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Cardinalities of Fuzzy Sets and Fuzzy Quantifiers over Residuated Lattices

Doctoral Thesis

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2005

To my wife Barbara and
our sons Lukáš and Daniel,

with love.

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Preface

During the last few years one could have already noticed a significant growth of the new calculation kinds that could, in general, be called the soft computing. This type of computing differs from the traditional calculations in its primary goal, i.e. the toleration of uncertainty and inaccuracy in order to reach quick, robust and cheap results. Thus the soft computing is used to find an approximate solution of an exactly formulated problem or, more frequently and most typically, to find the solution of the problem that is not formulated exactly itself. The new unconventional theories, methods, techniques and technologies in computer and information science, systems analysis, decision-making and control, expert systems, data modeling, engineering, etc., using techniques of soft computing together with their basic tools theory of fuzzy sets and fuzzy logic, are still investigated and developed.

Fuzzy set theory as well as fuzzy logic originated from ideas of L.A. Zadeh. The concept of fuzzy set together with the basic principles of fuzzy set theory was established in his seminal paper [119] published in 1965 and the basic ideas of fuzzy logic were elaborated in [121–123]. Thanks to the fact that fuzzy set theory and fuzzy logic could offer very strong tools for description of an uncertainty in common human reasoning, an intensive development took place in these areas. It was motivated not only by requirement to develop these sciences themselves, but, of course, to be able to solve practical problems. One of the topic of fuzzy logic that was and still is intensively investigated is the theory of fuzzy (generalized) quantifications. Fuzzy quantification is a construct that specifies the extend of validity of a predicate, where the range of predicate validity is often expressed roughly i.e. fuzzy. Fuzzy quantifiers are then elements of natural language which generate fuzzy quantifications. Fuzzy quantifiers could be seen in mathematics, but mainly in natural language. Everyone surely knows the expressions like “many happy people”, “nearly all members”, “some animals”, “few single women”, “nearly none mistake”, “about half of participants”, “about fifteen students” etc. The main goal of this thesis is to propose a new approach to fuzzy quantifications. The presented approach was originally motivated by looking for the suitable method of two time series comparing. After some critical remarks of Sigfried Gottwald to my previous definition of fuzzy quantifiers, the concept of fuzzy quantifiers in now based on the concept of equipollence of fuzzy sets. An introduction of equipollence of fuzzy sets was the second goal of this thesis. Many approaches to fuzzy quantifiers are based on the concept of scalar or fuzzy cardinalities. Therefore, it is interesting to investigate the relationship between equipollences and cardinalities of fuzzy sets. In order to study such relationships, it is necessary to generalize the concept of cardinality of fuzzy sets that is the third goal of this thesis. Each of the mentioned terms could be studied separately. Nevertheless, in this thesis we pay attention to such of their aspects that just contribute to our new approach to fuzzy quantifiers. The outline of this thesis is as follows.

Chapter 1 is a preliminary chapter devoted to the basic notions that are used in this thesis.

In Chapter 2, there are introduced equipollences (θ-equipollence and $\overline{\theta}$-equipollence) of fuzzy sets and shown some of their properties and representations (for the finite cases) with regard to needs of the following chapters.

Chapter 3 is devoted to cardinalities of finite fuzzy sets. There are introduced two types of fuzzy cardinalities, namely θ-cardinality and $\overline{\theta}$-cardinality with regard to the considered fuzzy sets ($\mathbf{L}$-sets or $\mathbf{L}^{\mathrm{d}}$-sets). Some relationships between cardinalities and equipollences are presented, too. The section 3.1 and the subsections 3.2.1 and 3.2.1 could be read independently on Chapter 2.

In Chapter 4, a concept of model of fuzzy quantifiers is introduced. The models of fuzzy quantifiers actually represent the semantical meaning of fuzzy quantifiers from a language of the first-ordered fuzzy logic with fuzzy quantifiers. The syntax and semantics of the first-ordered fuzzy logic with fuzzy quantifiers is also introduced and some tautologies of this logic are shown. This chapter, except for the part "Using cardinalities of fuzzy set", could be read only with the knowledge of the basic notions and the notions from the section 2.2.

Other properties and definitions of some notions from Chapter 1, which could help to easier readability of this thesis, are summarized in Appendix.

Conclusion contains a brief summary of the achieved results with a discussion on a further progress.

References contain a list of items which mostly inspired the author. All references are cited in the text.

List of symbols contains nearly all symbols that occur in the text to simplify the reader's orientation in them.

Index contains the most relevant terms.

Chapter 1

Preliminaries

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.

Aristotle (Metaphysica, 3-1078b, ca 330 BC)

This chapter is devoted to the mathematical background which is used in our work. The first section gives a brief survey of values structures to model the membership degrees of fuzzy sets and the truth values of logical formulas. For our modeling we chose the complete residuated lattices as a basic structure, because they seem to be suitable for general interpreting logical operations. Further, we introduce a dual structure to the residuated lattice, called dually residuated lattice. Using both types of residuated lattices we may establish fuzzy sets cardinality in a more general framework. The cardinality of fuzzy sets are then introduced in the chapter 3. A survey of fuzzy sets notions is given in the second section. In the fourth section the fuzzy algebras are introduced. They are used for modeling of fuzzy quantifiers in chapter 4. The last section is devoted to the convexity of fuzzy sets. The classical definition of fuzzy sets convexity is extended and also a convexity preservation by mappings, defined using the Zadeh extension principle, is investigated here.

1.1 Residuated and dually residuated lattices

In this thesis we suppose that the structure of truth values is a *residuated lattice*, i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \bot, \top \rangle$ is a lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \otimes, \top \rangle$ is a commutative monoid (i.e. $\otimes$ is associative, commutative and the identity

$a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.

$$a \leq b \to c \quad \text{iff} \quad a \otimes b \leq c \tag{1.1}$$

holds for each $a, b, c \in L$ ($\leq$ denotes the corresponding lattice ordering). The operations $\otimes$ and $\to$ are called *multiplication* and *residuum*, respectively. Note that the residuated lattices have been introduced by M. Ward and R. P. Dilworth in [96]. Since the operations $\wedge$ and $\otimes$ have a lot of common properties, which may be used for various alternative constructions, in this work we will denote them, in general, by the symbol θ. Thus, if we deal with the operation θ, then we will consider either the operation $\wedge$ or the operation $\otimes$, whereas none of them is specified. A residuated lattice is called *complete* or *linearly ordered*, if $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete or linearly ordered lattice, respectively. A residuated lattice $\mathbf{L}$ is *divisible*, if $a \otimes (a \to b) = a \wedge b$ holds for arbitrary $a, b \in L$. Further, a residuated lattice satisfies the *prelinearity axiom* (or also *Algebraic Strong de Morgan Law*), if $(a \to b) \vee (b \to a) = \top$ holds for arbitrary $a, b \in L$, and it keeps the *law of double negation*, if $(a \to \bot) \to \bot = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the axiom of prelinearity is called the BL-*algebra*, where 'BL' denotes "basic logic". This algebra has been introduced by P. Hájek as a basic structure for many-valued logic (see e.g. [41]). A divisible residuated lattice satisfying the low of double negation is called the MV-*algebra*, where 'MV' denotes "many valued". This algebra has been introduced by C.C. Chang in [10] as an algebraic system corresponding to the $\aleph_0$-valued propositional calculus. For interested readers, we refer them to [11]. An overview of basic properties of the (complete) residuated lattices may be found in [2, 43, 72, 74] or also in Appendix A.1.

Example 1.1.1. An algebra $\mathbf{L}_{\mathrm{B}} = \langle \{\bot, \top\}, \wedge, \vee, \to, \bot, \top \rangle$, where $\to$ is the classical implication (the multiplication $\otimes = \wedge$), is the simplest residuated lattice called the *Boolean algebra for classical logic*. In general, every Boolean algebra is a residuated lattice, if we put $a \to b = a' \vee b$, where a' denotes the complement of a.

Example 1.1.2. Let $n \geq 2$ be a natural number and $L = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. Then $\mathbf{L}_{n+1} = \langle L, \min, \max, \otimes, \to, 0, 1 \rangle$, where $a \otimes b = \max(a + b - 1, 0)$ and $a \to b = \min(1, 1 - a + b)$, is the residuated lattice called the $n+1$ *elements Łukasiewicz chain*. Note that this algebra is a subalgebra of the Łukasiewicz algebra on the unit interval, that will be defined later, and this is an example of the finite MV-algebra.

A very important group of complete residuated lattices is the group of residuated lattices on the unit interval which are determined by the left continuous t-norms. These residuated lattices will be denoted by $\mathbf{L}_T$, where T

denotes the considered left continuous t-norm. Here we will mention just three complete residuated lattices determined by well known left continuous t-norms, namely by the minimum, product and Łukasiewicz conjunction. Note that these residuated lattices are special cases of the BL-algebra. For more information about the t-norms and complete residuated lattices, determined by the left continuous t-norms, we refer to Appendix A.3 or to some specialized literature as e.g. [56].

Example 1.1.3. Let T_{M} be the minimum and $\rightarrow_{\mathrm{M}}$ be defined as follows

$$a \rightarrow_{\mathrm{M}} b = \begin{cases} 1, & a \leq b, \\ b, & \text{otherwise.} \end{cases} \tag{1.2}$$

Then $\mathbf{L}_{\mathrm{M}} = \langle [0,1], \wedge, \vee, \rightarrow_{\mathrm{M}}, 0, 1 \rangle$ is the complete residuated lattice called the *Gödel algebra*. The Gödel algebra is a special case of the more general algebra, called the *Heyting algebra*, used in the intuitionistic logic.

Example 1.1.4. Let T_{P} be the product t-norm and $\rightarrow_{\mathrm{P}}$ be defined as follows

$$a \rightarrow_{\mathrm{P}} b = \begin{cases} 1, & a \leq b, \\ \frac{b}{a}, & \text{otherwise.} \end{cases} \tag{1.3}$$

Then $\mathbf{L}_{\mathrm{P}} = \langle [0,1], \wedge, \vee, T_{\mathrm{P}}, \rightarrow_{\mathrm{P}}, 0, 1 \rangle$ is the complete residuated lattice called the *Goguen algebra* (or also the *product algebra*) and used in the product logic (see e.g. [41]).

Example 1.1.5. Let $T_{\mathrm{Ł}}$ be the Łukasiewics conjunction and $\rightarrow_{\mathrm{Ł}}$ is defined as follows

$$a \rightarrow_{\mathrm{Ł}} b = \min(1 - a + b, 1). \tag{1.4}$$

Then $\mathbf{L}_{\mathrm{Ł}} \langle [0,1], \wedge, \vee, T_{\mathrm{Ł}}, \rightarrow_{\mathrm{Ł}}, 0, 1 \rangle$ is the complete residuated lattice called the *Łukasiewicz algebra*. The Łukasiewicz algebra is a special case of infinite *MV-algebra* used in the Łukasiewicz logic (see e.g. [11, 41, 74]).

We may introduce additional operations of residuated lattices using the basic ones. Here we restrict ourselves on the operation biresiduum, which interprets the logical connection equivalence, and negation. The *biresiduum* in $\mathbf{L}$ is a binary operation $\leftrightarrow$ on L defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \tag{1.5}$$

The *negation* in $\mathbf{L}$ is a unary operation $\neg$ on L defined by $\neg a = a \rightarrow \bot$.

Example 1.1.6. For the Gödel, Goguen and Łukasiewicz algebra the operation of biresiduum is given by

$$a \leftrightarrow_{\mathrm{M}} b = \min(a, b), \tag{1.6}$$

$$a \leftrightarrow_{\mathrm{P}} b = \min(\frac{a}{b}, \frac{b}{a}), \tag{1.7}$$

$$a \leftrightarrow_{\mathrm{L}} b = 1 - |a - b|, \tag{1.8}$$

respectively, where we establish $\frac{a}{0} = 1$ for any $a \in [0, 1]$.

Example 1.1.7. For the Gödel, Goguen and Łukasiewicz algebra the operation of negation is given by

$$\neg_{\mathrm{M}} a = \neg_{\mathrm{P}} a = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.9}$$

$$\neg_{\mathrm{L}} a = 1 - a, \tag{1.10}$$

respectively. We can see that the Łukasiewicz negation, contrary to the Gödel and Goguen ones, is very natural from the practical point of view and therefore the Łukasiewicz algebra is useful and popular in applications.

The following theorem gives a basic list of the biresiduum properties, which are often used in our work.

Theorem 1.1.1. (basic property of biresiduum) *Let* $\mathbf{L}$ *be a residuated lattice. Then the following items hold for arbitrary* $a, b, c, d \in L$*:*

$$a \leftrightarrow a = \top, \tag{1.11}$$

$$a \leftrightarrow b = b \leftrightarrow a, \tag{1.12}$$

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq a \leftrightarrow c, \tag{1.13}$$

$$(a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \otimes c) \leftrightarrow (b \otimes d), \tag{1.14}$$

$$(a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \rightarrow c) \leftrightarrow (b \rightarrow d), \tag{1.15}$$

$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d), \tag{1.16}$$

$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \vee c) \leftrightarrow (b \vee d). \tag{1.17}$$

Moreover, let $\mathbf{L}$ *be a complete residuated lattice. Then the following items hold for arbitrary sets* $\{a_i \mid i \in I\}$*,* $\{b_i \mid i \in I\}$ *of elements from* L *over an arbitrary set of indices* I*:*

$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq (\bigwedge_{i \in I} a_i) \leftrightarrow (\bigwedge_{i \in I} b_i), \tag{1.18}$$

$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq (\bigvee_{i \in I} a_i) \leftrightarrow (\bigvee_{i \in I} b_i). \tag{1.19}$$

Proof. It could be found in e.g. [2, 74]. □

Obviously, the operation of multiplication may be extended also for an arbitrary finite number of arguments ($n \geq 1$) by $\bigotimes_{i=1}^{n} a_i = a_1 \otimes \cdots \otimes a_n$. If we put $\top$ as the result of multiplication for the case $n = 0$, then we will also use the notation $\bigotimes_{i \in I} a_i$ for an arbitrary finite index set I[1]. In our work an extension of this finite operation to the countable (i.e. finite or denumerable) number of arguments is needed. This extension may be done by applying infimum to the finite multiplication as follows. Let $\{a_i \mid i \in I\}$ be a set of elements from L over a countable set of indices I, then the *countable* multiplication is given by

$$\bigotimes_{i \in I} a_i = \bigwedge_{I' \in \mathrm{Fin}(I)} \bigotimes_{i \in I'} a_i, \tag{1.20}$$

where $\mathrm{Fin}(I)$ denotes the set of all finite subsets of the set of indices I. Sometimes, we will also write $\bigotimes_{i=1}^{\infty} a_i$ for a sequence $a_1, a_2, \ldots$ of elements from the support L. It is easy to see that we can write $\bigotimes_{i=1}^{\infty} a_i = \bigwedge_{t=1}^{\infty} \bigotimes_{k=1}^{t} a_k$. The following theorem gives some properties of the denumerable multiplication.

Theorem 1.1.2. *Let* $\mathbf{L}$ *be a complete residuated lattice. Then the following items hold for arbitrary elements* $a_1, a_2, \ldots$ *and* $b_1, b_2, \ldots$ *from* L *and a permutation* $\pi : N \to N$*:*

$$\bigotimes_{i=1}^{\infty} a_i = \bigotimes_{i=1}^{\infty} a_{\pi(i)}, \tag{1.21}$$

$$(\bigotimes_{i=1}^{m} a_i) \otimes (\bigotimes_{i=m+1}^{\infty} a_i) \leq \bigotimes_{i=1}^{\infty} a_i, \tag{1.22}$$

$$(\bigotimes_{i=1}^{\infty} a_i) \otimes (\bigotimes_{i=1}^{\infty} b_i) \leq \bigotimes_{i=1}^{\infty} (a_i \otimes b_i), \tag{1.23}$$

$$\bigotimes_{i=1}^{\infty} (a_i \leftrightarrow b_i) \leq (\bigotimes_{i=1}^{\infty} a_i) \leftrightarrow (\bigotimes_{i=1}^{\infty} b_i), \tag{1.24}$$

Moreover, if $\mathbf{L}$ *is an* MV*-algebra, then the inequalities in (1.22) and (1.23) may be replaced by the equalities.*

Proof. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be arbitrary elements from L and $\pi : N \to N$ be a permutation. Put $m_n = \max_{k=1,\ldots,n} \pi(k)$ for every $n \in N$. Since $\{a_{\pi(k)}\}_{k=1}^{n} \subseteq \{a_k\}_{k=1}^{m_n}$ holds for every $n \in N$, then we have $\bigotimes_{k=1}^{n} a_{\pi(k)} \geq$

[1] In this work, the empty set is considered as a finite set.

$\bigotimes_{k=1}^{m_n} a_k$ for every $n \in N$ (due to monotony of $\otimes$) and hence we obtain $\bigotimes_{i=1}^{\infty} a_{\pi(i)} = \bigwedge_{n=1}^{\infty} \bigotimes_{k=1}^{n} a_{\pi(k)} \geq \bigwedge_{n=1}^{\infty} \bigotimes_{k=1}^{m_n} a_k = \bigotimes_{i=1}^{\infty} a_i$. Analogously, we can put $n_m = \max_{k=1,\dots,m} \pi^{-1}(k)$ for every $m \in N$. Since $\{a_k\}_{k=1}^{m} \subseteq \{a_{\pi(k)}\}_{k=1}^{n_m}$ holds for every $m \in N$, then we have $\bigotimes_{k=1}^{m} a_k \geq \bigotimes_{k=1}^{n_m} a_{\pi(k)}$ for every $m \in N$ and hence we obtain $\bigotimes_{i=1}^{\infty} a_i \geq \bigotimes_{i=1}^{\infty} a_{\pi(i)}$. Thus, the equality (1.21) is proved. Due to (A.16), we have $(\bigotimes_{i=1}^{m} a_i) \otimes (\bigotimes_{i=m+1}^{\infty} a_i) = (a_1 \otimes \cdots \otimes a_m) \otimes \bigwedge_{t=m+1}^{\infty} \bigotimes_{i=m+1}^{t} a_i \leq \bigwedge_{t=m+1}^{\infty} (a_1 \otimes \cdots \otimes a_m) \otimes \bigotimes_{i=m+1}^{t} a_i = \bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} a_i = \bigotimes_{i=1}^{\infty} a_i$. Hence, the inequality (1.22) is proved. Analogously, we have $(\bigotimes_{i=1}^{\infty} a_i) \otimes (\bigotimes_{i=1}^{\infty} b_i) = (\bigwedge_{s=1}^{\infty} \bigotimes_{i=1}^{s} a_i) \otimes (\bigwedge_{r=1}^{\infty} \bigotimes_{i=1}^{r} b_i) \leq \bigwedge_{s=1}^{\infty} \bigwedge_{r=1}^{\infty} (\bigotimes_{i=1}^{s} a_i \otimes \bigotimes_{i=1}^{r} b_i) = \bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} (a_i \otimes b_i) = \bigotimes_{i=1}^{\infty} (a_i \otimes b_i)$, where the second equality follows from the fact that to each couple (r, s) there exists t such that $\bigotimes_{i=1}^{s} a_i \otimes \bigotimes_{i=1}^{r} b_i \geq \bigotimes_{i=1}^{t} (a_i \otimes b_i)$ and hence we obtain the inequality $\bigwedge_{s=1}^{\infty} \bigwedge_{r=1}^{\infty} (\bigotimes_{i=1}^{s} a_i \otimes \bigotimes_{i=1}^{r} b_i) \geq \bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} (a_i \otimes b_i)$. Similarly, to each t there exists a couple (r, s) such that $\bigotimes_{i=1}^{s} a_i \otimes \bigotimes_{i=1}^{r} b_i \leq \bigotimes_{i=1}^{t} (a_i \otimes b_i)$ and hence the opposite inequality is obtained. Thus, the inequality (1.23) is also true. Finally, due to (1.15) and (1.18), we have $\bigotimes_{i=1}^{\infty} (a_i \leftrightarrow b_i) = \bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} (a_i \leftrightarrow b_i) \leq \bigwedge_{t=1}^{\infty} \big((\bigotimes_{i=1}^{t} a_i) \leftrightarrow (\bigotimes_{i=1}^{t} b_i)\big) \leq (\bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} a_i) \leftrightarrow (\bigwedge_{t=1}^{\infty} \bigotimes_{i=1}^{t} b_i) = (\bigotimes_{i=1}^{\infty} a_i) \leftrightarrow (\bigotimes_{i=1}^{\infty} b_i)$ and thus the inequality (1.24) is also proved. The rest of the proof follows from the distributivity of $\otimes$ over $\wedge$, which holds in each MV-algebra. □

Remark 1.1.8. The equality (1.21) states that the countable multiplication is commutative. Unfortunately, the inequalities (1.22) and (1.23) state that the countable multiplication is not associative, in general. Particularly, the associativity of countable multiplication is satisfied, if $\mathbf{L}$ is a complete MV-algebra.

Remark 1.1.9. The inequalities (1.18) and (1.24) will be occasionally written in more general form (i.e. $\theta \in \{\wedge, \otimes\}$) to simplify expressions as follows

$$\bigominus_{i \in I} (a_i \leftrightarrow b_i) \leq (\bigominus_{i \in I} a_i) \leftrightarrow (\bigominus_{i \in I} b_i), \tag{1.25}$$

where countable sets $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ of elements from L are supposed.

In order to introduce the cardinalities of fuzzy sets, a dual structure to the residuated lattice is needed here. An algebra $\mathbf{L}^{\mathrm{d}} = (L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ is called the *dually residuated lattice*, if $\langle L, \wedge, \vee, \bot, \top \rangle$ is a lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \oplus, \bot \rangle$ is a commutative monoid and the (dual) adjointness property is satisfied, i.e.

$$a \leq b \oplus c \quad \text{iff} \quad a \ominus b \leq c \tag{1.26}$$

holds for each $a, b, c \in L$ ($\leq$ denotes the corresponding lattice ordering). A dually residuated lattice is called *complete*, if $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete lattice. The operations $\oplus$ and $\ominus$ will be called *addition* and *difference*, respectively. Note that we have not found references to the dually residuated lattices in this form. The dually residuated lattices generalize the bounded commutative dually residuated lattice ordered monoids (DRl-monoids for short) that are a special case of the dually residuated lattice ordered semigroup. For interested reader we refer to [18, 77–79, 88–90]. Analogously to the denotation θ, the operations $\vee$ and $\oplus$ will be denoted, in general, by the symbol $\overline{\theta}$ and if we deal with $\overline{\theta}$, then we will consider either the operation $\vee$ or the operation $\oplus$, whereas none of them is specified further. Moreover, if $\mathbf{L}_1$ and $\mathbf{L}_2^{\mathrm{d}}$ are a residuated and dually residuated lattices, respectively, then the couple $\langle \wedge_1, \vee_2 \rangle$ or the couple $\langle \otimes_1, \oplus_2 \rangle$ can be understood as a couple of the "dual" operations[2]. Therefore, the couple of operations θ and $\overline{\theta}$ will be also considered as the couple of dual operations. For instance, if $\theta = \wedge_1$ is supposed, then we have $\overline{\theta} = \overline{\wedge_1} = \vee_2$.

Example 1.1.10. The algebra $\mathbf{L}_{\mathrm{B}}^{\mathrm{d}} = \langle \{0, 1\}, \wedge, \vee, \ominus, 0, 1 \rangle$, where $a \ominus b = 0 \vee (a - b)$ (the addition $\oplus = \vee$), is the simplest dually residuated lattice called *the dual Boolean algebra*. It is the dual algebra to the Boolean algebra $\mathbf{L}_{\mathrm{B}}$ from Ex. 1.1.1.

Example 1.1.11. The algebra $\mathrm{R}_0^+ = \langle [0, \infty], \wedge, \vee, \oplus, \ominus, 0, \infty \rangle$, where ∞ is the symbol for infinity and for every $a, b \in [0, \infty]$ we have

$$a \oplus b = \begin{cases} a + b, & a, b \in [0, \infty), \\ \infty, & \text{otherwise} \end{cases}$$

and

$$a \ominus b = \begin{cases} 0 \vee (a - b), & a, b \in [0, \infty), \\ 0, & b = \infty, \\ \infty, & \text{otherwise,} \end{cases}$$

is the complete dually residuated lattice of *non-negative real numbers*.

The following three complete dually residuated lattices are dual to the residuated lattices from Ex. 1.1.3, 1.1.4 and 1.1.5. It is known that each t-norm has the dual operation called t-conorm. Hence, an analogical construction to the construction of complete residuated lattices, determined by the left continuous t-norms, leads to the dually residuated lattices determined by the right continuous t-conorms. Again, more information about

[2]These dual relations between the mentioned operations could be well seen, if a homomorphism between residuated and dually residuated lattices is introduced (see p. 11).

the mentioned problems could be found in Appendix C or in some specialized literature as e.g. [56].

Example 1.1.12. Let S_{M} be the maximum and $\ominus_{\mathrm{M}}$ be defined as follows

$$a \ominus_{\mathrm{M}} b = \begin{cases} 0, & a \leq b, \\ a, & \text{otherwise.} \end{cases}$$

Then $\mathbf{L}^{\mathrm{d}}_{\mathrm{M}} = \langle [0,1], \wedge, \vee, \ominus_{\mathrm{M}}, 0, 1 \rangle$ is the complete dually residuated lattice, which is dual to the Gödel algebra $\mathbf{L}_{\mathrm{M}}$, and it will be called the *dual Gödel algebra*.

Example 1.1.13. Let S_{P} be the probabilistic sum and $\ominus_{\mathrm{P}}$ be defined as follows

$$a \ominus_{\mathrm{P}} b = \begin{cases} 0, & a \leq b, \\ \frac{a-b}{1-b}, & \text{otherwise.} \end{cases}$$

Then $\mathbf{L}^{\mathrm{d}}_{\mathrm{P}} = \langle [0,1], \wedge, \vee, S_{\mathrm{P}}, \ominus_{\mathrm{P}}, 0, 1 \rangle$ is the complete dually residuated lattice, which is dual to the Goguen algebra $\mathbf{L}_{\mathrm{P}}$, and it will be called the *dual Goguen algebra*.

Example 1.1.14. Let $S_{\mathrm{Ł}}$ be the Łukasiewicz conjunction and $\ominus_{\mathrm{Ł}}$ be defined as follows

$$a \ominus_{\mathrm{Ł}} b = \max(a-b, 0).$$

Then $\mathbf{L}^{\mathrm{d}}_{\mathrm{Ł}} = \langle [0,1], \wedge, \vee, S_{\mathrm{Ł}}, \ominus_{\mathrm{Ł}}, 0, 1 \rangle$ is the complete dually residuated lattice, which is dual to the Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}$, and it will be called the *dual Łukasiewicz algebra*.

Analogously to the residuated lattices, an additional operations may be introduced in the dually residuated lattices using the basic ones. Here, we restrict ourselves to the operations of bidifference and negation. A *bidifference* (or better *absolute difference*) in $\mathbf{L}^{\mathrm{d}}$ is a binary operation $| \ominus |$ on L defined by

$$|a \ominus b| = (a \ominus b) \vee (b \ominus a). \tag{1.27}$$

The (dual) *negation* in $\mathbf{L}^{\mathrm{d}}$ is a unary operation $\neg$ on L defined by $\neg a = \top \ominus a$.

Example 1.1.15. For the dual Gödel, Goguen and Łukasiewicz algebra the operation of bidifference is given by, respectively:

$$|a \ominus_{\mathrm{M}} b| = \max(a, b), \tag{1.28}$$

$$|a \ominus_{\mathrm{P}} b| = \frac{|a-b|}{1-\min(a,b)}, \tag{1.29}$$

$$|a \ominus_{\mathrm{Ł}} b| = |a-b|, \tag{1.30}$$

where we establish $\frac{0}{0} = 0$.

The following theorem gives a list of the bidifference properties. It is interesting to compare the properties of biresiduum and bidifference in order to see some natural consequences of the dualism.

Theorem 1.1.3. (basic properties of bidifference) *Let $\mathbf{L}^{\mathrm{d}}$ be a dually residuated lattice. Then the following items hold for arbitrary $a, b, c, d \in L$:*

$$|a \ominus a| = \bot, \tag{1.31}$$
$$|a \ominus b| = |b \ominus a|, \tag{1.32}$$
$$|a \ominus c| \leq |a \ominus b| \oplus |b \ominus c|, \tag{1.33}$$
$$|(a \oplus b) \ominus (c \oplus d)| \leq |a \ominus c| \oplus |b \ominus d|, \tag{1.34}$$
$$|(a \ominus b) \ominus (c \ominus d)| \leq |a \ominus c| \oplus |b \ominus d|, \tag{1.35}$$
$$|(a \wedge b) \ominus (c \wedge d)| \leq |a \ominus c| \vee |b \ominus d|, \tag{1.36}$$
$$|(a \vee b) \ominus (c \vee d)| \leq |a \ominus c| \vee |b \ominus d|. \tag{1.37}$$

Let $\mathbf{L}^{\mathrm{d}}$ be a complete residuated lattice. Then the following items hold for arbitrary sets $\{a_i \mid i \in I\}$, $\{b_i \mid i \in I\}$ of elements from L over an arbitrary set of indices I:

$$|\bigwedge_{i\in I} a_i \ominus \bigwedge_{i \in I} b_i| \leq \bigvee_{i \in I} |a_i \ominus b_i|, \tag{1.38}$$
$$|\bigvee_{i\in I} a_i \ominus \bigvee_{i \in I} b_i| \leq \bigvee_{i \in I} |a_i \ominus b_i|. \tag{1.39}$$

Proof. We will prove only the equalities (1.36) and (1.37). The rest could be done by analogy. In the first case, it is sufficient to prove the inequality $(a \wedge b) \leq (|a \ominus c| \vee |b \ominus d|) \oplus (c \wedge d)$. The rest follows from adjointness and symmetry of the formula. Obviously, $(|a \ominus c| \vee |b \ominus d|) \oplus (c \wedge d) \geq ((a \ominus c) \vee (b \ominus d)) \oplus (c \wedge d) = (((a \ominus c) \vee (b \ominus d)) \oplus c) \wedge (((a \ominus c) \vee (b \ominus d)) \oplus d) \geq ((a \ominus c) \oplus c) \wedge ((b \ominus d) \oplus d) \geq a \wedge b$, where distributivity of $\oplus$ over $\wedge$ and (A.19) are used. In the second case, it is sufficient to prove the inequality $|a \ominus c| \oplus |b \ominus d| \oplus (c \ominus d) \geq a \ominus b$. Obviously, $|a \ominus c| \oplus |b \ominus d| \oplus (c \ominus d) \geq (a \ominus c) \oplus (d \ominus b) \oplus (c \ominus d) \geq (a \ominus d) \oplus (d \ominus b) \geq (a \ominus b)$, where (A.26) is applied twice. □

Similarly to the previous part, we extend the operation of addition to a countable number of arguments. This extension may be easily done by using of supremum applied to the finite addition as follows. Let $\{a_i \mid i \in I\}$ be a set of elements from L over a countable set of indices I, then the *countable* addition is given by

$$\bigoplus_{i \in I} a_i = \bigvee_{I' \in \mathrm{Fin}(I)} \bigoplus_{i \in I'} a_i, \tag{1.40}$$

where we establish $\bigoplus_{i\in\emptyset} a_i = \bot$ and $\bigoplus_{i\in I} a_i = \bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n$, whenever $I = \{1,\ldots,n\}$. Sometimes, we will also write $\bigoplus_{i=1}^{\infty} a_i$ for a sequence $a_1, a_2, \ldots$ of elements from L. Again, we can write $\bigoplus_{i=1}^{\infty} a_i = \bigvee_{t=1}^{\infty} \bigoplus_{k=1}^{t} a_k$. Note that MV-algebras are examples of residuated lattices, where we may define dually residuated lattices (on the same supports) using the operations $\otimes$, $\rightarrow$ and the least element $\bot$ (see e.g. [2,11,74]). Both structures are then isomorphic.

Theorem 1.1.4. *Let* $\mathbf{L}$ *be a complete dually residuated lattice. Then the following items hold for arbitrary elements* $a_1, a_2, \ldots$ *and* $b_1, b_2, \ldots$ *from* L *and a permutation* $\pi : N \rightarrow N$*:*

$$\bigoplus_{i=1}^{\infty} a_i = \bigoplus_{i=1}^{\infty} a_{\pi(i)}, \tag{1.41}$$

$$(\bigoplus_{i=1}^{m} a_i) \oplus (\bigoplus_{i=m+1}^{\infty} a_i) \geq \bigoplus_{i=1}^{\infty} a_i, \tag{1.42}$$

$$(\bigoplus_{i=1}^{\infty} a_i) \oplus (\bigoplus_{i=1}^{\infty} b_i) \geq \bigoplus_{i=1}^{\infty} (a_i \oplus b_i), \tag{1.43}$$

$$\bigoplus_{i\in I} |a_i \ominus b_i| \geq |\bigoplus_{i\in I} a_i \ominus \bigoplus_{i\in I} b_i|. \tag{1.44}$$

Moreover, if $\mathbf{L}$ *is an* MV*-algebra, then the inequalities (1.42) and (1.43) may be replaced by the equalities.*

Proof. It could be done by analogy to the proof of Theorem 1.1.2. □

Remark 1.1.16. The equality (1.41) states that the countable addition is commutative. Again, due to the inequalities (1.42) and (1.43), the countable addition is not associative, in general. Particularly, the associativity of the countable addition is satisfied, if $\mathbf{L}$ is a complete MV-algebra.

Remark 1.1.17. The inequalities (1.34)and (1.37) will be occasionally written in a more general form to simplify expressions as follows

$$\left| \overline{\bigominus_{i\in I}} a_i \ominus \overline{\bigominus_{i\in I}} b_i \right| \leq \overline{\bigominus_{i\in I}} |a_i \ominus b_i|, \tag{1.45}$$

where countable sets $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ of elements from L are supposed.

Let $\mathbf{L}_1$, $\mathbf{L}_2$ be (complete) residuated lattices and $\mathbf{L}_1^{\mathrm{d}}$, $\mathbf{L}_2^{\mathrm{d}}$ be (complete) dually residuated lattices. A mapping $h : L_1 \rightarrow L_2$ is a (*complete*) *homomorphism* $h : \mathbf{L}_1 \rightarrow \mathbf{L}_2$ of the (complete) residuated lattices $\mathbf{L}_1$ and $\mathbf{L}_2$, if h

preserves the structure, i.e. $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$ $(h(\bigwedge_1 b_i) = \bigwedge_2 h(b_i))$, $h(a \vee_1 b) = h(a) \vee_2 h(b)$ $(h(\bigvee_1 b_i) = \bigvee_2 h(b_i))$, $h(a \otimes_1 b) = h(a) \otimes_2 h(b)$ and $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$. Analogously, a (complete) homomorphism $h : \mathbf{L}_1^{\mathrm{d}} \to \mathbf{L}_2^{\mathrm{d}}$ of the dually (complete) residuated lattices can be introduced. Further, a mapping $h : L_1 \to L_2$ is a homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2^{\mathrm{d}}$ of the (complete) residuated lattice $\mathbf{L}_1$ to the (complete) dual residuated lattice $\mathbf{L}_2^{\mathrm{d}}$, if f preserves the operations of $\mathbf{L}_1$ to the corresponding dual operations of $\mathbf{L}_2^{\mathrm{d}}$, i.e. $h(a \wedge_1 b) = h(a) \vee_2 h(b)$ $(h(\bigwedge_1 b_i) = \bigvee_2 h(b_i))$, $h(a \vee_1 b) = h(a) \wedge_2 h(b)$ $(h(\bigvee_1 b_i) = \bigwedge_2 h(b_i))$, $h(a \otimes_1 b) = h(a) \oplus_2 h(b)$ and $h(a \rightarrow_1 b) = h(b) \ominus_2 h(a)$. Finally, a mapping $h : L_1 \to L_2$ is a homomorphism $h : \mathbf{L}_1^{\mathrm{d}} \to \mathbf{L}_2$ of the (complete) residuated lattice $\mathbf{L}_1^{\mathrm{d}}$ to the (complete) dual residuated lattice $\mathbf{L}_2$, if h preserves the operations of $\mathbf{L}_1^{\mathrm{d}}$ to the corresponding dual operations of $\mathbf{L}_2$, i.e. $h(a \wedge_1 b) = h(a) \vee_2 h(b)$ $(h(\bigwedge_1 b_i) = \bigvee_2 h(b_i))$, $h(a \vee_1 b) = h(a) \wedge_2 h(b)$ $(h(\bigvee_1 b_i) = \bigwedge_2 h(b_i))$, $h(a \oplus_1 b) = h(a) \otimes_2 h(b)$ and $h(a \ominus_1 b) = h(b) \rightarrow_2 h(a)$. Note that if some operation can be defined using other operations (e.g. $a \to b = \bigvee\{c \in L \mid a \otimes c \leq b\}$ in a complete residuated lattice), then to verify the homomorphism, it is sufficient to show that the mapping preserves the other operations (i.e. the preserving $\otimes$ and $\bigvee$ in the previous case). A homomorphism h is called the *monomorphism*, *epimorphism* and *isomorphism*, if the mapping h is the injection, surjection and bijection, respectively.

An example of the isomorphism from $\mathbf{L}_1$ onto $\mathbf{L}_2^{\mathrm{d}}$ could be easily construct on the unit interval $[0, 1]$ as the following theorem shows. Recall that between the t-norms and t-conorms there is very important relation showing that the t-norms and t-conorms are the dual operations on $[0, 1]$. Precisely, if T is a t-norm, then the corresponding t-conorm is given by

$$S(a, b) = 1 - T(1 - a, 1 - b). \tag{1.46}$$

Obviously, the dual operation to a dual operation is the original one.

Theorem 1.1.5. *Let $\mathbf{L}_T$ be a complete residuated lattice determined by a left continuous t-norm, $\mathbf{L}_S$ be a complete dually residuated lattice determined by a right continuous t-conorm. If S is the t-conorm such that T and S satisfy (1.46), then the mapping $h : [0, 1] \to [0, 1]$, given by $h(a) = 1 - a$, is a complete isomorphism from $\mathbf{L}_T$ onto $\mathbf{L}_S$.*

Proof. Let $\{a_i \in [0, 1] \mid i \in I\}$ be a nonempty family. Then $h(\bigwedge_{i \in I} a_i) = 1 - \bigwedge_{i \in I} a_i = \bigvee_{i \in I}(1 - a_i) = \bigvee_{i \in I} h(a)$. Similarly, we obtain $h(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} h(a_i)$. Further, we have $h(T(a, b)) = 1 - T(a, b) = S(1 - a, 1 - b) = S(h(a), h(b))$. The rest of the properties of h follows from the fact that the biresiduum $\rightarrow_T$ is uniquely determined by T and $\vee$. $\square$

1.2 Fuzzy sets

Let $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ be a complete residuated lattice and dually residuated lattice, respectively, and X be a nonempty set. An $\mathbf{L}$-*set* or dually an $\mathbf{L}^{\mathrm{d}}$-*set* in X is a mapping $A : X \to L$, where L is a support of $\mathbf{L}$ or $\mathbf{L}^{\mathrm{d}}$, respectively[3]. The set X is called the *universe of discovering* and $A(x)$ is called the *membership degree of x* in the $\mathbf{L}$-set or $\mathbf{L}^{\mathrm{d}}$-set A. In general, the $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets will be called the *fuzzy sets*. The set of all $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets in X will be denoted by $\mathcal{F}_{\mathbf{L}}(X)$ and $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$, respectively, or shortly $\mathcal{F}(X)$ for both cases, if the considered lattices are known or their specifications are not needed. Let A be a fuzzy set (i.e. an $\mathbf{L}$-set or $\mathbf{L}^{\mathrm{d}}$-set) in X, then the set $A_a = \{x \in X \mid A(x) \geq a\}$ is called the *a-cut of fuzzy set A*. Moreover, the set $A^{\mathrm{d}}_a = \{x \in X \mid A(x) \leq a\}$ is called the *dual a-cut of fuzzy set A*. The set $\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\}$ is called the *support* of the fuzzy set A. Obviously, it can be written $\mathrm{Supp}(A) = X \setminus A^{\mathrm{d}}_{\perp}$. The fuzzy set $\emptyset$ in X, defined by $\emptyset(x) = \perp$, is called the *empty* fuzzy set. A fuzzy set A in X, given by its characteristic mapping $\chi_A : X \to \{\perp, \top\}$, is called the *crisp set* and a fuzzy set A in X such that $A(x) > \perp$ for some $x \in X$ and $A(y) = \perp$ for all $y \in X$ with $y \neq x$ is called *singleton*. Recall that a set whose members can be labeled by the natural numbers is called a countable set. Countable sets are classified into finite or countable infinite (or also denumerable). A set which is not countable is called uncountable. We use also this terminology for classifying fuzzy sets (i.e. $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets). We say that a fuzzy set A is *countable* (i.e. *finite* or *denumerable*), if the set $\mathrm{Supp}(A)$ is countable (finite or denumerable). In the opposite case, we say that the fuzzy set is *uncountable*. Note that countable fuzzy sets can be also defined over uncountable universes. The set of all countable (finite) $\mathbf{L}$-sets or $\mathbf{L}^{\mathrm{d}}$-sets in a universe X will be denoted by $\mathcal{FC}_{\mathbf{L}}(X)$ or $\mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ ($\mathcal{FIN}_{\mathbf{L}}(X)$ or $\mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$), respectively, and sometimes we will use shortly $\mathcal{FC}(X)$ analogously as in the case $\mathcal{F}(X)$ ($\mathcal{FIN}(X)$). Let A, B be arbitrary fuzzy sets on X. We say, that A is *less than or equal* to B or B is *greater than or equal* to A, if $A(x) \leq B(x)$ holds for every $x \in X$. It is easy to see that this relation is a relation of partial ordering on the set $\mathcal{F}(X)$ and we say that $(\mathcal{F}(X), \leq)$ is a partial ordered set or shortly *po*-set of all fuzzy sets over X.

Example 1.2.1. Let $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ be residuated and dually residuated lattice over the unit interval, $X = \{x_1, \dots, x_5\}$ be a universe and

$$A = \{0.8/x_1, 0.3/x_2, 0.5/x_3, 0/x_4, 1/x_5\}$$

[3]The notion of $\mathbf{L}$-sets (precisely $\mathbf{L}$-fuzzy sets) was introduced by J.A. Goguen in [35].

be a finite fuzzy set in X, i.e. an $\mathbf{L}$-set or also $\mathbf{L}^{\mathrm{d}}$-set in X. Then, for example, $A_{0.5} = \{x_1, x_3, x_5\}$ and $A^{\mathrm{d}}_{0.5} = \{x_2, x_3, x_4\}$ are the a-cut and dual a-cut of the fuzzy set A, respectively. Moreover, obviously $A_0 = A^{\mathrm{d}}_1 = X$.

1.3 Fuzzy algebra

Let $\mathbf{L}$ be a complete residuated lattice and X be a nonempty universe. The operations with common sets as union, intersection, complement of set etc., can be naturally extended to the appropriate operations with $\mathbf{L}$-sets as follows

$$(A \cup B)(x) = A(x) \vee B(x), \tag{1.47}$$

$$(A \cap B)(x) = A(x) \wedge B(x), \tag{1.48}$$

$$\overline{A}(x) = A(x) \to \bot. \tag{1.49}$$

Moreover, using the operations $\otimes$ and $\to$, we can establish additional operations as follows

$$(A \otimes B)(x) = A(x) \otimes B(x), \tag{1.50}$$

$$(A \to B)(x) = A(x) \to B(x). \tag{1.51}$$

Obviously, we have $A \to \emptyset = \overline{A}$. A subset $\mathcal{A}$ of $\mathcal{F}_{\mathbf{L}}(X)$ will be called the *fuzzy algebra over* X, if the following conditions are satisfied

(*i*) $\emptyset$, $X \in \mathcal{A}$,

(*ii*) if $A \in \mathcal{A}$, then $\overline{A} \in \mathcal{A}$,

(*iii*) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.

Typical examples of fuzzy algebras are the set $\mathcal{F}_{\mathbf{L}}(X)$ and the power set of X, i.e. $\mathcal{P}(X)$. For interested reader we refer to [82, 95], where many examples and properties of fuzzy algebras could be found. Analogously, it could be defined operations with $\mathbf{L}^{\mathrm{d}}$-sets and a fuzzy algebra for $\mathbf{L}^{\mathrm{d}}$-sets, but we will not do it here.

1.4 Convex fuzzy sets

An important notion in fuzzy set theory is the notion of convex fuzzy set which has been studied in greater detail by R. Lowen [65] and Y.M. Liu [62]. The convex fuzzy sets are often constructed on a Cartesian product of the set of real numbers R^n and the definition is naturally derived from the convexity of a-cuts of fuzzy sets. In particular, a fuzzy set (precisely **L**-set) A over R^n is convex, if each a-cut of A is a convex subset of R^n. It is easy to see that an equivalent definition of convexity could be stated as follows. Let $\leq$ be the partial ordering on R^n defined by $(x_1,\dots,x_n) \leq (y_1,\dots,y_n)$, if $x_i \leq y_i$ for any $i = 1,\dots,n$. Then a fuzzy set A is convex, if $A(x) \wedge A(z) \leq A(y)$ holds for arbitrary $x, y, z \in R^n$ such that $x \leq y \leq z$. We can see that the convexity of fuzzy sets is closely related to a partial ordered set and to a suitable operation of the residuated lattice. A generalization of the notion of convex **L**-set may be given as follows.

Let **L** be a complete residuated lattice, $(X, \leq)$ be a partial ordered set and $\theta \in \{\wedge, \otimes\}$. We say that the **L**-set A over X is *θ-convex*, if $A(x)\theta A(z) \leq A(y)$ holds for arbitrary $x, y, z \in X$ such that $x \leq y \leq z$. The set of all θ-convex **L**-sets over X is denoted by $\mathcal{CV}^{\theta}_{\mathbf{L}}(X)$.

Example 1.4.1. Let $\mathbf{L}_{\text{Ł}}$ be the Łukasiewicz algebra (see Example 1.1.5) and $(N^{-0}_{\omega}, \leq)$ be the linearly ordered set of all natural numbers without zero extended by the first infinite cardinal, where $\leq$ is the common linear ordering of natural numbers ($n < \omega$ for any $n \in N$). An example of the θ-convex $\mathbf{L}_{\text{Ł}}$-set (called a *generalized extended natural number*) could be given by $A(n) = \frac{1}{n}$ for any $n \notin \{0, \omega\}$ and $A(\omega) = 0$. Let $a, b \in [0, 1]$ be two numbers such that $a > b$ and $2a - 1 < b$ (consider e.g. $a = 0.7$ and $b = 0.5$). Then an example of T_L-convex $\mathbf{L}_{\text{Ł}}$-set which is not $\wedge$-convex could be easily given by the following formula

$$A(n) = \begin{cases} a, & n \text{ is even number,} \\ b, & n \text{ is odd number.} \end{cases}$$

Obviously, the $\otimes$-convexity of **L**-sets does not correspond to the original idea of convex a-cuts, in general. On the other hand, the $\otimes$-convexity has “nicer” properties (due to distributivity $\otimes$ over $\vee$) than $\wedge$-convexity, if we deal with general residuated lattices.

Let $f : X_1 \times \cdots \times X_n \to Y$ be an arbitrary mapping. If we want to extend this mapping to a mapping $\widehat{f}^{\theta} : \mathcal{F}_{\mathbf{L}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{L}}(X_n) \to \mathcal{F}_{\mathbf{L}}(Y)$ assigning an **L**-set over Y to each n-tuple **L**-sets over $X_1,\dots,X_n$, we can use the Zadeh extension principle[4] defined (in the more general form) as follows

$$\widehat{f}^{\theta}(A_1,\dots,A_n)(y) = \bigvee_{\substack{(x_1,\dots,x_n)\in\prod_{i=1}^n X_i \\ f(x_1,\dots,x_n)=y}} A_1(x_1)\theta\cdots\theta A_n(x_n), \tag{1.52}$$

[4]Note that Zadeh extension principle was introduced in [121]. It was and still is a very powerful tool of the fuzzy set theory with many practical applications of fuzzy sets to various areas of research.

where $(A_1, \dots, A_n) \in \mathcal{F}_{\mathbf{L}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{L}}(X_n)$. Note that if $f(x_1, \dots, x_n) \neq y$ holds for every $x_i \in X_i$, $i = 1, \dots, n$, then $\widehat{f}^{\theta}(A_1, \dots, A_n)(y) = \bigvee \emptyset = \bot$. Now we may ask the question: When is the convexity of **L**-sets preserved by the Zadeh extension principle? Formally speaking, when $\widehat{f}^{\theta}(A_1, \dots, A_n) \in \mathcal{CV}_{\mathbf{L}}(Y)$ holds for any $(A_1, \dots, A_n) \in \mathcal{CV}_{\mathbf{L}}(X_1) \times \cdots \times \mathcal{CV}_{\mathbf{L}}(X_n)$. The following theorem states some necessary conditions for the preservation of the convexity of **L**-sets by $\widehat{f}^{\theta}$. Recall that a residuated lattice is divisible, if $a \wedge b = a \otimes (a \to b)$ is satisfied for any $a, b \in L$. In this case the distributivity of $\wedge$ over $\vee$ is fulfilled.

Theorem 1.4.1. *Let $(X_1, \leq_1), \dots, (X_n, \leq_n)$ be arbitrary linearly ordered sets and $(Y, \leq)$ be an arbitrary partial ordered set. Let $f : X_1 \times \cdots \times X_n \to Y$ be a surjective mapping such that for every $(x_1, \dots, x_n), (z_1, \dots, z_n) \in \prod_{i=1}^n X_i$ and for every $y \in Y$, where $f(x_1, \dots, x_n) \leq y \leq f(z_1, \dots, z_n)$, there exists $(y_1, \dots, y_n) \in \prod_{i=1}^n X_i$ satisfying the following conditions*

(*i*) $f(y_1, \dots, y_n) = y$,

(*ii*) $x_i \leq_i y_i \leq_i z_i$ *or* $z_i \leq_i y_i \leq_i x_i$ *hold for every* $i = 1, \dots, n$.

*Then $\widehat{f}^{\otimes}$ preserves $\otimes$-convexity of **L**-sets. Moreover, if **L** is divisible, then $\widehat{f}^{\wedge}$ preserves $\wedge$-convexity of **L**-sets.*

Proof. Let $f : X_1 \times \cdots \times X_n \to Y$ be a mapping satisfying the presumption of the theorem. Further, let $A_1, \dots, A_n$ be arbitrary θ-convex **L**-sets over $X_1, \dots, X_n$ and $x \leq y \leq z$ be arbitrary elements from Y. Since f is the surjective mapping, then there exist n-tuples $(x_1, \dots, x_n)$ and $(z_1, \dots, z_n)$ from $\prod_{i=1}^n X_i$ such that $f(x_1, \dots, x_n) = x$ and $f(z_1, \dots, z_n) = z$. Moreover, from the linearity of orderings $\leq_i$ we have $x_i \leq z_i$ or $z_i \leq x_i$ for every $i = 1, \dots, n$. With regard to the presumption there exists $(y_1, \dots, y_n) \in \prod_{i=1}^n (X_i)$ satisfying the conditions (*i*) and (*ii*). Since $A_1, \dots, A_n$ are the θ-convex **L**-sets, then we have $A_i(x_i) \theta A_i(z_i) \leq_i A_i(y_i)$ for every $i = 1, \dots, n$. Hence, for $\theta = \otimes$ we can write

$$\widehat{f}^{\otimes}(A_1, \dots, A_n)(y) = \bigvee_{\substack{(y'_1, \dots, y'_n) \in \prod_{i=1}^n X_i \\ f(y'_1, \dots, y'_n) = y}} \bigotimes_{i=1}^n A_i(y'_i) \geq$$

$$\bigvee_{\substack{(x_1, \dots, x_n) \in \prod_{i=1}^n X_i \\ f(x_1, \dots, x_n) = x}} \bigotimes_{i=1}^n A_i(x_i) \otimes \bigvee_{\substack{(z_1, \dots, z_n) \in \prod_{i=1}^n X_i \\ f(z_1, \dots, z_n) = z}} \bigotimes_{i=1}^n A_i(z_i) =$$

$$\widehat{f}^{\otimes}(A_1, \dots, A_n)(x) \otimes \widehat{f}^{\otimes}(A_1, \dots, A_n)(z).$$

If **L** is divisible and thus the distributivity of $\wedge$ over $\vee$ is held, then the proof of the $\wedge$-convexity preservation is analogical to the previous one, where it is sufficient to consider $\wedge$ instead of $\otimes$. □

Example 1.4.2. Let $+ : N \times N \to N$ be the common addition $+$ of natural numbers. Then by Zadeh extension principle we can extend the addition of natural numbers to the addition of θ-convex **L**-sets over N as follows

$$(A +^{\theta} B)(i) = \bigvee_{i_1+i_2=i} A(i_1)\theta A(i_2). \tag{1.53}$$

Later we will prove that this definition (in a more general form) is correct. It means that the sum of two θ-convex **L**-sets is again a θ-convex **L**-set.

Now we introduce a dual notion to the θ-convex **L**-sets for the $\mathbf{L}^{\mathrm{d}}$-sets. In order to establish this dual notion, we have to use the dual operations to $\wedge$ and $\otimes$. Let $\mathbf{L}^{\mathrm{d}}$ be a complete dually residuated lattice, $(X, \leq)$ be an ordered set and $\overline{\theta} \in \{\vee, \oplus\}$. We say that the $\mathbf{L}^{\mathrm{d}}$-set A over X is *$\overline{\theta}$-convex*, if $A(j) \leq A(i)\overline{\theta}A(k)$ holds for any $i \leq j \leq k$. The set of all $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-sets over X is denoted by $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(X)$.

Let $f : X_1 \times \cdots \times X_n \to Y$ be an arbitrary mapping and $A_i \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X_i)$, $i = 1, \ldots, n$, be arbitrary $\mathbf{L}^{\mathrm{d}}$-sets. The dual Zadeh extension principle is defined as follows

$$\widehat{f}^{\overline{\theta}}(A_1, \ldots, A_n)(y) = \bigwedge_{\substack{(x_1,\ldots,x_n)\in\prod_{i=1}^n X_i \\ f(x_1,\ldots,x_n)=y}} A_1(x_1)\overline{\theta}\cdots\overline{\theta}A_n(x_n). \tag{1.54}$$

We say that a dually residuated lattice $\mathbf{L}^{\mathrm{d}}$ is (dually) divisible, if $a \vee b = a \oplus (b \ominus a)$ holds for every $a, b \in L$.

Theorem 1.4.2. *Let $(X_1, \leq_1), \ldots, (X_n, \leq_n)$ be arbitrary linearly ordered sets and $(Y, \leq)$ be an arbitrary partial ordered set. Let $f : X_1 \times \cdots \times X_n \to Y$ be a surjective mapping satisfying the presumption of Theorem 1.4.1. Then $\widehat{f}^{\oplus}$ preserves the $\oplus$-convexity of the $\mathbf{L}^{\mathrm{d}}$-sets. Moreover, if $\mathbf{L}^{\mathrm{d}}$ is (dually) divisible, then $\widehat{f}^{\vee}$ preserves the $\vee$-convexity of the $\mathbf{L}^{\mathrm{d}}$-sets.*

Proof. It is analogical to the proof of Theorem 1.4.1. □

Chapter 2

Equipollence of fuzzy sets

Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be "no" that proof appears almost unnecessary.

George Cantor (In a letter to Dedekind dated 5 January 1874)

In the classical set theory we say that two sets are *equipollent* (or also bijective, equipotent, equinumerous etc.), if there exists a one-to-one correspondence (a bijection) between them (see e.g. [69, 83, 87]). The notion of equipollent sets is the important notion in the modern set theory. Using this notion we can compare the size of sets. In particular, we say that two sets X and Y have the same cardinality and write $|X| = |Y|$, if they are equipollent. In this case the notion of cardinality of sets has only a functional role. In order to express the cardinality of sets as a specified object itself, the equivalence of being equipollent on the class of all sets is introduced. This equivalence is called the *equipollence* (or also equipotence, equinumerosity etc.). Then the cardinality of a set A may be defined as its equivalence class under equipollence or as the suitable representant of this class[1] which is called the *cardinal number*.

In this chapter, we will attempt to introduce an analogical notion of equipollent sets for fuzzy sets. The notion of equipollent fuzzy sets is not new in fuzzy sets theory. Some classical-like approaches to the notion of

[1]Precisely, this representant is the least ordinal number, which is the element of the equivalence class under equipollence.

equipollence of fuzzy sets, which are based on various definitions of uniqueness of "many-valued" mappings, are presented by Sigfried Gottwald [36, 37] or Dieter Klaua [55]. These approaches are, however, rather purely theoretical and thus no much usable in practice. In the first book of Maciej Wygralak [104], that is a systematical treatise of the theory of cardinality of vague defined objects, the notion of equipollence (precisely equipotence) of fuzzy sets has been introduced using the a-cuts and the cardinality of sets in such a way to obtain the equality of cardinality of equipollent vague defined objects. Some modified definitions of equipollence of fuzzy sets may be also found in e.g. [20, 107–109].

In this work, we will define the notion of equipollent fuzzy sets (precisely $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets) using the classical bijections between their universes which are, however, evaluated by some degrees of their bijectivity. Thus this approach proposes to introduce a degree of equipollence of two fuzzy sets, that belongs to the support of the considered residuated lattice. Note that the equipollence of fuzzy sets is not investigated intensively here, but rather with regard to the further requirements. Therefore, there are various problems for solution that could be inspired by the classical set theory. Our original goal was to find a suitable tool that enable us to define fuzzy quantifiers models, introduced in Chapter 4, in a reasonable way. A relation between equipollency of fuzzy sets and their cardinalities is investigated in the next chapter. In the first part, we introduce different definitions of evaluated mappings (injections, surjections and bijections) between $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets, where each mapping is connected with a degree of the "existence" of this mapping. Contrary to the classical set theory, where all bijections between two sets, if there exist, have the same importance, the different bijections between two fuzzy sets can have different degrees of bijectivity and thus a different importance. If we want to define a degree of fuzzy sets equipollence, it is natural to use "the best degree" of a corresponding bijection between them. It motivates our definition of the notion of evaluated equipollence of fuzzy sets, which is established in the second section for $\mathbf{L}$-sets and in the third section for $\mathbf{L}^{\mathrm{d}}$-sets. Moreover, a relation between equipollence of countable fuzzy sets and similarity of special fuzzy sets is described. A preservation of degrees of equipollence by lattices homomorphisms is investigated in the last section.

2.1 Evaluated mappings between fuzzy sets

In this and the following sections, we will suppose that all considered universes are non-empty. Let $A \in \mathcal{F}_{\mathbf{L}}(X)$ and $B \in \mathcal{F}_{\mathbf{L}}(Y)$ be arbitrary $\mathbf{L}$-sets.

A mapping $f : X \to Y$ is called the $a^{\wedge}$*-mapping* from A to B, if

$$a = \bigwedge_{x \in X} (A(x) \to B(f(x))). \tag{2.1}$$

The value a is called the $\wedge$*-degree of the mapping* f between the **L**-sets A and B. Obviously, if x belongs to A in a high membership degree and $f(x)$ belongs to B in a low membership degree, then the mapping $f : X \to Y$ can not be the mapping between **L**-sets A and B in a high degree (compare with the crisp sets). This fact is well expressed by the formula $A(x) \to B(f(x))$. In order to find the degree of the mapping f between A and B, we apply the universal quantifier which is interpreted by infimum. Note that in the literature fuzzy mappings between fuzzy sets are often introduced as the $\top^{\wedge}$-mapping, which is equivalent to the following definition. A mapping f is a fuzzy mapping from A to B, if $A(x) \leq B(f(x))$ holds for every $x \in X$ (see e.g. [57, 72]). We use the denomination "$a^{\wedge}$-mapping", because it indicates the way of calculation of the value a, which is useful to distinguish other types of calculation of the value a being introduced below. It is easy to show that if f is an $a^{\wedge}$-mapping from A to B and g is a $b^{\wedge}$- mapping from B to C, then the $\wedge$-degree of the mapping $g \circ f$, i.e. the composition of the mappings f and g from A to C, is greater than or equal to the value $a \otimes b$. In fact, it follows from

$$a \otimes b = \bigwedge_{x \in X} (A(x) \to B(f(x)) \otimes \bigwedge_{y \in Y} (B(y) \to C(g(y))) \leq$$
$$\bigwedge_{x \in X} \big((A(x) \to B(f(x)) \otimes (B(f(x)) \to C(g(f(x))))\big) \leq$$
$$\bigwedge_{x \in X} (A(x) \to C(g \circ f(x)).$$

In the definition of the $a^{\wedge}$-mapping there are used the operations $\wedge$ and $\to$ of the complete residuated lattices. In order to introduce a dual notion to $a^{\wedge}$-mapping for $\mathbf{L}^{\mathrm{d}}$-sets, it is natural to replace (in a corresponding way) the mentioned operations by their dual operations. Thus, let $C \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $D \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be two $\mathbf{L}^{\mathrm{d}}$-sets. A mapping $f : X \to Y$ is called the $a^{\vee}$*-mapping* from C to D, if

$$a = \bigvee_{x \in X} (D(f(x)) \ominus C(x)). \tag{2.2}$$

The value a is called the $\vee$*-degree of the mapping* f between the $\mathbf{L}^{\mathrm{d}}$-sets C and D. Obviously, if f is an $a^{\vee}$-mapping from C to D and g is a $b^{\vee}$-

mapping from D to E, then similarly to the previous remark the $\vee$-degree of the mapping $g \circ f$ from C to E is less than or equal to $a \oplus b$. The following lemma shows how to simplify the calculation of the $\wedge$- and $\vee$-degree of the mapping f between fuzzy sets.

Lemma 2.1.1. *Let $A \in \mathcal{F}_{\mathbf{L}}(X)$, $B \in \mathcal{F}_{\mathbf{L}}(Y)$ and $C \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$, $D \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be fuzzy sets and $f : X \to Y$ be a mapping. Then the values*

$$a = \bigwedge_{x \in \mathrm{Supp}(A)} (A(x) \to B(f(x))) \quad \text{and} \quad b = \bigvee_{x \in f^{-1}(\mathrm{Supp}(B))} (D(f(x)) \ominus C(x))$$

are the $\wedge$- and $\vee$-degrees of the mapping f between fuzzy sets, respectively.

Proof. If $x \notin \mathrm{Supp}(A)$, then $A(x) \to B(f(x)) = \bot \to B(f(x)) = \top$ and analogously if $x \notin f^{-1}(\mathrm{Supp}(B))$, then $B(f(x)) \ominus A(x) = \bot \ominus A(x) = \bot$. Hence, these elements have no effect on the appropriate degree a or b of the mapping f and thus we can restrict the calculation on $\mathrm{Supp}(A)$ in the first case and $f^{-1}(\mathrm{Supp}(B))$ in the second case. □

Let $\triangle_1 \in \{\mathbf{L}_1, \mathbf{L}_1^{\mathrm{d}}\}$ and $\triangle_2 \in \{\mathbf{L}_2, \mathbf{L}_2^{\mathrm{d}}\}$. For any universe X a homomorphism $h : L_1 \to L_2$ from $\triangle_1$ to $\triangle_2$ determines a mapping $h^{\to} : \mathcal{F}_{\triangle_1}(X) \to \mathcal{F}_{\triangle_2}(X)$ that is given by $h^{\to}(A)(x) = h(A(x))$. Now for instance, if $f : X \to Y$ is an $a^{\wedge}$-mapping between $\mathbf{L}_1$-sets A and B and $h : \mathbf{L}_1 \to \mathbf{L}_2^{\mathrm{d}}$ is a homomorphism from $\mathbf{L}_1$ to $\mathbf{L}_2^{\mathrm{d}}$, then f is the $h(a)^{\wedge}$-mapping between $\mathbf{L}_2^{\mathrm{d}}$-sets $h^{\to}(A)$ and $h^{\to}(B)$.

Example 2.1.1. Let us suppose that the membership degrees of fuzzy sets are interpreted in the Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}$ and the dual Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}^{\mathrm{d}}$. Further, let us consider two universes $X = \{x_1, \dots, x_5\}$, $Y = \{y_1, \dots, y_5\}$ and the mapping $f : X \to Y$ given by $f(x_i) = y_i$. If we put

$$A = \{0.3/x_1, 0.8/x_2, 0.4/x_3, 0/x_4, 0/x_5\},$$
$$B = \{0/y_1, 0.6/y_2, 0.9/y_3, 0/y_4, 0.4/y_5\},$$

then f is the $0.7^{\wedge}$-mapping and $0.5^{\vee}$-mapping between the fuzzy sets A and B. According to Theorem 1.1.5 the mapping $h : [0,1] \to [0,1]$, given by $h(a) = 1 - a$, is the homomorphism of the Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}$ and the dual Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}^{\mathrm{d}}$. Then we have

$$h^{\to}(A) = \{0.7/x_1, 0.2/x_2, 0.6/x_3, 1/x_4, 1/x_5\},$$
$$h^{\to}(B) = \{1/y_1, 0.4/y_2, 0.1/y_3, 1/y_4, 0.6/y_5\}.$$

Thus, f is the $0.3^{\vee}$-mapping between $h^{\to}(A)$ and $h^{\to}(B)$. The same result could be obtained by using of h to the $0.7^{\wedge}$-mapping, i.e. $h(0.7) = 0.3$.

Let $A \in \mathcal{F}_{\mathbf{L}}(X)$, $B \in \mathcal{F}_{\mathbf{L}}(Y)$ and $C \in \mathcal{F}_{\mathbf{L}^d}(X)$, $D \in \mathcal{F}_{\mathbf{L}^d}(Y)$ be fuzzy sets. A mapping $f : X \to Y$ is the $a^\wedge$*-injection* or $a^\vee$*-injection* from A to B or from C to D, if f is the injection and simultaneously the $a^\wedge$-mapping or $a^\vee$-mapping, respectively. A mapping $f : X \to Y$ is the $a^\wedge$*-surjection* or $a^\vee$*-surjection* from A onto B or from C onto D, if f is the surjection[2] and simultaneously

$$a = \bigwedge_{y \in Y} \bigwedge_{\substack{x \in X \\ f(x)=y}} (A(x) \leftrightarrow B(y)) \quad \text{or} \quad a = \bigvee_{y \in Y} \bigvee_{\substack{x \in X \\ f(x)=y}} |C(x) \ominus D(y)|, \tag{2.3}$$

respectively. The value a is called the $\triangle$*-degree of the injection* (*surjection*) f or $\triangle$*-injectivity* (*surjectivity*) *degree of the mapping* f, where $\triangle \in \{\wedge, \vee\}$. We have established the $\triangle$-degree of injection as the $\triangle$-degree of mapping, because we suppose that the mapping is injective (between universes) and in this case a natural definition of the injectivity degree coincides with the definition of the mapping degree. In order to investigate consistently the injectivity of mappings between fuzzy sets, we would need to know relations (similarities) between the elements of universes. The sets enriched by a similarity relation are intensively investigated from the categorical point of view[3]. For the interested reader, we refer to e.g. [44,74,94]. In the case of the $\triangle$-degree of the surjection we have a different situation. For each $x \in X$ we have to consider two values, evaluating first the property that f assigns an element y from B to x from A, and second the property of the surjectivity of this assignment. The value $A(x) \to B(y)$ and dually $B(y) \ominus A(x)$ well express the first property which has been commented formerly. In the classical set theory, the property of the surjectivity is connected with the existence of at least one element from X to each element from Y with respect to the mapping f. In our case, the existence degree of x from A to y from B such that $f(x) = y$ seems to be well described by $B(y) \to A(x)$ and dually $A(x) \ominus B(y)$. For instance, if $A(x) = \bot$ and $B(f(x)) = \top$, then it is reasonable to expect the value $\bot$ as a surjectivity degree for x, which can be obtained by $B(f(x)) \to A(x) = \bot$. Clearly, the $\wedge$-degree of a surjection is less than or equal to the $\wedge$-degree of the mapping and dually the $\vee$-degree of a surjection is greater than or equal to the $\vee$-degree of a mapping.

In the classical set theory, a mapping $f : X \to Y$ is the bijection if and only if f is injective and simultaneously surjective. Hence, we can introduce

[2] We could consider more general surjection, for example, the surjection between supports of fuzzy sets, but it is only the technical matter that could complicate following definitions and proofs.

[3] Note that a notion of *monomorphism* is considered in the category theory and the monomorphisms in the category of sets are precisely the injections.

an equivalent notion for fuzzy sets. A mapping $f : X \to Y$ is the $a^{\wedge}$*-bijection* or $a^{\vee}$*-bijection* from $A \in \mathcal{F}_{\mathbf{L}}(X)$ onto $B \in \mathcal{F}_{\mathbf{L}}(Y)$ or from $C \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ onto $D \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(Y)$, if f is the bijection and

$$a = \bigwedge_{x \in X} (A(x) \leftrightarrow B(f(x))) \text{ or } \; a = \bigvee_{x \in X} |C(x) \ominus D(f(x))|, \qquad (2.4)$$

respectively. Obviously, these formulas are derived from (2.3), where the injectivity of f is assumed. The value a is called the $\triangle$*-degree of the bijection* f or the $\triangle$*-bijectivity degree of the mapping* f, where $\triangle \in \{\wedge, \vee\}$. Note that in the set theory there is also the other definition of bijection, namely, we say that $f : X \to Y$ is a bijection, if there exists a mapping $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$, where id_X and id_Y denote the identity mappings on X and Y, respectively. Let $\triangle \in \{\wedge, \vee\}$, then it is easy to show that f is an $a^{\triangle}$-bijection if and only if there exists a mapping $g : Y \to X$ such that $f \circ g = 1_Y$, $g \circ f = 1_X$ and $a = b \triangle c$, where b is the $\triangle$-degree of f and c is the $\triangle$-degree of g. In fact, if we suppose, for example, that $\triangle = \wedge$, then $b \wedge c = \bigwedge_{x \in X}(A(x) \to B(f(x))) \wedge \bigwedge_{y \in Y}(B(y) \to A(g(y))) = \bigwedge_{x \in X}(A(x) \to B(f(x))) \wedge \bigwedge_{x \in X}(B(f(x)) \to A(g(f(x)))) = \bigwedge_{x \in X}(A(x) \to B(f(x))) \wedge (B(f(x)) \to A(x)) = a$.

Example 2.1.2. Let us consider the fuzzy sets A, B and the mapping f from Ex. 2.1.1. Then f is the $0.5^{\wedge}$-bijection and the $0.5^{\vee}$-bijection.

Again, in order to find the $\wedge$- and $\vee$-degrees of a bijection between fuzzy sets, we need not to deal, in general, with the whole universe as follows.

Lemma 2.1.2. *Let $A \in \mathcal{F}_{\mathbf{L}}(X)$, $B \in \mathcal{F}_{\mathbf{L}}(Y)$ and $C \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$, $D \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be fuzzy sets, $f : X \to Y$ be a bijection and H be a subset of X such that* $\mathrm{Supp}(A) \cup f^{-1}(\mathrm{Supp}(B)) \subseteq H$. *Then the values*

$$a = \bigwedge_{x \in H} (A(x) \leftrightarrow B(f(x))) \quad \text{and} \quad b = \bigvee_{x \in H} |A(x) \ominus B(f(x))|,$$

are the $\wedge$- and $\vee$-degrees of the bijection f between fuzzy sets, respectively.

Proof. If $x \notin H$, then due to the presumption about H, we have $A(x) \leftrightarrow B(f(x)) = \perp \leftrightarrow \perp = \top$ and analogously $|A(x) \ominus B(f(x))| = |\perp \ominus \perp| = \perp$. Hence, these elements have no effect on the appropriate degrees a and b of the bijection f and thus we can restrict the calculation of a or b on H. □

Now we introduce alternative notions to the notions of $a^{\wedge}$-mapping and $a^{\vee}$-mapping (injection, surjection and bijection), that may be used for the countable fuzzy sets. Let $A \in \mathcal{FC}_{\mathbf{L}}(X)$, $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ and $C \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$,

$D \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be countable fuzzy sets. A mapping $f : X \to Y$ is the $a^{\otimes}$-*mapping* or $a^{\oplus}$-*mapping* between the fuzzy sets A and B or C and D, if

$$a = \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \to B(f(x))) \text{ or} \tag{2.5}$$

$$a = \bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} (D(f(x)) \ominus C(x)), \tag{2.6}$$

respectively, where $\mathrm{Fin}(X)$ denotes the set of all finite subsets in the universe X. The value a is called the $\triangle$-*degree of the mapping* f, where $\triangle \in \{\otimes, \oplus\}$. Recall that

$$\bigotimes_{i \in I} b_i = \bigwedge_{I' \in \mathrm{Fin}(I)} \bigotimes_{i \in I'} b_i \quad \text{and} \quad \bigoplus_{i \in I} b_i = \bigvee_{I' \in \mathrm{Fin}(I)} \bigoplus_{i \in I'} b_i$$

are defined for any countable set $\{b_i \mid i \in I\}$.

Lemma 2.1.3. *Let $A \in \mathcal{FC}_{\mathbf{L}}(X)$, $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ and $C \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$, $D \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be countable fuzzy sets and $f : X \to Y$ be a mapping. Then the values*

$$a = \bigotimes_{x \in \mathrm{Supp}(A)} (A(x) \to B(f(x))) \quad \text{and} \quad b = \bigoplus_{x \in f^{-1}(\mathrm{Supp}(A))} (B(f(x)) \ominus A(x))$$

are the $\otimes$- and $\oplus$-degrees of the mapping f between fuzzy sets, respectively.

Proof. Here, we will prove just the first statement, the second one could be done by analogy. Obviously, we have to prove that the following equality

$$\bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \to B(f(x))) = \bigwedge_{Z \in \mathrm{Fin}(\mathrm{Supp}(A))} \bigotimes_{x \in Z} (A(x) \to B(f(x)))$$

holds. Since $A' \in \mathrm{Fin}(\mathrm{Supp}(A))$ implies $A' \in \mathrm{Fin}(X)$, then the inequality $\leq$ is true. Further, let $Z \in \mathrm{Fin}(X)$ be a finite set, then $Z \cap \mathrm{Supp}(A) \in \mathrm{Fin}(\mathrm{Supp}(A))$. If $x \in Z \setminus \mathrm{Supp}(A)$, then $A(x) \to B(f(x)) = \bot \to B(f(x)) = \top$ and thus $\bigotimes_{x \in Z} A(x) \to B(f(x)) = \bigotimes_{x \in Z \cap \mathrm{Supp}(A)} A(x) \to B(f(x))$. Hence, the inequality $\geq$ is also true and the proof is complete. □

We say that f is the $a^{\otimes}$-*injection* or $a^{\oplus}$-*injection*, if f is the injection and simultaneously the $a^{\otimes}$-mapping or $a^{\oplus}$-mapping, respectively. Further, the mapping f is the $a^{\otimes}$-*surjection* or $a^{\oplus}$-*surjection*, if f is the surjection and simultaneously

$$\begin{aligned} a &= \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{y \in f(Z)} \bigotimes_{\substack{x \in Z \\ f(x) = y}} (A(x) \leftrightarrow B(y)) \quad \text{or} \\ a &= \bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{y \in f(Z)} \bigoplus_{\substack{x \in Z \\ f(x) = y}} (C(x) \ominus D(y)), \end{aligned} \tag{2.7}$$

respectively. Finally, the mapping f is the $a^{\otimes}$*-bijection* or $a^{\oplus}$*-bijection*, if f is the bijection and simultaneously

$$\begin{aligned} a &= \bigwedge_{Z\in \mathrm{Fin}(X)} \bigotimes_{x\in Z} (A(x) \leftrightarrow B(f(x))) \text{ or} \\ a &= \bigvee_{Z\in \mathrm{Fin}(X)} \bigoplus_{x\in Z} |C(x) \ominus D(f(x))|, \end{aligned} \tag{2.8}$$

respectively. According to type of the mapping f, the value a is called the $\triangle$*-degree of the injection* (*surjection, bijection*) f or the $\triangle$*-injectivity* (*surjectivity, bijectivity*) *degree of the mapping* f, where $\triangle \in \{\otimes, \oplus\}$.

Example 2.1.3. Again, let us consider the fuzzy sets A, B and the mapping f from Ex. 2.1.1. Then f is the $0^{\otimes}$-bijection and $1^{\oplus}$-bijection.

Lemma 2.1.4. *Let* $A \in \mathcal{FC}_{\mathbf{L}}(X)$*,* $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ *and* $C \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$*,* $D \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(Y)$ *be countable fuzzy sets,* $f : X \to Y$ *be a bijection and* H *be a subset of* X *such that* $\mathrm{Supp}(A) \cup f^{-1}(\mathrm{Supp}(B)) \subseteq H$*. Then the values*

$$a = \bigotimes_{x\in H} (A(x) \leftrightarrow B(f(x))) \quad \textit{and} \quad b = \bigoplus_{x\in H} |C(x) \ominus D(f(x))| \tag{2.9}$$

are the $\otimes$*- and* $\oplus$*-degrees of the bijection* f *between fuzzy sets, respectively.*

Proof. Here, we will prove just the first statement, the second one could be done analogously. The statement is true, if $\bigwedge_{Z\in \mathrm{Fin}(X)} \bigotimes_{x\in Z}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{Z\in \mathrm{Fin}(H)} \bigotimes_{x\in Z}(A(x) \leftrightarrow B(f(x)))$. Since $\mathrm{Fin}(H) \subseteq \mathrm{Fin}(X)$, then the inequality $\leq$ is fulfilled. For any $Z \in \mathrm{Fin}(X)$ we have $Z \cap H \in \mathrm{Fin}(H)$. If $x \in Z \setminus H$, then by the presumption about H we obtain $A(x) \leftrightarrow B(f(x)) = \bot \leftrightarrow \bot = \top$ and thus these values has no effect on the $\wedge$-degree of bijection f. Hence, we have $\bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x)) = \bigotimes_{x\in Z\cap H} A(x) \leftrightarrow B(f(x))$ and thus the inequality $\leq$ is true. □

2.2 θ-equipollent L-sets

In the following parts, we will suppose that there exists a bijection between the given universes. Let $\mathbf{L}$ be a complete residuated lattice and $A \in \mathcal{F}_{\mathbf{L}}(X)$, $B \in \mathcal{F}_{\mathbf{L}}(Y)$ be arbitrary $\mathbf{L}$-sets. We say that A is $a^{\wedge}$*-equipollent* with B (or also A and B are $a^{\wedge}$*-equipollent*), if

$$a = \bigvee \{b \mid \exists f : X \to Y \textit{ and } f \textit{ is a } b^{\wedge}\textit{-bijection between } A \textit{ and } B\}. \tag{2.10}$$

Further, let $A \in \mathcal{FC}_{\mathbf{L}}(X)$, $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ be arbitrary countable **L**-sets. We say that A is *$a^{\otimes}$-equipollent* with B (or also A and B are *$a^{\otimes}$-equipollent*), if

$$a = \bigvee\{b \mid \exists f : X \to Y \textit{ and } f \textit{ is a } b^{\otimes}\textit{-bijection between } A \textit{ and } B\}. \tag{2.11}$$

The value a is called the *θ-degree of (fuzzy) equipollence* of the **L**-sets A and B, respectively. Note that two **L**-sets may be a^{θ}-equipollent, however, there is not any a^{θ}-bijection between them. This fact is a straightforward consequence of the supremum. Hence, we can ask the following question. Let us suppose an arbitrary complete residuated lattice **L**. Then are there **L**-sets A and B, which are $\top^{\theta}$-equipollent, but $|A_a| \neq |B_a|$ for some $a \in L$? The answer is an open problem. On the other hand, we can easy construct infinite **L**-sets A and B such that $|A_a| = |B_a|$ for any $a \in L$, but A and B are not $\top^{\theta}$-equipollent. Hence, we can assert that the equipollent fuzzy sets in Wygralak's approach[4] are not $\top^{\theta}$- equipollent in our approach, in general.

Let us suppose that X and Y are infinite universes. In this case, there exists an infinite number of bijections between them and thus the practical computation of the degree of equipollence becomes impossible even for finite **L**-sets. Hence, it is desirable to show that the θ-degrees of equipollence for the finite fuzzy sets may be found over a finite number of bijections. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $B \in \mathcal{FIN}_{\mathbf{L}}(Y)$ be finite **L**-sets. Denote A_B and B_A subsets of X and Y, respectively, such that $\mathrm{Supp}(A) \subseteq A_B$, $\mathrm{Supp}(B) \subseteq B_A$ and $|A_B| = |B_A| = \max(|\mathrm{Supp}(A)|, |\mathrm{Supp}(B)|)$. Note that A_B and B_A are the least equipollent sets (with respect to the inclusion relation) up to bijection covering $\mathrm{Supp}(A)$ and $\mathrm{Supp}(B)$, which are of course finite. Finally, if X and Y are equipollent sets, then we denote $\mathrm{Bij}(X, Y)$ the set of all bijections between the sets X and Y.

Theorem 2.2.1. *Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $B \in \mathcal{FIN}_{\mathbf{L}}(Y)$ be finite **L**-sets (X and Y are bijective), A_B and B_A be sets defined above. Then the value*

$$a = \bigvee_{f \in \mathrm{Bij}(A_B, B_A)} \bigominus_{x \in A_B} A(x) \leftrightarrow B(f(x)) \tag{2.12}$$

*is the θ-degree of equipollence of the **L**-sets A and B.*

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $B \in \mathcal{FIN}_{\mathbf{L}}(Y)$ be two arbitrary **L**-sets. If $A = \emptyset$ or $B = \emptyset$, then clearly the statement is true. Let us suppose that

[4] Note that a basic definition of equipollence of fuzzy sets is given as follows. Fuzzy sets $A \in \mathcal{F}_{\mathbf{L}}(X)$ and $B \in \mathcal{F}_{\mathbf{L}}(Y)$ (X and Y are bijective) are equipollent, if $\bigwedge_{|A_a| \leq i} a = \bigwedge_{|B_a| \leq i} a$ and $\bigvee_{|A_a| \geq i} a = \bigvee_{|B_a| \geq i} a$ hold for any cardinal number i.

$A \neq \emptyset \neq B$ and consider a bijection $f \in \mathrm{Bij}(X,Y)$. Put $H_f = \{x \in A_B \mid f(x) \notin B_A\}$. If $H_f = \emptyset$ (i.e. $f(x) \in B_A$ for any $x \in A_B$ and thus $f(x) \notin B_A$ for any $x \in X \setminus A_B$), then we have for $\theta = \wedge$

$$\bigwedge_{x\in X}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{x\in A_B}(A(x) \leftrightarrow B(f(x)))\wedge \bigwedge_{x\in X\setminus A_B}(\bot \leftrightarrow \bot) = \bigwedge_{x\in A_B}(A(x) \leftrightarrow B(f(x))),$$

and for $\theta = \otimes$

$$\bigwedge_{Z\in\mathrm{Fin}(X)}\bigotimes_{x\in Z}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{Z\in\mathrm{Fin}(X)}\Big(\bigotimes_{x\in Z\cap A_B}(A(x) \leftrightarrow B(f(x)))\otimes \bigotimes_{x\in Z\setminus A_B}(A(x) \leftrightarrow B(f(x)))\Big) = \bigwedge_{Z\in\mathrm{Fin}(X)}\Big(\bigotimes_{x\in Z\cap A_B}(A(x) \leftrightarrow B(f(x)))\otimes \bigotimes_{x\in Z\setminus A_B}(\bot \leftrightarrow \bot)\Big) = \bigwedge_{Z\in\mathrm{Fin}(A_B)}\bigotimes_{x\in Z}(A(x) \leftrightarrow B(f(x))).$$

Now let us suppose that $H_f \neq \emptyset$ and put $H = f^{-1}(B_A) \setminus A_B$. Since H_f is a part of the finite set A_B and f is a bijection, then necessarily there exists a bijection[5] $h : H_f \to H$. Now let us establish a bijection $g : X \to Y$ as follows

$$g(x) = \begin{cases} f(x), & x \in X \setminus H_f \cup H, \\ f(h(x)), & x \in H_f, \\ f(h^{-1}(x)), & x \in H. \end{cases} \tag{2.13}$$

Obviously, g is a bijection such that $g(H_f) = f(H)$ and $g(H) = f(H_f)$. Hence, we obtain that $H_g = \emptyset$. Furthermore, we have for $\theta = \wedge$

$$\bigwedge_{x\in X} A(x) \leftrightarrow B(f(x)) = \bigwedge_{x\in X\setminus H_f\cup H}(A(x) \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in H_f}(A(x) \leftrightarrow \bot)\wedge \bigwedge_{x\in H}(\bot \leftrightarrow B(f(x))) = \bigwedge_{x\in X\setminus H_f\cup H}(A(x) \leftrightarrow B(f(x)))\wedge \bigwedge_{x\in H_f}\big((A(x) \leftrightarrow \bot) \wedge (\bot \leftrightarrow B(f(h(x))))\big) \leq \bigwedge_{x\in X\setminus H_f\cup H}(A(x) \leftrightarrow B(g(x))\wedge \bigwedge_{x\in H_f}(A(x) \leftrightarrow B(g(x)) \wedge \bigwedge_{x\in H}(\bot \leftrightarrow \bot) = \bigwedge_{x\in X}(A(x) \leftrightarrow B(g(x))$$

[5] Note that it is not true in the case, if infinite sets are used. In fact, if $A_B \subset X$ would be an infinite set, then there exists a bijection $f : X \to Y$ such that $A_B \setminus H_f \subset A_B$ would be bijective with A_B and $f(A_B \setminus H_f) = B_A$. Hence, there is no bijection between $H_f \neq \emptyset$ and $H = \emptyset$. Recall that a set is infinite if and only if it is bijective with some of its proper subset.

where clearly $A(x) \leftrightarrow B(g(x)) = \perp \leftrightarrow \perp = \top$ holds for any $x \in H$. Since $\mathrm{Supp}(A) \cup f^{-1}(\mathrm{Supp}(B)) \subseteq A_B \cup H$ holds, then due to Lemma 2.1.4, we have

$$\bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \leftrightarrow B(f(x))) = \bigotimes_{x \in A_B \cup H} (A(x) \leftrightarrow B(f(x))) =$$

$$\bigotimes_{x \in A_B \setminus H_f} (A(x) \leftrightarrow B(f(x))) \otimes \bigotimes_{x \in H_f} (A(x) \leftrightarrow \perp) \otimes \bigotimes_{x \in H} (\perp \leftrightarrow B(f(x))) =$$

$$\bigotimes_{x \in A_B \setminus H_f} (A(x) \leftrightarrow B(f(x))) \otimes \bigotimes_{x \in H_f} \big((A(x) \leftrightarrow \perp) \otimes (\perp \leftrightarrow B(f(h(x))))\big) \le$$

$$\bigotimes_{x \in A_B \setminus H_f} (A(x) \leftrightarrow B(f(x))) \otimes \bigotimes_{x \in H_f} \big((A(x) \leftrightarrow B(g(x))) \otimes \bigotimes_{x \in H} (\perp \leftrightarrow \perp) =$$

$$\bigotimes_{x \in A_B \cup H} (A(x) \leftrightarrow B(g(x))) = \bigwedge_{Z \in \mathrm{Fin}(X)} \big(\bigotimes_{x \in Z} A(x) \leftrightarrow B(g(x)).$$

Hence, to each bijection $f \in \mathrm{Bij}(X, Y)$ there exists a bijection $g \in \mathrm{Bij}(X, Y)$ such that $\bigwedge_{x \in X} (A(x) \leftrightarrow B(f(x))) \le \bigwedge_{x \in X} (A(x) \leftrightarrow B(g(x)))$ or analogously $\bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \leftrightarrow B(f(x))) \le \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \leftrightarrow B(g(x)))$ and moreover $H_g = \emptyset$, i.e. $g|_{A_B} \in \mathrm{Bij}(A_B, B_A)$, where $g|_{A_B}$ denotes the restriction of g on A_B. Then, due to the previous equalities for the case $H_g = \emptyset$, we obtain the desired statement and the proof is complete. □

Example 2.2.1. Let us suppose that $A = \{0.8/x, 0.6/y, 0.4/z, 0/r, \dots\}$ and $B = \{0.9/a, 0.5/b, 0.7/c, 0/d, \dots\}$ are **L**-sets over infinite universes, where the membership degrees are interpreted in the Goguen algebra $\mathbf{L}_\mathrm{P}$ (see Ex. 1.1.4). Recall that $a \leftrightarrow_\mathrm{P} b = \min(\frac{a}{b}, \frac{b}{a})$, where $\frac{0}{0} = 1$ and $\frac{a}{0} = \infty$ for $a \neq 0$. Since the supports of A and B are equipollent, then in order to find the θ-degree of equipollence of A and B, it is sufficient to consider the bijections between their supports (according to the previous theorem). Because the supports have 3 elements, we have to construct $3! = 6$ bijections. It is easy to verify that the mapping f given by $f(x) = a$, $f(y) = c$ and $f(z) = b$ defines the greatest θ-degree of bijection between A and B. Particularly, the **L**-sets A and B are $0.8^{\wedge}$-equipollent and $0.61^{\otimes}$-equipollent (a is rounded to two decimals), where $0.8 = \min(\frac{0.8}{0.9}, \frac{0.6}{0.7}, \frac{0{,}4}{0.5})$ and $0.61 \doteq \frac{0.8}{0.9} \cdot \frac{0.6}{0.7} \cdot \frac{0{,}4}{0.5}$.

The previous theorem shows that the equipollence of finite **L**-sets may be studied over suitable finite sets. A theoretical question is, if the same principle can be applied also for the denumerable sets. It means, if we can restrict ourselves to a suitable denumerable sets. One could seem that if A and B are denumerable, then it is sufficient to put $A_B = \mathrm{Supp}(A)$ and $B_A = \mathrm{Supp}(B)$, which is in accordance with the previous idea for finite **L**-sets (A_B and B_A are the least equipollent sets (with respect to the inclusion relation) covering Supp(A) and Supp(B), respectively). The following example shows that it is not true.

Example 2.2.2. Let A, B be two $\mathbf{L}_{\mathrm{L}}$-sets in the set of natural numbers N, which are given as follows. Put $A(n) = 1$, if n is even, $A(n) = 0.1$, otherwise, and $B(n) = 1$, if n is even, $B(n) = 0$, otherwise. Let us suppose that $A_B = \mathrm{Supp}(A)$ and $B_A = \mathrm{Supp}(B)$. Then we obtain that A and B are 0.1^-equipollent, if we use the same principle for denumerable **L**-sets as in the previous theorem. However, if we consider the identity mapping on N, then this mapping is 0.9-bijective. Hence, the $\wedge$-degree of equipollence of $\mathbf{L}_{\mathrm{L}}$-sets A and B has to be greater than or equal to 0.9. Thus the previous theorem can not be extended for denumerable sets, if we suppose that $A_B = \mathrm{Supp}(A)$ and $B_A = \mathrm{Supp}(B)$.

It is easy to see that the previous theorem is also true, if instead of A_B and B_A we consider arbitrary finite equipollent sets $X' \subseteq X$ and $Y' \subseteq Y$ covering the supports of A and B, respectively. The following theorem shows that this form of the previous theorem may be extended for arbitrary countable **L**-sets.

Theorem 2.2.2. *Let $A \in \mathcal{FC}_{\mathbf{L}}(X)$ and $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ be two countable* **L**-*sets. Then A is a^θ-equipollent with B if and only if there exist countable equipollent subsets $X' \subseteq X$ and $Y' \subseteq Y$ covering* Supp(A) *and* Supp(B), *respectively, such that the θ-degree of equipollence of the* **L**-*sets A and B is equal to*

$$a = \bigvee_{f \in \mathrm{Bij}(X', Y')} \bigominus_{x \in X'} A(x) \leftrightarrow B(f(x)), \tag{2.14}$$

where Bij(X', Y') *denotes the set of all bijections between X' and Y'.*

Proof. Due to the Theorem 2.2.1, the statement is true for arbitrary finite **L**-sets. Let $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$ be arbitrary countable **L**-sets such that at least one of them is denumerable. Obviously, if X is a denumerable universe or $A = \emptyset$ or $B = \emptyset$, then again the statement is also true. Thus let us suppose that X is an uncountable universe and $A \neq \emptyset \neq B$. Let $X_A \subset X$ and $Y_B \subset Y$ be arbitrary denumerable sets such that $X_A \cap \mathrm{Supp}(A) = \emptyset$ and $Y_B \cap \mathrm{Supp}(B) = \emptyset$. First, we will prove that if a is the $\wedge$-degree and b is the $\otimes$-degree of equipollence of the **L**-sets A and B, then we can write

$$a = \bigvee_{f \in \mathrm{Bij}(Z_A, Z_B)} \bigwedge_{x \in Z_A} A(x) \leftrightarrow B(f(x)), \tag{2.15}$$

$$b = \bigvee_{f \in \mathrm{Bij}(Z_A, Z_B)} \bigwedge_{Z \in \mathrm{Fin}(Z_A)} \bigotimes_{x \in Z} A(x) \leftrightarrow B(f(x)), \tag{2.16}$$

where $Z_A = X \setminus X_A$ and $Z_B = Y \setminus Y_B$. Obviously, the inequality $\geq$ is fulfilled in both cases. In order to show the opposite inequality, it is sufficient to prove, that to each $f \in \mathrm{Bij}(X, Y)$ there exists $g \in \mathrm{Bij}(Z_A, Z_B)$ such that

$$\bigwedge_{x\in X} A(x) \leftrightarrow B(f(x)) \leq \bigwedge_{x\in Z_A} A(x) \leftrightarrow B(g(x)),$$
$$\bigwedge_{Z\in \mathrm{Fin}(X)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x)) \leq \bigwedge_{Z\in \mathrm{Fin}(Z_A)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(g(x)).$$

Let $f \in \mathrm{Bij}(X, Y)$ be a bijection. Put $T_A = \{x \in Z_A \mid f(x) \in Y_B\}$ and $T_B = \{x \in Z_B \mid f^{-1}(x) \in X_A\}$. Due to Cantor-Bernstein theorem, we obtain that the sets $X_A \setminus f^{-1}(T_B)$ and $Y_B \setminus f(T_A)$ are equipollent. Now, we can choose two countable sets $U_A \subset Z_A$ and $U_B \subset Z_B$ such that $U_A \cap (\mathrm{Supp}(A) \cup f^{-1}(\mathrm{Supp}(B)) \cup T_A \cup f^{-1}(U_B)) = \emptyset$ and $U_B \cap (f(\mathrm{Supp}(A)) \cup \mathrm{Supp}(B) \cup T_B \cup f(U_A)) = \emptyset$, respectively. Clearly, it is possible, because Z_A and Z_B are the uncountable universes and all considered sets are just countable sets. Recall that the union of a finite number of countable sets is again countable set. Hence, there exist a bijection $u_B : S_f \cup f^{-1}(U_B) \to U_B$ and a bijection $u_A : U_A \to T_f \cup f(U_A)$. Moreover, we put a bijection $h : X_A \to Y_B$. Now, we can establish a bijection $g : X \to Y$ as follows

$$g(x) = \begin{cases} u_B(x), & x \in T_A \cup f^{-1}(U_B), \\ u_A(x), & x \in U_A, \\ h(x), & x \in X_A \\ f(x), & \text{otherwise.} \end{cases} \tag{2.17}$$

Obviously, we can write $A(x) \leftrightarrow B(f(x)) = A(x) \leftrightarrow \bot = A(x) \leftrightarrow B(g(x))$ for every $x \in T_A \cup f^{-1}(U_B)$ and therefore $\bigwedge_{x\in T_A\cup f^{-1}(U_B)}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{x\in T_A\cup f^{-1}(U_B)}(A(x) \leftrightarrow B(g(x)))$ and $\bigotimes_{x\in Z\cap(T_A\cup f^{-1}(U_B))} A(x) \leftrightarrow B(f(x)) = \bigotimes_{x\in Z\cap(T_A\cup f^{-1}(U_B))} A(x) \leftrightarrow B(g(x))$ holds for any $Z \in \mathrm{Fin}(X)$. Further, to each $x \in f^{-1}(T_B)$ there exists a unique $x' \in U_A$, where clearly $x' = u_A^{-1}(f(x))$, such that $A(x) \leftrightarrow B(f(x)) = \bot \leftrightarrow B(f(x)) = A(x') \leftrightarrow B(g(x'))$. Moreover, we have $A(x) \leftrightarrow B(f(x)) = \bot \leftrightarrow \bot = A(x) \leftrightarrow B(g(x))$ for each $x \in U_A \setminus u_A^{-1}(T_B)$. Hence, we obtain $\bigwedge_{x\in U_A\cup f^{-1}(T_B)}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{x\in U_A}(A(x) \leftrightarrow B(g(x)))$. Moreover, obviously to each $Z \in \mathrm{Fin}(X)$ we can define $Z' \in \mathrm{Fin}(X)$ such that we have $Z \setminus (U_A \cup f^{-1}(T_B)) = Z' \setminus U_A$ and $\bigotimes_{x\in Z\cap(U_A\cup f^{-1}(T_B))}(A(x) \leftrightarrow B(f(x))) = \bigotimes_{x\in Z'\cap U_A}(A(x) \leftrightarrow B(g(x)))$. In fact, it is sufficient to set

$$Z' = Z \setminus f^{-1}(T_B) \cup \{u_A^{-1}(f(x)) \mid x \in f^{-1}(T_B)\}. \tag{2.18}$$

Finally, we have $A(x) \leftrightarrow B(f(x)) = \bot \leftrightarrow \bot = A(x) \leftrightarrow B(g(x))$ for every $x \in X_A \setminus f^{-1}(T_B)$. Hence, we obtain $\bigwedge_{x\in X_A\setminus f^{-1}(T_B)}(A(x) \leftrightarrow B(f(x))) =$

$\bigwedge_{x\in X_A}(A(x) \leftrightarrow B(g(x))) = \top$ and $\bigotimes_{x\in Z\cap X_A\setminus f^{-1}(T_B)}(A(x) \leftrightarrow B(f(x))) = \bigotimes_{x\in Z\cap X_A}(A(x) \leftrightarrow B(g(x))) = \top$. Let us establish $S = Z_A \setminus (T_A \cup U_A \cup f^{-1}(U_B))$. It is easy to see that $f|_S = g|_S$. Then we have for $\theta = \wedge$

$$\bigwedge_{x\in X}(A(x) \leftrightarrow B(f(x))) = \bigwedge_{x\in S}(A(x) \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in T_A\cup f^{-1}(U_B)}(A(x) \leftrightarrow \bot)\wedge \bigwedge_{x\in U_A\cup f^{-1}(T_B)}(\bot \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in X_A\setminus f^{-1}(T_B)}\top = \bigwedge_{x\in S}(A(x) \leftrightarrow B(g(x)))\wedge \bigwedge_{x\in T_A\cup f^{-1}(U_B)}(A(x) \leftrightarrow B(g(x))) \wedge \bigwedge_{x\in U_A}(A(x) \leftrightarrow B(g(x))) \wedge \bigwedge_{x\in X_A}\top = \bigwedge_{x\in Z_A}(A(x) \leftrightarrow B(g(x))) \wedge \bigwedge_{x\in X_A}(A(x) \leftrightarrow B(g(x))) = \bigwedge_{x\in Z_A}(A(x) \leftrightarrow B(g(x)))$$

and thus (2.15) is true. Let $\varphi : \mathrm{Fin}(X) \to \mathrm{Fin}(X \setminus f^{-1}(T_B))$ be a mapping defined by $\varphi(Z) = Z'$, where Z' is given by (2.18). Now we can write for $\theta = \otimes$

$$\bigwedge_{Z\in\mathrm{Fin}(X)}\bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x))) = \bigwedge_{Z\in\mathrm{Fin}(X)}\Big(\bigotimes_{x\in Z\cap S}(A(x) \leftrightarrow B(f(x)))\otimes \bigotimes_{x\in Z\cap(T_A\cup f^{-1}(U_B))}(A(x) \leftrightarrow B(f(x))) \otimes \bigotimes_{x\in Z\cap(U_A\cup f^{-1}(T_B))}(A(x) \leftrightarrow B(f(x)))\otimes \bigotimes_{x\in Z\cap X_A\setminus f^{-1}(T_B)}\top\Big) = \bigwedge_{Z\in\mathrm{Fin}(X)}\Big(\bigotimes_{x\in\varphi(Z)\cap S}(A(x) \leftrightarrow B(g(x)))\otimes \bigotimes_{x\in\varphi(Z)\cap(T_A\cup f^{-1}(U_B))}(A(x) \leftrightarrow B(g(x))) \otimes \bigotimes_{x\in\varphi(Z)\cap U_A}(A(x) \leftrightarrow B(g(x)))\otimes \bigotimes_{x\in\varphi(Z)\cap X_A}\top\Big) = \bigwedge_{Z\in\mathrm{Fin}(X)}\Big(\bigotimes_{x\in Z\cap Z_A}(A(x) \leftrightarrow B(g(x)))\otimes \bigotimes_{x\in Z\cap X_A}(A(x) \leftrightarrow B(g(x))) = \bigwedge_{Z\in\mathrm{Fin}(Z_A)}\Big(\bigotimes_{x\in Z\cap Z_A}(A(x) \leftrightarrow B(g|_{Z_A}(x)))$$

where $g|_{Z_A}$ denotes the restriction g on Z_A. With regard to the definition of g we have $g_{Z_A} \in \mathrm{Bij}(Z_A, Z_B)$ and (2.16) is also true. Now, let us establish $X' = \mathrm{Supp}(A) \cup X_A$ and $Y' = \mathrm{Supp}(B) \cup Y_B$. Obviously, X' and Y' satisfy the presumptions of the theorem. Due to (2.15) and (2.16), we can restrict ourselves to the bijections from $\mathrm{Bij}(Z_A, Z_B)$. Let $f \in \mathrm{Bij}(Z_A, Z_B)$ be an arbitrary bijection, $H_f = \{x \in \mathrm{Supp}(A) \mid f(x) \notin \mathrm{Supp}(B)\}$ and $H_{f^{-1}} = \{y \in \mathrm{Supp}(B) \mid f^{-1}(y) \notin \mathrm{Supp}(A)\}$. If $H_f = H_{f^{-1}} = \emptyset$, then f is the bijection between $\mathrm{Supp}(A)$ and $\mathrm{Supp}(B)$ according to the Cantor-Bernstein theorem. Hence, we can easily construct a bijection g, where $f|_{\mathrm{Supp}(A)} = g|_{\mathrm{Supp}(A)}$, from

$\mathrm{Bij}(X', Y')$ such that $\bigwedge_{x\in Z_A} A(x) \leftrightarrow B(f(x)) = \bigwedge_{x\in X'} A(x) \leftrightarrow B(g(x))$ and $\bigwedge_{Z\in \mathrm{Fin}(Z_A)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x)) = \bigotimes_{x\in X'} A(x) \leftrightarrow B(g(x))$. Let $H_f \neq \emptyset$ or $H_{f^{-1}} \neq \emptyset$. Since H_f or $H_{f^{-1}}$ are countable sets, then there exist $S_A \subseteq X_A$ being equipollent with $f^{-1}(H_{f^{-1}})$ and $S_B \subseteq Y_B$ being equipollent with $f(H)$ such that $X_A \setminus S_A$ and $Y_B \setminus S_B$ are equipollent. Let $j : S_A \to f^{-1}(H_{f^{-1}})$, $i : f(H_f) \to S_B$ and $h : X_A \setminus S_A \to Y_B \setminus S_B$ be arbitrary bijections. If a set is empty, then we can consider the empty mapping. Now we can define a bijection $g : X' \to Y'$ as follows

$$g(x) = \begin{cases} f(x), & x \in \mathrm{Supp}(A) \setminus H_f, \\ i \circ f(x), & x \in H_f, \\ f \circ j(x), & x \in S_A, \\ h(x), & x \in X_A \setminus S_A. \end{cases} \tag{2.19}$$

Then we have for $\theta = \wedge$

$$\begin{aligned}\bigwedge_{x\in Z_A} (A(x) \leftrightarrow B(f(x))) &= \bigwedge_{x\in \mathrm{Supp}(A)\setminus H_f} (A(x) \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in H_f} (A(x) \leftrightarrow \bot) \wedge \\ &\bigwedge_{x\in f^{-1}(H_{f^{-1}})} (\bot \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in Z_A\setminus(\mathrm{Supp}(A)\cup f^{-1}(H_{f^{-1}}))} (\bot \leftrightarrow \bot) = \\ &\bigwedge_{x\in \mathrm{Supp}(A)\setminus H_f} (A(x) \leftrightarrow B(f(x))) \wedge \bigwedge_{x\in H_f} (A(x) \leftrightarrow B(i(f(x)))) \wedge \\ &\bigwedge_{x\in S_A} (A(x) \leftrightarrow B(j(f(x)))) \wedge \bigwedge_{x\in X_A\setminus S_A} (\bot \leftrightarrow \bot) = \bigwedge_{x\in X'} (A(x) \leftrightarrow B(g(x)))\end{aligned}$$

and analogously for $\theta = \otimes$ we obtain

$$\bigwedge_{Z\in \mathrm{Fin}(Z_A)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x))) = \bigwedge_{Z\in \mathrm{Fin}(X')} \bigotimes_{x\in Z} A(x) \leftrightarrow B(g(x))).$$

We have shown that for each $f \in \mathrm{Bij}(Z_A, Z_B)$ there exists $g \in \mathrm{Bij}(X', Y')$ defined by (2.19) such that $\bigwedge_{x\in Z_A} A(x) \leftrightarrow B(f(x)) = \bigwedge_{x\in X'} A(x) \leftrightarrow B(g(x))$ and $\bigwedge_{Z\in \mathrm{Fin}(Z_A)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x)) = \bigotimes_{x\in X'} A(x) \leftrightarrow B(g(x))$. Hence, we have

$$\bigvee_{f\in \mathrm{Bij}(Z_A,Z_B)} \bigwedge_{x\in Z_A} A(x) \leftrightarrow B(f(x)) = \bigvee_{g\in \mathrm{Bij}(X',Y')} \bigwedge_{x\in X'} A(x) \leftrightarrow B(g(x)),$$

$$\bigvee_{f\in \mathrm{Bij}(Z_A,Z_B)} \bigwedge_{Z\in \mathrm{Fin}(Z_A)} \bigotimes_{x\in Z} A(x) \leftrightarrow B(f(x)) = \bigvee_{g\in \mathrm{Bij}(X',Y')} \bigotimes_{x\in X'} A(x) \leftrightarrow B(g(x)).$$

From (2.15) and (2.16) we immediately obtain the desired statement and the proof is complete. □

The previous proof shows a possibility how to define the considered sets X' and Y' for denumerable **L**-sets as follows.

Corollary 2.2.3. *Let X be an uncountable universe, $A \in \mathcal{FC}_{\mathbf{L}}(X)$ and $B \in \mathcal{FC}_{\mathbf{L}}(Y)$ be denumerable **L**-sets, $X_A \subset X \setminus \mathrm{Supp}(A)$ and $Y_B \subset Y \setminus \mathrm{Supp}(B)$ be arbitrary denumerable sets. If we put $X' = \mathrm{Supp}(A) \cup X_A$ and $Y' = \mathrm{Supp}(B) \cup Y_B$, then the value*

$$a = \bigvee_{f \in \mathrm{Bij}(X',Y')} \bigominus_{x \in X'} A(x) \leftrightarrow B(f(x)) \tag{2.20}$$

is the θ-degree of equipollence of A and B.

As we have mentioned, in the classical set theory the equipollence of sets is an equivalence relation on the class of all sets. Recall that the relation $\equiv$ on the class of all sets $\mathbb{X}$, defined by $(X,Y) \in \equiv$ (denoted better by $X \equiv Y$) if and only if X and Y are equipollent, is an equivalence $\equiv$ on the class $\mathbb{X}$. Now in a similar way we will show that the equipollence of **L**-sets is actually a similarity relation on the class of all **L**-sets over a given universe, which is a natural extension of the classical notion of equivalence. Before we show it, let us recall the notion of similarity relation. Let **L** be a complete residuated lattice and X be a non-empty universe. A mapping $R : \mathcal{F}_{\mathbf{L}}(X) \times \mathcal{F}_{\mathbf{L}}(X) \to L$ is called the *fuzzy binary relation on* $\mathcal{F}_{\mathbf{L}}(X)$ on $\mathcal{F}_{\mathbf{L}}(X)$. Further, a fuzzy binary relation R on $\mathcal{F}_{\mathbf{L}}(X)$ is called the *similarity relation* or *similarity* for short on $\mathcal{F}_{\mathbf{L}}(X)$, if it satisfies the following axioms

$$R(A,A) = 1, \tag{\textit{reflexivity}}$$
$$R(A,B) = R(B,A), \tag{\textit{symmetry}}$$
$$R(A,B) \otimes R(B,C) \le R(A,C). \tag{\textit{thansitivity}}$$

In the literature (see e.g. [2, 5, 56, 57, 120]), we may find other names for the similarity relation as e.g. the likeness, **L**-equivalence or $\otimes$-equivalence. Note that the similarity relation is a very important notion in the fuzzy set theory. A simple example of the similarity relation between two **L**-sets A and B in X could be defined by

$$R(A,B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)), \tag{2.21}$$

where the value $R(A,B)$ shows how much A is similar to B. Obviously, this similarity relation could be obtained by the special case of an $a^{\wedge}$-bijection between **L**-sets on X, where the identity mapping is considered. Thus, the following theorems show how to define a similarity relation between **L**-sets from the **L**-sets equipollence point of view.

Theorem 2.2.4. *Let X be a non-empty universe. Then a fuzzy binary relation $\equiv^\wedge$ on the set $\mathcal{F}_{\mathbf{L}}(X)$, where $A \equiv^\wedge B = a$ if and only if A and B are $a^\wedge$-equipollent, is a similarity relation on $\mathcal{F}_{\mathbf{L}}(X)$.*

Proof. We have to verify the axioms of similarity relation. It is easy to see that the relation $\equiv^\wedge$ is reflexive and symmetric. Let us denote $\mathrm{Perm}(X) = \{f : X \to X \mid f \text{ is a bijection}\}$. Obviously, the formula (2.10) may be rewritten as follows

$$A \equiv^\wedge B = \bigvee_{f \in \mathrm{Perm}(X)} \bigwedge_{x \in X} A(x) \leftrightarrow B(f(x)). \tag{2.22}$$

Let $A, B, C \in \mathcal{F}_{\mathbf{L}}(X)$ be arbitrary $\mathbf{L}$-sets. Then we have

$$\begin{aligned}
&(A \equiv^\wedge B) \otimes (B \equiv^\wedge C) = \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigwedge_{x \in X} (A(x) \leftrightarrow B(f(x))) \otimes \bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{y \in X} (B(y) \leftrightarrow C(g(y))) = \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{x \in X} \bigwedge_{y \in X} \big((A(x) \leftrightarrow B(f(x))) \otimes (B(y) \leftrightarrow C(g(y)))\big) \leq \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{x \in X} \bigwedge_{y \in X} \big((A(x) \leftrightarrow B(f(x))) \otimes (B(y) \leftrightarrow C(g(y)))\big) \leq \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{x \in X} (A(x) \leftrightarrow C(g \circ f(x))) = \\
&\qquad \bigvee_{h \in \mathrm{Perm}(X)} \bigwedge_{x \in X} (A(x) \leftrightarrow C(h(x))) = A \equiv^\wedge C
\end{aligned}$$

and thus the fuzzy relation $\equiv^\wedge$ is also transitive. □

Theorem 2.2.5. *Let X be a non-empty universe. Then a fuzzy binary relation $\equiv^\otimes$ on the set $\mathcal{FC}_{\mathbf{L}}(X)$, where $A \equiv^\otimes B = a$ if and only if A and B are $a^\otimes$-equipollent, is a similarity relation on $\mathcal{FC}_{\mathbf{L}}(X)$.*

Proof. Obviously, the fuzzy relation $\equiv^\otimes$ is reflexive, symmetric and the formula (2.11) may be rewritten as follows

$$A \equiv^\otimes B = \bigvee_{f \in \mathrm{Perm}(X)} \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{x \in Z} (A(x) \leftrightarrow B(f(x))). \tag{2.23}$$

Let $A, B, C \in \mathcal{FC}_{\mathbf{L}}(X)$ be arbitrary countable **L**-sets. Then we have

$$
\begin{aligned}
&(A \equiv^{\otimes} B) \otimes (B \equiv^{\otimes} C) = \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigwedge_{Y \in \mathrm{Fin}(X)} \bigotimes_{y \in Y} (A(y) \leftrightarrow B(f(y))) \otimes \\
&\bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{z \in Z} (B(z) \leftrightarrow C(g(z))) \leq \\
&\bigvee_{f \in \mathrm{Perm}(X)} \bigvee_{g \in \mathrm{Perm}(X)} \bigwedge_{Y \in \mathrm{Fin}(X)} \bigwedge_{Z \in \mathrm{Fin}(X)} \bigotimes_{y \in Y} (A(y) \leftrightarrow B(f(y))) \otimes \\
&\bigotimes_{z \in Z} (B(z) \leftrightarrow C(g(z))) \leq \\
&\bigvee_{f,g \in \mathrm{Perm}(X)} \bigwedge_{Y \in \mathrm{Fin}(X)} \bigotimes_{y \in Y} \big(A(y) \leftrightarrow B(f(y)) \otimes B(f(y)) \leftrightarrow C(g \circ f(y))\big) = \\
&\bigvee_{h \in \mathrm{Perm}(X)} \bigwedge_{Y \in \mathrm{Fin}(X)} \bigotimes_{y \in Y} (A(y) \leftrightarrow C(h(y))) = A \equiv^{\otimes} C
\end{aligned}
$$

and thus the fuzzy relation $\equiv^{\otimes}$ is also transitive. □

Remark 2.2.3. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be two finite **L**-sets and Y be an arbitrary subset of X such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq Y$. Due to Theorem 2.2.1, we can write

$$
A \equiv^{\theta} B = \bigvee_{f \in \mathrm{Perm}(Y)} \bigominus_{x \in Y} (A(x) \leftrightarrow B(f(x))). \tag{2.24}
$$

Due to Theorem 2.2.2, if $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$ are countable **L**-sets, then there exists $Y \subseteq X$ such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq Y$ and (2.24) is satisfied.

Remark 2.2.4. If $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$, then the $\wedge$-degree is greater or equal to the $\otimes$-degree of equipollence of A and B, i.e. $A \equiv^{\wedge} B \geq A \equiv^{\otimes} B$. In fact, it follows from the inequality $\bigwedge_{y \in Y} A(y) \leftrightarrow B(f(y)) \geq \bigotimes_{y \in Y} A(y) \leftrightarrow B(f(y))$, where Y is a suitable countable set covering $\mathrm{Supp}(A) \cup \mathrm{Supp}(B)$ and $f \in \mathrm{Perm}(Y)$.

Remark 2.2.5. If X is a countable universe, then $\overline{A} \equiv^{\theta} \overline{B} \geq A \equiv^{\theta} B$ holds for arbitrary $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$, where $\overline{A}$ and $\overline{B}$ are the complements of **L**-sets A and B, respectively. In fact, if $f \in \mathrm{Perm}(X)$, then $\bigominus_{x \in X}(\overline{A}(x) \leftrightarrow \overline{B}(f(x))) = \bigominus_{x \in X}((A(x) \to \bot) \leftrightarrow (B(f(x)) \to \bot)) \geq \bigominus_{x \in X}((A(x) \leftrightarrow B(f(x))) \otimes (\bot \leftrightarrow \bot)) = \bigominus_{x \in X}((A(x) \leftrightarrow B(f(x)))$. Hence, we obtain $\overline{A} \equiv^{\theta} \overline{B} = \bigvee_{f \in \mathrm{Perm}(X)} \bigominus_{x \in X}(\overline{A}(x) \leftrightarrow \overline{B}(f(x))) \geq \bigvee_{f \in \mathrm{Perm}(X)} \bigominus_{x \in X}(A(x) \leftrightarrow B(f(x))) = A \equiv^{\theta} B$.

In this part we introduce special mappings (**L**-sets over the set of natural numbers extended by the first infinite cardinal) that will be needed in the following chapter being devoted to the cardinalities of fuzzy sets. Note that these mappings are used for the construction of fuzzy cardinality so-called *FGCount* (see e.g. [8,20,81,104,108,109,125], or the introduction of the next chapter). Further, we show that these mappings enable to give upper estimations of the θ-degrees of equipollence of **L**-sets and using these mappings we may even precisely calculate the $\wedge$-degree of the finite **L**-sets equipollence over complete linear residuated lattices. In general, it is easy to see that the calculation of the θ-degrees of **L**-sets equipollence is often time-consuming, even if quite small **L**-sets[6] are supposed, therefore, the mentioned attempts are very useful from the practical point of view.

Let X be a non-empty universe, $A \in \mathcal{FC}_{\mathbf{L}}(X)$. Let us denote N_ω the set of all natural numbers N extended by the first infinite cardinal ω (the cardinality of the natural numbers set) and $A_X = \{Y \subseteq X \mid \mathrm{Supp}(A) \subseteq Y\}$. We define a mapping $p^\theta_A : N_\omega \times A_X \to L$ as follows

$$p^\theta_A(i, Y) = \bigvee_{\substack{Z \subseteq Y \\ |Z| = i}} \bigominus_{z \in Z} A(z), \tag{2.25}$$

where $|Z|$ denotes the classical cardinality of the set Z. It is easy to see that $p^\theta_A(i, Y) = p^\theta_A(i, \mathrm{Supp}(A))$ for arbitrary $Y \in A_X$ and $i \in N_\omega$. In fact, if $Y \in A_X$ and $y \in Y \setminus \mathrm{Supp}(A)$, then $\bigominus_{x \in Y} A(x) = \bot$. Hence, we have the required equality[7]. Therefore, we put $p^\theta_A(i, \mathrm{Supp}(A)) = p^\theta_A(i)$ for short[8]. Obviously, $p^\theta_A(i) \geq p^\theta_A(j)$, whenever $i \leq j$, $p^\theta_A(0) = \bigominus_{z \in \emptyset} A(z) = \top$ and $p^\theta_A(1) = \bigvee\{A(x) \mid x \in \mathrm{Supp}(A)\}$. Moreover, if A is a finite **L**-set, then $p^\theta_A(i) = \bigvee \emptyset = \bot$ holds for every $i > |\mathrm{Supp}(A)|$. Note that the value $p^\theta_A(1)$ is called the *height* of the fuzzy set A. If we restrict ourselves to linearly ordered complete residuated lattices **L** and finite **L**-sets, then we can compute the values $p^\otimes_A(i)$ using the values $p^\wedge_A(i)$ as the following lemma shows.

Lemma 2.2.6. *Let* **L** *be a complete linearly ordered residuated lattice. Then*

$$p^\otimes_A(i) = \bigotimes_{k=0}^{i} p^\wedge_A(k) \tag{2.26}$$

[6] In this case we assume that the size of **L**-sets is only bound to the size (the cardinality) of their supports.

[7] We chose this definition with respect to the compatibility with the definition of the dual mapping $p^{\bar\theta}_A$ (see p. 43).

[8] Note that in the mentioned papers there is considered only the case $p^\wedge_A(i)$ which is denoted by $[A]_i$.

holds for any $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ *and* $i \in N_\omega$.

Proof. Let us suppose that A is a finite **L**-set over X and $\mathrm{Supp}(A) = \{x_1, \dots, x_n\}$. Due to the linearity of **L**, to each $0 < i \leq n$ there exists $x_{j_i} \in \mathrm{Supp}(A)$ such that $p_A^\wedge(i) = A(x_{j_i})$. Moreover, we have $p_A^\wedge(0) = \top$ and $p_A^\wedge(i) = \bot$ for any $i > |\mathrm{Supp}(A)|$. Since $A(x_{k_j}) \geq A(x_{k_{j+1}})$ holds for any $1 \leq j < n-1$, we can write for any $i \in N_\omega$

$$p_A^\otimes(i) = \bigvee_{\substack{Z \subseteq \mathrm{Supp}(A)\\ |Z|=i}} \bigotimes_{z\in Z} A(z) = \bigotimes_{k=0}^{i} A(x_{j_k}) = \bigotimes_{k=0}^{i} p_A^\wedge(k),$$

where clearly $p_A^\otimes(0) = p_A^\wedge(0)$, $p_A^\otimes(i) = (\bigotimes_{k=0}^{n} p_A^\wedge(k)) \otimes \bot = \bot$ hold for any $|\mathrm{Supp}(A)| < i < \omega$ and $p_A^\otimes(\omega) = \bigotimes_{k=0}^{\infty} p_A^\wedge(k) = \bigwedge_{t=0}^{\infty} \bigotimes_{k=0}^{t} p_A^\wedge(k) = \bot$. □

There is an interesting question, if the value $p_A^\theta(\omega)$ can be obtained by infimum of values $p_A^\theta(i)$, it means that the sequence $p_A^\theta(0), p_A^\theta(1), \dots$ converges to $p_A^\theta(\omega)$. Before we give an answer in the following lemma, we establish an important notion. Let **L** be a complete residuated lattice. We say that the complete residuated lattice **L** is *dense* with regard to the corresponding lattice ordering $\leq$, if for arbitrary $a, b \in L$ such that $a < b$ there exists $c \in L$ with $a < c < b$. Let us emphasize that the concept of a dense complete residuated lattice does not depend on its residuation, neither on its completeness, but only on its order $\leq$ defined by the lattice structure $(L, \wedge, \vee)$. For instance, the residuated lattices over $[0, 1]$ determined by the left continuous t-norms are dense with respect to the ordering of real numbers (the corresponding lattice ordering of the mentioned residuated lattices).

Lemma 2.2.7. *Let* **L** *be a complete linearly ordered residuated lattice that is dense with regard to* $\leq$. *Then* $p_A^\wedge(\omega) = \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$ *holds for any* $A \in \mathcal{FC}_{\mathbf{L}}(X)$.

Proof. If A is a finite **L**-set, then there exists i_0 such that $p_A^\wedge(i) = \bot$ for every $i > i_0$ and hence $p_A^\wedge(\omega) = \bot = \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$. Let us suppose that A is a denumerable **L**-set. Obviously, we have $p_A^\wedge(\omega) \leq p_A^\wedge(i)$ for every $i \in N$ and thus $p_A^\wedge(\omega) \leq \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$ is true. Before we prove the opposite inequality, we state the following claim. If $a < p_A^\wedge(i)$ holds for every $i \in N$, then $a \leq p_A^\wedge(\omega)$. From the presumption of the claim and the linearity of **L** it follows that for every $i \in N$ there exists a suitable subset $Y_i \subset \mathrm{Supp}(A)$ such that $a < \bigwedge_{y \in Y_i} A(y)$. Further, we establish $Y = \bigcup_{i \in N} Y_i$. Obviously, $|Y| = \omega$. Hence, we have $a \leq \bigwedge_{i=1}^{\infty} \bigwedge_{y\in Y_i} A(y) = \bigwedge_{y \in Y} A(y) \leq p_A^\wedge(\omega)$. Hence, we obtain the desired claim. Now let us suppose that $p_A^\wedge(\omega) < \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$ holds. Since **L** is dense with regard to $\leq$, then there exists $a \in L$ such that $p_A^\wedge(\omega) < a < \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$ and hence, due to the claim, we obtain a contradiction. Thus, $p_A^\wedge(\omega) = \bigwedge_{i=0}^{\infty} p_A^\wedge(i)$ and the proof is complete. □

The following lemma shows a relation between the θ-degree of the equipollence of **L**-sets A and B and the values of the mappings $p^\theta_A(i)$ and $p^\theta_B(i)$. In particular, there is defined upper estimations of the θ-equipollence degrees.

Lemma 2.2.8. *Let X be a non-empty universe. Then*

$$A \equiv^\theta B \leq p^\theta_A(i) \leftrightarrow p^\theta_B(i), \tag{2.27}$$

holds for arbitrary $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$ and $i \in N_\omega$.

Proof. Let $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$ be arbitrary **L**-sets. Let $f \in \mathrm{Perm}(X)$ be an arbitrary bijection (permutation) on the universe X. Then we can write (in the more general form) for any $i \in N_\omega$

$$p^\theta_A(i) \leftrightarrow p^\theta_B(i) = p^\theta_A(i, X) \leftrightarrow p^\theta_B(i, X) =$$
$$\Big(\bigvee_{\substack{Y\subseteq X\\ |Y|=i}} \bigominus_{y\in Y} A(y)\Big) \leftrightarrow \Big(\bigvee_{\substack{Z\subseteq X\\ |Z|=i}} \bigominus_{z\in Z} B(z)\Big) \geq$$
$$\bigwedge_{\substack{Y\subseteq X\\ |Y|=i}} \Big(\bigominus_{y\in Y} A(y) \leftrightarrow \bigominus_{z\in f(Y)} B(z)\Big) \geq \bigwedge_{\substack{Y\subseteq X\\ |Y|=i}} \bigominus_{y\in Y} (A(y) \leftrightarrow B(f(y))).$$

First, we suppose that $\theta = \wedge$. Then we have

$$p^\wedge_A(i) \leftrightarrow p^\wedge_B(i) \geq \bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{\substack{Y\subseteq X\\ |Y|=i}} \bigwedge_{y\in Y} (A(y) \leftrightarrow B(f(y))) \geq$$
$$\bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{y\in X} (A(y) \leftrightarrow B(f(y))) = A \equiv^\wedge B.$$

Further, we suppose that $\theta = \otimes$. Then, analogously, we can write

$$p^\otimes_A(i) \leftrightarrow p^\otimes_B(i) \geq \bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{\substack{Y\subseteq X\\ |Y|=i}} \bigotimes_{y\in Y} (A(y) \leftrightarrow B(f(y))) \geq$$
$$\bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{Y\in\mathrm{Fin}(X)} \bigotimes_{y\in Y} (A(y) \leftrightarrow B(f(y))) = A \equiv^\otimes B,$$

and the proof is complete. □

Let us denote p^θ_A, p^θ_B the corresponding **L**-sets over N_ω for countable **L**-sets $A, B \in \mathcal{FC}_{\mathbf{L}}(X)$. Further, we put $p^\theta_A \approx p^\theta_B = \bigwedge\{p^\theta_A(i) \leftrightarrow p^\theta_B(i) \mid i \in$

$N_\omega\}$. It is easy to verify that $\approx$ is a similarity relation on the set of all p^θ_A. Now we can formulate a consequence of the previous lemma as follows

$$A \equiv^\theta B \le p^\theta_A \approx p^\theta_B. \tag{2.28}$$

As we have mentioned, the previous inequality gives us, in the general case, just an upper estimation of the θ-degrees of the equipollence of **L**-sets. The following example shows that these estimations can be sometimes very poor.

Example 2.2.6. Let us suppose that $\mathbf{L}_\mathrm{M}$ is the Gödel algebra, defined in Ex. 1.1.3, and $A, B \in \mathcal{FC}_{\mathbf{L}_\mathrm{M}}(N)$ are $\mathbf{L}_\mathrm{M}$-sets, defined by $A(n) = 1 - \frac{1}{n}$ and $B(n) = 1$ for every $n \in N$. Recall, that $a \leftrightarrow_\mathrm{M} b = \min\{a, b\}$. Then we have

$$A \equiv^\wedge B = \bigvee_{f \in \mathrm{Perm}(N)} \bigwedge_{n=1}^{\infty} A(n) \leftrightarrow_\mathrm{M} B(f(n)) = \bigwedge_{n=1}^{\infty} A(n) = A(1) = 0.$$

However, for an arbitrary $i \in N$ we can state $p^\wedge_B(i) = 1$ and

$$p^\wedge_A(i) = \bigvee_{\substack{M \subset N \\ |M|=i}} \bigwedge_{n \in M} A(n) = \bigvee_{\substack{M \subset N \\ |M|=i}} A(\max(M)) = 1 - \bigwedge_{\substack{M \subset N \\ |M|=i}} \frac{1}{\max(M)} = 1.$$

Hence, we obtain that $p^\wedge_A \approx p^\wedge_B = \bigwedge_{i=1}^{\infty} p^\wedge_A(i) \leftrightarrow p^\wedge_B(i) = 1$ holds and thus $A \equiv^\wedge B < p^\wedge_A \approx p^\wedge_B$.

Obviously, the cause of a failure is in the application of supremum on the countable **L**-set A (in the calculation of the value $p^\wedge_A(i)$). If we restrict to the finite **L**-sets, then we can dispose of the mentioned failure for the case $\theta = \wedge$ as the following lemma shows.

Lemma 2.2.9. *Let* **L** *be a complete linearly ordered residuated lattice and* X *be a nonempty universe. Then we have*

$$A \equiv^\wedge B = p^\wedge_A \approx p^\wedge_B \tag{2.29}$$

for arbitrary $A, B \in \mathcal{FIN}_\mathbf{L}(X)$.

Proof. Due to Lemma 2.2.8, it is sufficient to prove that

$$A \equiv^\wedge B \ge \bigwedge_{i=0}^{\infty} p^\wedge_A(i) \leftrightarrow p^\wedge_B(i).$$

Let us denote $Z = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$ and suppose $|Z| = n$. If $n = 0$ then clearly the equality is true. Let us suppose that $n > 0$. Then from the linearity of **L** and the finiteness of A and B there exist two finite sequences

$\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ of the elements from Z such that $A(u_i) \geq A(u_{i+1})$ and $B(v_i) \geq B(v_{i+1})$ hold for every $i = 1, \ldots, n-1$. It is easy to see that $p_A^\wedge(i) = A(u_i)$ and $p_B^\wedge(i) = B(v_i)$. Moreover, clearly $p_A^\wedge(k) = p_B^\wedge(k) = 0$ holds for every $k > n$. Now let us define a bijection $g : X \to X$ as follows

$$g(x) = \begin{cases} v_i, & x = u_i \\ x, & \text{otherwise.} \end{cases} \tag{2.30}$$

Then we have

$$A \equiv^\wedge B = \bigvee_{f \in \mathrm{Perm}(X)} \bigwedge_{x \in X} A(x) \leftrightarrow B(f(x)) \geq \bigwedge_{x \in X} A(x) \leftrightarrow B(g(x)) =$$
$$\bigwedge_{i=1}^{n} A(u_i) \leftrightarrow B(v_i) = \bigwedge_{i=1}^{n} p_A^\wedge(i) \leftrightarrow p_B^\wedge(i) = \bigwedge_{i=0}^{\infty} p_A^\wedge(i) \leftrightarrow p_B^\wedge(i) = p_A^\wedge \approx p_B^\wedge$$

and the proof is complete. □

Example 2.2.7. Let us suppose that $A = \{0.3/x_1, 0.8/x_2, 0.4/x_3, 0/x_4, 0/x_5\}$ and $B = \{0/x_1, 0.6/x_2, 0.9/x_3, 0/x_4, 0.7/x_5\}$, where the membership degrees are interpreted in the Łukasiewicz algebra. In order to find the $\wedge$-degree of bijectivity of fuzzy sets A and B, we introduce first the corresponding mappings $p_A, p_B : N \to [0, 1]$. It is easy to see, that $p_A^\wedge$, $p_B^\wedge$ are the non-increasing mappings, where $p_A(0) = p_B(0) = 1$, $p_A^\wedge(1)$ and $p_B^\wedge(1)$ are the greatest membership degrees for fuzzy sets A and B, $p_A^\wedge(2)$ and $p_B^\wedge(2)$ are the second greatest membership degrees for fuzzy set A and B etc., respectively. Particulary, we may write the following sequences

$$p_A = \{1, 0.8, 0.4, 0.3, 0, 0, \ldots\},$$
$$p_B = \{1, 0.9, 0.7, 0.6, 0, 0, \ldots\}.$$

Using Lemma 2.2.9 we obtain $A \equiv^\wedge B = 0.7$.

2.3 $\overline{\theta}$-equipollent $\mathbf{L}^{\mathrm{d}}$-sets

Let $\mathbf{L}^{\mathrm{d}}$ be a complete dually residuated lattice and $A \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $B \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be arbitrary $\mathbf{L}^{\mathrm{d}}$-sets (again we will suppose that there exists a bijection between X and Y). We say that A is $a^\vee$*-equipollent* with B (or A and B are $a^\vee$*-equipollent*), if

$$a = \bigwedge \{b \mid \exists f : X \to Y \, and \; f \; is \; a \; b^\vee\text{-}bijection \; between \; A \; and \; B\}. \tag{2.31}$$

Further, let $A \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$, $B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be arbitrary countable $\mathbf{L}^{\mathrm{d}}$-sets. We say that A is $a^{\oplus}$-*equipollent* with B (or A and B are $a^{\oplus}$-*equipollent*), if

$$a = \bigwedge\{b \mid \exists f : X \to Y \textit{ and } f \textit{ is a } b^{\oplus}\textit{-bijection between } A \textit{ and } B\}. \quad (2.32)$$

The value a is called the $\overline{\theta}$-*degree of (fuzzy) equipollence* of A and B. The following theorems are dual to Theorems 2.2.1 and 2.2.2. Recall that A_B and B_A are subsets of X and Y, respectively, such that $\mathrm{Supp}(A) \subseteq A_B$, $\mathrm{Supp}(B) \subseteq B_A$ and $|A_B| = |B_A| = \max(|\mathrm{Supp}(A)|, |\mathrm{Supp}(B)|)$.

Theorem 2.3.1. *Let $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(Y)$ be two finite $\mathbf{L}^{\mathrm{d}}$-sets (X and Y are bijective). Then A is a^{θ}-equipollent with B if and only if the $\overline{\theta}$-degree of equipollency of the $\mathbf{L}^{\mathrm{d}}$-sets A and B is*

$$a = \bigwedge_{f \in \mathrm{Bij}(A_B, B_A)} \overline{\bigominus_{x \in A_B}} |A(x) \ominus B(f(x))|. \quad (2.33)$$

Proof. Analogously to the proof of Theorem 2.2.1, we can establish the set $H_f = \{x \in A_B \mid f(x) \notin B_A\}$ for every $f \in \mathrm{Bij}(A_B, B_A)$. If $H_f = \emptyset$, then we have for $\overline{\theta} = \vee$

$$\bigvee_{x \in X} |A(x) \ominus B(f(x))| = \bigvee_{x \in A_B} |A(x) \ominus B(f(x))| \vee \bigvee_{x \in X \setminus A_B} |\bot \ominus \bot| = \bigvee_{x \in A_B} |A(x) \ominus B(f(x))|$$

and for $\overline{\theta} = \oplus$ we obtain by analogy

$$\bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(f(x))| = \bigvee_{Z \in \mathrm{Fin}(A_B)} \bigoplus_{x \in Z} |A(x) \ominus B(f(x))|.$$

If $H_f \neq \emptyset$ and $H = f^{-1}(B_A) \setminus A_B$, then we define $g : X \to Y$ by (2.13), where $g(H_f) = f(H)$ and $g(H) = f(H_f)$. Hence, we obtain that $H_g = \emptyset$. Finally, we have for $\overline{\theta} = \vee$

$$\bigvee_{x \in X} |A(x) \ominus B(f(x))| = \bigvee_{x \in X \setminus H_f \cup H} |A(x) \ominus B(f(x))| \vee \bigvee_{x \in H_f} |A(x) \ominus \bot| \vee \bigvee_{x \in H} |\bot \ominus B(f(x))| = \bigvee_{x \in X \setminus H_f \cup H} |A(x) \ominus B(f(x))| \vee \bigvee_{x \in H_f} (|A(x) \ominus \bot| \vee |\bot \ominus B(f(h(x)))|) \geq \bigvee_{x \in X \setminus H_f \cup H} |A(x) \ominus B(f(x))| \vee \bigvee_{x \in H_f} |A(x) \ominus B(g(x))| \vee \bigvee_{x \in H} |\bot \ominus \bot| = \bigvee_{x \in X} |A(x) \ominus B(g(x))|,$$

where $|A(x) \ominus B(g(x))| = |\bot \ominus \bot| = \bot$ holds for any $x \in H$. By analogy, we obtain for $\overline{\theta} = \otimes$

$$\bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(f(x))| \geq \bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(g(x))|.$$

Hence, to each bijection $f \in \mathrm{Bij}(X, Y)$ there exists a bijection $g \in \mathrm{Bij}(X, Y)$ such that $\bigvee_{x \in X} |A(x) \ominus B(f(x))| \geq \bigvee_{x \in X} |A(x) \ominus B(g(x))|$ or analogously $\bigvee_{z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(f(x))| \geq \bigvee_{z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(g(x))|$ and moreover $H_g = \emptyset$, i.e. $g|_{A_B} \in \mathrm{Bij}(A_B, B_A)$. Then, due to the previous results for $H_g = \emptyset$, we obtain the desired statement and the proof is complete. □

Theorem 2.3.2. *Let* $A \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ *and* $B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(Y)$ *be two at most countable sets* $\mathbf{L}^{\mathrm{d}}$*-sets (X and Y are bijective). Then A is* $a^{\overline{\theta}}$*-equipollent with B if and only if there exist at most countable subsets* $X' \subseteq X$ *and* $Y' \subseteq Y$ *covering* $\mathrm{Supp}(A)$ *and* $\mathrm{Supp}(B)$ *such that they are equipollent and*

$$a = \bigwedge_{f \in \mathrm{Bij}(X', Y')} \overline{\bigominus_{x \in X'}} |A(x) \ominus B(f(x))|, \tag{2.34}$$

where $\mathrm{Bij}(X', Y')$ *denotes the set of all bijections between* X' *and* Y'*.*

Proof. It could be done by analogy to the proof of Theorem 2.2.2. □

Contrary to the θ-equipollence of **L**-sets, which is closely connected with a similarity relation on the class of all **L**-sets, an equipollence of $\mathbf{L}^{\mathrm{d}}$-sets enables us to introduce a fuzzy pseudo-metric on the class of $\mathbf{L}^{\mathrm{d}}$-sets over a given universe. Let $\mathbf{L}^{\mathrm{d}}$ be a complete dual residuated lattice. A fuzzy binary relation d on $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ is called the *fuzzy pseudo-metric* ($\mathbf{L}^{\mathrm{d}}$*-pseudo-metric* or $\oplus$*-pseudo-metric*) on $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$, if it satisfies the following axioms

$$d(A, A) = 0, \qquad (\textit{reflexivity})$$

$$d(A, B) = d(B, A), \qquad (\textit{symmetry})$$

$$d(A, B) \oplus d(B, C) \geq d(A, C). \qquad (\textit{tiangular inequality})$$

The following theorems show how to define a fuzzy pseudo metric on the set of all $\mathbf{L}^{\mathrm{d}}$-subsets from the $\mathbf{L}^{\mathrm{d}}$-sets equipollence point of view.

Theorem 2.3.3. *Let X be a nonempty universe. Then a fuzzy binary relation* $\equiv^{\vee}$ *on* $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$*, where* $A \equiv^{\vee} B = a$ *if and only if A and B are* $a^{\vee}$*-equipollent, is a fuzzy pseudo-metric on* $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$*.*

Proof. It is easy to see that the relation $\equiv^{\vee}$ is reflexive and symmetric. Obviously, the formula (2.31) may be rewritten as follows

$$A \equiv^{\vee} B = \bigwedge_{f \in \mathrm{Perm}(X)} \bigvee_{x \in X} |A(x) \ominus B(f(x))|. \tag{2.35}$$

Let $A, B, C \in \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary $\mathbf{L}^{\mathrm{d}}$-sets. Then we have

$$\begin{aligned}
&(A \equiv^{\vee} B) \oplus (B \equiv^{\vee} C) = \\
&\bigwedge_{f \in Perm(X)} \bigvee_{x \in X} |A(x) \ominus B(f(x))| \oplus \bigwedge_{g \in \mathrm{Perm}(X)} \bigvee_{y \in X} |A(y) \ominus B(g(y))| \geq \\
&\bigwedge_{f \in \mathrm{Perm}(X)} \bigwedge_{g \in \mathrm{Perm}(X)} \bigvee_{x \in X} \bigvee_{y \in X} |A(x) \ominus B(f(x))| \oplus |B(y) \ominus C(g(y))| \geq \\
&\bigwedge_{f \in \mathrm{Perm}(X)} \bigwedge_{g \in \mathrm{Perm}(X)} \bigvee_{x \in X} |A(x) \ominus B(f(x))| \oplus |B(f(x)) \ominus C(g(f(x)))| \geq \\
&\bigwedge_{f \in \mathrm{Perm}(X)} \bigwedge_{g \in \mathrm{Perm}(X)} \bigvee_{x \in X} |A(x) \ominus C(g \circ f(x))| = \\
&\bigwedge_{h \in \mathrm{Perm}(X)} |A(x) \ominus C(h(x))| = A \equiv^{\vee} C
\end{aligned}$$

and thus the triangular inequality is also satisfied. □

Theorem 2.3.4. *Let X be a nonempty universe. Then a fuzzy binary relation $\equiv^{\oplus}$ on $\mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$, where $A \equiv^{\oplus} B = a$ if and only if A and B are $a^{\oplus}$-equipollent, is a fuzzy pseudo-metric on $\mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$.*

Proof. It is analogical to the proof of the previous theorem and dual to the proof of Theorem 2.2.5. □

Again, the formula (2.32) may be rewritten as follows

$$A \equiv^{\oplus} B = \bigwedge_{f \in \mathrm{Perm}(X)} \bigvee_{Z \in \mathrm{Fin}(X)} \bigoplus_{x \in Z} |A(x) \ominus B(f(x))|. \tag{2.36}$$

Remark 2.3.1. Let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be two finite $\mathbf{L}^{\mathrm{d}}$-sets and Y be an arbitrary subset of X such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq Y$. Due to Theorem 2.3.1, we can write

$$A \equiv^{\overline{\theta}} B = \bigvee_{f \in \mathrm{Perm}(Y)} \overline{\bigominus_{x \in Y}} |A(x) \ominus B(f(x))|. \tag{2.37}$$

Due to Theorem 2.3.2, if $A, B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ are countable $\mathbf{L}^{\mathrm{d}}$-sets, then there exists $Y \subseteq X$ such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(B) \subseteq Y$ holds and (2.37) is satisfied.

Remark 2.3.2. If $A, B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$, then the $\vee$-degree is less or equal to $\oplus$-degree of equipollence of A and B, i.e. $A \equiv^{\vee} B \leq A \equiv^{\oplus} B$.

Now we introduce dual mappings to the mappings $p^{\wedge}_A$ and $p^{\oplus}_A$. Let X be a universe and $A \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$. Recall that $A_X = \{Y \subseteq X \mid \mathrm{Supp}(A) \subseteq Y\}$. Let us define a mapping $p^{\overline{\theta}}_A : N_\omega \times A_X \to L$ as follows

$$p^{\overline{\theta}}_A(i, Y) = \bigwedge_{\substack{Z \subseteq Y \\ |Z|=i}} \overline{\bigominus_{z \in Z}} A(z). \tag{2.38}$$

Contrary to the mapping p^{θ}_A, we have $p^{\overline{\theta}}_A(i, Y) \neq p^{\overline{\theta}}_A(i, Y')$, in general. Obviously, the equality is satisfied for such $\mathbf{L}^{\mathrm{d}}$-sets that have the universe as their supports. Again, we put $p^{\overline{\theta}}_A(i, \mathrm{Supp}(A)) = p^{\overline{\theta}}_A(i)$. Further, we have $p^{\overline{\theta}}_A(i, Y) \geq p^{\overline{\theta}}_A(j, Y)$, whenever $i \geq j$, $p^{\overline{\theta}}_A(0, Y) = \bot$ and $p^{\overline{\theta}}_A(1, Y) = \bigwedge\{A(x) \mid x \in Y\}$. Moreover, if the support of A is finite, then $p^{\overline{\theta}}_A(i, Y) = \bigwedge \emptyset = \top$ holds for every $i > |Y|$.

Lemma 2.3.5. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered dually residuated lattice. Then*

$$p^{\oplus}_A(i) = \bigoplus_{k=0}^{i} p^{\vee}_A(k) \tag{2.39}$$

holds for any $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $i \in N_\omega$.

The following lemma is dual to Lemma 2.2.7.

Lemma 2.3.6. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered dually residuated lattice that is dense with regard to $\leq$. Then $p^{\vee}_A(\omega) = \bigvee_{i=0}^{\infty} p^{\vee}_A(i)$ holds for any $A \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$.*

Let us denote $\mathrm{Eqp}(A, B)$ the set of all sets $Y \subseteq X$ covering $\mathrm{Supp}(A) \cup \mathrm{Supp}(B)$ and satisfying (2.37). Obviously, we have $\mathrm{Eqp}(A, B) \subseteq A_X \cap B_X$. The following lemma is a dual to Lemma 2.2.8 and using them we can obtain a lower estimation of the $\overline{\theta}$-equipollence degree.

Lemma 2.3.7. *Let X be a non-empty universe. Then*

$$A \equiv^{\overline{\theta}} B \geq |p^{\overline{\theta}}_A(i, Y) \ominus p^{\overline{\theta}}_B(i, Y)| \tag{2.40}$$

holds for arbitrary $A, B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$, $i \in N_\omega$ and $Y \in \mathrm{Eqp}(A, B)$.

Proof. Let $A, B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary $\mathbf{L}^{\mathrm{d}}$-sets, $Y \in \mathrm{Eqp}(A, B)$ and $f \in \mathrm{Perm}(Y)$. Then we can write for any $i \in N_\omega$

$$|p_A^{\overline{\theta}}(i, Y) \ominus p_A^{\overline{\theta}}(i, Y)| = |\bigwedge_{\substack{Z \subseteq Y \\ |Z|=i}} \overline{\bigominus_{z \in Z}} A(z) \ominus \bigwedge_{\substack{Z' \subseteq Y \\ |Z'|=i}} \overline{\bigominus_{z' \in Z'}} A(z')| \leq$$

$$\bigvee_{\substack{Z \subseteq Y \\ |Z|=i}} |\overline{\bigominus_{z \in Z}} A(z) \ominus \overline{\bigominus_{z' \in f(Z)}} B(z')| \leq \bigvee_{\substack{Z \subseteq Y \\ |Z|=i}} \overline{\bigominus_{z \in Z}} |A(z) \ominus B(f(z))|.$$

Since $Y \in \mathrm{Eqp}(A, B)$, then we have for $\overline{\theta} = \vee$ according to (2.37)

$$|p_A^{\vee}(i, Y) \ominus p_A^{\vee}(i, Y)| \leq \bigwedge_{f \in \mathrm{Perm}(Z)} \bigvee_{\substack{Z \subseteq Y \\ |Z|=i}} \bigvee_{z \in Z} |A(z) \ominus B(f(z))| \leq$$

$$\bigwedge_{f \in \mathrm{Perm}(Z)} \bigvee_{z \in Y} |A(z) \ominus B(f(z))| = A \equiv^{\vee} B.$$

Analogously, for $\overline{\theta} = \oplus$

$$|p_A^{\oplus}(i, Y) \ominus p_A^{\oplus}(i, Y)| \leq \bigwedge_{f \in \mathrm{Perm}(Z)} \bigvee_{\substack{Z \subseteq Y \\ |Z|=i}} \bigoplus_{z \in Z} |A(z) \ominus B(f(z))| \leq$$

$$\bigwedge_{f \in \mathrm{Perm}(Z)} \bigvee_{Z \subseteq \mathrm{Fin}(Y)} \bigoplus_{z \in Z} |A(z) \ominus B(f(z))| = A \equiv^{\oplus} B$$

and the proof is complete. □

Let us denote $p_{AY}^{\overline{\theta}}$, $p_{BY}^{\overline{\theta}}$ the corresponding $\mathbf{L}^{\mathrm{d}}$-sets over N_ω^{-0} for arbitrary $A, B \in \mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $Y \in \mathrm{Eqp}(A, B)$. Further, we establish $p_{AY}^{\overline{\theta}} \approx_{\mathrm{d}} p_{BY}^{\overline{\theta}} = \bigvee\{|p_A^{\overline{\theta}}(i, Y) \ominus p_B^{\overline{\theta}}(i, Y)| \mid i \in N_\omega\}$. It is easy to verify that $\approx_{\mathrm{d}}$ is a fuzzy pseudo-metric on $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(N_\omega^{-0})$. Now we may formulate a consequence of the previous lemma as follows

$$A \equiv^{\overline{\theta}} B \geq p_{AY}^{\overline{\theta}} \approx_{\mathrm{d}} p_{BY}^{\overline{\theta}}. \tag{2.41}$$

Lemma 2.3.8. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered dually residuated lattice and X be a nonempty universe. Then we have*

$$A \equiv^{\vee} B = p_{AY}^{\vee} \approx_{\mathrm{d}} p_{BY}^{\vee} \tag{2.42}$$

for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$, where $Y = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$.

Proof. It is analogical to the proof of Lemma 2.2.9. □

2.4 Lattices homomorphisms and fuzzy sets equipollence

In this section we investigate a relation between complete homomorphisms of (dually) residuated lattice and θ- or $\overline{\theta}$-equipollence of fuzzy sets. The following lemmas are the properties consequences of homomorphisms between various types of algebras. For an illustration we will prove just one part of the second lemma. The rest could be done analogously.

Lemma 2.4.1. *Let $g : \mathbf{L}_1 \to \mathbf{L}_2$ and $h : \mathbf{L}_1^{\mathrm{d}} \to \mathbf{L}_2^{\mathrm{d}}$ be arbitrary complete homomorphisms, X be a non-empty universe. Then*

(*i*) $g(A \equiv^{\theta_1} B) = g^{\to}(A) \equiv^{\theta_2} g^{\to}(B)$, *whenever* $\equiv^{\theta_1}$ *makes sense*[9],

(*ii*) $h(A \equiv^{\overline{\theta}_1} B) = h^{\to}(A) \equiv^{\overline{\theta}_2} h^{\to}(B)$, *whenever* $\equiv^{\overline{\theta}_1}$ *makes sense,*

(*iii*) $g(p_A^{\theta_1}(i, Y)) = p_{g^{\to}(A)}^{\theta_2}(i, Y)$,

(*iv*) $h(p_A^{\overline{\theta}_1}(i, Y)) = p_{h^{\to}(A)}^{\overline{\theta}_2}(i, Y)$

hold for arbitrary $A, B \in \mathcal{F}_{\mathbf{L}}(X)$ and $A \in \mathcal{FC}_{\mathbf{L}}(X)$, $\mathrm{Supp}(A) \subseteq Y \subseteq X$ and $i \in N_\omega$, respectively.

Recall that if $g : \mathbf{L}_1 \to \mathbf{L}_2^{\mathrm{d}}$ is a homomorphism of the corresponding algebras, then $g(a \leftrightarrow b) = |g(a) \ominus g(b)|$ and dually, if $h : \mathbf{L}_1^{\mathrm{d}} \to \mathbf{L}_2$ is a homomorphism of the corresponding algebras, then $h(|a \ominus b|) = h(a) \leftrightarrow h(b)$.

Lemma 2.4.2. *Let $g : \mathbf{L}_1 \to \mathbf{L}_2^{\mathrm{d}}$ and $h : \mathbf{L}_1^{\mathrm{d}} \to \mathbf{L}_2$ be arbitrary complete homomorphisms, X be a non-empty universe. Then*

(*i*) $g(A \equiv^{\theta_1} B) = g^{\to}(A) \equiv^{\overline{\theta}_2} g^{\to}(B)$, *whenever* $\equiv^{\theta_1}$ *makes sense,*

(*ii*) $h(A \equiv^{\overline{\theta}} B) = h^{\to}(A) \equiv^{\theta} h^{\to}(B)$, *whenever* $\equiv^{\overline{\theta}}$ *makes sense,*

(*iii*) $g(p_A^{\theta}(i, Y)) = p_{g^{\to}(A) \cap Y}^{\overline{\theta}}(i, Y)$,

(*iv*) $h(p_A^{\overline{\theta}}(i, Y)) = p_{h^{\to}(A) \cap Y}^{\theta}(i, Y)$

hold for arbitrary $A, B \in \mathcal{F}_{\mathbf{L}}(X)$ and $A \in \mathcal{FC}_{\mathbf{L}}(X)$, $\mathrm{Supp}(A) \subseteq Y \subseteq X$ and $i \in N_\omega$, respectively.

[9] For $\theta = \otimes$ we must suppose that A and B are at most countable $\mathbf{L}$-sets. In the opposite case $\equiv^{\otimes}$ has no sense.

Proof. Here, we will prove just the statement (i) and (iii), the rest could be proved analogously. First, let us suppose that $\theta = \wedge$. According to the properties of the homomorphism g, we have

$$g(A \equiv^{\wedge} B) = h\big(\bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{x\in X} (A(x) \leftrightarrow B(f(x)))\big) =$$

$$\bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{x\in X} g(A(x) \leftrightarrow B(f(x))) = \bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{x\in X} |g(A(x)) \ominus g(B(f(x)))| =$$

$$\bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{x\in X} |g^{\rightarrow}(A)(x) \ominus g^{\rightarrow}(B)(f(x))| = g^{\rightarrow}(A) \equiv^{\vee} g^{\rightarrow}(B).$$

Further, let us suppose that $\theta = \otimes$ and A, B are at most countable **L**-sets. Then we have

$$g(A \equiv^{\otimes} B) = h\big(\bigvee_{f\in\mathrm{Perm}(X)} \bigwedge_{Z\in\mathrm{Fin}(X)} \bigotimes_{x\in Z} (A(x) \leftrightarrow B(f(x)))\big) =$$

$$\bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{Z\in\mathrm{Fin}(X)} \bigoplus_{x\in Z} g(A(x) \leftrightarrow B(f(x))) =$$

$$\bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{Z\in\mathrm{Fin}(X)} \bigoplus_{x\in Z} |g(A(x)) \ominus g(B(f(x)))| =$$

$$\bigwedge_{f\in\mathrm{Perm}(X)} \bigvee_{Z\in\mathrm{Fin}(X)} \bigoplus_{x\in Z} |g^{\rightarrow}(A)(x) \ominus g^{\rightarrow}(B)(f(x))| = g^{\rightarrow}(A) \equiv^{\oplus} g^{\rightarrow}(B)$$

and (i) is proved. Let A be at most countable **L**-set, $Y \subseteq X$ such that $\mathrm{Supp}(A) \subseteq Y$ and $i \in N$. If $i > |Y|$, then we have $p^{\theta}_A(i, Y) = \bot$ and $p_{g^{\rightarrow}(A)\cap Y}(i, Y) = \top$. Hence, we obtain $g(p^{\theta}_A(i, Y)) = p_{g^{\rightarrow}(A)\cap Y}(i, Y)$. Let $i \le |Y|$ and $i < \omega$, then we have

$$g(p^{\theta}_A(i, Y)) = g(\bigvee_{\substack{Z\subseteq Y\\ |Z|=i}} \bigominus_{z\in Z} A(z)) = \bigwedge_{\substack{Z\subseteq Y\\ |Z|=i}} \overline{\bigominus}_{z\in Z} g(A(z)) = p^{\bar{\theta}}_{g^{\rightarrow}(A)\cap Y}(i, Y).$$

Furthermore, for $i = \omega$ we have

$$g(p^{\theta}_A(\omega, Y)) = g(\bigvee_{\substack{Z\subseteq Y\\ |Z|=\omega}} \bigominus_{z\in Z} A(z)) =$$

$$g(\bigvee_{\substack{Z\subseteq Y\\ |Z|=\omega}} \bigwedge_{Z'\in\mathrm{Fin}(Z)} \bigominus_{z\in Z'} A(z)) = \bigwedge_{\substack{Z\subseteq Y\\ |Z|=\omega}} \bigvee_{Z'\in\mathrm{Fin}(Z)} \overline{\bigominus}_{z\in Z'} g(A(z)) =$$

$$\bigwedge_{\substack{Z\subseteq Y\\ |Z|=\omega}} \overline{\bigominus}_{z\in Z} g^{\rightarrow}(A)(z) = p^{\bar{\theta}}_{g^{\rightarrow}(A)\cap Y}(\omega, Y).$$

Hence, the proof is complete. □

Remark 2.4.1. If just finite fuzzy sets are considered, then the completeness of homomorphism need not be assumed. It follows from Theorems 2.2.1 and 2.3.1.

Chapter 3

Cardinalities of finite fuzzy sets

Cardinality seems to be one of the most fascinating and enigmatic mathematical aspects of fuzzy sets. As concerns applications, let us mentioned communication with data bases and information/intelligent systems, modeling the meaning of imprecise quantifiers in natural language statements, computing with word, the computational theory of perceptions, decision-making in fuzzy environment, probabilities of fuzzy events, metrical analysis of grey images, etc.

Maciej Wygralak in [108]

In the previous chapter we have mentioned that the cardinality of a set is expressed by a cardinal number describing the "number of elements of the set". In the case of finite sets, the cardinal numbers are expressed by the natural numbers. If we want to introduce the notion of cardinality for finite fuzzy sets, we have to decide what are the suitable objects that could characterize the size of these fuzzy sets. Obviously, the situation is, contrary to the cardinality of sets, complicated by the graduation of membership of elements of fuzzy sets. A classical-like approaches to the cardinality of fuzzy sets are inspired by the cardinality of sets. The cardinality theories of fuzzy sets are built using the equivalence classes of fuzzy sets under the various types of equipollence of fuzzy sets (see e.g. [36, 37, 55]). These approaches, however, did not fit the applicability of cardinal theory of fuzzy sets and therefore they were not been studied more intensively. The main development of cardinal theory for fuzzy sets proceeded in the following two directions.

The first one contains so-called *scalar* approaches (mainly for finite fuzzy sets) which have a single ordinary cardinal number or a non-negative real number as the object expressing the size of fuzzy sets. A basic definition of the *scalar cardinality*, named the *power* of a finite fuzzy set, was proposed by A. De Luca and S. Terminy [13]. This definition is very simple and the power of a finite fuzzy set A is given by the sum of membership degrees of the fuzzy set A, i.e. $|A| = \sum_{x\in X} A(x)$. Some authors refer to $|A|$ as $\Sigma Count$ (*sigma count*). An extension of this definition to *p-power* of finite fuzzy sets was proposed by A. Kaufman [52] and S. Gottwald [38]. Another definition of scalar cardinality, based on *FECount* (see below), was introduced by D. Ralescu [81] and properties of scalar cardinality of fuzzy sets could be found in e.g. [16,17,38,102,125]. An axiomatic approach to the scalar cardinality of fuzzy sets was suggested by M. Wygralak [106,107]. The relationship between fuzzy mappings and scalar cardinality of fuzzy sets was stated in [9, 50, 85]. L.A. Zadeh in [125] has introduced a relative measure of scalar cardinality of finite fuzzy sets by $\Sigma Count(A|B) = \Sigma Count(A \cap B)/\Sigma Count(B)$ to model the truth values of formulas with fuzzy quantifiers.

In the second direction the objects expressing the size of fuzzy sets are constructed as more-less special fuzzy sets over the set of natural numbers N or universes containing a broader class of cardinal or ordinal numbers. A dominant role is then plaid by the convex fuzzy sets over N or over a class of common cardinals, which are called *generalized natural numbers* or *generalized cardinals* (see e.g. [16, 102–105]), respectively, where the membership degrees are constructed over $[0, 1]$. This type of cardinality of fuzzy sets is usually called *fuzzy cardinality*. The first definition of fuzzy cardinality of finite fuzzy sets, by means of functions from N to $[0, 1]$, was done due to Lotfi A. Zadeh [124] and it is based on the a-cuts of fuzzy sets. In particular, the power of finite fuzzy set A is introduced by $|A|_{\mathcal{F}}(i) = \bigvee\{a \mid |A_a| = i\}$. This definition seems to be reasonable, but one important property, which is derived from the cardinality of sets, is failed. This property is defined as $|A| + |B| = |A \cap B| + |A \cup B|$ and it is called *valuation property*[1]. In the Zadeh's case of fuzzy cardinality, the operation of addition is defined by his extension principle (see Section 1.4). It is easy to see that the Zadeh's fuzzy cardinality leads to the fuzzy sets, which are non-convex, in general. This lack was removed by L.A. Zadeh in [125], where he introduced three types of fuzzy cardinalities *FGCount*, *FLCount* and *FECount* (again the mappings from N to $[0, 1]$) for modeling fuzzy quantifiers in natural language. Note that the fuzzy cardinality *FECount* (but in a different notation) was also independently introduced by M. Wygralak in [101]. Keeping Zadeh's notation, $FGCount(A)(k) = \bigvee\{a \mid |A_a| \geq n\}$ expresses a degree to which A contains *at least* k elements. The dual variant $FLCount(A)(k) = 1 - FGCount(A)(k + 1)$ determines a degree to which A has *at most* k elements. A degree to which A has *exactly* k elements is then expressed by $FECount(A)(k) = FGCount(A)(k) \wedge FLCount(A)(k)$. Other approaches

[1]Note that this property is also referred to *additivity*, but this denotation will be used for a different property of fuzzy cardinality (see p. 79).

to the definition of fuzzy cardinality for finite fuzzy sets could be found in [15, 16, 81]. A generalization of *FGCount*, *FLCount* and *FECount* using t-norms and t-conorms was proposed in [108, 109] and singular fuzzy sets with regard to the triangular norm-based generalized cardinals were investigated in [19, 20]. An axiomatic approach to fuzzy cardinalities of finite fuzzy sets, defined by means of the generalized natural numbers, was proposed by J. Casasnovas and J. Torrens in [8]. Using the proposed system of axioms an infinite class of fuzzy cardinalities could be obtained, which contains a lot of the fuzzy cardinalities referred above. Further, a representation of axiomatic introduced fuzzy cardinalities by the couples of mappings, where one is a non-increasing mapping and the other one is a non-decreasing mapping, could be given in a similar way as in the case of the axiomatic approach to scalar cardinality. A definition of cardinality for (possibly infinite) fuzzy set, which actually generalizes *FGCount*, is suggested by A.P. Šostak in [92, 93]. Another extension of *FGCount* was proposed by P. Lubczonok in [66], where the unit interval is replaced by a totally ordered lattice **L**. Cardinalities of the resulting **L**-fuzzy sets are then defined as mappings from a class of cardinal numbers to the lattice **L**. As we have mentioned, the cardinalities of fuzzy sets are mainly constructed as the convex fuzzy sets. An approach to the non-convex cardinalities of fuzzy sets could be found in [14].

The approach to cardinalities of finite fuzzy sets (**L**-sets and $\mathbf{L}^{\mathrm{d}}$-sets), presented in this chapter, is axiomatic. As objects, representing the cardinalities of fuzzy sets, we chose convex fuzzy sets, which are an analogy to the generalized natural numbers. The structures of such convex fuzzy sets are introduced in the following section. In the second section we propose a system of axioms for so-called *θ-cardinalities* of **L**-sets. This system was motivated by the axiomatic system proposed by J. Casasnovas and J. Torrens in [8] and it is suggested to satisfy requirements that results from dealing with the **L**-sets. Actually, our proposed axiomatic system generalizes the original one. In the third section we introduce an axiomatic system for so-called *$\overline{\theta}$-cardinalities* of $\mathbf{L}^{\mathrm{d}}$-sets, which could be understood as a "dual" system to the axiomatic system for θ-cardinalities. It is easy to see that the operations of dually residuated lattices seem to be suitable for a generalization of $\Sigma Count$. Hence, the original idea was to introduce a dual axiomatic system to the previous one that enables to establish scalar cardinalities of fuzzy sets. The final system of axioms, however, gives a possibility to establish a broader class of cardinalities of fuzzy sets than only the scalar cardinalities. The scalar cardinalities, of course, for $\mathbf{L}^{\mathrm{d}}$-sets could be comprehended as some special $\overline{\theta}$-cardinalities having the form of singletons (see p. 83). Our investigation of cardinalities of finite **L**-sets and $\mathbf{L}^{\mathrm{d}}$-sets is largely restricted here as regards their properties and our primary goal is to study the relationships between the concepts of evaluated equipollence (or bijection) and cardinalities of fuzzy sets. In the first subsections of the mentioned sections we focus our attention on the representation of θ- and $\overline{\theta}$-cardinalities of **L**-sets and $\mathbf{L}^{\mathrm{d}}$-sets, respectively, and we show that both types of cardinalities may be represented by couples of suitable homomorphisms between residuated and dually residuated lattices in a similar way as in [8]. The second subsections are then devoted to various relationships between evaluated equipollencies or bijections and cardinalities of fuzzy sets and some further expressions of cardinalities by the mappings p^{θ} and $p^{\overline{\theta}}$. These relationships can be used, in some special cases, to establishing fuzzy quantifiers as will be shown in Subsection 4.3.2.

3.1 Structures for fuzzy sets cardinalities

In order to introduce cardinalities or better the "power" of fuzzy sets measures, we have to choose objects (special fuzzy sets) being suitable to model an analogy with cardinal numbers denoting the "size" of fuzzy sets. In the classical set theory the cardinalities of finite sets are expressed by cardinal (or equivalently ordinal) numbers, which are represented by natural numbers, and the cardinality of finite sets may be formally described as a mapping $|\ | : \mathrm{Fin} \to N$ assigning to each finite set the cardinal number corresponding to their numbers of elements. This mapping has the following properties. First, for arbitrary $A, B \in \mathrm{Fin}$ such that $A \cap B = \emptyset$ we have $|A \cup B| = |A| + |B|$ and then $|A| \leq |B|$, whenever there exists an injection from A to B. In order to be able to deal with the mentioned properties, we have to assume not only the set of natural numbers, but a richer structure of natural numbers. In particular, it seems to be reasonable to suppose a *linearly ordered commutative monoid* (or *loc*-monoid for short) *of natural numbers* denoted by $\mathbb{N} = (N, +, \leq, 0)$. Further, it is easy to see that the element 1 (characterizing the one element set) has a dominant role in $\mathbb{N}$, precisely, each element except of 0 may be uniquely determined by the finite repeated counting of 1. This element is the *unit*[2] of $\mathbb{N}$. Finally, if we extend the set N by the first infinite cardinal, denoted by ω (or also $\aleph_0$), to the set of extended natural numbers N_ω, then we obtain the *bounded loc-monoid of extended natural numbers* $\mathbb{N}_\omega = (N_\omega, +, \leq, 0, \omega)$, where the addition and ordering are naturally extended by $\omega + a = a + \omega = \omega$ and $a \leq \omega$ for any $a \in N_\omega$, respectively.

[2]The definition of the unit is not the same in different algebraical theories. Here we use the terminology of ordered algebraical structures and the unit is defined as an element u of the ordered algebraical structures G such that for each positive element $g \in G^+$ there is a natural $n \in N$ with $g \leq nu$ (see e.g. [11]).

Obviously, this structure could serve as a range of values for the cardinality of countable sets[3] and it is an example of the more general algebraic structure as follows. An algebraical structure $\mathbb{M} = (M, +, \leq, 0, m, u)$, where $(M, +, 0)$ is the commutative monoid (i.e. $+$ is associative, commutative and the identity $a + 0 = a$ holds for any $a \in M$), $(M \leq)$ is the linearly ordered set with 0 as the least element and m as the greatest element $(0 < m)$ and if $a \leq b$, then $a + c \leq b + c$ holds for any $c \in M$, $u \in M$ such that

(*i*) for every $a \in M$, $a \neq 0$, there exists $n \in N$ such that $nu = a$ and

(*ii*) $\bigvee_{n=1}^{\infty} nu = m$ (or $\lim_{n\to\infty} nu = m$),

where $1u = u$ and $nu = (n-1)u + u$ holds for any $n > 1$, will be called the *bounded linearly ordered commutative monoid with the unit u* or *bounded loc-monoid with the unit u* for short. This algebraical structure is very simple, even all elements are determined by the unit u, nevertheless, it seems to be suitable structure over which a more general construction of the rings of cardinalities of finite fuzzy sets could be done. Before showing a simple representation of bounded *loc*-monoids with units, let us give two examples of them. The first one could be used in practice for modeling of the relative measure of scalar cardinality for sets expressing how much a subset of a given set is contained in this set (a special case of $\Sigma Count(A/B)$ that was mentioned above). The second one will be used in the mentioned representation.

Example 3.1.1. Let $n \in N$, $n > 0$, be an arbitrary natural number. The structure $\mathbb{M}_n = (M_n, S_L, \leq, 0, 1, \frac{1}{n})$, where $M_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ and S_L is the Łukasiewicz t-conorm, is the bounded *loc*-monoid with the unit $\frac{1}{n}$. In particular, we have $\mathbb{M}_1 = (\{0,1\}, S_L, \leq, 0, 1, 1)$, where $S_L(0,0) = 0$, $S_L(0,1) = S_L(1,1) = 1$ and 1 is the unit, will be called the *trivial* upper bound *loc*-monoid with the unit 1.

Example 3.1.2. Let $n \in N_\omega$, $n > 0$, be an arbitrary natural number or $n = \omega$. Then we establish $\mathbb{N}_n = (N_n, \boxplus, \leq, 0, n, 1)$, where $N_n = \{0, \dots, n\}$, the relation $\leq$ is the common linear ordering of natural numbers from N_n and $a \boxplus b = \min(a + b, n)$ holds for each $a, b \in N_n$, where $+$ denotes the extended addition on N_ω.

The following lemma shows the mentioned representation of bounded *loc*-monoids with a unit u by the structures $\mathbb{N}_n$.

[3]Note that axiomatic systems for cardinalities of countable fuzzy sets were also one of the goals of this theses. Since their theory is not still finished, they are not presented here. Nevertheless, a consequence of this effort is the consideration of such bounded structure. Obviously, a generalization of this structure given below is also appropriate for the investigation of cardinalities of finite fuzzy sets.

Lemma 3.1.1. *Let* $\mathbb{M}$ *be a bounded loc-monoid with a unit. Then there exists a unique* $m \in N_\omega$ *such that* $\mathbb{M}$ *and* $\mathbb{N}_n$ *are isomorphic, i.e.* $\mathbb{M} \cong \mathbb{N}_n$.

Proof. It is easy to verify that the mapping $f : M \to N_n$, where we put $n = \min\{n' \mid n'u = m\}$, if it is possible, otherwise $n = \omega$, defined by $f(a) = k$ ($a < m$) if and only if $a = ku$ and $f(m) = n$, is an ordered homomorphism of the algebraic structures. The uniqueness of m follows from the fact that there exists a bijective mapping between two finite sets if and only if these sets have the same number of elements and two countable infinite sets are always bijective. □

According to this lemma, we can assume that each bounded *loc*-monoid with a unit has the form $\mathbb{N}_n$ for a suitable $n \in N_\omega$. In the following part we will introduce similar structures to the bounded *loc*-monoids with units, which objects are suitable for expressing of finite fuzzy cardinals. In particular, we will extend the bounded *loc*-monoids with units using Zadeh extension principle and show that the extended structures are the *bounded partial ordered commutative monoids* (without the units) or also *bounded poc-monoids* for short. In spite of the fact that the bounded *poc*-monoids do not possess all properties of $\mathbb{N}_n$, it seems to be acceptable for our investigation of cardinalities of finite $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets.

Let $\mathbf{L}$ be a complete residuated lattice and $\mathbb{N}_n$ be a bounded *loc*-monoid with the unit. Then we denote $\mathcal{CV}^\theta_{\mathbf{L}}(N_n)$ the set of all θ-convex $\mathbf{L}$-sets over N_n, where $\theta \in \{\wedge, \otimes\}$. Using the Zadeh extension principle we can extend the addition $\boxplus$ of $\mathbb{N}_n$ to the operation of addition $+^\theta$ on $\mathcal{CV}^\theta_{\mathbf{L}}(N_n)$. The following theorem shows properties of the extended operation $+^\theta$. Let us set $E(k) = \top$, if $k = 0$, and $E(k) = \bot$ otherwise. It is easy to see that E is the θ-convex $\mathbf{L}$-set and thus belongs to $\mathcal{CV}^\theta_{\mathbf{L}}(N_n)$.

Theorem 3.1.2. *Let* $\mathbf{L}$ *be a complete residuated lattice. Then the structure* $(\mathcal{CV}^\otimes_{\mathbf{L}}(N_n), +^\otimes)$ *is a commutative monoid with the neutral element* E*. Moreover, if* $\mathbf{L}$ *is divisible, then the structure* $(\mathcal{CV}^\wedge_{\mathbf{L}}(N_n), +^\wedge)$ *is a commutative monoid with the neutral element* E.

Proof. In order to prove that $+^\theta$ is defined correctly, i.e. $A +^\theta B \in \mathcal{CV}_{\mathbf{L}}(N_n)$ for any $A, B \in \mathcal{CV}_{\mathbf{L}}(N_n)$ and $n \in N_\omega$, it is sufficient (due to Theorem 1.4.1) to prove that for arbitrary $i \leq j \leq k$ from N_n and for arbitrary $i_1, i_2, k_1, k_2 \in N_n$ such that $i_1 \boxplus i_2 = i$ and $k_1 \boxplus k_2 = k$ there exist $j_1, j_2 \in N_n$ with $j_1 \boxplus j_2 = j$ and $i_t \leq j_t \leq k_t$ or $k_t \leq j_t \leq i_t$ hold for $t = 1, 2$. Note that the addition is a surjective mapping. Obviously, if $a \geq b$ and $a \neq \omega$ (it is necessary to suppose it in the case of N_ω), then we can put $a \boxminus b = c$ if and only if $a = b \boxplus c$. It establishes a partial subtraction on N_n. Now, let us suppose that $i_1 \leq k_1$,

$i_2 \leq k_2$ and $k_1 \neq \omega \neq k_2$. If $j \leq i_1 \boxplus k_2$, then it is sufficient to put $j_1 = i_1$ and $j_2 = j \boxminus i_1$. Obviously, $i_2 \leq j_2 \leq k_2$, because $i_1 \boxplus i_2 \leq j$. If $j > i_1 \boxplus k_2$, then we put $j_2 = k_2$ and $j_1 = j \boxminus k_2$. Again, we obtain $i_1 \leq j_1 \leq k_1$, because $j \leq k_1 \boxplus k_2$. Further, let us suppose (for N_ω) that $i_1 \leq k_1$, $i_2 \leq k_2$ and $k_1 = \omega$ and $k_2 \neq \omega$. If $j = \omega$, then we put $j_1 = i_1$ and $j_2 = \omega$. If $j \neq \omega$, then we put $j_1 = i_1$ and $j_2 = j \boxminus i_1$. The same results could be obtained for $k_1 \neq \omega$ and $k_2 = \omega$ or $k_1 \neq \omega$ and $k_2 \neq \omega$. Finally, the cases $k_1 \leq i_1$, $i_2 \leq k_2$ and $i_1 \leq k_1$, $k_2 \leq i_2$ could be proved by analogy and thus the addition $+^\otimes$ is closed on $\mathcal{CV}^\otimes_{\mathbf{L}}(N_n)$. If $\mathbf{L}$ is divisible, then also the addition $+^\wedge$ is closed on $\mathcal{CV}^\wedge_{\mathbf{L}}(N_n)$. Since the operations $\otimes$ and $\wedge$ are commutative, then $+^\otimes$ and $+^\wedge$ are commutative, too. Let A be a θ-convex $\mathbf{L}$-set. Then

$$(A +^\theta E)(i) = \bigvee_{\substack{i_1,i_2 \in N_n \\ i_1 \boxplus i_2 = i}} A(i_1)\theta E(i_2) = A(i)\theta E(0) = A(i)\theta \top = A(i).$$

holds for any $i \in N_n$. From the commutativity of $+^\theta$, we have $A +^\theta E = E +^\theta A = A$ and thus E is a neutral element in the groupoid $(\mathcal{CV}_{\mathbf{L}}(N_n), +^\theta)$. Finally, for arbitrary $A, B, C \in \mathcal{CV}_{\mathbf{L}}(N_n)$ and $i \in N_n$, we have

$$((A +^\otimes B) +^\otimes C)(i) = \bigvee_{\substack{j,k \in N_n \\ j \boxplus k = i}} (\bigvee_{\substack{j_1,j_2 \in N_n \\ j_1 \boxplus j_2 = j}} A(j_1) \otimes B(j_2)) \otimes C(k) =$$

$$\bigvee_{\substack{j,k \in N_n \\ j + k = i}} \bigvee_{\substack{j_1,j_2 \in M \\ j_1 \boxplus j_2 = j}} A(j_1) \otimes B(j_2) \otimes C(k) = \bigvee_{\substack{j_1,j_2,k \in N_n \\ j_1 \boxplus j_2 \boxplus k = i}} A(j_1) \otimes B(j_2) \otimes C(k) =$$

$$\bigvee_{\substack{j_1,j \in N_n \\ j_1 \boxplus j = i}} A(j_1) \otimes (\bigvee_{\substack{j_2,k \in N_n \\ j_2 \boxplus k = j}} B(j_2) \otimes C(k)) = (A +^\otimes (B +^\otimes C))(i)$$

and thus the operation $+^\otimes$ is associative. If $\mathbf{L}$ is divisible, the same result could be obtained by analogy for $+^\wedge$. Hence, $(\mathcal{CV}^\theta_{\mathbf{L}}(N_n), +^\theta)$ is a commutative monoid (for $+^\wedge$ the divisibility of $\mathbf{L}$ has to be assumed) and the proof is complete. □

In the next part, for simplicity, if a commutative monoid $(\mathcal{CV}^\theta_{\mathbf{L}}(N_n), +^\theta)$ is considered in this general form, then for $\theta = \wedge$ the divisibility of the complete residuated lattice $\mathbf{L}$ will always be assumed and therefore it will not be mentioned further.

Let $\mathbf{L}^{\mathrm{d}}$ be a complete dually residuated lattice and $\mathbb{N}_n$ be a bounded *loc*-monoid with the unit. Then we denote $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ the set of all $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-sets over N_n, where $\overline{\theta} \in \{\vee, \oplus\}$. Again, using the dual Zadeh extension principle, we can extend the addition $\boxplus$ of $\mathbb{N}_n$ to the operation of addition $+^{\overline{\theta}}$ on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$. Let us set $E(k) = \bot$, if $k = 0$, and $E(k) = \top$ otherwise. It is easy to see that E is the $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-set and thus belongs to $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$.

Theorem 3.1.3. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete dually residuated lattice. Then the structure $(\mathcal{CV}^{\oplus}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\oplus})$ is a commutative monoid with the neutral element E. Moreover, if $\mathbf{L}^{\mathrm{d}}$ is divisible, then the structure $(\mathcal{CV}^{\vee}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\vee})$ is also a commutative monoid with the neutral element E.*

Proof. It could be done by analogy to the proof of Theorem 3.1.2. □

Again, for simplicity, if a commutative monoid $(\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\overline{\theta}})$ is considered in this general form, then for $\overline{\theta} = \vee$ the dual divisibility of the complete dually residuated lattice $\mathbf{L}^{\mathrm{d}}$ will always be assumed and therefore it will not be mentioned further.

Let A, B be arbitrary fuzzy sets on X. Recall that A is *less than or equal* to B or B is *greater than or equal* to A, if $A(x) \leq B(x)$ holds for every $x \in X$. It is easy to see that this relation is a relation of partial ordering on the set $\mathcal{F}(X)$ and we say that $(\mathcal{F}(X), \leq)$ is a partial ordered set or shortly *po*-set. Moreover, if we put $\mathbf{0}(x) = \bot$ and $\mathbf{1}(x) = \top$ for every $x \in X$, then $\mathbf{0}$ is the least fuzzy set and $\mathbf{1}$ is the greatest fuzzy set in $\mathcal{F}(X)$. Let us define an operator ${}^c : \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ by $A^c(i) = \bigvee_{j \in N_n, j \leq i} A(j)$. Obviously, $A \leq A^c$ and $A^c = A^{cc}$ hold for every $A \in \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$. Hence, the operator ^{c} is a closure operator on $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$.

Lemma 3.1.4. *Let $(\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n), +^{\theta})$ be a commutative monoid and $\leq$ be the ordering relation on $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ defined above. Then $(\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n), \leq)$ is the po-set, where $\mathbf{0}$ is the least element and $\mathbf{1}$ is the greatest element of $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$. Moreover, we have*

(i) $\mathbf{0} +^{\theta} A = \mathbf{0}$,

(ii) $\mathbf{1} +^{\theta} A = A^c$,

(iii) *if* $A \leq B$, *then* $A +^{\theta} C \leq B +^{\theta} C$

for every $A, B, C \in \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$.

Proof. Evidently, the first part of this lemma is true, i.e. $(\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n), \leq)$ is the bounded *po*-set. Let $A \in \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ be arbitrary $\mathbf{L}$-set. Then

$$(\mathbf{0} +^{\theta} A)(i) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} \mathbf{0}(i_1) \theta A(i_2) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} \bot \theta A(i_2) = \bot$$

holds for every $i \in N_n$. Further, we have

$$(\mathbf{1} +^{\theta} A)(i) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} \mathbf{1}(i_1) \theta A(i_2) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} \top \theta A(i_2) = \bigvee_{\substack{k \in N_n \\ k \leq i}} A(k) = A^c(i)$$

for every $i \in N_n$. Finally, if $A, B, C \in \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ be arbitrary **L**-sets such that $A \leq b$, then

$$(A +^{\theta} C)(i) = \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} A(i_1) \theta C(i_2) \leq \bigvee_{\substack{i_1, i_2 \in N_n \\ i_1 + i_2 = i}} B(i_1) \theta C(i_2) = (B +^{\theta} C)(i)$$

holds for every $i \in N_n$, where we apply the inequality $A(i_1) \leq B(i_1)$. □

Again, we can define a dual operator $^c : \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ by $A^c(i) = \bigwedge_{j \in N_n, j \leq i} A(j)$. Obviously, $A^c \leq A$ and $A^c = A^{cc}$ hold for every $A \in \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$.

Lemma 3.1.5. *Let $(\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\theta})$ be a commutative monoid and $\leq$ be the ordering relation on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ defined above. Then $(\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n), \leq)$ is the bounded po-set, where $\mathbf{0}$ is the least element and $\mathbf{1}$ is the greatest element of $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$. Moreover, we have*

(i) $\mathbf{0} +^{\overline{\theta}} A = A^c$,

(ii) $\mathbf{1} +^{\overline{\theta}} A = \mathbf{1}$,

(iii) *if $A \leq B$, then $A +^{\overline{\theta}} C \leq B +^{\overline{\theta}} C$*

for every $A, B, C \in \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$.

Proof. It is analogical to the proof of the previous lemma. □

The following theorem summarizes the previous properties and defines the structures, which will be used for expressing the cardinalities of **L**-sets and $\mathbf{L}^{\mathrm{d}}$-sets.

Theorem 3.1.6. *Let $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ be complete residuated lattices. Then the structures $(\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n), +^{\theta}, \leq, \mathbf{0}, \mathbf{1})$ and $(\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\overline{\theta}}, \leq, \mathbf{0}, \mathbf{1})$ are the bounded poc-monoids, where for $\theta = \wedge$ and $\overline{\theta} = \vee$ the divisibility of $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ is assumed, respectively.*

Proof. It is a straightforward consequence of Theorems 3.1.2, 3.1.3 and Lemmas 3.1.4, 3.1.5. □

In the following text we denote $\mathcal{CV}^{\theta}_{\mathbf{L}}(\mathbb{N}_n) = (\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n), +^{\theta}, \leq, \mathbf{0}, \mathbf{1})$ and $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(\mathbb{N}_n) = (\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n), +^{\overline{\theta}}, \leq, \mathbf{0}, \mathbf{1})$. It is easy to see that these structures have no units. On the other hand, it is not complicated to show that each finite fuzzy set (**L**-set or $\mathbf{L}^{\mathrm{d}}$-set) could be determined as a sum of suitable singletons (an analogy to the unit). Further, let us note that the structure over the generalized natural numbers, which is often used to be expressed of cardinalities of finite fuzzy sets, is a substructure of $\mathcal{CV}^{T}_{\mathbf{L}_T}(\mathbb{N}_\omega)$, where T is a continuous t-norm (see e.g. [108,109]). Finally, let us stress that if a structure $\mathcal{CV}^{\theta}_{\mathbf{L}}(\mathbb{N}_n)$ or $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(\mathbb{N}_n)$ is considered in this general form, then for $\theta = \wedge$ or $\overline{\theta} = \vee$ the divisibility of $\mathbf{L}$ or $\mathbf{L}^{\mathrm{d}}$ will always be assumed, respectively.

3.2 θ-cardinalities of finite L-sets

As we have mentioned, the finite sets cardinality is the mapping $|\,| : \mathrm{Fin} \to N$, where N is the support of *loc*-monoid with the unit 1. It was a motivation to construct the cardinality of finite $\mathbf{L}$-sets as a mapping $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \widehat{N}$, where $\widehat{N}$ is the support of a structure of extended natural numbers. In our case, we put $\widehat{N} = \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ and the θ-cardinality of finite $\mathbf{L}$-sets is then an arbitrary mapping $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ satisfying a suitable set of axioms. The axioms are proposed in such a way to characterize properties of the well-known fuzzy cardinalities. Our axiomatic system generalizes, as we have mentioned, the system of axioms suggested by J. Casasnovas and J. Torrens in [8]. The essential difference is that the axiom of monotonicity is exchanged by the axioms of singleton independency and preservation of non-existence and existence. The Casasnovas-Torrens axiomatic system could be obtained, if we restrict ourselves to the residuated and dually residuated lattices over the unit interval and put $\theta = \wedge$.

3.2.1 Definition and representation

Let $\mathbb{N}_n$ be a bounded *loc*-monoid with the unit 1 and $i \in N_n$. Then $i \cdot 1$ is defined by induction as follows $0 \cdot 1 = 0$ and $i \cdot 1 = (i-1) \cdot 1 \boxplus 1$ for any $i > 0$. Now the θ-cardinalities of finite $\mathbf{L}$-sets are defined as follows.

Definition 3.2.1. *Let* $\mathbf{L}$ *and* $\mathbf{L}^{\mathrm{d}}$ *be a complete residuated and dually residuated lattice, respectively, with the same support and* $\mathcal{CV}^{\theta}_{\mathbf{L}}(\mathbb{N}_n)$ *be a bounded poc-monoid. A* θ-cardinality of finite $\mathbf{L}$-sets *on* $\mathcal{FIN}_{\mathbf{L}}(X)$ *is a mapping* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ *such that the following conditions are satisfied:*

(*i*) *For every* $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ *such that* $A \cap B = \emptyset$, *we have* $\mathbb{C}(A \cup B) = \mathbb{C}(A) +^{\theta} \mathbb{C}(B)$.

(*ii*) *For every* $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ *and* $i, j \in N_n$ *such that* $i > |\mathrm{Supp}(A)|$ *and* $j > |\mathrm{Supp}(B)|$, *we have* $\mathbb{C}(A)(i) = \mathbb{C}(B)(j)$.

(*iii*) *If* $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ *is a crisp set* ($A \subseteq X$), *then* $\mathbb{C}(A)(i) \in \{\bot, \top\}$ *holds for every* $i \in N_n$ *and, moreover, if* $|A| = i$, *then* $\mathbb{C}(A)(i \cdot 1) = \top$.

(*iv*) *Let* $a \in L$, *then* $\mathbb{C}(\{a/x\})(i) = \mathbb{C}(\{a/y\})(i)$ *holds for every* $x, y \in X$ *and* $i \in N_n$.

(*v*) *Let* $a, b \in L$, *then*

$$\mathbb{C}(\{a\overline{\theta}b/x\})(0) = \mathbb{C}(\{a/x\})(0)\theta\mathbb{C}(\{b/x\})(0), \tag{3.1}$$

$$\mathbb{C}(\{a\theta b/x\})(1) = \mathbb{C}(\{a/x\})(1)\theta\mathbb{C}(\{b/x\})(1). \tag{3.2}$$

The mentioned axioms are called the *additivity, variability, consistency, singleton independency, preservation of non-existence* and *existence*, respectively. Note that the terms of additivity, variability and consistency are used from [8]. Let us point out some motivation as well as the meaning for each one of these axioms. The additivity of θ-cardinality is the assumed property of the classical sets cardinality, where the θ-convex **L**-sets are considered as values of θ-cardinality instead of the cardinal numbers, and it seems quite natural. As regards the variability, the idea is that the element not belonging to the support of a finite **L**-set has no effect on its θ-cardinality. Hence, the θ-cardinality of any finite **L**-set as a θ-convex **L**-set must have the same membership degrees for all numbers from N_n greater than the cardinality of its support. The consistency requires that any θ-cardinality of a finite crisp set must take the values $\bot$ and $\top$, because each crisp set is a mapping from a universe to $\{\bot, \top\}$, and the value $\top$ on the concrete cardinal number of this set, if it belongs to N_n. Later we will show that if the cardinal number of a crisp set is greater than n, then the membership degree of any θ-cardinality in n is equal to $\top$. A consequence of the consistency is, moreover, the fact that the degree of the θ-cardinality of a finite **L**-set A in i actually expresses a measure of the "truth" of the statement that the **L**-set A has i elements. This is a different point of view to the cardinality of fuzzy sets than in the case of the $\overline{\theta}$-cardinalities, which will be introduced later (see p. 79). The singleton independency abstracts away from the concrete elements of a universe. If A is a **L**-sets and $x \in X$, then we may ask a question, whether the element x rather does not exist or rather does exist in the **L**-set A. Clearly, the non-existence of x in A will be more supported for $A(x)$ close to $\bot$ and the existence of x in A vice versa. Thus, the axiom of non-existence and existence preservations roughly says that the degree of non-existence of an element from a universe in an **L**-set will be decreasing, if the membership degree of this element in the **L**-set will be increasing[4] and similarly for the degree of existence. The following example generalizes the original definition of *FGCount* for **L**-sets. Further examples may be found in the end of this section or in e.g. [8].

[4] Recall that $a\overline{\theta}b \geq a \vee b$ and $a\theta b \leq a \wedge b$.

Example 3.2.1. A basic definition of the convex fuzzy sets cardinality called *FGCount* (over the unit interval) has been introduced by L.A. Zadeh in [125] (see the introduction to this chapter), where the a-cuts of fuzzy sets are used. Let $\mathbf{L}$ be a complete divisible residuated lattice. Then a $\wedge$-cardinality of finite fuzzy sets $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\wedge}_{\mathbf{L}}(N_n)$, constructed by the a-cuts of fuzzy sets, may be defined as follows

$$\mathbb{C}(A)(i) = \bigvee\{a \mid a \in L \text{ and } |A_a| \geq i\}. \tag{3.3}$$

Let us prove that $\mathbb{C}$ is the $\wedge$-cardinality of finite $\mathbf{L}$-sets. Obviously, $\mathbb{C}(A)(i) \geq \mathbb{C}(A)(j)$, whenever $i \leq j$, holds for arbitrary $i, j \in N_n$ and thus $\mathbb{C}(A)$ is the $\wedge$-convex $\mathbf{L}$-set for any $A \in \mathcal{FIN}_{\mathbf{L}}(X)$. If $A \cap B = \emptyset$ are two finite $\mathbf{L}$-sets, then $|(A \cup B)_a| = |A_a| + |B_a| \geq \min(|A_a| + |B_a|, n) = |A_a| \boxplus |B_a|$. Hence, if $|A_a| + |B_a| \geq i$, then also $|A_a| \boxplus |B_a| \geq i$, because $i \leq n$ according to the definition. Moreover, we have $|A_a| \leq |A_b|$, whenever $b \leq a$. Hence, for any $i \in N_n$ we have

$$(\mathbb{C}(A) +^{\wedge} \mathbb{C}(B))(i) = \bigvee_{\substack{k,l \in N_n \\ k \boxplus l = i}} (\mathbb{C}(A)(k) \wedge \mathbb{C}(B)(l)) =$$

$$\bigvee_{\substack{k,l \in N_n \\ k \boxplus l = i}} (\bigvee_{\substack{a \in L \\ |A_a| \geq k}} a) \wedge (\bigvee_{\substack{b \in L \\ |B_b| \geq l}} b) = \bigvee_{\substack{k,l \in N_n \\ k \boxplus l = i}} \bigvee_{\substack{a \in L \\ |A_a| \geq k}} \bigvee_{\substack{b \in L \\ |B_b| \geq l}} (a \wedge b) \leq$$

$$\bigvee_{\substack{a,b \in L \\ |A_{a \wedge b}| \boxplus |B_{a \wedge b}| \geq i}} (a \wedge b) = \bigvee_{\substack{c \in L \\ |A_c| + |B_c| \geq i}} c = \bigvee_{\substack{c \in L \\ |(A \cup B)_c| \geq i}} c = \mathbb{C}(A \cup B)(i)$$

Let $|(A \cup B)_c| \geq i$ for some $c \in L$. Then clearly there exist $k_c, l_c \in N_n$ such that $|A_c| \geq k_c$, $|B_c| \geq l_c$ and $k_c \boxplus l_c = i$. Therefore, we have

$$(\bigvee_{\substack{a \in L \\ |A_a| \geq k_c}} a) \wedge (\bigvee_{\substack{b \in L \\ |B_b| \geq l_c}} b) \geq c \wedge c = c.$$

Since to each $c \in L$ with $|(A \cup B)_c| \geq i$ there exist $k_c, l_c \in N_n$ with the considered properties, then we can write

$$\mathbb{C}(A \cup B)(i) = \bigvee_{\substack{c \in L \\ |(A \cup B)_c| \geq i}} c \leq \bigvee_{\substack{k,l \in N_n \\ k \boxplus l = i}} (\bigvee_{\substack{a \in L \\ |A_a| \geq k}} a) \wedge (\bigvee_{\substack{b \in L \\ |B_b| \geq l}} b) = (\mathbb{C}(A) +^{\wedge} \mathbb{C}(B))(i)$$

and thus the axiom of additivity is satisfied. Obviously, if $i > |\text{Supp}(A)|$, then also $i > |A_a|$ and thus $\mathbb{C}(A)(i) = \bigvee \emptyset = \bot$. Hence, the variability is fulfilled. Let $A \subseteq X$ be a crisp set. If $i \leq |A|$, then evidently $\mathbb{C}(A)(i) = \top$ and $\mathbb{C}(A)(i \cdot 1) = \top$ for $i = |A|$. Moreover, from the previous part we have

$\mathbb{C}(A)(i) = \bot$ for all $i > |A|$. Hence, the consistency is satisfied. It is easy to see that the singleton independency is also satisfied. Finally, we have $\mathbb{C}(\{c/x\})(0) = \bigvee\{a \in L \mid |\{c/x\}_a| \geq 0\} = \top$ for any $c \in L$, which implies the validity of the axiom of non-existence preservation, and $\mathbb{C}(\{c/x\})(1) = \bigvee\{a \in L \mid |\{c/x\}_a| \geq 1\} = c$ for any any $c \in L$, which implies the validity of the axiom of existence preservation. Thus, we have shown that $\mathbb{C}$ is really the $\wedge$-cardinality of finite **L**-sets. In particular, if we deal with the **L**-sets interpreted in some complete residuated lattice over $[0, 1]$ and we put $n = \omega$, then the $\wedge$-cardinality of A, where

$$A = \{0/x_1, 1/x_2, 0.9/x_3, 0/x_4, 0.6/x_5, 0.3/x_6, 0.2/x_7, 0.1/x_8, 0/x_9, 0/x_{10}, \dots\},$$

is the **L**-set from $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_\omega)$ as follows

$$\mathbb{C}(A) = \{1/0, 1/1, 0.9/2, 0.7/3, 0.6/4, 0.3/5, 0.2/6, 0.1/7, 0/8, \dots, 0/\omega\}.$$

Note that $\mathbb{C}(\emptyset)(0) = 1$ and $\mathbb{C}(\emptyset)(k) = 0$ for every $k > 0$.

In the previous example a special shape of the $\wedge$-cardinality for the empty set was shown. Shapes of θ-cardinality for $\emptyset$ are investigated in the following lemma. Recall that E denotes the neutral element and $\mathbf{1}$ denotes the greatest element of $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$.

Lemma 3.2.1. *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite* **L***-sets. Then $\mathbb{C}(\emptyset) = E$ or $\mathbb{C}(\emptyset) = \mathbf{1}$. Moreover, if $\mathbb{C}(\emptyset) = \mathbf{1}$, then $\mathbb{C}(A)$ is a closed* **L***-set*[5] *for any $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ and thus $\mathbb{C}(A)$ is a non-decreasing mapping, i.e. $\mathbb{C}(A)(i) \leq \mathbb{C}(A)(j)$, whenever $i \leq j$.*

Proof. Due to the consistency, we have $\mathbb{C}(\emptyset)(0) = \top$ and $\mathbb{C}(\emptyset)(i) \in \{\bot, \top\}$ for every $i > 0$ and from the variability we have then $\mathbb{C}(\emptyset)(i) = \mathbb{C}(\emptyset)(j)$ for every $i, j > 0$. Hence, we obtain either $\mathbb{C}(\emptyset)(i) = \bot$ for any $i > 0$ and thus $\mathbb{C}(\emptyset) = E$, or $\mathbb{C}(\emptyset)(i) = \top$ for any $i > 0$ and thus $\mathbb{C}(\emptyset) = \mathbf{1}$. Due to (ii) from Lemma 3.1.4, we have $\mathbb{C}(A) = \mathbb{C}(A \cup \emptyset) = \mathbb{C}(A) +^{\theta} \mathbf{1} = \mathbb{C}(A)^c$ and hence $\mathbb{C}(A)$ is a closed **L**-set. The rest of the proof is a straightforward consequence of the definition of the closed **L**-sets. □

Remark 3.2.2. According to the previous lemma, the **L**-set $\mathbb{C}(\emptyset)$ serves as a neutral element on the set $\mathbb{C}(\mathcal{FIN}_{\mathbf{L}}(X))$ of all images with regard to the mapping $\mathbb{C}$. In other words, the identity $\mathbb{C}(\emptyset) +^{\theta} A = A$ holds for any $A \in \mathbb{C}(\mathcal{FIN}_{\mathbf{L}}(X))$.

The following lemma shows how to find the θ-cardinality of finite **L**-sets.

[5] It means that $\mathbb{C}(A) = \mathbb{C}(A)^c$, where ^{c} is the closure operator defined on p. 56

Lemma 3.2.2. *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ *be a* θ*-cardinality of finite* $\mathbf{L}$*-sets and* $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ *such that* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$*. Then we have*

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\dots,i_m \in N_n \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{3.4}$$

for every $i \in N_n$*.*

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ be a finite $\mathbf{L}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$. Due to the additivity, we obtain

$$\begin{aligned}\mathbb{C}(A)(i) =& \mathbb{C}(\{A(x_1)/x_1\} \cup \cdots \cup \{A(x_m)/x_m\})(i) = \\ & (\mathbb{C}(\{A(x_1)/x_1\}) +^{\theta} \cdots +^{\theta} \mathbb{C}(\{A(x_m)/x_m\}))(i).\end{aligned}$$

for every $i \in N_n$. Applying the definition of $+^{\theta}$, we obtain the desired statement. □

It is easy to see that there exist a lot of combinations of elements $i_1, \dots, i_m$ from N_n satisfying the equality $i_1 \boxplus \cdots \boxplus i_m = i$, even if we want to find θ-cardinality of "small" finite $\mathbf{L}$-sets. A solution of this failure is given in the following theorem.

Theorem 3.2.3. *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ *be a* θ*-cardinality of finite* $\mathbf{L}$*-sets and* $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ *such that* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} \subseteq X$*. Then we have*

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{3.5}$$

for every $i \in N_n$*, where* $i \leq m$*. Moreover, if* $m < n$*, then* $\mathbb{C}(A)(i) = \bot$ *or* $\mathbb{C}(A)(i) = \top$ *holds for every* $m < i \leq n$*, respectively.*

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ be a finite $\mathbf{L}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$. First, we will suppose that $m < n$. Evidently, the statement is true, if $i \leq 1$. Further, let $1 < i \leq m$ and $i_1, \dots, i_m \in N_n$ be a finite sequence such that $i_1 \boxplus \cdots \boxplus i_m = i$. Put $I = \{i_k \mid i_k \in N_n,\ i_k \notin \{0,1\}\}$. Obviously, the statement is true, if $I = \emptyset$. Let $i_k \in I$ be an element and $i_k = r$. Then there exist at least $r - 1$ elements from $\{i_1, \dots, i_m\}$ that are equal to 0. In fact, if there exist $s < r - 1$ elements that are equal to 0, then also there exist $(m - 1) - s$ elements greater than 0 and different from i_k. Hence, we have $i_1 \boxplus \cdots \boxplus i_m = i_1 + \cdots + i_m \geq r + ((m - 1) - s) > r +$

$((m-1)-(r-1)) = m \geq i$, a contradiction. Thus, we can choose the elements $i_{k_1} = \cdots = i_{k_{r-1}} = 0$. Due to the variability and consistency we can write $\mathbb{C}(\{A(x_k)/x_k\})(i_k) = \mathbb{C}(\{A(x_k)/x_k\})(2) = \mathbb{C}(\{A(x_l)/x_l\})(2) = \mathbb{C}(\{\bot/x_k\})(2) = \mathbb{C}(\emptyset)(2) \in \{\bot, \top\}$, where $l \in \{k_1, \ldots, k_{r-1}\}$. Applying the θ-convexity, the preservation of existence and the fact that $\mathbb{C}(\{\bot/x_k\})(0) = \top$ we obtain

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_l)/x_l\})(0) = \\ \mathbb{C}(\{A(x_l)/x_l\})(2)\theta\mathbb{C}(\{A(x_l)/x_l\})(0) \leq \mathbb{C}(\{A(x_l)/x_l\})(1)$$

for every $l \in \{k_1, \ldots, k_{r-1}\}$ and

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k) = \mathbb{C}(\{\bot/x_k\})(2)\theta\top = \mathbb{C}(\{\bot/x_k\})(2)\theta\mathbb{C}(\{\bot/x_k\})(0) \leq \\ \mathbb{C}(\{\bot/x_k\})(1) = \mathbb{C}(\{\bot\theta A(x_k)/x_k\})(1) = \\ \mathbb{C}(\{\bot/x_k\})(1)\theta\mathbb{C}(\{A(x_x)/x_k\})(1) \leq \mathbb{C}(\{A(x_x)/x_k\})(1).$$

Since $\mathbb{C}(\{A(x_k)/x_k\})(i_k) \in \{\bot, \top\}$ and $\bot$, $\top$ are idempotent elements of $\mathbf{L}$ with respect to θ, then using the previous inequalities we can write

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\theta\cdots\theta\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(0) = \\ \mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\theta\cdots\theta \\ \mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(0) \leq \\ \mathbb{C}(\{A(x_x)/x_k\})(1)\theta\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1)\theta\cdots\theta\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(1).$$

Hence, we can create a new sequence of elements $i'_1, \cdots, i'_m \in N_n$ such that $i'_1 \boxplus \cdots \boxplus i'_m = i$, $I' = \{i'_k \mid i'_k \in N_n, i'_k \not\in \{0,1\}\} = I \setminus \{i_k\}$ and

$$\bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \bigodot_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i'_k).$$

As we can repeat the mentioned procedure as long as we remove all elements from I, the desired statement is true for $1 < i \leq m$. Further, we will suppose that $m \geq n$. If $i < n$, then we can apply the same procedure as in the previous case to obtain the desired statement. Let $i = n$ and $i_1 \boxplus \cdots \boxplus i_m = n$. If $i_{k_1}, \ldots, i_{k_r}$ be all elements from $i_1, \ldots, i_m$ that are equaled to 0 and $i_k > 1$, then we have (see above)

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\theta\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\theta\cdots\theta\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_r}\})(0) \leq \\ \mathbb{C}(\{A(x_x)/x_k\})(1)\theta\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1)\theta\cdots\theta\mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(1).$$

Moreover, the inequality $\mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \mathbb{C}(\{A(x_k)/x_k\})(1)$ holds for any $i_k \geq 1$ and thus we obtain

$$\bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \leq \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(1).$$

Hence, the first part of the theorem is proved. Let $m < n$. Then, due to the variability and Lemma 3.2.1, we have $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \bot$ or $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \top$ for any $i \in N_n$ such that $|\mathrm{Supp}(A)| \leq m < i \leq n$, respectively, and the second part of the theorem is proved, too. □

Remark 3.2.3. Obviously, if $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$, then we have

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_m = i}} \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{3.6}$$

for every $i \in N_n$, where $i \leq m$. Moreover, if $m < n$, then $\mathbb{C}(A)(i) = \bot$ or $\mathbb{C}(A)(i) = \top$ holds for every $m < i \leq n$, respectively, and

$$\mathbb{C}(A)(n) = \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_k)/x_k\})(1), \tag{3.7}$$

whenever $n \leq m$. In particular, if A is a crisp set and $n \leq |A|$, then $\mathbb{C}(A)(n) = \top$, as we have mentioned formerly.

In [8] there is shown a representation of fuzzy cardinalities of finite fuzzy sets using two mappings $f, g : [0,1] \to [0,1]$, where first one is non-decreasing, the second one is non-increasing and further $f(0), g(1) \in \{0,1\}$, $f(1) = 1$ and $g(0) = 0$. Before we introduce an analogical representation of the θ-cardinalities of finite **L**-sets, we establish several notions and prove a lemma. Let $\mathbf{L}_1$, $\mathbf{L}_2$ be (complete) residuated lattices and $h : L_1 \to L_2$ be a mapping. We say that h is the (*complete*) *θ-homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2$, if h is a (complete) homomorphism from the substructure $(L_1, \theta_1, \top_1)$ of the residuated lattice $\mathbf{L}_1$ to the substructure $(L_2, \theta_2, \top_2)$ of the residuated lattices $\mathbf{L}_2$, i.e. $h(a\theta_1 b) = h(a)\theta_2 h(b)$ and $h(\top_1) = \top_2$. Obviously, each (complete) homomorphism between complete residuated lattices is also a (complete) θ-homomorphism. Further, let $\mathbf{L}_1^{\mathrm{d}}$ and $\mathbf{L}_2$ be (complete) dually residuated lattice and residuated lattice, respectively, and $h : L_1 \to L_2$ be a mapping. We say that h is the (*complete*) *$\overline{\theta}_d$-homomorphism*, if h is a (complete) homomorphism from the substructure $(L_1, \overline{\theta}_1, \bot_1)$ of the dually residuated lattice $\mathbf{L}_1^{\mathrm{d}}$ to the substructure $(L_2, \theta_2, \top_2)$ of the residuated lattice $\mathbf{L}_2$, i.e.

$h(a\overline{\theta}_1 b) = h(a)\theta h(b)$ and $h(\bot_1) = \top_2$. Again, each (complete) homomorphism from a complete dually residuated lattice to a complete residuated lattice is also a (complete) $\overline{\theta}$-homomorphism.

Lemma 3.2.4. *Let $f, g : L \to L$ be a θ- and $\overline{\theta}_d$-homomorphism from $\mathbf{L}$ to $\mathbf{L}$ and from $\mathbf{L}^{\mathrm{d}}$ to $\mathbf{L}$, respectively, such that $f(\bot) \in \{\bot, \top\}$ and $g(\top) \in \{\bot, \top\}$. Further, let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ be a mapping defined by induction as follows*

$$\mathbb{C}_{f,g}(\{a/x\})(0) = g(a),\ \mathbb{C}_{f,g}(\{a/x\})(1) = f(a) \text{ and}$$
$$\mathbb{C}_{f,g}(\{a/x\})(k) = f(\bot),\ k > 1$$

hold for every singleton $\{a/x\} \in \mathcal{FIN}_{\mathbf{L}}(X)$ and

$$\mathbb{C}_{f,g}(A) = \mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\theta} \cdots +^{\theta} \mathbb{C}_{f,g}(\{A(x_m)/x_m\})$$

holds for every $A \in \mathcal{FIN}_{\mathbf{L}}(X)$, where $\mathrm{Supp}(A) = \{x_1, \ldots, x_m\}$*. Then the mapping $\mathbb{C}_{f,g}$ is a* θ-cardinality of finite $\mathbf{L}$-sets generated by the θ- and $\overline{\theta}_d$-homomorphisms f and g*, respectively.*

Proof. First, we will prove that the definition of the mapping $\mathbb{C}_{f,g}$ is correct. Let $\{a/x\}$ be a singleton from $\mathcal{FIN}_{\mathbf{L}}(X)$. If $n = 1$, then $\mathbb{C}_{f,g}(\{a/x\})$ is clearly a θ-convex $\mathbf{L}$-set. Let $n > 1$. Since $f(\bot) = f(\bot\theta a) = f(\bot)\theta f(a) \leq f(a)$ holds for any $a \in L$, then we have

$$\mathbb{C}_{f,g}(\{a/x\})(0)\theta\mathbb{C}_{f,g}(\{a/x\})(2) = g(a)\theta f(\bot) \leq f(a) = \mathbb{C}_{f,g}(\{a/x\})(1).$$

Furthermore, this inequality is trivially fulfilled for each triplet $0 < i \leq j \leq k$ from N_n. Hence, the mapping $\mathbb{C}_{f,g}$ assigns a θ-convex $\mathbf{L}$-sets to each singletons from $\mathcal{FIN}_{\mathbf{L}}(X)$. Since the sum of θ-convex of $\mathbf{L}$-sets is a θ-convex $\mathbf{L}$-set (according to Theorem 3.1.2), we obtain that the definition of $\mathbb{C}_{f,g}$ is correct. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be arbitrary disjoint $\mathbf{L}$-sets, where $\mathrm{Supp}(A) = \{x_1, \ldots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \ldots, y_s\}$. Due to the associativity of the operation $+^{\theta}$ and the definition of $\mathbb{C}_{f,g}$, we have

$$\begin{aligned}
&\mathbb{C}_{f,g}(A \cup B) = \\
&\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\theta} \cdots +^{\theta} \mathbb{C}_{f,g}(\{A(x_r)/x_r\}) +^{\theta} \mathbb{C}_{f,g}(\{B(y_1)/y_1\}) +^{\theta} \cdots \\
&\quad +^{\theta} \mathbb{C}_{f,g}(\{B(y_s)/y_s\}) = \big(\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\theta} \cdots +^{\theta} \mathbb{C}_{f,g}(\{A(x_r)/x_r\})\big) +^{\theta} \\
&\qquad \big(\mathbb{C}_{f,g}(\{B(y_1/y_1\}) +^{\theta} \cdots +^{\theta} \mathbb{C}_{f,g}(\{B(y_s)/y_s\})\big) = \mathbb{C}_{f,g}(A) +^{\theta} \mathbb{C}_{f,g}(B).
\end{aligned}$$

Hence, the mapping $\mathbb{C}_{f,g}$ satisfies the additivity. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ be an $\mathbf{L}$-set with $\mathrm{Supp}(A) = \{x_1, \ldots, x_r\}$. From the additivity of $\mathbb{C}_{f,g}$ we obtain

$$\mathbb{C}_{f,g}(A)(i) = \bigvee_{\substack{i_1,\ldots,i_r \in N_n \\ i_1+\cdots+i_r = i}} \mathbb{C}(\{A(x_1)/x_1)(i_1)\theta \cdots \theta\mathbb{C}(\{A(x_r)/x_r)(i_r). \tag{3.8}$$

Let us denote $\mathbf{i} \in N_n^r$ an r-dimensional vector of elements from N_n, i.e. $\mathbf{i} = (i_1, \dots, i_r)$, such that $i_1 \boxplus \cdots \boxplus i_r = i$. The set of all such vectors will be denoted by $\mathcal{I}$. Further, let us denote $K_\mathbf{i} = \{k \mid i_k = 0\}$, $L_\mathbf{i} = \{l \mid i_l = 1\}$ and $M_\mathbf{i} = \{m \mid i_m > 1\}$. Clearly, $K_\mathbf{i} \cup L_\mathbf{i} \cup M_\mathbf{i} = \{1, \dots, r\}$ and they are mutually disjoint. Finally, let us establish $a_{K_\mathbf{i}} = \Theta_{k \in K_\mathbf{i}} g(A(x_k))$, $a_{L_\mathbf{i}} = \Theta_{l \in L_\mathbf{i}} f(A(x_l))$, $a_{M_\mathbf{i}} = \Theta_{m \in M_\mathbf{i}} f(\bot)$. Recall that $\Theta_{a \in \emptyset} a = \top$ (see p. 5). Then the formula (3.8) can be rewritten as follows

$$\mathbb{C}_{f,g}(A)(i) = \bigvee_{\mathbf{i} \in \mathcal{I}} a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}}. \tag{3.9}$$

Now let us suppose that $i > r$, then necessarily $M_\mathbf{i} \neq \emptyset$ for every $\mathbf{i} \in \mathcal{I}$. Since $a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} \leq a_{M_\mathbf{i}} = f(\bot)$ holds for every $\mathbf{i} \in \mathcal{I}$, then $\mathbb{C}_{f,g}(A)(i) = \bigvee_{\mathbf{i} \in \mathcal{I}} a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} \leq \bigvee_{\mathbf{i} \in \mathcal{I}} f(\bot) = f(\bot)$. On the other hand, there exists $\mathbf{i} \in \mathcal{I}$ such that $K_\mathbf{i} = \emptyset$. Since $f(a) \geq f(\bot)$ holds for every $a \in L$, we have $a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} = \top \theta a_{L_\mathbf{i}} \theta f(\bot) \geq f(\bot)$ and thus $\mathbb{C}_{f,g}(A)(i) \geq f(\bot)$. Hence, $\mathbb{C}_{f,g}$ satisfies the axiom of variability. The axiom of consistency is a simple consequence of the previous consideration. In fact, let us suppose that $A \subseteq X$ is a crisp set. If $i > |A|$, then $\mathbb{C}_{f,g}(A)(i) = f(\bot) \in \{\bot, \top\}$ with regard to the presumption about the values of $f(\bot)$. If $i \leq |A|$, then for every $\mathbf{i} \in \mathcal{I}$ we have $a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} \in \{\bot, \top\}$, because $a_{K_\mathbf{i}}, a_{L_\mathbf{i}}, a_{M_\mathbf{i}} \in \{\bot, \top\}$, where for instance $a_{K_\mathbf{i}} = g(\top \theta \cdots \theta \top) = g(\top) \in \{\bot, \top\}$, if $K_\mathbf{i} \neq \emptyset$, and $a_{K_\mathbf{i}} = \top$, if $K_\mathbf{i} = \emptyset$. Hence, $\mathbb{C}(A)(i) = \bigvee_{\mathbf{i} \in \mathcal{I}} a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} \in \{\bot, \top\}$. Moreover, if we suppose $|A| = r \leq n$, then obviously there exists $L_\mathbf{i}$ such that $K_\mathbf{i} = M_\mathbf{i} = \emptyset$. Hence, we have $\mathbb{C}(A)(r) = \mathbb{C}(A)(r \cdot 1) \geq a_{K_\mathbf{i}} \theta a_{L_\mathbf{i}} \theta a_{M_\mathbf{i}} = a_{L_\mathbf{i}} = f(\top \theta \cdots \theta \top) = f(\top) = \top$ and therefore $\mathbb{C}(A)(r) = \top$. Further, if $|A| = r > n$, then $n = r \cdot 1$ and analogously we obtain $\mathbb{C}(A)(n) = \mathbb{C}(r \cdot 1) = a_{L_\mathbf{i}} = \top$, where again $L_\mathbf{i}$ is such set that $K_\mathbf{i} = M_\mathbf{i} = \emptyset$. The conditions of singleton independency and non-existence and existence preservations follow immediately from the definition of mappings f and g. Thus, we have shown that $\mathbb{C}_{f,g}$ is a θ-cardinality of finite **L**-sets. □

The denotations $K_\mathbf{i}$, $L_\mathbf{i}$, $M_\mathbf{i}$ for the sets and $a_{K_\mathbf{i}}$, $a_{L_\mathbf{i}}$, $a_{M_\mathbf{i}}$ for the values, established in the proof of the previous lemma, will be often used in the following text. For simplicity, their definitions will not eb mentioned further. Now we can proceed to the representation of the θ-cardinalities.

Theorem 3.2.5. (representation of θ-cardinalities) *Let* **L** *and* $\mathbf{L}^\mathrm{d}$ *be a complete residuated lattice and a complete dually residuated lattice, respectively, and* $\mathcal{CV}^\theta_\mathbf{L}(\mathbb{N}_n)$ *be a bounded poc-monoid. Further, let* $\mathbb{C} : \mathcal{FIN}_\mathbf{L}(X) \to \mathcal{CV}^\theta_\mathbf{L}(N_n)$ *be a mapping which satisfies the axiom of additivity. Then the following statements are equivalent:*

(*i*) $\mathbb{C}$ *is a* θ*-cardinality of finite* $\mathbf{L}$*-sets,*

(*ii*) *there exist a* θ*-homomorphism* $f : L \to L$ *and a* $\overline{\theta}_d$*-homomorphism* $g : L \to L$*, such that* $f(\bot) \in \{\bot, \top\}$*,* $g(\top) \in \{\bot, \top\}$ *and*

$$\mathbb{C}(\{a/x\})(0) = g(a),\ \mathbb{C}(\{a/x\})(1) = f(a),\ \mathbb{C}(\{a/x\})(k) = f(\bot)$$

hold for arbitrary $a \in L$*,* $x \in X$ *and* $k > 1$*.*

Proof. First, we will show that (*i*) implies (*ii*). Let $\mathbb{C}$ be a θ-cardinality of finite $\mathbf{L}$-sets. Let us define two mappings $f, g : L \to L$ as follows

$$f(a) = \mathbb{C}(\{a/x\})(1), \tag{3.10}$$

$$g(a) = \mathbb{C}(\{a/x\})(0). \tag{3.11}$$

Due to the axioms of existence and non-existence preservations by $\mathbb{C}$, we have

$$f(a\theta b) = \mathbb{C}(\{a\theta b/x\})(1) = \mathbb{C}(\{a/x\})(1)\theta\mathbb{C}(\{b/x\})(1) = f(a)\theta f(b),$$

$$g(a\overline{\theta}b) = \mathbb{C}(\{a\overline{\theta}b/x\})(0) = \mathbb{C}(\{a/x\})(0)\theta\mathbb{C}(\{b/x\})(0) = g(a)\theta g(b).$$

According to the consistency, we have $f(\top) = \mathbb{C}(\{\top/x\})(1) = \top$, $g(\bot) = \mathbb{C}(\{\bot/x\}(0) = \top$. Hence, we obtain that f is a θ-homomorphism and g is a $\overline{\theta}_d$-homomorphism of the relevant algebraic structures. Moreover, the values $f(\bot) = \mathbb{C}(\{\bot/x\})(1)$ and $g(\top) = \mathbb{C}(\{\top/x\})(0)$ belong to $\{\bot, \top\}$ from the condition of consistency. Finally, due to the variability, we have $f(\bot) = \mathbb{C}(\{\bot/x\})(1) = \mathbb{C}(\{a/x\})(k)$ for every $k > 1$. Second, we will show that (*ii*) implies (*i*). Let $\mathbb{C}_{f,g}$ be the θ-cardinality of finite $\mathbf{L}$-sets generated by the θ- and $\overline{\theta}_d$-homomorphisms f and g, respectively, defined in the previous lemma. Since $\mathbb{C}_{f,g}(\{a/x\}) = \mathbb{C}(\{a/x\})$ holds for any singleton from $\mathcal{FIN}_{\mathbf{L}}(X)$ and $\mathbb{C}$ satisfies the axiom of additivity, then also $\mathbb{C}_{f,g}(A) = \mathbb{C}(A)$ holds for any $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ and thus $\mathbb{C}$ is a θ-cardinality of finite $\mathbf{L}$-sets. □

According to the previous theorem, each θ-cardinality $\mathbb{C}$ of finite $\mathbf{L}$-sets is generated by a θ-homomorphism f and $\overline{\theta}_d$-homomorphism g satisfying the conditions of (ii), i.e. $\mathbb{C} = \mathbb{C}_{f,g}$. This representation enables us to investigate the θ-cardinalities from the perspective of various types of θ- and $\overline{\theta}$-homomorphisms.

In the set theory the cardinality of finite sets as $|\ | : \mathcal{FIN} \to N$ preserves the partial ordering of the set $\mathcal{FIN}$, determined by the inclusion relation, to the partial ordering of natural numbers, i.e. $A \subseteq B$ implies $|A| \leq |B|$. The last part of this subsection is now devoted to the question, if the θ-cardinalities of finite $\mathbf{L}$-sets fulfil this property. We will show that the preservation of the partial ordering is satisfied only for special types of the mentioned homomorphisms. First, let us introduce special types of the θ- and $\overline{\theta}_d$-homomorphisms. We say that a mapping $h : L_1 \to L_2$ is the (*complete*) *θ-po-homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2$, if h is a (complete) θ-homomorphism preserving the partial ordering of the considered lattices, i.e $h(a) \leq h(b)$, whenever $a \leq b$. Further, $h : L_1 \to L_2$ is the (*complete*) *$\overline{\theta}_d$-po-homomorphism* from $\mathbf{L}_1^{\mathrm{d}}$ to $\mathbf{L}_2$, if h is a (complete) $\overline{\theta}_d$-homomorphism reversing the partial ordering of the considered lattices, i.e $h(a) \leq h(b)$, whenever $a \geq b$. It is easy to see that a θ-*po*- or $\overline{\theta}$-*po*-homomorphism are, in some cases, already determined by the θ- or $\overline{\theta}$-homomorphism, respectively. For instance, if $\theta = \wedge$, then $a \leq_1 b$ implies $f(a) = f(a \wedge_1 b) = f(a) \wedge_2 f(b)$ and thus $f(a) \leq_2 f(b)$. Analogously, for $\overline{\theta} = \vee$.

Example 3.2.4. Let $f : L \to L$ be a mapping defined by $h(a) = a^n$, where $n \geq 1$. Then h is the $\otimes$-*po*-homomorphism. In fact, we have $h(a \otimes b) = (a \otimes b)^n = a^n \otimes b^n = h(a) \otimes h(b)$ and if $a \leq b$, then by monotony of $\otimes$ we obtain $f(a) = a^n \leq b^n = f(b)$. Further, let us suppose that $\mathbf{L}$ is a complete residuated lattice such that the law of double negation is satisfied, i.e. $\neg\neg a = (a \to 0) \to 0 = a$ holds for any $a \in L$. In this case it could be shown that there exists a complete dually residuated lattice, where the operation of the addition is given by $a \oplus b = \neg(\neg a \otimes \neg b)$ and the operation of difference is given by $a \ominus b = a \oplus \neg b$. Then a $\overline{\theta}$-*po*-homomorphism $h : L \to L$ could be defined by $h(a) = (\neg a)^n$. In fact, we have $h(a \oplus b) = \neg(a \oplus b) = \neg\neg(\neg a \otimes \neg b) = \neg a \otimes \neg b = h(a) \otimes h(b)$ and if $a \leq b$, then it could be shown that $\neg b \leq \neg a$ and thus $h(b) \leq h(a)$.

Example 3.2.5. Let us put $f(a) = \top_2$ for every $a \in L$, then we obtain another example of θ-*po*-homomorphism called *trivial θ-po-homomorphism* from $\mathbf{L}_1^{\mathrm{d}}$ to $\mathbf{L}_2$. In fact, we have $f(a\theta_1 b) = \top_2 = \top_2\theta_2\top_2 = f(a)\theta_2 f(b)$. Analogously, we can define a *trivial $\overline{\theta}$-po-homomorphism* putting $g(a) = \top_2$ for every $a \in L_1$. In these special cases, where a θ-cardinality of finite $\mathbf{L}$-sets is generated just by a θ-homomorphism f (g is trivial and thus it has no effect) or a $\overline{\theta}_d$-homomorphism g (f is trivial and thus it has no effect), we will denote this θ-cardinality by $\mathbb{C}_f$ or $\mathbb{C}_g$, respectively.

The following theorem shows, when the θ-cardinality preserves or reverses partial ordering of $\mathbf{L}$-sets.

Theorem 3.2.6. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite fuzzy sets generated by θ-po- and $\overline{\theta}$-po-homomorphisms f and g, respectively. Then*

(i) $\mathbb{C}_{f,g}$ preserves the partial ordering of $\mathbf{L}$-sets if and only if g is the trivial $\overline{\theta}_d$-po-homomorphism.

(*ii*) $\mathbb{C}_{f,g}$ *reverses the partial ordering of* $\mathbf{L}$*-sets if and only if* f *is the trivial* θ*-po-homomorphism.*

Proof. Here, we will prove just the first statement, the second one could be done by analogy. First, let us suppose that $\mathbb{C}_{f,g}$ preserves the partial ordering of $\mathbf{L}$-sets and g is a non-trivial $\overline{\theta}_d$-*po*-homomorphism. Since g is the non-trivial $\overline{\theta}_d$-*po*-homomorphism, then necessarily there exists $a \in L \setminus \{\bot\}$ such that $g(a) < \top$. Then we have $\{\bot/x\} < \{a/x\}$ for any $x \in X$ and thus $\mathbb{C}_{f,g}(\{\bot/x\})(0) = g(\bot) > g(a) = \mathbb{C}_{f,g}(\{a/x\})(0)$, a contradiction. Hence, the implication $\Rightarrow$ is true. Contrary, let f be a θ-*po*-homomorphism, g be the trivial $\overline{\theta}_d$-*po*-homomorphism, $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be arbitrary $\mathbf{L}$-sets and $i \in N_n$. If $A \leq B$, then $\mathbb{C}_{f,g}(\{A(x)/x\})(0) = g(A(x)) = \top = g(B(x)) = \mathbb{C}_{f,g}(\{B(x)/x\})(0)$ and $\mathbb{C}_{f,g}(\{A(x)/x\})(1) = f(A(x)) \leq f(B(x)) = \mathbb{C}_{f,g}(\{B(x)/x\})(1)$ hold for arbitrary $x \in X$. Moreover, $\mathbb{C}_{f,g}(\{A(x)/x\})(i) = \mathbb{C}_{f,g}(\{B(x)/x\})(i)$ holds for any $i > 1$ and thus $\mathbb{C}_{f,g}(\{A(x)/x\}) \leq \mathbb{C}_{f,g}(\{B(x)/x\})$ holds for any $x \in \mathrm{Supp}(B)$. The inequality $\mathbb{C}_{f,g}(A) \leq \mathbb{C}_{f,g}(B)$ is an immediate consequence of the additivity axiom and the statement (*iii*) from Lemma 3.1.4. Hence, the implication $\Leftarrow$ is true and the proof is complete. □

Remark 3.2.6. In the introduction we have mentioned the valuation property of the cardinality of sets. Let us define the generalized θ-intersection of fuzzy sets by $(A \cap^\theta B)(x) = A(x)\theta B(x)$ and the generalized $\overline{\theta}$-union of fuzzy sets by $(A \cup^{\overline{\theta}} B)(x) = A(x)\overline{\theta} B(x)$. Then it is easy to see that the valuation property in the following more general form

$$\mathbb{C}(A \cap^\theta B) +^\theta \mathbb{C}(A \cup^{\overline{\theta}} B) = \mathbb{C}(A) +^\theta \mathbb{C}(B) \tag{3.12}$$

is not satisfied, if arbitrary complete residuated and dually residuated lattices are supposed. In fact, from the verification of the valuation property for $i = 0$, i.e. $(\mathbb{C}(A \cap^\theta B) +^\theta \mathbb{C}(A \cup^{\overline{\theta}} B))(0) = (\mathbb{C}(A) +^\theta \mathbb{C}(B))(0)$, we obtain (due to the axiom of non-existence preservation) the equality $(a\theta b)\overline{\theta}(a\overline{\theta}b) = a\overline{\theta}b$, which is not evidently satisfied in arbitrary residuated and dually residuated lattices with the same support. On the other hand, this equality is true for $\theta = \wedge$ and $\overline{\theta} = \vee$, for example, and it is not complicated to prove that the valuation property is satisfied by a θ-cardinality, if the considered complete residuated and dually residuated lattices are linearly ordered and $\theta = \wedge$ and $\overline{\theta} = \vee$. An open problem is, if this property is valid also for other types of operations.

3.2.2 θ-cardinality and equipollence of L-sets

In the classical set theory the cardinality of sets is primarily connected with the equipollence of sets. Recall that two finite sets have the same cardinal number if and only if they are equipollent, that is, there exists a bijection between them. In order to investigate an analogy of the previous relationship for θ-cardinalities of finite **L**-sets, we have to establish a similarity relation on the set of all θ-convex **L**-sets over N_n. Let us define a relation on $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ as follows

$$K \approx L = \bigwedge_{i \in N_n} K(i) \leftrightarrow L(i). \tag{3.13}$$

It is easy to verify that the relation $\approx$ is a similarity relation on $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$. First, let us show a relation between the θ-cardinalities of finite **L**-sets and bijections of **L**-sets.

Theorem 3.2.7. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite **L**-sets generated by θ- and $\overline{\theta}_d$-homomorphisms f and g, respectively, $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $h \in \mathrm{Perm}(X)$. Then*

$$a^{\theta}_{gh} \theta a^{\theta}_{fh} \leq \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B), \tag{3.14}$$

where a^{θ}_{gh} or a^{θ}_{fh} is the θ-degree of the bijection h between $g^{\to}(A)$ and $g^{\to}(B)$ or $f^{\to}(A)$ and $f^{\to}(B)$, respectively.

Proof. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be arbitrary finite **L**-sets, where $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \dots, y_s\}$, $i \in N_n$ and $h \in \mathrm{Perm}(X)$. Put $\{x_1, \dots, x_m\} = \mathrm{Supp}(A) \cup h^{-1}(\mathrm{Supp}(B))$ and $\{y_1, \dots, y_m\} = h(\mathrm{Supp}(A)) \cup \mathrm{Supp}(B)$ such that $h(x_k) = y_k$ for any $x_k \in \{x_1, \dots, x_m\}$. If $i > m$, then we obtain $\mathbb{C}_{f,g}(A)(i) = \mathbb{C}_{f,g}(B)(i)$ and hence we have

$$\mathbb{C}_{f,g}(A)(i) \leftrightarrow \mathbb{C}_{f,g}(B)(i) = \top \geq a^{\theta}_{gh} \theta a^{\theta}_{fh}.$$

Let us suppose that $i \leq m$ and establish $\mathbf{i} \in \mathcal{O}$ if and only if $\mathbf{i} \in \mathcal{I}$ and simultaneously $M_{\mathbf{i}} = \emptyset$ (clearly $\mathcal{O} \subset \mathcal{I}$). Then due to Theorem 3.2.3, Lemma 2.1.2

and Lemma 2.1.4, we have

$$\mathbb{C}_{f,g}(A)(i) \leftrightarrow \mathbb{C}_{f,g}(B)(i) = \big(\bigvee_{\mathbf{i}\in\mathcal{O}} a_{K_\mathbf{i}}\theta a_{L_\mathbf{i}}\big) \leftrightarrow \big(\bigvee_{\mathbf{i}\in\mathcal{O}} b_{K_\mathbf{i}}\theta b_{L_\mathbf{i}}\big) \geq$$
$$\bigwedge_{\mathbf{i}\in\mathcal{O}}((a_{K_\mathbf{i}}\theta a_{L_\mathbf{i}}) \leftrightarrow (b_{K_\mathbf{i}}\theta b_{L_\mathbf{i}})) \geq \bigwedge_{\mathbf{i}\in\mathcal{O}}((a_{K_\mathbf{i}} \leftrightarrow b_{K_\mathbf{i}})\theta(a_{L_\mathbf{i}} \leftrightarrow b_{L_\mathbf{i}})) \geq$$
$$\bigwedge_{\mathbf{i}\in\mathcal{O}}\Big(\bigominus_{k\in K_\mathbf{i}} g(A(x_k)) \leftrightarrow \bigominus_{k\in K_\mathbf{i}} g(B(y_k))\Big)\theta\Big(\bigominus_{l\in L_\mathbf{i}} f(A(x_l)) \leftrightarrow \bigominus_{l\in L_\mathbf{i}} f(B(y_l))\Big) \geq$$
$$\bigwedge_{\mathbf{i}\in\mathcal{O}}\Big(\bigominus_{k\in K_\mathbf{i}} (g(A(x_k)) \leftrightarrow g(B(y_k)))\Big)\theta\Big(\bigominus_{l\in L_\mathbf{i}} (f(A(x_l)) \leftrightarrow f(B(y_l)))\Big) \geq$$
$$\bigominus_{k=1}^{m} \big(g(A(x_k)) \leftrightarrow g(B(h(x_k)))\big)\theta\big(f(A(x_k)) \leftrightarrow f(B(h(x_k)))\big) = a^\theta_{gh}\theta a^\theta_{fh},$$

where $a_{K_\mathbf{i}}$, $a_{L_\mathbf{i}}$ are defined on the page 66 and $b_{K_\mathbf{i}}$, $b_{L_\mathbf{i}}$ are established analogously. Applying the infimum to the inequalities for $i > m$ and the inequalities for $i \leq m$, we obtain (3.14). □

Remark 3.2.7. Obviously, the inequality $(g^\rightarrow(A) \equiv^\theta g^\rightarrow(B))\theta(f^\rightarrow(A) \equiv^\theta f^\rightarrow(B)) \leq \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B)$ can not be proved, because $\bigvee_{i\in I}(a_i\theta b_i) \leq (\bigvee_{i\in I} a_i)\theta(\bigvee_{i\in I} b_i)$ holds, in general. Nevertheless, if there is a bijection h such that $g^\rightarrow(A)(x) = g^\rightarrow(B)(h(x))$ and $f^\rightarrow(A)(x) = f^\rightarrow(B)(h(x))$, then $a^\theta_{gh} = \top$ and $a^\theta_{fh} = \top$ and hence we obtain $\mathbb{C}_{f,g}(A) = \mathbb{C}_{f,g}(B)$.

The failure mentioned in the previous remark can be eliminated, if we assume that one of θ- and $\overline{\theta}_d$-homomorphisms is trivial, as it is stated in the following corollary.

Corollary 3.2.8. *Let $f : L \to L$ be a θ-homomorphism and $g : L \to L$ be a $\overline{\theta}_d$-homomorphism. Then*

(i) $f^\rightarrow(A) \equiv^\theta f^\rightarrow(B) \leq \mathbb{C}_f(A) \approx \mathbb{C}_f(B)$ *and*

(ii) $g^\rightarrow(A) \equiv^\theta g^\rightarrow(B) \leq \mathbb{C}_g(A) \approx \mathbb{C}_g(B)$

hold for arbitrary $A, B \in \mathcal{FIN}_\mathbf{L}(X)$.

Proof. Let g be the trivial $\overline{\theta}$-homomorphism, i.e. $g(a) = \top$ holds for any $a \in L$, $A, B \in \mathcal{FIN}_\mathbf{L}(X)$ and $h \in \mathrm{Perm}(X)$ be an arbitrary mapping. Then obviously h is the $\top^\theta$-bijection between $g^\rightarrow(A)$ and $g^\rightarrow(B)$. Hence, we have $\mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B) \geq \bigvee_{h\in\mathrm{Perm}(X)} a^\theta_{gh}\theta a^\theta_{fh} = \bigvee_{h\in\mathrm{Perm}(X)} \top\theta a^\theta_{fh} = f^\rightarrow(A) \equiv^\theta f^\rightarrow(B)$ and the first statement is proved. Analogously, we could obtain the second statement and the proof is complete. □

Additional properties of θ- and $\overline{\theta}_d$-homomorphisms are supposed in the following two corollaries.

Corollary 3.2.9. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite $\mathbf{L}$-sets generated by θ- and $\overline{\theta}_d$-homomorphisms f and g, respectively, such that $f(a \leftrightarrow b) \leq f(a) \leftrightarrow f(b)$ and $g(|a \ominus b|) \leq g(a) \leftrightarrow g(b)$ hold for any $a, b \in L$. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $h \in \mathrm{Perm}(X)$. Then we have*

$$g(a_h)\theta f(b_h) \leq \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B), \tag{3.15}$$

where a_h and b_h are the θ- and $\overline{\theta}$-degrees of the bijection h between A and B, respectively.

Proof. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be arbitrary finite $\mathbf{L}$-sets, $h \in \mathrm{Perm}(X)$ and $H = \mathrm{Supp}(A) \cup h^{-1}(\mathrm{Supp}(B))$. Since the mapping g and f are the θ- and $\overline{\theta}_d$-homomorphisms, then we can write (due to Theorem 3.2.7)

$$\mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B) \geq$$

$$\bigominus_{x\in H} (g^{\to}(A)(x) \leftrightarrow g^{\to}(B)(h(x)))\theta \bigominus_{x\in H} (f^{\to}(A)(x) \leftrightarrow f^{\to}(B)(h(x))) \geq$$

$$\bigominus_{x\in H} g(|A(x) \ominus B(h(x))|)\theta \bigominus_{x\in H} f(A(x) \leftrightarrow B(h(x))) = g(a_h)\theta f(b_h)$$

and the proof is complete. □

Corollary 3.2.10. *Let $f : \mathbf{L} \to \mathbf{L}$ and $g : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}$ be arbitrary homomorphisms. Then*

(i) $f(A \equiv^{\theta} B) \leq \mathbb{C}_f(A) \approx \mathbb{C}_f(B)$ *and*

(ii) $g(A \equiv^{\overline{\theta}} B) \leq \mathbb{C}_g(A) \approx \mathbb{C}_g(B)$

hold for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$.

Proof. It is a straightforward consequence of Lemmas 2.4.1 and 2.4.2 and Corollary 3.2.8. □

Let $\mathbf{L}$ be a complete residuated lattice and $h : L \to L$ be an arbitrary mapping. We say that the mapping h is *k-θ-compatible with the biresiduum* $\leftrightarrow$ of $\mathbf{L}$, if $(a \leftrightarrow b)^k = (a \leftrightarrow b)\theta \cdots \theta(a \leftrightarrow b) \leq h(a) \leftrightarrow h(b)$ holds for arbitrary $a, b \in L$. Obviously, the kth power has a sense just for $\theta = \otimes$, because for $\theta = \wedge$ the definition of the k-$\wedge$-compatibility with the biresiduum coincides with $a \leftrightarrow b \le h(a) \leftrightarrow h(b)$. The θ-*po*-homomorphism and $\overline{\theta}_d$-*po*-homomorphism introduced on p. 68 are the examples of mappings being k-compatible with the biresiduum. For instance, if we consider the mapping $h : L \to L$ given by $h(a) = (\neg a)^k$, then we have $(a \leftrightarrow b)^k = (\neg a \leftrightarrow \neg b)^k \le (\neg a)^k \leftrightarrow (\neg b)^k = h(a) \leftrightarrow h(b)$ and thus h is the mapping which is k-$\otimes$-compatible with $\leftrightarrow$ of $\mathbf{L}$.

Corollary 3.2.11. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite $\mathbf{L}$-sets generated by θ- and $\overline{\theta}_d$-homomorphisms f and g, which are k-θ- and l-θ-compatible with $\leftrightarrow$, respectively. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ and $h \in \mathrm{Perm}(X)$. Then we have*

$$a_h^{k+l} \le \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B), \tag{3.16}$$

where a_h is the θ-degree of the bijection h between A and B. Furthermore, we have

$$(A \equiv^{\theta} B)^l \le \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B) \quad or \quad (A \equiv^{\theta} B)^k \le \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B),$$

whenever f or g is the trivial θ- or $\overline{\theta}_d$-homomorphism, respectively.

Proof. It is a straightforward consequence of the k-θ- and l-θ-compatibility of f and g with the biresiduum, Theorem 3.2.7 and Corollary 3.2.8. □

In the previous chapter we have established mappings $p^{\theta}_A, p^{\overline{\theta}}_A : N_\omega \times A_X \to L$, where L is a support of residuated or dually residuated lattice, respectively. In some special cases, these mappings may be used to establish θ-cardinalities of $\mathbf{L}$-sets. In the proof of Theorem 3.1.2 we have introduced the operation of partial subtraction on N_n. In particular, if $k, l \in N_n$ are arbitrary elements such that $k \le l$ and $l \ne \omega$ (supposing for $n = \omega$), then $l \boxminus k = m$ if and only if $k \boxplus m = l$. Since this operation coincides with the classical subtraction, we will write only $-$ instead of $\boxminus$ in the following parts.

Theorem 3.2.12. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ be a θ-cardinality of finite $\mathbf{L}$-sets generated by homomorphisms f, g and $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} = Y$. Then*

$$\mathbb{C}_{f,g}(A)(i) \le g(p^{\overline{\theta}}_A(m - i, Y))\theta f(p^{\theta}_A(i, Y)) \tag{3.17}$$

holds for any $i \in N_n$, where $i \le m$.

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ be an arbitrary finite $\mathbf{L}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} = Y$ and $i \in N_n$ such that $i \le m$. Recall that $\mathbf{i} \in \mathcal{O}$ if and only if $\mathbf{i} \in \mathcal{I}$ and simultaneously $M_{\mathbf{i}} = \emptyset$. Since g and f are the homomorphisms, then we have (due to Theorem 3.2.3)

$$\mathbb{C}(A)(i) = \bigvee_{\substack{i_1,\dots,i_m\in\{0,1\}\\ i_1\boxplus\cdots\boxplus i_m=i}} \bigominus_{k=1}^{m} \mathbb{C}(\{A(x_x)/x_k\})(i_k) = \bigvee_{i\in\mathcal{O}}(\bigominus_{k\in K_{\mathbf{i}}} g(A(x_k))\theta$$

$$\bigominus_{l\in L_{\mathbf{i}}} f(A(x_l))) = \bigvee_{i\in\mathcal{O}} g(\overline{\bigominus_{k\in K_{\mathbf{i}}}} A(x_k))\theta f(\bigominus_{l\in L_{\mathbf{i}}} f(A(x_l))) \leq \bigvee_{i\in\mathcal{O}} g(\overline{\bigominus_{k\in K_{\mathbf{i}}}} A(x_k))\theta$$

$$\bigvee_{i\in\mathcal{O}} f(\bigominus_{l\in L_{\mathbf{i}}} A(x_l)) = g(\bigwedge_{i\in\mathcal{O}} \overline{\bigominus_{k\in K_{\mathbf{i}}}} A(x_k))\theta f(\bigvee_{i\in\mathcal{O}} \bigominus_{l\in L_{\mathbf{i}}} A(x_l)) =$$

$$g(\bigwedge_{\substack{Z\subseteq Y\\ |Z|=m-i}} \overline{\bigominus_{z\in Z}} A(z))\theta f(\bigvee_{\substack{Z\subseteq Y\\ |Z|=i}} \bigominus_{z\in Z} f(A(z))) = g(p_A^{\overline{\theta}}(m-i,Y))\theta f(p_A^{\theta}(i,Y)),$$

where clearly $|K_{\mathbf{i}}| = m - i$ and $|L_{\mathbf{i}}| = i$ hold for any $\mathbf{i} \in \mathcal{O}$. □

The following theorem shows a simple method how to compute a θ-cardinality of finite $\mathbf{L}$-sets generated by homomorphisms, if a linearly ordered residuated lattice is considered. Note that the θ-cardinalities defined below are a generalization of *FECount*.

Theorem 3.2.13. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice and* $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ *be a θ-cardinality of finite* $\mathbf{L}$*-sets generated by homomorphisms* f, g *and* $A \in \mathcal{FIN}_{\mathbf{L}}(X)$ *with* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} = Y$. *Then*

$$\mathbb{C}_{f,g}(A)(i) = g(p_A^{\overline{\theta}}(m-i,Y))\theta f(p_A^{\theta}(i,Y)) \tag{3.18}$$

holds for any $i \in N_n$, *where* $i \leq m$.

Proof. Let $A \in \mathcal{FIN}(X)$ be an arbitrary finite $\mathbf{L}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} = Y$ and $i \in N_n$, where $i \leq m$. Due to Theorem 3.2.12, it is sufficient to show that $\mathbb{C}(A)(i) \geq g(p_A^{\overline{\theta}}(m-i,Y))\theta f(p_A^{\theta}(i,Y))$, i.e.

$$\bigvee_{i\in\mathcal{O}} \bigominus_{k\in K_{\mathbf{i}}} g(A(x_k))\theta \bigominus_{l\in L_{\mathbf{i}}} f(A(x_l)) \geq \bigvee_{i\in\mathcal{O}} \bigominus_{k\in K_{\mathbf{i}}} g(A(x_k))\theta \bigvee_{i\in\mathcal{O}} \bigominus_{l\in L_{\mathbf{i}}} f(A(x_l))$$

holds (see the proof of Theorem 3.2.12). Let us establish

$$a^g = \bigvee_{\mathbf{i}\in\mathcal{O}} \bigominus_{k\in K_{\mathbf{i}}} g(A(x_k)) = \bigvee_{\mathbf{i}\in\mathcal{O}} a_{K_{\mathbf{i}}} \quad\text{and}\quad a^f = \bigvee_{\mathbf{i}\in\mathcal{O}} \bigominus_{l\in L_{\mathbf{i}}} f(A(x_l)) = \bigvee_{\mathbf{i}\in\mathcal{O}} a_{L_{\mathbf{i}}}.$$

Obviously, the mentioned inequality is satisfied, if there exists $\mathbf{i} \in \mathcal{I}$ such that $a_{K_\mathbf{i}} = a^g$ and $a_{L_\mathbf{i}} = a^f$. Since the linearly ordered lattice is supposed and $\mathcal{O}$ is just a finite set, then there exist $K = \{p_1, \dots, p_{m-i}\}$ and $L = \{q_1, \dots, q_i\}$ such that $a^g = g(A(x_{p_1}))\theta \cdots \theta g(A(x_{p_{m-i}}))$ and $a^f = f(A(x_{q_1}))\theta \cdots \theta f(A(x_{q_i}))$, respectively. Let us denote the set of all such couples $\langle K, L\rangle$ by $\mathcal{T}$. Now the mentioned requirement can be stated as there exists $\mathbf{i} \in \mathcal{O}$ such that $\langle K_\mathbf{i}, L_\mathbf{i}\rangle \in \mathcal{T}$ or also there exists $\langle K, L\rangle \in \mathcal{T}$ such that $K \cap L = \emptyset$. Let us suppose that there is no couple with the required condition. Let $\langle K_0, L_0\rangle \in \mathcal{T}$ be a couple such that $t = |K_0 \cap L_0| \leq |K \cap L|$ for any $\langle K, L\rangle \in \mathcal{T}$. Let $K_0 = \{p_1 0, \dots, p^0_{m-i}\}$ and $L_0 = \{q_1 0, \dots, q^0_i\}$ and $r \in K_0 \cap L_0$. Since we suppose that $t > 0$, then necessary $|K_0 \cup L_0| < m$. Hence, there is an index $s \in \{1, \dots, m\}$ such that $s \notin K_0 \cup L_0$. Because of $\mathbf{L}$ is a linearly ordered lattice, we have $A(x_s) \geq A(x_r)$ or $A(x_s) \leq A(x_r)$. In the first case, we have $f(A(x_s)) \geq f(A(x_r))$ and thus,

$$f(A(x_{q_1 0}))\theta \cdots \theta f(A(x_s))\theta \cdots \theta f(A(x_{q^0_i})) \geq f(A(x_{q_1 0}))\theta \cdots \theta f(A(x_{q^0_i})),$$

where $f(A(x_r))$ is replaced by $f(A(x_s))$ on the left side of the inequality. If we establish $L = \{s\} \cup L_0/\{r\}$, then we obtain $\langle K_0, L\rangle \in \mathcal{T}$ and simultaneously $|K_0 \cap L| < t$, a contradiction. In the second case, we have $g(A(x_s)) \geq g(A(x_r))$ and thus,

$$g(A(x_{p_1 0}))\theta \cdots \theta g(A(x_s))\theta \cdots \theta g(A(x_{p^0_{m-i}})) \geq g(A(x_{p_1 0}))\theta \cdots \theta g(A(x_{p^0_{m-i}})),$$

where again $g(A(x_r))$ is replaced by $g(A(x_s))$ on the left side of the inequality. If we establish $K = \{s\} \cup K_0/\{r\}$, then we obtain $\langle K, L_0\rangle \in \mathcal{T}$ and simultaneously $|K \cap L_0| < t$, a contradiction. Thus, we have shown that there exists $\mathbf{i} \in \mathcal{O}$ such that $\langle K_\mathbf{i}, L_\mathbf{i}\rangle \in \mathcal{T}$ and the proof is complete. □

Corollary 3.2.14. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice and* $\mathbb{C}_{f,g} : \mathcal{FIN}_\mathbf{L}(X) \to \mathcal{CV}_\mathbf{L}(N_n)$ *be a* θ*-cardinality of finite* $\mathbf{L}$*-sets generated by homomorphisms* f, g *and* $A \in \mathcal{FIN}_\mathbf{L}(X)$ *with* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\} = Y$. *Then*

$$\mathbb{C}(A)(i) = p^\theta_{g^\rightarrow(A)\cap Y}(m-i, Y)\theta p^\theta_{f^\rightarrow(A)}(i, Y) \tag{3.19}$$

holds for any $i \in N_n$, *where* $i \leq m$.

Proof. It is a straightforward consequence of Lemmas 2.4.1, 2.4.2 and Theorem 3.2.13. □

Corollary 3.2.15. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice and* $\mathbb{C}_{f,g} : \mathcal{FIN}_\mathbf{L}(X) \to \mathcal{CV}_\mathbf{L}(N_n)$ *be a* θ*-cardinality of finite* $\mathbf{L}$*-sets generated by homomorphisms* f, g *and* $A \in \mathcal{FIN}_\mathbf{L}(X)$ *with* $|\mathrm{Supp}(A)| = m$. *Then*

$$\mathbb{C}(A)(i) = g(p^\wedge_A(i+1))\theta f(p^\wedge_A(i)) \tag{3.20}$$

holds for any $i \in N_n$, *where* $i \leq m$.

Proof. It is sufficient to prove that $p^{\vee}_A(m-i) = p^{\wedge}_A(i+1)$ holds for any $i \le N_n$, where $i \le m = |\mathrm{Supp}(A)|$. If $i = m$, then $p^{\vee}_A(0) = \bot = p^{\wedge}_A(m+1)$. Let us suppose that $i < m$ and $x_1, \dots, x_m$ is a sequence of the elements from Supp(A) such that $A(x_i) \ge A(x_{i+1})$. Since **L** is linearly ordered, then clearly we have $p^{\wedge}_A(i+1) = A(x_{i+1}) = A(x_{m-i}) = p^{\vee}_A(m-i)$ and the proof is complete. □

In the following example we use Theorem 3.2.13 to construct three types of θ-cardinalities, namely for the minimum, product and Łukasiewicz t-norms.

Example 3.2.8. Let us suppose that the membership degrees of **L**-sets are interpreted in the Goguen algebra $\mathbf{L}_{\mathrm{P}}$ or Łukasiewicz algebra $\mathbf{L}_{\mathrm{Ł}}$ (see Ex. 1.1.4 and 1.1.5), X is a non-empty universe for **L**-sets, $N_8 = \{0, \dots, 8\}$ is a universe for the values of θ-cardinalities and

$$A = \{0.5/x_1, 0.2/x_2, 0.9/x_3, 1/x_4, 0.8/x_5, 0.9/x_6, 1/x_7, 0, 6/x_8\}$$

is a finite **L**-set over X. Further, let us define $f, g : [0,1] \to [0,1]$ by $f(a) = a$ (the identity mapping) and $g(a) = 1 - a$ for any $a \in [0,1]$. Obviously, both mappings are homomorphisms of the corresponding structures, where the dual Goguen and Łukasiewicz algebras are assumed as the dual structures to $\mathbf{L}_{\mathrm{P}}$ and $\mathbf{L}_{\mathrm{Ł}}$ (see Theorem 1.1.5). Putting $\theta = T_{\mathrm{M}}$, $\theta = T_{\mathrm{P}}$ and $\theta = T_{\mathrm{Ł}}$, we can generate three types of the θ-cardinalities of finite **L**-sets which shall be denoted by $\mathbb{C}_{T_{\mathrm{M}}}$, $\mathbb{C}_{T_{\mathrm{P}}}$ and $\mathbb{C}_{T_{\mathrm{Ł}}}$, respectively. In particular, if T is one of the mentioned t-norms, then the T-cardinality of finite **L**-sets generated by f and g can be expressed (due to Theorem 3.2.13) as follows

$$\mathbb{C}_T(A)(i) = T(1 - p^S_A(8-i), p^T_A(i)), \tag{3.21}$$

where S is the corresponding t-conorm to T and $i \in N_8$. Moreover, due to Lemmas 2.2.6 and 2.3.5, we can determined the mappings p^T_A and p^S_A as the following sequences

$$p^T_A = \{p^{T_{\mathrm{M}}}_A(0), T(p^{T_{\mathrm{M}}}_A(0), p^{T_{\mathrm{M}}}_A(1)), \dots, T(p^{T_{\mathrm{M}}}_A(0), \dots, p^{T_{\mathrm{M}}}_A(8))\},$$
$$p^S_A = \{p^{S_{\mathrm{M}}}_A(0), S(p^{S_{\mathrm{M}}}_A(0), p^{S_{\mathrm{M}}}_A(1)), \dots, S(p^{S_{\mathrm{M}}}_A(0), \dots, p^{S_{\mathrm{M}}}_A(8))\},$$

where clearly we have

$$p^{T_{\mathrm{M}}}_A = \{1, 1, 1, 0.9, 0.9, 0.8, 0.6, 0.5, 0.2\},$$
$$p^{S_{\mathrm{M}}}_A = \{0, 0.2, 0.5, 0.6, 0.8, 0.9, 0.9, 1, 1\}.$$

The following three **L**-sets[6] are the values (fuzzy cardinals) of the appropriate T-cardinalities

$$\mathbb{C}_{T_{\mathrm{M}}}(A) = \{0/0, 0/1, 0.1/2, 0.1/3, 0.2/4, 0.4/5, 0.5/6, 0.5/7, 0.2/8\},$$

$$\mathbb{C}_{T_{\mathrm{P}}}(A) = \{0/0, 0/1, 0.00032/2, 0.00288/3, 0.02592/4, 0.10368/5, 0.15552/6, 0.15552/7, 0.03888/8\},$$

$$\mathbb{C}_{T_{\mathrm{L}}}(A) = \{0/0, 0/1, 0/2, 0/3, 0/4, 0/5, 0/6, 0/7, 0/8\}.$$

Note that the **L**-set A is an example of the singular **L**-set with regard to the T_{L}-cardinality, i.e. $\mathbb{C}_{T_{\mathrm{L}}}(A) = \emptyset$. The singular fuzzy set with regard to the norm-based cardinals are investigated in e.g. [19, 20].

Up till now all relationships between the evaluated equipollence and similarity of θ-cardinalities of **L**-sets have been in the form of the inequality. The following lemma shows that the equality can arise only in a very special case.

Corollary 3.2.16. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice and* $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}_{\mathbf{L}}(N_n)$ *be a* θ*-cardinality of finite* **L***-sets generated by homomorphisms* f, g*. Then*

(i) $(g^{\rightarrow}(A) \equiv^{\theta} g^{\rightarrow}(B))\theta(f^{\rightarrow}(A) \equiv^{\theta} f^{\rightarrow}(B)) \leq \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(B)$,

(ii) $g^{\rightarrow}(A) \equiv^{\wedge} g^{\rightarrow}(B) = \mathbb{C}_{g}(A) \approx \mathbb{C}_{g}(B)$,

(iii) $f^{\rightarrow}(A) \equiv^{\wedge} f^{\rightarrow}(B) = \mathbb{C}_{f}(A) \approx \mathbb{C}_{f}(B)$

hold for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$.

Proof. Let $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$ be arbitrary **L**-sets, $Y = \{x_1, \dots, x_m\} = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$ and $i \in N_n$. If $i > m$, then $\mathbb{C}_{f,g}(A)(i) = \mathbb{C}_{f,g}(B)(i)$ according to the axiom of variability and thus $\mathbb{C}_{f,g}(A)(i) \leftrightarrow \mathbb{C}_{f,g}(B)(i) = \top$. Further, let us suppose that $i \leq m$. Due to the properties of the homomorphisms g and f, Theorem 3.2.13 and Lemmas 2.2.8, 2.3.7, 2.4.1, 2.4.2, we have

$$\begin{aligned}
&\mathbb{C}_{f,g}(A)(i) \leftrightarrow \mathbb{C}_{f,g}(B)(i) = \\
&\quad (g(p_A^{\bar{\theta}}(m-i, Y))\theta f(p_A^{\theta}(i, Y))) \leftrightarrow (g(p_B^{\bar{\theta}}(m-i, Y))\theta f(p_B^{\theta}(i, Y))) \geq \\
&\quad (g(p_A^{\bar{\theta}}(m-i, Y)) \leftrightarrow g(p_B^{\bar{\theta}}(m-i, Y)))\theta(f(p_A^{\theta}(i, Y)) \leftrightarrow f(p_A^{\theta}(i, Y))) = \\
&\quad g(|p_A^{\bar{\theta}}(m-i, Y) \ominus p_B^{\bar{\theta}}(m-i, Y)|)\theta f(p_A^{\theta}(i, Y) \leftrightarrow p_B^{\theta}(i, Y)) \geq \\
&\quad g(A \equiv^{\bar{\theta}} B)\theta f(A \equiv^{\theta} B) = (g^{\rightarrow}(A) \equiv^{\theta} g^{\rightarrow}(B))\theta(f^{\rightarrow}(A) \equiv^{\theta} f^{\rightarrow}(B)).
\end{aligned}$$

[6]Precisely, the first **L**-set is the $\mathbf{L}_{\mathrm{P}}$-set and simultaneously the $\mathbf{L}_{\mathrm{L}}$-set, the second one is the $\mathbf{L}_{\mathrm{P}}$-set and the third one is the $\mathbf{L}_{\mathrm{L}}$-set.

Hence, we obtain the statement (i). If f is the trivial homomorphism, then clearly we have $\mathbb{C}_{f,g}(A)(i) \leftrightarrow \mathbb{C}_{f,g}(B)(i) = \top = g(p^{\vee}_A(i,Y)) \leftrightarrow g(p^{\vee}_B(i,Y)) = g(\top) \leftrightarrow g(\top)$ for any $i \in N_n$, where $i > m$. Hence, due to the properties of the homomorphism g, Theorem 3.2.13 and Lemmas 2.3.8 and 2.4.2, we can write

$$\mathbb{C}_g(A) \approx \mathbb{C}_g(B) = \bigwedge_{i=0}^{n}(\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) =$$
$$\bigwedge_{i=0}^{m}(\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) \wedge \bigwedge_{i=m+1}^{n}(\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) =$$
$$\bigwedge_{i=0}^{m}(g(p^{\vee}_A(m-i,Y)) \leftrightarrow g(p^{\vee}_B(m-i,Y)))\wedge$$
$$\bigwedge_{i=m+1}^{n}(g(p^{\vee}_A(i,Y) \leftrightarrow g(p^{\vee}_B(i,Y))) = g(\bigvee_{i=0}^{n}|p^{\vee}_A(i,Y)) \ominus p^{\vee}_B(i,Y)|) =$$
$$g(p_{AY} \approx_{\mathrm{d}} p_{BY}) = g(A \equiv^{\vee} B) = g^{\rightarrow}(A) \equiv^{\wedge} g^{\rightarrow}(B)$$

and (ii) is proved. Analogously, it could be done (iii) and the proof is complete. □

3.3 $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$- sets

In this section we attempt to define an axiomatic system for cardinalities of $\mathbf{L}^{\mathrm{d}}$-sets, which is, in a certain sense, dual to the previous system for the θ-cardinalities of $\mathbf{L}$-sets. These cardinalities are then called $\overline{\theta}$-cardinalities. As we have mentioned in the introduction to this chapter, the $\overline{\theta}$-cardinalities were originally proposed as a way to generalize the scalar cardinalities of fuzzy sets. The idea was based on the fact that the operation addition, used in computing of scalar cardinalities, may be described as an addition in dually residuated lattices (cf. Ex. 1.1.11 and $\Sigma Count$). Nevertheless, the axiomatic system presented below contains a broader class of $\overline{\theta}$-cardinalities of $\mathbf{L}^{\mathrm{d}}$-sets and the scalar cardinalities could be understood as elements of a subclass of the class of $\overline{\theta}$-cardinalities, which values ranges comprise just singletons. The following subsections keep the same outline as the previous one's and the statements have the "dual" forms. Therefore, some of the comments and proofs (primarily in the second subsection) are omitted.

3.3.1 Definition and representation

Recall that, for simplicity, the divisibility (and thus the distributivity of $\vee$ over $\wedge$ is held) of $\mathbf{L}^{\mathrm{d}}$ in the case $\mathcal{CV}^{\vee}_{\mathbf{L}^{\mathrm{d}}}(\mathbb{N}_n)$ is always assumed and it will not be mentioned in the following text. Now the $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets is defined as follows.

Definition 3.3.1. *Let* $\mathbf{L}$ *and* $\mathbf{L}^{\mathrm{d}}$ *be a complete residuated and dually residuated lattice, respectively, with the same supports and* $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(\mathbb{N}_n)$ *be a bounded poc-monoid. A* $\overline{\theta}$-fuzzy cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets *on* $\mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ *is a mapping* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ *such that the following condition are satisfied:*

(*i*) *For every* $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ *with* $A \cap B = \emptyset$*, we have* $\mathbb{C}(A \cup B) = \mathbb{C}(A) +^{\overline{\theta}} \mathbb{C}(B)$.

(*ii*) *For every* $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ *and* $i, j \in N_n$ *such that* $i > |\mathrm{Supp}(A)|$ *and* $j > |\mathrm{Supp}(B)|$*, we have* $\mathbb{C}(A)(i) = \mathbb{C}(B)(j)$*,*

(*iii*) *If* $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ *is a crisp set, then* $\mathbb{C}(A)(i) \in \{\bot, \top\}$ *holds for every* $i \in N_n$ *and, moreover, if* $|A| = i$*, then* $\mathbb{C}(A)(i \cdot 1) = \bot$*.*

(*iv*) *Let* $a \in L$*, then* $\mathbb{C}(\{a/x\})(i) = \mathbb{C}(\{a/y\})(i)$ *holds for every* $x, y \in X$ *and* $i \in N_n$*.*

(*v*) *Let* $a, b \in L$*, then*

$$\mathbb{C}(\{a\theta b/x\})(1) = \mathbb{C}(\{a/x\})(1)\overline{\theta}\mathbb{C}(\{b/x\})(1), \tag{3.22}$$

$$\mathbb{C}(\{a\overline{\theta} b/x\})(0) = \mathbb{C}(\{a/x\})(0)\overline{\theta}\mathbb{C}(\{b/x\})(0). \tag{3.23}$$

The mentioned axioms are again called the *additivity, variability, consistency, singleton independency*, *preservation of non-existence* and *existence*, respectively. The first two axioms have the same meaning as the axioms of additivity and variability for the θ-cardinalities. The axiom of consistency also states that the values of θ-cardinalities must belong to $\{\bot, \top\}$ for the crisp sets. However, if $A \subseteq X$ is a crisp set with $|A| = i$, then we have $\mathbb{C}(A)(i \cdot 1) = \bot$, contrary to $\mathbb{C}(A)(i \cdot 1) = \top$ for the θ-cardinalities. This value could be interpreted as a truth value[7] of the assertion that the crisp set A has not the cardinality which is equal to i. For instance, the cardinality of the empty set is equal to 0 and therefore $\mathbb{C}(\emptyset)(0) = \bot$ holds for any $\overline{\theta}$-cardinality. A consequence of the mentioned consideration is the fact that

[7]The term of the truth value could be contentious for the values of general dually residuated lattices. In this case, we use them in the common meaning.

the membership degree of the $\overline{\theta}$-cardinality of a finite $\mathbf{L}^{\mathrm{d}}$-set A in the value i expresses an extent of "truth" of the statement that the $\mathbf{L}^{\mathrm{d}}$-set A has not i elements. In other words, the value $\mathbb{C}(A)(i)$ describes the degree of "truth" of the statement that A has either more or less (but not equal) elements than i. Further, the axiom of singleton independency has the same meaning as in the case of the θ-cardinalities. The last two axioms are proposed to be dual to the corresponding axioms of preservation of non-existence and existence for the θ-cardinalities. Moreover, these axioms support our interpretation of the $\overline{\theta}$-cardinalities values. In fact, let us suppose that $a \leq b$ are elements of a linearly ordered dually residuated lattice and $\overline{\theta} = \vee$. From $a \leq b$ we can say that the $\mathbf{L}^{\mathrm{d}}$-set $B = \{b/x\}$ has more elements than the $\mathbf{L}^{\mathrm{d}}$-set $A = \{a/x\}$[8]. Hence, it is natural to expect that the degree of truth of the statement, that the $\mathbf{L}^{\mathrm{d}}$-set has no element, must be greater for B than A and thus $\mathbb{C}(A)(0) \leq \mathbb{C}(B)(0)$. It is easy to see that this inequality is, however, a simple consequence of the axiom of existence preservation. Analogously, the degree of truth of the statement, that the $\mathbf{L}$-set has not just one element, must be greater for A than B and thus $\mathbb{C}(A)(1) \geq \mathbb{C}(B)(1)$. Again, this inequality is a simple consequence of the axiom of non-existence preservation.

The following example of the $\vee$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets could be understood as dual to the $\wedge$-cardinality (constructed using the a-cuts of fuzzy sets) defined in Ex. 3.2.1 on p. 60.

Example 3.3.1. Let $\mathbf{L}$ be a complete residuated lattice and $\mathbf{L}^{\mathrm{d}}$ be a divisible complete dually residuated lattice with the same support L. Recall that A_a^d denotes a dual a-cut of fuzzy set A (see p. 12). Then a $\vee$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\vee}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ can be defined as follows

$$\mathbb{C}(A)(i) = \begin{cases} \bigwedge\{a \mid a \in L \text{ and } |X \setminus A_a^{\mathrm{d}}| \leq i\}, & i \neq n \\ \bot, & i = n. \end{cases} \tag{3.24}$$

Note that for $n = \infty$ we have $\mathbb{C}(A)(n) = \bigwedge\{a \mid a \in L \text{ and } |X \setminus A_a^{\mathrm{d}}| \leq n\} = \bot$ and thus the definition could be simplified. Let us show that $\mathbb{C}$ is really the $\vee$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets. Obviously, $\mathbb{C}(A)(i) \leq \mathbb{C}(A)(j)$, whenever $i \leq j$, holds for arbitrary $i, j \in N_n$ and thus $\mathbb{C}(A)$ is the $\vee$-convex $\mathbf{L}^{\mathrm{d}}$-set for every $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$. Let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary pairwise disjoint finite $\mathbf{L}^{\mathrm{d}}$-sets. If $i = n$, then we have

$$(\mathbb{C}(A) +^{\overline{\theta}} \mathbb{C}(B))(n) = \bigwedge_{\substack{k,l \in N_n \\ k \boxplus l = n}} (\mathbb{C}(A)(k) \vee \mathbb{C}(B)(l)) = \mathbb{C}(A)(n) \vee \mathbb{C}(B)(n) = \bot.$$

[8] It is a consequence of the fact that the element x belongs to B with the greater membership degree than to A.

Hence, we obtain $(\mathbb{C}(A) +^{\bar{\theta}} \mathbb{C}(B))(n) = \mathbb{C}(A \cup B)(n)$ for every disjoint $\mathbf{L}^{\mathrm{d}}$-sets $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$. Let us suppose that $i < n$. It is easy to see that $|X \setminus (A \cup B)^{\mathrm{d}}_c| = |X \setminus A^{\mathrm{d}}_c| + |X \setminus B^{\mathrm{d}}_c|$ holds for every disjoint $\mathbf{L}^{\mathrm{d}}$-sets $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $X \setminus A^{\mathrm{d}}_a \subseteq X \setminus A^{\mathrm{d}}_b$, whenever $a \geq b$. Furthermore, we have $i = k \boxplus l = k + l$ for $i < n$. Hence, we have

$$(\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(i) = \bigwedge_{\substack{k,l \in N_n \\ k \boxplus l = i}} (\mathbb{C}(A)(k) \vee \mathbb{C}(B)(l)) =$$

$$\bigwedge_{\substack{k,l \in N_n \\ k \boxplus l = i}} \Big(\bigwedge_{\substack{a \in L \\ |X \setminus A^{\mathrm{d}}_a| \leq k}} a \vee \bigwedge_{\substack{b \in L \\ |X \setminus B^{\mathrm{d}}_b| \leq l}} b \Big) = \bigwedge_{\substack{k,l \in N_n \\ k \boxplus l = i}} \bigwedge_{\substack{a \in L \\ |X \setminus A^{\mathrm{d}}_a| \leq k}} \bigwedge_{\substack{b \in L \\ |X \setminus B^{\mathrm{d}}_b| \leq l}} (a \vee b) \geq$$

$$\bigwedge_{\substack{a,b \in L \\ |X \setminus A^{\mathrm{d}}_{a \vee b}| \boxplus |X \setminus B^{\mathrm{d}}_{a \vee b}| \leq i}} (a \vee b) = \bigwedge_{\substack{c \in L \\ |X \setminus A^{\mathrm{d}}_c| \boxplus |X \setminus B^{\mathrm{d}}_c| \leq i}} c = \bigwedge_{\substack{c \in L \\ |X \setminus (A \cup B)^{\mathrm{d}}_c| \leq i}} c = \mathbb{C}(A \cup B)(i).$$

Conversely, let $|X \setminus (A \cup B)^{\mathrm{d}}_c| \leq i$ for some $c \in L$. Then obviously there exist $k_c, l_c \in N_n$ such that $|X \setminus A^{\mathrm{d}}_c| \leq k_c$, $|X \setminus B^{\mathrm{d}}_c| \leq l_c$ and $k_c \boxplus l_c = k_c + l_c = i$. Hence, we obtain the following inequality

$$\bigwedge_{\substack{a \in L \\ |X \setminus A^{\mathrm{d}}_a| \leq k_c}} a \vee \bigwedge_{\substack{b \in L \\ |X \setminus B^{\mathrm{d}}_b| \leq l_c}} b \leq c \vee c = c.$$

Since to each $c \in L$ with $|X \setminus (A \cup B)^{\mathrm{d}}_c| \leq i$ there exist $k_c, l_c \in N_n$ with the considered properties, then we can write

$$\mathbb{C}(A \cup B)(i) = \bigwedge_{\substack{c \in L \\ |X \setminus (A \cup B)^{\mathrm{d}}_c| \leq i}} c \geq$$

$$\bigwedge_{\substack{k,l \in N_n \\ k \boxplus l = i}} \Big(\bigwedge_{\substack{a \in L \\ |X \setminus A^{\mathrm{d}}_a| \leq k}} a \vee \bigwedge_{\substack{b \in L \\ |X \setminus B^{\mathrm{d}}_b| \leq l}} b \Big) = (\mathbb{C}(A) +^{\vee} \mathbb{C}(B))(i)$$

and thus the additivity of $\mathbb{C}$ is satisfied. Further, if $i > |\mathrm{Supp}(A)|$ then $|X \setminus A^{\mathrm{d}}_0| \leq i$ and thus $\mathbb{C}(A)(i) = \bot$. Hence, the variability is fulfilled. Let $A \subseteq X$ be a crisp set. If $i < |A|$, then we have $\mathbb{C}(A)(i) = \bigwedge\{a \in L \mid |X \setminus A^{\mathrm{d}}_a| \leq i\} = \top \in \{\bot, \top\}$. Moreover, if $|A| = i < n$, then $|X \setminus A^{\mathrm{d}}_0| = i \leq i$ and $\mathbb{C}(A)(i \cdot 1) = \mathbb{C}(A)(i) = \bot$. If $|A| = i \geq n$, then $\mathbb{C}(A)(i \cdot 1) = \mathbb{C}(A)(n) = \bot$. Hence, the axiom of consistency is also satisfied. The singleton independency is clearly fulfilled. Finally, we have $\mathbb{C}(\{a/x\})(0) = a$, because $|X \setminus A^{\mathrm{d}}_a| = 0$ and a is the least element with the desired property, and from the consistency we have $\mathbb{C}(\{a/x\})(1) = \bot$. Hence, the preservation of non-existence and the preservation of existence are also fulfilled.

Example 3.3.2. Let $\mathbf{L}_{\mathrm{L}}$ and $\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}$ be the complete residuated and dually residuated lattices determined by the Łukasiewicz t-norm and t-conorm and X is a non-empty universe. An S_{L}-cardinality $\mathbb{C}_{S_{\mathrm{L}}} : \mathcal{FIN}_{\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}}^{S_{\mathrm{L}}}(N_1)$ could be defined as follows

$$\mathbb{C}_{S_{\mathrm{L}}}(A)(i) = \begin{cases} \min(\sum_{k=1}^{m} A(x_k), 1), & i = 0, \\ 0, & i = 1, \end{cases} \tag{3.25}$$

where $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$[9]. Let us prove that the definition is correct. Obviously, $\mathbb{C}_{S_{\mathrm{L}}}(A)$ is trivially the S_{L}-convex $\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}$-set for every $A \in \mathcal{FIN}_{\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}}(X)$. Let $A, B \in \mathcal{FIN}_{\mathbf{L}_{\mathrm{L}}^{\mathrm{d}}}(X)$ be arbitrary disjoint $\mathbf{L}^{\mathrm{d}}$-sets. Then we have

$$(\mathbb{C}_{S_{\mathrm{L}}}(A) +^{S_{\mathrm{L}}} \mathbb{C}_{S_{\mathrm{L}}}(B))(0) = \bigwedge_{\substack{i_1, i_2 \in \{0,1\} \\ i_1 \boxplus i_2 = 0}} \min(\mathbb{C}_{S_{\mathrm{L}}}(A)(i_1) + \mathbb{C}_{S_{\mathrm{L}}}(B)(i_2), 1) =$$
$$\min(\mathbb{C}_{S_{\mathrm{L}}}(A)(0) + \mathbb{C}_{S_{\mathrm{L}}}(B)(0), 1) = \min(\sum_{k \in \mathrm{Supp}(A)} A(x_k) + \sum_{l \in \mathrm{Supp}(B)} B(x_l), 1) =$$
$$\min(\sum_{r \in \mathrm{Supp}(A \cup B)} (A \cup B)(x_r), 1) = \mathbb{C}_{S_{\mathrm{L}}}(A \cup B)(0)$$

and

$$(\mathbb{C}_{S_{\mathrm{L}}}(A) +^{S_{\mathrm{L}}} \mathbb{C}_{S_{\mathrm{L}}}(B))(1) = \bigwedge_{\substack{i_1, i_2 \in \{0,1\} \\ i_1 \boxplus i_2 = 1}} \min(\mathbb{C}_{S_{\mathrm{L}}}(A)(i_1) + \mathbb{C}_{S_{\mathrm{L}}}(B)(i_2), 1) =$$
$$\min(\mathbb{C}(A)(1) + \mathbb{C}(B)(1), 1) = 0 = \mathbb{C}_{S_{\mathrm{L}}}(A \cup B)(1)$$

and hence the axiom of additivity is satisfied. Obviously, $i > |\mathrm{Supp}(A)|$ and simultaneously $j > |\mathrm{Supp}(B)|$ hold just for $A = B = \emptyset$. Hence, we have $\mathbb{C}(A)(i) = \mathbb{C}(B)(j) = 0$ and thus the variability is verified. If $A \subseteq X$ is a crisp set, then $\mathbb{C}(A)(0)$ and $\mathbb{C}(A)(1)$ clearly belong to $\{0, 1\}$. Moreover, $\mathbb{C}_{S_{\mathrm{L}}}(\emptyset)(0 \cdot 1) = \mathbb{C}_{S_{\mathrm{L}}}(\emptyset)(0) = 0$ and if $|A| = i > 0$, then $\mathbb{C}(A)(i \cdot 1) = \mathbb{C}(A)(1) = 0$. Hence, the consistency is satisfied. Finally, we have $\mathbb{C}_{S_{\mathrm{L}}}(\{a/x\})(0) = a$ and $\mathbb{C}_{S_{\mathrm{L}}}(\{a/x\})(1) = 0$ for any $x \in X$. Hence, the singleton independency is fulfilled. Furthermore, we have

$$\mathbb{C}_{S_{\mathrm{L}}}(\{S_{\mathrm{L}}(a, b)/x\})(0) = \mathbb{C}_{S_{\mathrm{L}}}(\{\min(a + b, 1)/x\})(0) = \min(a + b, 1) =$$
$$\min(\mathbb{C}_{S_{\mathrm{L}}}(\{a/x\})(0) + \mathbb{C}_{S_{\mathrm{L}}}(\{b/x\})(0), 1) = S_{\mathrm{L}}(\mathbb{C}_{S_{\mathrm{L}}}(\{a/x\})(0), \mathbb{C}_{S_{\mathrm{L}}}(\{b/x\})(0))$$

and $\mathbb{C}(\{T_{\mathrm{L}}(a, b)/x\})(1) = 0 = S_{\mathrm{L}}(\mathbb{C}(\{a/x\})(1), \mathbb{C}(\{b/x\})(1))$. Hence, the axiom of preservation of existence and non-existence is also satisfied and

[9] Recall that if $\mathrm{Supp}(A) = \emptyset$, i.e. $A = \emptyset$, then $\min(\sum_{x \in \emptyset} A(x), 1) = 0$. See also p. 10.

$\mathbb{C}_{S_\mathrm{L}}$ is really the S_L-cardinality of finite $\mathbf{L}^\mathrm{d}$-sets. We can see that this S_L-cardinality is independent on the choice of the left continuous t-norm, because the axiom of non-existence preservation is trivially satisfied.

Obviously, in the second example the range of S_L-cardinality $\mathbb{C}_{S_\mathrm{L}}$ contains just the singletons with the same supports containing the element 0. In this case the S_L-cardinality of an $\mathbf{L}^\mathrm{d}$-set A is uniquely determined by the height of $\mathbb{C}_{S_\mathrm{L}}(A)$ or equivalently by the value $\mathbb{C}_{S_\mathrm{L}}(A)(0)$. It may be used for a natural construction of the scalar cardinalities as follows. Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^\mathrm{d}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^\mathrm{d}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^\mathrm{d}$-sets such that its range contains just singletons with the same supports. Further, let us denote $hg : \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^\mathrm{d}}(N_n) \to L$, where L is the support of $\mathbf{L}^\mathrm{d}$, the mapping assigning to each $\mathbf{L}^\mathrm{d}$-set its height. Then the mapping $\mathbb{C}_\mathrm{sc} : \mathcal{FIN}_{\mathbf{L}^\mathrm{d}}(X) \to L$, given by the composition $\mathbb{C}_\mathrm{sc} = hg \circ \mathbb{C}$, is called the *$\overline{\theta}$-scalar cardinality of finite $\mathbf{L}^\mathrm{d}$-sets*. Note that not all scalar cardinalities, introduced by Wygralak's axiomatic system in [107], may be also established as a little modification of the $\overline{\theta}$-scalar cardinalities (for $\mathbf{L}^\mathrm{d} = \mathrm{R}_0^+$). The reason is that the $\overline{\theta}$-scalar cardinalities have to satisfied the axioms of non-existence and existence preservations, which are much stronger than the Wygralak's axiom of singleton monotonicity (cf. [107,109]). On the other hand, using the presented modality we can define other forms of the scalar cardinalities than has been done up to now. However, in the following parts the $\overline{\theta}$-scalar cardinalities will not be the focus of our attention and we will deal only with the $\overline{\theta}$-cardinalities of finite $\mathbf{L}^\mathrm{d}$-sets, in general. The following lemma is dual to Lemma 3.2.1. Recall that E denotes the neutral element and $\mathbf{0}$ denotes the least element of the bounded *poc*-monoid $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^\mathrm{d}}(\mathbb{N}_n)$.

Lemma 3.3.1. *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^\mathrm{d}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^\mathrm{d}$-sets. Then $\mathbb{C}(\emptyset) = E$ or $\mathbb{C}(\emptyset) = \mathbf{0}$. Moreover, if $\mathbb{C}(\emptyset) = \mathbf{0}$, then $\mathbb{C}(A)$ is closed $\mathbf{L}^\mathrm{d}$-set[10] for any $A \in \mathcal{FIN}_{\mathbf{L}^\mathrm{d}}(X)$ and thus $\mathbb{C}(A)$ is a non-increasing mapping, i.e. $\mathbb{C}(A)(i) \leq \mathbb{C}(j)$, whenever $i \geq j$.*

Proof. The first statement is straightforward consequence of the axioms of variability and consistency. If $\mathbb{C}(\emptyset) = \mathbf{0}$, then from (i) of Lemma 3.1.5 we have $\mathbb{C}(A) = \mathbb{C}(A \cup \emptyset) = \mathbb{C}(A) +^{\overline{\theta}} \mathbb{C}(\emptyset) = \mathbb{C}(A) +^{\overline{\theta}} \mathbf{0} = \mathbb{C}(A)^c$. Hence, we obtain that $\mathbb{C}(A)$ is the closed $\mathbf{L}^\mathrm{d}$-set. The rest of the proof follows from the definition of the closed $\mathbf{L}^\mathrm{d}$-sets. □

Lemma 3.3.2. *Let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^\mathrm{d}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^\mathrm{d}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^\mathrm{d}$-sets and $A \in \mathcal{FIN}_{\mathbf{L}^\mathrm{d}}(X)$ such that* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$. *Then we*

[10] It means that $\mathbb{C}(A) = \mathbb{C}(A)^c$, where ^{c} is the dual closure operator defined on p. 57.

have

$$\mathbb{C}(A)(i) = \bigwedge_{\substack{i_1,\dots,i_m \in N_n \\ i_1 \boxplus \cdots \boxplus i_m = i}} \overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i_k), \tag{3.26}$$

for every $i \in N_n$.

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be a finite $\mathbf{L}^{\mathrm{d}}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$. Due to the additivity, we obtain

$$\begin{aligned}\mathbb{C}(A)(i) =& \mathbb{C}(\{A(x_1)/x_1\} \cup \cdots \cup \{A(x_m)/x_m\})(i) = \\ & (\mathbb{C}(\{A(x_1)/x_1\}) +^{\overline{\theta}} \cdots +^{\overline{\theta}} \mathbb{C}(\{A(x_m)/x_m\}))(i).\end{aligned}$$

for every $i \in N_n$. Applying the definition of $+^{\overline{\theta}}$, we obtain the desired statement. □

A stronger result showing how to compute $\overline{\theta}$-fuzzy cardinalities is given in the following theorem. It is dual to Theorem 3.2.3.

Theorem 3.3.3. *Let* $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ *be a* $\overline{\theta}$*-cardinality of finite* $\mathbf{L}^{\mathrm{d}}$*-sets and* $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ *with* $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$*. Then we have*

$$\mathbb{C}(A)(i) = \bigwedge_{\substack{i_1,\dots,i_m \in \{0,1\} \\ i_1 \boxplus \cdots \boxplus i_n = i}} \overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \tag{3.27}$$

for every $i \in N_n$*, where* $i \leq m$*. Moreover, if* $m < n$*, then* $\mathbb{C}(A)(i) = \bot$ *or* $\mathbb{C}(A) = \top$ *holds for every* $m < i \leq n$*, respectively.*

Proof. Let $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be a finite $\mathbf{L}^{\mathrm{d}}$-set with $\mathrm{Supp}(A) \subseteq \{x_1, \dots, x_m\}$. First, we will suppose that $m < n$. The statement is evidently true for $i \leq 1$. Further, let $1 < i \leq m$ and $i_1, \dots, i_m \in N_n$ be a finite sequence such that $i_1 \boxplus \cdots \boxplus i_m = i$. Put $I = \{i_k \mid i_k \in N_n,\ i_k \notin \{0,1\}\}$. The statement is true, if $I = \emptyset$. Let $i_k \in I$ be an element and $i_k = r$. Then there exist at least $r-1$ elements from $\{i_1, \dots, i_m\}$ that are equal to 0 (see the proof of Theorem 3.2.3 on p. 62). Thus we can choose the elements $i_{k_1} = \cdots = i_{k_{r-1}} = 0$. Due to the variability, consistency and Lemma 3.3.1, we can write $\mathbb{C}(\{A(x_k)/x_k\})(i_k) = \mathbb{C}(\{A(x_k)/x_k\})(2) = \mathbb{C}(\{A(x_l)/x_l\})(2) = \mathbb{C}(\{\bot/x_k\})(2) = \mathbb{C}(\emptyset)(2) \in \{\bot, \top\}$, where $l \in \{k_1, \dots, k_{r-1}\}$. Hence, we obtain $\mathbb{C}(\{\top_2/x\})(0) = \bot_2$. Now applying the $\overline{\theta}$-convexity of $\mathbf{L}^{\mathrm{d}}$-sets, the existence preservation and the fact that $\mathbb{C}(\{\bot/x_k\})(0) = \bot$, we obtain

$$\begin{aligned}&\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_l)/x_l\})(0) = \\ &\quad \mathbb{C}(\{A(x_l)/x_l\})(2)\overline{\theta}\mathbb{C}(\{A(x_l)/x_l\})(0) \geq \mathbb{C}(\{A(x_l)/x_l\})(1)\end{aligned}$$

for every $l \in \{k_1, \dots, k_{r-1}\}$ and

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k) = \mathbb{C}(\{\perp/x_k\})(2)\overline{\theta}\perp =$$
$$\mathbb{C}(\{\perp/x_k\})(2)\overline{\theta}\mathbb{C}(\{\perp/x_k\})(0) \geq$$
$$\mathbb{C}(\{\perp/x_k\})(1) = \mathbb{C}(\{\perp\theta A(x_k)/x_k\})(1) =$$
$$\mathbb{C}(\{\perp/x_k\})(1)\overline{\theta}\mathbb{C}(\{A(x_x)/x_k\})(1) \geq \mathbb{C}(\{A(x_k)/x_k\})(1).$$

Since $\mathbb{C}(\{A(x_k)/x_k\})(i_k) \in \{\perp, \top\}$ and $\perp$, $\top$ are the idempotent elements of $\mathbf{L}^{\mathrm{d}}$ with respect to $\overline{\theta}$, then using the previous inequalities we can write

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\overline{\theta}\cdots\overline{\theta}\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(0) =$$
$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\overline{\theta}\cdots\overline{\theta}$$
$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(0) \geq$$
$$\mathbb{C}(\{A(x_k)/x_k\})(1)\overline{\theta}\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1)\overline{\theta}\cdots\overline{\theta}\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_{r-1}}\})(1).$$

Hence, we can create a new sequence of elements $i'_1, \cdots, i'_m \in N_n$ such that $i'_1 \boxplus \cdots \boxplus i'_m = i$, $I' = \{i'_k \mid i'_k \in N_n, i'_k \notin \{0,1\}\} = I \setminus \{i_k\}$ and

$$\overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \geq \overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i'_k). \tag{3.28}$$

Obviously, the mentioned procedure can be repeated as long as all elements from I are removed. As to each sum $i_1 \boxplus \cdots \boxplus i_m = i$ there exists a sum $i'_1 \boxplus \cdots \boxplus i'_m = i$ such that $i'_k \in \{0,1\}$ and the inequality (3.28) is satisfied, the desired equality (3.27) is true for any $1 < i \leq m$. Further, we will suppose that $m \geq n$. If $i < n$, then we can apply the same procedure as in the previous case to obtain the desired statement. Let $i = n$ and $i_1 \boxplus \cdots \boxplus i_m = n$. Finally, if $i_{k_1}, \dots, i_{k_r}$ are all elements from $i_1, \dots, i_m$ that are equal to 0 and $i_k > 1$, then we have

$$\mathbb{C}(\{A(x_k)/x_k\})(i_k)\overline{\theta}\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(0)\overline{\theta}\cdots\overline{\theta}\mathbb{C}(\{A(x_{k_{r-1}})/x_{k_r}\})(0) \geq$$
$$\mathbb{C}(\{A(x_x)/x_k\})(1)\overline{\theta}\mathbb{C}(\{A(x_{k_1})/x_{k_1}\})(1)\overline{\theta}\cdots\overline{\theta}\mathbb{C}(\{A(x_{k_r})/x_{k_r}\})(1).$$

Since $\mathbb{C}(\{A(x_k)/x_k\})(i_k) \geq \mathbb{C}(\{A(x_k)/x_k\})(1)$ holds for any $i_k \geq 1$, then we have

$$\overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(i_k) \geq \overline{\bigominus_{k=1}^{m}} \mathbb{C}(\{A(x_k)/x_k\})(1).$$

Hence, the first part of the theorem is proved. Further, let us suppose that $m < n$. Due to the variability and Lemma 3.3.1, we have $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \bot$ or $\mathbb{C}(A)(i) = \mathbb{C}(\emptyset)(i) = \top$ for any $i \in N_n$ such that $m < i \leq n$, respectively, and the second part of this theorem is proved, too. $\square$

Before we introduce an analogical representation of the $\overline{\theta}$-cardinalities of finite $\mathbf{L}^{\rm d}$-sets, we will establish two types of homomorphisms. Let $\mathbf{L}_1^{\rm d}$ and $\mathbf{L}_2^{\rm d}$ be (complete) dually residuated lattices and $h : L_1 \to L_2$ be a mapping. We say that h is the (*complete*) $\overline{\theta}$*-homomorphism* from $\mathbf{L}_1^{\rm d}$ to $\mathbf{L}_2^{\rm d}$, if h is a (complete) homomorphism from the substructure $(L_1, \overline{\theta}_1, \bot_1)$ of the dually residuated lattice $\mathbf{L}_1^{\rm d}$ to the substructure $(L_2, \overline{\theta}_2, \bot_2)$ of the dually residuated lattices $\mathbf{L}_2^{\rm d}$, i.e. $h(a\overline{\theta}_1 b) = h(a)\overline{\theta}_2 h(b)$ and $h(\bot_1) = \bot_2$. Further, let $\mathbf{L}_1$ and $\mathbf{L}_2^{\rm d}$ be (complete) residuated and dually residuated lattices, respectively, $h : L_1 \to L_2$ be a mapping. We say that h is the (*complete*) θ_d*-homomorphism*, if h is a (complete) homomorphism from the substructure $(L_1, \theta_1, \top_1)$ of the residuated lattice $\mathbf{L}_1$ to the substructure $(L_2, \overline{\theta}, \bot_2)$ of the dually residuated lattice $\mathbf{L}_2$, i.e. $h(a\theta_1 b) = h(a)\overline{\theta} h(b)$ and $h(\top_1) = \bot_2$.

Lemma 3.3.4. *Let $f, g : L \to L$ be a $\overline{\theta}$- and θ_d-homomorphism from $\mathbf{L}^{\rm d}$ to $\mathbf{L}^{\rm d}$ and from $\mathbf{L}$ to $\mathbf{L}^{\rm d}$, respectively, such that $f(\top) \in \{\bot, \top\}$ and $g(\bot) \in \{\bot, \top\}$. Further, let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\rm d}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\rm d}}(N_n)$ be a mapping defined by induction as follows*

$$\mathbb{C}_{f,g}(\{a/x\})(0) = f(a),\ \mathbb{C}_{f,g}(\{a/x\})(1) = g(a) \ \textit{and}$$
$$\mathbb{C}_{f,g}(\{a/x\})(k) = g(\bot),\ k > 1$$

hold for every singleton $\{a/x\} \in \mathcal{FIN}_{\mathbf{L}}(X)$ and

$$\mathbb{C}_{f,g}(A) = \mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\overline{\theta}} \cdots +^{\overline{\theta}} \mathbb{C}_{f,g}(\{A(x_m)/x_m\})$$

holds for every $A \in \mathcal{FIN}_{\mathbf{L}^{\rm d}}(X)$, where $\mathrm{Supp}(A) = \{x_1, \dots, x_m\}$. *Then the mapping $\mathbb{C}_{f,g}$ is a* $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\rm d}$-sets generated by the $\overline{\theta}$- and θ_d-homomorphisms f and g, *respectively.*

Proof. First, we will prove that the definition of the mapping $\mathbb{C}_{f,g}$ is correct. Let $\{a/x\}$ be a singleton from $\mathcal{FIN}_{\mathbf{L}^{\rm d}}(X)$. If $n = 1$, then $\mathbb{C}_{f,g}(\{a/x\})$ is clearly a $\overline{\theta}$-convex $\mathbf{L}^{\rm d}$-set. Let $n > 1$. Since $g(\bot) = g(\bot\theta a) = g(\bot)\overline{\theta} g(a) \geq g(a)$ holds for any $a \in L$, then we have

$$\mathbb{C}_{f,g}(\{a/x\})(0)\overline{\theta}\mathbb{C}_{f,g}(\{a/x\})(2) = f(a)\overline{\theta} g(\bot) \geq g(a) = \mathbb{C}_{f,g}(\{a/x\})(1).$$

Furthermore, this inequality is trivially fulfilled for each triplet $0 < i \leq j \leq k$ from N_n. Hence, the mapping $\mathbb{C}_{f,g}$ assigns a $\overline{\theta}$-convex $\mathbf{L}^{\rm d}$-sets to each singletons from $\mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$. Since the sum of $\overline{\theta}$-convex of $\mathbf{L}^{\mathrm{d}}$-sets is again a $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-set (according to Theorem 3.1.3), we obtain that the definition of $\mathbb{C}_{f,g}$ is correct. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary disjoint $\mathbf{L}^{\mathrm{d}}$-sets, where $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \dots, y_s\}$. Due to the associativity of the operation $+^{\overline{\theta}}$ and the definition of $\mathbb{C}_{f,g}$, we have

$$\begin{aligned}
&\mathbb{C}_{f,g}(A \cup B) = \\
&\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\overline{\theta}} \cdots +^{\overline{\theta}} \mathbb{C}_{f,g}(\{A(x_r)/x_r\}) +^{\overline{\theta}} \mathbb{C}_{f,g}(\{B(y_1)/y_1\}) +^{\overline{\theta}} \cdots \\
&\quad +^{\overline{\theta}} \mathbb{C}_{f,g}(\{B(y_s)/y_s\}) = \big(\mathbb{C}_{f,g}(\{A(x_1)/x_1\}) +^{\overline{\theta}} \cdots +^{\overline{\theta}} \mathbb{C}_{f,g}(\{A(x_r)/x_r\})\big) +^{\overline{\theta}} \\
&\qquad \big(\mathbb{C}_{f,g}(\{B(y_1/y_1\}) +^{\overline{\theta}} \cdots +^{\overline{\theta}} \mathbb{C}_{f,g}(\{B(y_s)/y_s\})\big) = \mathbb{C}_{f,g}(A) +^{\overline{\theta}} \mathbb{C}_{f,g}(B).
\end{aligned}$$

Hence, the mapping $\mathbb{C}_{f,g}$ satisfies the additivity. Let $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be an $\mathbf{L}^{\mathrm{d}}$-set with $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$. From the additivity of $\mathbb{C}_{f,g}$ we obtain

$$\mathbb{C}(A)(i) = \bigwedge_{\substack{i_1,\dots,i_r \in N_n \\ i_1 \boxplus \cdots \boxplus i_r = i}} \mathbb{C}(\{A(x_1)/x_1)(i_1) \overline{\theta} \cdots \overline{\theta} \mathbb{C}(\{A(x_r)/x_r)(i_r). \tag{3.29}$$

Let us denote $\mathbf{i} \in N_n^r$ an r-dimensional vector of elements from N_n, i.e. $\mathbf{i} = (i_1, \dots, i_r)$, such that $i_1 \boxplus \cdots \boxplus i_r = i$. The set of all such vectors will be denoted by $\mathcal{I}$. Further, let us denote $K_{\mathbf{i}} = \{k \mid i_k = 0\}$, $L_{\mathbf{i}} = \{l \mid i_l = 1\}$ and $M_{\mathbf{i}} = \{m \mid i_m > 1\}$. Clearly, $K_{\mathbf{i}} \cup L_{\mathbf{i}} \cup M_{\mathbf{i}} = \{1, \dots, r\}$ and they are pairwise disjoint. Finally, let us establish $a_{K_{\mathbf{i}}} = \overline{\Theta}_{k \in K_{\mathbf{i}}} f(A(x_k))$, $a_{L_{\mathbf{i}}} = \overline{\Theta}_{l \in L_{\mathbf{i}}} g(A(x_l))$, $a_{M_{\mathbf{i}}} = \overline{\Theta}_{m \in M_{\mathbf{i}}} g(\perp)$. Recall that $\overline{\Theta}_{a \in \emptyset}\, a = \perp$ (see p. 10). Then (3.29) can be rewrite as follows

$$\mathbb{C}(A)(i) = \bigwedge_{\mathbf{i} \in \mathcal{I}} a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}}. \tag{3.30}$$

Now, let us suppose that $i > r$, then necessary $M_{\mathbf{i}} \neq \emptyset$ for every $\mathbf{i} \in \mathcal{I}$. Since $a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} \geq a_{M_{\mathbf{i}}} = g(\perp)$ holds for every $\mathbf{i} \in \mathcal{I}$, then $\mathbb{C}_{f,g}(A)(i) = \bigwedge_{\mathbf{i} \in \mathcal{I}} a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} \geq \bigwedge_{\mathbf{i} \in \mathcal{I}} g(\perp) = g(\perp)$. On the other hand, there exists $\mathbf{i} \in \mathcal{I}$ such that $K_{\mathbf{i}} = \emptyset$. Since $g(\perp) \geq g(a)$ holds for every $a \in L$, we have $a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} = \perp \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} g(\perp) \leq g(\perp)$ and thus $\mathbb{C}_{f,g}(A)(i) \leq g(\perp)$. Hence, $\mathbb{C}_{f,g}$ satisfies the axiom of variability. The axiom of consistency is a simple consequence of the previous consideration. In fact, let us suppose that $A \subseteq X$ is a crisp set. If $i > |A|$, then $\mathbb{C}(A)(i) = g(\perp) \in \{\perp, \top\}$ with regard to the presumption of the values of $g(\perp)$. If $i \leq |A|$, then for every $\mathbf{i} \in \mathcal{I}$ we have $a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} \in \{\perp, \top\}$, because $a_{K_{\mathbf{i}}}, a_{L_{\mathbf{i}}}, a_{M_{\mathbf{i}}} \in \{\perp, \top\}$, where for instance $a_{K_{\mathbf{i}}} = f(\top \overline{\theta} \cdots \overline{\theta} \top) = f(\top) \in \{\perp, \top\}$, if $K_{\mathbf{i}} \neq \emptyset$, and $a_{K_{\mathbf{i}}} = \perp$, if $K_{\mathbf{i}} = \emptyset$. Hence, $\mathbb{C}(A)(i) = \bigwedge_{\mathbf{i} \in \mathcal{I}} a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} \in \{\perp, \top\}$. Moreover, if we suppose $|A| =$

$r \leq n$, then, obviously, there exists $L_{\mathbf{i}}$ such that $K_{\mathbf{i}} = M_{\mathbf{i}} = \emptyset$. Hence, we have $\mathbb{C}(A)(r) = \mathbb{C}(A)(r \cdot 1) \leq a_{K_{\mathbf{i}}} \overline{\theta} a_{L_{\mathbf{i}}} \overline{\theta} a_{M_{\mathbf{i}}} = a_{L_{\mathbf{i}}} = g(\top\theta \cdots \theta\top) = g(\top) = \bot$ and therefore $\mathbb{C}(A)(r) = \bot$. Further, if $|A| = r > n$, then $n = r \cdot 1$ and, analogously, we obtain $\mathbb{C}(A)(n) = \mathbb{C}(r \cdot 1) = a_{L_{\mathbf{i}}} = \bot$, where again $L_{\mathbf{i}}$ is such set that $K_{\mathbf{i}} = M_{\mathbf{i}} = \emptyset$. The conditions of independency and preserving structure follow immediately from the definition of mappings f and g. Thus, we have shown that $\mathbb{C}_{f,g}$ is the θ-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets. □

Theorem 3.3.5. (representation of $\overline{\theta}$-cardinalities) *Let $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ be a complete residuated and dually residuated lattice, respectively, and $\mathcal{CV}^{\theta}_{\mathbf{L}}(\mathbb{N}_n)$ be a bounded poc-monoid. Further, let $\mathbb{C} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a mapping satisfying the axiom of additivity. Then the following statements are equivalent:*

- (i) *$\mathbb{C}$ is a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets,*
- (ii) *there exist a $\overline{\theta}$-homomorphism $f : L \to L$ and a θ_d-homomorphism $g : L \to L$, such that $f(\top) \in \{\bot, \top\}$, $g(\bot) \in \{\bot, \top\}$ and*

$$\mathbb{C}(\{a/x\})(0) = f(a),\ \mathbb{C}(\{a/x\})(1) = g(a),\ \mathbb{C}(\{a/x\})(k) = g(\bot)$$

hold for arbitrary $a \in L$, $x \in X$ and $k > 1$.

Proof. First, we will show that (i) implies (ii). Let $\mathbb{C}$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets. Let us define the mappings $f, g : L \to L$ as follows

$$f(a) = \mathbb{C}(\{a/x\})(0), \tag{3.31}$$
$$g(a) = \mathbb{C}(\{a/x\})(1). \tag{3.32}$$

Due to the axioms of the existence and non-existence preservations by $\mathbb{C}$, we have

$$= f(a\overline{\theta}b) = \mathbb{C}(\{a\overline{\theta}b/x\})(0) = \mathbb{C}(\{a/x\})(0)\overline{\theta}\mathbb{C}(\{b/x\})(0) = f(a)\overline{\theta}f(b),$$
$$g(a\theta b) = \mathbb{C}(\{a\overline{\theta}b/x\})(1) = \mathbb{C}(\{a/x\})(1)\overline{\theta}\mathbb{C}(\{b/x\})(1) = g(a)\overline{\theta}g(b).$$

According to the consistence, we have $f(\bot) = \mathbb{C}(\{\bot/x\})(0) = \bot$ and $g(\top) = \mathbb{C}(\{\top/x\}(1) = \bot$. Hence, we obtain that f is a $\overline{\theta}$-homomorphism and g is a θ_d-homomorphism of the relevant algebraic structures. Moreover, the values $f(\top) = \mathbb{C}(\{\top/x\})(0)$ and $g(\bot) = \mathbb{C}(\{\bot/x\})(1)$ belong to $\{\bot, \top\}$ with respect to the axiom of consistency. Finally, due to the variability, we have $g(\bot) = \mathbb{C}(\{\bot/x\})(1) = \mathbb{C}(\{a/x\})(k)$ for every $k > 1$. Second, we will show that (ii) implies (i). Let $\mathbb{C}_{f,g}$ be the $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by the homomorphisms f and g being defined in the previous lemma. Since $\mathbb{C}_{f,g}(\{a/x\}) = \mathbb{C}(\{a/x\})$ holds for any singleton from $\mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $\mathbb{C}$ satisfies the axiom of additivity, then also $\mathbb{C}_{f,g}(A) = \mathbb{C}(A)$ holds for any $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and thus $\mathbb{C}$ is the $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets. □

Analogously to the θ-cardinalities, each $\overline{\theta}$-cardinality $\mathbb{C}$ of finite $\mathbf{L}^{\mathrm{d}}$-set is generated by a $\overline{\theta}$-homomorphism f and θ_d-homomorphism g satisfying the conditions of (ii), i.e. $\mathbb{C} = \mathbb{C}_{f,g}$. Now we will look at the preservation of the partial ordering of $\mathbf{L}^{\mathrm{d}}$-sets by the $\overline{\theta}$-cardinalities. Again we have to establish special types of the $\overline{\theta}$- and θ_d-homomorphisms. We say that a mapping $h : L_1 \to L_2$ is the (*complete*) $\overline{\theta}$*-po-homomorphism* from $\mathbf{L}_1^{\mathrm{d}}$ to $\mathbf{L}_2^{\mathrm{d}}$, if h is a (complete) $\overline{\theta}$-homomorphism preserving the partial ordering of the considered lattices, i.e. $h(a) \leq h(b)$, whenever $a \leq b$. Further, $h : L_1 \to L_2$ is the (*complete*) θ_d*-po-homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2^{\mathrm{d}}$, if h is a (complete) θ_d-homomorphism reversing the partial ordering of the considered lattices, i.e. $h(a) \leq h(b)$, whenever $a \geq b$. Let us establish $f(a) = \perp_2$ for every $a \in L_1$, then we obtain an example of $\overline{\theta}$-*po*-homomorphism from $\mathbf{L}_1^{\mathrm{d}}$ to $\mathbf{L}_2^{\mathrm{d}}$ which will be called the *trivial* $\overline{\theta}$*-po-homomorphism*. Analogously, putting $g(a) = \perp_2$ we obtain the *trivial* θ_d*-po-homomorphism* from $\mathbf{L}_1$ to $\mathbf{L}_2^{\mathrm{d}}$. Further examples could be defined in a similar way as in the case of θ-*po*- and $\overline{\theta}_d$-*po*-homomorphisms (see p. 68). In these special cases, where a $\overline{\theta}$-cardinality of finite fuzzy sets is just generated by a non-trivial $\overline{\theta}$- or θ_{d}-homomorphism f (g is trivial and thus it has no effect) or g (f is trivial and thus it has no effect), we will denote this $\overline{\theta}$-cardinality by $\mathbb{C}_f$ or $\mathbb{C}_g$, respectively. The following theorem is a dual to Theorem 3.2.6

Theorem 3.3.6. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite fuzzy sets generated by $\overline{\theta}$-po- and θ_d-po-homomorphisms f and g, respectively. Then*

(*i*) *$\mathbb{C}_{f,g}$ preserves the partial ordering of $\mathbf{L}^{\mathrm{d}}$-sets if and only if g is the trivial θ_d-po-homomorphism.*

(*ii*) *$\mathbb{C}_{f,g}$ reverses the partial ordering of $\mathbf{L}^{\mathrm{d}}$-sets if and only if f is the trivial $\overline{\theta}$-po-homomorphism.*

Proof. We will prove just the first statement, the second one could be done by analogy. First, let us suppose that $\mathbb{C}_{f,g}$ preserves ordering and g is a non-trivial θ_d-*po*-homomorphism. Since g is the non-trivial mapping, then necessarily there exists $a \in L$ such that $g(a) > g(\top) = \perp$. If we put $\{\top/x\} > \{a/x\}$, where $x \in X$ is an arbitrary element from X, then $\mathbb{C}_{f,g}(\{\top/x\})(1) = g(\top) \geq g(a) = \mathbb{C}_{f,g}(\{a/x\})(1)$ holds with regard to the preservation of the partial ordering by $\mathbb{C}_{f,g}$, a contradiction. Hence, $\Rightarrow$ is true. Let f be a $\overline{\theta}$-*po*-homomorphism, g be the trivial θ_d-*po*-homomorphism, $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary fuzzy sets and $i \in N_n$. If $A \leq B$, then $\mathbb{C}_{f,g}(\{A(x)/x\})(1) = g(A(x)) = \perp = g(B(x)) = \mathbb{C}_{f,g}(\{B(x)/x\})(1)$ and $\mathbb{C}_{f,g}(\{A(x)/x\})(0) = f(A(x)) \leq f(B(x)) = \mathbb{C}_{f,g}(\{B(x)/x\})(0)$ hold for arbitrary $x \in X$. Moreover, $\mathbb{C}_{f,g}(\{A(x)/x\})(i) = \mathbb{C}_{f,g}(\{B(x)/x\})(i)$ holds for any $i > 1$ and thus

$\mathbb{C}_{f,g}(\{A(x)/x\}) \leq \mathbb{C}_{f,g}(\{B(x)/x\})$ holds for any $x \in \mathrm{Supp}(B)$. The inequality $\mathbb{C}_{f,g}(A) \leq \mathbb{C}_{f,g}(B)$ is an immediate consequence of the additivity axiom and the statement (*iii*) from Lemma 3.1.5. Hence, $\Leftarrow$ is also true and the proof is complete. □

3.3.2 $\overline{\theta}$-cardinality and equipollence of $\mathbf{L}^{\mathrm{d}}$-sets

In the previous section we have investigated a relation between the similarity of θ-cardinality and equipollency of $\mathbf{L}$-sets. To be able to study an analogy of such relation, we have to replace the similarity relation by the fuzzy pseudo-metric, because we deal with the $\mathbf{L}^{\mathrm{d}}$-sets. Let us define a fuzzy relation on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ as follows

$$K \approx_{\mathrm{d}} L = \bigvee_{i \in N_n} |K(i) \ominus L(i)|. \tag{3.33}$$

Obviously, the fuzzy relation $\approx_{\mathrm{d}}$ is a fuzzy pseudo-metric on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$.

Theorem 3.3.7. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by $\overline{\theta}$- and θ_d-homomorphisms f and g, respectively, $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $h \in \mathrm{Perm}(X)$. Then*

$$\mathbb{C}_{f,g}(A) \approx_d \mathbb{C}_{f,g}(B) \leq a^{\overline{\theta}}_{gh} \overline{\theta} a^{\overline{\theta}}_{fh}, \tag{3.34}$$

where $a^{\overline{\theta}}_{gh}$ or $a^{\overline{\theta}}_{fh}$ is the $\overline{\theta}$-degree of the bijection h between $g^{\rightarrow}(A)$ and $g^{\rightarrow}(B)$ or $f^{\rightarrow}(A)$ and $f^{\rightarrow}(B)$, respectively.

Proof. Let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary finite $\mathbf{L}^{\mathrm{d}}$-sets and $i \in N_n$ and $h \in \mathrm{Perm}(X)$. Further, let us suppose that $\mathrm{Supp}(A) = \{x_1, \dots, x_r\}$ and $\mathrm{Supp}(B) = \{y_1, \dots, y_s\}$. Put $\{x_1, \dots, x_m\} = \mathrm{Supp}(A) \cup h^{-1}(\mathrm{Supp}(B))$ and $\{y_1, \dots, y_m\} = h(\mathrm{Supp}(A)) \cup \mathrm{Supp}(B)$ such that $h(x_k) = y_k$ for any $x_k \in \{x_1, \dots, x_m\}$. If $i > m$, then we obtain $\mathbb{C}_{f,g}(A)(i) = \mathbb{C}_{f,g}(B)(i)$ and hence we have

$$|\mathbb{C}_{f,g}(A)(i) \ominus \mathbb{C}_{f,g}(B)(i)| = \bot \leq a^{\overline{\theta}}_{gh} \overline{\theta} a^{\overline{\theta}}_{fh}.$$

Let us suppose that $i \leq m$ and establish $\mathbf{i} \in \mathcal{O}$ if and only if $\mathbf{i} \in \mathcal{I}$ and simultaneously $M_{\mathbf{i}} = \emptyset$ (clearly $\mathcal{O} \subset \mathcal{I}$). Then due to Theorem 3.3.3, Lemma 2.1.2

and 2.1.4, we have

$$|\mathbb{C}_{f,g}(A)(i) \ominus \mathbb{C}_{f,g}(B)(i)| = |\big(\bigwedge_{\mathbf{i}\in\mathcal{O}} a_{K_\mathbf{i}} \overline{\theta} a_{L_\mathbf{i}}\big) \ominus \big(\bigwedge_{\mathbf{i}\in\mathcal{O}} b_{K_\mathbf{i}} \overline{\theta} b_{L_\mathbf{i}}\big)| \leq$$

$$\bigvee_{\mathbf{i}\in\mathcal{O}} |(a_{K_\mathbf{i}} \overline{\theta} a_{L_\mathbf{i}}) \ominus (b_{K_\mathbf{i}} \overline{\theta} b_{L_\mathbf{i}})| \leq \bigvee_{\mathbf{i}\in\mathcal{O}} |a_{K_\mathbf{i}} \ominus b_{K_\mathbf{i}}| \overline{\theta} |a_{L_\mathbf{i}} \ominus b_{L_\mathbf{i}})| \leq$$

$$\bigvee_{\mathbf{i}\in\mathcal{O}} \Big| \overline{\bigominus_{k\in K_\mathbf{i}}} f(A(x_k)) \ominus \overline{\bigominus_{k\in K_\mathbf{i}}} f(B(y_k))\Big| \,\overline{\theta}\, \Big|\overline{\bigominus_{l\in L_\mathbf{i}}} g(A(x_l)) \ominus \overline{\bigominus_{l\in L_\mathbf{i}}} g(B(y_l))\Big| \leq$$

$$\bigvee_{\mathbf{i}\in\mathcal{O}} \overline{\bigominus_{k\in K_\mathbf{i}}} |f(A(x_k)) \ominus f(B(y_k))| \,\overline{\theta}\, \overline{\bigominus_{l\in L_\mathbf{i}}} |g(A(x_l)) \ominus g(B(y_l))| \leq$$

$$\overline{\bigominus_{k=1}^{m}} |f(A(x_k)) \ominus f(B(h(x_k)))| \overline{\theta} |g(A(x_k)) \ominus g(B(h(x_k)))| = a_{gh}^{\overline{\theta}} \theta a_{fh}^{\overline{\theta}},$$

where $a_{K_\mathbf{i}}$, $a_{L_\mathbf{i}}$ and analogously $b_{K_\mathbf{i}}$, $b_{L_\mathbf{i}}$ were introduced on p. 87. Applying the infimum to the inequalities for $i > m$ and $i \leq m$, we obtain the desired inequality (3.34). □

Corollary 3.3.8. *Let $f : L \to L$ be a $\overline{\theta}$-homomorphism and $g : L \to L$ be a θ_d-homomorphism. Then*

(i) $\mathbb{C}_f(A) \approx_{\mathrm{d}} \mathbb{C}_f(B) \leq f^{\to}(A) \equiv^{\overline{\theta}} f^{\to}(B)$ *and*

(ii) $\mathbb{C}_g(A) \approx_{\mathrm{d}} \mathbb{C}_g(B) \leq g^{\to}(A) \equiv^{\overline{\theta}} g^{\to}(B)$

hold for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$.

Proof. Let g be the trivial θ_d-homomorphism, i.e. $g(a) = \bot$ holds for any $a \in L$, $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $h \in \mathrm{Perm}(X)$ be an arbitrary bijection. Then obviously h is the $\bot^{\overline{\theta}}$-bijection between $g^{\to}(A)$ and $g^{\to}(B)$. Hence, we have $\mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq \bigwedge_{h\in\mathrm{Perm}(X)} a_{gh}^{\overline{\theta}} \overline{\theta} a_{fh}^{\overline{\theta}} = \bigwedge_{h\in\mathrm{Perm}(X)} \bot \overline{\theta} a_{fh}^{\overline{\theta}} = f^{\to}(A) \equiv^{\overline{\theta}} f^{\to}(B)$ and the first statement is proved. Analogously we could obtain the second statement and the proof is complete. □

Corollary 3.3.9. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by $\overline{\theta}$- and θ_d-homomorphisms f and g, respectively, such that $f(|a \ominus b|) \geq |f(a) \ominus f(b)|$ and $g(a \leftrightarrow b) \geq |g(a) \ominus g(b)|$ hold for any $a, b \in L$. Further, let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $h \in \mathrm{Perm}(X)$. Then we have*

$$\mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq f(a_h) \overline{\theta} g(b_h), \tag{3.35}$$

where a_h and b_h are the $\overline{\theta}$- and θ_d-degrees of the bijection h between A and B, respectively.

Proof. Let $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ be arbitrary finite $\mathbf{L}^{\mathrm{d}}$-sets, $h \in \mathrm{Perm}(X)$ and $H = \mathrm{Supp}(A) \cup h^{-1}(\mathrm{Supp}(B))$. Since the mappings g and f are the $\overline{\theta}$- and θ_d-homomorphisms of the appropriate algebras, then we can write (due to Theorem 3.3.7)

$$\mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq$$

$$\overline{\bigominus_{x\in H}} |f^{\rightarrow}(A)(x) \ominus f^{\rightarrow}(B)(h(x))| \; \overline{\theta} \; \overline{\bigominus_{x\in H}} |g^{\rightarrow}(A)(x) \ominus g^{\rightarrow}(B)(h(x))| \leq$$

$$\overline{\bigominus_{x\in H}} f(|A(x) \ominus B(h(x))|) \; \overline{\theta} \; \overline{\bigominus_{x\in H}} g(A(x) \leftrightarrow B(h(x))) = f(a_h)\overline{\theta} g(b_h)$$

and the proof is complete. □

Corollary 3.3.10. *Let $f : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}^{\mathrm{d}}$ and $g : \mathbf{L} \to \mathbf{L}^{\mathrm{d}}$ be arbitrary homomorphisms. Then*

(i) $\mathbb{C}_f(A) \approx_{\mathrm{d}} \mathbb{C}_f(B) \leq f(A \equiv^{\overline{\theta}} B)$ *and*

(ii) $\mathbb{C}_g(A) \approx_{\mathrm{d}} \mathbb{C}_g(B) \leq g(A \equiv^{\theta} B)$

hold for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$.

Proof. It is a straightforward consequence of Lemmas 2.4.1 and 2.4.2 and Corollary 3.3.8. □

Let $\mathbf{L}^{\mathrm{d}}$ be a complete residuated lattice and $h : L \to L$ be an arbitrary mapping. We say that the mapping h is *k-$\overline{\theta}$-compatible with the bidifference* $|\oplus|$ of $\mathbf{L}^{\mathrm{d}}$, if $k|a \ominus b| = |a \ominus b|\overline{\theta} \cdots \overline{\theta}|a \ominus b| \geq |h(a) \ominus h(b)|$ holds for arbitrary $a, b \in L$. Obviously, for $\overline{\theta} = \vee$ the definition of the k-$\overline{\theta}$-compatibility with the bidifference coincides with $|a \ominus b| \geq |h(a) \ominus h(b)|$.

Corollary 3.3.11. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite fuzzy sets generated by $\overline{\theta}$- and θ_d-homomorphisms f and g that are k-$\overline{\theta}$- and l-$\overline{\theta}$-compatible with $|\oplus|$, respectively, $A, B \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ and $h \in$* Perm(X)*. Then*

$$\mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq (k + l)a_h, \tag{3.36}$$

where a_h is the $\overline{\theta}$-degree of the bijection h between A and B. Furthermore, we have

$$\mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq l(A \equiv^{\overline{\theta}} B) \quad \text{or} \quad \mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B) \leq k(A \equiv^{\overline{\theta}} B)$$

whenever f or g is the trivial θ- or $\overline{\theta}_d$-homomorphism, respectively.

Proof. It is a straightforward consequence of the k-$\overline{\theta}$- and l-$\overline{\theta}$-compatibility of f and g with the bidifference, Theorem 3.3.7 and Corollary 3.3.8. □

Theorem 3.3.12. *Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by homomorphisms f, g and $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ with* $\mathrm{Supp}(A) \subseteq \{x_1, \ldots, x_m\} = Y$. *Then*

$$f(p_A^{\overline{\theta}}(m-i, Y))\overline{\theta}g(p_A^{\theta}(i, Y)) \leq \mathbb{C}(A)(i) \tag{3.37}$$

holds for any $i \in N_n$, where $i \leq m$.

Proof. It is analogical to the proof of Theorem 3.2.12. □

Theorem 3.3.13. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered dually residuated lattice and $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by homomorphisms f, g and $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ with* $\mathrm{Supp}(A) \subseteq \{x_1, \ldots, x_m\} = Y$. *Then*

$$\mathbb{C}(A)(i) = f(p_A^{\overline{\theta}}(m-i, Y))\overline{\theta}g(p_A^{\theta}(i, Y)) \tag{3.38}$$

holds for any $i \in N_n$, where $i \leq m$.

Proof. It is analogical to the proof of Theorem 3.2.13. □

Corollary 3.3.14. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered residuated lattice and $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by homomorphisms f, g and $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ with* $\mathrm{Supp}(A) \subseteq \{x_1, \ldots, x_m\} = Y$. *Then*

$$\mathbb{C}(A)(i) = p^{\overline{\theta}}_{f^{\rightarrow}(A)\cap Y}(m-i, Y)\overline{\theta}p^{\overline{\theta}}_{g^{\rightarrow}(A)}(i, Y) \tag{3.39}$$

holds for any $i \in N_n$, where $i \leq m$.

Proof. It is a straightforward consequence of Lemmas 2.4.1, 2.4.2 and Theorem 3.3.13. □

Corollary 3.3.15. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered residuated lattice and $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by homomorphisms f, g and $A \in \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ with $|\mathrm{Supp}(A)| = m$. Then*

$$\mathbb{C}(A)(i) = f(p_A^{\vee}(m-i))\overline{\theta}g(p_A^{\vee}(m-i+1)) \tag{3.40}$$

holds for any $i \in N_n$, where $i \leq m$.

Proof. In the proof of Corollary 3.3.15 we have shown that $p^{\vee}_A(m-i) = p^{\wedge}_A(i+1)$ holds for any $i \leq N_n$, where $i \leq m = |\mathrm{Supp}(A)|$. Hence, we immediately obtain that $p^{\wedge}_A(i) = p^{\vee}_A(m-i+1)$ holds for any $i \leq m$, where clearly $p^{\wedge}(0) = \top = p^{\vee}_A(m+1)$. The rest follows from Theorem 3.3.13. □

Corollary 3.3.16. *Let $\mathbf{L}^{\mathrm{d}}$ be a complete linearly ordered residuated lattice and $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ be a $\overline{\theta}$-cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets generated by homomorphisms f and g. Then*

(i) $(g^{\to}(A) \equiv^{\overline{\theta}} g^{\to}(B))\overline{\theta}(f^{\to}(A) \equiv^{\overline{\theta}} f^{\to}(B)) \geq \mathbb{C}_{f,g}(A) \approx_{\mathrm{d}} \mathbb{C}_{f,g}(B)$,

(ii) $g^{\to}(A) \equiv^{\vee} g^{\to}(B) = \mathbb{C}_g(A) \approx_{\mathrm{d}} \mathbb{C}_g(B)$,

(iii) $f^{\to}(A) \equiv^{\vee} f^{\to}(B) = \mathbb{C}_f(A) \approx_{\mathrm{d}} \mathbb{C}_f(B)$

hold for arbitrary $A, B \in \mathcal{FIN}_{\mathbf{L}}(X)$.

Proof. It is analogical to the proof of Corollary 3.2.16. □

Chapter 4

Fuzzy quantifiers

Most of our discussion centers around the problem whether it is possible to set up a formal calculus which would enable us to prove all true propositions involving the new quantifiers. Although this problem is not solved in its full generality, yet it is clear from the partial results (...) that the answer to the problem is essentially negative. In spite of this negative result we believe that some at least of the generalized quantifiers deserve a closer study and some deserve to be included into systematic expositions of symbolic logic. This belief is based on the conviction that the construction of formal calculi is not the unique and even not the most important goal of symbolic logic.

Andrzej Mostowski in [70]

...the concept of a fuzzy quantifiers is related in an essential way to the concept of cardinality - or, more generally, the concept of measure - of fuzzy sets. More specifically, a fuzzy quantifier may be viewed as a fuzzy characterization of the absolute or relative cardinality of a collection of fuzzy sets.

Lotfi A. Zadeh in [127]

In natural language as well as formal language of mathematics, logic and computer science, *quantification* is a construct that specifies the extent of validity of a predicate, that is the extent to which a predicate holds over a range of things. A language element which generates a quantification is called a *quantifier* [100]. There is no doubt that natural language quantifiers like "all", "many", "few", "between five and ten", "much more than twenty", "at least fifty", etc. play a very important role in natural language, because they enable us to talk about properties of collections. These constructs, from the technical point of view we can speak about "second-order" constructs, describing properties of collections rather than individuals extend the expressive power of natural language far beyond that of propositional logic. The first real analysis of quantification originates, due to Aristotle, from the ancient Greece. This famous philosopher and scientist founded formal principles of reasoning and created the particular type of logic, which is called *Aristotelian logic*[1]. His treatises on logic are collectively called *Organon* (Greek word for *Instrument*). The basic unit of reasoning in Aristotelian logic is the *syllogism*[2]. Syllogisms consist of three parts: a *major premiss*, a *minor premiss* and a *conclusion*, each of which have a form of the following *quantified sentences*:

(A) *universal affirmative*: All S are P,
(E) *universal negative*: No S is P,
(I) *particular affirmative*: Some S is P,
(O) *particular negative*: Some S is not P,

where the letters in the parentheses denote the types of sentences. Each of quantified sentences (propositions) contains a *subject* (an individual entity, for instance "Socrates", or a class of entities, for instance "all men") and a *predicate* (a property or attribute or mode of existence which a given subject may or may not possess, for instance "mortal"), which create so-called *terms*. An example of the syllogism of the type AAA is as follows

All men are mortal (Major premiss)
All Greeks are men (Minor premiss)
Therefore, all Greeks are mortal (Conclusion).

Aristotle recognized logical relationships between various types of quantified sentences (for instance, the contradictory propositions have the opposite truth value) and described them. Later the collection of these relationships was embodied in a square diagram, which is called the *Aristotelian square of opposition*, see Fig. 4.1.

As could be shown a utilization of individual subjects (e.g. "Socrates", "Plato" etc.) in the quantified sentences seems to be rather awkward as e.g. All Socrates are men. This fact, clearly, explains why Aristotle excluded the individual subjects from his logic. However, the inability to deal with individual subjects is considered to be one of the greatest flaws of Aristotelian logic [60]. Another factor that slowed down the progress of semantics for universal and existential quantification was understanding the quantifiers (*all, some* and *no*) as not logical abstractions in their own right, but only

[1]It later developed into what became known as traditional logic or term logic [99].
[2]Therefore, the Aristotelian logic is also referred to the *syllogistic logic* [97].

as structural elements of quantified sentences. Nevertheless, logicians fell under the influence of Aristotelian principles for more than 2,000 years. For interested readers we refer to e.g. [4,84] or also to [60,75,76,86,97,98], where other references may be found.

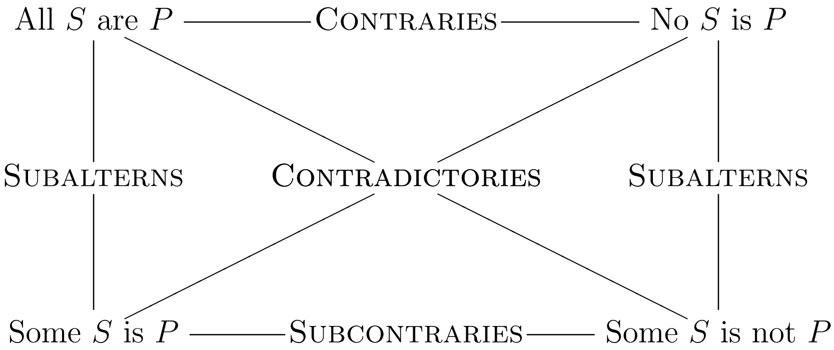


Figure 4.1: Aristotelian square of opposition.

As the first, who realized the undesirable effect of dividing of quantified propositions into subjects and predicates, which limited usability of Aristotelian logic, was German logician Gottlob Frege. In order to eliminate the outlined obstacles, he replaced subjects and predicates by functions and arguments and this analysis permitted him to found the modern discipline of logic. In his *Begriffsschrift* (Concept Script) (see [22]), which could be taken as the most significant publication since Aristotle, and later in *Grundgesetze der Arithmetik I. and II.* (see [23, 24]) he developed a formal logical system, where universal and existential quantifiers are its individual elements, which grounds the modern predicate calculus. G. Frege regarded the quantifiers as functions (notions) of second-level and therefore his logical calculus was the second-order.

A first attempt to formalize some simpler parts of natural language is due to Bertrand Russell (see e.g. [84]). In his theory the statements of natural language are analyzed as suitable combinations of formulas with classical logical quantifiers. Although, his theory enables to describe some special cases of more general natural language quantifiers, there is no apparent generalization Russell's approach to a broader class of natural language quantifiers.

On spite of the fact that the predicate logic became a powerful tool in mathematics, philosophy, linguistics etc., its expressive ability, nevertheless, was limited. The account consisted in a simplicity (triviality) of the classical quantifiers. A pioneering work on generalized quantifiers was published in the fifties of the last century by Andrzej Mostowski [70]. In this paper he introduced operators, which represent natural generalization of the classical logical quantifiers. Let us give a sketch of his original definition. A mapping $F : \prod_{i \in N} M \to \{\bot, \top\}$ is called *propositional function* on a set M, if there is a finite set of natural numbers $K \subset N$ such that $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots) \in \prod_{i \in N} M$ and $x_i = y_i$ for any $i \in K$, then $F(\mathbf{x}) = F(\mathbf{y})$. The smallest set K with the previous property is called *support* of F. If the support of F has only one element, then F is a function of one argument. Let π be a bijection from M onto M'. If $\mathbf{x} = (x_1, x_2, \dots) \in \prod_{i \in N} M$, then we denote by $\pi(\mathbf{x})$ the sequence $(\pi(x_1), \pi(x_2), \dots)$, and if F is a propositional function on M, then we denote by F_π the propositional function on M' such that $F_\pi(\pi(\mathbf{x})) = F(\mathbf{x})$. Let $\mathbb{F}_M$ be a set of all propositional functions of one argument on a set M. A (*generalized*) *quantifier limited to M* is a mapping $Q_M : \mathbb{F}_M \to \{\bot, \top\}$ such that the invariance property $Q_M(F) = Q_M(F_\pi)$ holds for any $F \in \mathbb{F}_M$ and permutation π on M. It is easy to see that this definition of generalized quantifiers is actually based on the cardinality. In fact, if we put $m = |F^{-1}(\top)|$ and $n = |F^{-1}(\bot)|$ (clearly $m + n = |M|$), then the quantifier Q_M can be defined as

$$Q_M(F) = \xi(|F^{-1}(\top)|, |F^{-1}(\bot)|), \tag{4.1}$$

where ξ is a suitable mapping from the set of all pairs of (finite or transfinite) cardinal numbers (m, n) with $m+n = |M|$ to the set of truth values $\{\bot, \top\}$. This definition is correct with regard to the invariance property of generalized quantifiers. A (*generalized*) *unlimited quantifier* is then introduced as a mapping which assigns a quantifier Q_M limited to M to each set M and which satisfies the equation $Q_M(F) = Q_{M'}(F_\pi)$ for each propositional function F on M with one argument and for each bijection $\pi : M \to M'$. For example, the existential quantifier as a quantifier limited on M is defined by

$$\xi_\exists(m, n) = \begin{cases} \top, & m \neq \emptyset, \\ \bot, & \text{otherwise,} \end{cases} \tag{4.2}$$

and analogously, the universal quantifier as a quantifier limited on M is defined by

$$\xi_\forall(m, n) = \begin{cases} \top, & m = |M|, \\ \bot, & \text{otherwise,} \end{cases} \tag{4.3}$$

for any cardinal numbers m, n such that $m + n = |M|$. Other examples can be introduced by laying down various conditions on the cardinal numbers m and n. For instance, replacing $m = |M|$ in (4.3) by $m = m_0$, we obtain the quantifier "there is exactly m_0", or by $m < \omega$, we obtain the quantifier "for at most finitely many". Clearly, new quantifiers limited on M (and similarly for unlimited) may be obtained by applying the Boolean operations to other quantifiers limited on M (or unlimited). For instance, if Q_{1M}, Q_{2M} are quantifiers limited on M, then we again obtain a quantifier Q_M limited on M by putting $Q_M(F) = Q_{1M}(F) \wedge Q_{2M}(F)$ for each propositional function F on M. Further, Mostowski introduced a language with generalized quantifiers in the same way as the language of the first-ordered logic, only instead of the symbols of classical quantifies new symbols $Q_1, \dots, Q_n$ for generalized quantifiers are assumed. The rules of building formulas by means of the symbols $\exists, \forall$ are replaced by the rule: if φ is a formula and x is a variable, then $(Q_k x)\varphi$ is a formula ($k = 1, \dots, s$). For introducing the truth values of formulas with generalized quantifiers in accordance with Mostowski's concept, we will use some notions whose precise definitions may be found in the following section. Let $\mathbf{M}$ be a structure for a given language (extended by the mappings $Q_{1M}, \dots, Q_{nM}$) with the domain M, v be an $\mathbf{M}$-evaluation ($v(x) \in M$ for any variable x of a given language) and φ be a formula. Let us denote $||\varphi||_{\mathbf{M},v}$ the truth value of the formula φ under v and $||\varphi||_{\mathbf{M},v(x/m)}$ the truth value of the formula φ under $v(x/m)$, where $v(x/m)$ denotes an $\mathbf{M}$-evaluation v' such that $v'(x) = m$ and $v'(y) = v(y)$ for any $y \neq x$. Finally, let φ be a formula with a free variable x and v be an $\mathbf{M}$-evaluation. Then a propositional function of one argument $F_{\mathbf{M},v,x} : \prod_{i\in N} M \to \{\perp, \top\}$, which corresponds to the formula φ, the $\mathbf{M}$-evaluation v and the variable x, may be defined as follows $F_{\mathbf{M},v,x}(m_1, \dots, m, \dots) = ||\varphi||_{\mathbf{M},v(x/m)}$ for any $(m_1, \dots, m, \dots) \in \prod_{i\in N} M$. Now the truth value of a formula $(Q_i x)\varphi$ under an $\mathbf{M}$-evaluation v is defined as follows

$$||(Q_i x)\varphi||_{\mathbf{M},v} = Q_{Mi}(F_{\mathbf{M},v,x}), \tag{4.4}$$

where Q_{Mi} is the corresponding quantifier to Q_i in the structure $\mathbf{M}$ and $F_{\mathbf{M},v,x}$ is the propositional function corresponding to the formula φ, the $\mathbf{M}$-evaluation v and the variable x. Note that not all generalized quantifiers have a positive solution of completeness problem for them as Mostowski also demonstrated in [70]. Nevertheless, the completeness problem for various types of generalized quantifiers is still not closed[3].

Mostowski's approach to generalized quantifiers significantly influenced further development of a theory of generalized quantifiers. The notion of generalized quantifier was first considerably generalized by Per Lindström in [61]. It is easy to see that Mostowskian generalized quantifiers can be also formulated as unary second-ordered predicates. In fact, let $\mathcal{P}(M)$ denote the set of all subsets of M. Then a generalized quantifier limited on M is a mapping $Q_M : \mathcal{P}(M) \to \{\perp, \top\}$ such that $Q_M(X) = Q_M(Y)$ holds for any

[3]And also for various types of fuzzy quantifiers.

equipollent sets X and Y from $\mathcal{P}(M)$, or equivalently a subset $Q_M \subseteq \mathcal{P}(M)$ such that if $X \in Q_M$ and $Y \in \mathcal{P}(M)$ is equipollent with X, then $Y \in Q_M$ (the invariance property). Hence, an immediate generalization may be given in such a way that unary predicates are replaced by n-ary predicates, in general. Thus, Lindström defined a generalized quantifier limited to M of the type $\langle k_1, \ldots, k_n \rangle$ as a mapping[4]

$$Q_M : \mathcal{P}(M^{k_1}) \times \cdots \times \mathcal{P}(M^{k_n}) \to \{\bot, \top\}, \tag{4.5}$$

where M^{k_i} denotes the k_i-ary Cartesian product over M (0-ary relations are identified with the truth value $\bot$ or $\top$), such that $Q_M(R_1, \ldots, R_n) = Q_M(S_1, \ldots, S_n)$ holds for any isomorphic relation structure $(M, R_1, \ldots, R_n)$ and $(M, S_1, \ldots, S_n)$. Obviously, Mostovkian quantifiers are of the type $\langle 1 \rangle$.

The study of generalized quantifiers in connection with natural language has been set in motion by Richard Montague [68]. Montague grammars made significant contributions to the formal semantics of quantifiers in natural language. His approach, however, is not explicitly connected with the definitions of generalized quantifiers. The first systematic study of natural language quantifiers in Mostowski and Lindström's conceptions is due to Jon Barwise and Robin Cooper [1]. Other early publications belong to James Higginbotham and Robert May [45] and Edward L. Keenan and Jonathan Stavi [53]. Natural language quantifiers are often modeled as quantifiers of the type $\langle 1, 1 \rangle$ limited to M as e.g.

$$\begin{aligned}
\mathsf{all}_M(X,Y) = \top \quad &\text{iff} \quad X \subseteq Y,\\
\mathsf{some}_M(X,Y) = \top \quad &\text{iff} \quad X \cap Y \neq \emptyset,\\
\mathsf{no}_M(X,Y) = \top \quad &\text{iff} \quad X \cap Y = \emptyset,\\
\mathsf{most}_M(X,Y) = \top \quad &\text{iff} \quad 2 \cdot |X \cap Y| > |X|,\\
\mathsf{at\ least\ ten}_M(X,Y) = \top \quad &\text{iff} \quad |X \cap Y| \geq 10,\\
\mathsf{all\ except\ five}_M(X,Y) = \top \quad &\text{iff} \quad |X \setminus Y| \leq 5,
\end{aligned}$$

where $X, Y \subseteq M$. For instance, if we have a proposition: *Most students attended the party*, then X expresses the set of all students from a universe M and Y expresses the set of students attended the party (cf. (4.7)). A survey of work on generalized quantifiers in natural language with a substantial list of references may be found in e.g. [54, 58].

The modeling of natural language quantifiers as fuzzy quantifiers was first pointed out by Lotfi A. Zadeh in [125]. In this paper he developed a method for dealing with fuzzy quantifiers which later led to a theory of "commonsense knowledge" and computational theory of "dispositions" [126–128].

[4]This is an alternative definition of the original one, see [21].

According to Zadeh, quantifications like e.g. "many happy people", "nearly all members", "some animals", "few single women", "nearly none mistake", "about half of participants", "about fifteen students" etc., are fuzzily defined in nature. Noticing the relation between quantifiers and cardinalities, he proposed to treat fuzzy quantifiers, which are closely related to fuzzy cardinalities, as fuzzy numbers[5]. Zadeh distinguish two kinds of fuzzy quantifiers. Fuzzy quantifiers of the first kind (or also absolute quantifiers) are defined on $[0, \infty)$ and characterize linguistic terms such as *about* 10, *much more than* 100, *at least about* 5 etc. Fuzzy quantifiers of the second kind (or also relative quantifiers) are defined on $[0, 1]$ and characterize linguistic terms such as *for nearly all*, *almost all*, *about half*, *most* etc. Note that some linguistic terms such as e.g *many* and *few* can be characterized in either sense, depending on the context.

Fuzzy quantifiers of the first kind relate to propositions of the canonical form

$$p_1 : \textit{There are } QA's, \tag{4.6}$$

where Q is a fuzzy number on $[0, \infty)$ and A is a fuzzy set, say $A : M \to [0, 1]$, that describes how elements from a universe M possess a considered property[6]. For instance, if we have a proposition: *There are about* 10 *students in a given class whose fluency in English is high*, then Q is a fuzzy number expressing the linguistic term *about ten* and A is a fuzzy set from the set of all students in a given class (individuals) to $[0, 1]$ expressing how individuals possess the property *high-fluency in English.* Fuzzy quantifiers of the second kind relate to propositions of the canonical form

$$p_2 : QA's \textit{ are } B\textit{'s}, \tag{4.7}$$

where Q is a fuzzy number on $[0, 1]$ and analogously, A and B are fuzzy sets, say $A, B : M \to [0, 1]$, express how individuals possess considered properties. For instance, if we have a proposition: *Almost all young students in a given class are students whose fluency in English is high*, then Q is a fuzzy number on $[0, 1]$ expressing the linguistic term *almost all*, A and B are fuzzy sets from the set of all students in a given class (individuals) to $[0, 1]$, whereas A expresses how individuals possess the property *young* and B expresses how individuals possess the property *high-fluency in English.* There are several

[5] A fuzzy number over R is a fuzzy set $A : R \to [0, 1]$, which is normal (i.e. the height of A is 1), A_a must be closed interval for any $a \in (0, 1]$ and the support of A must be bounded (see e.g. [57, 71, 72]).

[6] In other words, A is a fuzzy set that represents a fuzzy predicate as *tall*, *expensive*, *young*, *clever* etc.

approaches how to find the degrees $\tau(p_1)$, $\tau(p_2)$ to which propositions p_1, p_2 are true. The first one was proposed by Zadeh in [125] and it is based on the identification of those propositions with the following propositions

$$p_1 : \Sigma Count(A) \textit{ is } Q, \tag{4.8}$$
$$p_2 : \Sigma Count(B/A) \textit{ is } Q, \tag{4.9}$$

where $\Sigma Count(A)$ and $\Sigma Count(B/A)$ are the scalar cardinality and relative measure of scalar cardinality[7], respectively. The truth values of the propositions p_1, p_2 are then defined as

$$\tau(p_1) = Q(\Sigma Count(A)), \tag{4.10}$$
$$\tau(p_2) = Q(\Sigma Count(B/A)), \tag{4.11}$$

respectively. The second approach is based on the fuzzy cardinality *FGCount* and it was again introduced by Zadeh in [125]. In this case the truth values of p_1 and p_2 are found in the same way as was described above, where the fuzzy cardinality *FGCount* is transformed to the scalar cardinality $\Sigma Count$ using the formula $\Sigma Count(A) = \sum_{i=1}^{\infty} FGCount(A)(i)$. However, a more fundamental approach based on *FGCount*, which is restricted to non-decreasing fuzzy quantifiers, was proposed by Ronald R. Yager in [111, 116]. For instance, the truth value of p_1 is found by the formula

$$\tau(p_1) = \bigvee_{i=0,\dots,|M|} (Q(i) \wedge FGCount(A)(i)). \tag{4.12}$$

The third approach, which tries to overcome the restriction to non-decreasing quantifiers, was established by Anca L. Ralescu in [80] and it is based on the fuzzy cardinality *FECount*. This approach concentrates to the propositions with fuzzy quantifiers of the first type and the truth value of a proposition p_1 is found by the same formula as (4.12), where the fuzzy cardinality *FGCount* is replaced by the fuzzy cardinality *FECount*. The final approach, which is together with the previous approaches very popular in practice, was proposed by Yager in [115, 116] and it is based on so-called ordered weighted averaging (OWA) operators (see also [117, 118]). An interesting relationships between the *FGCount* approach and the Sugeno integral and also the OWA approach and the Choquet integral may be found in [6, 7].

Recall that Mostowkian generalized quantifiers limited to M can be defined as mappings $Q : \mathcal{P}(M) \to \{\bot, \top\}$. A fuzzy extension of this idea was proposed by Helmut Thiele in [91], where he introduced the notion of general

[7] For definitions, see the introduction to Chapter 3 on p. 50.

fuzzy quantifier (limited to M) as a mapping $Q : \mathcal{F}(M) \to [0, 1]$. Since this definition is too wide, he established several restrictions to it and obtained various types of general fuzzy quantifiers as e.g. the *cardinal* or *extensional* quantifiers. The former fuzzy quantifier is closely related to the isomorphism of fuzzy sets. In particular, fuzzy sets A, B on M are isomorphic ($A \equiv_{iso} B$) if and only if there exists a bijection f on X such that $F(x) = G(f(m))$ for any $m \in M$. Then Q is a cardinal quantifier, if $A \equiv_{iso} B$ implies $Q(A) = Q(B)$. It is well known that the modeling of classical quantifiers on the unit interval $[0, 1]$ is connected with the operations min and max, which are special examples of T-norms and S-conorms. Consequently, Radko Mesiar and Helmut Thiele in [67] proposed the concepts of T-quantifiers and S-quantifiers as a generalization of the classical quantifiers from the T-norms and S-conorms perspective. An investigation of expressions as *Usually* φ, φ *is probable*, *For many x,* φ etc., from the logical point of view, was studied by Petr Hájek in [41] and the quantifier *Many* in [42]. Hájek treats the expressions as formulas with generalized quantifiers and also as formulas with modalities. The second approach seems to be not unnatural, because general modalities may be considered as hidden quantifiers (see [41]). He also introduces several logics with some generalized quantifiers and shows their completeness. Generalized quantifiers as regular generalized operations on the residuated lattice $\mathbf{L}$, i.e. $Q : \mathcal{P}(L) \to L$ satisfying reasonable properties derived from $\bigvee$ and $\bigwedge$, was defined by Vilém Novák in [72]. In [73] Novák investigates linguistic quantifiers in fuzzy logic. A general theory of fuzzy quantifiers in natural language was proposed and intensively developed by Ingo Glöckner in [25–34]. Glöckner's approach is based on a generalization of the Linström's definition of generalized quantifiers. In particular, he introduces two types of generalized quantifiers, namely *semi-fuzzy quantifiers* and *fuzzy quantifiers* on a base $M \neq \emptyset$, which are defined as mappings $Q : \mathcal{P}(M)^n \to [0, 1]$ and $Q : \mathcal{F}(M)^n \to [0, 1]$, respectively, where n denotes the arity of Q. Glöckner also proposed an axiomatic system, which is called *determined fuzzification schemes* (DSF) and ensured a correctness of extensions of semi-fuzzy quantifiers to fuzzy quantifiers. Various plausible models of fuzzy quantifiers are then a result of DSF. Moreover, a correctness of familiar approaches may be verified from the DSF point of view. An overview of (not only) fuzzy quantifiers with other references may be found in [31, 33, 41, 63, 64, 73].

The aim of this chapter is to introduce a general model of fuzzy quantifier that can be used for the modeling of the large scale of generalized logical quantifiers as well as linguistic quantifiers, and to show how to introduce the syntax and semantics of the first-ordered logic with fuzzy quantifiers. Our approach to the modeling of fuzzy quantifiers is motivated by Thiele's approach that is, of course, a generalization of Mostowski's approach. Similarly to Thiele's fuzzy quantifiers, we will only deal with fuzzy quantifiers of the type $\langle 1 \rangle$. Contrary to Thiele's definition, our model of a fuzzy quantifier[8] Q over $\mathcal{M}$ is a mapping $\sharp Q : \mathcal{M} \to L$, where $\mathcal{M}$ is a non-empty fuzzy algebra over M and L is a support of complete residuated lattices $\mathbf{L}$. The main difference between our and Thiele's approach is that the relation $\equiv_{iso}$ on $\mathcal{F}_{\mathbf{L}}(X)$ (actually classical equivalence on $\mathcal{F}_{\mathbf{L}}(X)$), representing an "objectivity" of fuzzy quantifier (originally proposed by Mostowski), is replaced by the similarity relation $\equiv^\theta$ on $\mathcal{F}_{\mathbf{L}}(X)$ defined in Section 2.2. In general, the similarity relation $\equiv^\theta$ is more sensitive than the equivalence $\equiv_{iso}$ and therefore, it seems to be more convenient for an expression of the presumption of an "objectivity" of fuzzy quantifiers. For instance, the fuzzy sets $A = \{1/x\}$ and $B = \{0.99/x\}$ are not equivalent according to Thiele's definition (i.e. $(A \equiv_{iso} B) = 0$) and thus $Q(A)$ may be substantially different from $Q(B)$. However, this is a contradiction with our perception of fuzzy quantifiers, the fuzzy sets A and B are intuitively very close[9]. After the preliminary and motivation sections, in the third section we will introduce a general $\mathbf{L}_k^\theta$-model of fuzzy quantifiers, based on the similarity relation $\equiv^\theta$, and show several examples. Since the similarity relation $\equiv^\theta$ is closely related to some special cases of θ-cardinalities, $\mathbf{L}_k^\theta$-models of fuzzy quantifiers may be also constructed using θ-cardinalities (in the second part of the third section). The fourth section is devoted to constructions of new fuzzy quantifiers and their corresponding models using the logical connectives. It is an analogy to Mostowki's constructions mentioned above. It seems to be reasonable to distinguish fuzzy quantifiers into several types as it is done in the fifth section. In the next section we will introduce structures of fuzzy quantifiers. The fuzzy quantifiers from these structures are then implemented to first-ordered fuzzy logic. The syntax and semantics of fuzzy logic with fuzzy quantifiers (from the mentioned structure) are built in the seventh section. The truth values of formulas with fuzzy quantifiers are determined according to the types of fuzzy quantifiers. From the practical point of view, however, the computing of the truth values according to the definition may be rather complicated. A solution of this computing problem for special types of fuzzy quantifiers is presented in the last section. In this chapter we will not present some examples of fuzzy quantifiers applications. Several applications in the decision making (in Economics) may be found in [46–49]. Moreover, in [47,48] there is proved a relation between the Sugeno integral and the truth values of formulas with the special fuzzy quantifiers.

[8] We use the term "model of a fuzzy quantifier", because it interprets a meaning of fuzzy quantifier of a language for the first-ordered logic in a given structure for this language.

[9] A different situation is, when we model fuzzy quantifiers over crisp sets. In this case, our and Thiele's concept of fuzzy quantifiers are identical.

4.1 Fuzzy logic: syntax and semantics

In this section the syntax and semantics of a first-ordered fuzzy logic (many-valued predicate logic) will be introduced. Since the aim of this chapter is to investigate fuzzy quantifiers and mainly their models, we will not mention the notions as syntactical consequence (provability), theory etc. and the interested readers are referred to the special literature as e.g. [41, 74].

A language $\mathcal{J}$ of the first-ordered fuzzy logic consists of a non-empty set of *predicate symbols* $P, R, \ldots$, each with a positive natural number as its *arity*, a (possible empty) set of *functional symbols* $F, G, \ldots$, again each with its arity, and a (possibly empty) set of *object constants* $a, b, \ldots$. *Logical symbols* are *object variables* $x, y, \ldots$, *logical connectives* &, $\wedge$, $\vee$ and $\Rightarrow$ (*strong conjunction, conjunction, disjunction* and *implication*), *truth constants* $\bot$ and $\top$ (*false* and *true* and possibly other truth constants) and *quantifiers* $\forall$ and $\exists$ (*universal* and *existential*). Finally, the auxiliary symbols as (,) and $'$ are considered. Note that we use two types of conjunctions & and $\wedge$, where the former is interpreted by $\otimes$ and the later by $\wedge$ (see [41, 74]).

Atomic terms are object variables and constants. If F is an n-ary functional symbol and $t_1, \ldots, t_n$ terms, then the expression $F(t_1, \ldots, t_n)$ is a term. *Atomic formulas* have the form $P(t_1, \ldots, t_n)$, where P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms. If φ, ψ are formulas and x is an object variable, then $\varphi \& \psi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $(\forall x)\varphi$, $(\exists x)\varphi$, $\bot$, $\top$ are formulas. It means that each formula results from atomic formulas by iterated use of this rule. From the mentioned logical connectives we can define other one's. In particular, we establish $\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (*equivalence*) and $\neg\varphi = \varphi \Rightarrow \bot$ (*negation*).

Let $(\forall x)\varphi$ and $(\exists x)\varphi$ be formulas. The *scope of the quantifiers* $(\forall x)$ and $(\exists x)$ is the formula φ. If the variable x is in the scope of $(\forall x)$ or $(\exists x)$, then it is *bound* φ. If x is not in the scope of $(\forall x)$ or $(\exists x)$, then it is *free* in φ. For example, we have $(\forall x)(\exists y)(x \leq y) \Rightarrow (\exists y)(x \leq y)$. Here, if we put $\varphi = \psi = (x \leq y)$, then the scope of $(\forall x)$ is φ and the scope of $(\exists y)$ is φ and simultaneously ψ. Hence, y is bound and not free, but x is both free (in ψ) and bound (in φ).

Let φ be a formula with a variable x and t be a term. By $\varphi(x/t)$ we denote a formula in which all the free occurrences of the variable x are replaced by the term t. The term t is *substitutable* for the variable x of the formula φ, if there is no variable y of the term t such that x is in the scope of $(\forall y)$ or $(\exists y)$ in φ. For example, we have $\varphi = (\forall x)(x \leq y)$ and $t = x + x$ is a term. Then t is not substitutable for y. In fact, x is the variable of t and simultaneously y is in the scope of $(\forall x)$ in φ. On the other hand, if we put $t = z + a$, then t is substitutable for x and y, and thus we can write $\varphi(x/t) = (\forall x)(x \leq y)$

(no change, because x is not free in φ) and $\varphi(y/t) = (\forall x)(x \leq z + a)$ (y is free in φ).

Let $\mathcal{J}$ be a language and let $\mathbf{L}$ be a complete residuated lattice. An $\mathbf{L}$-*structure* $\mathbf{M} = \langle M, (r_P)_P, (f_F)_F, (m_a)_a \rangle$ for $\mathcal{J}$ has a non-empty *domain* M, an n-ary fuzzy relation $r_P : M^n \to L$ is given for each predicate symbol P, where L is the support of $\mathbf{L}$, an n-ary function $f_F : M^n \to M$ is given (it is not fuzzy) for each functional symbol F and m_a is an element from M for each object constant a. Let $\mathbf{M}$ and $\mathbf{M}'$ be two $\mathbf{L}$-structures for $\mathcal{J}$. We say that $\mathbf{M}'$ is an $\mathbf{L}$-*substructure* of $\mathbf{M}$, if $M' \subseteq M$, r'_P is the restriction of r_P on M'^n for each n-ary predicate symbol P of $\mathcal{J}$, and $f'_F : M'^n \to M'$ is the restriction of f_F on M'^n for each n-ary functional symbol F of $\mathcal{J}$ and $(m_a)_a = (m_{a'})_{a'}$, i.e. if m_a is given for a in $\mathbf{M}$ and $m_{a'}$ is given for a' in $\mathbf{M}'$, then $m_a = m_{a'}$.

Let $\mathcal{J}$ be a language and $\mathbf{M}$ be an $\mathbf{L}$-structure for $\mathcal{J}$. An $\mathbf{M}$-*evaluation of object variables* is a mapping v assigning to each object variable x an element $v(x) \in M$, i.e. $v : \mathcal{OV} \to M$, where $\mathcal{OV}$ is the set of all object variables. The set of all $\mathbf{M}$-evaluations is denoted by $\mathcal{V}_\mathbf{M}$. Let v, v' be two $\mathbf{M}$-evaluations, then $v \equiv_x v'$ means that $v(y) = v'(y)$ for each variable y distinct from x. If v is an $\mathbf{L}$-evaluation, then the set of all $\mathbf{M}$-evaluations v', which are $v \equiv_x v'$, is denoted by $\mathcal{V}_\mathbf{M}(x, v)$. Let $\mathbf{M}'$ be a $\mathbf{L}$-substructure of $\mathbf{M}$. Then clearly any $\mathbf{M}'$-evaluation v is also an $\mathbf{M}$-evaluation.

The value of an atomic term under an $\mathbf{M}$-evaluation v is defined by $||x||_{\mathbf{M},v} = v(x)$, $||a||_{\mathbf{M},v} = m_a$. If $t = F(t_1, \ldots, t_n)$ is a term, then the value of t under an $\mathbf{M}$-evaluation is defined by $||t||_{\mathbf{M},v} = f_F(||t_1||_{\mathbf{M},v}, \ldots, ||t_n||_{\mathbf{M},v})$. Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, \bot, \top \rangle$ be a complete residuated lattice. A *truth value* $||\varphi||^\mathbf{L}_{\mathbf{M},v}$ of φ under an $\mathbf{M}$-evaluation v is defined as follows:

$$
\begin{aligned}
&||P(t_1, \ldots, t_n)||^\mathbf{L}_{\mathbf{M},v} = r_P(||t_1||_{\mathbf{M},v}, \ldots, ||t_n||_{\mathbf{M},v}),\\
&||\bot||^\mathbf{L}_{\mathbf{M},v} = \bot, \quad ||\top||^\mathbf{L}_{\mathbf{M},v} = \top,\\
&||\varphi \& \psi||^\mathbf{L}_{\mathbf{M},v} = ||\varphi||^\mathbf{L}_{\mathbf{M},v} \otimes ||\varphi||^\mathbf{L}_{\mathbf{M},v},\\
&||\varphi \wedge \psi||^\mathbf{L}_{\mathbf{M},v} = ||\varphi||^\mathbf{L}_{\mathbf{M},v} \wedge ||\varphi||^\mathbf{L}_{\mathbf{M},v},\\
&||\varphi \vee \psi||^\mathbf{L}_{\mathbf{M},v} = ||\varphi||^\mathbf{L}_{\mathbf{M},v} \vee ||\varphi||^\mathbf{L}_{\mathbf{M},v},\\
&||\varphi \Rightarrow \psi||^\mathbf{L}_{\mathbf{M},v} = ||\varphi||^\mathbf{L}_{\mathbf{M},v} \to ||\varphi||^\mathbf{L}_{\mathbf{M},v},\\
&||(\forall x)\varphi||^\mathbf{L}_{\mathbf{M},v} = \bigwedge\{||\varphi||^\mathbf{L}_{\mathbf{M},v'} \mid v' \in \mathcal{V}_\mathbf{M}(x, v)\},\\
&||(\exists x)\varphi||^\mathbf{L}_{\mathbf{M},v} = \bigvee\{||\varphi||^\mathbf{L}_{\mathbf{M},v'} \mid v' \in \mathcal{V}_\mathbf{M}(x, v)\}.
\end{aligned}
$$

Let $\mathbf{M}'$ be a $\mathbf{L}$-substructure of $\mathbf{M}$ and v be an $\mathbf{M}'$-evaluation. Then we have $||\varphi||^\mathbf{L}_{\mathbf{M}',v} = ||\varphi||^\mathbf{L}_{\mathbf{M},v}$. Let $\mathbf{L}$ be a complete residuated lattice, φ be a formula of a language $\mathcal{J}$ and $\mathbf{M}$ be an $\mathbf{L}$-structure for $\mathcal{J}$. The *truth value* of φ in $\mathbf{M}$ is

$$||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \bigwedge\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \mid v \in \mathcal{V}_{\mathbf{M}}\}. \tag{4.13}$$

A formula φ of a language $\mathcal{J}$ is an $\mathbf{L}$-*tautology*, if $||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \top$ for each $\mathbf{L}$-structure $\mathbf{M}$.

4.2 Motivation

As was mentioned, the first-ordered fuzzy logic deals with the classical quantifiers for all and there is, which are interpreted in some $\mathbf{L}$-structure by the corresponding lattice operations $\wedge$ and $\vee$, respectively. Before we introduce a generalization of the classical quantifiers which leads to the fuzzy quantifiers, we will motivate our approach. Let us start with an example illustrating the practical application of fuzzy logic and mainly the universal quantifier in a decision-making.

Example 4.2.1. Let us consider objects $O_1, \dots, O_m$ and criteria $C_1, \dots, C_n$ and suppose that our goal is to find "the best" object, it means, the object satisfying all criteria in high level of their satisfactions. In order to model this decision problem using the first-ordered fuzzy logic notions, we have to extent the set of criteria by m "abstract" criteria, which are totally satisfied for all objects (i.e. in the degree $\top$) and thus they have no effect on our decision-making. Now our language $\mathcal{J}$ has one binary predicate symbol P, no functional symbols and n object constants $o_1, \dots, o_n$ (symbolized the objects $O_1, \dots, O_n$). Let $\mathbf{L}$ be a complete linearly ordered residuated lattice, our $\mathbf{L}$-structure $\mathbf{M} = \langle M, r_P, (m_{o_k})_{o_k}\rangle$ for $\mathcal{J}$ has $M = \{1, \dots, m+n\}$, the fuzzy relation $r_P : M \times M \to L$ which is given by the following matrix

r_P	1	$\cdots$	m	$m+1$	$\cdots$	$m+n$
1	$\top$	$\cdots$	$\top$	$a_{1,m+1}$	$\cdots$	$a_{1,m+n}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
m	$\top$	$\cdots$	$\top$	$a_{m,m+1}$	$\cdots$	$a_{m,m+n}$
$m+1$	$\top$	$\cdots$	$\top$	$\top$	$\cdots$	$\top$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$m+n$	$\top$	$\cdots$	$\top$	$\top$	$\cdots$	$\top$,

where we establish

$$r_P(k,l) = a_{k,l} \quad \text{iff} \quad \text{“}O_k \text{ satisfies } C_{l-m} \text{ in the degree } a_{k,l}\text{”}$$

for each $k = 1, \dots, m$ and $l = m+1, \dots, m+n$, and $m_{o_k} = k$ for $k = 1 \dots, m$. The value representing the (global) satisfaction of all criteria by an object O_k can be defined as the truth value of a suitable formula with the general quantifier in our **L**-structure **M** as follows. If v is an **M**-evaluation, then the truth value $||(\forall x)P(o_k, x)||^{\mathbf{L}}_{\mathbf{M},v}$ is given by

$$||(\forall x)P(o_k, x)||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{v' \in \mathcal{V}_{\mathbf{M}}(x,v)} r_P(k, v'(x)) = \bigwedge_{l=m+1}^{m+n} r_P(k, l). \tag{4.14}$$

Because $P(o_k, x)$ contains just one variable x, the truth value of $(\forall x)P(o_k, x)$ is independent of v. Hence, we have $||(\forall x)P(o_k, x)||^{\mathbf{L}}_{\mathbf{M},v} = ||(\forall x)P(o_k, x)||^{\mathbf{L}}_{\mathbf{M}}$ and thus the global satisfaction of all criteria by an object O_k in the **L**-structure **M** is really given by the formula (4.14). Thus the best object is that with the greatest truth value of the formula $(\forall x)P(o_k, x)$ in **M**. Note that there exist more sophisticated methods for solving our decision problem in practice.

It is easy to see that our solution of the mentioned decision problem has a drawback. In particular, just one wrong satisfaction of some criterion influences a wrong global satisfaction of all criteria. Obviously, the obtained decision-making could be understood as correct, i.e. in accordance with our comprehension of the good decision-making, if the same or similar (a little better) result is obtained, when one or two wrong truth values of the formula $\varphi = S(o, x)$ are removed. In the opposite case, the obtained decision-making seems to be failed (not correct). Let us suppose that the elimination of just one value from the domain M of an **L**-structure **M** (we assume an **L**-substructure $\mathbf{M}'$ of **M** having one value less than **M**)[10] results in a much better truth value of the formula $(\forall x)\varphi$ in $\mathbf{M}'$ than **M**. In this case we should decide for the purpose of the decision-making correctness, whether we want to keep the original result or to accept new much better result, when the only one value is removed! Obviously, our toleration of the considered formula truth value is influenced by the selected **L**-substructures $\mathbf{M}'$ of **M** as well as by a nature of the considered quantifier. For example, in the case of the quantifier for all there is clearly tolerated just the original **L**-structure **M** and for the quantifier there is then clearly all **L**-substructures of **M** are respected (suppose that they exist). Analogously, for a fuzzy quantifier for nearly all, which could be applied instead of the classical one for all in our example, there is tolerated the **L**-structure **M** and also the **L**-substructures

[10]Obviously, each value of M can not be removed from M to obtain an **L**-substructure of the **L**-structure **M**, in general. For instance, the value 1 can not be and $m + n$ can be removed in the example 4.2.1.

being closed to **M** (e.g. they have the similar cardinal numbers of their domains) or for a fuzzy quantifier for nearly none a toleration could be modeled in such a way that either the small **L**-substructures are just preferred (i.e. the considered formula with the quantifier for nearly none is true only in an **L**-substructure with a small number of values) or this formula is not true for any **M**-evaluation.

The previous reasoning immediately leads to investigation of toleration of the **L**-substructures of the original **L**-structure **M** with regard to the considered quantifier. If we make an effort to model a tolerance under all **L**-substructures of **M**, some difficulties could arise. First, for some languages $\mathcal{J}$ and their **L**-structures the sets of all **L**-substructures are often very poor (even it can have just one element **M**) and thus modeling of the tolerance is practically impossible. Further, it seems to be reasonable also to suppose (for some types of quantifiers as e.g. for not all) the **L**-substructure with the empty domain, which is not respected. Finally, for introducing an "internal" negation of quantifiers[11], we need a complementary **L**-substructure $\mathbf{M}''$ to each **L**-substructure $\mathbf{M}'$ of **M** (it means the **L**-substructure $\mathbf{M}''$ of **M** with the domain $M'' = M \setminus M'$). This is the reason why we just restrict ourselves to a structure of subsets (or more general fuzzy subsets) of the domain of **M** in spite of the fact that for some subsets of M there are no **L**-substructures of **M**. To achieve our aim we choose a fuzzy algebra over the domain of **M** (see p. 13) and over that fuzzy algebra a tolerance with regard to a considered fuzzy quantifier will be defined. One could be surprised why "fuzzy". In practice, we often work in a fuzzy environment as e.g. not all criteria have the same weight or not all data have the same reliability. Consequently, we should take into account this fuzzy environment and thus the toleration of subsets should also respect the "importance" of their elements in this fuzzy environment[12].

Thus, it is natural to establish the measure of tolerance with respect to a fuzzy quantifier for an **L**-structure as a mapping from a fuzzy algebra to the support of **L**. What properties should have this mapping? Of course, it depends on the nature of fuzzy quantifier. However, one property plays a major role for us. It is an "objectivity" of the tolerance, where the objectivity is expressed by a preservation of a fuzzy sets similarity. It means, if two fuzzy sets are similar in someway, then also our toleration should be similar for these fuzzy sets. The similarity of two sets or fuzzy sets can be defined, for example, using similarity of the (fuzzy) cardinals or by the equipollence

[11]For example, using this type of negation together with so-called "external" negation we may defined the existential quantifier from the general one and vice-versa.

[12]Note that a fuzzy environment is interpreted in an **L**-structure for $\mathcal{J}$ as not fuzzy.

of (fuzzy) sets (see e.g. [67,91]). For our modeling of tolerance we choose the θ-equipollence of fuzzy sets (precisely $\mathbf{L}$-set), because it seems to be more universal than a similarity of fuzzy cardinals. Moreover, using similarity of fuzzy cardinals we would be (more-less) restricted to dealing with the finite fuzzy sets and thus with the finite $\mathbf{L}$-structures.

4.3 $\mathbf{L}_k^\theta$-models of fuzzy quantifiers

In the first part of this section we are going to introduce a general model of fuzzy quantifiers and to show several examples. The second part is then devoted to the constructions of fuzzy quantifiers models by the equipollence and cardinalities of fuzzy sets.

4.3.1 Definition and several examples

In order to define a general model of fuzzy quantifiers, we have to make some preparations. The fuzzy quantifiers will be denoted by the symbols $Q, Q_1, Q_2, \dots$ In the specific cases the mentioned (general) symbols for fuzzy quantifiers may be replaced by specific one's as $\forall, \exists, \forall^{fin}, \exists^{\leq k}, Q^{\approx m}, Q^{fna}$ etc. or by adequate linguistic expressions as for all, there exists, for at most finitely many, there exist at most k, about m, for nearly all etc., giving a more information capability for the fuzzy quantifiers. Note that for the practical building of a fuzzy logic with fuzzy quantifiers, of course, just the specific cases of fuzzy quantifiers play the important role.

As we have mentioned, if $\mathbf{M}$ is an $\mathbf{L}$-structure, then the meaning of a fuzzy quantifier will be interpreted in $\mathbf{M}$ using a suitable mapping from a fuzzy algebra over the domain of $\mathbf{M}$ to the support of $\mathbf{L}$, where our toleration of the elements of the given fuzzy algebra is expressed with regard to the nature of the considered fuzzy quantifier. Obviously, each fuzzy quantifier model is dependent on the considered $\mathbf{L}$-structure $\mathbf{M}$ and the fuzzy algebra $\mathcal{M}$ over the domain of $\mathbf{M}$. However, to introduce the model of a fuzzy quantifier an $\mathbf{L}$-structure $\mathbf{M}$ is not needed to be assumed and it is sufficient to suppose just a fuzzy algebra $\mathcal{M}$ over M, where we can imagine that M is the domain of a potential $\mathbf{L}$-structure. For simplicity, therefore, we omit the presumption of an $\mathbf{L}$-structure and suppose just a fuzzy algebra $\mathcal{M}$ over a set M in the following parts.

Finally, let us recall that the equipollence $\equiv^\wedge$ is a similarity relation which could be established in each fuzzy algebra, but $\equiv^\otimes$ only in the fuzzy algebras over the countable (finite or denumerable) universes. Moreover, $a^k = \bigotimes_{i=1}^k a_i$, where $a_i = a$ for all $i = 1, \dots, k$, and $a^0 = \top$ for any $a \in L$.

Definition 4.3.1. ($\mathbf{L}_k^\theta$-model of fuzzy quantifier) *Let* $\mathbf{L}$ *be a complete residuated lattice,* k *be a natural number (possibly zero),* $\mathcal{M}$ *be a fuzzy algebra on a non-empty universe* M*, where for* $\theta = \otimes$ *the fuzzy algebra* $\mathcal{M}$ *is over the countable set* M*, and* Q *be a fuzzy quantifier. Then a mapping* $\sharp Q : \mathcal{M} \to L$ *is called the* $\mathbf{L}_k^\theta$-model of the fuzzy quantifier Q over $\mathcal{M}$*, if*

$$(A \equiv^\theta B)^k \leq \sharp Q(A) \leftrightarrow \sharp Q(B) \tag{4.15}$$

holds for arbitrary $A, B \in \mathcal{M}$*.*

Let us make several remarks on our definition of $\mathbf{L}_k^\theta$-model of fuzzy quantifier. Obviously, if the θ-degree of equipollence of fuzzy sets A and B is high, then the values of the mapping $\sharp Q$ in A and B have to be also close, which corresponds with the mentioned objectivity of the tolerance for the fuzzy quantifier Q. The closeness may be weakened by the parameter k, but this parameter is important, when we create new $\mathbf{L}_k^\otimes$-models. Further, in the definition of fuzzy quantifiers models we also admit the case $k = 0$. This case clearly leads to the constant models of fuzzy quantifiers, that is, if $\sharp Q$ is an $\mathbf{L}_0^\theta$-model of fuzzy quantifier Q over $\mathcal{M}$, then $\sharp Q(A) = a$ holds for any $A \in \mathcal{M}$. A fuzzy quantifier Q having the $\mathbf{L}_0^\theta$-models, which are given by $\sharp Q(A) = a$ and $A \in \mathcal{M}$ for an arbitrary fuzzy algebra $\mathcal{M}$, will be denoted by the specific symbol $\mathbf{a}$ and called the *constant fuzzy quantifier*. Note that the constant fuzzy quantifiers are not especially interesting in practice, but we suppose them here mainly from the theoretical point of view. An example of the useful constant fuzzy quantifier is $\perp$ that may be used (in some cases) to introduce a negation of the fuzzy quantifiers. Obviously, the set of all fuzzy quantifiers over the fuzzy algebra can be understood as the set of all k-extensional fuzzy sets over $\mathcal{M}$ with regard to the similarity relation $\equiv^\theta$. Recall that a fuzzy set A over X is called k*-extensional* with respect to a similarity relation R over X, if $A(x) \otimes R^k(x, y) \leq A(y)$ holds for every $x, y \in X$. For the case $k = 1$, we say, that fuzzy set A is extensional with respect to the similarity relation R.

If the fuzzy algebra $\mathcal{M}$, over that an $\mathbf{L}_k^\theta$-model of fuzzy quantifier is defined, is known, then we will write that $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q (over $\mathcal{M}$ is omitted). Moreover, if we deal with the general operation θ, then we will not mention that the supposed fuzzy algebra $\mathcal{M}$ is defined over a non-empty countable universe M. Before we show several examples of $\mathbf{L}_k^\theta$-models of fuzzy quantifiers, we state three useful lemmas.

Lemma 4.3.1. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $k \leq l$ *and* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M*. If a mapping* $\sharp Q : \mathcal{M} \to L$ *is the* $\mathbf{L}_k^\theta$*-model of a fuzzy quantifier* Q*, then also* $\sharp Q$ *is an* $\mathbf{L}_l^\theta$*-model of the fuzzy quantifier* Q*.*

Proof. Let $\sharp Q$ be an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q. Obviously, if the inequality (4.15) is satisfied for k, then we have $(A \equiv^\theta B)^l \leq (A \equiv^\theta B)^k \leq \sharp Q(A) \leftrightarrow \sharp Q(B)$. Therefore, $\sharp Q$ is an $\mathbf{L}_l^\theta$-model of the fuzzy quantifier Q. □

Lemma 4.3.2. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $k \leq l$ *and* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M*. If a mapping* $\sharp Q : \mathcal{M} \to L$ *is the* $\mathbf{L}^\wedge$*-model of a fuzzy quantifier* Q*, then* $\sharp Q$ *is also an* $\mathbf{L}_k^\otimes$*-model of the fuzzy quantifier* Q*.*

Proof. Due to Remark 2.2.4 on p. 34, we have $A \equiv^\wedge B \geq A \equiv^\otimes B$. The rest follows from the inequality $A \equiv^\otimes B \geq (A \equiv^\otimes B)^k$. □

Lemma 4.3.3. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M} \subseteq \mathcal{P}(M)$ *be a subalgebra over a non-empty universe* M *and* $\sharp Q : \mathcal{M} \to L$ *be a mapping. If* $\sharp Q(A) = \sharp Q(B)$ *holds for arbitrary equipollent sets* $A, B \in \mathcal{M}$*, then* $\sharp Q$ *is an* $\mathbf{L}_k^\theta$*-model of a fuzzy quantifier* Q*.*

Proof. Let $\sharp Q : \mathcal{M} \to L$ be a mapping such that $\sharp Q(A) = \sharp Q(B)$ holds, whenever there exists a bijection $f : A \to B$. Let $A, B \in \mathcal{M}$ be arbitrary sets. Obviously, the inequality (4.15) is true, if there exists a bijection between sets A and B. Let us suppose that there is no bijection between A and B. We have to prove that the mentioned inequality is also satisfied. According to the definition of bijection, for every bijection $\pi : M \to M$ there is a couple $(x_\pi, y_\pi) \in M \times M$ such that $\pi(x_\pi) = y_\pi$ and $A(x_\pi) \neq B(y_\pi)$. In the opposite case, we can construct a bijection between A and B and this is a contradiction with the presumption. Then we have

$$A \equiv^\theta B = \bigvee_{\pi \in Perm(M)} \bigwedge_{x \in M} A(x) \leftrightarrow B(\pi(x)) \leq \bigvee_{\pi \in Perm(M)} A(x_\pi) \leftrightarrow B(y_\pi) = \bot,$$

since A, B are the crisp sets and $\bot \leftrightarrow \top = \bot$. Hence, the inequality (4.15) is trivially true and $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier Q. □

Lemma 4.3.4. *Let* $\mathbf{L}$ *be a residuated lattice,* $\mathcal{M} \subseteq \mathcal{P}(M)$ *be a subalgebra over a non-empty finite universe* M *and* $\sharp Q : \mathcal{M} \to L$ *be a mapping. Then* $\sharp Q(A) = \sharp Q(B)$ *holds for arbitrary equipollent sets* $A, B \in \mathcal{M}$*, if and only if* $\sharp Q$ *is an* $\mathbf{L}_k^\theta$*-model of a fuzzy quantifier* Q*.*

Proof. According to Lemma 4.3.3, it is sufficient to prove the necessary condition of this lemma. Let $\sharp Q$ be an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q. If $f : A \to B$ is a bijection between finite sets A and B, then obviously this bijection can be extended to a bijection $f^* : M \to M$, where $f^*|_A(a) = f(a)$ holds for any $a \in A$. Moreover, clearly $A(x) = B(f^*(x))$ holds for any $x \in M$. Hence, we have

$$A \equiv^\theta B = \bigvee_{\pi \in Perm(M)} \bigwedge_{x \in M} A(x) \leftrightarrow B(\pi(x)) \geq \bigwedge_{x \in M} A(x) \leftrightarrow B(f^*(x)) = \top$$

and thus $(A \equiv^\theta B)^k = \top$, which implies $\sharp Q(A) \leftrightarrow \sharp Q(B) \geq \top$ or equivalently $\sharp Q(A) = \sharp Q(B)$. □

In the following three examples of $\mathbf{L}_k^\theta$-models of fuzzy quantifiers, we will suppose that $\mathbf{L}$ is a complete residuated lattice and $\mathcal{M} = \mathcal{P}(M)$ is the common Boolean algebra of all subsets of $M \neq \emptyset$.

Example 4.3.1. The mappings from $\mathcal{M}$ to L, given by

$$\forall A \in \mathcal{M} : \sharp\exists(A) = \begin{cases} \bot, & \textit{if } A = \emptyset, \\ \top, & \textit{otherwise}, \end{cases} \tag{4.16}$$

$$\forall A \in \mathcal{M} : \sharp\forall(A) = \begin{cases} \top, & \textit{if } A = M, \\ \bot, & \textit{otherwise}, \end{cases} \tag{4.17}$$

define the $\mathbf{L}_k^\theta$-models of the classical quantifiers there exists and for all. Later we will show that using these models the classical quantifiers may be interpreted in an $\mathbf{L}$-structure. To verify that these mappings are really $\mathbf{L}_k^\theta$-models of fuzzy quantifiers, it is sufficient to prove (due to Lemma 4.3.3) that $\sharp Q(A) = \sharp Q(B)$ holds for arbitrary $A, B \in \mathcal{M}$ such that $|A| = |B|$, i.e. they have the same cardinality. But this is, however, an immediate consequence of the definitions of mappings $\sharp\exists$ and $\sharp\forall$. We can ask, whether we are able to display graphically the mentioned models of quantifiers. In order to achieve it, we restrict ourselves to complete residuated lattices on $[0, 1]$ and finite universes. Let us define a mapping $g : \mathcal{M} \to [0, 1]$, where $\mathcal{M}$ is the power set of a finite set M, by $g(A) = \frac{|A|}{|M|}$ for all $A \in \mathcal{M}$ (the relative cardinality of the subsets of M) and mappings $te, fa : [0, 1] \to [0, 1]$ as follows

$$\forall x \in [0, 1] : \ te(x) = \begin{cases} 0, & x = 0, \\ 1, & \text{otherwise}, \end{cases} \tag{4.18}$$

$$\forall x \in [0, 1] : \ fa(x) = \begin{cases} 1, & x = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.19}$$

These mappings are displayed on Fig. 4.2. Now it is easy to see that $\sharp\exists = te \circ g$ and $\sharp\forall = fa \circ g$ define the $\mathbf{L}_k^\theta$-models of the quantifiers $\exists$ and $\forall$. Obviously, we

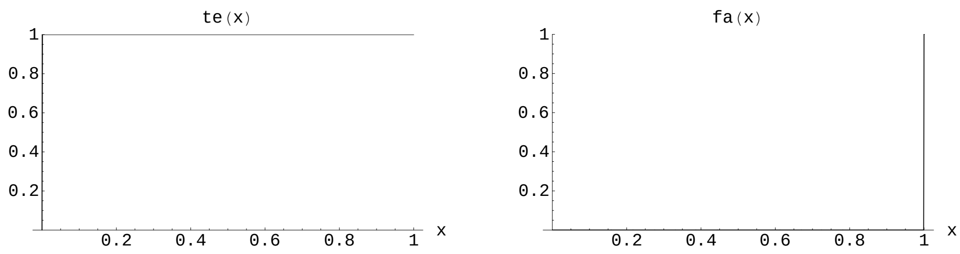


Figure 4.2: Mappings describing the $\mathbf{L}_k^\theta$-models of $\exists$ and $\forall$

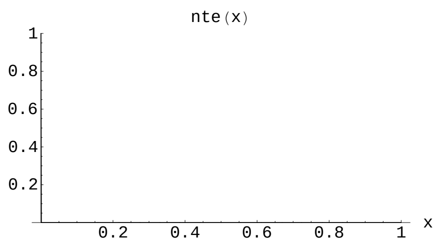
can use the mappings te, fa to define the $\mathbf{L}_k^\theta$-models of $\exists$ and $\forall$ even in more general way (of course, the finite universes have to be supposed), because the relative cardinality g can be established for the different finite algebras of subsets. Hence, it seems to be reasonable to consider the mappings te, fa as (more universal) $\mathbf{L}_k^\theta$-models of the quantifiers $\exists$ and $\forall$. Note that this technique, where first a suitable mapping $g : \mathcal{M} \rightarrow [0, 1]$ (e.g. the relative cardinality of subsets or fuzzy subsets) is established and then a model of fuzzy quantifier Q in the form of a mapping $h_Q : [0, 1] \rightarrow [0, 1]$ is used, is very convenient for the practical computation of the $\mathbf{L}_k^\theta$-models of fuzzy quantifiers Q values. Analogously, the truth value of a formula $p : QA's\ are\ B's$, where Q is a fuzzy quantifier of the second type, is found in Zadeh approach (cf. (4.11)).

Example 4.3.2. The negation of the classical quantifiers there exists and for all are quantifiers that could be called there exists none and not for all. Their $\mathbf{L}_k^\theta$-models are then defined as follows

$$\forall A \in \mathcal{M}(X) : \sharp\exists^{not}(A) = \begin{cases} \top, & if\ A = \emptyset, \\ \bot, & otherwise, \end{cases} \tag{4.20}$$

$$\forall A \in \mathcal{M}(X) : \sharp\forall^{not}(A) = \begin{cases} \bot, & if\ A = M, \\ \top, & otherwise. \end{cases} \tag{4.21}$$

Obviously, the $\mathbf{L}_k^\theta$-models of the quantifiers there exists none and not for all can be obtained by some negation of the classical one's. In particular, we can write $\sharp\exists^{not}(A) = \sharp\exists(A) \rightarrow \bot$ and $\sharp\forall^{not}(A) = \sharp\forall(A) \rightarrow \bot$ for any $A \in \mathcal{M}$. Again, we can use the mappings nte and nfa, which are displayed on Fig. 4.3, to define these negations of the classical quantifiers in the composition with the mapping g established in the previous example.



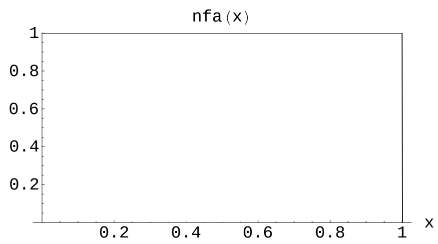


Figure 4.3: Mappings describing the $\mathbf{L}_k^\theta$-models of $\exists^{not}$ and $\forall^{not}$

Example 4.3.3. Let us denote $\mathrm{Fin}(M) = \{A \subseteq M \mid A \text{ is a finite set}\}$. Then the mappings

$$\forall A \in \mathcal{M} : \ \sharp\forall^{\mathit{infin}}(A) = \begin{cases} \bot, & \mathit{if}\ A \in \mathrm{Fin}(M), \\ \top, & \mathit{otherwise}, \end{cases} \tag{4.22}$$

$$\forall A \in \mathcal{M} : \ \sharp\forall^{\mathit{fin}}(A) = \begin{cases} \top, & \mathit{if}\ A \in \mathrm{Fin}(M), \\ \bot, & \mathit{otherwise} \end{cases} \tag{4.23}$$

define $\mathbf{L}_k^\theta$-models of "fuzzy" quantifiers called for all except a finite number and for at most finitely many. Again, it is easy to see that if there exists a bijection between two sets from $\mathcal{M}$, then the values of these sets are identical in the mappings $\forall^{\mathit{infin}}$ and $\forall^{\mathit{fin}}$. Note that there is no bijection between a finite set and an infinite set. Obviously, the quantifier for at most finitely many is a negation of the quantifier for all except a finite number. Particulary, we can write for these models $\forall^{\mathit{fin}}(A) = \forall^{\mathit{infin}}(A) \to \bot$ for any $A \in \mathcal{M}$. Obviously, these quantifiers can not be reasonably displayed on a figure.

As could be seen, all the mentioned examples have been introduced by two classes of subsets of X and two truth values $\bot$ and $\top$. From this point of view these quantifiers are rather generalizations of the classical quantifiers than the literally fuzzy quantifiers. The following examples demonstrate fuzzy quantifiers, where $\mathbf{L}_k^\theta$-models of fuzzy quantifiers are constructed using the θ- and $\overline{\theta}$-cardinalities of finite fuzzy sets.

Example 4.3.4. Let us suppose that $M = \{x_1, \dots, x_n\}$, $\mathbf{L}_{\text{Ł}}$ and $\mathbf{L}_{\text{Ł}}^{\text{d}}$ are the Łukasiewicz and dual Łukasiewicz algebra, $\mathcal{M} = [0,1]^M$ is the common fuzzy algebra of all fuzzy subsets of M and $\mathbb{C}_{S_{\text{Ł}}}$ is the $S_{\text{Ł}}$-cardinality of finite fuzzy sets defined in Ex 3.3.2. Let us denote $\frac{A}{n}$ the fuzzy set over M, given by

$\frac{A}{n}(x_i) = \frac{A(x_i)}{n}$, where $A \in \mathcal{F}_{\mathbf{L}_Ł}(M)$. Now the following mappings

$$\forall A \in \mathcal{M}: \quad \sharp Q^{fna}(A) = \mathbb{C}_{S_Ł}(\tfrac{A}{n})(0) = \frac{1}{n}\sum_{x_i \in M} A(x_i), \tag{4.24}$$

$$\forall A \in \mathcal{M}: \quad \sharp Q^{fnn}(A) = \mathbb{C}_{S_Ł}(\tfrac{\overline{A}}{n})(0) = \frac{1}{n}\sum_{x_i \in M} (1 - A(x_i)), \tag{4.25}$$

where $\overline{A}(x) = 1 - A(x)$, define $\mathbf{L}_{Ł}{}_k^\theta$-models of the fuzzy quantifiers, that will be called for nearly all and for nearly none, respectively. Let P be a property for the object from X. Then the fuzzy quantifier for nearly all specifies a great amount (including M) of the objects from M possessing the property P and the fuzzy quantifier for nearly none specifies a small amount (including $\emptyset$) of objects from M possessing the property P. Let us show that the definitions of $\mathbf{L}_{Ł}{}_1^\theta$-models of the fuzzy quantifiers for nearly all and for nearly none are correct. Due to Lemma 2.2.9 on p. 38, we have for $\theta = \wedge$

$$\begin{aligned}
&\sharp Q^{fna}(A) \leftrightarrow \sharp Q^{fna}(B) = \\
&\quad 1 - \left|\mathbb{C}_{S_Ł}(\tfrac{A}{n}) - \mathbb{C}_{S_Ł}(\tfrac{B}{n})\right| = 1 - \left|\sum_{i=1}^{n} \frac{A(x_i)}{n} - \sum_{j=1}^{n} \frac{B(x_j)}{n}\right| = \\
&\quad 1 - \frac{1}{n}\left|\sum_{i=1}^{n} p_A^\wedge(i) - \sum_{i=1}^{n} p_B^\wedge(i)\right| = 1 - \frac{1}{n}\left|\sum_{i=1}^{n} (p_A^\wedge(i) - p_B^\wedge(i))\right| \geq \\
&\quad 1 - \frac{1}{n}\sum_{i=1}^{n} |p_A^\wedge(i) - p_B^\wedge(i)| \geq 1 - \bigvee_{i=0}^{\infty} |p_A^\wedge(i) - p_B^\wedge(i)| = \\
&\quad \bigwedge_{i=0}^{\infty} 1 - |p_A^\wedge(i) - p_B^\wedge(i)| = \bigwedge_{i=0}^{\infty} p_A^\wedge(i) \leftrightarrow p_B^\wedge(i) = A \equiv^\wedge B,
\end{aligned}$$

where $p_A^\wedge$ and $p_B^\wedge$ are mappings defined on p. 35. Hence, the mapping $\sharp Q^{fna}$ is an $\mathbf{L}_{Ł}{}_1^\wedge$-model of the fuzzy quantifier for nearly all. Due to Lemmas 4.3.1 and 4.3.2, we obtain that $\sharp Q^{fna}$ is also an $\mathbf{L}_{Ł}{}_k^\theta$-model of fuzzy quantifier for nearly all. Analogously, it could be shown that $\sharp Q^{fnn}$ is an $\mathbf{L}_{Ł}{}_k^\theta$-model of the fuzzy quantifier for nearly none. Later we will show that it is a consequence of internal negation of fuzzy quantifiers (see Theorem 4.4.3 on p. 129). It can be seen that the mappings, displayed in Fig. 4.4, in the composition with the mapping g, given by $g(A) = \frac{\sum_{i=1}^{n} A(x_i)}{n}$, define the $\mathbf{L}_{Ł}{}_k^\theta$-models of these fuzzy quantifiers. As we have said, this approach is very convenient for the practical computation and moreover, it gives a possibility to graphically display the fuzzy quantifiers models. Another advantage of this approach is that new models of fuzzy quantifiers may be obtained using some suitable,

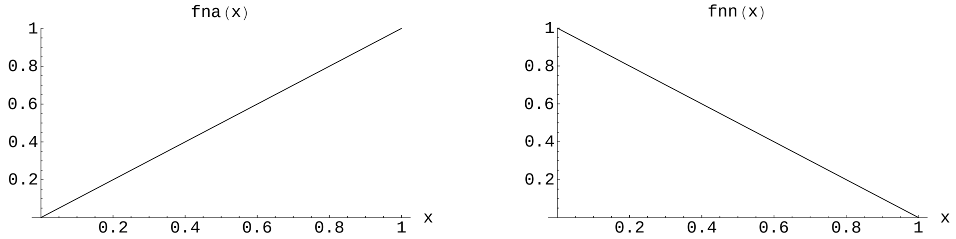


Figure 4.4: Mappings describing the $\mathbf{L}_k^\theta$-models of Q^{fna} and Q^{fnn}

often simple modification of the (displayed) mappings. For example, let us suppose that $\sharp Q(A) = h \circ g(A)$, where $g : \mathcal{M} \to [0,1]$ and $h : [0,1] \to [0,1]$. Putting $h^n(x) = h(x) \otimes \cdots \otimes h(x) = (h(x))^n$, we again obtain an $\mathbf{L}_k^\theta$-model, given by $h^n \circ g$, of a fuzzy quantifier. On Fig. 4.5 there are displayed modified models of fuzzy quantifier Q^{fna} for $k = 3, 4$.

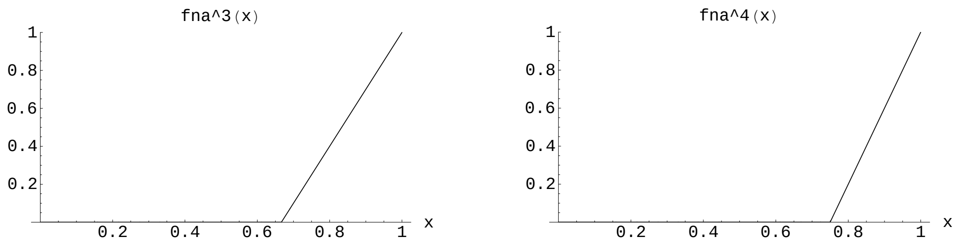


Figure 4.5: Mappings describing the modified $\mathbf{L}_k^\theta$-models of Q^{fna}

Example 4.3.5. Again, let us suppose that $M = \{x_1, \dots, x_n\}$, $\mathbf{L}_{\text{Ł}}$ and $\mathbf{L}_{\text{Ł}}^{\text{d}}$ are the Łukasiewicz and dual Łukasiewicz algebra, $\mathcal{M} = [0,1]^M$ is the common fuzzy algebra of all fuzzy subsets of M. Then the mappings

$$\forall A \in \mathcal{M}: \quad \sharp Q^{m\lesssim}(A) = p_A^{\otimes}(m) = \bigvee_{\substack{Y \subseteq M \\ |Y| = m}} \bigotimes_{y \in Y} A(y), \tag{4.26}$$

$$\forall A \in \mathcal{M}: \quad \sharp Q^{\lesssim m}(A) = 1 - p_A^{\oplus}(n - m) = 1 - \bigwedge_{\substack{Y \subseteq M \\ |Y| = m}} \bigoplus_{y \in Y} A(y), \tag{4.27}$$

where $1 \leq m \leq n$, $\otimes = T_{\text{Ł}}$ and $\oplus = S_{\text{Ł}}$, define $\mathbf{L}_k^\theta$-models of fuzzy quantifiers that could be called for about or more than m and for about or less m, respectively. Let $A \in \mathcal{M}$ be a crisp set and suppose that $|A| \geq m$. Then there is $Y \subseteq A$ such that $|Y| = m$ and thus $\sharp Q^{m\lesssim}(A) = 1$. Further, we obtain $\sharp Q^{m\lesssim}(A) = 0$ for $|A| < m$, because for each $|Y| = m$ there is $y \in Y$ such that $A(y) = 0$. Hence, it seems that the model of fuzzy quantifier for about or more than m is reasonably introduced . Analogously, we could show that $\sharp Q^{\lesssim m}(A) = 1$ for $|A| \leq m$ and $\sharp Q^{\lesssim m}(A) = 0$ for $|A| > m$. Later we will show that these definitions are correct (see Ex. 4.3.8 on p. 121).

4.3.2 Constructions of fuzzy quantifiers models

In the previous examples the $\mathbf{L}_k^\theta$-models of fuzzy quantifiers were constructed using the cardinality or equivalently (in some cases) equipollence of fuzzy sets. Here we will show how to establish $\mathbf{L}_k^\theta$-models of fuzzy quantifiers using equipollence and cardinality of fuzzy sets, in general. Of course, these approaches are not the only one's how to establish the $\mathbf{L}_k^\theta$-models of fuzzy quantifiers.

Using equipollence of fuzzy sets

As we have already mentioned, the set of all fuzzy quantifiers is equal to the set of all k-extensional fuzzy sets with respect to the similarity relation $\equiv^\theta$. One technique, how to introduce extensional fuzzy sets for a given similarity relation R, is to establish $A_x(y) = R(x, y)$ for all $x \in X$. The fuzzy set A_x over X is then the extensional fuzzy set with respect to the similarity relation R. In fact, from $R(x, y) \otimes R(y, z) \leq R(x, z)$ we obtain $R(y, z) \leq A_x(y) \rightarrow A_x(z)$ and, analogously, from $R(x, z) \otimes R(z, y) \leq R(x, y)$ we obtain $R(y, z) \leq A_x(z) \rightarrow A_x(y)$ (by adjointness) and thus $R(y, z) \leq A_x(y) \leftrightarrow A_x(z)$. Hence, we can immediately define a class of $\mathbf{L}_1^\theta$-models fuzzy quantifiers as follows.

Theorem 4.3.5. *Let $\mathbf{L}$ be a complete residuated lattice, $\mathcal{M}$ be a fuzzy algebra over a non-empty universe M and $K \in \mathcal{M}$. Then $\sharp Q_K(A) = (A \equiv^\theta K)^k$ defines an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier generated by the equipollence of fuzzy sets and the fuzzy set K.*

Proof. First, let us suppose that $k = 1$. Due to Theorems 2.2.4, 2.2.5 and the adjointness property, we have $A \equiv^\theta B \leq Q_K(A) \rightarrow Q_K(B)$ and similarly $A \equiv^\theta B \leq Q_K(B) \rightarrow Q_K(A)$. Hence, we obtain $A \equiv^\theta B \leq Q_K(A) \leftrightarrow Q_K(B)$ for arbitrary $A, B \in \mathcal{M}$ and thus $\sharp Q_K$ is an L_1^θ-model of a fuzzy quantifier Q_K. Further, let $k > 0$ be an arbitrary natural number and $\sharp Q_K(A) = (A \equiv^\theta K)^k$ be defined for any $A \in \mathcal{M}$. According to the previous part of the proof, we have $A \equiv^\theta B \leq (A \equiv^\theta K) \leftrightarrow (B \equiv^\theta K)$. Hence, we can write $(A \equiv^\theta B)^k \leq ((A \equiv^\theta B) \leftrightarrow (B \equiv^\theta K))^k \leq (A \equiv^\theta B)^k \leftrightarrow (B \equiv^\theta K)^k = Q_K(A) \leftrightarrow Q_K(B)$ and the proof is complete. □

Remark 4.3.6. If we deal with a complete linearly ordered residuated lattice and a finite non-empty universe is assumed, then using Lemma 2.2.9 we may construct $\mathbf{L}_k^\theta$-models of fuzzy quantifiers. In particular, if $K \in \mathcal{FIN}_{\mathbf{L}}(X)$ is a fuzzy set, then $\sharp Q_K(A) = p_A^\theta \approx p_K^\theta$.

Example 4.3.7. Let us suppose that $M = \{x_1, \dots, x_n\}$, $\mathbf{L}$ is the Łukasiewicz algebra, $\mathcal{M} = [0,1]^X$ is the common fuzzy algebra of fuzzy subsets over the unit interval and $K \in \mathcal{M}$. Then

$$\sharp Q_K(A) = \bigwedge_{i=1}^{n} 1 - |p_A(i) - p_K(i)| \tag{4.28}$$

is the $\mathbf{L}^\wedge$-model of a fuzzy quantifier Q_K, where $p_A = (A(x_{i_1}), \dots, A(x_{i_n}))$ is an ordered vector of membership degrees of the fuzzy set A such that $A(x_{i_j}) \geq A(x_{i_k})$, whenever $j < k$, and analogically for the vector p_K.

In every fuzzy algebra we can use the operation $\cup$ to established the corresponding partial ordering $\leq$. Particularly, we put $A \subseteq B$ if and only if $A \cup B = B$. Obviously, this definition is equivalent to $A \subseteq B$ if and only if $A(x) \leq B(x)$ holds for any $x \in M$. An algebra with the mentioned partial ordering will be denoted by $(\mathcal{M}, \subseteq)$. Because of $\sharp Q : \mathcal{M} \to L$ is a mapping from a fuzzy algebra $\mathcal{M}$ to the support of a residuated lattice, we can ask a question, if there exist $\mathbf{L}_k^\theta$-models of fuzzy quantifiers that preserve or reverse partial ordering between structures $(\mathcal{M}, \subseteq)$ and $(L, \leq)$, where $\leq$ is the corresponding lattice ordering. Moreover, it seems to be natural that such models do exist. If we want, for example, to model the fuzzy quantifier for nearly all, then the fuzzy sets near to a given universe would have greater values in a model of this fuzzy quantifier than are the values of fuzzy sets being dissimilar to the universe. The following theorem shows two natural examples of such models.

Theorem 4.3.6. *Let $\sharp Q_K : \mathcal{M} \to L$ be an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier generated by the equipollence of fuzzy sets and a fuzzy sets K. Then*

(i) if $K = M$, then $\sharp Q_K$ preserves the partial ordering and

(ii) if $K = \emptyset$, then $\sharp Q_K$ reverses the partial ordering.

Proof. First, let $K = M$ and $A \subseteq B$ be fuzzy sets from $\mathcal{M}$. Since $M(x) = \top$ for any $x \in M$ and $a \leftrightarrow \top = a$ for any $a \in L$, then we have

$$\sharp Q_M(A) = (M \equiv^\theta A) = \bigvee_{f \in \mathrm{Perm}(M)} \bigwedge_{x \in M} M(x) \leftrightarrow A(f(x))) = \bigwedge_{x \in M} A(x) \leq$$
$$\bigwedge_{y \in M} B(y) = \bigvee_{g \in \mathrm{Perm}(M)} \bigwedge_{y \in M} M(y) \leftrightarrow B(g(y)) = \sharp Q_M(B).$$

Second, let $K = \emptyset$ and $A \subseteq B$ be fuzzy sets of $\mathcal{M}$. Since $\emptyset(x) = \bot$ and $A(x) \to \bot \geq B(x) \to \bot$ hold for any $x \in M$, then we have

$$\sharp Q_\emptyset(A) = (A \equiv^\theta \emptyset) = \bigvee_{f \in \mathrm{Perm}(M)} \bigwedge_{x \in M} K(x) \leftrightarrow A(f(x)) = \bigwedge_{x \in M} (A(x) \to \bot) \geq$$
$$\bigwedge_{x \in M} (B(x) \to \bot) = \bigvee_{f \in \mathrm{Perm}(M)} \bigwedge_{x \in M} \emptyset(x) \leftrightarrow B(f(x)) = (B \equiv^\theta \emptyset) = \sharp Q_\emptyset(B)$$

and the proof is complete. □

Using cardinalities of fuzzy set

In the previous chapter we have investigated the relation between cardinalities of finite fuzzy sets and equipollence of fuzzy sets. The results may be used to establish $\mathbf{L}_k^\theta$-models of fuzzy quantifiers. A relation between θ-cardinalities (or $\overline{\theta}$-cardinalities) and θ-equipollence (or θ-equipollence) of $\mathbf{L}$-sets (or $\mathbf{L}^{\mathrm{d}}$-sets) is conditioned by the presumption of homomorphisms that are compatible with biresiduum (see p. 72) and bidifference (see p. 92). The following theorem shows a possibility to define $\mathbf{L}_k^\theta$-models of fuzzy quantifiers using special classes of θ-cardinalities of finite fuzzy set.

Theorem 4.3.7. *Let $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ be complete residuated and dually residuated lattices, respectively, with the same support, $\mathcal{M}$ be a fuzzy algebra over a finite non-empty universe M and $K \in \mathcal{M}$. Let $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}}(M) \to \mathcal{CV}_{\mathbf{L}}^\theta(N_n)$ be a θ-cardinality of finite fuzzy sets generated by θ- and $\overline{\theta}_d$-homomorphisms f and g which are k-θ- and l-θ-compatible with biresiduum of $\mathbf{L}$, respectively, and $i \in N_n$. Then the mapping $\sharp Q_K : \mathcal{M} \to L$, given by $\sharp Q_K(A) = \mathbb{C}_{g,f}(A)(i) \leftrightarrow \mathbb{C}_{g,f}(K)(i)$ or $\sharp Q_K(A) = \mathbb{C}_{g,f}(A) \approx \mathbb{C}_{g,f}(K)$, is*

(i) an $\mathbf{L}_l^\theta$-model of a fuzzy quantifier generated by the θ-cardinality $\mathbb{C}_{f,g}$ and fuzzy set K, if f is the trivial θ-homomorphism,

(ii) an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier generated by the θ-cardinality $\mathbb{C}_{f,g}$ and fuzzy set K, if g is the trivial $\overline{\theta}_d$-homomorphism.

Proof. Here, we will prove just the statement (i), the statement (ii) could be done by analogy. Let $K \in \mathcal{M}$ be an arbitrary fuzzy set and g be a $\overline{\theta}$-homomorphism being l-θ-compatible with $\leftrightarrow$. Due to Corollary 3.2.11, we have $\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i) \geq \mathbb{C}_g(A) \approx \mathbb{C}_g(B) \geq (A \equiv^\theta B)^l$. Hence, it is sufficient to show that $\sharp Q_K(A) \leftrightarrow \sharp Q_K(B) \geq \mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)$ in the first case or $\sharp Q_K(A) \leftrightarrow \sharp Q_K(B) \geq \mathbb{C}_g(A) \approx \mathbb{C}_g(B)$ in the second case. Let

$A, B \in \mathcal{M}$ be arbitrary fuzzy sets. Then we have in the first case

$$\begin{aligned}
&\sharp Q_K(A) \leftrightarrow \sharp Q_K(B) = \\
&\quad (\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \leftrightarrow (\mathbb{C}_g(B)(i) \leftrightarrow \mathbb{C}_g(K)(i)) = \\
&\quad ((\mathbb{C}_g(A)(i) \rightarrow \mathbb{C}_g(K)(i)) \wedge (\mathbb{C}_g(K)(i) \rightarrow \mathbb{C}_g(A))) \leftrightarrow \\
&\quad ((\mathbb{C}_g(B)(i) \rightarrow \mathbb{C}_g(K)(i)) \wedge (\mathbb{C}_g(K)(i) \rightarrow \mathbb{C}_g(B)(i))) \geq \\
&\quad ((\mathbb{C}_g(A)(i) \rightarrow \mathbb{C}_g(K)(i)) \leftrightarrow (\mathbb{C}_g(B)(i) \rightarrow \mathbb{C}_g(K)(i))) \wedge \\
&\quad ((\mathbb{C}_g(K)(i) \rightarrow \mathbb{C}_g(A)(i)) \leftrightarrow (\mathbb{C}_g(K)(i) \rightarrow \mathbb{C}_g(B)(i))) \geq \\
&\quad ((\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) \otimes (\mathbb{C}_g(K)(i) \leftrightarrow \mathbb{C}_g(K)(i))) \wedge \\
&\quad ((\mathbb{C}_g(K)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \otimes (\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i))) = \\
&\qquad\qquad (\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) \otimes \top = \mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)
\end{aligned}$$

and in the second case

$$\begin{aligned}
&\sharp Q_K(A) \leftrightarrow \sharp Q_K(B) = \\
&\quad (\mathbb{C}_g(A) \approx \mathbb{C}_g(K)) \leftrightarrow (\mathbb{C}_g(B) \approx \mathbb{C}_g(K)) = \\
&\quad \big(\bigwedge_{i \in N_n} (\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \big) \leftrightarrow \big(\bigwedge_{i \in N_n} (\mathbb{C}_g(B)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \big) \geq \\
&\quad \bigwedge_{i \in N_n} \big((\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \leftrightarrow (\mathbb{C}_g(B)(i) \leftrightarrow \mathbb{C}_g(K)(i)) \big) \geq \\
&\quad \bigwedge_{i \in N_n} (\mathbb{C}_g(A)(i) \leftrightarrow \mathbb{C}_g(B)(i)) = \mathbb{C}_g(A) \approx \mathbb{C}_g(B).
\end{aligned}$$

Hence, the proof is complete. □

Example 4.3.8. Let us suppose that $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ are complete linearly ordered residuated and dually residuated lattices with the same support L, $M = \{x_1, \dots, x_n\}$ and $\mathcal{M} = \mathcal{F}_{\mathbf{L}}(M)$. Further, let us suppose that $\mathbb{C}_f, \mathbb{C}_g : \mathcal{F}_{\mathbf{L}}(M) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ are θ-cardinalities of finite fuzzy sets generated by homomorphisms $f : \mathbf{L} \to \mathbf{L}$ and $g : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}$ being k-θ-compatible with biresiduum of $\mathbf{L}$. Then the mappings

$$\forall A \in \mathcal{M} : \quad \sharp Q^{m\lesssim}_{M}(A) = \mathbb{C}_f(A)(m), \tag{4.29}$$

$$\forall A \in \mathcal{M} : \quad \sharp Q^{\lesssim m}_{\emptyset}(A) = \mathbb{C}_g(A)(n-m), \tag{4.30}$$

where $1 \leq m \leq n$, are other $\mathbf{L}^{\theta}_{k}$-models of the fuzzy quantifiers that have been called for about or more than m and for about or less m, respectively. It is easy to show that $\mathbb{C}_f(M)(m) = f(p^{\theta}_{M}(m, M)) = \top$ and $\mathbb{C}_g(\emptyset)(n-m) = g(p^{\overline{\theta}}_{\emptyset}(n-m, M)) = \top$ hold for any $m \in \{1, \dots, n\}$. Since all presumptions of the previous theorem are fulfilled, then $Q_M^{m\lesssim}(A) = \mathbb{C}_f(A)(m) \leftrightarrow \mathbb{C}_f(A)(m) = \mathbb{C}_f(M)(m)$ and $Q_\emptyset^{\lesssim m}(A) = \mathbb{C}_g(A)(n-m) \leftrightarrow \mathbb{C}_g(\emptyset)(n-m) = \mathbb{C}_g(A)(n-m)$. Hence, the definitions of $\mathbf{L}_k^\theta$-models of fuzzy quantifiers $Q_M^{m\lesssim}$ and $Q_\emptyset^{\lesssim m}$ are correct. If we suppose that $\mathbf{L}$ and $\mathbf{L}^{\mathrm{d}}$ are the Łukasiewicz and dual Łukasiewicz algebra, $f(x) = x$ and $g(x) = 1 - x$ (both mappings are 1-T_{L}-compatible with $\leftrightarrow_{\mathrm{L}}$), then we obtain the $\mathbf{L}_k^\theta$-models of fuzzy quantifiers for about or more than m and for about or less m from Ex. 4.3.5.

Since we suppose that the truth values structure for fuzzy logic is a complete residuated lattice, it is natural to define fuzzy quantifiers model by θ-cardinalities of fuzzy sets what we have been done just now. However, we can ask, whether fuzzy quantifier models can be established using $\overline{\theta}$-cardinalities of fuzzy sets, too. An answer is given in the following theorem.

Theorem 4.3.8. *Let* $\mathbf{L}$ *and* $\mathbf{L}^{\mathrm{d}}$ *be complete residuated and dually residuated lattices, respectively, with the same support,* $\mathcal{M}$ *be a fuzzy algebra over a finite non-empty universe* M *and* $K \in \mathcal{M}$. *Let* $\mathbb{C}_{f,g} : \mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(M) \to \mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}^{\overline{\theta}}(N_n)$ *be a* $\overline{\theta}$*-cardinality of finite fuzzy sets generated by* $\overline{\theta}$*- and* θ_d*-homomorphisms* f *and* g *which are* k-$\overline{\theta}$*- and* l-$\overline{\theta}$*-compatible with the bidifference of* $\mathbf{L}^{\mathrm{d}}$*, respectively,* $h : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}$ *be a homomorphism which is* $\top$-θ*-compatible with the biresiduum of* $\mathbf{L}$ *and* $i \in N_n$. *Then the mapping* $\sharp Q_K^h : \mathcal{M} \to L$*, given by* $\sharp Q_K^h(A) = h(|\mathbb{C}_{g,f}(A)(i) \ominus \mathbb{C}_{g,f}(K)(i)|)$ *or* $\sharp Q_K^h(A) = h(\mathbb{C}_{g,f}(A) \approx_{\mathrm{d}} \mathbb{C}_{g,f}(K))$*, is*

(*i*) *an* $\mathbf{L}_l^\theta$*-model of a fuzzy quantifier generated by the* $\overline{\theta}$*-cardinality* $\mathbb{C}_{f,g}$*, fuzzy set* K *and* h*, if* f *is the trivial* $\overline{\theta}$*-homomorphism,*

(*ii*) *an* $\mathbf{L}_k^\theta$*-model of a fuzzy quantifier generated by the* $\overline{\theta}$*-cardinality* $\mathbb{C}_{f,g}$*, fuzzy set* K *and* h*, if* g *is the trivial* θ_d*-homomorphism.*

Proof. It is analogical to the proof of Theorem 4.3.7, where Corollary 3.3.11 is used. □

Example 4.3.9. Let us suppose that $\mathbb{C}_{S_{\mathrm{L}}}$ is the S_{L}-cardinality of finite fuzzy sets defined in Ex 3.3.2. This S_{L}-cardinality of finite fuzzy sets is generated by the identity S_{L}-homomorphism $1_{[0,1]} : \mathbf{L}_{\mathrm{L}}^{\mathrm{d}} \to \mathbf{L}_{\mathrm{L}}^{\mathrm{d}}$ (i.e. $1_{[0,1]}(a) = a$ for any $a \in [0,1]$) and the trivial $T_{\mathrm{L}d}$-homomorphism $g : \mathbf{L}_{\mathrm{L}} \to \mathbf{L}_{\mathrm{L}}^{\mathrm{d}}$ (i.e. $g(a) = 0$ for any $a \in [0,1]$) which are evidently 1-S_{L}-compatible with the bidifference. Further, let $h : [0,1] \to [0,1]$ be given by $h(a) = 1 - a$. Then $h(a) \leftrightarrow h(b) = 1 - |h(a) - h(b)| = 1 - |(1-a) - (1-b)| = 1 - |a - b| = a \leftrightarrow b$ and thus $h : \mathbf{L}_{\mathrm{L}}^{\mathrm{d}} \to \mathbf{L}_{\mathrm{L}}$ is the homomorphism being 1-T_{L}-compatible with biresiduum. Let $\mathcal{M}$ be a fuzzy algebra over $\mathbf{L}_{\mathrm{L}}$ and non-empty set M. Then according to

Theorem 4.3.8, the mapping $\sharp Q_K^h : \mathcal{M} \to [0,1]$, given by

$$\sharp Q_K^h(A) = 1 - (\mathbb{C}_{S_{\mathrm{L}}}(A) \approx_{\mathrm{d}} \mathbb{C}_{S_{\mathrm{L}}}(K)), \tag{4.31}$$

is the $\mathbf{L}_{\mathrm{L1}}^{S_{\mathrm{L}}}$-model of a fuzzy quantifier generated by the S_{L}-cardinality $\mathbb{C}_{S_{\mathrm{L}}}$ (or also $\mathbb{C}_{1_{[0,1]}}$), fuzzy set K and the homomorphism h. Let us suppose that $K = M$. Then we can write $\sharp Q_M^h(A) = 1 - (\mathbb{C}_{S_{\mathrm{L}}}(A) \approx_{\mathrm{d}} \mathbb{C}_{S_{\mathrm{L}}}(M)) = 1 - (|\mathbb{C}_{S_{\mathrm{L}}}(A)(0) - 1| \vee |0 - 0|) = \mathbb{C}_{S_{\mathrm{L}}}(A)(0)$. Comparing this definition of a fuzzy quantifier model with the $\mathbf{L}_k^\theta$-models of fuzzy quantifiers introduced in Ex. 4.3.4, we obtain modified models of Q^{fna} and Q^{fnn} as follows

$$\forall A \in \mathcal{M} : \quad \sharp Q^{fna*}(A) = \mathbb{C}_{S_{\mathrm{L}}}(A)(0), \tag{4.32}$$

$$\forall A \in \mathcal{M} : \quad \sharp Q^{fnn*}(A) = \mathbb{C}_{S_{\mathrm{L}}}(\overline{A})(0). \tag{4.33}$$

Obviously, we have $\sharp Q^{fna}(A) = \sharp Q^{fna*}(A^*)$ and $\sharp Q^{fnn}(A) = \sharp Q^{fnn*}(\overline{A^*})$, where $A^* = \frac{A}{n}$. Since $A \equiv^{S_{\mathrm{L}}} B \geq A^* \equiv^{S_{\mathrm{L}}} B^*$ holds for any $A, B \in \mathcal{M}$ and $\sharp Q^{fna*}$,$\sharp Q^{fnn*}$ are $\mathbf{L}_{\mathrm{L1}}^{S_{\mathrm{L}}}$-models of fuzzy quantifiers, we easily obtain that $\sharp Q^{fna}$, $\sharp Q^{fnn}$ are also $\mathbf{L}_{\mathrm{L1}}^{S_{\mathrm{L}}}$-models of fuzzy quantifiers. Obviously, this technique may be used for constructing and verifying the fuzzy quantifiers models.

In the previous part, we have shown how to define monotone $\mathbf{L}_k^\theta$-models of fuzzy quantifiers generated by equipollence of fuzzy sets and a fuzzy set. Analogously, we can define monotone fuzzy quantifiers models using cardinalities of fuzzy sets as follows. Recall that $f : \mathbf{L} \to \mathbf{L}$ is a θ-*po*-homomorphism, if f is a θ-homomorphism preserving partial ordering of the lattices and analogously $g : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}$ is a $\overline{\theta}_d$-*po*-homomorphism, if g is a $\overline{\theta}_d$-homomorphism reversing partial ordering of the lattices.

Theorem 4.3.9. *Let $\sharp Q_K : \mathcal{M} \to L$ be an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier generated by a θ-cardinality $\mathbb{C}_f$ or $\mathbb{C}_g$ and a fuzzy set K, where f is a θ-po-homomorphism and g is a $\overline{\theta}_d$-po-homomorphism which are compatible with biresiduum of* $\mathbf{L}$. *Then we have*

- *(i) if $K = M$, then $\sharp Q_K$ preserves the partial ordering and*
- *(ii) if $K = \emptyset$, then $\sharp Q_K$ reverses the partial ordering.*

Proof. Let f be θ-*po*-homomorphism and g be a $\overline{\theta}_d$-*po*-homomorphism which are compatible with biresiduum. Due to Theorem 3.2.6, we have that the θ-cardinality $\mathbb{C}_f$ preserves and $\mathbb{C}_g$ reverses the partial ordering. Here we will suppose $\sharp Q_K = \mathbb{C}_{f,g}(A) \approx \mathbb{C}_{f,g}(K)$ as the definition of fuzzy quantifiers models. The proof for the second definition of fuzzy quantifiers models is analogical. First, let us suppose that $K = M = \{x_1, \dots, x_n\}$ and $A, B \in \mathcal{M}$ be two fuzzy sets such that $A \subseteq B$. Then for $\mathbb{C}_f : \mathcal{F}_{\mathbf{L}}(M) \to \mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ we have $\mathbb{C}_f(A) \leq \mathbb{C}_f(B) \leq \mathbb{C}_f(M)$ and hence

$$\sharp Q_M(A) = \mathbb{C}_f(A) \approx \mathbb{C}_f(M) = \bigwedge_{i \in N_n} (\mathbb{C}_f(A)(i) \leftrightarrow \mathbb{C}_f(M)(i)) =$$
$$\bigwedge_{i \in N_n} (\mathbb{C}_f(M)(i) \to \mathbb{C}_f(A)(i)) \leq \bigwedge_{i \in N_n} (\mathbb{C}_f(M)(i) \to \mathbb{C}_f(B)(i)) =$$
$$\bigwedge_{i \in N_n} (\mathbb{C}_f(B)(i) \to \mathbb{C}_f(M)(i)) = \mathbb{C}_f(B) \approx \mathbb{C}_f(M) = \sharp Q_M(B),$$

where the monotony of biresiduum is applied (see e.g. Theorem A.1.2). The proof for $\mathbb{C}_g$ could be done by analogy. Second, let us suppose that $K = \emptyset$ and $A, B \in \mathcal{M}$ such that $A \subseteq B$. Then similarly to the previous part we have for $\mathbb{C}_f$ $(\mathbb{C}_f(\emptyset) \leq \mathbb{C}_f(A))$

$$\sharp Q_\emptyset(A) = \mathbb{C}_f(A) \approx \mathbb{C}_f(\emptyset) = \bigwedge_{i \in N_n} (\mathbb{C}_f(A)(i) \leftrightarrow \mathbb{C}_f(\emptyset)(i)) =$$
$$\bigwedge_{i \in N_n} (\mathbb{C}_f(A)(i) \to \mathbb{C}_f(\emptyset)(i)) \geq \bigwedge_{i \in N_n} (\mathbb{C}_f(B)(i) \to \mathbb{C}_f(\emptyset)(i)) =$$
$$\bigwedge_{i \in N_n} (\mathbb{C}_f(B)(i) \to \mathbb{C}_f(\emptyset)(i)) = \mathbb{C}_f(B) \approx \mathbb{C}_f(\emptyset) = \sharp Q_\emptyset(B).$$

Again, the proof for $\mathbb{C}_g$ could be done by analogy. □

Theorem 4.3.10. *Let $\sharp Q^h_K : \mathcal{M} \to L$ be an $\mathbf{L}^{\theta}_k$-model of a fuzzy quantifier generated by a $\overline{\theta}$-cardinality $\mathbb{C}_f$ or $\mathbb{C}_g$, a fuzzy set K and a homomorphism $h : \mathbf{L}^{\mathrm{d}} \to \mathbf{L}$ being compatible with biresiduum, where f is a θ-po-homomorphism and g is a $\overline{\theta}_d$-po-homomorphism which are compatible with bidifference of $\mathbf{L}^{\mathrm{d}}$. Then we have*

- *(i) if $K = M$, then $\sharp Q^h_K$ preserves the partial ordering and*
- *(ii) if $K = \emptyset$, then $\sharp Q^h_K$ reverses the partial ordering.*

Proof. It is analogical to the proof of the previous theorem, where Theorem 3.3.6 and the monotony of bidifference (see e.g. Theorem A.2.2) are applied. □

4.4 Logical connections for fuzzy quantifiers

One could see that the constructions and verifications of new fuzzy quantifiers models are often laborious. Therefore, it would be useful to have some tools using them the building of fuzzy quantifiers models becomes more effective. A natural way, how to introduce new and often more complex fuzzy quantifiers and mainly their models, could be based on an approach, where new fuzzy quantifiers and their models are defined using already known and often simpler fuzzy quantifiers and their models. This approach to the fuzzy quantifiers building from other one's often occurs in natural language, too. For example, the fuzzy quantifier at least five and at most ten[13] comprises two fuzzy quantifiers, namely at least five and at most ten, which are connected by the logical connection and. Thus we can formally write

$$\text{at least five and at most ten} = (\text{at least five})\ \text{and}\ (\text{at most ten}). \quad (4.34)$$

Further example could be done by using of a negation, say not, of the fuzzy quantifier at least five as follows

$$\text{not (at least five)} = \text{at most four}. \quad (4.35)$$

Note that we said "a negation", because we will define two different types of fuzzy quantifiers negations, namely "external" and "internal" negation, respectively. It seems to be reasonable to use an analogical approach to defining the formulas in a (fuzzy) logic, when more complex formulas of the logic are derived from the atomic one's using the logical connectives, truth constants and quantifiers (by iterated use of the known rule). In the following part we will first propose four binary logical connectives for fuzzy quantifiers, that are analogical to the binary logical connectives for formulas, and then two unary logical connectives, using them we will construct the negations of fuzzy quantifiers.

Let $\&$, $\wedge$, $\vee$ and $\Rightarrow$ be the symbols of binary *logical connectives for fuzzy quantifiers* which have the same forms and meanings as in the case of formulas. In our work we will suppose only one form of these logical connectives, because the meaning of the considered connectives will always be unmistakable. Moreover, we will deal with the first-ordered logic extended by fuzzy quantifiers and thus the connections for fuzzy quantifiers will have just a specific character (contrary to the logical connections of the second-order). Let Q_1 and Q_2 be two fuzzy quantifiers. Then $Q_1 \& Q_2$, $Q_1 \wedge Q_2$, $Q_1 \vee Q_2$ and $Q_1 \Rightarrow Q_2$ are again the fuzzy quantifiers. In order to be able to deal with such derived fuzzy quantifiers, we have to introduce their models. Again, we can use an analogy with the logical connectives interpretations in residuated lattices. Let $\sharp Q_1$ and $\sharp Q_2$ be an $\mathbf{L}^{\theta}_{k_1}$-model and an $\mathbf{L}^{\theta}_{k_2}$-model of

[13] Note that the connective "and" may be replaced by the connective "but".

fuzzy quantifiers Q_1 and Q_2, respectively. Then

$$\sharp(Q_1 \& Q_2) = \sharp Q_1 \otimes \sharp Q_2,\ \ \sharp(Q_1 \wedge Q_2) = \sharp Q_1 \wedge \sharp Q_2,$$
$$\sharp(Q_1 \vee Q_2) = \sharp Q_1 \vee \sharp Q_2,\ \ \sharp(Q_1 \Rightarrow Q_2) = \sharp Q_1 \rightarrow \sharp Q_2,$$

where the operations $\otimes, \wedge, \vee, \rightarrow$ are the extended operations between fuzzy sets (see Section 1.3 on p. 13), define $\mathbf{L}^\theta_k$-models of the corresponding fuzzy quantifiers. An example of the $\mathbf{L}^\theta_k$-models of the fuzzy quantifier for about half and modified fuzzy quantifier for nearly all could be seen on Fig. 4.6. The former is the conjunction of the fuzzy quantifiers, say, for about and more forty percents and for about and less sixty percents. The later is the square of the fuzzy quantifier for nearly all, where the strong conjunction is applied.[14]. The

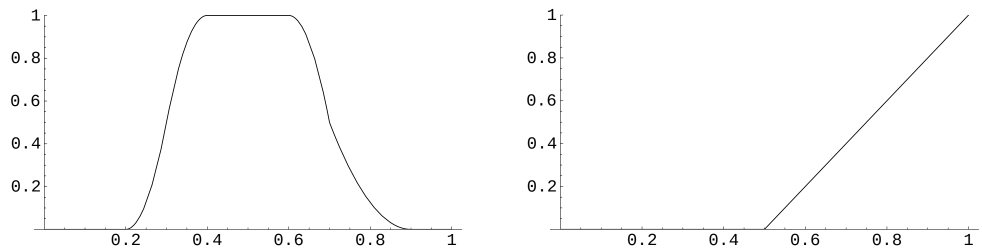


Figure 4.6: Mappings describing $\mathbf{L}^\theta_k$-models of Q^{ah} and $Q^{fna} \& Q^{fna}$

following theorem shows the correctness of our definition of the preceding $\mathbf{L}^\theta_k$-models which were derived using the corresponding operations with fuzzy sets.

Theorem 4.4.1. *Let $\mathbf{L}$ be a complete residuated lattice, $\mathcal{M}$ be a fuzzy algebra over a non-empty universe M and $\sharp Q_i : \mathcal{M} \rightarrow L$ be $\mathbf{L}^\theta_{k_i}$-models of fuzzy quantifiers Q_i, where $i = 1, 2$. Then*

(i) $\sharp(Q_1 \wedge Q_2)$ *and* $\sharp(Q_1 \vee Q_2)$ *are the* $\mathbf{L}^\theta_{k_1 \vee k_2}$*-models of fuzzy quantifiers* $Q_1 \wedge Q_2$ *and* $Q_1 \vee Q_2$*, respectively, and*

(ii) $\sharp(Q_1 \& Q_2)$ *and* $\sharp(Q_1 \Rightarrow Q_2)$ *are the* $\mathbf{L}^\theta_{k_1 + k_2}$*-models of fuzzy quantifiers* $Q_1 \& Q_2$ *and* $Q_1 \Rightarrow Q_2$*, respectively.*

Proof. Here, we will prove the cases $\sharp(Q_1 \wedge Q_2)$ and $\sharp(Q_1 \Rightarrow Q_2)$. The rest could be done by analogy. Let $A, B \in \mathcal{M}$. Since $\sharp Q_i$, $i = 1, 2$, is the

[14]Note that an analogical construction is used in fuzzy syllogisms with fuzzy quantifiers (see e.g. [127–130])

$\mathbf{L}^{\theta}_{k_i}$-models of fuzzy quantifiers Q_i , then we have

$$(A \equiv^{\theta} B)^{k_1} \leq \sharp Q_1(A) \leftrightarrow \sharp Q_1(B),$$
$$(A \equiv^{\theta} B)^{k_2} \leq \sharp Q_2(A) \leftrightarrow \sharp Q_2(B).$$

Hence, applying the operation of meet, we obtain

$$\begin{aligned}(A \equiv^{\theta} B)^{k_1 \vee k_2} =&(A \equiv^{\theta} B)^{k_1} \wedge (A \equiv^{\theta} B)^{k_2} \leq \\ &(\sharp Q_1(A) \leftrightarrow \sharp Q_1(B)) \wedge (\sharp Q_2(A) \leftrightarrow \sharp Q_2(B)) \leq \\ &(\sharp Q_1(A) \wedge \sharp Q_2(A)) \leftrightarrow (\sharp Q_1(B) \wedge \sharp Q_2(B)) = \\ &\sharp(Q_1 \wedge Q_2)(A) \leftrightarrow \sharp(Q_1 \wedge Q_2)(B)\end{aligned}$$

and thus the mapping $\sharp(Q_1 \cap Q_2)$ is the $\mathbf{L}^{\theta}_{k_1 \vee k_2}$-model of the fuzzy quantifier $Q_1 \wedge Q_2$. Analogously, applying the operation of multiplication, we obtain

$$\begin{aligned}(A \equiv^{\theta} B)^{k_1 + k_2} =&(A \equiv^{\theta} B)^{k_1} \otimes (A \equiv^{\theta} B)^{k_2} \leq \\ &(\sharp Q_1(A) \leftrightarrow \sharp Q_1(B)) \otimes (\sharp Q_2(A) \leftrightarrow \sharp Q_2(B)) \leq \\ &(\sharp Q_1(A) \rightarrow \sharp Q_2(A)) \leftrightarrow (\sharp Q_1(B) \rightarrow \sharp Q_2(B)) = \\ &\sharp(Q_1 \Rightarrow Q_2)(A) \leftrightarrow \sharp(Q_1 \Rightarrow Q_2)(B)\end{aligned}$$

and thus the mapping $\sharp(Q_1 \Rightarrow Q_2)$ is the $\mathbf{L}^{\theta}_{k_1 + k_2}$-model of the fuzzy quantifier $Q_1 \Rightarrow Q_2$. □

If one of the fuzzy quantifiers connected by a logical connective is a constant fuzzy quantifier, then we can state a simple consequence of the previous theorem.

Corollary 4.4.2. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M}$ *be a fuzzy algebra over a non-empty* M*,* $\sharp Q$ *be an* $\mathbf{L}^{\theta}_{k}$*-model of a fuzzy quantifier and* $\mathbf{a}$ *be the constant fuzzy quantifier. Then* $\sharp(Q \wedge \mathbf{a})$*,* $\sharp(Q \vee \mathbf{a})$*,* $\sharp(Q \& \mathbf{a})$*,* $\sharp(Q \Rightarrow \mathbf{a})$ *and* $\sharp(\mathbf{a} \Rightarrow Q)$ *define the* $\mathbf{L}^{\theta}_{k}$*-models of the corresponding fuzzy quantifiers.*

Now we will introduce negations of fuzzy quantifiers. Let us suppose that we want to find a relation between the models of the classical quantifiers for all and there exists. Then it is easy to see that just one type of negation of a quantifier is not sufficient to express it. It motivates us to define two types of fuzzy quantifiers negations, which allow us to introduce the relevant relations between fuzzy quantifiers, and using their interpretations to make natural transformations between fuzzy quantifiers models. Note that negations of fuzzy quantifiers are more natural and practically useful in the residuated lattices, where the law of double negation is fulfilled.

The *external negation*, denoted by the symbol $\sim$, of a fuzzy quantifier Q is the fuzzy quantifier $\sim Q$. If $\sharp Q$ is the $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q, then $\sharp{\sim}Q(A) = \sharp Q(A) \to \bot$ defines the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\sim Q$. In fact, due to the previous corollary, the mapping $\sharp(Q \Rightarrow \bot)$, where $\bot$ is the constant fuzzy quantifier, defines the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $Q \Rightarrow \bot$. Since $\sharp(Q \Rightarrow \bot)(A) = \sharp Q(A) \to \bot$, then also $\sharp{\sim}Q$ is an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\sim Q$ and thus our definition of the $\mathbf{L}_k^\theta$-model of a fuzzy quantifier with the external negation is correct. One could be surprised, why the external negation of a fuzzy quantifier Q is not defined by $\sim Q := Q \Rightarrow \bot$. The reason is that the constant fuzzy quantifier $\bot$ is not supposed, in general, in a language for fuzzy quantifiers (see the next section). In the opposite case, that is, if $\bot$ belongs to the set of so-called atomic fuzzy quantifiers of the given language, then the mentioned definition is possible, because both $\mathbf{L}_k^\theta$-models are identical. An example of the $\mathbf{L}_{\text{Ł}k}^\theta$-models of a fuzzy quantifier and its external negation could be seen on Fig. 4.7. In particular, assuming that the mapping h defines an $\mathbf{L}_{\text{Ł}k}^\theta$-model of a fuzzy quantifier Q, then the mapping *enh* is the $\mathbf{L}_{\text{Ł}k}^\theta$-model of its external negation.

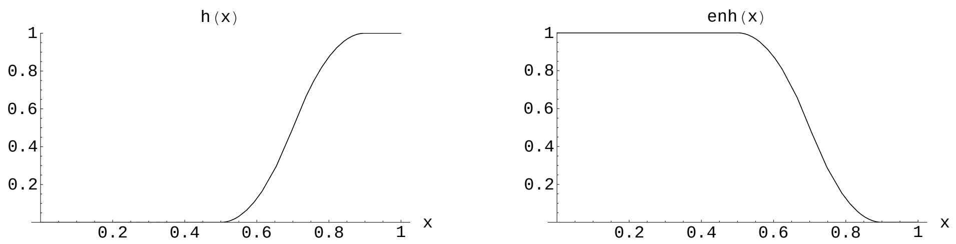


Figure 4.7: Mappings describing $\mathbf{L}_{\text{Ł}k}^\theta$-models of Q and $\sim Q$

Example 4.4.1. As we have mentioned in Ex. 4.3.2, 4.3.3 and Ex. 4.3.4, we have $\sharp\exists^{not} = \sharp{\sim}\exists$, $\sharp\forall^{not} = \sharp{\sim}\forall$, $\sharp\forall^{infin} = \sharp{\sim}\forall^{fin}$ and $\sharp Q^{fnn} = \sharp{\sim}Q^{fna}$ for the $\mathbf{L}_k^\theta$-models of the fuzzy quantifiers of $\exists^{not}$, $\forall^{not}$, $\forall^{infin}$ and Q^{fnn}. Hence, the fuzzy quantifiers $\exists^{not}$, $\forall^{not}$, $\forall^{infin}$ and Q^{fnn} could be defined using the external negation of the fuzzy quantifiers $\exists$, $\forall$, $\forall^{fin}$ and Q^{fna} as follows $\exists^{not} := {\sim}\exists$, $\forall^{not} := {\sim}\forall$, $\forall^{infin} := {\sim}\forall^{fin}$ and $Q^{fnn} := {\sim}Q^{fna}$.

The *internal negation*, denoted by the symbol $\lhd$, of a fuzzy quantifier Q is the fuzzy quantifier $\lhd Q$. If $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier Q, then $\sharp\lhd Q(A) = \sharp Q(\overline{A})$, where $\overline{A}$ denotes the complementary fuzzy set to A, defines the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\lhd Q$. Note that every fuzzy algebra has a complement to each element (see p. 13). The following theorem shows that our definition of the $\mathbf{L}_k^\theta$-model of a fuzzy quantifier with the internal negation is correct, i.e. if $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q, then also the mapping $\sharp\lhd Q$ is an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\lhd Q$.

Theorem 4.4.3. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M *and* $\sharp Q : \mathcal{M} \to L$ *be an* $\mathbf{L}_k^\theta$*-model of a fuzzy quantifier* Q*. Then* $\sharp\lhd Q$ *is an* $\mathbf{L}_k^\theta$*-model of the fuzzy quantifier* $\lhd Q$*.*

Proof. Let $\sharp Q$ be an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q and $A, B \in \mathcal{M}$ be arbitrary fuzzy sets. Suppose that if the operation $\theta = \otimes$ is considered, then we deal with a countable universe X. First, we put $k = 1$. Then we have

$$A \equiv^\theta B = \bigvee_{f \in \mathrm{Perm}(X)} \bigominus_{x \in X} A(x) \leftrightarrow B(f(x)) =$$

$$\bigvee_{\pi \in Perm(X)} \bigominus_{x \in X} (A(x) \leftrightarrow B(f(x)))\theta(\bot \leftrightarrow \bot) \leq$$

$$\bigvee_{f \in \mathrm{Perm}(X)} \bigominus_{x \in X} (A(x) \to \bot) \leftrightarrow (B(f(x)) \to \bot) =$$

$$\overline{A} \equiv^\theta \overline{B} \leq \sharp Q(\overline{A}) \leftrightarrow \sharp Q(\overline{B}) = \sharp\lhd Q(A) \leftrightarrow \sharp\lhd Q(B).$$

If $k > 1$, then analogously we obtain

$$(A \equiv^\theta B)^k \leq (\overline{A} \equiv^\theta \overline{B})^k \leq \sharp Q(\overline{A}) \leftrightarrow \sharp Q(\overline{B}) \leq \sharp\lhd Q(A) \leftrightarrow \sharp\lhd Q(B)$$

and the proof is complete. □

As could be shown, the $\mathbf{L}_k^\theta$-model of a fuzzy quantifier with the internal negation is established by some internal transformation of $\mathbf{L}_k^\theta$-models of the original one. This is the difference from the external negation that results from the "external" operation residuum applying to the values of $\mathbf{L}_k^\theta$-models. An example of the $\mathbf{L}_{Ł}{}_k^\theta$-models of a fuzzy quantifier (the mapping h) and its internal negation (the mapping inh) could be seen on Fig.4.8. Compare the mappings enh from Fig. 4.7 and inh.

Example 4.4.2. It is easy to see that $\sharp\forall^{not} = \sharp\lhd\exists$, $\sharp\exists^{not} = \sharp\lhd\forall$, $\sharp Q^{fnn} = \sharp\lhd Q^{fna}$ hold for the $\mathbf{L}_k^\theta$-models of the fuzzy quantifiers $\forall^{not}$, $\exists^{not}$, Q^{fnn} and vice-versa. Hence, the fuzzy quantifiers $\forall^{not}$, $\exists^{not}$, Q^{fnn} could be defined as follows $\forall^{not} := \lhd\exists$, $\exists^{not} := \lhd\forall$, $Q^{fnn} := \sharp\lhd Q^{fna}$. Furthermore, it is interesting to notice that the fuzzy quantifier for nearly none could be derived from the fuzzy quantifier for nearly all applying the external as well as internal negation, i.e. $Q^{fnn} := \sim Q^{fna}$ and simultaneously $Q^{fnn} := \lhd Q^{fna}$.

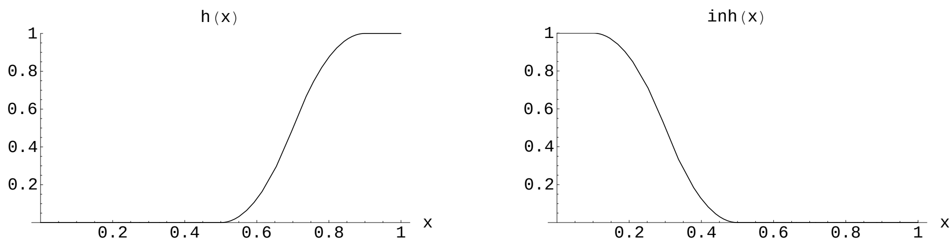


Figure 4.8: Mappings describing $\mathbf{L}_{\text{Ł}_k}^{\theta}$-models of Q and $\lhd Q$

In the following part we will give several statements about the models of complex fuzzy quantifiers. If negations are applied more than once to a fuzzy quantifier, then we will often write the resulting fuzzy quantifier without the parentheses, e.g. we will write $\sim\lhd\sim\lhd Q$ instead of $\sim(\lhd(\sim(\lhd Q)))$. The parentheses, however, have to be respected to find the $\mathbf{L}_k^{\theta}$-model of a fuzzy quantifier with the negations, e.g. $\sharp\sim\lhd\sim\lhd Q(A) = \sharp\sim(\lhd(\sim(\lhd Q)))(A) = (\sharp Q(\overline{\overline{A}}) \to \bot) \to \bot$ holds for any $A \in \mathcal{M}$. An immediate consequence of the definitions of the external and internal negations may be stated as follows.

Corollary 4.4.4. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M *and* $\sharp Q$ *be an* $\mathbf{L}_k^{\theta}$*-model of a fuzzy quantifier* Q*. Then* $\sharp\lhd\sim Q = \sharp\sim\lhd Q$ *is the* $\mathbf{L}_k^{\theta}$*-model of the fuzzy quantifier* $\lhd\sim Q$ *or equivalently of the fuzzy quantifier* $\sim\lhd Q$*.*

An example of the simultaneous application of both negation types could be done by the classical quantifiers, i.e. $\forall := \lhd\sim\exists$ and vice-versa. If an $\mathbf{L}_{\text{Ł}_k}^{\theta}$-model of a fuzzy quantifier Q is introduced using the mapping h, displayed on Fig. 4.9, then the $\mathbf{L}_{\text{Ł}_k}^{\theta}$-model of the fuzzy quantifier $\sim\lhd Q$ is introduced by the mapping *einh*.

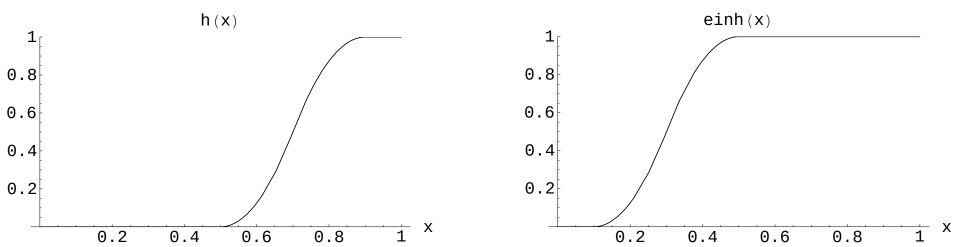


Figure 4.9: Mappings describing $\mathbf{L}_k^{\theta}$-models of Q and $\lhd\sim Q$

The relationships between external and internal negations and their combination may be described in a square of opposition that is an analogy to the Aristotelian square displayed on Fig. 4.1. Our square of opposition has a modern form (see also e.g. [31,33]) and it is displayed on Fig. 4.10. Examples of these negations derived from the classical quantifier for all are illustrated in parentheses.

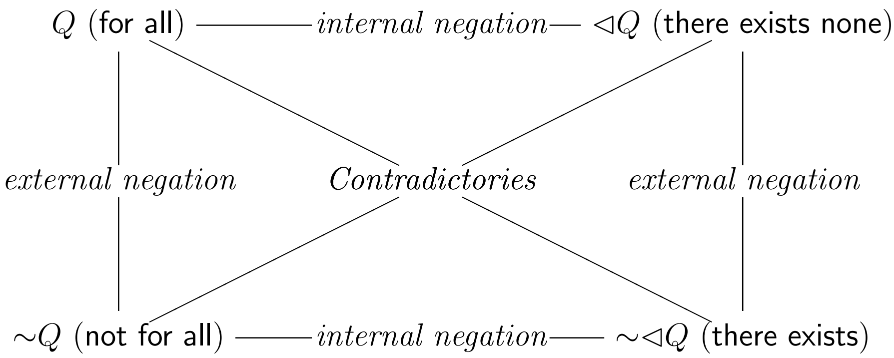


Figure 4.10: Modern square of opposition

The following theorem shows how the ordering of $\mathbf{L}_k^\theta$-models is changed, when the negations $\lhd$ and $\sim$ are applied. Recall that $\sharp Q$ is a fuzzy set over $\mathcal{M}$ and thus $\sharp Q \geq \sharp Q'$, if $\sharp Q(A) \geq \sharp Q'(A)$ holds for any $A \in \mathcal{M}$.

Theorem 4.4.5. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M *and* $\sharp Q, \sharp Q' : \mathcal{M} \to L$ *be* $\mathbf{L}_k^\theta$, $\mathbf{L}_{k'}^\theta$*-models of fuzzy quantifiers* Q, Q'*, respectively, such that* $\sharp Q \geq \sharp Q'$*. Then we have*

(i) $\sharp\lhd Q \leq \sharp\lhd Q'$ *and* $\sharp{\sim}Q \leq \sharp{\sim}Q'$,

(ii) $\sharp Q \leq \sharp\lhd\lhd Q$ *and* $\sharp Q \leq \sharp{\sim}{\sim}Q$,

(iii) $\sharp\lhd Q = \sharp\lhd\lhd\lhd Q$ *and* $\sharp{\sim}Q = \sharp{\sim}{\sim}{\sim}Q$.

Proof. It is a straightforward consequence of the monotony of residuum (see e.g. Theorem A.1.2). □

The following theorem shows an influence of the external and internal negations to fuzzy quantifiers derived by the binary logical connections.

Theorem 4.4.6. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathcal{M}$ *be a fuzzy algebra over a non-empty universe* M *and* $\sharp Q_1, \sharp Q_2 : \mathcal{M} \to L$ *be* $\mathbf{L}_{k_1}^\theta$, $\mathbf{L}_{k_2}^\theta$*-models of fuzzy quantifiers* Q_1, Q_2*, respectively. Then we have*

(i) $\sharp{\sim}(Q_1 \& Q_2) = \sharp(Q_1 \Rightarrow {\sim}Q_2)$,

(ii) $\sharp{\sim}(Q_1 \wedge Q_2) \geq \sharp({\sim}Q_1 \vee {\sim}Q_2)$,

(iii) $\sharp{\sim}(Q_1 \vee Q_2) = \sharp({\sim}Q_1 \wedge {\sim}Q_2)$,

(iv) $\sharp{\sim}(Q_1 \Rightarrow Q_2) \geq \sharp(Q_1 \& {\sim}Q_2)$.

(v) $\sharp\lhd(Q_1 \& Q_2) = \sharp(\lhd Q_1 \& \lhd Q_2)$,

(vi) $\sharp\lhd(Q_1 \wedge Q_2) = \sharp(\lhd Q_1 \wedge \lhd Q_2)$,

(vii) $\sharp\lhd(Q_1 \vee Q_2) = \sharp(\lhd Q_1 \vee \lhd Q_2)$,

(viii) $\sharp\lhd(Q_1 \Rightarrow Q_2) = \sharp(\lhd Q_1 \Rightarrow \lhd Q_2)$.

Moreover, if $\mathbf{L}$ *is an* MV*-algebra, then the inequalities in (ii) and (iv) may be replaced by the equalities.*

Proof. Let $\sharp Q_1, \sharp Q_2$ be $\mathbf{L}_k^\theta$-models of fuzzy quantifiers Q_1, Q_2, respectively, and $A \in \mathcal{M}$. Then we have $\sharp{\sim}(Q_1 \& Q_2)(A) = \sharp(Q_1 \& Q_2)(A) \to \bot = \sharp Q_1(A) \otimes \sharp Q_2(A) \to \bot = \sharp Q_1(A) \to (\sharp Q_2(A) \to \bot) = \sharp Q_1(A) \to \sharp{\sim}Q_2(A) = \sharp(Q_1 \Rightarrow {\sim}Q_2)(A)$. Hence, the first statement is true. Further, we have $\sharp{\sim}(Q_1 \wedge Q_2)(A) = \sharp(Q_1 \wedge Q_2)(A) \to \bot = (\sharp Q_1(A) \wedge \sharp Q_2(A)) \to \bot \geq (\sharp Q_1(A) \to \bot) \vee (\sharp Q_2(A) \to \bot) = \sharp{\sim}Q_1(A) \vee \sharp{\sim}Q_2(A) = \sharp({\sim}Q_1 \vee {\sim}Q_2)(A)$. Hence, the second statement is true. Because the proofs of the third and forth statements are analogical, we omit them. Further, we have $\sharp\lhd(Q_1 \& Q_2)(A) = \sharp Q_1 \& Q_2(\overline{A}) = \sharp Q_1(\overline{A}) \otimes \sharp Q_2(\overline{A}) = \sharp\lhd Q_1(A) \otimes \sharp\lhd Q_2(A) = \sharp\lhd Q_1 \& \lhd Q_2(A)$. Hence, the fifth statement is true. The remaining statements could be proved by analogy. Moreover, if $\mathbf{L}$ is an MV-algebra, then the statements follow from the equalities $(a \wedge b) \to c = (a \to c) \vee (b \to c)$ and $a \otimes (b \to \bot) = (a \to b) \to \bot$ that hold in each MV-algebra[15]. □

4.5 Fuzzy quantifiers types

In the previous sections we did not require any special properties from the fuzzy quantifiers. In order to establish the semantics of fuzzy logic with fuzzy quantifiers, we need to suppose that the structures of fuzzy quantifiers contain just fuzzy quantifiers with some property of their models. This requirement seems to be very natural also from the practical point of view, because each "reasonable" fuzzy quantifier is interpreted in an $\mathbf{L}$-structure either by just one model (as e.g. $\forall$, $\exists$), $\forall^{fin}$ etc. or by a class of models having, however, a common property (as e.g. Q^{fna} or Q^{fnn}, which models are under the influence

[15] Note that the equalities are a consequence of the law of double negation (see p. 2). For instance, we have $a \otimes (b \to \bot) = ((a \otimes (b \to \bot)) \to \bot) \to \bot = (a \to ((b \to \bot) \to \bot)) \to \bot = (a \to b) \to \bot$.

of a subjective understanding of the fuzzy quantifier meanings). Therefore, we introduce a notion of the *fuzzy quantifier type* that will characterize a general property of $\mathbf{L}_k^\theta$-models of this fuzzy quantifier.

An $\mathbf{L}_k^\theta$-model $\sharp Q$ of a fuzzy quantifier Q is monotonous, if $\sharp Q$ preserves or reverses the partial ordering between the ordering structures $(\mathcal{M}, \subseteq)$ and $(L, \leq)$. We say that a fuzzy quantifier is of the type at least or also al for short, if each of its $\mathbf{L}_k^\theta$-models preserves the partial ordering. It means that if a fuzzy quantifier of the type al is supposed, then each of its $\mathbf{L}_k^\theta$-models will preserve the partial ordering. Further, we say that a fuzzy quantifier is of the type at most or also am for short, if each of its $\mathbf{L}_k^\theta$-models reverses the partial ordering. Again, it means that if a fuzzy quantifier of the type am is supposed, then each of its $\mathbf{L}_k^\theta$-models will reverse the partial ordering. Finally, a fuzzy quantifier is of the type general or also g for short, if none of its $\mathbf{L}_k^\theta$-models is monotonous, that is, if none of its $\mathbf{L}_k^\theta$-models preserves or reverses the partial ordering. Thus, we do not admit such fuzzy quantifiers that some of their models preserve and some of their models reverse the partial ordering or some of their models preserve the partial ordering and some of their models are not monotonic etc. An exception is made by the fuzzy quantifiers having the constant models and thus they are of the type al and simultaneously am[16]. An example of such fuzzy quantifiers is given by the constant fuzzy quantifiers (see p. 111).

Let us denote by $\mathbb{T}$ the set of the mentioned types, i.e. $\mathbb{T} = \{\mathsf{al}, \mathsf{am}, \mathsf{g}\}$. The fact that a fuzzy quantifier Q is of the type $\iota \in \mathbb{T}$ will be denoted by the couple (Q, ι). The fuzzy quantifiers (Q, ι), where $\iota \in \mathbb{T} \setminus \{\mathsf{g}\}$, will be called the *basic fuzzy quantifiers*. The fuzzy quantifiers (Q, g) will be occasionally called the *non-basic fuzzy quantifiers*. Several examples of the basic fuzzy quantifiers could be given using Theorems 4.3.6, 4.3.9 and 4.3.10. In particular, a fuzzy quantifier Q_K is of the type al, if $K = M$, and of the type am, if $K = \emptyset$.

In the subsection 4.4 we have shown that new fuzzy quantifiers can be established using the logical connectives for fuzzy quantifiers. Now we can ask, what types of the fuzzy quantifiers are obtained after applying the logical connectives. In other words, is it possible to uniquely determine the types of the complex fuzzy quantifiers from the types of the atomic one's? Unfortunately, the influence of the logical connectives on types preservation is not uniquely determined, in general. In fact, if (Q_1, g) and (Q_2, al), then it is not difficult to find the $\mathbf{L}_k^\theta$-models of Q_1 and Q_2 (of course, respecting

[16]Note that the truth values of formulas with fuzzy quantifiers of different types are derived by different rules. Therefore, we will have to prove a correctness of those rules for formulas with the fuzzy quantifiers having the type al and am at the same time.

the types of Q_1 and Q_2) such that the resulting type of the fuzzy quantifier $Q_1 \vee Q_2$ is g and then to find other $\mathbf{L}_k^\theta$-models of Q_1 and Q_2 such that the resulting type of the fuzzy quantifier $Q_1 \vee Q_2$ is al. On the other hand, for the basic fuzzy quantifiers we can state the following lemmas and corollary.

Lemma 4.5.1. *Let Q_1, Q_2 be fuzzy quantifiers. If Q_1, Q_2 are of the type* al *or* am*, then $Q_1 \& Q_2$, $Q_1 \wedge Q_2$ and $Q_1 \vee Q_2$ are of the type* al *or* am*, respectively. Moreover, if Q_1 is of the type* al *and Q_2 is of the type* am*, then $Q_1 \Rightarrow Q_2$ is of the type* am *and $Q_2 \Rightarrow Q_1$ is of the type* al.

Proof. It is a straightforward consequence of the monotony of the operations $\otimes, \wedge, \vee$ and $\rightarrow$. □

Lemma 4.5.2. *Let Q be a fuzzy quantifier. If Q is of the type* al*, then $\lhd Q$ and $\sim Q$ is of the type* am *and conversely. Moreover, if Q possesses one of these types, then $\sim\lhd Q$ is of the same type.*

Proof. Let us suppose that $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier Q of the type al, i.e. $\sharp Q(A) \leq \sharp Q(B)$, whenever $A \subseteq B$. Obviously, if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$. Hence, we obtain $\sharp\lhd Q(A) = \sharp Q(\overline{A}) \geq \sharp Q(\overline{B}) = \sharp\lhd Q(B)$, whenever $A \subseteq B$. Moreover, $\sharp\sim Q(A) = \sharp Q(A) \rightarrow \bot \geq \sharp Q(B) \rightarrow \bot = \sharp\sim Q(B)$ and thus the first statement is proved. Further, we have $\sharp\sim\lhd Q(A) = \sharp\sim Q(\overline{A}) = \sharp Q(\overline{A}) \rightarrow \bot \leq \sharp Q(\overline{B}) \rightarrow \bot = \sharp\sim Q(\overline{A}) = \sharp\sim\lhd Q(B)$ and hence the second statement is true for the model of the type al. The analogical proof could be done, if an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier of the type am is considered. □

Corollary 4.5.3. *Let Q be a fuzzy quantifier. If Q is of the type* al*, then $\lhd\lhd Q$ and $\sim\sim Q$ is of the type* al *and, analogously, for the type* am.

Proof. It is a straightforward consequence of the previous lemma. □

4.6 Structures of fuzzy quantifiers

In the previous sections we established the logical connectives for fuzzy quantifiers and then the types of fuzzy quantifiers, namely at least, at most and general. In this part we will introduce a structure of fuzzy quantifiers on the basis of which the fuzzy quantifiers can be implemented into the language of the first-ordered fuzzy logic. As we have mentioned, not all fuzzy quantifiers are suitable for the interpretation in **L**-structures. Therefore, we have to restrict ourselves to a class of the basic fuzzy quantifiers and the fuzzy quantifiers which are derived from the basic one's using the logical connectives. First, we will define a class of fuzzy quantifiers which are derived from the fuzzy quantifiers of a given set $\mathcal{Q}$. The elements of the set $\mathcal{Q}$ play a role of "atoms". For example, the classical quantifiers could be understood as the atoms and then we could define a class of all fuzzy quantifiers over the set $\mathcal{Q} = \{\forall, \exists\}$. Obviously, this class is very abstract and nearly all of their (fuzzy) quantifiers are useless. Therefore, the fuzzy quantifiers structure will be defined as a special subset of the class of fuzzy quantifiers over $\mathcal{Q}$, where, moreover, each fuzzy quantifier is connected with some fuzzy quantifier type. In our example, a fuzzy quantifiers structure over the $\mathcal{Q}$ could be the set $\{(\forall, \mathsf{al}), (\exists, \mathsf{al})\}$, where to the classical quantifiers the type at least is assigned (cf. Ex. 4.3.1). Note that the formal assignment (fuzzy quantifier, its type) is very important to uniquely determine the evaluation rules, which are applied to formulas with the fuzzy quantifiers from this structure. In other words, we can not find the truth value of a formula with the fuzzy quantifiers, when the type of the used fuzzy quantifier is not specified[17].

Let $\mathcal{Q}$ be a non-empty finite set of fuzzy quantifiers. Then we can introduce the fuzzy quantifiers over $\mathcal{Q}$ using the following inductive definition. *Atomic fuzzy quantifiers* over $\mathcal{Q}$ are all fuzzy quantifiers from $\mathcal{Q}$. If Q_1, Q_2 and Q are fuzzy quantifiers over $\mathcal{Q}$, then also $Q_1 \& Q_2$, $Q_1 \wedge Q_2$, $Q_1 \vee Q_2$, $Q_1 \Rightarrow Q_2$ and $\sim Q$, $\rhd Q$ are fuzzy quantifiers over $\mathcal{Q}$. The set of all fuzzy quantifiers over $\mathcal{Q}$ will be denoted by $\mathcal{F}_{\mathcal{Q}}$. If $Q \in \mathcal{F}_{\mathcal{Q}}$ and the set $\mathcal{Q}$ is known, then we will say, for simplicity, that Q is a fuzzy quantifier instead of Q is a fuzzy quantifier over $\mathcal{Q}$. Let $Q \in \mathcal{F}_{\mathcal{Q}}$ be a fuzzy quantifier, then $Aq(Q) \subseteq \mathcal{F}_{\mathcal{Q}}$ is the subset of all atomic fuzzy quantifiers from $\mathcal{F}_{\mathcal{Q}}$ included in Q. This assignment defines a mapping $Aq : \mathcal{F}_{\mathcal{Q}} \to 2^{\mathcal{Q}}$.

Example 4.6.1. Let us suppose that $Q^{m\precsim}$ and $Q^{\precsim m}$ are fuzzy quantifiers for about or more than m and for about or less than m, respectively, belonging to $\mathcal{F}_{\mathcal{Q}}$. Then for the (non-atomic) fuzzy quantifier $Q := Q^{m\precsim} \wedge Q^{\precsim m}$, which is derived from $Q^{m\precsim}$ and $Q^{\precsim m}$ using the logical connective $\wedge$, we have $Aq(Q) = \{Q^{m\precsim}, Q^{\precsim m}\}$.

Let $Q \in \mathcal{F}_{\mathcal{Q}}$ be a fuzzy quantifier. Then obviously there exists a finite sequence $Q_1, Q_2, \ldots, Q_n$, where $Q := Q_n$, of fuzzy quantifiers from $\mathcal{F}_{\mathcal{Q}}$ such that for each $i \leq n$ the fuzzy quantifier Q_i is atomic (i.e. $Q_i \in \mathcal{Q}$), or there exist $k, l < i$ and a binary logical connective for fuzzy quantifiers such that Q_i is derived by the considered binary logical connective applied on Q_k and Q_l, or there exist $k < i$ and a unary logical connective for fuzzy quantifier such that Q_i is derived by the considered unary logical connective applied

[17]Note that for some complex fuzzy quantifiers there could be a problem to uniquely determine their (natural) types and therefore, we have to establish them. On other hand, if the type of a fuzzy quantifier is known, then it will be used.

on Q_k. The finite sequence $Q_1, Q_2, \dots, Q_n$, where $Q := Q_n$, defined above is called the *formation sequence* of the fuzzy quantifier Q.

Example 4.6.2. Let $Q := \sim((Q_1 \wedge Q_2) \Rightarrow (Q_3 \vee Q_4))$, then the formation sequence of Q can be given as follows $Q_1 := Q_1$, $Q_2 := Q_2$, $Q_3 := Q_1 \wedge Q_2$, $Q_4 := Q_3$, $Q_5 := Q_4$, $Q_6 := Q_3 \vee Q_4$, $Q_7 := Q_3 \Rightarrow Q_6$ and $Q_8 := \sim Q_7$, where clearly $Q := Q_8$. It is easy to see that there is more than one formation sequence, in general, but for each two formation sequences there exists a unique correspondence. In particular, if $Q_1, \dots, Q_n$ and $Q'_1, \dots, Q'_m$ are two formation sequences of Q, then $n = m$ and there exists a permutation f on $\{1, \dots, n\}$ such that $Q_i := Q'_{f(i)}$ holds for any $i = 1, \dots, n$. The proof could be done by induction on the complexity of the fuzzy quantifier Q.

Let $\mathcal{Q}$ be a non-empty set of fuzzy quantifiers. A subset $\mathcal{S}_{\mathcal{Q}} \subseteq \mathcal{F}_{\mathcal{Q}} \times \mathbb{T}$ is the *structure of fuzzy quantifiers* over $\mathcal{Q}$, if

(*i*) for every $(Q, \iota_1), (Q, \iota_2) \in \mathcal{S}_{\mathcal{Q}}$ we have $\iota_1 = \iota_2$,

(*ii*) for every $(Q, \iota) \in \mathcal{S}_{\mathcal{Q}}$ such that $Q \in \mathcal{Q}$ we have $\iota \in \{\mathsf{al}, \mathsf{am}\}$ and

(*iii*) for every $(Q, \iota) \in \mathcal{S}_{\mathcal{Q}}$ there exist $(Q_1, \iota_1), \dots, (Q_n, \iota_n) \in \mathcal{S}_{\mathcal{Q}}$ such that $Q_1, \dots, Q_n$ is a formation sequence of Q.

A structure of fuzzy quantifiers $\mathcal{S}_{\mathcal{Q}}$ is called *atomic*, if each fuzzy quantifier of $\mathcal{S}_{\mathcal{Q}}$ is the atomic fuzzy quantifier, i.e. $Q \in \mathcal{Q}$ for all $(Q, \iota) \in \mathcal{S}_{\mathcal{Q}}$. Moreover, a structure of fuzzy quantifiers $\mathcal{S}_{\mathcal{Q}}$ is called *basic*, if each fuzzy quantifier of $\mathcal{S}_{\mathcal{Q}}$ is the basic fuzzy quantifier, i.e. $\iota \in \{\mathsf{al}, \mathsf{am}\}$ for all $(Q, \iota) \in \mathcal{S}_{\mathcal{Q}}$. Clearly, each atomic structure is also the basic one.

Example 4.6.3. Let us suppose that $\forall, \exists, Q^{m\precsim}, Q^{\precsim m} \in \mathcal{Q}$. Then $\mathcal{S}_{\mathcal{Q}_1} = \{(\forall, \mathsf{al}), (\exists, \mathsf{al})\}$ is an atomic structure and

$$\mathcal{S}_{\mathcal{Q}_2} = \{(Q^{m\precsim}, \mathsf{al}), (Q^{\precsim m}, \mathsf{am}), (Q^{m\precsim} \wedge Q^{\precsim m}, \mathsf{g}), (\sim(Q^{m\precsim} \wedge Q^{\precsim m}), \mathsf{g})\}$$

is a (non-atomic) structure of fuzzy quantifiers. On the other hand, the set $\mathcal{S}_{\mathcal{Q}_3} = \{(\forall, \mathsf{al}), (\sim(\lhd\forall), \mathsf{al})\}$ is not a structure of fuzzy quantifiers, because (*iii*) is not satisfied, i.e. $(\lhd\forall, \iota) \notin \mathcal{S}_{\mathcal{Q}_3}$ for any $\iota \in \mathbb{T}$.

One could notice that in spite of $\exists := \sim(\lhd\forall)$ the set $\mathcal{S}_{\mathcal{Q}_3}$ is not a structure of fuzzy quantifiers over $\mathcal{Q}$. Let us make a short explanation of this strangeness. In each fuzzy quantifier structure we have to strictly distinguish, whether the fuzzy quantifier is the atomic or non-atomic fuzzy quantifier, because this fact will have an immediate impact on the interpretation of fuzzy quantifiers meanings in an **L**-structure. In particular, each atomic fuzzy quantifier of $\mathcal{S}_{\mathcal{Q}}$ will be interpreted by an atomic $\mathbf{L}_k^{\theta}$-model, i.e. its model is not derived from the others. On the other hand, each $\mathbf{L}_k^\theta$-model of non-atomic fuzzy quantifiers of $\mathcal{S}_\mathcal{Q}$ will be derived (by induction) from the $\mathbf{L}_k^\theta$-models of the corresponding atomic fuzzy quantifiers and thus they are dependent on some atomic models. Thus, if we return to the mentioned case, then the model of $(\exists, \mathsf{al})$ is introduced directly ($\exists$ is atomic), but the model of $(\sim\triangleleft\forall, \mathsf{al})$ has to be introduced by induction ($\sim\triangleleft\forall$ is non-atomic). However, it is impossible, because the model of $(\sim\triangleleft\forall, \mathsf{al})$ is derived from a model of $(\triangleleft\forall, \iota)$ for a suitable $\iota \in \mathbb{T}$, which is not supposed ($\triangleleft\forall \notin \mathcal{S}_{\mathcal{Q}3}$).

4.7 Fuzzy logic with fuzzy quantifiers: syntax and semantics

In this section the syntax and semantics of the first-ordered fuzzy logic will be extended by fuzzy quantifiers, namely, the classical quantifiers $\forall$ and $\exists$ will be replaced by fuzzy quantifiers from a structure for fuzzy quantifiers.

4.7.1 Syntax

Let us suppose that $\mathcal{S}_\mathcal{Q} = \{(\forall, \mathsf{al}), (\exists, \mathsf{al})\}$ is the structure of (fuzzy) quantifiers over $\mathcal{Q} = \{\forall, \exists\}$. Then a language of the first-ordered fuzzy logic $\mathcal{J}$ can be expressed as a system $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ of the same predicate symbols, functional symbols etc. as in the language $\mathcal{J}$, where the quantifiers symbols $\forall$ and $\exists$ are replaced by two new symbols $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, each one for a fuzzy quantifier of $\mathcal{S}_\mathcal{Q}$, e.g. $\mathfrak{Q}_1$ for $(\forall, \mathsf{al})$ and $\mathfrak{Q}_2$ for $(\exists, \mathsf{al})$. Now we can deal with the modified language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ of the first-ordered fuzzy logic with the fuzzy quantifiers of $\mathcal{S}_\mathcal{Q}$ instead of the original language $\mathcal{J}$. Obviously, both languages are equivalent. In the following part this approach will be generalized.

Let $\mathcal{S}_\mathcal{Q}$ be a structure of fuzzy quantifiers over a non-empty finite set $\mathcal{Q}$ and $\mathcal{J}$ be a language of the first-ordered fuzzy logic (with the classical quantifiers). Then the language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ of the first-ordered fuzzy logic with the fuzzy quantifiers of $\mathcal{S}_\mathcal{Q}$ over $\mathcal{J}$ consists of the same predicate symbols, functional symbols, object constants, object variables, logical connectives, truth constants and auxiliary symbols as in the language $\mathcal{J}$, but the quantifiers symbols $\forall$ and $\exists$ are replaced by new symbols $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots$ each for just one fuzzy quantifier from $\mathcal{S}_\mathcal{Q}$.

If $\mathfrak{Q}$ is assigned to the couple (Q, ι), then we will write $\mathfrak{Q} := (Q, \iota)$. Further, if $\mathfrak{Q}_1 := (Q_1, \iota_1)$, $\mathfrak{Q}_2 := (Q_2, \iota_2)$ and $\mathfrak{Q} := (Q_1 \circledast Q_2, \iota)$, where $\circledast$ is one of the binary logical connectives for fuzzy quantifiers, then we will also write $\mathfrak{Q} := \mathfrak{Q}_1 \circledast \mathfrak{Q}_2$ and analogously for $\mathfrak{Q} := (Q, \iota)$ and $\mathfrak{Q}' := (\ltimes Q, \iota')$, where $\ltimes$ is one of the fuzzy quantifier negations, we will write $\mathfrak{Q}' := \ltimes\mathfrak{Q}$.

The terms and atomic formulas are defined in the same way as in the case of the first-ordered fuzzy logic with the classical fuzzy quantifiers. If φ, ψ are formulas and x is an object variable, then $\varphi \& \psi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $\bot$, $\top$ and $(\mathfrak{Q}x)\varphi$ are formulas for all fuzzy quantifiers of the language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$. If $(\mathfrak{Q}x)\varphi$ be a formula, then the scope of the fuzzy quantifier $(\mathfrak{Q}x)$ is the formula φ. The definitions of free and bound variables of a formula and the substitutability of a term for a variable in a formula are again the same as in the case of the first-ordered fuzzy logic with the classical fuzzy quantifiers (see p. 105).

4.7.2 Semantics

In this part we are going to introduce the semantics of the first-ordered fuzzy logic with fuzzy quantifiers. In particular, each **L**-structure **M** for a language $\mathcal{J}$ will be extended by the $\mathbf{L}_k^\theta$-models for all fuzzy quantifiers of a language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$. Moreover, we have to add also a fuzzy algebra $\mathcal{M}$. It seems to be reasonable to suppose that all $\mathbf{L}_k^\theta$-models will be defined over the same operation θ. In order to indicate this fact, we will write the $\mathbf{L}^\theta$-structure.

We say that a mapping $\sharp Q : \mathcal{M} \to L$ is an $\mathbf{L}_k^\theta$-model of a fuzzy quantifier $\mathfrak{Q} := (Q, \iota)$ over $\mathcal{M}$, if the mapping $\sharp Q$ is an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier Q over $\mathcal{M}$ respecting the type ι.

Let $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ be a language and **L** be a complete residuated lattice. Then an $\mathbf{L}^\theta$-structure $\mathbf{M} = \langle M, \mathcal{M}, (r_P)_P, (f_F)_F, (m_a)_a, (\sharp Q)_\mathfrak{Q}\rangle$ for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ has a non-empty domain M, a fuzzy algebra $\mathcal{M}$ over the domain M, an n-ary fuzzy relation $r_P : M^n \to L$ for each predicate symbol P, where L is the support of **L**, an n-ary function $f_F : M^n \to M$ (it is not fuzzy) for each functional symbol F, an element m_a from M for each object constant a and an $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\mathfrak{Q}$ over $\mathcal{M}$ for each fuzzy quantifier $\mathfrak{Q}$. Moreover, the $\mathbf{L}_k^\theta$-model of each fuzzy quantifier $\mathfrak{Q}$ results from the iterated use the following rule (put $\circledast \in \{\&, \wedge, \vee, \Rightarrow\}$ and $\ltimes \in \{\sim, \lhd\}$):

- (i) if $\mathfrak{Q} := (Q, \iota)$ and Q is atomic, then $\sharp Q$ is the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\mathfrak{Q}$,
- (ii) if $\mathfrak{Q} := (Q_1 \circledast Q_2, \iota)$ and the mapping $\sharp Q_1 \circledast \sharp Q_2$ respects the type ι, then $\sharp Q_1 \circledast \sharp Q_2$ is the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\mathfrak{Q}$,
- (iii) if $\mathfrak{Q} := (\ltimes Q, \iota)$ and the mapping $\sharp \ltimes Q$ respects the type ι, then $\sharp \ltimes Q$ is the $\mathbf{L}_k^\theta$-model of the fuzzy quantifier $\mathfrak{Q}$.

Obviously, the $\mathbf{L}_{k_i}^\theta$-models of atomic fuzzy quantifiers from $\mathcal{S}_\mathcal{Q}$ play the major role in the $\mathbf{L}^\theta$-structure for the language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, because they are defined in such a way that the $\mathbf{L}_k^\theta$-models of the non-atomic fuzzy quantifiers (derived from them) respect their types. It is easy to show that not all $\mathbf{L}_{k_i}^\theta$-models of the atomic fuzzy quantifiers define an $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$, in general, even there are languages $\mathcal{J}(\mathcal{S}_Q)$ without any $\mathbf{L}$-structure for $\mathcal{J}(\mathcal{S}_Q)$. In fact, it is sufficient to suppose that $\mathcal{S}_Q = \{(Q, \mathsf{al}), (\sim Q, \mathsf{g})\}$. Then, due to Lemma 4.5.2, the fuzzy quantifier $\sim Q$ is of the type am, but we suppose that $\sim Q$ is of the type g, a contradiction, and thus $\mathcal{J}(\mathcal{S}_Q)$ has no $\mathbf{L}$-structure. We say that a language $\mathcal{J}(\mathcal{S}_Q)$ is *regular*, if there is an $\mathbf{L}$-structure for it. In the following parts we will suppose that each language $\mathcal{J}(\mathcal{S}_Q)$ is regular.

As the $\mathbf{M}$-evaluations of object variables, the values of terms and the truth values of formulas without quantifiers under $\mathbf{M}$-evaluation are defined in the same way as in the case of the first-ordered fuzzy logic, we will introduce only the truth values of formulas with the regular fuzzy quantifiers. First of all, we make a short comment to our approach.

Contrary to the formulas with the basic fuzzy quantifiers, where we are able to establish the rules for deriving their truth values under an $\mathbf{M}$-evaluation (in a natural way), there is no unique natural evaluation rule using that we could find the truth values of formulas with fuzzy quantifiers of the type general. In other words, formulas with different fuzzy quantifiers, however, of the type g, may have many different evaluation rules, i.e. different ways how to find their truth values under an $\mathbf{M}$-evaluation. Nevertheless, it seems to be reasonable to use the truth values of the formulas with the basic fuzzy quantifiers to determine the truth values of the problematic formulas, because as we know each fuzzy quantifier of the type g is derived from them. Hence, if we have a formula $(Q_1 \circledast Q_2 x)\varphi$ with a fuzzy quantifier of the type g, then we could split this formula to the formulas $(Q_1 x)\varphi$ and $(Q_2 x)\varphi$ and the same logical connective $\circledast$ (but for formulas), which is considered between them, i.e. we put $(Q_1 \circledast Q_2 x)\varphi = (Q_1 x)\varphi \circledast (Q_2 x)\varphi$. Thus, the truth value of the formula $(Q_1 \circledast Q_2)\varphi$ under an $\mathbf{M}$-evaluation could be found as the truth value of the formula $(Q_1 x)\varphi \circledast (Q_2 x)\varphi$ under this $\mathbf{M}$-evaluation.

Example 4.7.1. Let us consider the fuzzy quantifier at least five and at most ten being clearly of the type g, the basic fuzzy quantifiers at least five, at most ten and a formula $\varphi :=$"*x passes the exam from the Statistics*" with the free variable x for students. Due to the previous consideration, we put (according to (4.34))

$$(\text{at least five and at most ten } x)\varphi = (\text{at least five } x)\varphi \wedge (\text{at most ten } x)\varphi.$$

Hence, if a and b are the truth values of the formulas (at least five $x)\varphi$ and (at most ten $x)\varphi$ under an $\mathbf{M}$-evaluation v, respectively, then the truth value of the formula (at least five but at most ten $x)\varphi$ under this $\mathbf{M}$-evaluation v is found as $a \wedge b$.

The other question is how to proceed in the cases of the external and internal negations. Here the answer is more complicated than in the previous case, because both negations have absolutely different effect on the formulas with such fuzzy quantifiers. Recall that a negation of a basic fuzzy quantifier is again basic fuzzy quantifier (due to Lemma 4.5.2), therefore, it is sufficient to be interested in the formulas $(\ltimes(Q_1 \circledast Q_2)x)\varphi$ that are of the type g. In order to define the truth values of formulas with such fuzzy quantifiers, we use the results of Theorem 4.4.6. In particular, from the equality of the complex fuzzy quantifiers models we will suppose that these complex fuzzy quantifiers are equivalent, e.g. $\sim(\mathfrak{Q}_1\&\mathfrak{Q}_2) \equiv (\mathfrak{Q}_1 \Rightarrow \sim\mathfrak{Q}_2)$ or $\lhd(\mathfrak{Q}_1\&\mathfrak{Q}_2) \equiv (\lhd\mathfrak{Q}_1\&\lhd\mathfrak{Q}_2)$. To find the truth value of formulas with such quantifiers we use the formulas with the equivalent forms of the fuzzy quantifier, e.g. we use $((\mathfrak{Q}_1 \Rightarrow \sim\mathfrak{Q}_2)x)\varphi$ instead of $(\sim(\mathfrak{Q}_1\&\mathfrak{Q}_2)x)\varphi$. In the cases of the fuzzy quantifiers, where we have just the inequalities between their models, i.e. $\sharp\sim(Q_1 \wedge Q_2) \geq \sharp(\sim Q_1 \vee \sim Q_2)$ and $\sharp\sim(Q_1 \Rightarrow Q_2) \geq \sharp(Q_1\&\sim Q_2)$, we have to establish the equivalence of these complex fuzzy quantifiers by the method "ad-hoc". Otherwise, there are missing the reasonable rules how to derive the truth value of formulas with such fuzzy quantifiers (recall that they are of the type g). As we have mentioned, a natural meaning of the fuzzy quantifiers negations is given mainly in the residuated lattice, where the law of double negation is satisfied as e.g. in MV-algebras. This is a positive argument for the mentioned "ad-hoc" equivalences, because then these equivalences are established correctly (with regard to the previous consideration) just in MV-algebras.

Let $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ be a language and $\mathbf{M}$ be an $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$. Then the truth value $||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ of a formula $(\mathfrak{Q}x)\varphi$ with a fuzzy quantifier $\mathfrak{Q}$ under an $\mathbf{M}$-evaluation v is defined by iterated use of the following steps (put $\circledast \in \{\&, \wedge, \vee, \Rightarrow\}$):

(i) if $\mathfrak{Q} := (Q, \mathsf{al})$, then

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v(x)\in\mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \vee \sharp Q(\emptyset),$$

(ii) if $\mathfrak{Q} := (Q, \mathsf{am})$, then

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset),$$

(iii) if $\mathfrak{Q} := (Q, \mathsf{g})$ and $\mathfrak{Q} := \mathfrak{Q}_1 \circledast \mathfrak{Q}_2$, then we have

$$||((\mathfrak{Q}_1 \circledast \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(\mathfrak{Q}_1x)\varphi \circledast (\mathfrak{Q}_2x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

(iv) if $\mathfrak{Q} := (Q, \mathsf{g})$ and $\mathfrak{Q} :=\lhd (\mathfrak{Q}_1 \circledast \mathfrak{Q}_2)$, then we have

$$||(\lhd(\mathfrak{Q}_1 \circledast \mathfrak{Q}_l)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||((\lhd\mathfrak{Q}_1 \circledast \lhd\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

(v) if $\mathfrak{Q} := (Q, \mathsf{g})$ and $\mathfrak{Q} := {\sim}(\mathfrak{Q}_1 \circledast \mathfrak{Q}_2)$, then we have

$$||({\sim}(\mathfrak{Q}_1 \& \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||((\mathfrak{Q}_1 \Rightarrow {\sim}\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$
$$||({\sim}(\mathfrak{Q}_1 \wedge \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(({\sim}\mathfrak{Q}_1 \vee {\sim}\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$
$$||({\sim}(\mathfrak{Q}_1 \vee \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(({\sim}\mathfrak{Q}_1 \wedge {\sim}\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$
$$||({\sim}(\mathfrak{Q}_1 \Rightarrow \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(({\sim}\mathfrak{Q}_1 \& {\sim}\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}.$$

As can be shown, the rules for deriving the truth values of formulas with the non-basic fuzzy quantifiers are a little complicated, but they reflect our proceed consideration. Now, for instance, if we suppose a non-basic fuzzy quantifier $\mathfrak{Q} := ({\sim}(Q_1 \& Q_2), \mathsf{g})$, where we have $\mathfrak{Q}_1 := (Q_1, \mathsf{al})$ and $\mathfrak{Q}_2 := (Q_2, \mathsf{am})$, then the truth value of the formula $(\mathfrak{Q}x)\varphi$ over an $\mathbf{M}$-evaluation v is given by

$$||({\sim}(\mathfrak{Q}_1 \& \mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||((\mathfrak{Q}_1 \Rightarrow {\sim}\mathfrak{Q}_2)x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} =$$
$$||(\mathfrak{Q}_1 x)\varphi \Rightarrow ({\sim}\mathfrak{Q}_l x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(\mathfrak{Q}_1 x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \rightarrow ||({\sim}\mathfrak{Q}_2 x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

where the truth value of the formula $(\mathfrak{Q}_1 x)\varphi$ is derived by (i) and the truth value of the formula $(\mathfrak{Q}_2 x)\varphi$ by (ii).

In Ex. 4.3.1 on p. 113 we have proposed some $\mathbf{L}^{\theta}_{k}$-models of the classical quantifiers $\forall$ and $\exists$, where models were defined over the power set $\mathcal{P}(M)$. In the following example we will show that using these $\mathbf{L}^{\theta}_{k}$-models and introduced rules for the basic fuzzy quantifiers we obtain the standard rules for deriving the truth values of formulas with the classical quantifiers of the first-ordered predicate fuzzy logic.

Example 4.7.2. Let $(\forall, \mathsf{al}), (\exists, \mathsf{al}) \in \mathcal{S}_{\mathcal{Q}}$, $\mathbf{M}$ be an $\mathbf{L}^{\theta}$-structure for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ such that $\mathcal{M} = \mathcal{P}(M)$, φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. Now, according to the rule for al, we have for $\exists := (\exists, \mathsf{al})$

$$||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\sharp\exists(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp\exists(\emptyset) =$$
$$\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\top \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \bot =$$
$$\bigvee_{\substack{Y \in \mathcal{M}\setminus\{\emptyset\} \\ |Y|=1}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\top \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) = \bigvee_{v' \in \mathcal{V}_{\mathbf{M}}(x,v)} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'},$$

which is the standard rule for the truth values of formulas with the existential quantifier. Further, we have for $\forall := (\forall, \mathsf{al})$

$$||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp\forall(Y)) \wedge \sharp\forall(\emptyset) =$$

$$\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset,M\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\bot\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in M}} (\top\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \bot =$$

$$= \bot \vee \bigwedge_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} = \bigwedge_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'},$$

which gives the standard rule for truth values of formulas with the universal quantifier.

Remark 4.7.3. It is easy to see that the existential quantifier can not be correctly established for any fuzzy algebra (contrary to the general quantifier). Indeed, let us suppose an $\mathbf{L}^\theta$-structure $\mathbf{M}$, where the fuzzy algebra $\mathcal{M}$ has the support $\{\emptyset, M\}$ and $|M| > 1$. Then $\mathbf{M}$ can not be an $\mathbf{L}^\theta$-structure for a language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ containing $\exists$, because there is no $\mathbf{L}^\theta_k$-model $\sharp\exists : \mathcal{M} \to L$ expressing the existential quantifier. For instance, if $\sharp\exists : \mathcal{M} \to L$ is defined analogously as in Ex. 4.3.1, then we obtain

$$||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'\in\mathrm{Supp}(Y)}} (\top\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \vee \bot = \bigwedge_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

which is not possible to accept. Therefore, if we deal with a language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ containing some fuzzy quantifiers, which are dependent on the form of used fuzzy algebras, then the language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ has the $\mathbf{L}^\theta$-structures with the suitable fuzzy algebra to the $\mathbf{L}^\theta_k$-models of the fuzzy quantifiers from $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ be correctly established. For example, if $\exists \in \mathcal{J}(\mathcal{S}_\mathcal{Q})$, then we can suppose only such $\mathbf{L}^\theta$-structures, whose fuzzy algebras have the following form. To each element $m \in M$ there exists $A \in \mathcal{M}$ such that $\mathrm{Supp}(A) = \{m\}$.

Example 4.7.4. Let $(\forall, \mathsf{al}), (\exists, \mathsf{al}), (\forall^{not}, \mathsf{am}), (\exists^{not}, \mathsf{am}) \in \mathcal{S}_\mathcal{Q}$, $\mathbf{M}$ be an $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, where $\mathcal{M} = \mathcal{F}(M)$, φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. Suppose that all quantifiers are interpreted in the $\mathbf{L}^\theta$-structure $\mathbf{M}$ analogously as in Ex. 4.3.1 and 4.3.2. Then we have $||(\exists^{not} x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(\forall x)\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ and $||(\forall^{not} x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(\exists x)\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v}$. In fact, we can write

$$||(\exists^{not}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{Y\in\mathcal{M}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp\exists^{not}(Y)) \wedge \sharp\exists^{not}(\emptyset) =$$

$$\bigwedge_{Y\in\mathcal{M}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \bot) \wedge \top = \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in M}} ||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||(\forall x)\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v}.$$

Analogously, the second equality could be done. Note that $||\neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = ||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ holds in every MV-algebra (see Theorem 4.7.9). Since we can establish $\exists^{not} := \sim\exists$ and $\forall^{not} := \sim\forall$ (see Ex. 4.4.1), we obtain

$$||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||\neg((\forall x)\neg\varphi)||^{\mathbf{L}}_{\mathbf{M},v}, \tag{4.36}$$

$$||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||\neg((\exists x)\neg\varphi)||^{\mathbf{L}}_{\mathbf{M},v} \tag{4.37}$$

in MV-algebras.

As we have mentioned there are the fuzzy quantifiers being of the types al and simultaneously am. In order to our definition of evaluation rules (i) and (ii) be correct, we have to show that it does not depend on the choice of the evaluation rule for the mentioned fuzzy quantifiers.

Lemma 4.7.1. *Let* $\mathbf{M}$ *be an* $\mathbf{L}^\theta$*-structure for* $\mathcal{J}(\mathcal{S}_Q)$*,* $(\mathbf{a},\mathsf{al}),(\mathbf{b},\mathsf{am}) \in \mathcal{S}_Q$ *be constant fuzzy quantifiers,* φ *be a formula with the free variable* x *and* v *be an* $\mathbf{M}$*-evaluation. Putting* $\mathfrak{Q}_\mathbf{a} := (\mathbf{a},\mathsf{al})$ *and* $\mathfrak{Q}_\mathbf{b} := (\mathbf{b},\mathsf{am})$*, we have* $||(\mathfrak{Q}_\mathbf{a}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = a$ *and* $||(\mathfrak{Q}_\mathbf{b}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = b$*.*

Proof. Let us suppose that φ is a formula with the free variable x and v is an $\mathbf{M}$-evaluation. Then we have for $\mathfrak{Q}_\mathbf{a} := (\mathbf{a},\mathsf{al})$

$$||(\mathfrak{Q}_\mathbf{a}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (a \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee a = a.$$

and for $\mathfrak{Q}_\mathbf{b} := (\mathbf{b},\mathsf{am})$

$$||(\mathfrak{Q}_\mathbf{b}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to b) \wedge b = b,$$

because $b = \top \to b \leq ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to b$ for any $\mathbf{M}$-evaluation v'. □

Theorem 4.7.2. *Let* $\mathfrak{Q}$ *be a fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_Q)$ *such that it is of the type* al *and simultaneously* am*. If* φ *is a formula with the free variable* x*, then it does not depend on the choice of the evaluation rule (i.e. for* al *or* am*) to derive the truth value of the formula* $(\mathfrak{Q}x)\varphi$*.*

Proof. It is a straightforward consequence of the previous lemma. □

Let $\mathfrak{Q}, \exists, \forall \in \mathcal{J}(\mathcal{S}_Q)$, where $\mathfrak{Q} := (Q, \mathsf{al})$ is an arbitrary fuzzy quantifier such that $\sharp Q(\emptyset) = \bot$ holds in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$. Further, let $\mathbf{M}$ be an $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$, where the $\mathbf{L}^\theta_k$-models of $\forall$ and $\exists$ are defined analogously as above[18], φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. Then obviously we can write

$$||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq ||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}. \tag{4.38}$$

Now let us suppose that also $\mathfrak{Q}', \exists^{not}, \forall^{not} \in \mathcal{J}(\mathcal{S}_Q)$, where $\mathfrak{Q}' := (Q', \mathsf{am})$ is an arbitrary fuzzy quantifier such that $\sharp Q(\emptyset) = \top$ holds in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$, $\exists^{not} := (\exists^{not}, \mathsf{am})$ and $\forall^{not} := (\forall^{not}, \mathsf{am})$. If the $\mathbf{L}^\theta_k$-models of $\forall^{not}$ and $\exists^{not}$ are defined analogously as in Ex. 4.3.2, then we can write

$$||(\exists^{not} x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq ||(\mathfrak{Q}'x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq ||(\forall^{not} x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}. \tag{4.39}$$

This fact can be extended to the cases, where one model of a basic fuzzy quantifier $\mathfrak{Q}$ is greater than or equal to another model of a basic fuzzy quantifier $\mathfrak{Q}'$ with the same type.

Theorem 4.7.3. *Let $\mathbf{M}$ be an arbitrary $\mathbf{L}^\theta$-structure for a language $\mathcal{J}(\mathcal{S}_Q)$, $(Q, \iota), (Q', \iota) \in \mathcal{S}_Q$ be basic fuzzy quantifiers with the same type ι such that $\sharp Q \geq \sharp Q'$ in $\mathbf{M}$, φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. Putting $\mathfrak{Q} := (Q, \iota)$ and $\mathfrak{Q}' := (Q', \iota)$, we have*

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \geq ||(\mathfrak{Q}'x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}. \tag{4.40}$$

Proof. Let φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. First, we assume that $\mathfrak{Q} := (Q, \mathsf{al}), \mathfrak{Q}' := (Q', \mathsf{al})$ and $\sharp Q \geq \sharp Q'$ in $\mathbf{M}$. Since $\sharp Q(A) \geq \sharp Q'(A)$ holds for any $A \in \mathcal{M}$, then we have

$$||(Qx)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) \geq$$

$$\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\sharp Q'(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q'(\emptyset) = ||(Q'x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

where the inequality follows from the isotonicity of $\otimes$. Analogical result could be obtained, when the fuzzy quantifiers of the type am are supposed. Again, it follows from the isotonicity of the residuum in the second argument. □

[18] It means that the fuzzy algebra of $\mathbf{M}$ has the form introduced in Remark 4.7.3 and the mappings $\sharp\exists$ and $\sharp\forall$ are defined over that analogously as in Ex. 4.3.1. Moreover, for the fuzzy quantifier $\mathfrak{Q}$ we do not suppose a special $\mathbf{L}^\theta_k$-model.

Let $\mathbf{M} = \langle M, \mathcal{M}, (r_P)_P, (f_F)_F, (m_a)_a, (\sharp Q)_{\mathfrak{Q}}\rangle$ be an $\mathbf{L}^\theta$-structure for a language $\mathcal{J}(\mathcal{S}_\mathcal{Q})$. An $\mathbf{L}$-substructure $\mathbf{M}_Y = \langle Y, (r_{P,Y})_P, (f_{F,Y})_F, (m_a)_a\rangle$ of $\mathbf{M}$, where $Y \subseteq M$, $r_{P,Y}$ and $f_{F,Y}$ are the corresponding restrictions of r_P, f_F on Y, respectively, and $m_a \in Y$ for each object constant from $\mathcal{J}$, is an $\mathbf{L}$-structure for $\mathcal{J}$. Obviously, $\mathbf{M}_Y$ can not be established for each Y, in general. For example, $\mathcal{J}$ has to possess no logical constant. In the case, that $\mathbf{M}_Y$ may be established for every $Y \in \mathcal{M}$, we can state a theorem which demonstrates our idea of fuzzy quantifiers introduction mentioned in the section Motivation. Recall that if a formula has just one free variable x, then we obtain $||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M}} = ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}$ for an arbitrary $\mathbf{M}$-evaluation v. Note that this theorem is not true, if we suppose formulas with more than one free variable.

Theorem 4.7.4. *Let* $\mathbf{L}$ *be an* MV*-algebra,* $\mathbf{M}$ *be an* $\mathbf{L}^\theta$*-structure for* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ *such that* $\mathbf{M}_Y$ *is the* $\mathbf{L}$*-structure for the language* $\mathcal{J}$ *for any non-empty subset* $Y \subseteq \mathcal{M}$ *and* φ *be a formula with just one free variable* x*. If* $\mathfrak{Q} := (Q, \mathsf{al})$ *is a fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$*, then we have*

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M}} = \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} (||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M}_Y} \otimes \sharp Q(Y)) \vee \sharp Q(\emptyset). \tag{4.41}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$ *is a fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$*, then we have*

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M}} = \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} (||(\exists x)\varphi||^{\mathbf{L}}_{\mathbf{M}_Y} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset). \tag{4.42}$$

Proof. Here we will show only the first statement, the second one could be done by analogy. Let $\mathfrak{Q} \in \mathcal{J}(\mathcal{S}_\mathcal{Q})$, where $\mathfrak{Q} := (Q, \mathsf{al})$, be a fuzzy quantifier, φ be a formula with just one free variable x and v be an $\mathbf{M}$-evaluation. Since $\bigwedge_{i\in I}(a \otimes b_i) = a \otimes \bigwedge_{i\in I} b_i$ holds in each MV-algebra, then we have

$$\begin{aligned}
&||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M}} = ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) = \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} (\sharp Q(Y) \otimes \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) = \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} (\sharp Q(Y) \otimes \bigwedge_{v''\in\mathcal{V}_{\mathbf{M}_Y}(x,v')} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v''}) \vee \sharp Q(\emptyset) = \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} (||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M}} \otimes \sharp Q(Y)) \vee \sharp Q(\emptyset),
\end{aligned}$$

where $v' : \mathcal{OV} \to Y$ is an arbitrary $\mathbf{M}_Y$-evaluation. □

The last part of this section is devoted to **L**-tautologies of the first-ordered fuzzy logic with fuzzy quantifiers. The choice of formulas, which are the **L**-tautologies, was motivated by the **L**-tautologies with the classical quantifiers (see e.g. [41]). Further, we restricted ourselves to the basic fuzzy quantifies of an arbitrary fuzzy quantifiers structure over $\mathcal{Q}$. Since the classical quantifiers are of the type al, a similarity to the standard **L**-tautologies could be observed in the cases of **L**-tautologies with fuzzy quantifiers of the type al. The **L**-tautologies whit fuzzy quantifiers am result from the considered **L**-tautologies with the previous type. Recall that we suppose only regular languages $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, i.e. the languages having at least one $\mathbf{L}^\theta$-structure[19].

Theorem 4.7.5. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathfrak{Q}, \sim\mathfrak{Q}, \sim\sim\mathfrak{Q}$ *be basic fuzzy quantifiers from* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$*,* φ *be a formula with the free variable* x*. Then we have*

$$\models (\mathfrak{Q}x)\varphi \Rightarrow (\sim\sim\mathfrak{Q}x)\varphi, \tag{4.43}$$

$$\models (\sim\mathfrak{Q}x)\varphi \Rightarrow \neg((\mathfrak{Q}x)\varphi), \tag{4.44}$$

$$\models (\sim\sim\mathfrak{Q}x)\varphi \Rightarrow \neg\neg((\mathfrak{Q}x)\varphi). \tag{4.45}$$

Proof. Obviously, the statement (4.43) follows from Theorems 4.4.5 and 4.7.3. Further, let us suppose that $\mathfrak{Q} := (Q, \mathsf{al})$. Since $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ is regular, then we have $\sim\mathfrak{Q} := (\sim Q, \mathsf{am})$, which follows from Lemma 4.5.2. If φ is a formula with the free variable x, $\mathbf{M}$ is an $\mathbf{L}^\theta$-structure and v is $\mathbf{M}$-evaluation, then we have

$$\begin{aligned}
&||\neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_\mathbf{M}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp Q(Y)\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset)\Big) \to \bot = \\
&\Big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \Big(\bigwedge_{\substack{v'\in\mathcal{V}_\mathbf{M}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp Q(Y)\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v'})\Big) \to \bot\Big) \wedge (\sharp Q(\emptyset)\to\bot) \geq \\
&\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_\mathbf{M}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ((\sharp Q(Y)\otimes||\varphi(x)||^{\mathbf{L}}_{\mathbf{M},v'})\to\bot) \wedge \sharp\sim Q(\emptyset) = \\
&\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_\mathbf{M}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to (\sharp Q(Y)\to\bot)) \wedge \sharp\sim Q(\emptyset) = \\
&\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_\mathbf{M}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp\sim Q(Y)) \wedge \sharp\sim Q(\emptyset) = ||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}$$

[19] In the opposite case, it has no sense to investigate some tautologies.

Hence, we obtain $\top = ||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to ||\neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = ||(\sim\mathfrak{Q}x)\varphi \Rightarrow \neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v}$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ and $\mathbf{M}$-evaluation v and thus the statement is true for $\mathfrak{Q} := (Q, \mathsf{al})$. Now let us suppose that $\mathfrak{Q} := (Q, \mathsf{am})$. Again, due to the regularity of $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and Lemma 4.5.2, we have $\sim\mathfrak{Q} := (\sim Q, \mathsf{al})$ and thus, analogously to the previous case we can write

$$
\begin{aligned}
&||\neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\quad \Big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset)\Big) \to \bot \geq \\
&\quad \Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \Big(\bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y)\Big) \to \bot\Big) \vee (\sharp Q(\emptyset) \to \bot) = \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big((||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y)) \to \bot\big) \vee \sharp{\sim}Q(\emptyset) \geq \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big((\sharp Q(Y) \to \bot) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}\big) \vee \sharp Q(\emptyset) = \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp{\sim}Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) = ||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},
\end{aligned}
$$

where we use $a \otimes (b \to c) \leq (a \to b) \to c$. This inequality follows from $a \otimes (a \to b) \otimes (b \to c) \leq a \otimes c \leq c$, where (A.7) of Theorem A.1.1 is used, and adjointness. Thus, we have $\top = ||(\sim\mathfrak{Q}x)\varphi \Rightarrow \neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v}$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ and $\mathbf{M}$-evaluation v and the statement is true for $\mathfrak{Q} := (Q, \mathsf{am})$. The last statement is an immediate consequence of (4.44). □

Theorem 4.7.6. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathfrak{Q}, \sim\mathfrak{Q}$ *be fuzzy quantifiers from* $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ *such that* $\mathfrak{Q}$ *be of the type* al*,* φ *be a formula with the free variable* x *and* ν *be a formula, where* x *is not free. Then we have*

$$\models (\mathfrak{Q}x)(\nu \Rightarrow \varphi) \Rightarrow (\nu \Rightarrow (\mathfrak{Q}x)\varphi), \tag{4.46}$$

$$\models (\nu \& (\mathfrak{Q}x)\varphi) \Rightarrow (\mathfrak{Q}x)(\nu \& \varphi), \tag{4.47}$$

$$\models (\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu) \Rightarrow ((\sim\mathfrak{Q}x)\varphi \Rightarrow \nu). \tag{4.48}$$

Proof. Let $\mathfrak{Q} := (Q, \mathsf{al})$ be a fuzzy quantifier of $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$, φ be a formula with the free variable x and ν be a formula, where x is not free in ν. Further, let $\mathbf{M}$ be an arbitrary $\mathbf{L}^{\theta}$-structure for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and v be an $\mathbf{M}$-evaluation. Since x is not free in ν, then $||\nu||^{\mathbf{L}}_{\mathbf{M},v} = ||\nu||^{\mathbf{L}}_{\mathbf{M},v'}$ holds for any $v' \in \mathcal{V}_{\mathbf{M}}(x, v)$. Hence, we can put $||\nu||^{\mathbf{L}}_{\mathbf{M},v} = a$. Now we have

$$||\nu \Rightarrow (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = a \rightarrow \Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \vee \sharp Q(\emptyset)\Big) \geq$$

$$(a \rightarrow \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v})) \vee (a \rightarrow \sharp Q(\emptyset)) \geq$$

$$\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes (a \rightarrow ||\varphi||^{\mathbf{L}}_{\mathbf{M},v})) \vee \sharp Q(\emptyset) = ||(\mathfrak{Q}x)(\nu \Rightarrow \varphi)||^{\mathbf{L}}_{\mathbf{M},v},$$

where the inequalities $b \otimes (a \rightarrow c) \leq a \rightarrow (b \otimes c)$ and $b \leq a \rightarrow b$ are applied. Hence, we obtain $||(\mathfrak{Q}x)(\nu \Rightarrow \varphi) \Rightarrow (\nu \Rightarrow (\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and $\mathbf{M}$-evaluation v and thus the first statement is proved. Further, we have

$$||(\mathfrak{Q}x)(\varphi \& \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \otimes a)) \vee \sharp Q(\emptyset) \geq$$

$$\big(a \otimes \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v})\big) \vee \big(a \otimes \sharp Q(\emptyset)\big) =$$

$$a \otimes \Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \vee \sharp Q(\emptyset)\Big) = ||\nu \& (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}.$$

Hence, we obtain $||(\nu \& (\mathfrak{Q}x)\varphi) \Rightarrow ((\mathfrak{Q}x)\nu \& \varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and $\mathbf{M}$-evaluation v and the second statement is true. Since $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ is regular, then we have $\sim\mathfrak{Q} := (\sim Q, \mathsf{am})$. Now we can write

$$||(\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} =$$

$$\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes (||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \rightarrow a)) \vee \sharp Q(\emptyset) \leq$$

$$\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} ((\sharp Q(Y) \rightarrow ||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \rightarrow a) \vee ((\sharp Q(\emptyset) \rightarrow \bot) \rightarrow \bot) \leq$$

$$\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} ((||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \rightarrow \sharp{\sim}Q(Y)) \rightarrow a) \vee (\sharp{\sim}Q(\emptyset) \rightarrow a) \leq$$

$$\Big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \rightarrow \sharp{\sim}Q(Y)) \wedge \sharp{\sim}Q(\emptyset)\Big) \rightarrow a =$$

$$||(\sim\mathfrak{Q}x)\varphi \Rightarrow \nu||^{\mathbf{L}}_{\mathbf{M},v},$$

where $a \otimes (b \to c) \leq (a \to b) \to$ and $a \to (b \to c) = b \to (a \to c)$ are applied. Hence, we obtain $||(\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu) \Rightarrow ((\sim\mathfrak{Q}x)\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^\theta$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and thus the last statement is proved. □

Theorem 4.7.7. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathfrak{Q}, \sim\mathfrak{Q}$ *be fuzzy quantifiers from* $\mathcal{J}(\mathcal{S}_Q)$ *such that* $\mathfrak{Q}$ *be of the type* am*,* φ *be a formula with the free variable* x *and* ν *be a formula, where* x *is not free. Then we have*

$$\models (\nu \& (\mathfrak{Q}x)\varphi) \Rightarrow (\mathfrak{Q}x)(\nu \Rightarrow \varphi), \tag{4.49}$$

$$\models (\mathfrak{Q}x)(\nu \& \varphi) \Rightarrow (\nu \Rightarrow (\mathfrak{Q}x)\varphi), \tag{4.50}$$

$$\models (\sim\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu) \Rightarrow ((\mathfrak{Q}x)\varphi \Rightarrow \nu). \tag{4.51}$$

Proof. Let $\mathfrak{Q} := (Q, \mathsf{am})$ be a fuzzy quantifier of $\mathcal{J}(\mathcal{S}_Q)$, φ be a formula with the free variable x and ν be a formula, where x is not free in ν. Further, let $\mathbf{M}$ be an arbitrary $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$ and v be an $\mathbf{M}$-evaluation. Analogously to the proof of the previous theorem, we can put $||\nu||^{\mathbf{L}}_{\mathbf{M},v} = a$. Then we have

$$\begin{aligned}
&||(\mathfrak{Q}x)(\nu \Rightarrow \varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\quad \bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} ((a \to ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \to \sharp Q(Y)) \wedge \sharp Q(\emptyset) \geq \\
&\quad \bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (a \otimes (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y))) \wedge \sharp Q(\emptyset) \geq \\
&\quad a \otimes \Big(\bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y))\Big) \wedge (a \otimes \sharp Q(\emptyset)) \geq \\
&\quad a \otimes \Big(\bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset)\Big) = \\
&\quad ||\nu||^{\mathbf{L}}_{\mathbf{M},v} \otimes ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||\nu \& (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},
\end{aligned}$$

where $a \otimes (b \to c) \leq (a \to b) \to c$ and $a \otimes b \leq b$ are applied. Hence, we obtain $||(\nu \& (\mathfrak{Q}x)\varphi) \Rightarrow (\mathfrak{Q}x)(\nu \Rightarrow \varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^\theta$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the first statement is proved. Further, we have

$$
\begin{aligned}
&||(\mathfrak{Q}x)(\nu\&\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\qquad \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big((a\otimes||\varphi||^{\mathbf{L}}_{\mathbf{M},v})\to\sharp Q(Y)\big)\wedge\sharp Q(\emptyset) = \\
&\qquad \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big(a\to(||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to\sharp Q(Y))\big)\wedge\sharp Q(\emptyset) \leq \\
&\qquad \big(a\to \bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to\sharp Q(Y))\big)\wedge(a\to\sharp Q(\emptyset)) = \\
&\qquad a\to\big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to\sharp Q(Y))\wedge\sharp Q(\emptyset)\big) = \\
&\quad ||\nu||^{\mathbf{L}}_{\mathbf{M},v}\to||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = ||\nu\Rightarrow(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},
\end{aligned}
$$

where $(a\otimes b)\to c = a\to(b\to c)$ and $b\leq a\to b$ are applied. Hence, we obtain $||(\mathfrak{Q}x)(\nu\&\varphi)\Rightarrow(\nu\Rightarrow(\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v}=\top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the second statement is also proved. Since $\mathcal{J}(\mathcal{S}_Q)$ is regular, then we have $\sim\!\mathfrak{Q} := (\sim\! Q,\mathsf{al})$. Now we can write

$$
\begin{aligned}
&||(\mathfrak{Q}x)\varphi\Rightarrow\nu||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\qquad \big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to\sharp Q(Y))\wedge\sharp Q(\emptyset)\big)\to a \geq \\
&\qquad \big((\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to\sharp Q(Y)))\to a\big)\vee(\sharp Q(\emptyset)\to a) \geq \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big((\sharp\!\sim\! Q(Y)\to||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v})\to a\big)\vee\sharp\!\sim\! Q(\emptyset) \geq \\
&\qquad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp\!\sim\! Q(Y)\otimes(||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v}\to a))\vee\sharp\!\sim\! Q(\emptyset) = \\
&\quad ||(\sim\!\mathfrak{Q}x)(\neg\varphi\Rightarrow\nu)||^{\mathbf{L}}_{\mathbf{M},v},
\end{aligned}
$$

where $(a\to b)\to c\geq((b\to\bot)\to(a\to\bot))\to c$ and then $a\otimes(b\to c)\leq(a\to b)\to c$ are applied. Hence, we obtain $||(\sim\!\mathfrak{Q}x)(\neg\varphi\Rightarrow\nu)\Rightarrow((\mathfrak{Q}x)\varphi\Rightarrow\nu)||^{\mathbf{L}}_{\mathbf{M},v}=\top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the last statement is proved. □

Theorem 4.7.8. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\mathfrak{Q}$ *be a basic fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_Q)$*,* φ *be a formula with the free variable* x *and* ν *be a* *formula, where x is not free. If $\mathfrak{Q} := (Q, \mathsf{al})$ and $\sharp Q(\emptyset) = \bot$ holds in each $\mathbf{L}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, then*

$$\models (\mathfrak{Q}x)(\nu \wedge \varphi) \Rightarrow (\nu \wedge (\mathfrak{Q}x)\varphi). \tag{4.52}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$ and $\sharp Q(\emptyset) = \top$ holds in each $\mathbf{L}^\theta$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, then

$$\models (\nu \vee (\mathfrak{Q}x)\varphi) \Rightarrow (\mathfrak{Q}x)(\neg\nu \wedge \varphi). \tag{4.53}$$

Proof. Let $\mathfrak{Q}$ be a basic fuzzy quantifier of $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, φ be a formula with the free variable x and ν be a formula, where x is not free in ν. Further, let $\mathbf{M}$ be an arbitrary $\mathbf{L}$-structure for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ and v be an $\mathbf{M}$-evaluation. Put $||\nu||^{\mathbf{L}}_{\mathbf{M},v} = a$. If $\mathfrak{Q} := (Q, \mathsf{al})$ such that $\sharp Q(\emptyset) = \bot$ in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$, then we have

$$\begin{aligned}
&||(\mathfrak{Q}x)(\varphi \wedge \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \wedge a)) \vee \bot \leq \\
&\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} ((\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \wedge (\sharp Q \otimes a)) \leq \\
&a \wedge \big(\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \vee \bot \big) = ||\nu \wedge (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}$$

Hence, we obtain $||(\mathfrak{Q}x)(\nu \wedge \varphi) \Rightarrow (\nu \wedge (\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^\theta$-structure $\mathbf{M}$ and $\mathbf{M}$-evaluation v and thus the first statement is proved. Now let us suppose $\mathfrak{Q} := (Q, \mathsf{am})$ such that $\sharp Q(\emptyset) = \top$ in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$. Then we have

$$\begin{aligned}
&||\nu \vee (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = a \vee \big(\bigwedge_{Y \in \mathcal{M}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \wedge \top \big) \leq \\
&\bigwedge_{Y \in \mathcal{M}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} \big(a \vee (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \big) \leq \\
&\bigwedge_{Y \in \mathcal{M}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} \big((a \to \bot) \to \bot) \vee (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \big) \leq \\
&\bigwedge_{Y \in \mathcal{M}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v'(x) \in \mathrm{Supp}(Y)}} \big(((a \to \bot) \wedge ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \to \sharp Q(Y) \big) = ||(\mathfrak{Q}x)(\neg\nu \wedge \varphi)||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}$$

Hence, we obtain $||(\nu\vee(\mathfrak{Q}x)\varphi) \Rightarrow (\mathfrak{Q}x)(\neg\nu\wedge\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the second statement is proved. □

Theorem 4.7.9. *Let* $\mathbf{L}$ *be a complete* MV*-algebra,* $\mathfrak{Q}$*,* $\sim\mathfrak{Q}$ *be basic fuzzy quantifiers of* $\mathcal{J}(\mathcal{S}_Q)$*,* φ *be a formula with the free variable* x*. Then we have*

$$\models \neg((\mathfrak{Q}x)\varphi) \Leftrightarrow (\sim\mathfrak{Q}x)\varphi. \tag{4.54}$$

Proof. First, let us recall that $\bigvee_{i\in I}(a_i \to b) = (\bigwedge_{i\in I} a_i) \to b$ holds in every MV-algebra. Let us suppose that $\mathfrak{Q} := (Q, \mathsf{al})$ and thus $\sim\mathfrak{Q} := (\sim Q, \mathsf{am})$. If φ is a formula with the free variable x, $\mathbf{M}$ is an $\mathbf{L}$-structure for $\mathcal{J}(\mathcal{S}_Q)$ and v is an $\mathbf{M}$-evaluation, then we have

$$||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} =$$
$$\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp{\sim}Q(Y)) \wedge \sharp{\sim}Q(\emptyset) =$$
$$\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to (\sharp Q(Y) \to \bot)) \wedge (\sharp Q(\emptyset) \to \bot) =$$
$$\Big(\Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \otimes \sharp Q(Y))\Big) \to \bot\Big) \wedge (\sharp Q(\emptyset) \to \bot) =$$
$$\Big(\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \otimes \sharp Q(Y)) \wedge \sharp Q(\emptyset)\Big) \to \bot = ||\neg((\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v}.$$

Hence, we obtain $||\neg((\mathfrak{Q}x)\varphi) \Leftrightarrow (\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and thus the statement is true for $\mathfrak{Q} := (Q, \mathsf{al})$. The proof for $\mathfrak{Q} := (Q, \mathsf{am})$ could be done by analogy, where the inequality $(a \to b) \to \bot = a \otimes (b \to \bot)$ is applied. □

Theorem 4.7.10. *Let* $\mathbf{L}$ *be a complete* MV*-algebra,* $\mathfrak{Q}$ *be a basic fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_Q)$*,* φ *be a formula with the free variable* x *and* ν *be a formula, where* x *is not free. If* $\mathfrak{Q} := (Q, \mathsf{al})$*, then we have*

$$\models (\mathfrak{Q}x)(\nu \vee \varphi) \Rightarrow (\nu \vee (\mathfrak{Q}x)\varphi). \tag{4.55}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$*, then we have*

$$\models (\mathfrak{Q}x)(\nu \vee \varphi) \Rightarrow (\nu \wedge (\mathfrak{Q}x)\varphi). \tag{4.56}$$

Proof. Let φ be a formula with the free variable x and ν be a formula, where x is not free in ν. Further, let $\mathbf{M}$ be an arbitrary $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ and v be an $\mathbf{M}$-evaluation. Put $||\nu||^{\mathbf{L}}_{\mathbf{M},v} = a$. Now, if $\mathfrak{Q} := (Q, \mathsf{al})$, then we have

$$||(\mathfrak{Q}x)(\nu \vee \varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} (\sharp Q(Y) \otimes (a \vee ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'})) \vee \sharp Q(\emptyset) \leq$$

$$\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} (a \vee (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'})) \vee \sharp Q(\emptyset) =$$

$$a \vee \Big(\bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset)\Big) = ||\nu \vee (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

where the distributivity of $\vee$ over $\wedge$ is used. Hence, we obtain $||(\mathfrak{Q}x)(\nu \vee \varphi) \Rightarrow (\nu \vee (\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v}$ for any $\mathbf{L}^\theta$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ and $\mathbf{M}$-evaluation v and thus the first statement is proved. Let us suppose that $\mathfrak{Q} := (Q, \mathsf{am})$. Then we have

$$||(\mathfrak{Q}x)(\nu \vee \varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} ((a \vee ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \to \sharp Q(Y)) \wedge \sharp Q(\emptyset) =$$

$$\bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} \big((a \to \sharp Q(Y)) \wedge (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y))\big) \wedge \sharp Q(\emptyset) \leq$$

$$\bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} (a \wedge (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y))\big) \wedge \sharp Q(\emptyset) =$$

$$a \wedge \Big(\bigwedge_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v' \in \mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x) \in Y}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset)\Big) = ||\nu \wedge (\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v},$$

where the distributivity of $\wedge$ over $\vee$ is used. Hence, we obtain $||(\mathfrak{Q}x)(\nu \vee \varphi) \Rightarrow (\nu \wedge (\mathfrak{Q}x)\varphi)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^\theta$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_\mathcal{Q})$ and $\mathbf{M}$-evaluation v and the second statement is also proved. □

Remark 4.7.5. The statement (4.56) is also true, if we suppose only complete divisible residuated lattices.

Theorem 4.7.11. *Let* $\mathbf{L}$ *be a complete* MV*-algebra,* $\mathfrak{Q}$ *be a basic fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$*,* φ *be a formula with the free variable* x *and* ν *be a formula, where* x *is not free in* ν*. If* $\mathfrak{Q} := (Q, \mathsf{al})$ *such that* $\sharp Q(\emptyset) = \bot$ *in each* $\mathbf{L}^\theta$*-structure for* $\mathcal{J}(\mathcal{S}_\mathcal{Q})$*, then we have*

$$\models (\nu \& (\mathfrak{Q}x)\varphi) \Leftrightarrow (\mathfrak{Q}x)(\nu \& \varphi). \tag{4.57}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$ *such that* $\sharp Q(\emptyset) = \top$ *in each* $\mathbf{L}^\theta$*-structure for* $\mathcal{J}(\mathcal{S}_Q)$*, then we have*

$$\models (\mathfrak{Q}x)(\nu \& \varphi) \Leftrightarrow (\nu \Rightarrow (\mathfrak{Q}x)\varphi). \tag{4.58}$$

Proof. The proof of (4.57) is analogical to the proof of (4.47), where $\bigvee_{i\in I}(a \otimes b_i) = a \otimes (\bigvee_{i\in I} b_i)$ is applied. Similarly, the proof of (4.58) is analogical to the proof of (4.50), where $\bigvee_{i\in I}(a \to b_i) = a \to \bigwedge_{i\in I} b_i$ is applied. □

Theorem 4.7.12. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\forall, \exists, \mathfrak{Q}, \sim\mathfrak{Q}$ *be fuzzy quantifiers from* $\mathcal{J}(\mathcal{S}_Q)$ *such that* $\mathfrak{Q}$ *be of the type* al *and* $\sharp Q(\emptyset) = \bot$ *holds in every* $\mathbf{L}^\theta$*-structure for* $\mathcal{J}(\mathcal{S}_Q)$*,* φ *and* ν *be formulas with the free variable* x*. Then we have*

$$\models (\forall x)\varphi \Rightarrow (\mathfrak{Q}x)\varphi, \tag{4.59}$$

$$\models (\mathfrak{Q}x)\varphi \Rightarrow (\exists x)\varphi, \tag{4.60}$$

$$\models (\mathfrak{Q}x)(\varphi \Rightarrow \nu) \Rightarrow ((\forall x)\varphi \Rightarrow (\mathfrak{Q}x)\nu), \tag{4.61}$$

$$\models (\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu) \Rightarrow ((\sim\mathfrak{Q}x)\varphi \Rightarrow (\exists x)\nu). \tag{4.62}$$

Proof. Let $\mathfrak{Q} := (Q, \mathsf{al})$ be a fuzzy quantifier from $\mathcal{J}(\mathcal{S}_Q)$ such that $\sharp Q(\emptyset) = \bot$ in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$. Then, clearly, we have $\sim\mathfrak{Q} := (\sim Q, \mathsf{am})$ with $\sharp\sim Q(\emptyset) = \top$ in every $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$. Further, let φ and ν be formulas with the free variable x, $\mathbf{M}$ be an arbitrary $\mathbf{L}^\theta$-structure for $\mathcal{J}(\mathcal{S}_Q)$ and v be an arbitrary $\mathbf{M}$-evaluation. The first two statements are immediate consequences of Theorem 4.7.3 (see also the inequality (4.38)). Further, we have

$$\begin{aligned}
&||(\mathfrak{Q}x)(\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to ||\nu||^{\mathbf{L}}_{\mathbf{M},v'})) \vee \sharp Q(\emptyset) \leq \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to (\sharp Q(Y) \otimes ||\nu||^{\mathbf{L}}_{\mathbf{M},v'})) \leq \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} ((\bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in M}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \to (\sharp Q(Y) \otimes ||\nu||^{\mathbf{L}}_{\mathbf{M},v'})) \leq \\
&\quad (\bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in M}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \to (\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes ||\nu||^{\mathbf{L}}_{\mathbf{M},v'})) = \\
&\quad ||(\forall x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to ||(\mathfrak{Q}x)\nu||^{\mathbf{L}}_{\mathbf{M},v} = ||(\forall x)\varphi \Rightarrow (\mathfrak{Q}x)\nu||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}$$

Hence, we obtain $||(\sim\mathfrak{Q}x)(\varphi \Rightarrow \nu) \Rightarrow ((\forall x)\varphi \Rightarrow (\sim\mathfrak{Q}x)\nu)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the third statement is proved. Finally, we have

$$
\begin{aligned}
&||(\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \otimes (||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to ||\nu||^{\mathbf{L}}_{\mathbf{M},v'})) \leq \\
&\quad \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} ((||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp{\sim}Q(Y)) \to \bigvee_{v'\in\mathcal{V}_{\mathbf{M}}(x,v} ||\nu||^{\mathbf{L}}_{\mathbf{M},v'}) \leq \\
&\quad (\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp{\sim}Q(Y))) \to (\bigvee_{v'\in\mathcal{V}_{\mathbf{M}}(x,v} ||\nu||^{\mathbf{L}}_{\mathbf{M},v'}) = \\
&\quad ||(\sim\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to ||(\exists x)\nu||^{\mathbf{L}}_{\mathbf{M},v} = ||(\sim\mathfrak{Q}x)\varphi \Rightarrow (\exists x)\nu||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}
$$

Hence, we obtain $||(\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu) \Rightarrow ((\sim\mathfrak{Q}x)\varphi \Rightarrow (\exists x)\nu)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_Q)$ and $\mathbf{M}$-evaluation v and the last statement is proved. □

Theorem 4.7.13. *Let* $\mathbf{L}$ *be a complete residuated lattice,* $\forall^{not}, \exists^{not}, \mathfrak{Q}, \sim\mathfrak{Q}$ *be fuzzy quantifiers from* $\mathcal{J}(\mathcal{S}_Q)$ *such that* $\mathfrak{Q}$ *be of the type* am *and* $\sharp Q(\emptyset) = \top$ *holds in every* $\mathbf{L}^{\theta}$*-structure for* $\mathcal{J}(\mathcal{S}_Q)$*,* φ *and* ν *be formulas with the free variable* x*. Then we have*

$$\models (\exists^{not}x)\varphi \Rightarrow (\mathfrak{Q}x)\varphi, \tag{4.63}$$
$$\models (\mathfrak{Q}x)\varphi \Rightarrow (\forall^{not}x)\varphi, \tag{4.64}$$
$$\models ((\exists^{not}x)\varphi \& (\mathfrak{Q}x)\nu) \Rightarrow (\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu), \tag{4.65}$$
$$\models (\sim\mathfrak{Q}x)(\neg\varphi \Rightarrow \neg\nu) \Rightarrow ((\mathfrak{Q}x)\varphi \Rightarrow (\forall^{not}x)\nu). \tag{4.66}$$

Proof. Let $\mathfrak{Q} := (Q, \mathsf{am})$ be a fuzzy quantifier from $\mathcal{J}(\mathcal{S}_Q)$ such that $\sharp Q(\emptyset) = \top$ in every $\mathbf{L}^{\theta}$-structure for $\mathcal{J}(\mathcal{S}_Q)$. Obviously, we have $\sim\mathfrak{Q} := (\sim Q, \mathsf{al})$ with $\sharp{\sim}Q(\emptyset) = \bot$ in every $\mathbf{L}^{\theta}$-structure for $\mathcal{J}(\mathcal{S}_Q)$. Further, let φ and ν be formulas with the free variable x, $\mathbf{M}$ be an arbitrary $\mathbf{L}^{\theta}$-structure for $\mathcal{J}(\mathcal{S}_Q)$ and v be an arbitrary $\mathbf{M}$-evaluation. Again, the first two statements are immediate consequences of Theorem 4.7.3 (see also the inequality (4.39)).

Since $||((\exists^{not}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)}(||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \bot)$, then we have

$$
\begin{aligned}
&||(\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} \big(((||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot) \to ||\nu||^{\mathbf{L}}_{\mathbf{M},v'}) \to \sharp Q(Y)\big) \geq \\
&\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} \big((||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot) \otimes (||\nu||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y))\big) \geq \\
&\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} \big((\bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'\in M}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot)) \otimes (||\nu||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y))\big) \geq \\
&\bigwedge_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot) \otimes \bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (||\nu||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y)) = \\
&\quad ||(\exists^{not}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \otimes ||(\mathfrak{Q}x)\nu||^{\mathbf{L}}_{\mathbf{M},v} = ||(\exists^{not}x)\varphi \& (\mathfrak{Q}x)\nu||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}
$$

Hence, we obtain $||(\exists^{not}x)\varphi\&(\mathfrak{Q}x)\nu \Rightarrow (\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and $\mathbf{M}$-evaluation v and thus the third statement is also proved. Since $||((\forall^{not}x)\neg\nu||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{v'\in\mathcal{V}_{\mathbf{M}}(x,v)}(||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to \bot)$, then we have

$$
\begin{aligned}
&||{\sim}\mathfrak{Q}x)(\neg\varphi \Rightarrow \nu)||^{\mathbf{L}}_{\mathbf{M},v} = \\
&\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp{\sim}Q(Y) \otimes (||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to ||\nu||^{\mathbf{L}}_{\mathbf{M},v})) \leq \\
&\bigvee_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} \big((\sharp{\sim}Q(Y) \to ||\neg\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \to \bigvee_{v\in\mathcal{V}_{\mathbf{M}}(x,v)} (||\neg\nu||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot)\big) \leq \\
&\big(\bigwedge_{Y\in\mathcal{M}\setminus\{\emptyset\}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in Y}} (\sharp Q(Y) \to ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'})\big) \to \big(\bigvee_{v\in\mathcal{V}_{\mathbf{M}}(x,v)} (||\neg\nu||^{\mathbf{L}}_{\mathbf{M},v'} \to \bot)\big) = \\
&\qquad ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \to ||(\forall^{not}x)\neg\nu||^{\mathbf{L}}_{\mathbf{M},v} = ||(\mathfrak{Q}x)\varphi \Rightarrow (\forall^{not}x)\neg\nu||^{\mathbf{L}}_{\mathbf{M},v}.
\end{aligned}
$$

Hence, we obtain $|| \models ({\sim}\mathfrak{Q}x)(\neg\varphi \Rightarrow \neg\nu) \Rightarrow ((\mathfrak{Q}x)\varphi \Rightarrow (\forall^{not}x)\nu)||^{\mathbf{L}}_{\mathbf{M},v} = \top$ for any $\mathbf{L}^{\theta}$-structure $\mathbf{M}$ for $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ and $\mathbf{M}$-evaluation v and thus the last statement is also proved. □

4.8 Calculation of truth values of formulas with basic fuzzy quantifiers

In this section we will deal with finite **L**-structures, i.e. their domains are finite, where **L** is a complete linearly ordered residuated lattice. In order to compute the truth value of a formula with a basic fuzzy quantifier, we have to use a procedure that generates either all or many fuzzy subsets of a domain M. Note that the set of all supports of fuzzy sets from $\mathcal{M}$ is usually equal to the power set of M. Obviously, for greater cardinalities of **L**-structures domains we will face a problem of the "real time". A solution that gives a possibility to compute the mentioned truth values of formulas with basic fuzzy quantifiers, if we restrict ourselves to fuzzy quantifiers over special fuzzy algebras, will be given in Theorem 4.8.2. In the other cases, we may apply Lemma 4.8.1 to find lower and upper limits between which the truth value can be expected. Before we state the mentioned lemma and theorem we will introduce several notions.

Let **L** be a complete linearly residuated lattice, **M** be a **L**-structure for $\mathcal{J}(\mathcal{S}_Q)$ with the domain $M = \{m_1, \dots, m_n\}$ and $\sharp Q : \mathcal{M} \to L$ be an $\mathbf{L}_k^\theta$-model of a basic fuzzy quantifier $\mathfrak{Q}$ from $\mathcal{J}(\mathcal{S}_Q)$. If φ is a formula with a free variable x and v is an **M**-evaluation, then we establish

$$\boldsymbol{\varphi}_v = (||\varphi||^{\mathbf{L}}_{\mathbf{M},v_1}, ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_2}, \dots, ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_n}), \quad v_i \in \mathcal{V}_{\mathbf{M}}(x, v), \tag{4.67}$$

where $||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i} \geq ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_j}$, whenever $i \leq j$. Further, we put for the fuzzy quantifier $\mathfrak{Q} := (Q, \mathsf{al})$

$$\mathbf{q}^{\mathrm{al}}_{\min} = (q_1, \dots, q_n), \quad q_i = \begin{cases} \bigwedge_{\substack{A \in \mathcal{M} \setminus \{\emptyset\} \\ |\mathrm{Supp}(A)| = i}} \sharp Q(A), & \text{if there is } |\mathrm{Supp}(A)| = i; \\ \bot, & \text{otherwise.} \end{cases}$$

$$\mathbf{q}^{\mathrm{al}}_{\max} = (q_1, \dots, q_n), \quad q_i = \begin{cases} \bigvee_{\substack{B \in \mathcal{M} \setminus \{\emptyset\} \\ |\mathrm{Supp}(B)| = i}} \sharp Q(B), & \text{if there is } |\mathrm{Supp}(B)| = i; \\ \bot, & \text{otherwise.} \end{cases}$$

If $\mathbf{q}^{\mathrm{al}}_{\min} = \mathbf{q}^{\mathrm{al}}_{\max}$, then we will write $\mathbf{q}^{\mathrm{al}}$ for short. Analogously, we put for the fuzzy quantifier $\mathfrak{Q} := (Q, \mathsf{am})$

$$\mathbf{q}^{\mathrm{am}}_{\min} = (q_1, \dots, q_n), \quad q_i = \begin{cases} \bigwedge_{\substack{A \in \mathcal{M} \setminus \{\emptyset\} \\ |\mathrm{Supp}(A)| = i}} \sharp Q(A), & \text{if there is } |\mathrm{Supp}(A)| = i; \\ \top, & \text{otherwise.} \end{cases}$$

$$\mathbf{q}^{\mathrm{am}}_{\max} = (q_1, \dots, q_n), \quad q_i = \begin{cases} \bigvee_{\substack{B \in \mathcal{M} \setminus \{\emptyset\} \\ |\mathrm{Supp}(B)| = i}} \sharp Q(B), & \text{if there is } |\mathrm{Supp}(B)| = i; \\ \top, & \text{otherwise.} \end{cases}$$

Again, if $\mathbf{q}^{\mathrm{am}}_{\min} = \mathbf{q}^{\mathrm{am}}_{\max}$, then we will write $\mathbf{q}^{\mathrm{am}}$ for short. If for any $i = 1, \ldots, n$ there is $A \in \mathcal{M}$ such that $|\mathrm{Supp}(A)| = i$, then clearly we have $\mathbf{q}^{\mathrm{al}}_{\min} = \mathbf{q}^{\mathrm{am}}_{\min}$ and $\mathbf{q}^{\mathrm{al}}_{\max} = \mathbf{q}^{\mathrm{am}}_{\max}$.

Let $\mathbf{x}$, $\mathbf{y}$ be vectors of L^n. Then the product of vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as follows

$$\mathbf{x} \otimes \mathbf{y} = \bigvee \{x_i \otimes y_i \mid i = 1, \ldots, n\}. \tag{4.68}$$

Analogously, the residuum of vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \to \mathbf{y} = \bigwedge \{x_i \to y_i \mid i = 1, \ldots, n\}. \tag{4.69}$$

Now we can state the mentioned lemma that shows us how to construct the boundaries of an interval, where the truth value of a formula with basic fuzzy quantifier lies. Recall that consequences of the linearity of residuated lattices are the equality $\bigwedge_{i=1}^{n}(a_i \otimes b) = (\bigwedge_{i=1}^{n} a_i) \otimes b$ and $\bigvee_{i=1}^{n}(a_i \to b) = (\bigwedge_{i=1}^{n} a_i) \to b$.

Lemma 4.8.1. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice,* $\mathfrak{Q}$ *be a basic fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_Q)$*,* $\mathbf{M}$ *be an* $\mathbf{L}$*-structure for* $\mathcal{J}(\mathcal{S}_Q)$ *with the finite domain* M*,* φ *be a formula with the free variable* x *and* v *be an* $\mathbf{M}$*-evaluation. If* $\mathfrak{Q} := (Q, \mathsf{al})$*, then we have*

$$(\mathbf{q}^{\mathrm{al}}_{\min} \otimes \boldsymbol{\varphi}_v) \vee \sharp Q(\emptyset) \leq ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq (\mathbf{q}^{\mathrm{al}}_{\max} \otimes \boldsymbol{\varphi}_v) \vee \sharp Q(\emptyset). \tag{4.70}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$*, then we have*

$$(\boldsymbol{\varphi}_v \to \mathbf{q}^{\mathrm{am}}_{\min}) \wedge \sharp Q(\emptyset) \leq ||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} \leq (\boldsymbol{\varphi}_v \to \mathbf{q}^{\mathrm{am}}_{\max}) \wedge \sharp Q(\emptyset). \tag{4.71}$$

Proof. Let us suppose that $M = \{m_1, \ldots, m_n\}$ is the domain of the $\mathbf{L}$-structure $\mathbf{M}$, $\mathfrak{Q}$ is a basic fuzzy quantifier from $\mathcal{J}(\mathcal{S}_Q)$, φ is a formula with the free variable x and v be an $\mathbf{M}$-evaluation. Put $I \subseteq \{1, \ldots, n\}$ such that

$i \in I$, if there is $A \in \mathcal{M}$ with $|\mathrm{Supp}(A)| = i$. If $\mathfrak{Q} := (Q, \mathsf{al})$, then we have

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{i\in I} \bigvee_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (\sharp Q(Y) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) \geq$$

$$\bigvee_{i\in I} \bigvee_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ((\bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \sharp Q(Y)) \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) =$$

$$\bigvee_{i\in I}(\bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \sharp Q(Y) \otimes \bigvee_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'}) \vee \sharp Q(\emptyset) =$$

$$\bigvee_{i\in I}(q_i \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i}) \vee \sharp Q(\emptyset) = \bigvee_{i=1}^{n}(q_i \otimes ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i}) \vee \sharp Q(\emptyset) =$$

$$(\mathbf{q}^{\mathrm{al}}_{\min} \otimes \boldsymbol{\varphi}_v) \vee \sharp Q(\emptyset),$$

where clearly

$$\bigvee_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} = \min(||\varphi||^{\mathbf{L}}_{\mathbf{M},v_1}, \dots, ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i}) = ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i}.$$

Analogously, the second inequality could be shown and thus the first statement is proved. If $\mathfrak{Q} := (Q, \mathsf{am})$, then we have

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{i\in I} \bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} (||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to \sharp Q(Y)) \wedge \sharp Q(\emptyset) \geq$$

$$\bigwedge_{i\in I} \bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigvee_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} \big(||\varphi||^{\mathbf{L}}_{\mathbf{M},v'} \to (\bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \sharp Q(Y))\big) \wedge \sharp Q(\emptyset) =$$

$$\bigwedge_{i\in I} \big((\bigvee_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \bigwedge_{\substack{v'\in\mathcal{V}_{\mathbf{M}}(x,v)\\ v'(x)\in\mathrm{Supp}(Y)}} ||\varphi||^{\mathbf{L}}_{\mathbf{M},v}) \to (\bigwedge_{\substack{Y\in\mathcal{M}\setminus\{\emptyset\}\\ |Supp(Y)|=i}} \sharp Q(Y))\big) \wedge \sharp Q(\emptyset) =$$

$$\bigwedge_{i\in I}(||\varphi(x)||_{\mathbf{M},v_i} \to q_i) \wedge \sharp Q(\emptyset) = \bigwedge_{i=1}^{n}(||\varphi(x)||_{\mathbf{M},v_i} \to q_i) \wedge \sharp Q(\emptyset) =$$

$$(\boldsymbol{\varphi}_v \to \mathbf{q}^{\mathrm{am}}_{\min}) \wedge \sharp Q(\emptyset).$$

Analogously, the second inequality could be shown and thus the second statement is proved. □

Theorem 4.8.2. *Let* $\mathbf{L}$ *be a complete linearly ordered residuated lattice,* $\mathfrak{Q}$ *be a basic fuzzy quantifier from* $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$, $\mathbf{M}$ *be an* $\mathbf{L}$*-structure for* $\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$ *with* *the finite domain M such that $\mathcal{M} \subseteq \mathcal{P}(M)$, φ be a formula with the free variable x and v be an $\mathbf{M}$-evaluation. If $\mathfrak{Q} := (Q, \mathsf{al})$, then we have*

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = (\mathbf{q}^{\mathrm{al}} \otimes \boldsymbol{\varphi}_v) \vee \sharp Q(\emptyset). \tag{4.72}$$

If $\mathfrak{Q} := (Q, \mathsf{am})$, then we have

$$||(\mathfrak{Q}x)\varphi||^{\mathbf{L}}_{\mathbf{M},v} = (\boldsymbol{\varphi}_v \to \mathbf{q}^{\mathrm{am}}) \wedge \sharp Q(\emptyset). \tag{4.73}$$

Proof. Due to Lemma 4.8.1, it is sufficient to prove that $\mathbf{q}^{\mathrm{al}}_{\min} = \mathbf{q}^{\mathrm{al}}_{\max}$ and $\mathbf{q}^{\mathrm{am}}_{\min} = \mathbf{q}^{\mathrm{am}}_{\max}$. However, it follows from Lemma 4.3.3 and the presumption that $\mathcal{M} \subseteq \mathcal{P}(M)$. Indeed, if $\mathcal{M} \subseteq \mathcal{P}(M)$, then $\sharp Q(Y_1) = \sharp Q(Y_2)$ holds, whenever $|Y_1| = |Y_2|$, Recall that $\mathrm{Supp}(Y) = Y$ for any crisp set. Hence, we have

$$q_i = \bigwedge_{\substack{Y \in \mathcal{M}\setminus\{\emptyset\} \\ |Y|=i}} \sharp Q(Y) = \bigvee_{\substack{Y \in \mathcal{M}\setminus\{\emptyset\} \\ |Y|=i}} \sharp Q(Y)$$

for any $i \in I$ (I was defined in the proof of the previous lemma) and thus we obtain $\mathbf{q}^{\mathrm{al}}_{\min} = \mathbf{q}^{\mathrm{al}}_{\max}$ and $\mathbf{q}^{\mathrm{am}}_{\min} = \mathbf{q}^{\mathrm{am}}_{\max}$. □

Example 4.8.1. Let us consider the following finite sequences

$$\begin{aligned} \mathbf{x} &:= \{2.7, 3.8, 5.1, 2.7, 6.9\}, \\ \mathbf{y} &:= \{2.1, 2.3, 5.5, 3.2, 6.3\}. \end{aligned}$$

and suppose that our goal is to find a better sequence, where "better sequence" is that having nearly all values greater than the other sequence. For this purpose it seems to be suitable to use the fuzzy quantifier for nearly all which is of the type al. Recall that we consider fuzzy quantifiers being of the type $\langle 1 \rangle$, it means unary fuzzy quantifier, but this decision making would rather require a binary fuzzy quantifier. Therefore, we will suppose that the domain of our structure is a set of corresponding pairs from the vectors $\mathbf{x}$ and $\mathbf{y}$, i.e. we put $M = \{(x_i, y_i) \mid x_i \in \mathbf{x} \text{ and } y_i \in \mathbf{y}\}$. Hence, our language $\mathcal{J}$ has one unary predicate symbol P, one symbol for the fuzzy quantifier for nearly all, say $\mathfrak{Q}^{fna}$, no functional symbols and no constants. Let $\mathbf{L}_{\mathrm{P}}$ be the Goguen algebra ($T_{\mathrm{P}} = \cdot$). Our $\mathbf{L}_{\mathrm{P}}$-structure is the couple $\mathbf{M} = \langle M, \mathcal{M}, r_P, \sharp Q^{fna} \rangle$, where $\mathcal{M} = \mathcal{P}$, $r_P : M \to [0,1]$ is defined as follows

$$r_P(m) = \begin{cases} 1, & x > y, \\ 2.5^{-|x-y|}, & \text{otherwise}, \end{cases} \tag{4.74}$$

for any $m = (x, y) \in M$, and the $\mathbf{L}^{\theta}_{\mathrm{P}k}$-model of $\forall^{fna} := (Q^{fna}, \mathsf{al})$ is established as follows

$$\forall A \in \mathcal{M} : \quad \sharp Q^{fna}(A) = \frac{|A|}{|M|}. \tag{4.75}$$

According to Lemma 4.3.3, this definition is correct. Note that r_P is defined as the restriction of a fuzzy ordering relation[20] on the set of real numbers to M. Let v be an $\mathbf{M}$-evaluation, then truth value of the formula $(\mathfrak{Q}^{fna}x)P(x)$ is given by

$$||(\mathfrak{Q}^{fna}x)P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v} = \bigvee_{Y \in \mathcal{M}\setminus\{\emptyset\}} \bigwedge_{\substack{v \in \mathcal{V}_{\mathbf{M}}(x,v) \\ v(x) \in \mathrm{Supp}(Y)}} \sharp Q^{fna}(Y) \cdot ||P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v}. \tag{4.76}$$

Since the presumptions of Theorem 4.8.2 are satisfied, we can use (4.72) instead of (4.76) to find the truth value $||(\mathfrak{Q}^{fna}x)P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v}$. Let us suppose that $v_i(x) = m_i$ and an $\mathbf{L}_{\mathrm{P}}$-evaluation v is given. Then we have the vector

$$(||P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v_1}, \dots, ||P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v_5}) = (1, 1, 0.76, 0.71, 1).$$

Thus, we can establish the following vectors

$$\boldsymbol{\varphi}_v := (1, 1, 1, 0.69, 0.57),$$
$$\mathbf{q}^{\mathrm{al}} := (1/5, 2/5, 3/5, 4/5, 1)$$

and the truth value of the formula $(\mathfrak{Q}^{fna}x)P(x)$ under the $\mathbf{M}$-evaluation v is then given by

$$||(\mathfrak{Q}^{fna}x)P(x)||^{\mathbf{L}_{\mathrm{P}}}_{\mathbf{M},v} = \boldsymbol{\varphi}_v \cdot \mathbf{q}^{\mathrm{al}} = \max(0.2, 0.4, 0.6, 0.55, 0.57) = 0.6.$$

[20] We use Bodenhofer's definition of fuzzy ordering, which is called T-E-ordering, see e.g. [56].

Chapter 5

Conclusion

In this thesis we set out three goals, to establish a notion of equipollence of fuzzy sets, to propose axiomatic systems for cardinalities of fuzzy sets and to introduce syntax and semantics of first-ordered fuzzy logic with fuzzy quantifiers. We can say that all goals were more-less fulfilled and they are a contribution to the relevant theories. Let us summarize the results of this thesis and give some comments to them, which could show their further progress.

For needs of the goals of this thesis we used the notion of fuzzy sets together for $\mathbf{L}$-sets, whose membership degrees are interpreted in complete residuated lattices, and $\mathbf{L}^{\mathrm{d}}$-sets, whose membership degrees are interpreted in complete residuated lattices. A more general convexity of fuzzy sets was introduced and some necessary conditions for (classical) mappings were stated in order to the corresponding fuzzy mappings, obtained by more general Zadeh extension principle, preserve the convexity. These and other notions are introduced in Chapter 1.

In Chapter 2 we defined evaluated mappings (injections, surjections, bijections) between $\mathbf{L}$-sets and then $\mathbf{L}^{\mathrm{d}}$-sets. The evaluated bijections were used for introducing the notions of equipollent $\mathbf{L}$-sets and $\mathbf{L}^{\mathrm{d}}$-sets. Further, we proved that θ-equipollence of $\mathbf{L}$-sets is a similarity relation on the set of all $\mathbf{L}$-sets and $\overline{\theta}$-equipollence of $\mathbf{L}^{\mathrm{d}}$-sets is then a fuzzy pseudo-metrics. Finally, we showed some relationships between equipollences of countable $\mathbf{L}$-sets or $\mathbf{L}^{\mathrm{d}}$-sets and similarity relation or fuzzy pseudo-metrics of special $\mathbf{L}$-sets or $\mathbf{L}^{\mathrm{d}}$-sets defined on the set of all natural numbers extended by the first infinite cardinal ω, respectively. The equipollences of fuzzy sets did not be systematically investigated, we restricted our choice of their properties to the needs of the following chapters. Therefore, there is an open field for a further research as e.g. to describe, if there exists, a relationship between equipollence of fuzzy sets and equipollence of their a-cuts.

Two axiomatic systems for cardinalities of finite fuzzy sets were introduced in Chapter 3. In particular, the first one generalized the axiomatic systems proposed by J. Casasnovas and J. Torrens in [8]. The second axiomatic system was then defined as a "dual" system to the first one, which could give a possibility to describe also some class of scalar cardinalities. We proved that cardinalities of both axiomatic systems may be represented by homomorphisms between substructures of residuated and dually residuated lattices and further, we showed some relationships between cardinalities and evaluated bijections, and also equipollences of fuzzy sets. Building of cardinality theory based on the proposed axiomatic systems could be a topic for further research. Since cardinalities of fuzzy sets are dependent on homomorphisms, the study of such homomorphisms could be a topic of rather algebraic research.

Based on the θ-equipollence of $\mathbf{L}$-sets a definition of $\mathbf{L}_k^\theta$-model of fuzzy quantifiers was introduced in Chapter 4. Further, some constructions of fuzzy quantifiers models were established using θ-equipollence of $\mathbf{L}$-sets and some special cardinalities of finite fuzzy sets. Constructions of new fuzzy quantifiers and their corresponding models using the logical connectives were also proposed. Structure of fuzzy quantifiers, containing only fuzzy quantifiers of special types, was defined and fuzzy quantifiers from such structure were implemented into language of the first-ordered fuzzy logic with fuzzy quantifiers (of this structure). Syntax and semantics of the first-ordered fuzzy logic with fuzzy quantifiers were suggested and several examples of $\mathbf{L}$-tautologies were proved. Finally, a method how to practically compute or at least to estimate truth values of formulas with fuzzy quantifiers (for the finite cases) was proposed. The presented results could be a base for another investigation on the field of fuzzy quantifiers in fuzzy logic. For instance, Hajek's results presented in [41] could be an inspiration for it. Further, relationships between fuzzy quantifiers and fuzzy integrals seem to be interesting. Finally, our approach to fuzzy quantification offers the new possibilities of the fuzzy logic application in decision making, image processing, etc. New applications could be also a topic for research.

Appendix A

Selected subjects

A survey of properties of (complete) residuated and dually residuated lattices, which are chosen with regard to the requirements of this thesis, is given in the first two sections. The proofs are omitted and the interested readers are referred to the special literature mentioned in the first chapter. The third section is devoted to the t-norms and t-conorms.

A.1 Some properties of residuated lattices

Theorem A.1.1. (basic properties)
Let $\mathbf{L}$ *be residuated lattice. Then the following items hold for every* $a, b, c \in L$*:*

$$a \otimes (a \to b) \leq b, \quad a \leq a \to (a \otimes b), \quad a \leq (a \to b) \to b, \tag{A.1}$$

$$a \leq b \quad \textit{iff} \quad a \to b = \top \tag{A.2}$$

$$a \to a = \top, \quad a \to \top = \top, \quad \bot \to a = \top, \tag{A.3}$$

$$a \otimes \bot = \bot, \quad \top \to a = a, \tag{A.4}$$

$$a \otimes b \leq a \wedge b, \tag{A.5}$$

$$(a \otimes b) \to c = a \to (b \to c), \tag{A.6}$$

$$(a \to b) \otimes (b \to c) \leq a \to c, \tag{A.7}$$

$$a \to b \textit{ is the greatest element of } \{c \mid a \otimes c \leq b\}, \tag{A.8}$$

$$a \otimes b \textit{ is the least element of } \{c \mid a \leq b \to c\}. \tag{A.9}$$

Theorem A.1.2. (monotony of $\otimes$ and $\to$)
Let $\mathbf{L}$ *be residuated lattice. Then the following items hold for every* $a, b, c \in L$*:*

$$a \leq b \quad \textit{implies} \quad a \otimes c \leq b \otimes c, \tag{A.10}$$

$$a \leq b \quad \textit{implies} \quad c \to a \leq c \to b, \tag{A.11}$$

$$a \leq b \quad \textit{implies} \quad b \to c \leq a \to c. \tag{A.12}$$

Theorem A.1.3. (distributivity of $\otimes$, $\to$ over $\wedge$, $\vee$)
Let $\mathbf{L}$ *be a complete residuated lattice. Then the following items hold for every* $a \in L$ *and every set* $\{b_i \mid i \in I\}$ *of elements from* L *over an arbitrary set of indices* I*:*

$$a \otimes (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \otimes b_i), \tag{A.13}$$

$$a \to (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \to b_i), \tag{A.14}$$

$$(\bigvee_{i \in I} b_i) \to a = \bigwedge_{i \in I} (b_i \to a), \tag{A.15}$$

$$a \otimes (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} (a \otimes b_i), \tag{A.16}$$

$$\bigvee_{i \in I} (a \to b_i) \leq a \to (\bigvee_{i \in I} b_i), \tag{A.17}$$

$$\bigvee_{i \in I} (b_i \to a) \leq (\bigwedge_{i \in I} b_i) \to a. \tag{A.18}$$

A.2 Some properties of dually residuated lattices

Theorem A.2.1. (basic properties)
Let $\mathbf{L}^{\mathrm{d}}$ *be a dually residuated lattice. Then the following identities hold for every* $a, b, c \in L$*:*

$$a \leq (a \ominus b) \oplus b, \quad (a \oplus b) \ominus a \leq b, \quad (a \oplus b) \ominus b \leq a, \tag{A.19}$$

$$a \leq b \quad \textit{iff} \quad a \ominus b = \bot, \tag{A.20}$$

$$a \ominus a = \bot, \quad a \ominus \top = \bot, \quad \bot \ominus a = \bot, \tag{A.21}$$

$$\top \oplus a = \top, \quad a \ominus \bot = a, \tag{A.22}$$

$$a \vee b \leq a \oplus b, \tag{A.23}$$

$$a \ominus (b \oplus c) = (a \ominus b) \ominus c, \tag{A.24}$$

$$a \ominus c \leq (a \ominus b) \oplus (b \ominus c), \tag{A.25}$$

$$a \ominus b \textit{ is the least element of } \{c \mid a \leq b \oplus c\}, \tag{A.26}$$

$$a \oplus b \textit{ is the greatest element of } \{c \mid c \ominus a \leq b\}. \tag{A.27}$$

Theorem A.2.2. (monotony of $\oplus$ and $\ominus$)
Let $\mathbf{L}^{\mathrm{d}}$ *be a dually residuated lattice. Then the following identities hold for*

every $a, b, c \in L$*:*

$$a \leq b \quad \textit{implies} \quad a \oplus c \leq a \oplus b, \tag{A.28}$$

$$a \leq b \quad \textit{implies} \quad a \ominus c \leq b \ominus c, \tag{A.29}$$

$$a \leq b \quad \textit{implies} \quad c \ominus b \leq c \ominus a. \tag{A.30}$$

Theorem A.2.3. (distributivity of $\oplus$, $\ominus$ over $\wedge$, $\vee$)
Let $\mathbf{L}^{\mathrm{d}}$ *be a complete dually residuated lattice. Then the following identities hold for every* $a, b \in L$ *and sets* $\{a_i \mid i \in I\}$*,* $\{b_i \mid i \in I\}$ *of elements from* L *over arbitrary set of indices* I*:*

$$a \oplus \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \oplus b_i), \tag{A.31}$$

$$(\bigvee_{i \in I} b_i) \ominus a = \bigvee_{i \in I} (b_i \ominus a), \tag{A.32}$$

$$a \ominus \bigwedge_{i \in I} b_i = \bigvee_{i \in I} (a \ominus b_i), \tag{A.33}$$

$$a \ominus \bigvee_{i \in I} b_i \leq \bigvee_{i \in I} (a \ominus b_i), \tag{A.34}$$

$$\bigvee_{i \in I} (a \oplus b_i) \leq a \oplus \bigvee_{i \in I} b_i, \tag{A.35}$$

$$(\bigwedge_{i \in I} b_i) \ominus a \leq \bigwedge_{i \in I} (b_i \ominus a). \tag{A.36}$$

A.3 t-norms and t-conorms

A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is called t-norm, if

$$T(a, b) = T(b, a), \tag{A.37}$$

$$T(a, T(b, c)) = T(T(a, b), c), \tag{A.38}$$

$$T(a, b) \leq T(a, c), \text{ whenever } b \leq c, \tag{A.39}$$

$$T(a, 1) = a \tag{A.40}$$

is fulfilled for any $a, b, c \in [0, 1]$. Moreover, a t-norm T is left continuous, if

$$\bigvee_{i \in I} T(a_i, b) = T(\bigvee_{i \in I} a_i, b) \tag{A.41}$$

holds for any index set I. The following theorem is very useful for a construction of complete residuated lattices on $[0, 1]$, if left continuous t-norms are considered. The proof could be found in e.g. [2, 56].

Theorem A.3.1. *If T is a left continuous t-norm then letting*

$$a \to_T b = \bigvee\{c \in [0,1] \mid T(a,c) \leq b\}, \tag{A.42}$$

a structure $\mathbf{L} = \langle [0,1], \min, \max, T, \to_T, 0, 1\rangle$ is a complete residuated lattice. Conversely, if $\mathbf{L} = \langle [0,1], \min, \max, T, \to, 0, 1\rangle$ is a residuated lattice then T is a left continuous t-norm.

Example A.3.1. The following are the three basic left continuous t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ given by, respectively:

$$\begin{aligned} T_{\mathbf{M}}(a,b) &= \min(a,b) && (\textit{minimum}) \\ T_{\mathbf{P}}(a,b) &= a \cdot b && (\textit{product}) \\ T_{\mathbf{L}}(a,b) &= \max(a+b-1,0) && (\textit{Łukasiewicz t-norm}) \end{aligned}$$

A binary operation $S : [0,1] \times [0,1] \to [0,1]$ is called t-conorm, if

$$S(a,b) = S(b,a), \tag{A.43}$$

$$S(a,S(b,c)) = S(S(a,b),c), \tag{A.44}$$

$$S(a,b) \leq S(a,c), \text{ whenever } b \leq c, \tag{A.45}$$

$$S(a,0) = a \tag{A.46}$$

is fulfilled for any $a,b,c \in [0,1]$. Moreover, a t-conorm S is right continuous, if

$$\bigwedge_{i\in I} S(a_i,b) = S(\bigwedge_{i\in I} a_i, b) \tag{A.47}$$

holds for any index set I. The following theorem is very useful for a construction of complete dually residuated lattices on $[0,1]$, if right continuous t-conorms are considered. The proof could be done by analogy to the proof of Theorem A.3.1.

Theorem A.3.2. *If S is a right continuous t-conorm then letting*

$$a \ominus_T b = \bigwedge\{c \in [0,1] \mid S(a,b) \geq c\}, \tag{A.48}$$

a structure $\mathbf{L} = \langle [0,1], \min, \max, S, \ominus_S, 0, 1\rangle$ is a complete dual residuated lattice. Conversely, if $\mathbf{L} = \langle [0,1], \min, \max, S, \ominus, 0, 1\rangle$ is a dual residuated lattice then S is a right continuous t-conorm.

Example A.3.2. The following are the three basic right continuous t-norms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ and $S_{\mathbf{L}}$ given by, respectively:

$$\begin{aligned} S_{\mathbf{M}}(a,b) &= \max(a,b) && (\textit{maximum}) \\ S_{\mathbf{P}}(a,b) &= a + b - a \cdot b && (\textit{probabilistic sum}) \\ S_{\mathbf{L}}(a,b) &= \min(a+b,1) && (\textit{Łukasiewicz t-conorm}) \end{aligned}$$

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List of Symbols

$\bot, \top$	truth constants (false,true), page 105
$\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(X)$	set of all $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-sets over X, page 16
$(\mathcal{F}(X), \leq)$	*po*-set of all fuzzy sets over X, page 12
(Q, ι)	fuzzy quantifier Q of the type $\iota \in \mathbb{T} = \{\mathsf{al}, \mathsf{am}, \mathsf{g}\}$, page 133
$(L, \overline{\theta}, \bot)$	substructure of a dually residuated lattice $\mathbf{L}^{\mathrm{d}}$, page 64
$(L, \theta, \top)$	substructure of a residuated lattice $\mathbf{L}$, page 64
$+^{\overline{\theta}}$	extended addition on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ (also $+^{\vee}$ and $+^{\oplus}$, page 55
$+^{\theta}$	extended additions on $\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ (also $+^{\wedge}$ and $+^{\otimes}$), page 54
$a, b, \ldots$	object constants, page 105
$\mathcal{A}$	fuzzy algebra, page 13
$\approx_{\mathrm{d}}$	fuzzy pseudo-metric on $\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$, page 90
$\bigoplus_{i=1}^{\infty} a_i$	addition of a denumerable number of arguments, page 10
$\bigoplus_{i=1}^{n} a_i$	addition of a finite number of arguments, page 10
$\bigoplus_{i\in I} a_i$	addition over a countable set of indices I, page 10
$\bigotimes_{i=1}^{\infty} a_i$	multiplication of a denumerable number of arguments, page 5
$\bigotimes_{i=1}^{n} a_i$	multiplication of a finite number of arguments, page 5
$\bigotimes_{i\in I} a_i$	multiplication over a countable set of indices I, page 5
$\mathrm{Bij}(X)$	set of all bijection between X and Y, page 25
$\leftrightarrow_{\mathrm{M}}, \leftrightarrow_{\mathrm{P}}, \leftrightarrow_{\text{Ł}}$	Gödel, Goguen, Łukasiewicz biresiduum, page 4
$\boxminus$	partial subtraction on N_n, page 54
$\boxplus$	addition in $\mathbb{N}_n$, i.e. $a \boxplus b = \min(a+b, n)$, page 53
$\&, \wedge, \vee, \Rightarrow$	logical connectives (strong conjunction, conjunction, disjunction, implication), page 105
$\Leftrightarrow, \neg$	logical connectives (equivalence, negation), page 105
$\mathbf{q}^{\mathrm{al}}_{\min}, \mathbf{q}^{\mathrm{al}}_{\max}$	vectors of min, max values of fuzzy quantifiers of the type al (also $\mathbf{q}^{\mathrm{am}}_{\min}, \mathbf{q}^{\mathrm{am}}_{\max}, \mathbf{q}^{\mathrm{al}}, \mathbf{q}^{\mathrm{am}}$), page 158

$\boldsymbol{\varphi}_v$ vector defined by $\boldsymbol{\varphi}_v = (||\varphi||^{\mathbf{L}}_{\mathbf{M},v_1}, ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_2}, \ldots, ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_n})$, where $v_i \in \mathcal{V}_{\mathbf{M}}(x,v)$ and $||\varphi||^{\mathbf{L}}_{\mathbf{M},v_i} \geq ||\varphi||^{\mathbf{L}}_{\mathbf{M},v_j}$, whenever $i \leq j$, page 157

$\cap$, $\cup$, $\overline{}$ operations of union, intersection, complement of sets (or fuzzy sets), page 13

$\cap^{\theta}, \cup^{\overline{\theta}}$ operations of θ-intersection, $\overline{\theta}$-union of fuzzy sets, page 69

$\mathbb{C}$ θ-($\overline{\theta}$-)cardinality of finite $\mathbf{L}$-($\mathbf{L}^{\mathrm{d}}$-)sets, page 58

$\mathbb{C}_T$ T-cardinality of finite $\mathbf{L}$-sets (also $\mathbb{C}_{T_{\mathrm{M}}}$, $\mathbb{C}_{T_{\mathrm{P}}}$, $\mathbb{C}_{T_{\mathrm{L}}}$), page 76

$\mathbb{C}_{S_{\mathrm{L}}}$ S_{L}-cardinality of finite $\mathbf{L}^{\mathrm{d}}_{\mathrm{L}}$-sets, page 82

$\mathbb{C}_{f,g}$ θ-($\overline{\theta}$-)cardinality of finite $\mathbf{L}$-($\mathbf{L}^{\mathrm{d}}$-)sets generated by the θ-($\overline{\theta}$-) and $\overline{\theta}_d$-(θ_d-)homomorphisms f and g, page 65

$\mathbb{C}_f, \mathbb{C}_g$ θ-($\overline{\theta}$-)cardinality of finite $\mathbf{L}$-($\mathbf{L}^{\mathrm{d}}$-)sets generated just by θ-($\overline{\theta}$-)homomorphism f, $\overline{\theta}$-(θ_d-)homomorphism g, page 68

$\mathcal{CV}^{\theta}_{\mathbf{L}}(X)$ set of all θ-convex $\mathbf{L}$-sets over X, page 14

$\mathcal{CV}^{\theta}_{\mathbf{L}}(N_n)$ set of all θ-convex $\mathbf{L}$-sets over N_n, page 54

$\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(N_n)$ set of all $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-sets over N_n, page 55

$\mathcal{CV}^{\overline{\theta}}_{\mathbf{L}^{\mathrm{d}}}(\mathbb{N}_n)$ bounded *poc*-monoids of $\overline{\theta}$-convex $\mathbf{L}^{\mathrm{d}}$-sets over N_n, page 57

$\mathcal{CV}^{\theta}_{\mathbf{L}}(\mathbb{N}_n)$ bounded *poc*-monoids of θ-convex $\mathbf{L}$-sets over N_n, page 57

$\ominus_{\mathrm{M}}, \ominus_{\mathrm{P}}, \ominus_{\mathrm{L}}$ Gödel, Goguen, Łukasiewicz difference, page 8

$\equiv^{\vee}, \equiv^{\oplus}, \equiv^{\overline{\theta}}$ fuzzy pseudo-metric on $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ for $\vee$ and fuzzy pseudo-metrics on $\mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ for $\oplus, \overline{\theta}$, page 41

$\emptyset$ empty set or empty fuzzy set ($\mathbf{L}$-set, $\mathbf{L}^{\mathrm{d}}$-set), page 12

$\equiv_x$ equivalence on the set of all $\mathbf{M}$-evaluations, page 106

$\equiv_{iso}$ equivalence relation on $\mathcal{F}(X)$, page 103

$\mathcal{FC}_{\mathbf{L}^{\mathrm{d}}}(X)$ set of all countable $\mathbf{L}^{\mathrm{d}}$-sets over X, page 12

$\mathcal{FC}_{\mathbf{L}}(X)$ set of all countable $\mathbf{L}$-sets over X, page 12

$\mathcal{FC}(X)$ set of all countable fuzzy sets over X, page 12

$\mathcal{FIN}_{\mathbf{L}^{\mathrm{d}}}(X)$ set of all finite $\mathbf{L}^{\mathrm{d}}$-sets over X, page 12

$\mathcal{FIN}_{\mathbf{L}}(X)$ set of all finite $\mathbf{L}$-sets over X, page 12

$\mathcal{FIN}(X)$ set of all finite fuzzy sets over X, page 12

$\mathrm{Fin}(X)$ set of all finite subset of X, page 5

$\mathcal{F}_{\mathbf{L}}(X), \mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$ set of all $\mathbf{L}$-sets, $\mathbf{L}^{\mathrm{d}}$-sets over X, page 12

$\forall$, $\exists$ universal, existential quantifier (for all, there exists), page 105

$\forall^{infin}, \forall^{fin}$ generalized quantifiers (for all except a finite number, for at most finitely many), page 115

$\forall^{not}, \exists^{not}$ generalized quantifiers (not for all, there exists none), page 114

$\mathcal{F}_{\mathcal{Q}}$	set of all fuzzy quantifiers over $\mathcal{Q}$, page 135
$\mathcal{F}(X)$	set of all fuzzy sets over X, page 12
$\|\|x\|\|_{\mathbf{M},v}, \|\|a\|\|_{\mathbf{M},v}$	values of atomic terms under an $\mathbf{M}$-evaluation v, page 106
$\|\|\varphi\|\|^{\mathbf{L}}_{\mathbf{M},v}$	truth value of φ under an $\mathbf{M}$-evaluation v, page 106
$\|\|\varphi\|\|^{\mathbf{L}}_{\mathbf{M}}$	truth value of formula φ in $\mathbf{M}$, page 107
$\mathbf{i}$	$\mathbf{i} = (i_1, \dots, i_r)$ denotes r-dimensional vector elements from N_n such that $i_1 \boxplus \cdots \boxplus i_r = i$, page 66
$\equiv^{\wedge}, \equiv^{\otimes}, \equiv^{\theta}$	similarity relation on $\mathcal{F}_{\mathbf{L}}(X)$ for $\wedge$ and similarity relations on $\mathcal{FC}_{\mathbf{L}}(X)$ for $\otimes, \theta$, page 33
$\mathcal{J}$	language of the first-ordered fuzzy logic, page 105
$\mathcal{J}(\mathcal{S}_{\mathcal{Q}})$	language of the first-ordered fuzzy logic with the fuzzy quantifiers of $\mathcal{S}_{\mathcal{Q}}$ over $\mathcal{J}$, page 137
$\mathbf{L}$	(complete) residuated lattice, page 1
$\mathbf{L}_{n+1}$	$n+1$ elements Łukasiewicz chain, page 2
$\mathbf{L}_{\mathrm{B}}$	Boolean algebra for classical logic, page 2
$\mathbf{L}^{\mathrm{d}}$	(complete) dually residuated lattice, page 6
$\mathbf{L}^{\mathrm{d}}_{\mathrm{B}}$	dual Boolean algebra, page 7
$\mathbf{L}^{\mathrm{d}}_{\mathrm{M}}, \mathbf{L}^{\mathrm{d}}_{\mathrm{P}}, \mathbf{L}^{\mathrm{d}}_{\mathrm{Ł}}$	dual Gödel, Goguen, Łukasiewicz algebra, page 8
R^{+}_{0}	dually residuated of non-negative real numbers, page 7
$\mathbf{L}_{\mathrm{M}}, \mathbf{L}_{\mathrm{P}}, \mathbf{L}_{\mathrm{Ł}}$	Gödel, Goguen, Łukasiewicz algebra, page 3
$\leftrightarrow, a \leftrightarrow b$	biresiduum, biresiduum of a and b, page 3
$\leq$	partial ordering, page 12
$\overline{\Theta}_{i \in I}\, a_i$	common symbol for $\bigvee_{i \in I} a_i$ and $\bigoplus_{i \in I} a_i$, where I is a countable set of indices, page 10
$\Theta_{i \in I}\, a_i$	common symbol for $\bigwedge_{i \in I} a_i$ or $\bigotimes_{i \in I} a_i$, where I is a countable set of indices, page 6
$\mathbb{N}$	*loc*-monoid of natural numbers, page 52
$\mathbb{N}_{\omega}$	bounded *loc*-monoid of extended natural numbers, page 52
$\mathbf{0}$	the least fuzzy set in $\mathcal{F}(X)$, page 56
$\mathbf{1}$	the greatest fuzzy set in $\mathcal{F}(X)$, page 56
$\mathbf{a}, \mathbf{b}, \dots$	constant fuzzy quantifier, page 111
$\mathbf{x} \otimes \mathbf{y}, \mathbf{x} \to \mathbf{y}$	product and residuum of vectors $\mathbf{x}$ and $\mathbf{y}$, page 158
$\mathcal{I}$	set of all vectors $\mathbf{i}$, page 66
$\mathcal{O}$	set of all vectors $\mathbf{i}$ such that $M_{\mathbf{i}} = \emptyset$, page 70
$\mathcal{P}(X)$	set of all subsets of X (power set of X), page 13

$\Sigma Count$	example of scalar cardinality (*sigma count*), page 50
$FGCount$	example of fuzzy cardinality (also $FLCount$ and $FECount$), page 50
max	maximum, page 2
$\mathbf{M}$	$\mathbf{L}$-structure for a language, page 106
$\mathbb{M}$	bounded *loc*-monoid with a unit, page 53
$\mathbb{M}_n$	bounded *loc*-monoid over $M_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ with the unit $\frac{1}{n}$, page 53
min	minimum, page 2
$\mathbb{N}_n$	bounded *loc*-monoid over $N_n = \{0, \dots, n\}$ with the unit 1, page 53
$\neg$	negation on $\mathbf{L}$, i.e. $\neg a = a \to \bot$, or $\mathbf{L}^{\mathrm{d}}$, i.e. $\neg a = \top \ominus a$, page 3
$\neg_{\mathrm{M}}, \neg_{\mathrm{P}}, \neg_{\mathrm{Ł}}$	Gödel, Goguen, Łukasiewicz negation, page 4
$\omega, \aleph_0$	the first infinite cardinal, page 52
$\ominus$	difference, page 6
$\top$	the greatest element of a lattice, page 1
$\oplus$	addition, page 6
$\overline{\theta}$	common symbol for $\vee$ and $\oplus$, page 7
$\otimes$	multiplication, page 1
$\mathcal{OV}$	the set of all object variables, page 106
$\mathrm{Perm}(X)$	set of all bijections (permutations) on a set X, page 33
$\prod_{i=1}^{n} X_i$	finite product of sets X_i (also $X_1 \times \cdots \times X_n$), page 15
$\prod_{i \in N} X_i$	infinite product of sets X_i, page 98
$Q, Q_1, Q_2, \dots$	fuzzy quantifiers, page 110
Q^{fna}, Q^{fnn}	fuzzy quantifiers (for nearly all, for nearly none), page 115
$Q^{m\lesssim}, Q^{\lesssim m}$	fuzzy quantifiers (for about or more than m, for about or less m), page 117
$\mathcal{Q}$	set of all atomic fuzzy quantifiers, page 135
$\to_{\mathrm{M}}, \to_{\mathrm{P}}, \to_{\mathrm{Ł}}$	Gödel, Goguen, Łukasiewicz residuum, page 3
$\to$	residuum, page 1
$\mathbb{C}_{\mathrm{sc}}$	$\overline{\theta}$-scalar cardinality of finite $\mathbf{L}^{\mathrm{d}}$-sets, page 83
$\sharp Q_K$	model of fuzzy quantifier generated by equipollence (or θ-cardinality) and a fuzzy set K , page 118
$\sharp Q_K^h$	model of fuzzy quantifier generated by $\overline{\theta}$-cardinality, fuzzy set K and h, page 122

$\sharp Q$	$\mathbf{L}_k^{\theta}$-model of a fuzzy quantifier Q, page 111
$\mathcal{S}_{\mathcal{Q}}$	structure of fuzzy quantifiers over $\mathcal{Q}$, page 136
$\mathrm{Supp}(A)$	support of a fuzzy set A, page 12
θ	common symbol for $\wedge$ and $\otimes$, page 2
$\approx$	similarity relation on the set of all p_A^{θ} (or also $\mathcal{CV}_{\mathbf{L}}^{\theta}(N_n)$), page 38
$X, Y, \ldots$	universes of discovering, page 12
$\mathcal{V}_{\mathbf{M}}$	set of all $\mathbf{M}$-evaluations, page 106
$\mathcal{V}_{\mathbf{M}}(x, v)$	set of all $\mathbf{M}$-evaluation v' such that $v \equiv_c v'$, page 106
$\varphi, \psi, \ldots$	formulas, page 105
$\vee \bigvee$	supremum (join) in lattice, page 1
$\wedge \bigwedge$	infimum (meet) in lattice, page 1
$\widehat{f}^{\overline{\theta}}, \widehat{f}^{\oplus}, \widehat{f}^{\vee}$	mapping established using Zadeh extension principle which extends a mapping f, page 16
$\widehat{f}^{\theta}, \widehat{f}^{\otimes}, \widehat{f}^{\wedge}$	mapping established using Zadeh extension principle (for the operations $\theta, \otimes, \wedge$) which extends a mapping f, page 14
$\perp$	the smallest element of a lattice, page 1
$A(x)$	membership degree of x in a fuzzy set ($\mathbf{L}$-set, $\mathbf{L}^{\mathrm{d}}$-set) A, page 12
$A, B, C, \ldots$	fuzzy sets ($\mathbf{L}$-sets, $\mathbf{L}^{\mathrm{d}}$-sets), page 12
$A \otimes B, A \rightarrow B$	multiplication, residuum of $\mathbf{L}$-sets A and B, page 13
A^c	closure of θ-(or $\overline{\theta}$)-convex $\mathbf{L}$-(or $\mathbf{L}^{\mathrm{d}}$)-set A over N_n, page 56
A_a	a-cut of fuzzy set A, page 12
A_a^{d}	dual a-cut of fuzzy set A, page 12
A_X	set of all subsets of a set X containing the support of a fuzzy set A, page 35
$Aq(Q)$	set of all atomic fuzzy quantifiers from $\mathcal{F}_{\mathcal{Q}}$ included in Q, page 135
d	fuzzy pseudo-metric on $\mathcal{F}_{\mathbf{L}^{\mathrm{d}}}(X)$, page 41
E	neutral element of $(\mathcal{CV}_{\mathbf{L}}^{\theta}(N_n), +^{\theta})$ and $(\mathcal{CV}_{\mathbf{L}^{\mathrm{d}}}^{\overline{\theta}}(N_n), +^{\overline{\theta}})$, page 54
$F, G, \ldots$	functional symbols, page 105
$f, g, h, \ldots$	mappings, page 10
$g \circ f$	composition of mappings f and g, page 19
$h^{\rightarrow}$	mapping from $\triangle_1 \in \{\mathbf{L}_1, \mathbf{L}_1^{\mathrm{d}}\}$ to $\triangle_2 \in \{\mathbf{L}_2, \mathbf{L}_2^{\mathrm{d}}\}$ determined by a mapping $h : L_1 \rightarrow L_2$, page 20
I	set of indices, page 4

$K_{\mathbf{i}}, L_{\mathbf{i}}, M_{\mathbf{i}}$	sets defined by $K_{\mathbf{i}} = \{k \mid i_k = 0\}$, $L_{\mathbf{i}} = \{l \mid i_l = 1\}$, $M_{\mathbf{i}} = \{m \mid i_m > 1\}$, page 66
L	support of $\mathbf{L}$ or $\mathbf{L}^{\mathrm{d}}$, page 1
N	set of all natural numbers, page 5
N_{ω}^{-0}	set of all natural numbers without zero extended by the first infinite cardinal, page 14
$P, R, \ldots$	predicate symbols, page 105
$p_A^{\bar{\theta}}$	mapping $N_\omega \times A_X \to L$ defined by $p_A^{\bar{\theta}}(i, Y) = \bigwedge\{\overline{\Theta}_{z\in Z} A(z) \mid Z \subseteq Y, \|Z\| = i\}$, page 43
p_A^{θ}	mapping $N_\omega \times A_X \to L$ defined by $p_A^{\theta}(i, Y) = \bigvee\{\Theta_{z\in Z} A(z) \mid Z \subseteq Y, \|Z\| = i\}$, page 35
R^n	Cartesian product of the set of real numbers, page 13
v	$\mathbf{M}$-evaluation, page 106
$x, y, \ldots$	object variables, page 105
$X \setminus Y$	relative complement of Y in X, page 23
$\sim Q$	external negation of a fuzzy quantifier Q, page 127
$\lhd Q$	internal negation of a fuzzy quantifier Q, page 128
$\|X\|$	(classical) cardinality of a set X, page 17
$\|\ominus_{\mathrm{M}}\|, \|\ominus_{\mathrm{P}}\|, \|\ominus_{\mathrm{L}}\|$	Gödel, Goguen and Łukasiewicz bidifference, page 8
$\| \ominus \|, \|a \ominus b\|$	bidifference (absolute difference), bidifference of a and b, page 8
al, am, g	types of fuzzy quantifiers (at least, at most, general), page 132

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