assignment $p \in Asg(\mathcal{M}^T)$. Let A_o be an extension of T. Then $T \vdash A_o$ and so, $T' \vdash A_o$, too. By (ŁFT6) and Rule (R) we have $T' \vdash A_o \equiv \top$ and so $T \vdash A_o \equiv \top$. Using (2.7.1) we obtain

$$\mathcal{M}_p^T(A_o) = |\top| = \mathbf{1}_T.$$

This means that $\mathcal{M}^T$ is a safe model of T. □

Theorem 15
For every theory T and formula A_o

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

PROOF: The implication left-to-right is the soundness theorem.

Conversely, let us consider the canonical model $\mathcal{M}^T$ of T and let $\mathcal{M}_p^T(A_o) = \mathbf{1} = |\top|$ for some assignment p. This means that $T \vdash A_o \equiv \top$, i.e. $T \vdash A_o$. Hence $T \not\vdash A_o$ means that $\mathcal{M}_p^T(A_o) \neq \mathbf{1}_T$. □

Chapter 3

Trichotomous evaluative linguistic expressions

We continue with a chapter where we will speak about evaluative linguistic expressions which occur in many applications of fuzzy logic. In this thesis we will use these expressions in definitions of ten intermediate quantifiers introduced in the following chapter.

3.1 Syntactical characterization

Trichotomous evaluative linguistic expressions are expressions of a natural language, for example, *small, medium, big, about fourteen, very short, more or less deep, quite roughly strong* which will be denoted by $\langle$TE-adjective$\rangle$. Other ones are *very small, very big, extremely thin*, etc. These expressions contain *linguistic hedges* which are divided into two groups. The first are linguistic hedges *extremely, significantly, very* (narrowing) and the second *more or less, roughly, quite roughly, very roughly* (widening) or *rather* (specifying). Notice a special empty hedge which makes it possible to deal with evaluative expressions, e.g., "large" and "very large" in a unified way.

Definition 31 (Linguistic hedge)
The $\langle$linguistic hedge$\rangle$ is an intensifying adverb making the meaning of the evaluative

expressions either more or less specific. We distinguish the following:

$$\langle \textit{linguistic hedge} \rangle := \textsf{empty hedge} \mid \langle \textit{narrowing hedge} \rangle \mid \langle \textit{widening hedge} \rangle \mid \langle \textit{specifying hedge} \rangle$$

Definition 32 (Evaluative linguistic expression)
An evaluative linguistic expression *is either of the following:*

(i) ⟨*numeral*⟩ *is a name of some element from the considered scale*[*)]

(ii) Simple evaluative expression, *which is one of the linguistic expressions:*

(a) ⟨*trichotomous evaluative expression*⟩ := ⟨*linguistic hedge*⟩⟨*TE-adjective*⟩

(b) ⟨*fuzzy quantity*⟩ := ⟨*linguistic hedge*⟩⟨*numeral*⟩

(iii) Negative evaluative expression, *which is an expression*

not ⟨*trichotomous evaluative expression*⟩

(iv) Compound evaluative expression, *which is either of the following:*

(a) ⟨*trichotomous evaluative expression*⟩ or ⟨*trichotomous evaluative expression*⟩

(b) ⟨*trichotomous evaluative expression*⟩ and ⟨*negative evaluative expression*⟩

The connective "and" in the compound expression (iii)(b) can be replaced by the connective "but".

In the sequel, we will consider only simple expressions with the following syntactical structure:

$$\langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle \tag{3.1.1}$$

where ⟨TE-adjective⟩ is an evaluative adjective which also includes a class of gradable adjectives. A typical feature of TE-adjectives is that they form pairs of antonyms (e.g., small–big) completed by a middle member (medium). Canonical TE-adjectives are *small, medium, big.* In a concrete situation, of course, they can be replaced by more proper adjectives such as *short, medium short, long*, etc.

*) *Fuzzy quantities require a concrete semantics, i.e. a concrete scale. From the point of logic, it is a constant in a language expanded by the names of all elements of a given model.*

We will also consider *negative evaluative expressions*

$$\textsf{not}\ (\textsf{empty hedge}\langle \text{TE-adjective}\rangle). \tag{3.1.2}$$

The expressions in (3.1.1) and (3.1.2) will be used in the definition of ten intermediate quantifiers in Chapter 4.

3.2 Formal theory of evaluative expressions

The meaning of evaluative expressions is formalized within a formal logical theory T^{Ev} which is a special theory of Ł-FTT. The main ideas for its construction are covered in [30, 33, 40]. The meaning of an evaluative linguistic expression is constructed as a special formula representing *intension* whose interpretation in a model is a function from a set of possible worlds (in our theory, we prefer to speak about *contexts*) into a set of fuzzy sets. For each possible world, the intension determines the corresponding extension, which is a fuzzy set in some universe constructed using a specific *horizon* which can be shifted along the latter. All the details of the formal theory T^{Ev} and motivation can be found in [37, 36].

3.2.1 Language of T^{Ev}

First, we define a formal language J^{Ev} of the theory T^{Ev}. Its special symbols are:

(i) A constant (formula) $F \in Form_{(oo)o}$ for additional fuzzy equality on truth values.

(ii) A special constant $\bar{\boldsymbol{\nu}}_{oo}$ for the standard (i.e. empty) hedge

(iii) A constant † which represents the formula

$$\dagger := \iota_{o(oo)}\, \lambda z\, (\neg z \equiv z).$$

Remember that $\iota_{o(oo)}$ is the description operator.

We introduce a definition of a new formula for a specific fuzzy equality on truth values:

$$\sim_{(oo)o} := \lambda z\, \lambda t\, F_{(oo)o}\, tz. \tag{3.2.1}$$

Its properties are characterized by axioms (EV1-EV9) which will be introduced below.

3.2.2 Context of T^{Ev}

Context in T^{Ev} is understood as a formula $w_{\alpha o}$ whose interpretation is a function $w : L \longrightarrow M_\alpha$. Hence, the context determines in M_α a triple of elements $\langle v_L, v_S, v_R\rangle$ where $v_L, v_S, v_R \in M_\alpha$ and $v_L = \mathcal{M}_p(w\bot)$, $v_S = \mathcal{M}_p(w\dagger)$, $v_R = \mathcal{M}_p(w\top)$.

Remark 6

For the theory of intermediate quantifiers, we may consider only abstract expressions such as "very small" which contain no specification of " what is indeed small"[†)]*. Consequently, they have only one (abstract) context and so, their intension actually coincides with their extension.*

3.2.3 Horizon and hedges

The fuzzy equality $\sim$ makes it possible to introduce three horizons:

$$\begin{aligned} LH_{oo} &:= \lambda z_o\, (\bot \sim z_o),\\ MH_{oo} &:= \lambda z_o\, (\dagger \sim z_o),\\ RH_{oo} &:= \lambda z_o\, (\top \sim z_o). \end{aligned}$$

The left horizon LH is a function assigning to each z_o a truth degree of the fuzzy equality with $\bot$; similarly right RH and middle MH horizons.

We introduce special formulas of type oo so that we will denote $\boldsymbol{\nu}$ and call *abstract hedges* or simply hedges. To characterize their properties, we will define the following auxiliary formulas of type $o(oo)$:

$$H^1 \equiv \lambda \boldsymbol{\nu}\, (\boldsymbol{\nu}\, z \wedge \neg \boldsymbol{\nu}\, t), \tag{3.2.2}$$

$$H^2 \equiv \lambda \boldsymbol{\nu}\, (((t \Rightarrow z) \Rightarrow (\boldsymbol{\nu}\, t \Rightarrow t))\,\&((z \Rightarrow t) \Rightarrow (t \Rightarrow \boldsymbol{\nu}\, t))), \tag{3.2.3}$$

$$H^3 \equiv \lambda \boldsymbol{\nu}\, (\boldsymbol{\Delta}(z \Rightarrow t) \Rightarrow (\boldsymbol{\nu}\, z \Rightarrow \boldsymbol{\nu}\, t)) \tag{3.2.4}$$

where symbols $z, t, u, v \in \mathit{Form}_o$ are variables of type o.

Then we introduce a formula $\mathit{Hedge} \in \mathit{Form}_{o(oo)}$ saying that $\boldsymbol{\nu} \in \mathit{Form}_{oo}$ is a hedge:

$$\begin{aligned} \mathit{Hedge} \equiv \lambda \boldsymbol{\nu}\, ((\exists t)(\exists z)(\boldsymbol{\Delta}((t \Rightarrow z) \wedge (t \not\equiv z) \wedge (H^1 \boldsymbol{\nu})))\\ \wedge (\exists z)\boldsymbol{\Delta}(\forall t)(H^2 \boldsymbol{\nu}) \wedge (\forall z)(\forall t)(H^3 \boldsymbol{\nu})). \end{aligned} \tag{3.2.5}$$

[†)] *For example, a "very small animal" suggests considering various sizes of animals depending on context. Expressions of the form '$\mathcal{A}$ is ⟨noun⟩', where $\mathcal{A}$ is an evaluative expression, are called* evaluative (linguistic) predications.

The meaning of (3.2.5) is as follows: formula H^1 expresses that the hedge $\boldsymbol{\nu}$ sends a truth value z to top and a truth value t to bottom. Using H^3, which expresses monotonicity, it is obvious that also all bigger (smaller) truth values are mapped to the top (bottom). Finally, formula H^2 requires existence of an "inner truth value" splitting the behavior of the hedge $\boldsymbol{\nu}$ into two cases so that modification of truth values is "small" if they are "big", and "big" if they are "small". We say that a formula $\boldsymbol{\nu} \in Form_{oo}$ is a hedge if $T^{\mathrm{Ev}} \vdash Hedge\ \boldsymbol{\nu}$.

Definition 33 (Function interpreting hedges)
The interpretation of hedges are functions $\nu : L \to L$ on truth values having the following properties:

(i) There are $a, c \in L$ such that $a < c, \nu(a) = \mathbf{0}$ and $\nu(c) = \mathbf{1}$.

(ii) For all $x, y \in L$, $x \le y$ implies $\nu(x) \le \nu(y)$.

(iii) There is $b, a \le b \le c$ such that:

1. *$x \le b$ implies $\nu(x) \le x$,*

2. *$b \le x$ implies $x \le \nu(x)$ for all $x \in L$.*

We can see that ν is monotone, it sends some truth value to the top and some other truth value to the bottom, and there is an inner truth value b. If the interpretation of $\boldsymbol{\nu}$ has these properties, then formulas H^1, H^2, H^3 are true in degree $\mathbf{1}$.

Let us consider two hedges $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}_1 \wedge Hedge\, \boldsymbol{\nu}_2$. We define a relation of partial ordering of hedges by

$$\preceq := \lambda p_{oo}\, \lambda q_{oo}\, ((\forall z)(p_{oo} z \Rightarrow q_{oo} z)). \tag{3.2.6}$$

If $T^{\mathrm{Ev}} \vdash \boldsymbol{\nu}_1 \preceq \boldsymbol{\nu}_2$ then we say that hedge $\boldsymbol{\nu}_1$ has a *narrowing effect* with respect to $\boldsymbol{\nu}_2$, and $\boldsymbol{\nu}_2$ has a *widening effect* with respect to $\boldsymbol{\nu}_1$.

Before defining a natural hedge, we introduce special crisp formulas:

$$\Upsilon_{oo} \equiv \lambda z_o(\neg \boldsymbol{\Delta}(\neg z_o)), \qquad \text{(nonzero truth value)}$$
$$\hat{\Upsilon}_{oo} \equiv \lambda z_o(\neg \boldsymbol{\Delta}(z_o \vee \neg z_o)). \qquad \text{(general truth value)}$$

Definition 34 (Natural hedge)
Let $\boldsymbol{\nu} \in Form_{oo}$ *be a formula such that* $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$. *We say that* $\boldsymbol{\nu}$ *is a* natural hedge *if*

$$T^{\mathrm{Ev}} \vdash (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(LH\ u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(MH\ u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(RH\ u)$$

and introduce a special formula

$$NatHedge \equiv \lambda \boldsymbol{\nu}\,(Hedge\,\boldsymbol{\nu} \wedge ((\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(LH\ u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(MH\ u) \wedge (\exists u)\hat{\Upsilon}\,\boldsymbol{\nu}(RH\ u))). \tag{3.2.7}$$

One of the hedges plays the central role in the theory of evaluative expressions. This hedge is on surface level *empty*. We will denote it by a special constant $\bar{\boldsymbol{\nu}} \in Form_{oo}$ and call it a *standard hedge*.

We introduce the following special hedges: $\{Ex, Si, Ve, ML, Ro, QR, VR\}$ (*extremely, significantly, very, more or less, roughly, quite roughly, very roughly*, respectively) which are ordered as follows:

$$Ex \preceq Si \preceq Ve \preceq \bar{\boldsymbol{\nu}} \preceq ML \preceq Ro \preceq QR \preceq VR \tag{3.2.8}$$

Hedges Ex, Si, Ve have a narrowing effect with respect to the empty hedge and ML, Ro, QR, VR have a widening effect.

3.2.4 Axioms of $T^{\mathbf{Ev}}$

The theory T^{Ev} has 11 special axioms which characterize properties of both constants, properties of contexts (see below) and properties of special formulas which represent linguistic hedges.

Definition 35
The following formulas are the axioms of T^{Ev}.

(EV1) $(\exists z)\boldsymbol{\Delta}(\neg z \equiv z)$

(EV2) $(\bot \equiv w^{-1}\bot_w) \wedge (\dagger \equiv w^{-1}\dagger_w) \wedge (\top \equiv w^{-1}\top_w)$.

(EV3) $t \sim t$,

(EV4) $t \sim u \equiv u \sim t$,

(EV5) $(t \sim u \,\&\, u \sim z) \Rightarrow t \sim z$,

(EV6) $\neg(\bot \sim \dagger)$,

(EV7) $\boldsymbol{\Delta}((t \Rightarrow u) \,\&\, (u \Rightarrow z)) \Rightarrow (t \sim z \Rightarrow t \sim u)$,

(EV8) $t \equiv t' \,\&\, z \equiv z' \Rightarrow (t \sim z \Rightarrow t' \sim z')$,

(EV9) $(\exists u)\hat{\Upsilon}(\bot \sim u) \wedge (\exists u)\hat{\Upsilon}(\dagger \sim u) \wedge (\exists u)\hat{\Upsilon}(\top \sim u)$,

(EV10) $\mathit{NatHedge}\, \bar{\boldsymbol{\nu}} \,\&\, (\exists \boldsymbol{\nu})(\exists \boldsymbol{\nu}')(\mathit{Hedge}\, \boldsymbol{\nu} \,\&\, \mathit{Hedge}\, \boldsymbol{\nu}' \,\&\, (\boldsymbol{\nu}_1 \preceq \bar{\boldsymbol{\nu}} \wedge \bar{\boldsymbol{\nu}} \preceq \boldsymbol{\nu}_2))$,

(EV11) $(\forall z)((\Upsilon \bar{\boldsymbol{\nu}}(LH\, z)) \vee (\Upsilon \bar{\boldsymbol{\nu}}(MH\, z)) \vee (\Upsilon \bar{\boldsymbol{\nu}}(RH\, z)))$.

The first axiom of T^{Ev} is an axiom assuring the existence of the middle truth value. In the semantics of Ł-FTT the interpretation of † is equal to truth value 0.5. The second axiom assures that the assignment of $\bot$ to $\bot_w$ and $\top$ to $\top_w$ is one-to-one. Axioms (EV3)-(EV5) state that $\sim$ is a fuzzy equality. Axiom (EV6) expresses that falsity and medium truth are not equal in the sense of $\sim$. Axiom (EV7) expresses compatibility of $\sim$ with classical ordering of truth values. Axiom (EV8) expresses that $\sim$ is strongly extensional. Axiom (EV9) assures that $\sim$ is not crisp, i.e. we can find a truth value u that is not fully $\sim$-equal to either of $\bot$, † and $\top$. Axiom (EV10) expresses that $\bar{\boldsymbol{\nu}}$ is a natural hedge lying "among" other hedges so that some hedges have a narrowing effect and some widening with respect to $\bar{\boldsymbol{\nu}}$. Axiom (EV11) is used to prove that the scale is covered by the fundamental evaluative trichotomy (for details see [37]).

3.2.5 Properties of †

Using the properties of Ł-FTT and axiom (EV1), we may prove the properties below.

Lemma 27
Let T^{Ev} be a formal theory and $A_o \in \mathit{Form}_o$. Then the following is true:

(a) $T^{\mathrm{Ev}} \vdash A_o \wedge \neg A_o \Rightarrow \dagger$,

(b) $T^{\mathrm{Ev}} \vdash \dagger \Rightarrow A_o \vee \neg A_o$.

PROOF: (a)

(L1) $T^{\mathrm{Ev}} \vdash (A_o \Rightarrow \dagger) \vee (\dagger \Rightarrow A_o)$, is an instance of Lemma 15

(L2) $T^{\mathrm{Ev}} \vdash (A_o \Rightarrow \dagger) \vee (\neg A_o \Rightarrow \dagger)$, from (L1) by (ŁFT9), (EV1) and using (R),

(L3) $T^{\mathrm{Ev}} \vdash \neg(A_o \,\&\, \dagger) \vee \neg(\neg A_o \,\&\, \dagger)$, from (L2) using Lemma 8(a), (EV1) and by Rule (R),

(L4) $T^{\mathrm{Ev}} \vdash \neg((A_o \,\&\, \dagger) \wedge (\neg A_o \,\&\, \dagger))$, from (L3) using Lemma 9(a) and by Lemma 6(a),

(L5) $T^{\mathrm{Ev}} \vdash \neg(\dagger \,\&(A_o \wedge \neg A_o))$, from (L4) using (P30) and by Lemma 6(a),

(L6) $T^{\mathrm{Ev}} \vdash (A_o \wedge \neg A_o) \Rightarrow \dagger$, form (L5) using Lemma 8(a) and by Lemma 6(a).

(b) Hence from (a) by axiom (ŁFT9), using Lemma 9(a) and by Rule (R). □

Definition 36
The following formulas represent intensions of simple evaluative expressions (3.1.1):

(i) $Sm := \lambda \boldsymbol{\nu}\ \lambda z\, (\boldsymbol{\nu}(LH\, z))$,

(ii) $Me := \lambda \boldsymbol{\nu}\ \lambda z\, (\boldsymbol{\nu}(MH\, z))$,

(iii) $Bi := \lambda \boldsymbol{\nu}\ \lambda z\, (\boldsymbol{\nu}(RH\, z))$.

Note that the structure of these formulas represents construction of corresponding extensions, whose interpretation in a model is schematically depicted in Figure 3.1. To simplify the explanation, we will often use a general metavariable *Ev* standing for intensions (i)–(iii) above.

A specific role in this theory is played by formulas $Sm\,\boldsymbol{\Delta}$, $Me\,\boldsymbol{\Delta}$, $Bi\boldsymbol{\Delta}$, where the connective $\boldsymbol{\Delta}$ has been used as a specific hedge that can be taken as a linguistic hedge "utmost" (or, alternatively a "limit"). This makes it possible to also include classical quantifiers in our theory without the necessity to introduce them as a special case different from the rest of the theory.

3.3 Canonical model

A *canonical model* of T^{Ev} is based on the frame

$$\mathcal{M}^0 = \langle (M_\gamma, =_\gamma)_{\gamma \in Types}, \mathcal{L}_\Delta \rangle$$

where $\mathcal{L}_\Delta$ is the standard Łukasiewicz$_\Delta$-algebra and $M_\epsilon = \mathbb{R}$ (set of real numbers). Interpretation of the constant $\mathcal{M}^0(\dagger) = 0.5$.

Interpretation of special formulas of T^{Ev} in the canonical model, together with construction of extensions of evaluative expressions, is schematically depicted in Figure 3.1. According to our theory, it is easy to see that the kernel of the fuzzy set

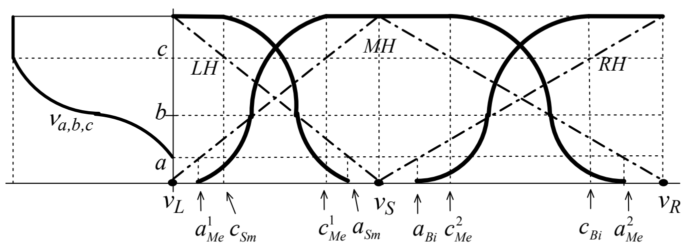


Figure 3.1: Scheme of the construction of extensions of evaluative expressions ($\nu_{a,b,c}$ is a function interpreting hedge $\boldsymbol{\nu}$ in $\mathcal{M}^0$; it is turned 90° counterclockwise).

is shortened or prolonged if a hedge with a narrowing effect (such as *very*) or that with a widening effect (such as *roughly*) is present (linguistic arguments for such behavior are given in [22]).

The following properties of evaluative expressions will be used in the sequel:

Theorem 16 ([37])
Let $\boldsymbol{\nu}$ be a hedge. Then

(a) $T^{\mathrm{Ev}} \vdash (Sm\,\boldsymbol{\nu})\bot$,

(b) $T^{\mathrm{Ev}} \vdash (Bi\,\boldsymbol{\nu})\top$,

(c) $T^{\mathrm{Ev}} \vdash (\forall z)((Sm\,\boldsymbol{\nu})z \Rightarrow \neg(Bi\,\boldsymbol{\nu})z)$,

(d) $T^{\mathrm{Ev}} \vdash (\forall z)((Bi\,\boldsymbol{\nu})z \Rightarrow \neg(Sm\,\boldsymbol{\nu})z)$.

Chapter 4

Intermediate Quantifiers and Generalized Syllogisms

The formal theory of *intermediate quantifiers* was originally introduced by Novák in [38]. According to its basic idea, these quantifiers are classical general or existential quantifiers, but the universe of quantification is modified and the modification can be imprecise.

The main goal of this chapter is to prove all 105 valid forms of the generalized intermediate syllogisms. By 105 forms we mean 24 traditional, 69 Thompson's which were introduced in [47], and 12 new syllogisms which were proposed by Peterson in [45]. First, we introduce definitions of all ten generalized intermediate quantifiers and then we show that all the 105 forms of the generalized syllogisms are formally valid in our theory of intermediate quantifiers. First, we start with basic definitions.

4.1 Theory of intermediate quantifiers

We introduce the theory of intermediate quantifiers T^{IQ} which is a special theory of Ł-FTT extending the theory T^{Ev} of evaluative linguistic expressions introduced in the previous chapter.

We must consider a measure on fuzzy sets for the theory of intermediate quantifiers. In T^{IQ}, we will represent it syntactically by considering a special formula μ of type $o(o\alpha)(o\alpha)$ whose interpretation are values taken from the set of truth values.

4.1.1 Definition of the measure

Definition 37
Let $R \in Form_{o(o\alpha)(o\alpha)}$ be a formula. Put

$$\mu := \lambda z_{o\alpha}\, \lambda x_{o\alpha}\, (R z_{o\alpha}) x_{o\alpha}. \tag{4.1.1}$$

We say that the formula $\mu \in Form_{o(o\alpha)(o\alpha)}$ represents a measure on fuzzy sets *in the universe of type $\alpha \in Types$ if it has the following properties:*

(M1) $\mathbf{\Delta}(x_{o\alpha} \equiv z_{o\alpha}) \equiv ((\mu z_{o\alpha}) x_{o\alpha} \equiv \top)$,

(M2) $\mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \Rightarrow ((\mu z_{o\alpha}) x_{o\alpha} \Rightarrow (\mu z_{o\alpha}) y_{o\alpha})$,

(M3) $\mathbf{\Delta}(z_{o\alpha} \not\equiv \emptyset_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})(z_{o\alpha} - x_{o\alpha}) \equiv \neg(\mu z_{o\alpha}) x_{o\alpha})$,

(M4) $\mathbf{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha}) x_{o\alpha} \Rightarrow (\mu y_{o\alpha}) x_{o\alpha})$.

One can see that the measure is normed with respect to a distinguished fuzzy set $z_{o\alpha}$. Remember that the operations on fuzzy sets included in the previous definition were introduced in Definition 23.

4.1.2 Definition of the theory $T^{\mathbf{IQ}}$

For the following definition, we have to consider a set of selected types $\mathcal{S}$ to which our theory will be confined. The reason is to avoid possible difficulties with interpretation of the formula μ for complex types which may correspond to sets of very large, possibly non-measurable cardinalities. This means that our theory is not fully general. However, we do not see it as a limitation, because one can hardly imagine the meaning of "most X's" over a set of inaccessible cardinality. On the other hand, our theory works whenever there is a model in which we can define a measure in the sense of Definition 37. The theory T^{IQ} defined below is thus parametrized by the set $\mathcal{S}$.

Let us introduce the following special formula representing a fuzzy set of all measurable fuzzy sets in the given type α:

$$\mathbf{M}_{o(o\alpha)} := \lambda z_{o\alpha}\, (\mathbf{\Delta}(\mu z_{o\alpha}) z_{o\alpha} \,\&\, (\forall x_{o\alpha})(\forall y_{o\alpha})((\mathrm{M2}) \,\&\, (\mathrm{M4})) \,\&\, (\forall x_{o\alpha}))(\mathrm{M3}) \tag{4.1.2}$$

where (M2)–(M4) are the respective axioms from Definition 37.

Definition 38
Let $\mathcal{S} \subseteq$ *Types be a distinguished set of types and* $\{R \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ *be a set of new constants. The* theory of intermediate quantifiers $T^{IQ}[\mathcal{S}]$ *w.r.t.* $\mathcal{S}$ *is a formal theory of L-FTT defined as follows:*

(i) The language of $T^{IQ}[\mathcal{S}]$ *is*

$$J^{Ev} \cup \{R_{o(o\alpha)(o\alpha)} \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}.$$

(ii) Special axioms of $T^{IQ}[\mathcal{S}]$ *are those of* T^{Ev} *and*

$$(\exists z_{o\alpha})\mathbf{M}_{o(o\alpha)} z_{o\alpha}, \qquad \alpha \in \mathcal{S}. \tag{4.1.3}$$

Remark 7
In definition of intermediate quantifiers, we consider special context, namely identical function $w_{oo} := \lambda x_o x_o$*. To simplify the notation in this case, we will omit the variable* w *for context in the corresponding formula: for example* $(Ve\,Bi)x_o$ *means that* x_o *is "very big" in the abstract context* w_{oo}*.*

4.1.3 Definition of intermediate generalized quantifiers

Intermediate quantifiers were formally defined by Novák in [38]. The following definition is a slight modification of the original definition by considering strong conjunction instead of the ordinary one.

Definition 39
Let $Ev \in Form_{oo}$ *be intension of some evaluative expression. Furthermore, let* $z \in Form_{o\alpha}$*,* $x \in Form_{\alpha}$ *be variables and* $A, B \in Form_{o\alpha}$ *be formulas,* $\alpha \in \mathcal{S}$*, such that*

$$T^{IQ}[S] \vdash \mathbf{M}_{o(o\alpha)} B_{o\alpha}$$

holds true. Then a type $\langle 1, 1\rangle$ *of intermediate generalized quantifier interpreting the sentence*

> *"⟨Quantifier⟩ B's are A"*

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\, \&(\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B)z)), \tag{4.1.4}$$

$$(Q^{\exists}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\, \&(\exists x)(zx \wedge Ax)) \wedge Ev((\mu B)z)). \tag{4.1.5}$$

4.1.4 Definition of intermediate generalized quantifiers with presupposition

The *presupposition* plays a very important role in proving intermediate generalized syllogisms. When a presupposition is needed, the previous definition must be slightly modified.

Definition 40
Let $T^{IQ}[\mathcal{S}]$ be a theory of intermediate quantifiers in the sense of Definition 38 and Ev, z, x, A, B be the same as in Definition 39. Then an intermediate generalized quantifier with presupposition is the formula

$$({}^*Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\,\&(\exists x)zx\,\&(\forall x)(z\,x \Rightarrow Ax)) \wedge Ev((\mu B)z)).$$

Note that only non-empty subsets of B are considered in this definition. We will now introduce definitions of several specific intermediate quantifiers based on the analysis provided by Peterson in his book [45].

Note that each formula above consists of three parts:

$$\underbrace{(\exists z)((\mathbf{\Delta}(z \subseteq B)}_{\text{"the greatest" part of } B\text{'s}} \quad \& \quad \underbrace{(\forall x)(z\,x \Rightarrow Ax))}_{\text{each } z\text{'s has } A} \quad \wedge \quad \underbrace{Ev((\mu B)z))}_{\text{size of } z \text{ is evaluated by } Ev} \tag{4.1.6}$$

4.1.5 Definition of ten intermediate generalized quantifiers

Definition 41
Let $T^{IQ}[\mathcal{S}]$ be a theory of intermediate quantifiers. Let $z \in Form_{o\alpha}$, $x \in Form_{\alpha}$ and $A, B \in Form_{o\alpha}$ be the same as in Definition 39. Then the following special

intermediate quantifiers can be introduced:

$$\mathbf{A:}\ \textit{All } B \textit{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),$$

$$\mathbf{E:}\ \textit{No } B \textit{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),$$

$$\mathbf{P:}\ \textit{Almost all } B \textit{ are } A := Q^{\forall}_{Bi\,Ex}(B, A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$
(*extremely big part of B has A*)

$$\mathbf{B:}\ \textit{Few } B \textit{ are } A\ (:= \textit{Almost all } B \textit{ are not } A) := Q^{\forall}_{Bi\,Ex}(B, \neg A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$
(*extremely big part of B does not have A*)

$$\mathbf{T:}\ \textit{Most } B \textit{ are } A := Q^{\forall}_{Bi\,Ve}(B, A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$
(*very big part of B has A*)

$$\mathbf{D:}\ \textit{Most } B \textit{ are not } A := Q^{\forall}_{Bi\,Ve}(B, \neg A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$
(*very big part of B does not have A*)

$$\mathbf{K:}\ \textit{Many } B \textit{ are } A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$
(*not small part of B does has A*)

$$\mathbf{G:}\ \textit{Many } B \textit{ are not } A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, \neg A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$
(*not small part of B does not have A*)

$$\mathbf{I:}\ \textit{Some } B \textit{ are } A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax),$$

$$\mathbf{O:}\ \textit{Some } B \textit{ are not } A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax).$$

Let us emphasize that there are two meanings of the quantifier "Most", namely "more than half" and "close to all". In our theory, we construe "Most" in the second meaning.

Remark 8

*Letters denoting specific quantifiers are introduced as follows: letters **A**, **I** denote formulas without negation, i.e. they are affirmatives; in Latin "***A***ff***I***rmo". Letters **E**,**O** denote formulas with negation, i.e. in Latin "n***E***g***O***".*

Other new letters are determined by the position of the respective quantifiers in the generalized Aristotle square (see [45]). Since "Almost-all" and "Few" are called **P**redominant statements, *the corresponding quantifiers are denoted by* ***P*** *and* ***B*** *respectively (the sound "B" is close to "P"). Similarly, "Most" is called a* Majori**T**y statement *and so, quantifiers "Most" and "Most B are not A" are denoted by* ***T*** *and* ***D***, *respectively. Finally, "Many" is called a* Common statement *and so, quantifiers "Many" and "Many B are not A" denoted by* ***K*** *and* ***G***, *respectively.*

Remark 9
By ***A**, ***E**, ***P**, ***B**, ***T**, ***D**, ***K**, ***G** *we denote quantifiers which contain a presupposition. Analogously, the specific quantifiers "Most", "Many", etc. with a presupposition will be written as "*Most", "*Many", etc.*

The following theorem demonstrates that an important role in our theory is played by monotonicity.

Theorem 17 (Valid implications; [38])
The following sets of implications are provable in T^{IQ}*:*

(a) $T^{IQ} \vdash \mathbf{A} \Rightarrow \mathbf{P}$, $\quad T^{IQ} \vdash \mathbf{P} \Rightarrow \mathbf{T}$, $\quad T^{IQ} \vdash \mathbf{T} \Rightarrow \mathbf{K}$.

(b) $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{B}$, $\quad T^{IQ} \vdash \mathbf{B} \Rightarrow \mathbf{D}$, $\quad T^{IQ} \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

4.2 Generalized Aristotle's syllogisms

As mentioned in the previous comment, Peterson demonstrated in his book [45] 105 generalized syllogisms which should be valid. In the section below, we syntactically prove that all the syllogisms are valid in T^{IQ}. (All syntactical proofs of intermediate generalize syllogisms are presented in [27]).

Definition 42
A classical syllogism *(or* logical appeal*) is a special kind of logical argument in which the* conclusion *is inferred from two* premises*: the* major premise *(first) and* minor premise *(second). The syllogisms will be written as triples of formulas* $\langle P_1, P_2, C\rangle$*. An* intermediate syllogism *is obtained from any classical syllogism (valid or not) when replacing one or more of its formulas by formulas containing intermediate quantifiers.*

Remark 10

In this section, we will consider a theory T being some extension of T^{IQ}. This will mostly be just the theory T^{IQ} but in a few cases more properties are required.

Definition 43

We say that syllogism $\langle P_1, P_2, C\rangle$ is strongly valid *in T if $T \vdash P_1 \,\&\, P_2 \Rightarrow C$, or equivalently, if $T \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$. We say that $\langle P_1, P_2, C\rangle$ is* weakly valid[*)] *in T if $T \cup \{P_1, P_2\} \vdash C$.*

4.2.1 Classification of syllogisms

We know from the classical theory of syllogisms that the latter are divided into four figures. Suppose that Q_1, Q_2, Q_3 are intermediate quantifiers and $X, Y, M \in Form_{o\alpha}$ are formulas representing properties. Then the following figures can be considered:

Figure I	**Figure II**	**Figure III**	**Figure IV**
Q_1 M is Y	Q_1 Y is M	Q_1 M is Y	Q_1 Y is M
Q_2 X is M	Q_2 X is M	Q_2 M is X	Q_2 M is X
Q_3 X is Y	Q_3 X is Y	Q_3 X is Y	Q_3 X is Y

Remark 11

Notation: If we will work with some syllogism, then instead of the tabular which is as follows

All M are Y
All X are M
All X are Y

we will use the **AAA**-*I. Similarly for other syllogisms.*

4.2.2 List of 93 Thompson's intermediate generalized syllogisms

Below, we introduce 93 Thompson's intermediate generalized syllogisms (24 traditional and 69 Thompson's) and prove that they are strongly valid either just in $T^{\mathrm{IQ}}[S]$ for some $\mathcal{S}$ or in its slight extension T.

First, we start with a table presenting the traditional Aristotle's syllogisms.

[*)] *This definition is introduced for completeness because in the future we may find syllogisms which will be weakly valid in our theory.*

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAA	EAE	A(*A)I	(*A)AI
EAE	AEE	IAI	AEE
AII	EIO	AII	IAI
EIO	AOO	E(*A)O	E(*A)O
A(*A)I	E(*A)O	OAO	EIO
E(*A)O	A(*E)O	EIO	A(*E)O

Remember that we can find in old publications notations of classical syllogisms by the names of the people. Namely, for example the first syllogism **AAA**-I has notation using the name B**a**rb**ara**, the syllogism **EAE**-II has notation by C**e**s**are**, etc. For better working we will use the notation introduced in the table above.

We continue with syllogisms which contain intermediate quantifier *most*.

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAT	***AED***	A(*T)I	AED
ATT	***ADD***	E(*T)O	E(*T)O
A(*T)I	A(*D)O	(*T)AI	(*T)AI
EAD	***EAD***	(*D)AO	
ETD	***ETD***		
E(*T)O	***ETO***		

The following are syllogisms with intermediate quantifiers *most* and *many*.

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAK	***AEG***	A(*K)I	***AEG***
ATK	***ADG***	E(*K)O	E(*K)O
A(*K)I	A(*G)O	(*K)AI	(*K)AI
AKK	***AGG***	(*G)AO	
EAG	***EAG***		
ETG	***ETG***		
E(*K)O	E(*K)O		
EKG	***EKG***		

We finish with intermediate syllogisms containing intermediate quantifiers *almost all, most, many* and *few*.

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAP	***AEB***	**(*P)AI**	***AEB***
APP	***ABB***	**E(*P)O**	**(*P)AI**
APT	***ABD***	**(*B)AO**	**E(*P)O**
APK	***ABG***	**A(*P)I**	
API	**A(*B)O**		
EAB	***EAB***		
EPB	***EPB***		
EPD	***EPD***		
EPG	***EPG***		
E(*P)O	**E(*P)O**		

4.3 Valid intermediate generalized syllogisms based on $T^{\mathbf{IQ}}$

In this section, we will give formal proofs of all the intermediate syllogisms summarized above. $X, Y, M \in Form_{o\alpha}$ are formulas representing properties and $x \in Form_{\alpha}$ is a variable of type α. We will also fix the set S and write T^{IQ} instead of $T^{\mathrm{IQ}}[S]$.

4.3.1 Figure I

Theorem 18
The following syllogisms are strongly valid in T^{IQ}:

$$\boldsymbol{AAA}\text{-}I: \begin{array}{l} \textit{All } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{All } X \textit{ are } Y \end{array} \qquad \boldsymbol{AAT}\text{-}I: \begin{array}{l} \textit{All } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Most } X \textit{ are } Y \end{array}$$

$$\boldsymbol{AAK}\text{-}I: \begin{array}{l} \textit{All } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Many } X \textit{ are } Y \end{array} \qquad \boldsymbol{AAP}\text{-}I: \begin{array}{l} \textit{All } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Almost all } X \textit{ are } Y \end{array}$$

PROOF: Using Definition 41 we can formally write syllogism **AAA**-I as

$$\begin{array}{c} (\forall x)(Mx \Rightarrow Yx) \\ (\forall x)(Xx \Rightarrow Mx) \\ \hline (\forall x)(Xx \Rightarrow Yx). \end{array}$$

The strong validity results from axiom (ŁFT8) then by Lemma 12(d) and (g):

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Yx) \Rightarrow ((Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow Yx))$$

By the rule of generalization and by (PP5) we have

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx) \Rightarrow ((\forall x)(Xx \Rightarrow Mx) \Rightarrow (\forall x)(Xx \Rightarrow Yx))$$

which is just strong validity of **AAA**-I.

From the strong validity of syllogism **AAA**-I by Theorem 17(a) we obtain strong validity of other three syllogisms **AAT**-I, **AAK**-I, and **AAP**-I. □

Theorem 19
All the syllogisms below are strongly valid in T^{IQ}:

$$\textbf{\textit{EAE}}\text{-}I: \begin{array}{l} \textit{No } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{No } X \textit{ are } Y \end{array} \qquad \textbf{\textit{EAB}}\text{-}I: \begin{array}{l} \textit{No } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Few } X \textit{ are } Y \end{array}$$

$$\textbf{\textit{EAD}}\text{-}I: \begin{array}{l} \textit{No } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Most } X \textit{ are not } Y \end{array} \qquad \textbf{\textit{EAG}}\text{-}I: \begin{array}{l} \textit{No } M \textit{ are } Y \\ \textit{All } X \textit{ are } M \\ \hline \textit{Many } X \textit{ are not } Y \end{array}$$

PROOF: The first syllogism can be formally written as

$$\begin{array}{c} (\forall x)(Mx \Rightarrow \neg Yx) \\ (\forall x)(Xx \Rightarrow Mx) \\ \hline (\forall x)(Xx \Rightarrow \neg Yx). \end{array}$$

Then the strong validity of **EAE**-I follows immediately from the provable formula

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \mathbin{\&} (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow \neg Yx);$$

then we proceed analogously as in the proof of **AAA**-I.

Analogously, we can also prove strong validity of other three syllogisms **EAB**-I, **EAD**-I, and **EAG**-I from **EAE**-I. □

Now we introduce two classical syllogisms (on the left is the linguistic form, on the right is the corresponding formal expression).

Theorem 20
The following syllogisms are strongly valid in T^{IQ}:

$$\textbf{\textit{EIO}}\text{-I: } \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Some } X \textit{ are } M\end{array}}{\textit{Some } X \textit{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow \neg Yx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

$$\textbf{\textit{AII}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Some } X \textit{ are } M\end{array}}{\textit{Some } X \textit{ are } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge Yx)}$$

PROOF: Using (P4) we know that $T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx))$. Then by the rule of generalization and by (PP5) and (PP6) we obtain strong validity of the classical syllogism **EIO**-I. Analogously we can also prove strong validity of **AII**-I. □

Theorem 21
All the syllogisms below are strongly valid in T^{IQ}:

$$\textbf{\textit{ATT}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are } Y} \qquad \textbf{\textit{AKK}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Many } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are } Y}$$

$$\textbf{\textit{APP}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Almost all } X \textit{ are } Y} \qquad \textbf{\textit{APT}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are } Y}$$

$$\textbf{\textit{ATK}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are } Y} \qquad \textbf{\textit{APK}}\text{-I: } \frac{\begin{array}{l}\textit{All } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are } Y}$$

PROOF: Analogously as above, we can formally write the first syllogism as follows:

$$\textbf{ATT}\text{-I:}\quad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu X)z))\end{array}}{(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)((\mu X)z)).}$$

Let us denote by $Ez := (Bi\ Ve)((\mu X)z)$. By the axiom (ŁFT8), using Lemma 12(d) and (g), using (PP5) we can prove that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow (\forall x)(zx \Rightarrow Yx)).$$

By Lemma 18(a,b) we obtain that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx) \Rightarrow \{((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow ((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge Ez)\}. \tag{4.3.1}$$

Finally, by the rule of generalization with respect to $(\forall z)$ and by (PP5), (PP6) and by Lemma 12(g) we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow Yx)\,\&(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow (\exists z)(\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge Ez).$$

If we replace Ez by $(Bi\ Ve)((\mu X)z)$, then we obtain strong validity of **ATT**-I. If we put $Ez := (Bi\,Ex)((\mu X)z)$, we obtain strong validity of **APP**-I; by putting $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu X)z)$ we obtain strong validity of **AKK**-I. Using Theorem 17(a), we can prove strong validity of **APT**-I and **APK**-I from the syllogism **APP**-I. Analogously, we obtain strong validity **ATK**-I from **ATT**-I. □

Theorem 22

The following syllogisms are strongly valid in T^{IQ}:

$$\textbf{\textit{EPB}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Few } X \textit{ are } Y} \qquad \textbf{\textit{EPD}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

$$\textbf{\textit{EPG}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y} \qquad \textbf{\textit{ETD}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

$$\textbf{\textit{ETG}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y} \qquad \textbf{\textit{EKG}}\text{-}I:\quad \frac{\begin{array}{l}\textit{No } M \textit{ are } Y\\ \textit{Many } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

PROOF: Syllogism **EPB**-I can be formally written as follows:

$$\textbf{EPB}\text{-I:}\quad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow \neg Yx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\,Ex)((\mu X)z))\end{array}}{(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\,Ex)((\mu X)z)).}$$

Similarly as above, we start with a provable formula

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx).$$

Using the same properties as in the proof above we obtain strong validity of EPB-I.

By Theorem 17(a) we obtain strong validity of other syllogisms **EPD**-I and **EPG**-I from **EPB**-I. Similarly, we obtain strong validity of **ETD**-I and hence, by Theorem 17(b), we also get strong validity of **ETG**-I and **EKG**-I. □

4.3.2 Figure I — syllogisms with presupposition:

In this subsection, we introduce syllogisms which require presupposition. First, we start with two classical syllogisms which require a presupposition in a minor premise.

Theorem 23
The following syllogisms are strongly valid in T^{IQ}:

$$\textbf{E}(^*\textbf{A})\textbf{O}\text{-I:}\quad \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ ^*\text{All } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow \neg Yx)\\ (\forall x)(Xx \Rightarrow Mx)\,\&(\exists x)Xx\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

$$\textbf{A}(^*\textbf{A})\textbf{I}\text{-I:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ ^*\text{All } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y} \qquad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\forall x)(Xx \Rightarrow Mx)\,\&(\exists x)Xx\end{array}}{(\exists x)(Xx \wedge Yx)}$$

PROOF: From

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx)\,\&(Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow \neg Yx)$$

by (P11) and using (ŁFT8) we obtain

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx)\,\&(Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow (Xx \wedge \neg Yx)).$$

Thus, by (PP5) and (PP6) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow \neg Yx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \Rightarrow ((\exists x)Xx \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is by Lemma 12(g) equivalent with

$$T^{\mathrm{IQ}} \vdash ((\forall x)(Mx \Rightarrow \neg Yx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)).$$

This means that syllogism **E**(***A**)**O**-I is strongly valid.

Analogously, we can prove the second classical syllogism **A**(***A**)**I**-I.

□

The following syllogisms also require a presupposition.

Theorem 24
All the syllogisms below are strongly valid in T^{IQ}:

E(***T**)**O**-I:
No M are Y
*Most X are M
Some X are not Y

E(***K**)**O**-I:
No M are Y
*Many X are M
Some X are not Y

E(***P**)**O**-I:
No M are Y
*Almost all X are M
Some X are not Y.

PROOF: The first syllogism with a presupposition can be written as follows:

E(***T**)**O**-I:
$(\forall x)(Mx \Rightarrow \neg Yx)$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu X)z))$
$(\exists x)(Xx \wedge \neg Yx)$.

Put $Ez := (Bi\ Ve)((\mu X)z)$. From

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx)$$

and thus by (P3)

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Xx) \,\&\, (zx \Rightarrow Mx) \Rightarrow ((zx \Rightarrow Xx) \,\&\, (zx \Rightarrow \neg Yx)).$$

Hence by (ŁFT8) and (P7) we get

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Xx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx)).$$

By Lemma 12(g) we can rewrite this formula as follows:

$$T^{\mathrm{IQ}} \vdash (zx \Rightarrow Xx) \Rightarrow \{(Mx \Rightarrow \neg Yx) \Rightarrow ((zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx)))\}. \quad (4.3.2)$$

By the rule of generalization and by (PP5) and (PP6) we obtain that

$$T^{\mathrm{IQ}} \vdash (\forall x)(zx \Rightarrow Xx) \Rightarrow \{(\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge \neg Yx)))\}. \quad (4.3.3)$$

By ($\mathrm{P}_{\boldsymbol{\Delta}}1$) and by (ŁFT8) we have

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(\forall x)(zx \Rightarrow Xx) \Rightarrow \{(\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge \neg Yx)))\} \quad (4.3.4)$$

which is equivalent with

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(z \subseteq X) \Rightarrow \{(\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow ((\exists x)zx \Rightarrow (\exists x)(Xx \wedge \neg Yx)))\}. \quad (4.3.5)$$

By Lemma 12(d) and (g) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow \{(\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\forall x)(zx \Rightarrow Mx) \,\&\, (\exists x)zx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)\}. \quad (4.3.6)$$

By Lemma 10(d) and by (ŁFT8) we obtain that

$$T^{\mathrm{IQ}} \vdash (\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\forall x)(zx \Rightarrow Mx) \,\&\, (\exists x)zx) \Rightarrow ((\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)). \quad (4.3.7)$$

By Lemma 10(d) and by (ŁFT8) we get

$$T^{\mathrm{IQ}} \vdash ((\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\forall x)(zx \Rightarrow Mx) \,\&\, (\exists x)zx) \wedge Ez) \Rightarrow ((\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)). \quad (4.3.8)$$

Finally, by the rule of generalization with respect to $(\forall z)$ and by (PP6) we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\forall x)(zx \Rightarrow Mx) \,\&\, (\exists x)zx) \wedge Ez) \Rightarrow ((\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)) \quad (4.3.9)$$

which is by Lemma 12(d') equivalent with

$$T^{\mathrm{IQ}} \vdash (\forall x)(Mx \Rightarrow \neg Yx) \Rightarrow$$
$$((\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)\,\&(\exists x)zx) \wedge Ez) \Rightarrow (\exists x)(Xx \wedge \neg Yx)). \quad (4.3.10)$$

If we replace Ez by $(Bi\ Ve)((\mu X)z)$, then we obtain strong validity of **E**(***T**)**O**-I.

If we replace Ez in the proof above by $\neg\, Sm(\bar{\boldsymbol{\nu}})((\mu X)z)$, then we get strong validity of **E**(***K**)**O**-I. Analogously by putting $(Bi\ Ex)((\mu X)z)$, we get strong validity of **E**(***P**)**O**-I.

□

Theorem 25
The following syllogisms are strongly valid in T^{IQ}:

$$\mathbf{A}(^{*}\mathbf{T})\mathbf{I}\text{-I:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Most } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y} \qquad \mathbf{A}(^{*}\mathbf{K})\mathbf{I}\text{-I:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Many } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y}$$

$$\mathbf{A}(^{*}\mathbf{P})\mathbf{I}\text{-I:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Almost all } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y}$$

PROOF: We have

$$\mathbf{A}(^{*}\mathbf{T})\mathbf{I}\text{-I:}\quad \frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu X)z))\end{array}}{(\exists x)(Xx \wedge Yx).}$$

The proof is constructed similarly as that of Theorem 24 but the initial formula is

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Yx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Yx).$$

Analogously, we may prove strong validity of other two syllogisms **A**(***K**)**I**-I and **A**(***P**)**I**-I. □

4.3.3 Figure II

The second figure will be opened with two classical syllogisms.

Theorem 26
The classical syllogisms below are strongly valid in T^{IQ}:

$$\textit{\textbf{EIO}-II:}\quad \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Some } X \textit{ are } M\end{array}}{\textit{Some } X \textit{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists x)(Xx \wedge Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

$$\textit{\textbf{AOO}-II:}\quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Some } X \textit{ are not } M\end{array}}{\textit{Some } X \textit{ are not } Y} \qquad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\exists x)(Xx \wedge \neg Mx)\end{array}}{(\exists x)(Xx \wedge \neg Yx)}$$

PROOF: By Lemma 12(j) using Lemma 7(b) and by (R) we have that

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \Rightarrow (Mx \Rightarrow \neg Yx).$$

By (P4) we derive

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow \neg Yx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx)).$$

From two previous formula by (ŁFT8)

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((Xx \wedge Mx) \Rightarrow (Xx \wedge \neg Yx)).$$

Then by (PP5) and (PP6) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \Rightarrow ((\exists x)(Xx \wedge Mx) \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is just strong validity of syllogism **EIO**-II. The second syllogism **AOO**-II is obtained similarly. □

Theorem 27
The following syllogisms are strongly valid in T^{IQ}:

$$\textbf{\textit{AEE}}\text{-}II:\ \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{No } X \textit{ are } M\end{array}}{\textit{No } X \textit{ are } Y}
\qquad
\textbf{\textit{EAE}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{All } X \textit{ are } M\end{array}}{\textit{No } X \textit{ are } Y}$$

$$\textbf{\textit{AEB}}\text{-}II:\ \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{No } X \textit{ are } M\end{array}}{\textit{Few } X \textit{ are } Y}
\qquad
\textbf{\textit{AED}}\text{-}II:\ \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{No } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

$$\textbf{\textit{AEG}}\text{-}II:\ \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{No } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y}
\qquad
\textbf{\textit{EAB}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{All } X \textit{ are } M\end{array}}{\textit{Few } X \textit{ are } Y}$$

$$\textbf{\textit{EAD}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{All } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}
\qquad
\textbf{\textit{EAG}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{All } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y}$$

PROOF: The classical syllogism can be written as follows:

$$\frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\forall x)(Xx \Rightarrow \neg Mx)\end{array}}{(\forall x)(Xx \Rightarrow \neg Yx).}$$

The strong validity of **AEE**-II it is obtained by contraposition from

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow Mx)\,\&\,(Xx \Rightarrow \neg Mx) \Rightarrow (Xx \Rightarrow \neg Yx).$$

Then using Lemma 12(g) and by (PP5).

Analogously, we get strong validity of the syllogism **EAE**-II. From **AEE**-II by Theorem 17(b) we may prove strong validity of other three syllogisms **AEB**-II, **AED**-II and **AEG**-II. Finally, from the strong validity of **EAE**-II by Theorem 17(b) we can prove strong validity of **EAB**-II, **EAD**-II and **EAG**-II. □

Theorem 28

All the syllogisms below are strongly valid in T^{IQ}*:*

$$\textbf{\textit{ABB}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Few } X \textit{ are } M\end{array}}{\textit{Few } X \textit{ are } Y} \qquad \textbf{\textit{ADD}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Most } X \textit{ are not } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

$$\textbf{\textit{AGG}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Many } X \textit{ are not } M\end{array}}{\textit{Many } X \textit{ are not } Y} \qquad \textbf{\textit{ABD}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Few } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y}$$

$$\textbf{\textit{ABG}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Few } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y} \qquad \textbf{\textit{ADG}}\text{-}II: \quad \frac{\begin{array}{l}\textit{All } Y \textit{ are } M\\ \textit{Most } X \textit{ are not } M\end{array}}{\textit{Many } X \textit{ are not } Y}$$

PROOF: Using Definition 41, the first syllogism **ABB**-II can be written as follows:

$$\frac{\begin{array}{l}(\forall x)(Yx \Rightarrow Mx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Mx)) \wedge (Bi\ Ex)((\mu X)z))\end{array}}{(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ex)((\mu X)z)).}$$

Let us denote $Ez := (Bi\ Ex)((\mu X)z)$. By contraposition and using (PP5) we can prove that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow Mx) \Rightarrow ((\forall x)(zx \Rightarrow \neg Mx) \Rightarrow (\forall x)(zx \Rightarrow \neg Yx)).$$

Using the same steps as in the proof of **ATT**-I we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow Mx)\,\&(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Mx)) \wedge Ez) \Rightarrow$$
$$((\exists z)(\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx) \wedge Ez)). \quad (4.3.11)$$

If we replace Ez by $(Bi\ Ex)((\mu X)z)$ then we obtain strong validity of the syllogism **ABB**-II.

If we put $Ez := (Bi\ Ve)((\mu X)z)$ in the prove above, we obtain **ADD**-II and finally by putting $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu X)z)$, we get strong validity of **AGG**-II. From syllogism **ABB**-II by Theorem 17(b) we obtain strong validity of syllogisms **ABD**-II and **ABG**-II. Analogously, from **ADD**-II by Theorem 17(b) we get strong validity of **ADG**-II. □

Theorem 29
The following syllogisms are strongly valid in T^{IQ}:

$$\textbf{\textit{ETD}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y} \qquad \textbf{\textit{EPB}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Few } X \textit{ are } Y}$$

$$\textbf{\textit{EKG}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Many } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y} \qquad \textbf{\textit{ETG}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Most } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y}$$

$$\textbf{\textit{EPD}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Most } X \textit{ are not } Y} \qquad \textbf{\textit{EPG}}\text{-}II:\ \frac{\begin{array}{l}\textit{No } Y \textit{ are } M\\ \textit{Almost all } X \textit{ are } M\end{array}}{\textit{Many } X \textit{ are not } Y}$$

PROOF: Analogously as above, we have

$$\textbf{ETD}\text{-II:}\ \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu X)z))\end{array}}{(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ve)((\mu X)z)).}$$

Let us denote $(Bi\ Ve)((\mu X)z)$ by Ez. Using contraposition and by (PP5) we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \Rightarrow ((\forall x)(zx \Rightarrow Mx) \Rightarrow (\forall x)(zx \Rightarrow \neg Yx)).$$

Thus using the same steps as in the proof of **ATT**-I we have that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx)\,\&(\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow Mx)) \wedge Ez) \Rightarrow$$
$$(\exists z)(\boldsymbol{\Delta}(z \subseteq X)\,\&(\forall x)(zx \Rightarrow \neg Yx) \wedge Ez). \quad (4.3.12)$$

Replacing Ez by $(Bi\ Ve)((\mu X)z)$, we obtain strong validity of **ETD**-II.

If we put $Ez := (Bi\ Ex)((\mu X)z)$, we get **EPB**-II and finally if we put $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu X)z)$, we obtain strong validity of syllogism **EKG**-II. From **ETD**-II by Theorem 17(b) we obtain strong validity of **ETG**-II. Finally, we conclude from **EPB**-II by Theorem 17(b) that **EPD**-II and **EPG**-II are also strongly valid. □

4.3.4 Figure II — syllogisms with presupposition

Analogously as in Figure I, we close this section with syllogisms requiring a presupposition. We start with two classical syllogisms. Their proofs are constructed in a similar way as in Figure I.

Theorem 30
The classical syllogisms below are strongly valid in T^{IQ}:

$$\mathbf{E}(^*\mathbf{A})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ ^*\text{All } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \mathbf{A}(^*\mathbf{E})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ ^*\text{No } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: After rewriting, the first syllogism takes the form

$$\mathbf{E}(^*\mathbf{A})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

We start with a provable formula

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \,\&\, (Xx \Rightarrow Mx) \Rightarrow (Xx \Rightarrow (Xx \wedge \neg Yx)).$$

Then by Lemma 12(g), using (PP5), (PP6) and once more by Lemma 12(g) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \Rightarrow ((\exists x)Xx \Rightarrow (\exists x)(Xx \wedge \neg Yx))$$

which is equivalent with

$$T^{\mathrm{IQ}} \vdash ((\forall x)(Yx \Rightarrow \neg Mx) \,\&\, (\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx) \Rightarrow (\exists x)(Xx \wedge \neg Yx).$$

This means that syllogism **E**(****A**)**O**-II is strongly valid.

Analogously, we obtain strong validity of the second syllogism **A**(****E**)**O**-II. □

Theorem 31
All the syllogisms below are strongly valid in T^{IQ}:

$$\mathbf{E}(^*\mathbf{K})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ ^*\text{Many } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \mathbf{E}(^*\mathbf{T})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ ^*\text{Most } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\mathbf{E}(^*\mathbf{P})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ ^*\text{Almost all } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \mathbf{A}(^*\mathbf{B})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ ^*\text{Few } X \text{ are } M\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\mathbf{A}(^*\mathbf{D})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ ^*\text{Most } X \text{ are not } M\end{array}}{\text{Some } X \text{ are not } Y} \qquad \mathbf{A}(^*\mathbf{G})\mathbf{O}\text{-II:}\quad \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ ^*\text{Many } X \text{ are not } M\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: Syllogism **E**(***T**)**O**-II can be rewritten as follows:

$$\begin{array}{l} (\forall x)(Yx \Rightarrow \neg Mx) \\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu X)z)) \\ \hline (\exists x)(Xx \wedge \neg Yx). \end{array}$$

The proof is constructed analogously as the proof of **E**(***T**)**O**-I, but the initial formula is

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx).$$

Similarly as in the proof of **E**(***T**)**O**-I, we get strong validity of **E**(***T**)**O**-II.

Analogously, we may also prove strong validity of other syllogisms **E**(***K**)**O**-II and **E**(***P**)**O**-II. When proving strong validity of syllogisms **A**(***B**)**O**-II, **A**(***D**)**O**-II and **A**(***G**)**O**-II, we have to start with the formula

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow Mx) \,\&\, (zx \Rightarrow \neg Mx) \Rightarrow (zx \Rightarrow \neg Yx)$$

and then proceed analogously as above. □

4.3.5 Figure III

The third figure will be opened with four classical syllogisms.

Theorem 32
The following syllogisms are strongly valid in T^{IQ}:

$$\textit{\textbf{IAI}}\text{-III:}\quad \begin{array}{l} \textit{Some } M \textit{ are } Y \\ \textit{All } M \textit{ are } X \\ \hline \textit{Some } X \textit{ are } Y \end{array} \qquad \textit{\textbf{OAO}}\text{-III:}\quad \begin{array}{l} \textit{Some } M \textit{ are not } Y \\ \textit{All } M \textit{ are } X \\ \hline \textit{Some } X \textit{ are not } Y \end{array}$$

$$\textit{\textbf{AII}}\text{-III:}\quad \begin{array}{l} \textit{All } M \textit{ are } Y \\ \textit{Some } M \textit{ are } X \\ \hline \textit{Some } X \textit{ are } Y \end{array} \qquad \textit{\textbf{EIO}}\text{-III:}\quad \begin{array}{l} \textit{No } M \textit{ are } Y \\ \textit{Some } M \textit{ are } X \\ \hline \textit{Some } X \textit{ are not } Y \end{array}$$

PROOF: The first syllogism can be rewritten as follows:

$$\begin{array}{l} (\exists x)(Mx \wedge Yx) \\ (\forall x)(Mx \Rightarrow Xx) \\ \hline (\exists x)(Xx \wedge Yx). \end{array}$$

From

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx) \Rightarrow ((Mx \wedge Yx) \Rightarrow (Xx \wedge Yx))$$

by (PP5), (PP6) and by Lemma 12(d) we get

$$T^{\mathrm{IQ}} \vdash (\exists x)(Mx \wedge Yx) \Rightarrow ((\forall x)(Mx \Rightarrow Xx) \Rightarrow (\exists x)(Xx \wedge Yx))$$

which is just strong validity of the syllogism **IAI**-III.

In the same way we can prove strong validity of the other three syllogisms **OAO**-III, **AII**-III and **EIO**-III. □

4.3.6 Figure III — syllogisms with presupposition:

We continue with syllogisms which require a presupposition. First, we start with two classical syllogisms with a presupposition included in the minor premise.

Theorem 33
The classical syllogisms below are strongly valid in T^{IQ}:

$$\mathbf{A}({}^*\mathbf{A})\mathbf{I}\text{-III:}\ \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ {}^*\text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y} \qquad \mathbf{E}({}^*\mathbf{A})\mathbf{O}\text{-III:}\ \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ {}^*\text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: The first syllogism can be rewritten as

$$\frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\forall x)(Mx \Rightarrow Xx)\,\&\,(\exists x)Mx\end{array}}{(\exists x)(Xx \wedge Yx).}$$

This immediately results from the instance of (P4) which is as follows:

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Yx)\,\&\,(Mx \Rightarrow Xx) \Rightarrow (Mx \Rightarrow (Xx \wedge Yx)).$$

Then by Lemma 12(g) and by (PP5) and (PP6) we obtain the strong validity of **A**(*__A__)**I**-III.

Analogously, we can prove strong validity of syllogism **E**(*__A__)**O**-III. □

Theorem 34
All the syllogisms below are strongly valid in T^{IQ}:

$$(^*\mathbf{T})\mathbf{AI}\text{-III:}\quad \frac{\begin{array}{l}\text{*Most } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}\qquad (^*\mathbf{P})\mathbf{AI}\text{-III:}\quad \frac{\begin{array}{l}\text{*Almost all } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

$$(^*\mathbf{K})\mathbf{AI}\text{-III:}\quad \frac{\begin{array}{l}\text{*Many } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}\qquad \mathbf{A}(^*\mathbf{T})\mathbf{I}\text{-III:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Most } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

$$\mathbf{A}(^*\mathbf{K})\mathbf{I}\text{-III:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Many } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}\qquad \mathbf{A}(^*\mathbf{P})\mathbf{I}\text{-III:}\quad \frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{*Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

PROOF: Analogously as above, we can write the first syllogism as

$$\frac{\begin{array}{l}(\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)((\mu M)z))\\ (\forall x)(Mx \Rightarrow Xx)\end{array}}{(\exists x)(Xx \wedge Yx).}$$

Let us put $Ez := (Bi\ Ve)((\mu M)z)$. The proof is analogous to the proof of $\mathbf{E}(^*\mathbf{T})\mathbf{O}$-I and thus, we will continue in a succinct way.

Let us start with a provable formula

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx).$$

Then by Lemma 12(g) and using (P3) we obtain

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx) \Rightarrow ((zx \Rightarrow Mx) \mathbin{\&} (zx \Rightarrow Yx) \Rightarrow (zx \Rightarrow Xx) \mathbin{\&} (zx \Rightarrow Yx)).$$

Now using the same properties as in $\mathbf{E}(^*\mathbf{T})\mathbf{O}$-I, we conclude that

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \mathbin{\&} (\exists x)zx \mathbin{\&} (\forall x)(zx \Rightarrow Yx)) \wedge Ez) \mathbin{\&} (\forall x)(Mx \Rightarrow Xx) \\ \Rightarrow (\exists x)(Xx \wedge Yx). \quad (4.3.13)$$

Replacing Ez by $(Bi\ Ve)((\mu M)z)$ we obtain the strong validity of $(^*\mathbf{T})\mathbf{AI}$-III.

Analogously, we can prove strong validity of $\mathbf{A}(^*\mathbf{T})\mathbf{I}$-III. If we put $Ez := (Bi\ Ex)((\mu M)z)$ in the proof above, then we obtain strong validity of $(^*\mathbf{P})\mathbf{AI}$-III and, hence, of $\mathbf{A}(^*\mathbf{P})\mathbf{I}$-III. If put $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu M)z)$, then we get strong validity of $(^*\mathbf{K})\mathbf{AI}$-III and thus strong validity of $\mathbf{A}(^*\mathbf{K})\mathbf{I}$-III as well. □

Theorem 35
The following syllogisms are strongly valid in T^{IQ}:

$$(^*\mathbf{B})\mathbf{AO}\text{-III:}\quad \frac{\begin{array}{l}\text{*Few } M \text{ are } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y} \qquad (^*\mathbf{D})\mathbf{AO}\text{-III:}\quad \frac{\begin{array}{l}\text{*Most } M \text{ are not } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

$$(^*\mathbf{G})\mathbf{AO}\text{-III:}\quad \frac{\begin{array}{l}\text{*Many } M \text{ are not } Y\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: The first syllogism can be rewritten as follows:

$$\frac{\begin{array}{l}(\exists z)((\mathbf{\Delta}(z \subseteq M)\,\&(\exists x)zx\,\&(\forall x)(zx \Rightarrow \neg Yx)) \wedge (Bi\ Ex)((\mu M)z))\\ (\forall x)(Mx \Rightarrow Xx)\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

The proof proceeds analogously as the proof above but the initial formula is

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx)\,\&(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx).$$

Then by (P3) we derive

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx)\,\&(zx \Rightarrow Mx)\,\&(zx \Rightarrow \neg Yx) \Rightarrow (zx \Rightarrow Xx)\,\&(zx \Rightarrow \neg Yx).$$

Further steps are similar as above.

Analogously, we get strong validity of other two syllogisms (*__D__)**AO**-III and (*__G__)**AO**-III. □

Theorem 36
The following syllogisms are strongly valid in T^{IQ}:

$$\mathbf{E}(^*\mathbf{T})\mathbf{O}\text{-III:}\quad \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{*Most } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y} \qquad \mathbf{E}(^*\mathbf{P})\mathbf{O}\text{-III:}\quad \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{*Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\mathbf{E}(^*\mathbf{K})\mathbf{O}\text{-III:}\quad \frac{\begin{array}{l}\text{No } M \text{ are } Y\\ \text{*Many } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

PROOF: The first syllogism **E**(**T*)**O**-III can be rewritten as follows:

$$\frac{\begin{array}{l}(\exists x)(Mx \Rightarrow \neg Yx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ve)((\mu M)z))\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

the proof is constructed analogously as the proof of **E**(**T*)**O**-I.

Analogously, we get strong validity of other two syllogisms **E**(**P*)**O**-III and **E**(**K*)**O**-III. □

4.3.7 Figure IV

Theorem 37
The classical syllogisms below are strongly valid in T^{IQ}:

$$\textbf{EIO}\text{-IV}: \frac{\begin{array}{l}\text{No } Y \text{ are } M\\ \text{Some } M \text{ are } X\end{array}}{\text{Some } X \text{ are not} Y} \qquad \textbf{IAI}\text{-IV}: \frac{\begin{array}{l}\text{Some } Y \text{ are } M\\ \text{All } M \text{ are } X\end{array}}{\text{Some } X \text{ are } Y}$$

PROOF: By contraposition, using (P4) and by (ŁFT8) we prove

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((Mx \wedge Xx) \Rightarrow (Xx \wedge \neg Yx)).$$

Then using (PP5) and (PP6) we obtain strong validity of **EIO**-IV.

Analogously, we can prove that **IAI**-IV is strongly valid. □

Theorem 38
The syllogisms below are strongly valid in T^{IQ}:

$$\textbf{AEE}\text{-IV}: \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{No } X \text{ are } Y} \qquad \textbf{AEB}\text{-IV}: \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Few } X \text{ are } Y}$$

$$\textbf{AED}\text{-IV}: \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Most } X \text{ are not } Y} \qquad \textbf{AEG}\text{-IV}: \frac{\begin{array}{l}\text{All } Y \text{ are } M\\ \text{No } M \text{ are } X\end{array}}{\text{Many } X \text{ are not } Y}$$

PROOF: Strong validity of the classical syllogism **AEE**-IV follows from a provable formula

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow Mx) \mathbin{\&} (Mx \Rightarrow \neg Xx) \Rightarrow (Xx \Rightarrow \neg Yx). \tag{4.3.14}$$

Then by Lemma 12(g) and using (PP5),(PP6) we obtain the strong validity of **AEE**-IV.

From the validity of **AEE**-IV by Theorem 17(b), we may also prove strong validity of the three syllogisms **AEB**-IV, **AED**-IV and **AEG**-IV. □

4.3.8 Figure IV — syllogisms with presupposition

The last figure with a presupposition will be divided into two groups. The first contains all the valid syllogisms with presupposition such that only non-empty subsets of Y are considered. The second one assumes that only non-empty subsets of M are considered.

Theorem 39
The following classical syllogism is strongly valid in T^{IQ}*:*

	*All Y are M	$(\forall x)(Yx \Rightarrow Mx) \mathbin{\&} (\exists x)Yx$
(***A**)**AI**-IV:	All M are X	$(\forall x)(Mx \Rightarrow Xx)$
	Some X are Y	$(\exists x)(Xx \wedge Yx)$.

PROOF: By (ŁFT8), using (P11) and by (PP5),(PP6) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow Mx) \mathbin{\&} (\forall x)(Mx \Rightarrow Xx) \mathbin{\&} (\exists x)Yx \Rightarrow ((\exists x)(Xx \wedge Yx))$$

which is just strong validity of our syllogism. □

Theorem 40
The following syllogisms are strongly valid in T^{IQ}*:*

	*Most Y are M		*Almost all Y are M
(***T**)**AI**-IV:	All M are X	(***P**)**AI**-IV:	All M are X
	Some X are Y		Some X are Y

	*Many Y are M
(***K**)**AI**-IV:	All M are X
	Some X are Y

PROOF: The syllogism (*$\mathbf{T}$)$\mathbf{AI}$-IV can be written as

$$\begin{array}{l}(\exists z)((\boldsymbol{\Delta}(z \subseteq Y) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)((\mu Y)z)) \\ (\forall x)(Mx \Rightarrow Xx) \\ \hline (\exists x)(Xx \wedge Yx).\end{array}$$

We start with a provable formula

$$T^{\mathrm{IQ}} \vdash (zx \Rightarrow Mx) \,\&\, (Mx \Rightarrow Xx) \Rightarrow (zx \Rightarrow Xx)$$

and thus by (P3) we have

$$T^{\mathrm{IQ}} \vdash (Mx \Rightarrow Xx) \,\&\, (zx \Rightarrow Yx) \,\&\, (zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Yx) \,\&\, (zx \Rightarrow Xx).$$

Then we continue analogously as in the proof of $\mathbf{E}$(*$\mathbf{T}$)$\mathbf{O}$-I. Analogously, we obtain strong validity of (*$\mathbf{P}$)$\mathbf{AI}$-IV and (*$\mathbf{K}$)$\mathbf{AI}$-IV. □

Theorem 41
The following classical syllogism is strongly valid in T^{IQ}:

$$\mathbf{E}(^*\mathbf{A})\mathbf{O}\text{-IV:}\quad \begin{array}{l}\text{No } Y \text{ are } M \\ ^*\text{All } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y\end{array} \qquad \begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx) \\ (\forall x)(Mx \Rightarrow Xx) \,\&\, (\exists x)Mx \\ \hline (\exists x)(Xx \wedge \neg Yx)\end{array}$$

PROOF: By contraposition, using (P7), by Lemma 12(g) and by (PP5), (PP6) we get that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Yx \Rightarrow \neg Mx) \,\&\, (\forall x)(Mx \Rightarrow Xx) \,\&\, (\exists x)Mx \Rightarrow (\exists x)(Xx \wedge \neg Yx)$$

which is just strong validity of $\mathbf{E}$(*$\mathbf{A}$)$\mathbf{O}$-IV.

□

Theorem 42
All the syllogisms below are strongly valid in T^{IQ}:

$$\mathbf{E}(^*\mathbf{T})\mathbf{O}\text{-IV:}\quad \begin{array}{l}\text{No } Y \text{ are } M \\ ^*\text{Most } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y\end{array} \qquad \mathbf{E}(^*\mathbf{P})\mathbf{O}\text{-IV:}\quad \begin{array}{l}\text{No } Y \text{ are } M \\ ^*\text{Almost all } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y\end{array}$$

$$\mathbf{E}(^*\mathbf{K})\mathbf{O}\text{-IV:}\quad \begin{array}{l}\text{No } Y \text{ are } M \\ ^*\text{Many } M \text{ are } X \\ \hline \text{Some } X \text{ are not } Y\end{array}$$

PROOF: Analogously as above, the first syllogism can be written as follows:

$$\frac{\begin{array}{l}(\forall x)(Yx \Rightarrow \neg Mx)\\ (\exists z)((\mathbf{\Delta}(z \subseteq M)\,\&\,(\exists x)zx\,\&\,(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ve)((\mu M)z))\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

The proof of this syllogism is obtained by (ŁFT8), using Lemma 12(d) and (g) from

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx)\,\&\,(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow \neg Yx)$$

and, hence, by Lemma 12(g) and by (P3) from

$$T^{\mathrm{IQ}} \vdash (Yx \Rightarrow \neg Mx) \Rightarrow ((zx \Rightarrow Xx)\,\&\,(zx \Rightarrow Mx) \Rightarrow (zx \Rightarrow Xx)\,\&\,(zx \Rightarrow \neg Yx)).$$

Then we continue using the same steps as in the proof of validity of **E**(***T**)**O**-I and we conclude that **E**(***T**)**O**-IV is strongly valid. Similarly, we may prove strong validity of other two syllogisms **E**(***K**)**O**-IV and **E**(***P**)**O**-IV. □

We close this section with a classical syllogism which is strongly valid and only non-empty subsets of X are considered. The proof is analogous to the proof of the validity of syllogism **E**(***A**)**O**-IV.

Theorem 43
The following syllogism is strongly valid in T^{IQ}:

	All Y are M	$(\forall x)(Yx \Rightarrow Mx)$
A(***E**)**O**-IV:	*No M are X	$(\forall x)(Mx \Rightarrow \neg Xx)\,\&\,(\exists x)Xx$
	Some X are not Y	$(\exists x)(Xx \wedge \neg Yx)$

4.4 Valid generalized intermediate syllogisms based on the extension of T^{IQ}

In this section we show that twelve new non-trivial intermediate syllogisms which are generalization of the corresponding ones presented in Peterson's book are strongly valid in our theory. Their non-triviality consists in the use of two non-classical intermediate quantifiers and thus, their validity is by no means obvious.

We start with a definition where we will introduce extension $T[M, M']$ of the theory T^{IQ} in which all 12 non-trivial intermediate generalized syllogisms will be proved.

Definition 44
Let $M, M' \in Form_{o\alpha}$ be formulas. By $T[M, M']$ we denote a theory being extension of T^{IQ} such that

(a) $T[M, M'] \vdash M \equiv M'$,

(b) $T[M, M'] \vdash (\exists x_\alpha)\boldsymbol{\Delta} Mx$ *and* $T[M, M'] \vdash (\exists x_\alpha)\boldsymbol{\Delta} M'x$.

Lemma 28
Let $T[M, M']$ be a theory, $z, z' \in Form_{o\alpha}$ and E be one of the following formulas:

$$\text{either} \qquad E := \lambda M_{o\alpha}\lambda z_{o\alpha}(Bi\ Ex)((\mu M)z), \tag{4.4.1}$$

$$\text{or} \qquad E := \lambda M_{o\alpha}\lambda z_{o\alpha}(Bi\ Ve)((\mu M)z), \tag{4.4.2}$$

$$\text{or} \qquad E := \lambda M_{o\alpha}\lambda z_{o\alpha}\neg(Sm(\bar{\boldsymbol{\nu}}))((\mu M)z). \tag{4.4.3}$$

Then

$$T[M, M'] \vdash (\exists z)(\exists z')\boldsymbol{\Delta}((EM)z \,\&\, (EM')z' \,\&\, (\exists x)(zx \,\&\, z'x)). \tag{4.4.4}$$

PROOF: We have from Theorem 16(a),(b) that $T[M, M'] \vdash (Bi\,\boldsymbol{\nu})\top$ and also $T[M, M'] \vdash (\neg\, Sm\,\bar{\boldsymbol{\nu}})\top$. Hence, by (M1) and by (R) we obtain that $T[M, M'] \vdash (Bi\ Ve)((\mu M)M)$, $T[M, M'] \vdash (Bi\ Ex)((\mu M)M)$ and $T[M, M'] \vdash \neg\, Sm(\bar{\boldsymbol{\nu}})(\mu M)M)$. Thus

$$T[M, M'] \vdash (EM)M. \tag{4.4.5}$$

Analogously, for M' we obtain

$$T[M, M'] \vdash (EM')M'. \tag{4.4.6}$$

From the assumptions (a), (b) we conclude (using rule (R)) that

$$T[M, M'] \vdash (\exists x)(Mx \,\&\, M'x). \tag{4.4.7}$$

From (4.4.5),(4.4.6) and (4.4.7) and using Rule (N) it results that

$$T[M, M'] \vdash \boldsymbol{\Delta}((EM)M \,\&\, (EM')M' \,\&\, (\exists x)(Mx \,\&\, M'x))$$

which implies

$$T[M, M'] \vdash (\exists z)(\exists z')\boldsymbol{\Delta}((EM)z \,\&\, (EM')z' \,\&\, (\exists x)(zx \,\&\, z'x)).$$

□

According to this lemma, extension $T[M, M']$ of the theory T^{IQ} can be introduced, which, as seen below, makes it possible to prove strong validity of all the non-trivial syllogisms.

In fact, we only assume in $T[M, M']$ that the basic fuzzy sets used in the syllogisms are normal. The condition (a) is needed for formal reasons only and it does not constitute any actual limitation.

4.4.1 Figure III — twelve non-trivial strongly valid syllogisms based on $T[M, M']$

We introduce below a list of the non-trivial intermediate generalized syllogisms which were proposed by Peterson in [14].

First, we present intermediate generalized quantifiers with "Most":

TTI-III
DTO-III

We continue with intermediate generalized quantifiers with "Most", "Many" and "Few":

PPI-III
TPI-III
KPI-III
PTI-III
PKI-III
BKO-III
DPO-III
GPO-III
BTO-III
BKO-III

Theorem 44
The following syllogisms are strongly valid in $T[M, M']$:

$$\textbf{TTI}\text{-III: }\begin{array}{l}\text{Most } M \text{ are } Y\\ \text{Most } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{PPI}\text{-III: }\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Almost all } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

$$\textbf{TPI}\text{-III: }\begin{array}{l}\text{Most } M \text{ are } Y\\ \text{Almost all } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{PTI}\text{-III: }\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Most } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

$$\textbf{PKI}\text{-III: }\begin{array}{l}\text{Almost all } M \text{ are } Y\\ \text{Many } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array} \qquad \textbf{KPI}\text{-III: }\begin{array}{l}\text{Many } M \text{ are } Y\\ \text{Almost all } M' \text{ are } X\\ \hline \text{Some } X \text{ are } Y\end{array}$$

PROOF: The first syllogism can be rewritten as follows:

$$\textbf{TTI}\text{-III: }\begin{array}{l}(\exists z)((\mathbf{\Delta}(z \subseteq M)\,\&(\forall x)(zx \Rightarrow Yx)) \wedge (Bi\ Ve)((\mu M)z))\\ (\exists z')((\mathbf{\Delta}(z' \subseteq M')\,\&(\forall x)(z'x \Rightarrow Xx)) \wedge (Bi\ Ve)((\mu M')z'))\\ \hline (\exists x)(Xx \wedge Yx).\end{array}$$

Put $T' = T[M, M'] \cup \{(EM)r\,\&(EM')r'\,\&(\exists x)(rx\,\&\,r'x)\}$ where $r, r' \notin J(T[M, M'])$ are new constants of type $o\alpha$. Then

$$T' \vdash (\exists x)(rx\,\&\,r'x).$$

Using (P1), then by Lemma 18(f) and using (P5) and by Lemma 12(g) we get

$$T' \vdash (r'x \Rightarrow Xx)\,\&(rx \Rightarrow Yx) \Rightarrow ((rx\,\&\,r'x) \Rightarrow (Xx \wedge Yx)).$$

Once more by Lemma 12(g), then by the rule of generalization and using (P5) and (P6) we obtain

$$T' \vdash (\forall x)(r'x \Rightarrow Xx)\,\&(\forall x)(rx \Rightarrow Yx) \Rightarrow ((\exists x)(rx\,\&\,r'x) \Rightarrow (\exists x)(Xx \wedge Yx)).$$

By Lemma 12(d) and (g) then by weakening of $\&$ and $\wedge$, we derive

$$\begin{aligned} T' \vdash \{(\mathbf{\Delta}(r \subseteq M)\,\&(\forall x)(rx \Rightarrow Yx) \wedge (EM)r)\,\& \\ (\mathbf{\Delta}(r' \subseteq M')\,\&(\forall x)(r'x \Rightarrow Xx) \wedge (EM')r')\} \Rightarrow \\ ((\exists x)(rx\,\&\,r'x) \Rightarrow (\exists x)(Xx \wedge Yx)). \quad (4.4.8)\end{aligned}$$

By Lemma 12(d) from $T' \vdash (\exists x)(rx \,\&\, r'x)$ and one more by Lemma12(g) we get that

$$T' \vdash (\boldsymbol{\Delta}(r \subseteq M) \,\&(\forall x)(rx \Rightarrow Yx) \wedge (EM)r) \Rightarrow$$
$$(\boldsymbol{\Delta}(r' \subseteq M') \,\&(\forall x)(r'x \Rightarrow Xx) \wedge (EM')r') \Rightarrow ((\exists x)(Xx \wedge Yx)). \quad (4.4.9)$$

By the deduction theorem we obtain

$$T[M, M'] \vdash \boldsymbol{\Delta}((EM)r \,\&(EM')r' \,\&(\exists x)(rx \,\&\, r'x)) \Rightarrow$$
$$\{(\boldsymbol{\Delta}(r \subseteq M) \,\&(\forall x)(rx \Rightarrow Yx) \wedge (EM)r) \Rightarrow$$
$$(\boldsymbol{\Delta}(r' \subseteq M') \,\&(\forall x)(r'x \Rightarrow Xx) \wedge (EM')r') \Rightarrow ((\exists x)(Xx \wedge Yx))\}. \quad (4.4.10)$$

Replace r, r' by variables v, v' not occurring in the proof of (4.4.10) and obtain

$$T[M, M'] \vdash \boldsymbol{\Delta}((EM)v \,\&(EM')v' \,\&(\exists x)(vx \,\&\, v'x)) \Rightarrow$$
$$\{(\boldsymbol{\Delta}(v \subseteq M) \,\&(\forall x)(vx \Rightarrow Yx) \wedge (EM)v) \Rightarrow$$
$$(\boldsymbol{\Delta}(v' \subseteq M') \,\&(\forall x)(v'x \Rightarrow Xx) \wedge (EM')v') \Rightarrow ((\exists x)(Xx \wedge Yx))\}. \quad (4.4.11)$$

Let us denote $T'' = T[M, M'] \cup \{(EM)v \,\&(EM')v' \,\&(\exists x)(vx \,\&\, v'x)\}$. Then by the deduction theorem (by Remark 5) we derive from (4.4.11) that

$$T'' \vdash \{(\boldsymbol{\Delta}(v \subseteq M) \,\&(\forall x)(vx \Rightarrow Yx) \wedge (EM)v) \Rightarrow$$
$$(\boldsymbol{\Delta}(v' \subseteq M') \,\&(\forall x)(v'x \Rightarrow Xx) \wedge (EM')v')\} \Rightarrow ((\exists x)(Xx \wedge Yx)). \quad (4.4.12)$$

By the rule of the generalization with respect to $\forall v, \forall v'$ and by quantifier properties we obtain

$$T'' \vdash (\exists v)(\boldsymbol{\Delta}(v \subseteq M) \,\&(\forall x)(vx \Rightarrow Yx) \wedge (EM)v) \Rightarrow$$
$$(\exists v')(\boldsymbol{\Delta}(v' \subseteq M') \,\&(\forall x)(v'x \Rightarrow Xx) \wedge (EM')v') \Rightarrow ((\exists x)(Xx \wedge Yx)). \quad (4.4.13)$$

By Theorem 6 and using Rule (R) we get that

$$T'' \vdash (\exists z)(\boldsymbol{\Delta}(z \subseteq M) \,\&(\forall x)(zx \Rightarrow Yx) \wedge (EM)z) \Rightarrow$$
$$(\exists z')(\boldsymbol{\Delta}(z' \subseteq M') \,\&(\forall x)(z'x \Rightarrow Xx) \wedge (EM')z') \Rightarrow ((\exists x)(Xx \wedge Yx)). \quad (4.4.14)$$

Again by the deduction theorem we get

$$T[M, M'] \vdash \boldsymbol{\Delta}((EM)v \,\&(EM')v' \,\&(\exists x)(vx \,\&\, v'x)) \Rightarrow$$
$$\{(\exists z)(\boldsymbol{\Delta}(z \subseteq M) \,\&(\forall x)(zx \Rightarrow Yx) \wedge (EM)z) \Rightarrow$$
$$(\exists z')(\boldsymbol{\Delta}(z' \subseteq M') \,\&(\forall x)(z'x \Rightarrow Xx) \wedge (EM')z') \Rightarrow ((\exists x)(Xx \wedge Yx))\} \quad (4.4.15)$$

By the rule of generalization with respect to $(\forall v)$ and $(\forall v')$ and using the quantifier properties we conclude that

$$T[M, M'] \vdash (\exists v)(\exists v')\mathbf{\Delta}((EM)v \mathbin{\&} (EM')v' \mathbin{\&} (\exists x)(vx \mathbin{\&} v'x)) \Rightarrow \{(\exists z)(\mathbf{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx) \wedge (EM)z) \Rightarrow (\exists z')(\mathbf{\Delta}(z' \subseteq M') \mathbin{\&} (\forall x)(z'x \Rightarrow Xx) \wedge (EM')z') \Rightarrow ((\exists x)(Xx \wedge Yx))\}. \quad (4.4.16)$$

Finally, by Lemma 28 we conclude that

$$T[M, M'] \vdash (\exists z)(\mathbf{\Delta}(z \subseteq M) \mathbin{\&} (\forall x)(zx \Rightarrow Yx) \wedge (EM)z) \Rightarrow (\exists z')(\mathbf{\Delta}(z' \subseteq M') \mathbin{\&} (\forall x)(z'x \Rightarrow Xx) \wedge (EM')z') \Rightarrow ((\exists x)(Xx \wedge Yx)). \quad (4.4.17)$$

By putting $(EM)z := (Bi\ Ve)((\mu M)z)$ and $(EM')z' := (Bi\ Ve)((\mu M')z')$, we obtain the strong validity of **TTI**-III. If we put $(EM)z := (Bi\ Ex)((\mu M)z)$ and $(EM')z' := (Bi\ Ex)((\mu M')z')$, we obtain strong validity of **PPI**-III. From the strong validity of **TTI**-III and using Theorem 17(a) we get the strong validity of **PTI**-III. Analogously, we may prove the strong validity of **TPI**-III. By putting $(EM)z := (Bi\ Ex)((\mu M)z)$ and $(EM')z' := \neg Sm(\bar{\boldsymbol{\nu}})((\mu M')z')$, we may prove strong validity of **PKI**-III. Analogously, we prove the strong validity of **KPI**-III. □

Theorem 45
The following syllogisms are strongly valid in $T[M, M']$:

BTO-III:
Almost all M are Y
Most M' are X
———
Some X are not Y

DTO-III:
Most M are not Y
Most M' are X
———
Some X are not Y

BPO-III:
Few M are Y
Almost all M' are X
———
Some X are not Y

DPO-III:
Most M are not Y
Almost all M' are X
———
Some X are not Y

GPO-III:
Many M are not Y
Almost all M' are X
———
Some X are not Y

BKO-III:
Few M are Y
Many M' are X
———
Some X are not Y

PROOF: The proof of strong validity of all the syllogisms can be constructed analogously as in the proof above but with the following starting formula:

$$T' \vdash (rx \Rightarrow \neg Yx) \mathbin{\&} (r'x \Rightarrow Xx) \Rightarrow ((rx \mathbin{\&} r'x) \Rightarrow (Xx \wedge \neg Yx))$$

By the same steps as above, we obtain that

$$T[M, M'] \vdash (\exists z)(\boldsymbol{\Delta}(z \subseteq M)\,\boldsymbol{\&}(\forall x)(zx \Rightarrow \neg Yx) \wedge (EM)z) \Rightarrow$$
$$(\exists z')(\boldsymbol{\Delta}(z' \subseteq M')\,\boldsymbol{\&}(\forall x)(z'x \Rightarrow Xx) \wedge (EM')z') \Rightarrow ((\exists x)(Xx \wedge \neg Yx)). \quad (4.4.18)$$

By putting $(EM)z := (Bi\ Ve)((\mu M)z)$ and $(EM')z' := (Bi\ Ve)((\mu M)z')$, we obtain strong validity of **DTO**-III. By Theorem 17(b), we obtain strong validity of **BTO**-III. If we replace $Ez := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu M)z)$ and $(EM')z' := (Bi\ Ex)((\mu M')z')$, we obtain strong validity of **GPO**-III. From this by Theorem 17(b), we get the strong validity of **DPO**-III and thus the strong validity of **BPO**-III. Analogously, by putting $(EM)z := (Bi\ Ex)((\mu M)z)$ and $(EM')z' := \neg\, Sm(\bar{\boldsymbol{\nu}})((\mu M')z')$, we may prove the strong validity of **BKO**-III. □

4.4.2 List of all 105 strongly valid syllogisms in $T[M, M']$

We closed this section with a list of all 105 syllogisms (24 traditional syllogisms, 69 Thompson's and 12 Peterson's) which are strongly valid in the theory $T[M, M']$.

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAA	**EAE**	**A(*A)I**	**(*A)AI**
EAE	**AEE**	**IAI**	**AEE**
AII	**EIO**	**AII**	**IAI**
EIO	**AOO**	**E(*A)O**	**E(*A)O**
A(*A)I	**E(*A)O**	**OAO**	**EIO**
E(*A)O	**A(*E)O**	**EIO**	**A(*E)O**

Figure I	***Figure II***	***Figure III***	***Figure IV***
AAT	***AED***	**A(*T)I**	**AED**
ATT	***ADD***	**E(*T)O**	**E(*T)O**
A(*T)I	**A(*D)O**	**(*T)AI**	**(*T)AI**
EAD	***EAD***	**(*D)AO**	
ETD	***ETD***		
E(*T)O	***ETO***		

Figure I	*Figure II*	*Figure III*	*Figure IV*
AAK	***AEG***	**A(*K)I**	***AEG***
ATK	***ADG***	**E(*K)O**	**E(*K)O**
A(*K)I	**A(*G)O**	**(*K)AI**	**(*K)AI**
AKK	***AGG***	**(*G)AO**	
EAG	***EAG***	**TTI**	
ETG	***ETG***	**DTO**	
E(*K)O	**E(*K)O**		
EKG	***EKG***		

Figure I	*Figure II*	*Figure III*	*Figure IV*
AAP	***AEB***	**(*P)AI**	***AEB***
APP	***ABB***	**E(*P)O**	**(*P)AI**
APT	***ABD***	**(*B)AO**	**E(*P)O**
APK	***ABG***	**A(*P)I**	
API	**A(*B)O**	**PPI**	
EAB	***EAB***	**TPI**	
EPB	***EPB***	**KPI**	
EPD	***EPD***	**PTI**	
EPG	***EPG***	**PKI**	
E(*P)O	**E(*P)O**	**BPO**	
		DPO	
		GPO	
		BTO	
		BKO	

4.5 Interpretation

In this section we will present four examples of syllogisms and show their validity in a simple model with a finite set M_ϵ of elements. The frame of the constructed model as follows:

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

where $M_o = [0,1]$ is support of the standard Łukasiewicz MV$_\Delta$-algebra. The fuzzy equality $=_o$ is the Łukasiewicz biresiduation $\leftrightarrow$. Furthermore, $M_\epsilon = \{u_1, \ldots, u_r\}$ is a finite set with a fixed numbering of its elements and $=_\epsilon$ is defined by

$$[u_i =_\epsilon u_j] = (1 - \min\left(1, \frac{|i-j|}{s}\right))$$

for a fixed natural number $s \leq r$. This is a separated fuzzy equality w.r.t. the Łukasiewicz conjunction $\otimes$. It can be verified that all the logical axioms of Ł-FTT are true in the degree 1 in $\mathcal{M}$ (all the considered functions are weakly extensional w.r.t. $\mathcal{M}(\equiv)$). Moreover, $\mathcal{M}$ is nontrivial because $1 - \frac{|i-j|}{s} \in (0,1)$ implies $\frac{|i-j|}{s} \in (0,1)$ and thus, taking the assignment p such that $p(x_\epsilon) = u_i$, $p(y_\epsilon) = u_j$ and considering $A_o := x_\epsilon \equiv y_\epsilon$, we obtain $\mathcal{M}_p(A_o \vee \neg A_o) \in (0,1)$.

To make $\mathcal{M}$ a model of T^{Ev} and T^{IQ}, we define interpretation of $\sim$ by $\mathcal{M}(\sim) = \leftrightarrow^2$, $\mathcal{M}(\dagger) = 0.5$ and put $\mathcal{M}(\boldsymbol{\nu})$ equal to a function $\nu_{a,b,c}$ which is a simple partially quadratic function given in [37]. In Figure 4.1, extensions of several evaluative expressions used below are depicted. It can be verified that $\mathcal{M} \models T^{\mathrm{Ev}}$.

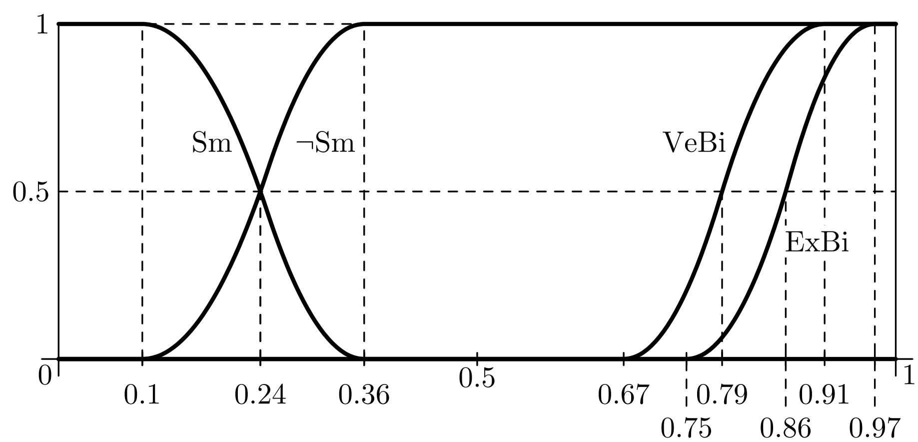


Figure 4.1: Shapes of the extensions of evaluative expressions in the context $[0,1]$ used in the examples below

The distinguished set $\mathcal{S} \subset$ *Types* is defined as follows: $\alpha \in \mathcal{S}$ iff α is a type not containing the type o of truth values. This means that all sets M_α for $\alpha \in \mathcal{S}$ are finite.

Let $A \underset{\sim}{\subset} M_\alpha$, $\alpha \in \mathcal{S}$ be a fuzzy set. We will put

$$|A| = \sum_{u \in \mathrm{Supp}(A)} A(u), \qquad u \in M_\alpha. \tag{4.5.1}$$

Furthermore, for fuzzy sets $A, B \underset{\sim}{\subseteq} M_\alpha$, $\alpha \in \mathcal{S}$ we define

$$F_R(B)(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases} \qquad (4.5.2)$$

Interpretation of constants $R \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in S$ is defined by $\mathcal{M}(R) = F_R$ where $F_R : \mathcal{F}(M_\alpha) \times \mathcal{F}(M_\alpha) \longrightarrow L$ is the function (4.5.2). It can be verified that axioms (M1)–(M4) are true in the degree 1 in $\mathcal{M}$. Thus, $\mathcal{M} \models T^{\mathrm{IQ}}$ and also $\mathcal{M} \models T[M, M']$.

We will demonstrate on concrete examples below how some of the syllogisms proved above behave on this model.

4.5.1 Example of strongly valid syllogism of Figure I

Let us consider the following syllogism:

ATT-I:
All women are well dressed
Most people at the party are women
———
Most people at the party are well dressed

Let M_ϵ be a set of people. Let $\mathrm{Wom}_{o\epsilon}$ be a formula representing "women" which is interpreted by $\mathcal{M}(\mathrm{Wom}_{o\epsilon}) = W \subseteq M_\epsilon$ where W is a classical set. Furthermore, let $\mathrm{Peop}_{o\epsilon}$ be a formula representing "people at the party" interpreted by $\mathcal{M}(\mathrm{Peop}_{o\epsilon}) = P \subseteq M_\epsilon$ where P is a classical set. Finally, let $\mathrm{Dress}_{o\epsilon}$ be a formula interpreted by $\mathcal{M}(\mathrm{Dress}_{o\epsilon}) = D \subseteq M_\epsilon$.

Major premise: "All women are well dressed" From the assumption

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{Wom}_{o\epsilon}(x_\epsilon) \Rightarrow \mathrm{Dress}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\mathrm{Wom}_{o\epsilon})(m) \rightarrow \mathcal{M}(\mathrm{Dress}_{o\epsilon})(m)) = 1$$

we conclude that $W \subseteq D$.

Minor premise: "Most people at the party are women" The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Peop}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Wom}_{o\epsilon})) \wedge (Bi\ Ve)((\mu \mathrm{Peop}_{o\epsilon}) z_{o\epsilon}))) = 1 \qquad (4.5.3)$$

leads to the requirement to find the greatest subset $\mathcal{M}(z_{o\epsilon}) = W' \subseteq P$ such that:

$$\mathcal{M}(\mathbf{\Delta}(z_{o\epsilon} \subseteq \text{Peop}_{o\epsilon})) = 1, \tag{4.5.4}$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \text{Wom}_{o\epsilon}) = 1, \tag{4.5.5}$$

$$\mathcal{M}((Bi\ Ve)((\mu \text{Peop}_{o\epsilon}) z_{o\epsilon})) = 1. \tag{4.5.6}$$

One can verify that this holds if $W' = W$.

From (4.5.6) and the interpretation of evaluative expressions (see Figure 4.1) it follows that $\mathcal{M}((\mu \text{Peop}_{o\epsilon}) z_{o\epsilon}) = F_R(P, W) \geq 0.91$. Thus, for example, if $|P| = 100$ then $|W| \geq 91$.

Conclusion: "Most people at the party are well dressed" The conclusion is the formula

$$Q^{\forall}_{Bi\ Ve}(\text{Peop}_{o\epsilon}, \text{Dress}_{o\epsilon}) :=$$
$$(\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \text{Peop}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \text{Dress}_{o\epsilon})) \wedge (Bi\ Ve)((\mu \text{Peop}_{o\epsilon}) z_{o\epsilon})). \tag{4.5.7}$$

Because we are dealing with classical sets, we conclude that finding a truth value of (4.5.7) requires to find a set $\mathcal{M}(z_{o\epsilon}) = D'$, where $D' \subseteq P$ and $D' \subseteq D$, which maximizes the truth value

$$\mathcal{M}((Bi\ Ve)((\mu \text{Peop}_{o\epsilon}) z_{o\epsilon})). \tag{4.5.8}$$

But from the first premise we know that $W \subseteq D$. From the fact that $F_R(P, W)$ provides the truth value 1 in (4.5.6) and from $D' \subseteq P$, we conclude that $W \subseteq D'$. Hence, $\mathcal{M}((\mu \text{Peop}_{o\epsilon}) z_{o\epsilon}) = F_R(P, D')$ provides the truth value 1 in (4.5.8). Consequently,

$$\mathcal{M}(Q^{\forall}_{Bi\ Ve}(\text{Peop}_{o\epsilon}, \text{Dress}_{o\epsilon})) = 1.$$

From $\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) \leq \mathcal{M}(C)$ it follows that this syllogism is strongly valid in our model.

For example, if $|P| = 100$ then the quantifier "most" means at least 91 people. By the discussed syllogism, if we know that all women are well dressed and most people at the party are women, then we conclude that at least 91 people at the party are well dressed.

4.5.2 Example of strongly valid syllogism of Figure II

$$\textbf{ETO}\text{-II:}\ \frac{\begin{array}{l}\text{No lazy people pass exam}\\ \text{Most students pass exam}\end{array}}{\text{Some students are not lazy people}}$$

Assume the same model and the definition of measure as above. Let M_ϵ be a set of people. Let $\mathrm{LP}_{o\epsilon}$ be a formula "lazy people" with the interpretation $\mathcal{M}(\mathrm{LP}_{o\epsilon}) = L \subseteq M_\epsilon$ where L is a classical set. Let $\mathrm{St}_{o\epsilon}$ be a formula "students" interpreted by $\mathcal{M}(\mathrm{St}_{o\epsilon}) = S \subseteq M_\epsilon$ where S is a classical set. Finally, let $\mathrm{Exam}_{o\epsilon}$ be a formula "students who pass exams" with the interpretation $\mathcal{M}(\mathrm{Exam}_{o\epsilon}) = E \subseteq M_\epsilon$ where E is a classical set.

Major premise: "No lazy people pass exam" From the assumption

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{LP}_{o\epsilon}(x_\epsilon) \Rightarrow (\neg \mathrm{Exam}_{o\epsilon}(x_\epsilon)))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\mathrm{LP}_{o\epsilon}(m)) \to (1 - \mathcal{M}(\mathrm{Exam}_{o\epsilon}(m)))) = 1 \quad (4.5.9)$$

we conclude that $L \subseteq M_\epsilon - E$, i.e., $E \subseteq M_\epsilon - L$.

Minor premise: "Most students pass exam" The assumption

$$(\exists z_{o\epsilon})((\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{St}_{o\epsilon}) \mathbin{\&} (\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Exam}_{o\epsilon})) \wedge (Bi\ Ve)(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon})) = 1 \quad (4.5.10)$$

means to find the greatest subset $\mathcal{M}(z_{o\epsilon}) = E' \subseteq S$ such that

$$\mathcal{M}(\mathbf{\Delta}(z_{o\epsilon} \subseteq \mathrm{St}_{o\epsilon})) = 1, \quad (4.5.11)$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Exam}_{o\epsilon})) = 1, \quad (4.5.12)$$

$$\mathcal{M}((Bi\ Ve)(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon}) = 1. \quad (4.5.13)$$

This holds if $E' \subseteq S \cap E$. Furthermore, from (4.5.13) and the interpretation of evaluative expressions (Figure 4.1), we conclude that $\mathcal{M}(\mu(\mathrm{St}_{o\epsilon}) z_{o\epsilon}) = F_R(S, E') \geq 0.91$, which means that $|S \cap E| \geq 0.91|S|$. Thus, for example, if $|S| = 100$ then $|S \cap E| \geq 91$.

Conclusion: "Some students are not lazy people" The conclusion is the formula

$$Q^{\exists}_{Bi\mathbf{\Delta}}(\mathrm{St}_{o\epsilon}, \mathrm{LSt}_{o\epsilon}) := (\exists x_\epsilon)(\mathrm{St}_{o\epsilon}(x_\epsilon) \wedge (\neg \mathrm{LSt}_{o\epsilon}(x_\epsilon))). \tag{4.5.14}$$

The interpretation $\mathcal{M}(\mathrm{St}_{o\epsilon}(x_\epsilon) \wedge \neg \mathrm{LSt}_{o\epsilon}(x_\epsilon)) = S \cap (M_\epsilon - L)$. From both premises we obtain $E' \subseteq (M_\epsilon - L)$ and $E' \subseteq S$ thus $E' \subseteq S \cap (M_\epsilon - L)$ which means that $S \cap (M_\epsilon - L) \neq \emptyset$ and we conclude that

$$\mathcal{M}(Q^{\exists}_{Bi\mathbf{\Delta}}(\mathrm{St}_{o\epsilon}, \mathrm{LSt}_{o\epsilon})) = 1.$$

Similarly as above from $\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) \leq \mathcal{M}(C)$, it follows that the syllogism **ETO**-II is strongly valid in our model. In the example, we see even more — that at least 91 students of 100 are not lazy people.

4.5.3 Example of strongly valid syllogism of Figure-III

	Almost all old people are ill
PPI-III:	Almost all old people have gray hair
	Some people with gray hair are ill

Suppose the same frame and the measure as above. Let M_ϵ be a set of people. We consider four people with the following age: u_1 (40 years), u_2 (70 years), u_3 (82 years), u_4 (95 years). Now we define interpretation of the formulas from our syllogism as follows: Let $\mathrm{Old}_{o\epsilon}$ be a formula "old people " with interpretation $\mathcal{M}(\mathrm{Old}_{o\epsilon}) = O \underset{\sim}{\subset} M_\epsilon$ defined by

$$O = \{^{0.3}/_{u_1}, ^{0.55}/_{u_2}, ^{0.8}/_{u_3}, ^{1}/_{u_4}\}.$$

Note that this fuzzy set is normal.

Let $\mathrm{Gr}_{o\epsilon}$ be a formula "people with gray hair" with the interpretation $\mathcal{M}(\mathrm{Gr}_{o\epsilon}) = G \underset{\sim}{\subset} M_\epsilon$ defined by

$$G = \{^{0.3}/_{u_1}, ^{0.55}/_{u_2}, ^{0.85}/_{u_3}, ^{0.9}/_{u_4}\}.$$

Finally, let $\mathrm{Ill}_{o\epsilon}$ be a formula "Ill people " with interpretation $\mathcal{M}(\mathrm{Ill}_{o\epsilon}) = I \underset{\sim}{\subset} M_\epsilon$ defined by

$$I = \{^{0.2}/_{u_1}, ^{0.4}/_{u_2}, ^{0.75}/_{u_3}, ^{0.95}/_{u_4}\}.$$

Major premise: "Almost all old people are ill" Let the assumption be

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})\,\&(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Ill}_{o\epsilon})) \wedge (Bi\ Ex)((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon}))) = a \in (0,1]. \quad (4.5.15)$$

This leads to the requirement to find the biggest fuzzy set $\mathcal{M}(z_{o\epsilon}) = X \underset{\sim}{\subsetneq} M_\epsilon$ such that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})) = 1, \quad (4.5.16)$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Ill}_{o\epsilon})) = b, \quad (4.5.17)$$

$$\mathcal{M}((Bi\ Ex)((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon})) = c, \quad (4.5.18)$$

where $b \wedge c = a$. From (4.5.18) and Figure 4.1 it follows that

$$\mathcal{M}((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon}) = F_R(O, X) > 0.75.$$

It can be verified that only the fuzzy set $O \underset{\sim}{\subsetneq} M_\epsilon$ has the properties above and gives us the greatest degree in (4.5.15). Thus we conclude that for $\mathcal{M}(z_{o\epsilon}) = O \underset{\sim}{\subsetneq} M_\epsilon$ we have that $c = 1$, $b = 0.85$ and hence

$$\mathcal{M}(Q^{\forall}_{Bi\ Ex}(\mathrm{Old}_{o\epsilon}, \mathrm{Ill}_{o\epsilon})) = a = 0.85. \quad (4.5.19)$$

Minor premise: "Almost all old people have gray hair" From the assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})\,\&(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Gr}_{o\epsilon}) \wedge (Bi\ Ex)((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon}))) = a' \in (0,1]. \quad (4.5.20)$$

Analogously as above, this means to find the biggest fuzzy set $\mathcal{M}(z_{o\epsilon}) = Y \underset{\sim}{\subsetneq} M_\epsilon$ such that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Old}_{o\epsilon})) = 1, \quad (4.5.21)$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Gr}_{o\epsilon})) = b', \quad (4.5.22)$$

$$\mathcal{M}((Bi\ Ex)((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon})) = c' \quad (4.5.23)$$

where $b' \wedge c' = a'$. From (4.5.23) and Figure 4.1 it results that

$$\mathcal{M}((\mu \mathrm{Old}_{o\epsilon})z_{o\epsilon}) = F_R(O, Y) > 0.75.$$

Analogously as above, it can be verified that only the fuzzy set $O \underset{\sim}{\subsetneq} M_\epsilon$ has the properties above and gives us the greatest degree in (4.5.20). Thus we obtain that $c' = 1$, $b' = 0.9$ and hence

$$\mathcal{M}(Q^{\forall}_{Bi\ Ex}(\mathrm{Old}_{o\epsilon}, \mathrm{Gr}_{o\epsilon})) = a' = 0.9. \quad (4.5.24)$$

Conclusion: "Some people with gray hair are ill" The conclusion is the formula

$$Q^{\exists}_{Bi\boldsymbol{\Delta}}(\mathrm{Gr}_{o\epsilon}, \mathrm{Ill}_{o\epsilon}) := (\exists x_\epsilon)(\mathrm{Gr}_{o\epsilon}(x_\epsilon) \wedge \mathrm{Ill}_{o\epsilon}(x_\epsilon)) \tag{4.5.25}$$

which is interpreted by

$$\mathcal{M}(Q^{\exists}_{Bi\boldsymbol{\Delta}}(\mathrm{Gr}_{o\epsilon}, \mathrm{Ill}_{o\epsilon})) = \bigvee_{m \in M_\epsilon} (\mathcal{M}(\mathrm{Gr}_{o\epsilon}(m)) \wedge \mathcal{M}(\mathrm{Ill}_{o\epsilon}(m))) = 0.9. \tag{4.5.26}$$

From (4.5.19),(4.5.24) and (4.5.26) we can see that $\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = 0.75 \leq \mathcal{M}(C) = 0.9$ which means that the syllogism above is strongly valid in our model.

We continue with two examples of invalid syllogisms which are formalizations of invalid syllogisms introduced in Peterson's book. By *invalid* syllogism we mean that there is a model in which premises are true in the degree 1 but the truth degree of the conclusion is smaller than 1 (i.e. such a syllogism is not even weakly valid).

4.5.4 First example of invalid syllogism

	Most bushes in the park are in blossom.
TAT-III:	All bushes in the park are perennial.
	Most perennial in the park are in blossom.

Suppose the same frame as above. Let M_ϵ be a set of "vegetables in the park". Let $\mathrm{Bushe}_{o\epsilon}$ be a formula "bushes in the park" with interpretation $\mathcal{M}(\mathrm{Bushe}_{o\epsilon}) = B \subseteq M_\epsilon$ where B is a classical set of 100 bushes. Furthermore, let $\mathrm{Bl}_{o\epsilon}$ be a formula "in blossom" with interpretation $\mathcal{M}(\mathrm{Bl}_{o\epsilon}) = F \subseteq M_\epsilon$ where F is a classical set of 95 vegetables in blossom. Finally, let $\mathrm{Per}_{o\epsilon}$ be a formula "perennial" with interpretation $\mathcal{M}(\mathrm{Per}_{o\epsilon}) = P \subseteq M_\epsilon$ where P is a classical set of 120 perennial.

Major premise "Most bushes in the park are in blossom" The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Bushe}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)((\mu \mathrm{Bushe}_{o\epsilon})z_{o\epsilon}))) = 1 \tag{4.5.27}$$

means to find the biggest subset $\mathcal{M}(z_{o\epsilon}) = F' \subseteq B$ such that:

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Bushe}_{o\epsilon})) = 1, \tag{4.5.28}$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Bl}_{o\epsilon})) = 1, \tag{4.5.29}$$

$$\mathcal{M}((Bi\ Ve)((\mu \mathrm{Bushe}_{o\epsilon})z_{o\epsilon})) = 1. \tag{4.5.30}$$

It can be verified that this holds if $F' = F$.

From (4.5.30) and Figure 4.1 it follows that $\mathcal{M}((\mu\text{Bushe}_{o\epsilon})z_{o\epsilon}) = F_R(B, F) \geq 0.91$. This means that if $|B| = 100$, then $|F| \geq 91$.

Minor premise "All bushes in the park are perennial" The assumption

$$\mathcal{M}((\forall x_\epsilon)(\text{Bushe}_{o\epsilon}(x_\epsilon) \Rightarrow \text{Per}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\text{Bushe}_{o\epsilon}(m) \to \mathcal{M}(\text{Per}_{o\epsilon}(m))) = 1 \quad (4.5.31)$$

means that $B \subseteq P$ and hence $F' \subseteq P$.

Conclusion "Most perennial in the park are in blossom" The conclusion is the following formula:

$$Q^{\forall}_{Bi\ Ve}(\text{Per}_{o\epsilon}, \text{Bl}_{o\epsilon}) := (\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{Per}_{o\epsilon})\,\&\,(\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Bl}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)((\mu\text{Per}_{o\epsilon})z_{o\epsilon})). \quad (4.5.32)$$

From the first premise for $\mathcal{M}(z_{o\epsilon}) = F'$ we get

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \text{Bl}_{o\epsilon})) = 1.$$

From the second one we obtain that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \text{Per}_{o\epsilon})) = 1. \quad (4.5.33)$$

From Figure 4.1 and from

$$\mathcal{M}((\mu\text{Per}_{o\epsilon})z_{o\epsilon}) = F_R(P, F') = 0.83$$

we obtain that

$$\mathcal{M}((Bi\ Ve)((\mu\text{Per}_{o\epsilon})z_{o\epsilon})) < 1.$$

Consequently, we conclude that

$$M(Q^{\forall}_{Bi\ Ve}(\text{Per}_{o\epsilon}, \text{Bl}_{o\epsilon})) < 1$$

which means that the syllogism **TAT**-III does not hold in our model, and thus, it is invalid.

4.5.5 Second example of invalid syllogism

	Most good dancers at the party are young people.
TAK-III:	All good dancers at the party are very nice dressed.
	Most very nice dressed dancers in the party are young people.

Suppose the same frame as above. Let M_ϵ be a set of "dancers at the party". Let us consider the following four dancers: d_1 (35 years), d_2 (45 years), d_3 (60 years), d_4 (70 years). Now we define interpretation of the formulas from our syllogism: Let $\mathrm{Dance}_{o\epsilon}$ be a formula "good dancers at the party" with interpretation $\mathcal{M}(\mathrm{Dance}_{o\epsilon}) = D \underset{\sim}{\subset} M_\epsilon$ defined by

$$D = \{^{0.7}/d_1, {}^{0.3}/d_2, {}^{0.1}/d_3, {}^{0.05}/d_4\}.$$

Furthermore, let $\mathrm{Young}_{o\epsilon}$ be a formula "young people" with the interpretation $\mathcal{M}(\mathrm{Young}_{o\epsilon}) = Y \underset{\sim}{\subset} M_\epsilon$ defined by

$$Y = \{^{0.9}/d_1, {}^{0.8}/d_2, {}^{0.75}/d_3, {}^{0.6}/d_4\}.$$

Finally, let $\mathrm{VeDr}_{o\epsilon}$ be a formula " very nice dressed" with the interpretation $\mathcal{M}(\mathrm{VeDr}_{o\epsilon}) = V \underset{\sim}{\subset} M_\epsilon$ defined by

$$V = \{^{0.95}/d_1, {}^{0.9}/d_2, {}^{0.85}/d_3, {}^{0.7}/d_4\}.$$

Major premise "Most good dancers at the party are young people" The assumption

$$\mathcal{M}((\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Dance}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Young}_{o\epsilon}x_\epsilon)) \wedge (Bi\ Ve)((\mu\mathrm{Dance}_{o\epsilon})z_{o\epsilon}))) = 1 \quad (4.5.34)$$

means to find the biggest fuzzy subset $D' \underset{\sim}{\subset} M_\epsilon$ such that $\mathcal{M}(z_{o\epsilon}) = D'$ and the following holds:

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{Dance}_{o\epsilon})) = 1, \quad (4.5.35)$$

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon}x_\epsilon \Rightarrow \mathrm{Young}_{o\epsilon})) = 1, \quad (4.5.36)$$

$$\mathcal{M}((Bi\ Ve)((\mu\mathrm{Dance}_{o\epsilon})z_{o\epsilon})) = 1. \quad (4.5.37)$$

It can be verified that this holds if $D' = D$.

From (4.5.37) and Figure 4.1 it follows that $\mathcal{M}((\mu\mathrm{Dance}_{o\epsilon})z_{o\epsilon}) = F_R(D, D') \geq 0.91$.

Minor premise "All good dancers at the party are very nice dressed" The assumption is

$$\mathcal{M}((\forall x_\epsilon)(\mathrm{Dance}_{o\epsilon}(x_\epsilon) \Rightarrow \mathrm{VeDr}_{o\epsilon}(x_\epsilon))) = \bigwedge_{m \in M_\epsilon} (\mathcal{M}(\mathrm{Dance}_{o\epsilon}(m) \rightarrow \mathcal{M}(\mathrm{VeDr}_{o\epsilon}(m))) = 1. \quad (4.5.38)$$

Conclusion "Most very nice dressed dancers in the party are young people" The conclusion is the following formula:

$$Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(\mathrm{VeDr}_{o\epsilon}, \mathrm{Young}_{o\epsilon}) := (\exists z_{o\epsilon})((\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{VeDr}_{o\epsilon}) \,\&\, (\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Young}_{o\epsilon} x_\epsilon)) \wedge (\neg(Sm\,\bar{\boldsymbol{\nu}}))((\mu \mathrm{VeDr}_{o\epsilon}) z_{o\epsilon})). \quad (4.5.39)$$

From the first premise for $\mathcal{M}(z_{o\epsilon}) = D' = D$ we have

$$\mathcal{M}((\forall x_\epsilon)(z_{o\epsilon} x_\epsilon \Rightarrow \mathrm{Young}_{o\epsilon})) = 1.$$

From the second one we obtain that

$$\mathcal{M}(\boldsymbol{\Delta}(z_{o\epsilon} \subseteq \mathrm{VeDr}_{o\epsilon})) = 1 \quad (4.5.40)$$

because (4.5.38) is equivalent with (4.5.40). From Figure 4.1 and from

$$\mathcal{M}((\mu \mathrm{VeDr}_{o\epsilon}) z_{o\epsilon}) = F_R(V, D') = 0.34$$

we obtain that

$$\mathcal{M}(\neg(Sm\,\bar{\boldsymbol{\nu}})((\mu \mathrm{VeDr}_{o\epsilon}) z_{o\epsilon})) < 1.$$

Consequently, we conclude that

$$M(Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(\mathrm{VeDr}_{o\epsilon}, \mathrm{Young}_{o\epsilon})) < 1$$

which means that the syllogism **TAK**-III does not hold in our model and thus, it is invalid.

Chapter 5

Analysis of generalized Aristotle's square in Ł-FTT

The main goal of this chapter is to find relationships between intermediate generalized quantifiers by generalizing the complete square of opposition which was first studied in classical logic by Thompson in [47]. First, we will recall the classical Aristotle's square (or simply *square*), which works with two quantifiers only — the *universal and existential* which are interpreted as *all* and *some*. Then we will continue with a construction of the generalized complete square of opposition. This generalized square is constructed using three *generalized intermediate quantifiers*, namely *few, many* and *most*.

5.1 Aristotle's square and modern square in classical logic

There are a lot of publications related to this area. Remember the work of Parsons [41], Peterson [45] and many others (see [1, 50]). The square consists of the relations among generalized intermediate quantifiers. In correspondence with the classical square, we will consider the relations of contrary, contradictory and subcontrary.

5.1.1 The basic definitions in classical logic

First, we will recall the definitions of *contradictories, contraries, subcontraries* and *subalterns* in classical logic.

Definition 45 (Contrary and Contradictory)

- *We say that two formulas are* contradictory *iff in any model they both cannot be true and they both cannot be false.*

- *We say that two formulas are* contraries *iff in any model they both cannot be true but both can be false.*

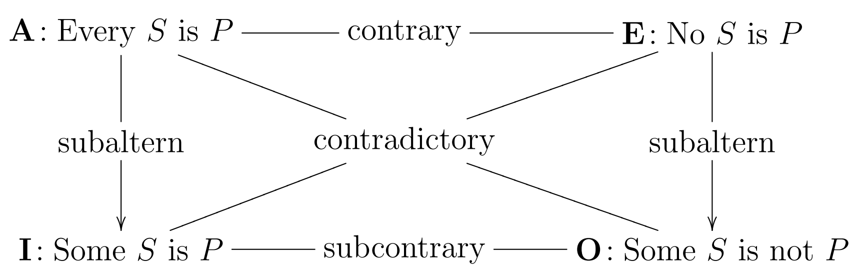
- *We say that two formulas are* subcontraries *iff in any model they both cannot be false but both can be true.*

- *A formula is a* subaltern *of another one called* superaltern *iff in any model it must be true if its superaltern is true. At the same time, the superaltern must be false if the subaltern is false.*

5.1.2 Traditional square of opposition

The diagram introduced below is called Aristotle's (classical) square of opposition which contains the four categorical proposition.

Aristotle's square: The following diagram will is called *Aristotle's square*.

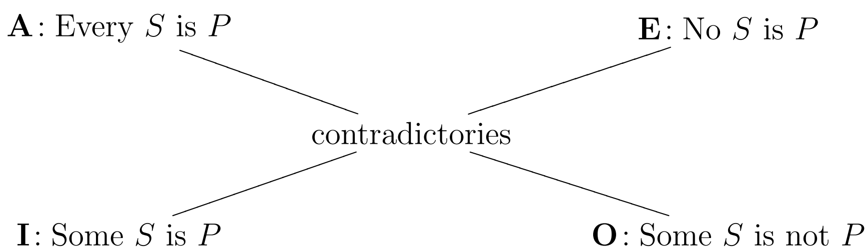


We can see that **A**, **O** and **E**, **I** are contradictories, which means that **A** and **O** forms entail each others' negations, the same as **E** and **I** formulas do. This means that the negation of **A** entails the unnegated **O**. Analogously for the second pair **E**, **I**. In our theory, it is not allowed that **A**, **O** and **E**, **I** entail each other's negations, because the formula **O** is not defined as negation of the formula **A**, the same for the second pair. As we can see below, the definition of contradictory in Ł-FTT will be introduced using $\boldsymbol{\Delta}$ connective.

The formulas **I**, **O** are subcontraries, which means that the negation of **I** entails the unnegation of **E**. Analogously for the second pair, namely negation of **O** entails the unnegation of **A**. The formula **I** is subaltern of **A**, which means that **A** entails **I** and thus **O** is subaltern **E**, which means that the formula **E** entails the formula **O**.

5.1.3 Modern square of opposition

If we translate Aristotle's formulas **A**, **E**, **I** and **O** in a standard way into the notation of the first-order logic, we obtain the *modern square* which is as follows:



such that

$$\textbf{A}: \text{All } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow Ax), \tag{5.1.1}$$

$$\textbf{E}: \text{No } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow \neg Ax), \tag{5.1.2}$$

$$\textbf{I}: \text{Some } B \text{ are } A \qquad (\exists x)(Bx \wedge Ax), \tag{5.1.3}$$

$$\textbf{O}: \text{Some } B \text{ are not } A \qquad (\exists x)(Bx \wedge \neg Ax). \tag{5.1.4}$$

From this situation we can see that $(\forall x)(Bx \Rightarrow Ax)$ and $(\forall x)(Bx \Rightarrow \neg Ax)$ are both **true** when $\neg(\exists x)Bx$ is true and this is when there are no B. However, "All B are A" in the modern interpretation does not imply that there are elements in B, and so it does not imply that some B are A. Thus each of the four propositions must be taken to **presuppose** that the formulas A, B are assigned **non empty sets**.

Now we have two possibilities. First, a presupposition is assumed and second, instead of presupposing that there are Bs, we can assert it by inserting the formula $(\exists x)Bx$ to the Aristotelian propositions. Thus the scheme above can be rewritten as follows:

$$\textbf{A}: \text{All } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow Ax) \wedge (\exists x)Bx, \tag{5.1.5}$$

$$\textbf{E}: \text{No } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow \neg Ax), \tag{5.1.6}$$

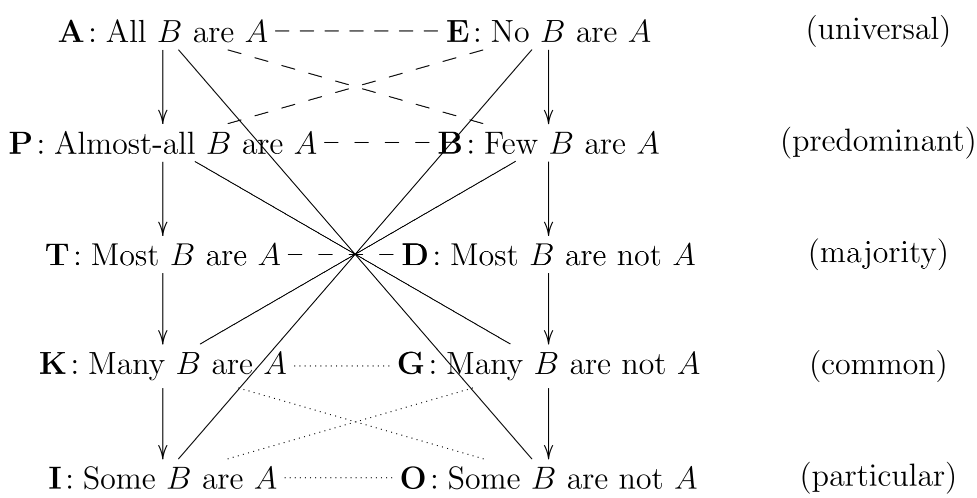
$$\textbf{I}: \text{Some } B \text{ are } A \qquad (\exists x)(Bx \wedge Ax), \tag{5.1.7}$$

$$\textbf{O}: \text{Some } B \text{ are not } A \qquad (\exists x)(Bx \wedge \neg Ax) \vee \neg(\exists x)Bx. \tag{5.1.8}$$

Remark 12
Remember that ***A** *and* ***O** *denote formulas which contain presupposition.*

5.2 Thompson's complete square of opposition

In this section, we will first present the *complete square* of opposition in classical logic which is generalization of the classical Aristotle's square. Its first version was introduced by P. Peterson in (1979) in [44]. He published the square of opposition with the intermediate quantifiers "Allmost-all" and "Many". Thompson (1982) extended the approach by the intermediate quantifier "Most" in [47] and introduced a complete square of opposition with contradictions, contraries and subalternations as follows:



The straight lines mark contradictories, the dashed lines contraries, the dotted lines sub-contraries. The arrows indicate subalterns.

5.3 Generalized definitions in Ł-FTT

The main idea is to find proper definitions of contraries, contradictories and sub-contraries in Ł-FTT in order to show that the *generalized complete square* using the generalized intermediate quantifiers defined in the previous chapter can be constructed in Ł-FTT as well.

5.3.1 Contraries and subcontraries in Ł-FTT

Definition 46 (Contraries in Ł-FTT)
Let P_1 and P_2 be from $Form_o$. We say that P_1 and P_2 are contraries *in T^{IQ} (denoted by Contr(P_1, P_2)) if in every model $\mathcal{M} \models T^{IQ}$ the following is true:*

$$\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = \mathcal{M}(\bot).$$

Alternatively we can say that P_1 and P_2 are contraries, if $T^{\mathrm{IQ}} \vdash P_1 \,\&\, P_2 \equiv \bot$.

Definition 47 (Sub-contraries in Ł-FTT)
We say that P_1 and P_2 are sub-contraries *in T^{IQ} (denoted by Sub-contr(P_1, P_2)) if in every model $\mathcal{M} \models T^{IQ}$ the following is true:*

$$\mathcal{M}(P_1) \oplus \mathcal{M}(P_2) = \mathcal{M}(\top).$$

Analogously as above P_1 and P_2 are sub-contraries, if $T^{IQ} \vdash P_1 \boldsymbol{\nabla} P_2$.

Remark 13
Remember that $\otimes$ is the Łukasiewicz conjunction and $\oplus$ is the Łukasiewicz disjunction defined in Chapter 1.

5.3.2 Contradictories in Ł-FTT

The first idea how to define contradictories in Ł-FTT was to say that P_1 and P_2 are contradictories if they are both contraries and subcontraries. But later we will want to show that the classical quantifiers **A** and **O** as well as **E** and **I** are contradictories. However, we may prove that

$$\neg\mathbf{A} \equiv \neg(\forall x)(Bx \Rightarrow Ax) \equiv (\exists x)(Bx \,\&\, Ax) \not\equiv \mathbf{O}$$

(in the definition of **O**, the conjunction $\wedge$ is used). Therefore, we will define the contradictories in Ł-FTT using the delta connective.

Definition 48 (Contradictories in Ł-FTT)
Let $P_1, P_2 \in Form_o$. We say that P_1 and P_2 are contradictories *in T^{IQ} (denoted by Contrad(P_1, P_2)) if in every model $\mathcal{M} \models T^{IQ}$ it holds that:*

(a) $\mathcal{M}(\boldsymbol{\Delta} P_1) \otimes \mathcal{M}(\boldsymbol{\Delta} P_2) = \mathcal{M}(\bot)$ *as well as*

(b) $\mathcal{M}(\boldsymbol{\Delta} P_1) \oplus \mathcal{M}(\boldsymbol{\Delta} P_2) = \mathcal{M}(\top)$.

Alternatively we can say that P_1 and P_2 are contradictories, if $T^{IQ} \vdash \boldsymbol{\Delta} P_1 \,\&\, \boldsymbol{\Delta} P_2 \equiv \bot$ and $T^{IQ} \vdash \boldsymbol{\Delta} P_1 \boldsymbol{\nabla} \boldsymbol{\Delta} P_2$.

5.3.3 Subalterns in Ł-FTT

Definition 49 (Subalterns in Ł-FTT)
Let S and A be from $Form_o$. We say that A is a subaltern *of S in T^{IQ} (denoted by $Subaltern(A,S)$) if in every model $\mathcal{M} \models T^{IQ}$ the following holds:*

$$\mathcal{M}(A) \leq \mathcal{M}(S).$$

We will call S as superaltern *of A. Alternatively we can say that A is a subaltern of S, if $T^{IQ} \vdash A \Rightarrow S$.*

5.4 Properties of classical quantifiers in Ł-FTT

This section contains formal proofs of properties of all the intermediate generalized quantifiers described in the complete square of opposition. $z \in Form_{o\alpha}$, $x \in Form_{\alpha}$ are variables and $A, B \in Form_{o\alpha}$ are formulas, $\alpha \in \mathcal{S}$. On many places, we write simply "by properties of Ł-FTT" because it is not possible to present explicitly all the formal properties used in the proofs. We will also fix the set S and write T^{IQ} instead of $T^{\mathrm{IQ}}[S]$.

5.4.1 Contraries and sub-contraries of classical quantifiers in $T^{\mathbf{IQ}}$

Lemma 29
The following is true:

$$T^{IQ} \vdash {}^{*}\mathbf{A} \mathbin{\&} \mathbf{E} \equiv \bot. \tag{5.4.1}$$

PROOF: Remember that

$$\mathbf{A} := (\forall x)(Bx \Rightarrow Ax) \quad \text{and} \quad \mathbf{E} := (\forall x)(Bx \Rightarrow \neg Ax).$$

From

$$\vdash (Bx \Rightarrow Ax) \mathbin{\&} (Bx \Rightarrow \neg Ax) \Rightarrow (Bx \Rightarrow (Ax \mathbin{\&} \neg Ax)),$$

Lemma 12(g), using (PP5),(PP6) and by (P20) we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow Ax) \mathbin{\&} (\forall x)(Bx \Rightarrow \neg Ax) \Rightarrow ((\exists x)Bx \Rightarrow \bot).$$

From this, ones again by Lemma 12(g) we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow Ax) \mathbin{\&} (\exists x)Bx \mathbin{\&} (\forall x)(Bx \Rightarrow \neg Ax) \Rightarrow \bot.$$

The opposite implication is provable by Lemma 11(a). Finally, by Lemma 11(e), by (ŁFT7) and using Lemma 6(a), we get (5.4.1). □

Theorem 46 (Contrary in $T^{\mathbf{IQ}}$)
*(a) Contr(***A**, **E**) in T^{IQ},*

*(b) if $T^{IQ} \vdash (\exists x)Bx$ then Contr(**A**, **E**) in T^{IQ}.*

PROOF: Immediately from the previous lemma. □

Thus we may verify that Contr(***A**, **E**) is a special case of the strongly valid syllogism **E**(***A**)**O**-III which was proved in Theorem 33.

Lemma 30
*Let **O**, **I** be generalized intermediate quantifiers defined above. Then the following is true:*

$$T^{IQ} \vdash {}^*\mathbf{O} \nabla \mathbf{I}.$$

PROOF: Remember that

$$\mathbf{O} := (\exists x)(Bx \wedge \neg Ax) \quad \text{and} \quad \mathbf{I} := (\exists x)(Bx \wedge Ax).$$

From Lemma 29 we know that

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow Ax) \,\&\, (\exists x)Bx \,\&\, (\forall x)(Bx \Rightarrow \neg Ax) \Rightarrow \bot.$$

By (ŁFT9) we have

$$T^{\mathrm{IQ}} \vdash \top \Rightarrow \neg((\forall x)(Bx \Rightarrow Ax) \,\&\, (\exists x)Bx \,\&\, (\forall x)(Bx \Rightarrow \neg Ax)).$$

By Lemma 6(e), MP, Lemma 9(c) Lemma 6(a) we obtain

$$T^{\mathrm{IQ}} \vdash \neg(\forall x)(Bx \Rightarrow Ax) \nabla \neg(\exists x)Bx \nabla \neg(\forall x)(Bx \Rightarrow \neg Ax).$$

By quantifier properties we have

$$T^{\mathrm{IQ}} \vdash (\exists x)\neg(Bx \Rightarrow Ax) \nabla \neg(\exists x)Bx \nabla (\exists x)\neg(Bx \Rightarrow \neg Ax).$$

From this by Lemma 8(a), Lemma 7(b) and using rule (R) we derive

$$T^{\mathrm{IQ}} \vdash (\exists x)(Bx \,\&\, \neg Ax) \nabla \neg(\exists x)Bx \nabla (\exists x)(Bx \,\&\, Ax). \tag{5.4.2}$$

By (P5), rule of generalization and (PP6) we have

$$T^{\mathrm{IQ}} \vdash (\exists x)(Bx \,\&\, \neg Ax) \Rightarrow (\exists x)(Bx \wedge \neg Ax) \tag{5.4.3}$$

Analogously for the second formula

$$T^{\mathrm{IQ}} \vdash (\exists x)(Bx \,\&\, Ax) \Rightarrow (\exists x)(Bx \wedge Ax). \tag{5.4.4}$$

From (5.4.3) by $T^{\mathrm{IQ}} \vdash \neg(\exists x)Bx \Rightarrow \neg(\exists x)Bx$ and using Lemma 18(f) we get

$$T^{\mathrm{IQ}} \vdash (\exists x)(Bx \,\&\, \neg Ax) \,\nabla\, \neg(\exists x)Bx \Rightarrow (\exists x)(Bx \wedge \neg Ax) \,\nabla\, \neg(\exists x)Bx. \tag{5.4.5}$$

From (5.4.5) and (5.4.4) by Lemma 18(f) we obtain

$$\begin{aligned} T^{\mathrm{IQ}} \vdash (\exists x)(Bx \,\&\, \neg Ax) \,\nabla\, \neg(\exists x)Bx \,\nabla\, (\exists x)(Bx \,\&\, Ax) \Rightarrow \\ (\exists x)(Bx \wedge \neg Ax) \,\nabla\, \neg(\exists x)Bx \,\nabla\, (\exists x)(Bx \wedge Ax). \end{aligned} \tag{5.4.6}$$

From (5.4.2), (5.4.6) by MP we obtain

$$T^{\mathrm{IQ}} \vdash {}^{*}\mathbf{O} \,\nabla\, \mathbf{I}.$$

□

Theorem 47 (Sub-contrary in $T^{\mathbf{IQ}}$)

(a) *Sub-contr*$({}^{*}\mathbf{O}, \mathbf{I})$ *in* T^{IQ},

(b) *if* $T^{IQ} \vdash (\exists x)Bx$, *then Sub-contr*$(\mathbf{O}, \mathbf{I})$ *in* T^{IQ}.

PROOF: (a) Immediately from the lemma above. Property (b) is provable using Theorem 46(b) by the same steps as in Lemma 30. □

5.4.2 Contradictories of classical quantifiers in $T^{\mathbf{IQ}}$

In this subsection we show that there is a connection between the pairs of formulas $\mathbf{A}$, $\mathbf{O}$ and $\mathbf{E}$, $\mathbf{I}$.

Lemma 31

Let $\mathbf{A}$, $\mathbf{E}$ *and* $\mathbf{I}, \mathbf{O}$ *be generalized quantifiers introduced above. Then there is no model such that*

(a) $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\bot)$ *and* $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\bot)$,

(b) $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\top)$ *and* $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\top)$,

(c) $\mathcal{M}(\mathbf{E}) = \mathcal{M}(\bot)$ *and* $\mathcal{M}(\mathbf{I}) = \mathcal{M}(\bot)$,

(d) $\mathcal{M}(\mathbf{E}) = \mathcal{M}(\top)$ *and* $\mathcal{M}(\mathbf{I}) = \mathcal{M}(\top)$.

PROOF: (a) Let there be a model $\mathcal{M} \models T^{\mathrm{IQ}}$ such that $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\bot)$. Then

$$\mathcal{M}((\forall x)(Bx \wedge \neg Ax)) \leq \mathcal{M}((\exists x)(Bx \wedge \neg Ax)) = 0.$$

Thus from this assumption we know that $\mathcal{M}(B \wedge \neg A)_x[m] = 0$ for all $m \in M$. Thus $\mathcal{M}(B_x[m]) = 0$ or $\mathcal{M}(\neg A_x[m]) = 0$ and so $\mathcal{M}(A_x[m]) = 1$ for all $m \in M$. Consequently $\mathcal{M}((\forall x)(Bx \Rightarrow Ax)) = 1$ i.e, $\mathcal{M}(\mathbf{A}) = 1$.

(b) Let there be a model $\mathcal{M} \models T^{\mathrm{IQ}}$ such that $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\top)$. Then there exists $m \in M$ such that $\mathcal{M}(B \wedge \neg A)_x[m] = 1$. Then $\mathcal{M}(B_x[m]) = 1$ and $\mathcal{M}(\neg A_x[m]) = 1$ and hence $\mathcal{M}(A_x[m]) = 0$. Thus there exists $m \in M$ such that $\mathcal{M}(B \Rightarrow A)_x[m] = 0$. Because

$$\mathcal{M}((\forall x)(Bx \Rightarrow Ax)) \leq \mathcal{M}((\exists x)(Bx \Rightarrow Ax)) = 0$$

then $\mathcal{M}(\forall x)(Bx \Rightarrow Ax) = 0$ i.e. $\mathcal{M}(\mathbf{A}) = 0$.

(c) and (d) can be proved analogously as (a),(b). □

Theorem 48 (Contradictory in $T^{\mathbf{IQ}}$)

(a) *Contrad*$(\mathbf{A}, \mathbf{O})$ *in* T^{IQ},

(b) *Contrad*$(\mathbf{E}, \mathbf{I})$ *in* T^{IQ}.

PROOF: This results from the previous lemma. □

5.4.3 Subalterns of classical quantifiers in $T^{\mathbf{IQ}}$

Lemma 32
The following is true:

(a) $T^{IQ} \vdash {}^*\mathbf{A} \Rightarrow \mathbf{I}$,

(b) $T^{IQ} \vdash \mathbf{E} \Rightarrow {}^*\mathbf{O}$.

PROOF: (a) From (P11) we have

$$T^{\mathrm{IQ}} \vdash (Bx \Rightarrow Ax) \Rightarrow (Bx \Rightarrow (Bx \wedge Ax)).$$

By (PP5) and using (PP6) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow Ax) \Rightarrow ((\exists x)Bx \Rightarrow (\exists x)(Bx \wedge Ax)). \tag{5.4.7}$$

By Lemma 12(g) we get (a).

(b) By the same steps as in (a) we get

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow \neg Ax) \Rightarrow ((\exists x)Bx \Rightarrow (\exists x)(Bx \wedge \neg Ax)). \tag{5.4.8}$$

From (5.4.8) by Lemma 8(b), using Lemma 7(b) and by (R) we obtain

$$T^{\mathrm{IQ}} \vdash (\forall x)(Bx \Rightarrow \neg Ax) \Rightarrow (\neg(\exists x)Bx \, \nabla (\exists x)(Bx \wedge \neg Ax)).$$

□

Theorem 49 (Subalterns in $T^{\mathbf{IQ}}$)

(a) Subaltern($^*\mathbf{A}, \mathbf{I}$) *in* T^{IQ},

(b) if $T^{IQ} \vdash (\exists x)Bx$, *then Subaltern*($\mathbf{A}, \mathbf{I}$) *in* T^{IQ},

(c) Subaltern($\mathbf{E}, {}^*\mathbf{O}$) *in* T^{IQ},

(d) if $T^{IQ} \vdash (\exists x)Bx$, *then Subaltern*($\mathbf{E}, \mathbf{O}$) *in* T^{IQ}.

PROOF: (a) and (c) follow from the previous lemma. Let $T^{\mathrm{IQ}} \vdash (\exists x)Bx$. Then the property (b) follows from (5.4.7) by Lemma 12(d) and by MP. Analogously from (5.4.8) we obtain (d). □

5.5 Properties of classical quantifiers based on the extension of $T^{\mathbf{IQ}}$

5.5.1 Contraries and sub-contraries of classical quantifiers in $T[B, B']$

As it will be seen later, when we work with the quantifiers $\mathbf{P}, \mathbf{B}$, etc., we will need a special theory $T[B, B']$ as the extension of the theory T^{IQ} analogously as in subsection 4.4.1. We show that in the theory $T[B, B']$ also holds the results presented in the previous subsection.

Remark 14
The theory $T[B, B']$ is specified by the distinguished formulas B, B' which are, in fact, those occurring in the definitions of the intermediate quantifiers. When looking for a connection between intermediate generalized quantifiers we must realize that they are defined on the same universe. This formally means that they are represented by equivalent formulas B and B'. Their equivalence is supposed in Definition 44(a). Furthermore, we suppose that both formulas B and B' are normal fuzzy sets (it is supposed in Definition 44(b)). Remember that the following formula is provable analogously in the theory $T[B, B']$ as in Lemma 28:

$$T[B, B'] \vdash (\exists z)(\exists z')\boldsymbol{\Delta}((EB)z \,\&\, (EB')z' \,\&\, (\exists x)(zx \,\&\, z'x)). \tag{5.5.1}$$

Lemma 33
The following is true:

(a) $T[B, B'] \vdash {}^{*}\mathbf{A} \,\&\, \mathbf{E} \equiv \bot$,

(b) $T[B, B'] \vdash \mathbf{A} \,\&\, \mathbf{E} \equiv \bot$.

PROOF: Let $\mathbf{A}, \mathbf{E}$ be as follows:

$$\mathbf{A} := (\forall x)(Bx \Rightarrow Ax) \quad \text{and} \quad \mathbf{E} := (\forall x)(B'x \Rightarrow \neg Ax).$$

Because $T[B, B']$ is the extension of T^{IQ} from Lemma 29 we get

$$T[B, B'] \vdash (\forall x)(Bx \Rightarrow Ax) \,\&\, (\exists x)Bx \,\&\, (\forall x)(Bx \Rightarrow Ax) \Rightarrow \bot.$$

Thus by $T[B, B'] \vdash B \equiv B'$, using Rule (R) we get (a). From (a) by the assumption that $T[B, B'] \vdash (\exists x)\boldsymbol{\Delta}Bx$ and also $T[B, B'] \vdash (\exists x)Bx$, by Lemma 12(g) and by MP we obtain (b). □

Theorem 50 (Contrary)
(a) Contr$({}^{*}\mathbf{A}, \mathbf{E})$ *in* $T[B, B']$,

(b) Contr$(\mathbf{A}, \mathbf{E})$ *in* $T[B, B']$.

PROOF: Immediately as in the lemma above. □

Lemma 34
The following is true:

(a) $T[B,B'] \vdash {}^{*}\mathbf{O}\,\nabla\,\mathbf{I}$,

(b) $T[B,B'] \vdash \mathbf{O}\,\nabla\,\mathbf{I}$.

PROOF: (a) Let **O** and **I** be as follows:

$$\mathbf{I} := (\exists x)(B'x \wedge Ax) \quad \text{and} \quad \mathbf{O} := (\exists x)(Bx \wedge \neg Ax)$$

where $T[B,B'] \vdash B \equiv B'$. Then from Lemma 33(a) we have

$$T[B,B'] \vdash (\forall x)(Bx \Rightarrow Ax)\,\&(\exists x)Bx\,\&(\forall x)(B'x \Rightarrow \neg Ax) \Rightarrow \bot.$$

Then we continue with the same steps as in the proof of Lemma 30 and we conclude that $T[B,B'] \vdash {}^{*}\mathbf{O}\,\nabla\,\mathbf{I}$.

(b) Let **O** and **I** are as above. Then from Lemma 33(b) we have

$$T[B,B'] \vdash (\forall x)(Bx \Rightarrow Ax)\,\&(\forall x)(B'x \Rightarrow \neg Ax) \Rightarrow \bot.$$

Then we continue with the same steps as in (a).

□

Theorem 51 (Sub-contrary)
(a) *Sub-contr*$({}^{*}\mathbf{O}, \mathbf{I})$ *in* $T[B,B']$,

(b) *Sub-contr*$(\mathbf{O}, \mathbf{I})$ *in* $T[B,B']$.

PROOF: Immediately from the lemma above. □

5.5.2 Contradictories of classical quantifiers in $T[B,B']$

In this subsection we show that there is a connection between the formulas **A**, **O** as well as between **E**, **I** in the theory $T[B,B']$.

Let us put:

A :All B are A	$(\forall x)(Bx \Rightarrow Ax)$,	(5.5.2)
E :No B are A	$(\forall x)(Bx \Rightarrow \neg Ax)$,	(5.5.3)
I :Some B' are A	$(\exists x)(B'x \wedge Ax)$,	(5.5.4)
O :Some B' are not A	$(\exists x)(B'x \wedge \neg Ax)$.	(5.5.5)

Lemma 35
Let $\mathbf{A}$, $\mathbf{E}$ *and* $\mathbf{I}, \mathbf{O}$ *be generalized quantifiers introduced above. Then there is no model of* $T[B, B']$ *such that*

(a) $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\bot)$ *and* $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\bot)$*,*

(b) $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\top)$ *and* $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\top)$*,*

(c) $\mathcal{M}(\mathbf{E}) = \mathcal{M}(\bot)$ *and* $\mathcal{M}(\mathbf{I}) = \mathcal{M}(\bot)$*,*

(d) $\mathcal{M}(\mathbf{E}) = \mathcal{M}(\top)$ *and* $\mathcal{M}(\mathbf{I}) = \mathcal{M}(\top)$*.*

PROOF: (a) Let there be a model $\mathcal{M} \models T$ such that $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\bot)$. Then

$$\mathcal{M}((\forall x)(B'x \wedge \neg Ax)) \leq \mathcal{M}((\exists x)(B'x \wedge \neg Ax)) = 0.$$

Thus from this assumption we know that $\mathcal{M}(B' \wedge \neg A)_x[m] = 0$ for all $m \in M$. Thus $\mathcal{M}(B'_x[m]) = 0$ or $\mathcal{M}(\neg A_x[m]) = 0$ and so $\mathcal{M}(A_x[m]) = 1$ for all $m \in M$. Consequently using the assumption of $T \vdash B \equiv B'$ we get that $\mathcal{M}((\forall x)(Bx \Rightarrow Ax)) = 1$ i.e, $\mathcal{M}(\mathbf{A}) = 1$.

(b) Let there be a model $\mathcal{M} \models T$ such that $\mathcal{M}(\mathbf{O}) = \mathcal{M}(\top)$. Then there exists $m \in M$ such that $\mathcal{M}(B' \wedge \neg A)_x[m] = 1$. Then $\mathcal{M}(B'_x[m]) = 1$ and $\mathcal{M}(\neg A_x[m]) = 1$ and hence $\mathcal{M}(A_x[m]) = 0$. By the assumption of $T \vdash B \equiv B'$ we get that $\mathcal{M}(B_x[m]) = 1$. Thus there exists $m \in M$ such that $\mathcal{M}(B \Rightarrow A)_x[m] = 0$. Because

$$\mathcal{M}((\forall x)(Bx \Rightarrow Ax)) \leq \mathcal{M}((\exists x)(Bx \Rightarrow Ax)) = 0$$

then $\mathcal{M}(\forall x)(Bx \Rightarrow Ax) = 0$ i.e. $\mathcal{M}(\mathbf{A}) = 0$.

□

Theorem 52 (Contradictory)
(a) Contrad$(\mathbf{A}, \mathbf{O})$ *in* $T[B, B']$*,*

(b) Contrad$(\mathbf{E}, \mathbf{I})$ *in* $T[B, B']$*.*

PROOF: It results from the previous lemma. □

5.5.3 Subalterns of classical quantifiers in $T[B, B']$

Lemma 36
The following is true:

(a) $T[B, B'] \vdash {}^*\mathbf{A} \Rightarrow \mathbf{I}$,

(b) $T[B, B'] \vdash \mathbf{A} \Rightarrow \mathbf{I}$,

(c) $T[B, B'] \vdash \mathbf{E} \Rightarrow {}^*\mathbf{O}$,

(d) $T[B, B'] \vdash \mathbf{E} \Rightarrow \mathbf{O}$.

PROOF: By Lemma 32(a) and using the assumption that $T[B, B']$ is the extension of T^{IQ} we know that

$$T[B, B'] \vdash (\forall x)(Bx \Rightarrow Ax) \Rightarrow ((\exists x)Bx \Rightarrow (\exists x)(Bx \wedge Ax)). \tag{5.5.6}$$

By the assumption that $T[B, B'] \vdash B \equiv B'$, by (R) and using Lemma 12(g) we get (a). From (a) by Lemma 12(g), using the assumption $T[B, B'] \vdash (\exists x)Bx$ and by MP we obtain (b). Analogously (using the assumption that $T[B, B'] \vdash B \equiv B'$) by Lemma 32(b) we may prove (c). From the provable formula (5.4.8) by the assumption that $T[B, B'] \vdash B \equiv B'$ and so $T[B, B'] \vdash (\exists x)Bx$ we obtain (d). □

Theorem 53 (Subalterns)

(a) Subaltern$({}^*\mathbf{A}, \mathbf{I})$ *in* $T[B, B']$,

(b) Subaltern$(\mathbf{A}, \mathbf{I})$ *in* $T[B, B']$,

(c) Subaltern$(\mathbf{E}, {}^*\mathbf{O})$ *in* $T[B, B']$,

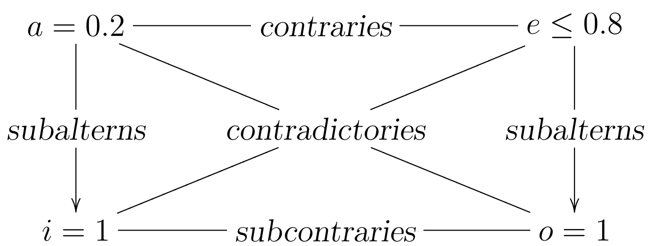
(d) Subaltern$(\mathbf{E}, \mathbf{O})$ *in* $T[B, B']$.

PROOF: From the previous lemma. □

5.5.4 Example of Aristotle's square interpreted in Ł-FTT

Example 8
Let there be a model $\mathcal{M} \models T[B, B']$ *such that* $\mathcal{M}(\mathbf{A}) = a = 0.2$. *Then from Contr*$(\mathbf{A}, \mathbf{E})$ *it follows that* $\mathcal{M}(\mathbf{E}) = e \leq 0.8$. *From Contrad*$(\mathbf{A}, \mathbf{O})$ *it follows that* $\mathcal{M}(\mathbf{O}) = o = 1$. *Consequently,* $\mathbf{E}$ *is subaltern of* $\mathbf{O}$. $\mathbf{I}$ *is superaltern of* $\mathbf{A}$ *thus*

$\mathcal{M}(\mathbf{I}) = i \geq 0.2$ but $\mathbf{I}$ is contradictory with $\mathbf{E}$ thus $\mathcal{M}(\mathbf{I}) = i = 1$. Finally, $\mathbf{I}$ is sub-contrary with $\mathbf{O}$ because $\mathcal{M}(\mathbf{O}\,\boldsymbol{\nabla}\,\mathbf{I}) = 1$ and $\mathbf{I}$ is superaltern of $\mathbf{A}$. These results are summarized in the following scheme:



5.6 Properties of Generalized quantifiers in Ł-FTT

5.6.1 Basic properties

Lemma 37
Let $\mathbf{P}, \mathbf{B}, \mathbf{T}, \mathbf{D}, \mathbf{K}, \mathbf{G}$ be the generalized quantifiers introduced above. Then the following is true:

(a) $T^{IQ} \vdash {}^*\mathbf{P} \Rightarrow \mathbf{P}$, $\qquad T^{IQ} \vdash {}^*\mathbf{B} \Rightarrow \mathbf{B}$,

(b) $T^{IQ} \vdash {}^*\mathbf{T} \Rightarrow \mathbf{T}$, $\qquad T^{IQ} \vdash {}^*\mathbf{D} \Rightarrow \mathbf{D}$,

(c) $T^{IQ} \vdash {}^*\mathbf{K} \Rightarrow \mathbf{K}$, $\qquad T^{IQ} \vdash {}^*\mathbf{G} \Rightarrow \mathbf{G}$.

PROOF: (a) Put $Ez := (Bi\, Ex)((\mu B)z)$. Then using the properties of Ł-FTT we have

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax) \Rightarrow \boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax).$$

Using $T^{\mathrm{IQ}} \vdash A\,\&\,B \Rightarrow A$ and other properties we obtain

$$T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)\,\&(\exists x)zx \Rightarrow \boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)$$

and hence

$$T^{\mathrm{IQ}} \vdash (\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(zx \Rightarrow Ax) \,\&\, (\exists x)zx) \wedge Ez \Rightarrow (\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(zx \Rightarrow Ax)) \wedge Ez. \quad (5.6.1)$$

By the rule of generalization with respect to $(\forall z)$ and using (PP6) we obtain

$$T^{\mathrm{IQ}} \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(zx \Rightarrow Ax) \,\&\, (\exists x)zx) \wedge Ez) \Rightarrow (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(zx \Rightarrow Ax)) \wedge Ez). \quad (5.6.2)$$

Analogously we may prove the second property. If we put $Ez := (Bi\ Ve)((\mu B)z)$ we obtain (b) and by putting $Ez := \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)$ we get (c). □

5.6.2 Contraries in generalized quantifiers in $T[B, B']$

In the following work we will see a very interesting connection with subsection 4.4.1 from the previous chapter which contains the twelve non-trivial generalized intermediate syllogisms. Further, we will work with a special theory $T[B, B']$ which was specified in subsection 5.5 in Remark 14.

Lemma 38
Let $\mathbf{P}, \mathbf{B}, \mathbf{T}, \mathbf{D}, \mathbf{K}, \mathbf{G}$ *be intermediate quantifiers. Let* $T[B, B']$ *be a theory introduced above. Then the following is true:*

(a) $T[B, B'] \vdash \mathbf{B} \,\&\, \mathbf{P} \equiv \bot$,

(b) $T[B, B'] \vdash \mathbf{D} \,\&\, \mathbf{T} \equiv \bot$,

(c) $T[B, B'] \vdash \mathbf{G} \,\&\, \mathbf{K} \equiv \bot$.

PROOF: Recall that

$$\mathbf{B} := (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \,\&\, (\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ex)((\mu B)z)), \quad (5.6.3)$$

$$\mathbf{P} := (\exists z')((\boldsymbol{\Delta}(z' \subseteq B') \,\&\, (\forall x)(z'x \Rightarrow Ax)) \wedge (Bi\ Ex)((\mu B')z')). \quad (5.6.4)$$

The proof will be constructed analogously to the proof of the strong validity of the syllogisms **BPO**-III. Put $M = B$,$M' = B'$,$Y = A$ and $X = A$. Put $T' = T[B, B'] \cup \{(EB)r \,\&\, (EB')r' \,\&\, (\exists x)(rx \,\&\, r'x)\}$ where $r, r' \notin J(T[B, B'])$ are new constants of type $o\alpha$. Then

$$T' \vdash (rx \Rightarrow \neg Ax) \,\&\, (r'x \Rightarrow Ax) \Rightarrow ((rx \,\&\, r'x) \Rightarrow (Ax \,\&\, \neg Ax)).$$

Thus by the property (P22) (it is $\vdash Ax \mathbin{\&} \neg Ax \equiv \bot$) and using rule (R) we obtain

$$T' \vdash (rx \Rightarrow \neg Ax) \mathbin{\&} (r'x \Rightarrow Ax) \Rightarrow ((rx \mathbin{\&} r'x) \Rightarrow \bot).$$

Analogously as in the proof of **BPO**-III where $Xx \wedge \neg Yx$ is replaced by $\bot$, we obtain

$$T[B,B'] \vdash (\exists z)((\boldsymbol{\Delta}(z \subseteq B) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Ax)) \wedge (EB)z) \Rightarrow (\exists z')((\boldsymbol{\Delta}(z' \subseteq B') \mathbin{\&} (\forall x)(z'x \Rightarrow Ax)) \wedge (EB')z') \Rightarrow \bot \quad (5.6.5)$$

which is by Lemma 12(g) equivalent with

$$T[B,B'] \vdash \{(\exists z)((\boldsymbol{\Delta}(z \subseteq B) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Ax)) \wedge (EB)z) \mathbin{\&} (\exists z')((\boldsymbol{\Delta}(z' \subseteq B') \mathbin{\&} (\forall x)(z'x \Rightarrow Ax)) \wedge (EB')z')\} \Rightarrow \bot \quad (5.6.6)$$

The opposite implication is provable by Lemma 11(a). Finally, by Lemma 11(e), by (ŁFT7) and using Lemma 6(a) we get

$$T[B,B'] \vdash \{(\exists z)((\boldsymbol{\Delta}(z \subseteq B) \mathbin{\&} (\forall x)(zx \Rightarrow \neg Ax)) \wedge (EB)z) \mathbin{\&} (\exists z')((\boldsymbol{\Delta}(z' \subseteq B') \mathbin{\&} (\forall x)(z'x \Rightarrow Ax)) \wedge (EB')z')\} \equiv \bot. \quad (5.6.7)$$

By putting $(EB)z := (Ex\,Bi)((\mu B)z)$ and $(EB')z' := (Ex\,Bi)((\mu B')z')$ we get the property (a). If we denote $(EB)z := (Ve\,Bi)((\mu B)z)$ and also $(EB')z' := (Ve\,Bi)((\mu B')z')$ we obtain (b) and finally by putting $(EB)z := \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)$ and also $(EB')z' := \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B')z')$ we have (c). □

Theorem 54 (Contrary)

(a) Contr$(\mathbf{B},\mathbf{P})$ *in* $T[B,B']$,

(b) Contr$(\mathbf{D},\mathbf{T})$ *in* $T[B,B']$,

(c) Contr$(\mathbf{G},\mathbf{K})$ *in* $T[B,B']$.

PROOF: It results from the previous lemma. □

From these results it follows that there exists a connection with the non-trivial syllogisms from Figure-III. Namely, Contr$(\mathbf{B},\mathbf{P})$ is a special case of the strong validity of the syllogism **BPO**-III. Analogously for the pairs Contr$(\mathbf{D},\mathbf{T})$ which is a special case of **DTO**-III.

Here arises the first question for the future work because the previous proof suggests that it will be possible to construct a proof of the syllogism **GKO**-III which is not included in the list of the 105 valid forms.

Remark 15
Below we will write $Contr(\mathbf{B}, \mathbf{P})$ *or* $Contr(\mathbf{P}, \mathbf{B})$ *which have the same meaning because* $\mathbin{\&}$ *is a commutative in Ł-FTT. Analogously for other generalized quantifiers.*

Lemma 39
The following is true:

(a) $T[B, B'] \vdash \mathbf{G} \mathbin{\&} \mathbf{P} \equiv \bot$,

(b) $T[B, B'] \vdash \mathbf{K} \mathbin{\&} \mathbf{B} \equiv \bot$.

PROOF: The proof is constructed analogously as in Lemma 38. By putting $(EB)z := \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)$ and $(EB')z' := (Ex\,Bi)((\mu B')z')$ we obtain (a) and if we denote $(EB)z := (Ex\,Bi)((\mu B)z)$ and $(EB')z' := \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B')z')$, we get (b). □

Theorem 55 (Contrary)
(a) $Contr(\mathbf{G}, \mathbf{P})$ *in* $T[B, B']$,

(b) $Contr(\mathbf{K}, \mathbf{B})$ *in* $T[B, B']$.

PROOF: This results from the previous lemma. □

Obviously as above, we can see that Contr$(\mathbf{G}, \mathbf{P})$ is a special case of the strong validity of the syllogism **GPO**-III. Analogously for the second pair, we can see the connection with the syllogism **BKO**-III.

We continue with lemmas that are provable using the monotonicity of the intermediate quantifiers introduced in Theorem 17.

Lemma 40
The following is true:

(a) $T[B, B'] \vdash \mathbf{E} \mathbin{\&} \mathbf{K} \equiv \bot$,

(b) $T[B, B'] \vdash \mathbf{E} \mathbin{\&} \mathbf{T} \equiv \bot$,

(c) $T[B, B'] \vdash \mathbf{E} \mathbin{\&} \mathbf{P} \equiv \bot$,

(d) $T[B, B'] \vdash \mathbf{A} \mathbin{\&} \mathbf{G} \equiv \bot$,

(e) $T[B, B'] \vdash \mathbf{A} \mathbin{\&} \mathbf{D} \equiv \bot$, *(f)* $T[B, B'] \vdash \mathbf{A} \,\&\, \mathbf{B} \equiv \bot$.

PROOF: (a) Using Lemma 39(b) and by $T[B, B'] \vdash \mathbf{E} \Rightarrow \mathbf{B}$.

(b) From (a) by $T[B, B'] \vdash \mathbf{T} \Rightarrow \mathbf{K}$.

(c) From (b) by $T[B, B'] \vdash \mathbf{P} \Rightarrow \mathbf{T}$.

(d) Using Lemma 39(a) and by $T[B, B'] \vdash \mathbf{A} \Rightarrow \mathbf{P}$.

(e) From (d) by $T[B, B'] \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

(f) From (e) by $T[B, B'] \vdash \mathbf{B} \Rightarrow \mathbf{D}$. □

Theorem 56 (Contrary)
(a) *Contr*$(\mathbf{E}, \mathbf{K})$ *in* $T[B, B']$,

(b) *Contr*$(\mathbf{E}, \mathbf{T})$ *in* $T[B, B']$,

(c) *Contr*$(\mathbf{E}, \mathbf{P})$ *in* $T[B, B']$,

(d) *Contr*$(\mathbf{A}, \mathbf{G})$ *in* $T[B, B']$,

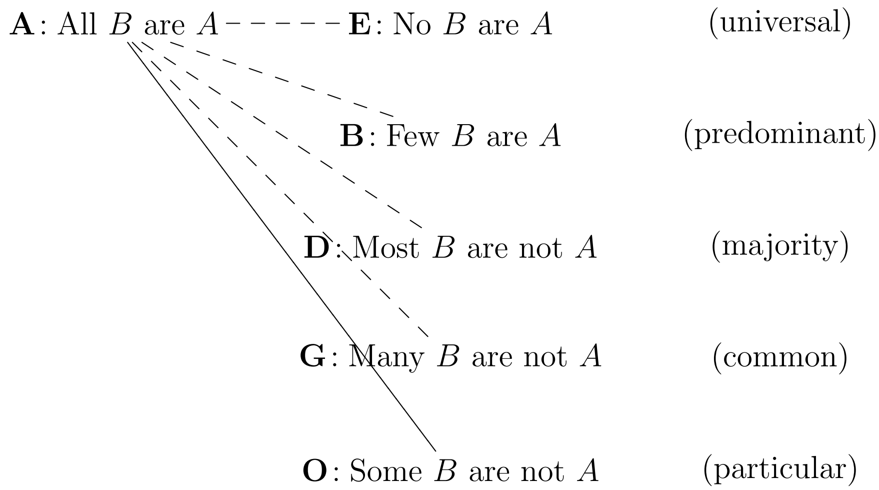
(e) *Contr*$(\mathbf{A}, \mathbf{D})$ *in* $T[B, B']$,

(f) *Contr*$(\mathbf{A}, \mathbf{B})$ *in* $T[B, B']$.

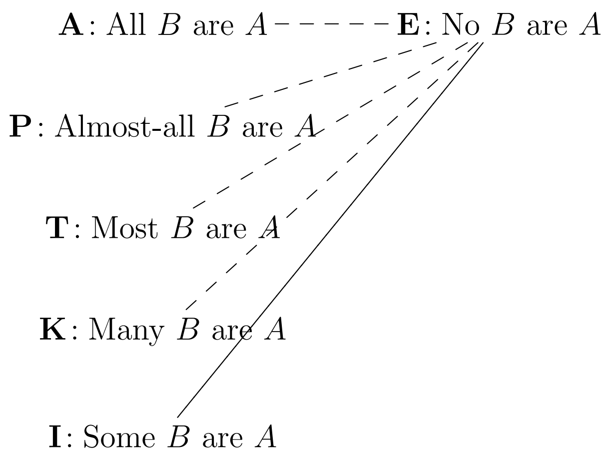
PROOF: This results from the previous lemma. □

Above we could see a connection with the syllogisms from Figure-III. There is a second question: is it possible to prove the syllogims **EPO**-III, **ETO**-III, **EKO**-III (remember that we proved the strong validity of all syllogisms with presupposition in Theorem 36), finally, is it possible to prove the syllogisms **BAO**-III, **DAO**-III, **GAO**-III? (we proved (*$\mathbf{B}$)**AO**-III, (*$\mathbf{D}$)**AO**-III, (*$\mathbf{G}$)**AO**-III in Theorem 35).

For reader's convenience we may depict the results from Theorem 56(d) and (e),(f) in the following figure. Remember that Contr$(\mathbf{A}, \mathbf{E})$ was proved in Theorem 50 and Contrad$(\mathbf{A}, \mathbf{O})$ was proved in Theorem 48(a).



We continue with the figure summarizing results $(a), (b), (c)$ from Theorem 56. The Contrad($\mathbf{E}, \mathbf{I}$) was proved in Theorem 48(b).



Lemma 41
The following is true:

(a) $T[B, B'] \vdash \mathbf{B} \,\&\, \mathbf{T} \equiv \bot$,

(b) $T[B, B'] \vdash \mathbf{P} \,\&\, \mathbf{D} \equiv \bot$,

(c) $T[B, B'] \vdash \mathbf{K} \,\&\, \mathbf{D} \equiv \bot$,

(d) $T[B, B'] \vdash \mathbf{T} \,\&\, \mathbf{G} \equiv \bot$.

PROOF: (a) From Lemma 38(b) by $T[B, B'] \vdash \mathbf{B} \Rightarrow \mathbf{D}$.

(b) From Lemma 38(b) by $T[B, B'] \vdash \mathbf{P} \Rightarrow \mathbf{T}$.

(c) From Lemma 38(c) and by by $T[B, B'] \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

(d) From Lemma 38(c) by $T[B, B'] \vdash \mathbf{T} \Rightarrow \mathbf{K}$. □

Theorem 57 (Contrary)
(a) Contr$(\mathbf{B}, \mathbf{T})$ *in* $T[B, B']$*,*

(b) Contr$(\mathbf{P}, \mathbf{D})$ *in* $T[B, B']$*,*

(c) Contr$(\mathbf{K}, \mathbf{D})$ *in* $T[B, B']$*,*

(d) Contr$(\mathbf{T}, \mathbf{G})$ *in* $T[B, B']$*.*

PROOF: This results from the previous lemma. □

5.6.3 Subalterns in Ł-FTT

Lemma 42
Let $\mathbf{A}, \mathbf{P}, \mathbf{T}, \mathbf{K}, \mathbf{I}$ *be generalized intermediate quantifiers. Then in every model* $\mathcal{M} \models T^{IQ}$ *the following is true*

$$\mathcal{M}(\mathbf{A}) \leq \mathcal{M}(\mathbf{P}) \leq \mathcal{M}(\mathbf{T}) \leq \mathcal{M}(\mathbf{K}) \leq \mathcal{M}(\mathbf{I}).$$

PROOF: This follows from Lemma 17 on monotonicity of intermediate quantifiers. □

Theorem 58 (Subalternations in $T^{\mathbf{IQ}}$)
(a) Subaltern$(\mathbf{A}, \mathbf{P})$ *in* T^{IQ}*,*

(b) Subaltern$(\mathbf{P}, \mathbf{T})$ *in* T^{IQ}*,*

(c) Subaltern$(\mathbf{T}, \mathbf{K})$ *in* T^{IQ}*,*

(d) Subaltern$(\mathbf{K}, \mathbf{I})$ *in* T^{IQ}*.*

PROOF: Obvious. □

Theorem 59 (Subalternations in the extension of $T^{\mathbf{IQ}}$)
(a) Subaltern$(\mathbf{A}, \mathbf{P})$ *in* $T[B, B']$*,*

(b) Subaltern$(\mathbf{P}, \mathbf{T})$ *in* $T[B, B']$*,*

(c) *Subaltern*($\mathbf{T}, \mathbf{K}$) *in* $T[B, B']$,

(d) *Subaltern*($\mathbf{K}, \mathbf{I}$) *in* $T[B, B']$.

PROOF: It follows from the fact that $T[B, B']$ is the extension of T^{IQ} and the quantifiers $\mathbf{A}, \mathbf{P}, \mathbf{T}, \mathbf{K}, \mathbf{I}$ are defined above the same universe B. □

Corollary 2
$\mathbf{A}$ *is subaltern of* $\mathbf{P}, \mathbf{T}, \mathbf{K}, \mathbf{I}$ *and* $\mathbf{I}$ *is superaltern of* $\mathbf{A}, \mathbf{P}, \mathbf{T}, \mathbf{K}$.

Lemma 43
Let $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{G}, \mathbf{O}$ *be generalized intermediate quantifiers. Then in every model* $\mathcal{M} \models T^{IQ}$ *the following is true:*

$$\mathcal{M}(\mathbf{E}) \leq \mathcal{M}(\mathbf{B}) \leq \mathcal{M}(\mathbf{D}) \leq \mathcal{M}(\mathbf{G}) \leq \mathcal{M}(\mathbf{O})$$

PROOF: Analogously as above, this follows from the monotonicity of intermediate quantifiers. □

Theorem 60 (Subalternations in $T^{\mathbf{IQ}}$)
(a) *Subaltern*($\mathbf{E}, \mathbf{B}$) *in* T^{IQ},

(b) *Subaltern*($\mathbf{B}, \mathbf{D}$) *in* T^{IQ},

(c) *Subaltern*($\mathbf{D}, \mathbf{G}$) *in* T^{IQ},

(d) *Subaltern*($\mathbf{G}, \mathbf{O}$) *in* T^{IQ}.

PROOF: Obvious. □

Theorem 61 (Subalternations in the extension of $T^{\mathbf{IQ}}$)
(a) *Subaltern*($\mathbf{E}, \mathbf{B}$) *in* $T[B, B']$,

(b) *Subaltern*($\mathbf{B}, \mathbf{D}$) *in* $T[B, B']$,

(c) *Subaltern*($\mathbf{D}, \mathbf{G}$) *in* $T[B, B']$,

(d) *Subaltern*($\mathbf{G}, \mathbf{O}$) *in* $T[B, B']$.

PROOF: Obviously as in Theorem 59. □

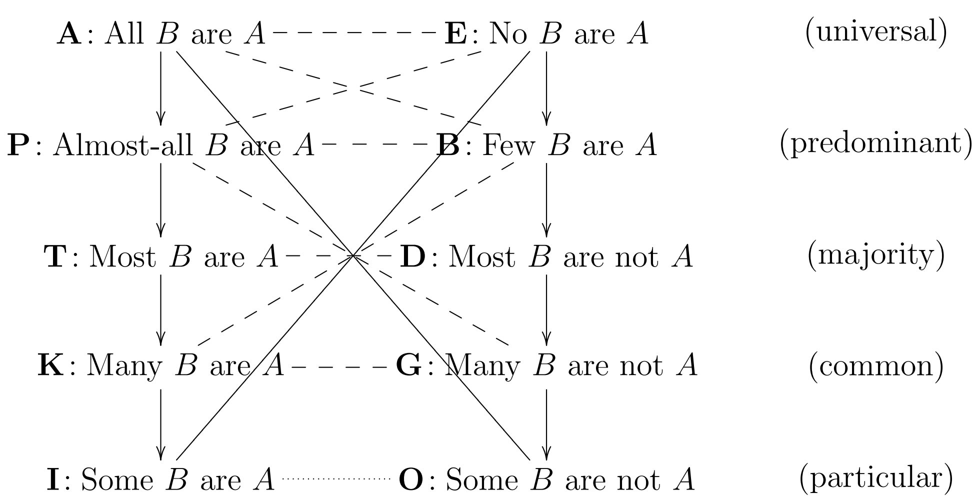
Corollary 3
$\mathbf{E}$ *is subaltern of* $\mathbf{B}, \mathbf{D}, \mathbf{G}, \mathbf{O}$ *and* $\mathbf{O}$ *is superaltern of* $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{G}$.

5.7 Generalized complete square of opposition in Ł-FTT

We finish this chapter with a figure depicting the generalized complete square with contradictions, contraries, sub-contraries and subalterns which generalizes the classical complete square of opposition presented by Thompson's in [47] and also by Peterson in [45].

5.7.1 Generalized complete square in Ł-FTT

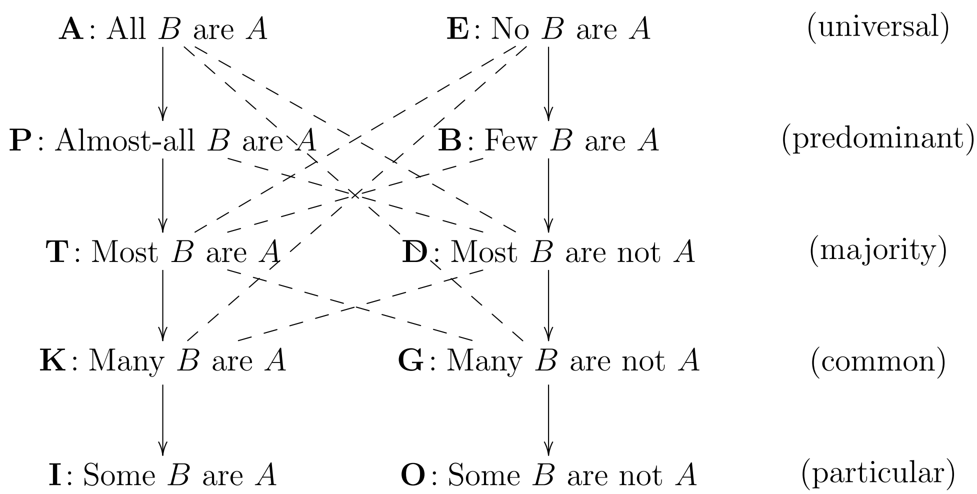
Remember that the straight lines mark contradictories, the dashed lines contraries, the dotted lines sub-contraries. The arrows indicate subalterns.



Furthermore, we will focus on the properties of the formulas which have different properties in comparison with the classical Thompson's square of opposition. Namely, we showed that Contr(**P**, **G**) and Contr(**B**, **K**) (they are in classical complete square contradictory), Contr(**K**, **G**) (in classical complete square they are subcontrary). The question if there exists a relation between the forms **K** and **O** and also between **G** and **I** is the topic of future work.

From the results of Theorem 56 and Theorem 57 we conclude that there are many other relations among intermediate generalized quantifiers not introduced either in [45] or in [47]. For reader's convenience, we summarize only the properties which are not included in the previous figure.

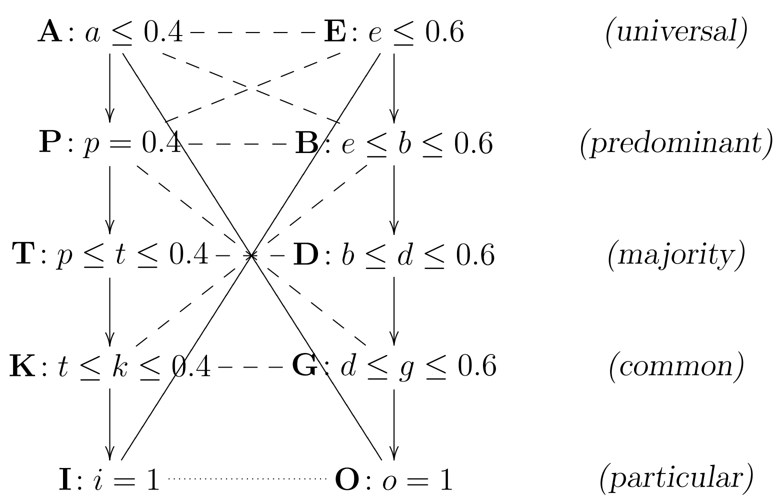
5.7.2 Extension of the generalized complete square of opposition in Ł-FTT



5.7.3 Example of the generalized complete square of opposition in Ł-FTT

Example 9

Let there be a model $\mathcal{M} \models T[B, B']$ such that $\mathcal{M}(\mathbf{P}) = p = 0.4$ and put $\mathcal{M}(\mathbf{A}) = a$, $\mathcal{M}(\mathbf{E}) = e$, $\mathcal{M}(\mathbf{B}) = b$, $\mathcal{M}(\mathbf{T}) = t$, $\mathcal{M}(\mathbf{D}) = d$, $\mathcal{M}(\mathbf{K}) = k$, $\mathcal{M}(\mathbf{G}) = g$, $\mathcal{M}(\mathbf{I}) = i$, $\mathcal{M}(\mathbf{O}) = o$ where $a, e, p, b, t, d, k, g, i, o \in [0, 1]$. Then by the definitions of contrary, contradictory, subcontrary and subalterns, we may construct the generalized complete square of opposition in $T[B, B']$ as follows:



We may see that all a formulas which are connected by dashed lines are contrary, the formulas **A**, **O** *and* **E**, **I** *are contradictory and* **I**, **O** *are sub-contrary. The subalterns are also valid.*

Chapter 6

Conclusion

The main results of this thesis were divided into three main parts. First, in Chapter 2 we have introduced a Łukasiewicz fuzzy type theory based on a linearly ordered MV_{Δ}-algebra. Many of its formal properties were described and the completeness theorem was proved. It is important to say that the Łukasiewicz fuzzy type is higher order logic which is sufficiently powerful to be applied in modeling of the vagueness phenomenon, or in modeling of the commonsense reasoning.

In Chapter 3, we have developed the formal theory T^{Ev} of evaluative linguistic expressions which is a special theory of Ł-FTT. In Chapter 4, we continue developing the formal theory T^{IQ} of intermediate quantifiers introduced in [38]. This theory is proposed as an extension of the theory T^{Ev}. Both theories are special theories of Ł-FTT. Following the book of P. Peterson [45], who informally demonstrated that 105 syllogisms generalizing the classical Aristotle's ones are valid, we have formally proved that all of them are also valid in our theory, and even in a strong sense, which means that the implication $P_1 \,\&\, P_2 \Rightarrow C$ is provable and thus the truth value of C in any model is greater or equal to the truth value of $P_1 \,\&\, P_2$.

In Chapter 5, we proposed a definitions of contrary, sub-contrary and contradictory in Ł-FTT which are appropriate for description of the relationships between intermediate generalized quantifiers. These relations are shown by generalizing the complete square of opposition, which was studied first in classical logic by Thompson in [47].

There are several interesting problems to be solved in the future:

(a) What are the general properties of intermediate quantifiers from the point of view of the classical theory of general quantifiers (see classical literature [19, 43, 49]; fuzzy logic literature [8, 11, 18])? We have already mentioned that they are *isomorphism-invariant* $\langle 1, 1\rangle$ quantifiers. More properties, e.g. *extension, conservativity*, and other ones are to be studied.

(b) What further kinds of generalized quantifiers can be defined and what corresponding generalized syllogisms can be formed? The possible syllogisms were mentioned in subsection 5.6.2.

(c) How the concept of possible worlds can be incorporated (cf. [24, 37]) because, as could be seen in our examples above, we are dealing with properties whose extensions depend on possible worlds.

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Index