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EQ-LOGICS

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I hereby state that this submitted thesis is my original author work and that I elaborated it myself. I properly cite all references and other sources that I used to work up the thesis. Those references and other sources are given in the list of references.

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(podpis)

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Summary

Mathematical logic has been for many years developed on the basis of implication as the main connective. In eighties, new direction of the development has been initiated which is called equational logic. This logic is based on equality as the main connective. It brought an idea to develop also fuzzy logic on the basis of fuzzy equality (equivalence) as the principal connective.

The goal of the thesis is to propose a new class of many-valued logics called EQ-logics which are based on a special algebra of truth values called EQ-algebra, and thus to show a possible direction in the development of mathematical fuzzy logics by starting with equivalence instead of implication. The author introduces the formal theory of EQ-logic, studies its properties and also outlines a possible continuation in the future.

The structure of the work is as follows. Chapter 1 is devoted to introduction and motivation of the study and it also briefly recalls the simplest implication-based fuzzy logic whose properties are used later in this thesis. Chapter 2 contains overview of three basic truth structures for EQ-logics. This chapter provides properties and examples of each of these special classes of EQ-algebras.

In Chapter 3, four kinds of propositional EQ-logic are introduced. Namely, the basic one and its extensions—involutive EQ-logic, prelinear and finally EQ-logic which is equivalent to MTL-logic. Their properties are studied and the completeness theorems in each of them are proved. Chapter 4 focuses on extension of EQ-logic by the delta connective. The resulting EQ_{Δ}-logics are demonstrated to have reasonable properties including completeness. Introducing Δ-connective in EQ-logic makes it possible to prove generalized deduction theorem and it opens the door to development of the predicate first-order EQ-logic. Chapter 3 and 4 form the main part of this thesis.

Finally, Chapter 5 outlines how the predicate version of EQ-logic should be developed. The last chapter summarizes the results presented in the thesis and briefly discusses them.

Keywords: EQ-algebra; EQ-logic; Fuzzy equality; Delta connective; Mathematical fuzzy logic.

Anotace

Matematická logika se mnoho let rozvíjela na základě implikace. V osmdesátých letech se však objevil nový směr v rozvoji logiky, vznikla tzv. ekvacionální logika. Tato logika je založená na rovnosti. To vedlo k myšlence rozvinout fuzzy logiku, která byla doposud rozvíjena s pomocí implikace, na základě rovnosti (ekvivalence).

Cílem této disertace je navrhnout novou třídu vícehodnotových logik, které se nazývají EQ-logiky. Jsou založené na speciální algebře pravdivostních hodnot — EQ-algebře. Tato disertace se zároveň snaží ukázat možný směr v rozvoji matematických fuzzy logik vycházející z ekvivalence namísto implikace. Autor představuje formální teorii EQ-logik, studuje její vlastnosti a ukazuje její pokračování do budoucna.

Struktura práce je následující. Kapitola 1 se věnuje úvodu a motivaci do studia a zároveň připomíná nejjednodušší logiku založenou na implikaci, jejíž vlastnosti jsou dále v disertační práci použity. Kapitola 2 zahrnuje přehled tří základních pravdivostních struktur pro EQ-logiky. Vlastnosti a příklady každé této speciální třídy EQ-algeber jsou v této kapitole vzpomenuty.

V kapitole 3 jsou představeny čtyři druhy výrokových EQ-logik, a to základní EQ-logika a její rozšíření — involutivní EQ-logika, prelineární a nakonec EQ-logika, která je ekvivalentní s MTL-logikou. Jsou studovány jejich vlastnosti a jsou dokázány věty o úplnosti každé z těchto logik. Kapitola 4 se zaměřuje na rozšíření EQ-logiky delta spojkou. Tato kapitola uvádí, že EQ_{Δ}-logika má rozumné vlastnosti včetně úplnosti. Zavedení delta spojky v EQ-logice umožňuje dokázat zobecněnou větu o dedukci, což otevírá dveře k rozvoji EQ-logiky prvního řádu. Kapitoly 3 a 4 jsou ústřední části této disertační práce.

Konečně, kapitola 5 naznačuje, jak by měla být predikátová EQ-logika rozvinuta. Poslední kapitola shrnuje výsledky této disertační práce.

Klíčová slova: EQ-algebra; EQ-logika; Fuzzy rovnost; Delta spojka; Matematická fuzzy logika.

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Chapter 1

Introduction

1.1 Motivation

There are two basic directions in the development of mathematical logic. The first one (called implication-based) is based on implication as the principal connective and the basic inference rule is modus ponens (from A and $A \Rightarrow B$ infer B). The second direction, which may be called equality-based, is developed on the basis of logical equivalence (i.e. equality between truth values) as the fundamental connective and the basic inference rules are equanimity (from A and $A \equiv B$ infer B) and Leibniz (from $A \equiv B$ infer $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$) ones. Although the latter direction is not so popular as the first one, it gains step by step still more and more interest too. It illustrates, for example, the book developing classical boolean logic on the basis of equivalence as the main but not sole connective [36].

One of the main reason for its rising popularity is that equality (equivalence) seems to be more essential than implication. This idea was already supported by Ramsey [34] and Leibniz [3] who proclaimed:

"A fully satisfactory logical calculus must be an equational one".

They even tried to construct foundations of mathematics using equality only. Unfortunately, it is not possible in classical propositional or predicate logic. Nevertheless, Henkin presented in [24] that it is possible in type theory, i.e. higher-order logic introduced by Russell in [35] and further elaborated by Church, Henkin and Andrews (see [1, 4, 23]).

Another reason, why the equality-based logic is interesting for mathematicians, is based on the idea that formal proofs can be more effectively formed in an equational style. Many arguments supporting such approach have been described in [17]. Namely, the equational style allows to develop and present calculations in a rigorous manner, without too complexity and detail overwhelming. Thus, proofs in this style are well readable and relatively easy to construct.

This gives rise to an idea whether also fuzzy (many-valued) logic could be developed on the basis of generalized equality called fuzzy equality as the main connective. Apparently, there are two approaches which have already been developed in fuzzy type theory. In the first approach (cf. [27]), the structure of truth values is formed by residuated lattice (see [14, 37]) and equivalence is interpreted by a biresiduation $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ which is a derived operation. However, it seems to be unnatural to interpret the basic connective by a derived algebraic operation.

The second approach successfully solves this fundamental methodological discrepancy by introducing a specific kind of algebra called EQ-algebra [28, 30, 33]. Unlike the residuated lattices, the basic operation in it is a fuzzy equality ($\sim$) while implication is derived from it. More precisely, it is a lower semilattice endowed by two operations of multiplication ($\otimes$) and already mentioned fuzzy equality. Its axioms reflect basic properties which fuzzy equality should have to fit the supporting structure — the ordered set. However, the original EQ-algebras allowed also distinct elements to be equal in the degree $\mathbf{1}$, which is unacceptable for logic and only the, so called, *good* EQ-algebras must be considered, i.e. algebras in which each element a is equal to $\mathbf{1}$ in the degree a. This axiom implies that the algebra is separated (two elements equal in the degree $\mathbf{1}$ must be identical). Let us notice that one of the important algebraic consequences of goodness axiom is fact that each good EQ-algebra gives rise to a BCK-algebra [25, 26].

The implication operation interpreting implication in logic is in EQ-algebra derived from the fuzzy equality as $a \rightarrow b = (a \wedge b) \sim a$ and relaxed from the multiplication so that it is no more the residuation. The role of multiplication in EQ-algebras thus becomes only auxiliary and enables us to fuse statements about ordering and equality. Moreover, it can be non-commutative. EQ-algebras generalize residuated lattices in the sense that each residuated lattice gives rise to an EQ-algebra but not vice-versa. When we check properties of the implication in relation with the already developed fuzzy type theory based on EQ-algebras [31, 32], we conclude that it preserves all the essential ones (including, e.g. the modus ponens and the exchange principle). However, it is possible to find EQ-algebras with the same implication but different fuzzy equalities and so, the fuzzy equality, in general, cannot be reconstructed from the implication (in special cases, however, like in all linearly ordered EQ-algebras it is possible).

Evidently, good EQ-algebras seem to be suitable algebraic structures for development of the fuzzy equality-based logics. The logics based on these algebras are called EQ-logics. Although EQ-logics have many interesting properties (including completeness) they are still limited. Namely, the generalized deduction theorem does not hold in them (of course, except for the residuated EQ(MTL)-logic). This deficiency has fatal consequences, e.g. for the development of the first-order EQ-logic since its completeness can hardly be provable without the deduction theorem. The situation is different in EQ_Δ-logics, where we already are able to prove the Δ-deduction theorem. Thus, unlike the residuated fuzzy logics where the Δ-connective is interesting but dispensable option, the role of it in fuzzy equality-based logics is much deeper. We conclude that the general fuzzy equivalence is not sufficient and a crisp equivalence is necessary for well behaving logic. From algebraic point of view, the key lays in the extensionality of operations: if $a = b$ and $*$ is any binary operation then $a * c = b * c$ for any c. If we consider fuzzy equality instead, then we derive the following inequality

$$a \sim b \leq a * c \sim b * c.$$

In EQ-algebras, this property is satisfied both for $* = \wedge, \sim$. Nevertheless, if we put $* = \otimes$, the EQ-algebra becomes residuated. We can make it weaker by introducing the Δ-operation and assuming $\Delta(a \sim b) \leq a * c \sim b * c$. Then this is strong enough to assure validity of the Δ-deduction theorem.

1.2 MTL-logic

In this section, we are going to review a well-known fuzzy logic – the MTL-logic (Monoidal t-norm-based logic). We recall this logic because of presenting EQ-logic which is equivalent to MTL-logic and thus, we will refer to its axioms and properties particularly. If the reader is familiar with this logic then he can skip this section.

MTL-logic was introduced by Esteva and Godo in [13] and further studied in [12, 20]. It is the logic of left-continuous t-norms (see [6]) and it is a basis of a wide class of fuzzy logic called core fuzzy logics [22].

The logic MTL has three binary connectives $\wedge, \&, \Rightarrow$ and the truth constant $\perp$. Further connectives are defined as follows:

$$A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A), \tag{1.2.1}$$

$$A \equiv B := (A \Rightarrow B) \& (B \Rightarrow A), \tag{1.2.2}$$

$$\neg A := A \Rightarrow \perp, \tag{1.2.3}$$

$$\top := \perp \Rightarrow \perp. \tag{1.2.4}$$

Let us note that the symbol ":= " means "is defined by".

The following formulas are logical axioms of MTL-logic:

(A1) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

(A2) $(A \& B) \Rightarrow A$

(A3) $(A \& B) \Rightarrow (B \& A)$

(A4) $(A \wedge B) \Rightarrow A$

(A5) $(A \wedge B) \Rightarrow (B \wedge A)$

(A6) $(A \& (A \Rightarrow B)) \Rightarrow (A \wedge B)$

(A7a) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \& B) \Rightarrow C)$

(A7b) $((A \& B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

(A8) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

(A9) $\perp \Rightarrow A$

The deduction rule is *modus ponens* (from A and $A \Rightarrow B$ infer B).

Now we recall some theorems of MTL-logic which we will refer to in this thesis.

Lemma 1
The following properties are provable in MTL-logic.

(T1) $\vdash A \Rightarrow (B \Rightarrow A)$

(T2) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$

(T3) $\vdash A \Rightarrow A$

(T4) $\vdash (A \& (A \wedge B)) \Rightarrow B$

(T5) $\vdash A \Rightarrow (B \Rightarrow (A \& B))$

(T6) $\vdash (A \Rightarrow B) \Rightarrow ((A \& C) \Rightarrow (B \& C))$

(T7) $\vdash ((A \Rightarrow B) \& (C \Rightarrow D)) \Rightarrow ((A \& C) \Rightarrow (B \& D))$

(T8) $\vdash (A \wedge B) \Rightarrow B$

(T9) $\vdash (A \& B) \Rightarrow (A \wedge B)$

(T10) $\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

(T11) $\vdash ((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$

(T12) $\vdash A \Rightarrow (A \wedge A)$

(T13) $\vdash \top$

(T14) $\vdash A \Rightarrow (\top \& A)$

(T15) $\vdash ((A \wedge B) \wedge C) \equiv (A \wedge (B \wedge C))$

(T16) $\vdash ((A \& B) \& C) \equiv (A \& (B \& C))$

(T17) $\vdash A \equiv A$

(T18) $\vdash (A \equiv B) \Rightarrow (B \equiv A)$

(T19) $\vdash (A \equiv B) \& (B \equiv C) \Rightarrow (A \equiv C)$

(T20) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$

(T21) $\vdash (A \equiv B) \Rightarrow ((A \& C) \equiv (B \& C))$

(T22) $\vdash (A \equiv B) \Rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$

(T23) $\vdash (A \equiv B) \Rightarrow ((C \Rightarrow A) \equiv (C \Rightarrow B))$

(T24) $\vdash (A \equiv B) \equiv ((A \Rightarrow B) \wedge (B \Rightarrow A))$

The structure of truth values for MTL-logic is formed by the following algebra:

Definition 1
An MTL-algebra is a structure $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ *such that:*

(i) $\langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ *is a bounded lattice,*

(ii) $\langle L, \otimes, \mathbf{1}\rangle$ *is a commutative monoid,*

(iii) $a \leq b \rightarrow c$ *iff* $a \otimes b \leq c$ *for all* $a, b, c \in L$, *(adjunction)*

(iv) $(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$. *(prelinearity)*

Chapter 2

Algebras for EQ-logics

In this second chapter, we give basic definitions and properties of the structures of truth values for EQ-logics – EQ-algebras, EQ$_\Delta$-algebras and lattice EQ$_\Delta$-algebras. We also present several examples of these algebras.

2.1 EQ-algebras

EQ-algebras were initiated by Novák in [28]. Later, in [11, 33] the authors elaborated them further in more detail.

2.1.1 Definitions and basic properties

Definition 2
A non-commutative EQ-algebra $\mathcal{E}$ is an algebra of type (2, 2, 2, 0), i.e.

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle ,$$

where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1} \rangle$ *is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$) where the ordering $a \leq b$ is defined in the standard way as $a \wedge b = a$,*

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ *is a monoid and $\otimes$ is isotone w.r.t. $\leq$, i.e. $a \leq b$ implies both $a \otimes c \leq b \otimes c$ as well as $c \otimes a \leq c \otimes b$,*

(E3) $a \sim a = \mathbf{1}$, *(reflexivity axiom)*

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$, *(substitution axiom)*

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$, *(congruence axiom)*

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ *(monotonicity axiom)*

(E7) $a \otimes b \leq a \sim b$. *(boundedness axiom)*

The operation $\wedge$ is called *meet* (infimum), $\otimes$ is called *multiplication* and $\sim$ is the *fuzzy equality*. EQ-algebra is *commutative* if $\otimes$ is a commutative operation.

Note that the original definition of EQ-algebras also includes the following monotonicity axiom

$$(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c). \tag{2.1.1}$$

However, in [11], it has been shown that it can be derived from the other ones and therefore we can omit it from the list of axioms of EQ-algebras.

We also put, for $a, b \in E$

$$a \to b = (a \wedge b) \sim a. \tag{2.1.2}$$

The derived operation $\to$ is called *implication*. Using (2.1.2), we may rewrite (E6) and (2.1.1) as

$$a \to (b \wedge c) \leq a \to b\,, \tag{2.1.3}$$

$$a \to b \leq (a \wedge c) \to b\,, \tag{2.1.4}$$

respectively. Thus, axiom (E6) and formula (2.1.1) express isotonicity of $\to$ w.r.t. the second variable and antitonicity of $\to$ w.r.t. the first variable. Note also that the substitution axiom (E4) can be seen as a special form of extensionality (see [19]).

The following theorem demonstrates that $\sim$ has basic properties of the fuzzy equality, namely reflexivity, symmetry and transitivity, and that the implication is also transitive. Note that their proofs (see [33]) do not require commutativity of $\otimes$.

Theorem 1

Let $\mathcal{E}$ be an EQ-algebra. The following holds for all $a, b, c \in E$:

(a) Symmetry: $a \sim b = b \sim a$,

(b) Transitivity: $(a \sim b) \otimes (b \sim c) \leq a \sim c$,

(c) Transitivity of implication: $(a \to b) \otimes (b \to c) \leq a \to c$.

Let $\mathcal{E}$ contain also the bottom element $\mathbf{0}$. Then we put

$$\neg a = a \sim \mathbf{0}, \qquad a \in E \tag{2.1.5}$$

and call $\neg a$ a *negation* of $a \in E$.

We distinguish several special classes of EQ-algebras.

Definition 3
Let $\mathcal{E}$ be an EQ-algebra and $a, b, c, d \in E$. We say that $\mathcal{E}$ is:

(i) separated *if for all $a \in E$,*

(E8) $a \sim b = \mathbf{1}$ *implies* $a = b$.

(ii) good *if*

(E9) $a \sim \mathbf{1} = a$.

(iii) residuated *if for all $a, b, c \in E$,*

(E10) $(a \otimes b) \wedge c = a \otimes b$ *iff* $a \wedge ((b \wedge c) \sim b) = a$.

(iv) involutive *(IEQ-algebra) if for all $a \in E$,*

(E11) $\neg\neg a = a$.

(v) prelinear *if for all $a, b \in E$,*

(E12) $\sup\{a \to b, b \to a\} = \mathbf{1}$.

(vi) lattice ordered *if it has also the binary operation join (supremum) $\vee$ so that $\langle E, \wedge, \vee \rangle$ is a lattice.*

(vii) lattice EQ-algebra *(ℓEQ-algebra) if it is a lattice and for all $a, b, c, d \in E$,*

(E13) $((a \vee b) \sim c) \otimes (d \sim a) \leq (d \vee b) \sim c$.

The following lemmas reflect some properties of EQ-algebras useful in the sequel. Their proofs do not require commutativity of $\otimes$ again (see [11, 33]).

Lemma 2
Let $\mathcal{E}$ be an EQ-algebra. For all $a, b \in E$ such that $a \leq b$ it holds that

(a) $a \to b = \mathbf{1}$,

(b) $a \sim b = b \to a$,

(c) $c \to a \leq c \to b$ *and* $b \to c \leq a \to c$.

Lemma 3
Let $\mathcal{E}$ *be a good EQ-algebra. For all* $a, b, c \in E$ *it holds that*

(a) $a \otimes b \leq a$, $a \otimes b \leq a \wedge b$, $c \otimes (a \wedge b) \leq (c \otimes a) \wedge (c \otimes b)$,

(b) $a \sim b \leq a \to b$, $a \to a = \mathbf{1}$,

(c) $(a \to b) \otimes (b \to a) \leq a \sim b$,

(d) $a = b$ *implies* $a \sim b = \mathbf{1}$,

(e) $a = \mathbf{1} \to a$ *and* $a \to \mathbf{1} = \mathbf{1}$,

(f) $a \otimes (a \sim b) \leq b$,

(g) $b \leq a \to b$,

(h) $a \leq b \to c$ *iff* $b \leq a \to c$.

Obviously, if $\mathcal{E}$ is separated then $a \to b = \mathbf{1}$ implies $a \leq b$. Thus, it follows from Lemma 3(d) that EQ-algebra is separated iff for all $a, b \in E$ it holds that

$$a = b \quad \text{iff} \quad a \sim b = \mathbf{1} . \tag{2.1.6}$$

It is easy to see that all *good EQ-algebras are separated*.

The following theorem shows that EQ-algebra enriched by the adjunction condition leads to a residuated EQ-algebra that is commutative (i.e. with commutative multiplication).

Theorem 2 ([11])
Let $\mathcal{E}$ *be a residuated EQ-algebra. Then its multiplication* $\otimes$ *is commutative.*

In good EQ-algebras, only one implication of the adjunction condition is satisfied:

$$a \leq b \to c \quad \text{implies} \quad a \otimes b \leq c. \tag{2.1.7}$$

There are several conditions which make good EQ-algebra residuated (see [10, Proposition 6]). We mention only one of them which we will need later.

Lemma 4
Let $\mathcal{E}$ be an EQ-algebra. The following statements are equivalent:

(a) *$\mathcal{E}$ is residuated,*

(b) *$\mathcal{E}$ is good and for all $a, b, c \in \mathcal{E}$ it holds that*

$$a \to b \leq (a \otimes c) \to (b \otimes c). \tag{2.1.8}$$

The following lemma has been proved in [33].

Lemma 5
Let $\mathcal{E}$ be an IEQ-algebra. Then

(a) *$\mathcal{E}$ is a good ℓEQ-algebra with join defined by*

$$a \vee b = \neg(\neg a \wedge \neg b).$$

(b) $a \sim b = \neg a \sim \neg b$.

(c) $a \otimes \neg a = \mathbf{0}$.

The following lemma illustrates properties of $\vee$ which will be helpful to prove substitution property mentioned in Lemma 7.

Lemma 6
Let $\mathcal{E}$ be a prelinear and good ℓEQ-algebra. For all $a, b, c \in E$ it holds that

(a) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(b) $a \sim b \leq (a \vee c) \sim (b \vee c)$.

PROOF: (a) See Lemma 8 in [10]. (b) is proven in Proposition 4 in [11]. □

Lemma 7
Let $\mathcal{E}$ be a prelinear and good ℓEQ-algebra. Then the substitution axioms (E4) and (E13) are equivalent to the following one:

(E14) $((((a \wedge b) \vee c) \sim d) \otimes (f \sim c)) \otimes (e \sim a) \leq d \sim (f \vee (b \wedge e))$

PROOF: By (E13) and Lemma 6(a), we have

$$\begin{aligned}(((a \wedge b) \vee c) \sim d) \otimes (f \sim c) &= ((c \vee (a \wedge b)) \sim d) \otimes (f \sim c) \\ &\leq (f \vee (a \wedge b)) \sim d = ((f \vee a) \wedge (f \vee b)) \sim d.\end{aligned}$$

Thus using isotonicity of $\otimes$, Lemma 6(b), (E4) and again Lemma 6(a), we get (E14):

$$\begin{aligned}&((((a \wedge b) \vee c) \sim d) \otimes (f \sim c)) \otimes (e \sim a) \leq (((f \vee a) \wedge (f \vee b)) \sim d) \otimes (e \sim a) \\ &\quad \leq (((f \vee a) \wedge (f \vee b)) \sim d) \otimes ((f \vee e) \sim (f \vee a)) \leq d \sim ((f \vee e) \wedge (f \vee b)) \\ &\quad = d \sim (f \vee (e \wedge b)) = d \sim (f \vee (b \wedge e)).\end{aligned}$$

Conversely, let us use (E14) to prove (E4) and (E13). Firstly, we prove (E4):

$$\begin{aligned}((a \wedge b) \sim c) \otimes (d \sim a) &= ((((a \wedge b) \vee \mathbf{0}) \sim c) \otimes (\mathbf{0} \sim \mathbf{0})) \otimes (d \sim a) \\ &\leq c \sim (\mathbf{0} \vee (b \wedge d)) = c \sim (d \wedge b).\end{aligned}$$

Now, we prove (E13):

$$\begin{aligned}((a \vee b) \sim c) \otimes (d \sim a) &= ((((a \wedge \mathbf{1}) \vee b) \sim c) \otimes (b \sim b)) \otimes (d \sim a) \\ &\leq c \sim (b \vee (\mathbf{1} \wedge d)) = (d \vee b) \sim c.\end{aligned}$$

□

Remark 1
On the basis of this lemma, we can use axiom (E14) instead of two axioms (E4) and (E13) in the definition of prelinear and good ℓEQ-algebras.

Similarly as in the case of residuated lattices, we say that an EQ-algebra is *representable* if it is subdirectly embeddable into a product of linearly ordered EQ-algebras. The following theorems characterize the class of representable good EQ-algebras (see [10]).

Theorem 3
Let $\mathcal{E}$ be a good EQ-algebra. Then the following properties are equivalent:

(a) $\mathcal{E}$ is representable.

(b) $\mathcal{E}$ *satisfies the formula*

$$(a \to b) \vee (d \to (d \otimes (c \to (b \to a) \otimes c))) = \mathbf{1} \tag{2.1.9}$$

for all $a, b, c, d \in E$.

Theorem 4
If a good EQ-algebra $\mathcal{E}$ *satisfies (2.1.9) then it is prelinear.*

2.1.2 Examples

In this section, we give several examples of EQ-algebras.

Example 1
Consider $E_5 = \{\mathbf{0} < a < b < c < \mathbf{1}\}$ *to be a five-element chain and put*

$\otimes$	**0**	**a**	**b**	**c**	**1**
0	**0**	**0**	**0**	**0**	**0**
a	**0**	**0**	**0**	**0**	a
b	**0**	**0**	**0**	**0**	b
c	**0**	a	a	c	c
1	**0**	a	b	c	**1**

$\sim$	**0**	**a**	**b**	**c**	**1**
0	**1**	b	a	**0**	**0**
a	b	**1**	b	a	a
b	a	b	**1**	b	b
c	**0**	a	b	**1**	c
1	**0**	a	b	c	**1**

$\to$	**0**	**a**	**b**	**c**	**1**
0	**1**	**1**	**1**	**1**	**1**
a	b	**1**	**1**	**1**	**1**
b	a	b	**1**	**1**	**1**
c	**0**	a	b	**1**	**1**
1	**0**	a	b	c	**1**

It can be verified that $\langle E_5, \wedge, \otimes, \sim, \mathbf{1}\rangle$ *is a linearly ordered good EQ-algebra. Moreover, this algebra is non-residuated, since* $a = c \otimes b \leq a$, *but* $c \nleq b \to a = b$.

Example 2
Example of a finite non-trivial non-residuated IEQ-algebra is depicted in 2.1. Product and fuzzy equality are defined as follows:

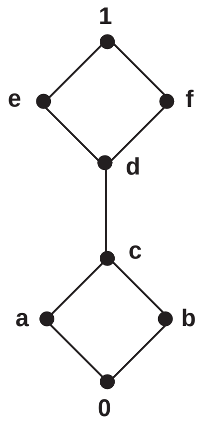

Figure 2.1: Eight element EQ-algebra.

$\otimes$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
a	**0**	**0**	**0**	**0**	**0**	**0**	**0**	a
b	**0**	**0**	**0**	**0**	**0**	**0**	**0**	b
c	**0**	**0**	**0**	**0**	**0**	**0**	**0**	c
d	**0**	**0**	**0**	**0**	d	d	d	d
e	**0**	**0**	**0**	**0**	d	e	d	e
f	**0**	**0**	**0**	a	d	d	d	f
1	**0**	a	b	c	d	e	f	**1**

$\sim$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**1**	e	f	d	c	a	b	**0**
a	e	**1**	d	f	c	a	c	a
b	f	d	**1**	e	c	c	b	b
c	d	f	e	**1**	c	c	c	c
d	c	c	c	c	**1**	f	e	d
e	a	a	c	c	f	**1**	d	e
f	b	c	b	c	e	d	**1**	f
1	**0**	a	b	c	d	e	f	**1**

Implication is given in the following table:

$\rightarrow$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
a	e	**1**	e	**1**	**1**	**1**	**1**	**1**
b	f	f	**1**	**1**	**1**	**1**	**1**	**1**
c	d	f	e	**1**	**1**	**1**	**1**	**1**
d	c	c	c	c	**1**	**1**	**1**	**1**
e	a	a	c	c	f	**1**	f	**1**
f	b	c	b	c	e	e	**1**	**1**
1	**0**	a	b	c	d	e	f	**1**

Note that the multiplication $\otimes$ is not commutative. Indeed, $f \otimes c = a$ but $c \otimes f = \mathbf{0}$. This algebra is also non-residuated since, for instance, $\mathbf{0} = c \otimes e \leq \mathbf{0}$, but $c \not\leq e \rightarrow \mathbf{0}$. Notice also that the first column of $\sim$ gives negation, thus we can check that $\neg\neg x = x$.

There are many other examples of non-trivial finite EQ-algebras, including linearly ordered ones (see [33] for commutative EQ-algebras or [11] for non-commutative ones). Note that in general, neither non-linear, nor linear EQ-algebras coincide with residuated lattices.

2.2 EQ_{Δ}-algebras

Below, we are going to enlarge EQ-algebra by additional operation Δ. This algebra will become a structure of truth values for basic $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic. Its definition is motivated by Novák's definition of a delta operation in EQ-algebras (see [32]), where it was introduced for the first time.

2.2.1 Definition

Definition 4
An EQ_{Δ}-algebra is a structure

$$\mathcal{E}_{\boldsymbol{\Delta}} = \langle E, \wedge, \otimes, \sim, \Delta, \mathbf{1} \rangle ,$$

which is a good non-commutative EQ-algebra expanded by a unary operation Δ in which the following formulas are true:

(EΔ1) $\Delta \mathbf{1} = \mathbf{1}$

(EΔ2) $\Delta a \leq \Delta\Delta a$

(EΔ3) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$

(EΔ4) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$

(EΔ5) $\Delta a = \Delta a \otimes \Delta a$

(EΔ6) $\Delta(a \sim b) \leq (a \otimes c) \sim (b \otimes c)$

(EΔ7) $\Delta(a \sim b) \leq (c \otimes a) \sim (c \otimes b)$

Notice that the last two axioms (EΔ6), (EΔ7) guarantee good behavior of the multiplication with respect to the crisp equality. They are especially needed for proving deduction theorem and subsequently also for developing predicate version of EQ-logic.

2.2.2 Basic properties

Lemma 8
EQ$_\Delta$-algebra becomes residuated if Δ operation in (EΔ6) or (EΔ7) is omitted.

PROOF: Assume that (EΔ6) does not contain Δ. Using Lemma 4, we need only to show that $\mathcal{E}_\Delta$ satisfies (2.1.8). Thus, from (2.1.2), (EΔ6) without Δ, Lemma 3(b), monotonicity of $\to$ and isotonicity of $\otimes$ we get

$$a \to b = (a \wedge b) \sim a \le ((a \wedge b) \otimes c) \sim (a \otimes c) = (a \otimes c) \sim ((a \wedge b) \otimes c)$$
$$\le (a \otimes c) \to ((a \wedge b) \otimes c) \le (a \otimes c) \to (b \otimes c).$$

The second part of the proof can be proven in the similar way (using also Theorem 2). □

The following are properties of EQ$_\Delta$-algebras.

Lemma 9
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. For all $a, b \in E$ it holds that

(a) $\Delta a \le a$,

(b) If $a \le b$ *then* $\Delta a \le \Delta b$,

(c) $\Delta(a \to b) \le \Delta a \to \Delta b$.

PROOF:

(a) By (E9) and (EΔ6) we have

$$\Delta a = \Delta(a \sim \mathbf{1}) \le (a \otimes \mathbf{1}) \sim (\mathbf{1} \otimes \mathbf{1}) = a \sim \mathbf{1} = a.$$

(b) and (c) have been already proved in [32]. □

Example 3
The following is an example of a finite non-trivial EQ$_\Delta$-algebra. The lattice structure is depicted in Figure 2.2. The multiplication and fuzzy equality are defined as follows:

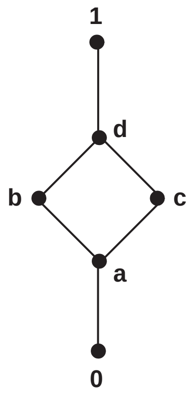

Figure 2.2: Six element EQ_Δ*-algebra.*

$\otimes$	**0**	**a**	**b**	**c**	**d**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**
a	**0**	**0**	**0**	**0**	**0**	*a*
b	**0**	**0**	**0**	**0**	*a*	*b*
c	**0**	**0**	**0**	*c*	*c*	*c*
d	**0**	**0**	**0**	*c*	*c*	*d*
1	**0**	*a*	*b*	*c*	*d*	**1**

$\sim$	**0**	**a**	**b**	**c**	**d**	**1**
0	**1**	*d*	*c*	*b*	*a*	**0**
a	*d*	**1**	*c*	*b*	*a*	*a*
b	*c*	*c*	**1**	*a*	*b*	*b*
c	*b*	*b*	*a*	**1**	*c*	*c*
d	*a*	*a*	*b*	*c*	**1**	*d*
1	**0**	*a*	*b*	*c*	*d*	**1**

Using (2.1.2) we obtain the implication:

$\rightarrow$	**0**	**a**	**b**	**c**	**d**	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**
a	*d*	**1**	**1**	**1**	**1**	**1**
b	*c*	*c*	**1**	*c*	**1**	**1**
c	*b*	*b*	*b*	**1**	**1**	**1**
d	*a*	*a*	*b*	*c*	**1**	**1**
1	**0**	*a*	*b*	*c*	*d*	**1**

The Δ *operation is defined by* $\Delta(\mathbf{1}) = \mathbf{1}$ *and* $\Delta(x) = \mathbf{0}$ *otherwise.*

Notice that the multiplication $\otimes$ *is not commutative. Indeed, for example* $b \otimes d = a$ *but* $d \otimes b = \mathbf{0}$*. Moreover, this algebra is also non-residuated since, e.g.,* $c = d \otimes d \leq c$*, but* $d \nleq d \rightarrow c = c$*.*

2.3 Lattice EQ$_\Delta$-algebras

In this section we introduce the set of truth values for prelinear EQ$_{\boldsymbol{\Delta}}$-logic. It is formed by a lattice ordered EQ$_\Delta$-algebra which keeps substitution property (E13).

2.3.1 Definition

Definition 5

A lattice EQ$_\Delta$-algebra (ℓEQ$_\Delta$-algebra) is an algebra $\mathcal{E}_\Delta = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1}\rangle$ *where* $\langle E, \wedge, \vee, \otimes, \sim, \mathbf{0}, \mathbf{1}\rangle$ *is a good non-commutative and bounded ℓEQ-algebra ($\mathbf{0}$ and $\mathbf{1}$ are bottom and top elements, respectively) expanded by a unary operation* $\Delta : E \longrightarrow E$ *fulfilling axioms (EΔ1)–(EΔ7) plus*

(EΔ8) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,

(EΔ9) $\Delta a \vee \neg\Delta a = 1$.

2.3.2 Basic properties

Note that on the basis of Lemma 7, we will consider only axiom (E14) instead of (E4) and (E13) in prelinear ℓEQ$_\Delta$-algebras.

Lemma 10

Let $\mathcal{E}_\Delta$ be an prelinear ℓEQ$_\Delta$-algebras. For all $a, b \in E$ it holds that

(a) $\Delta\mathbf{0} = \mathbf{0}$,

(b) $\Delta(\neg a) \leq \neg(\Delta a)$,

(c) $\Delta(a \vee b) = \Delta a \vee \Delta b$,

(d) $\Delta(a \to b) \vee \Delta(b \to a) = 1$,

(e) $\Delta \otimes \Delta(a \to b) \leq \Delta b$.

PROOF: (a) is obvious. (b) follows from axiom (EΔ3) and (a):

$$\Delta(\neg a) = \Delta(a \sim \mathbf{0}) \leq \Delta a \sim \Delta\mathbf{0} = \Delta a \sim \mathbf{0} = \neg(\Delta a).$$

(c) Due to Lemma 9(b), it holds that $\Delta a \leq \Delta(a \vee b)$ and $\Delta b \leq \Delta(a \vee b)$, and hence, $\Delta a \vee \Delta b \leq \Delta(a \vee b)$. A converse inequality is axiom (EΔ8).

(d) follows immediately from prelinearity, (c) and (EΔ1).

(e) Using Lemma 9(c) and 3(h) it follows that $\Delta a \leq \Delta(a \to b) \to \Delta b$. This proof is completed using (2.1.7). □

The following theorem characterizes a representable class of ℓEQ$_\Delta$-algebras.

Theorem 5
Let $\mathcal{E}_\Delta$ be ℓEQ_Δ-algebra. Then the following properties are equivalent:

(a) $\mathcal{E}_\Delta$ is subdirectly embeddable into a product of linearly ordered ℓEQ_Δ-algebras (i.e., $\mathcal{E}_\Delta$ is representable).

(b) $\mathcal{E}_\Delta$ satisfies condition (2.1.9) for all $a, b, c, d \in E$.

PROOF: This theorem was proven in [11] for good EQ$_\Delta$-algebras. We will therefore follow this proof and concentrate on different points only. Obviously, if $\mathcal{E}_\Delta$ is representable then it satisfies condition (2.1.9) (since either $x \to y = \mathbf{1}$ or $y \to x = \mathbf{1}$ holds for all x, y in linearly ordered EQ$_\Delta$-algebra). Conversely, due to Theorem 4, $\mathcal{E}_\Delta$ is prelinear. Then, the proof proceeds in the same way as in Theorem 11 in [11]. It is only sufficient to verify that axioms of ℓEQ$_\Delta$-algebra hold also in the factor algebra $\mathcal{E}_\Delta / \approx_F$ where $\approx_F$ is a congruence defined for a given filter F as $a \approx_F b$ iff $a \sim b \in F$.

We will demonstrate it, e.g. for (EΔ6) and (EΔ9). Then

$$\Delta([a]_F \sim [b]_F) = [\Delta(a \sim b)]_F \leq [(a \otimes c) \sim (b \otimes c)]_F = ([a]_F \otimes [c]_F) \sim ([b]_F \otimes [c]_F)$$

because

$$[\Delta(a \sim b)]_F \leq [(a \otimes c) \sim (b \otimes c)]_F \quad \text{iff} \quad \Delta(a \sim b) \to ((a \otimes c) \sim (b \otimes c)) = \mathbf{1} \in F$$

which holds because (EΔ6) holds in $\mathcal{E}_\Delta$ (see also Lemma 2(a)). Now, we verify it for (EΔ9). From (EΔ9) follows that $\Delta a \vee \neg\Delta a \in F$, hence from Lemma 3(g) and properties of the filter we have

$$\Delta a \vee \neg\Delta a \leq (\Delta a \vee \neg\Delta a) \sim \mathbf{1} \in F$$

and consequently

$$\Delta[a]_F \vee \neg\Delta[a]_F = [\Delta a \vee \neg\Delta a]_F = [\mathbf{1}]_F.$$

Similarly for the other axioms. □

2.3.3 Examples

In this subsection, we give some interesting examples of lattice EQ_Δ-algebras.

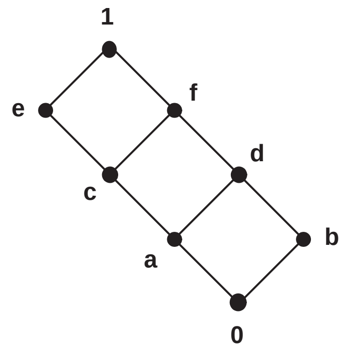

Figure 2.3: Eight element $\ell\mathrm{EQ}_\Delta$-algebra.

Example 4
Consider an example of a finite non-trivial EQ_Δ-algebra whose lattice structure is depicted in the Figure 2.3. The following multiplication and fuzzy equality defined as follows:

$\otimes$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
a	**0**	**0**	**0**	a	**0**	a	a	a
b	**0**	**0**	b	**0**	b	**0**	b	b
c	**0**	**0**	**0**	c	**0**	c	c	c
d	**0**	**0**	b	a	b	a	d	d
e	**0**	a	**0**	c	a	e	c	e
f	**0**	**0**	b	c	b	c	f	f
1	**0**	a	b	c	d	e	f	**1**

$\sim$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**1**	b	e	b	**0**	b	**0**	**0**
a	b	**1**	**0**	d	e	d	a	a
b	e	**0**	**1**	**0**	b	**0**	b	b
c	b	d	**0**	**1**	a	f	e	c
d	**0**	e	b	a	**1**	a	d	d
e	b	d	**0**	f	a	**1**	c	e
f	**0**	a	b	e	d	c	**1**	f
1	**0**	a	b	c	d	e	f	**1**

Using (2.1.2) we obtain the implication:

$\to$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
a	b	**1**	b	**1**	**1**	**1**	**1**	**1**
b	e	e	**1**	e	**1**	e	**1**	**1**
c	b	d	b	**1**	d	**1**	**1**	**1**
d	**0**	e	b	e	**1**	e	**1**	**1**
e	b	d	b	f	d	**1**	f	**1**
f	**0**	a	b	e	d	e	**1**	**1**
1	**0**	a	b	c	d	e	f	**1**

The Δ operation is defined by

$$\Delta(x) = \begin{cases} \mathbf{1} & \text{if } x = \mathbf{1} \\ e & \text{if } x = e \\ b & \text{if } x = b, d, f \\ \mathbf{0} & \text{if } x = \mathbf{0}, a, c \end{cases}$$

Note that the multiplication $\otimes$ is not commutative. Moreover, this algebra is non-residuated since, e.g., $\mathbf{0} = a \otimes d \leq \mathbf{0}$, but $a \nleq d \to \mathbf{0} = \mathbf{0}$. Notice also that the algebra is prelinear (unlike the algebra from Example 3), since it satisfies the assumptions of Theorem 4.

Example 5

Consider $(E_\Delta)_1 = \langle E_1 = \{0 < a < b < 1\}, \wedge_1, \vee_1, \otimes_1, \sim_1, \Delta_1, \mathbf{0}, \mathbf{1}\rangle$ to be a four-element linearly ordered EQ$_\Delta$-algebra and let $(E_\Delta)_2 = \langle E_2 = \{0 < a < b < c < 1\}, \wedge_2, \vee_2, \otimes_2, \sim_2, \Delta_2, \mathbf{0}, \mathbf{1}\rangle$ be a five-element linearly ordered EQ$_\Delta$-algebra with multiplications and the fuzzy equalities defined as follows:

$\otimes_1$	**0**	**a**	**b**	**1**
0	**0**	**0**	**0**	**0**
a	**0**	**0**	a	a
b	**0**	**0**	b	b
1	**0**	a	b	**1**

$\sim_1$	**0**	**a**	**b**	**1**
0	**1**	**0**	**0**	**0**
a	**0**	**1**	a	a
b	**0**	a	**1**	b
1	**0**	a	b	**1**

$\otimes_2$	**0**	**a**	**b**	**c**	**1**
0	**0**	**0**	**0**	**0**	**0**
a	**0**	**0**	**0**	a	a
b	**0**	**0**	**0**	b	b
c	**0**	**0**	**0**	c	c
1	**0**	a	b	c	**1**

$\sim_2$	**0**	**a**	**b**	**c**	**1**
0	**1**	b	a	**0**	**0**
a	b	**1**	b	a	a
b	a	b	**1**	b	b
c	**0**	a	b	**1**	c
1	**0**	a	b	c	**1**

The Δ operation is (just as in all linearly ordered EQ-algebras) defined by $\Delta(\mathbf{1}) = \mathbf{1}$ and $\Delta(x) = \mathbf{0}$ otherwise. Note that both algebras are non-commutative, non-residuated and good.

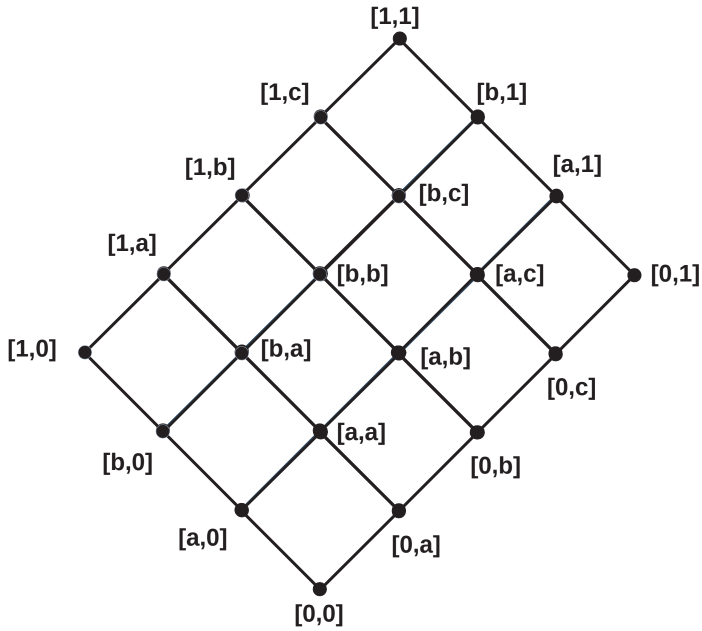

Figure 2.4: Twenty element ℓEQ_Δ-algebra.

Then a direct product $(E_\Delta)_1 \times (E_\Delta)_2$ provides a twenty-element non-linear ℓEQ_Δ-algebra which is also non-commutative (see below), non-residuated and good. Its lattice structure is depicted in Figure 2.4. Δ operation is defined as follows:

$$\Delta(x) = \begin{cases} [\mathbf{1},\mathbf{1}] & \textit{if } x = [\mathbf{1},\mathbf{1}] \\ [\mathbf{0},\mathbf{1}] & \textit{if } x = [\mathbf{0},\mathbf{1}], [a,\mathbf{1}], [b,\mathbf{1}] \\ [\mathbf{1},\mathbf{0}] & \textit{if } x = [\mathbf{1},\mathbf{0}], [\mathbf{1},a], [\mathbf{1},b], [\mathbf{1},c] \\ [\mathbf{0},\mathbf{0}] & \textit{otherwise} \end{cases}$$

Due to the high number of elements we will not state how the remaining operations of this algebra exactly look like. This can be easily found in the standard *way (see e.g. [2]). However, let us present, at least, a small illustration for each operation, for example:*

$$[b,a]\otimes[a,c]=[\mathbf{0},a] \quad \textit{but} \quad [a,c]\otimes[b,a]=[a,\mathbf{0}]$$

$$[\mathbf{1},\mathbf{0}]\sim[b,c]=[b,\mathbf{0}].$$

Chapter 3

EQ-logic

In this third chapter, we introduce a specific formal logic which is developed on the basis of fuzzy equality and in which the implication is derived from the latter. Moreover, the fusion connective (strong conjunction) is non-commutative. We call this logic EQ-logic.

We present four kinds of EQ-logics. First of all, we introduce basic EQ-logic which is rich enough to enjoy the completeness property. Moreover, we formulate three interesting extensions. The first one is IEQ-logic which is EQ-logic with the law of double negation. The second one adds prelinearity which allows us to prove a stronger variant of the completeness property. Finally, we introduce EQ-logic which is equivalent with MTL-logic.

3.1 Basic EQ-logic

The simplest logic definable on the basis of EQ-algebras will be called basic EQ-logic.

3.1.1 Syntax

Definition 6 (Language)
A propositional language J of the basic EQ-logic consists of:

(i) Propositional variables $p, q, \dots$.

(ii) Binary connectives $\wedge$ (conjunction), $\&$ (fusion), $\equiv$ (equality or equivalence).

(iii) The truth (logical) constant $\top$.

(iv) Auxiliary symbols: brackets.

Remark 2
We will use one more symbol which will occur in Leibniz rule (defined in Subsection 3.1.3), namely symbol $\mathbf{p}$ *being inside expressions such as* $C[\mathbf{p} := A]$*. It is so-called metavariable (syntactic variable), i.e. symbol outside the language that we can use to refer to or point, generally, to any variable. Thus,* $\mathbf{p}$ *names any variable* $p, q, \ldots$.

Definition 7 (Formulas)
Let J *be a propositional language. The formulas are finite sequences of symbols from* J *defined as follows:*

(i) Each propositional variable or the logical constant $\top$ *is a formula.*

(ii) If A, B *are formulas, then* $A \wedge B$, $A \,\&\, B$, *and* $A \equiv B$ *are formulas.*

The set of all formulas for the given language J is denoted by F_J.

Implication is a derived connective defined by:

$$A \Rightarrow B := (A \wedge B) \equiv A. \tag{3.1.1}$$

3.1.2 Semantics

In this section we will define semantics of the basic EQ-logic. First of all, we must characterize the set of truth values for basic EQ-logic.

As indicated in the previous chapter, the structure of truth values for basic EQ-logic is formed by a good non-commutative EQ-algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle.$$

Definition 8 (Truth evaluation)
A truth evaluation *is a function* $e : F_J \longrightarrow E$ *defined as follows: If* $p \in F_J$ *is a propositional variable then* $e(p) \in E$*. Furthermore,*

$$\begin{aligned} e(\top) &= \mathbf{1}, \\ e(A \wedge B) &= e(A) \wedge e(B), \\ e(A \,\&\, B) &= e(A) \otimes e(B), \\ e(A \equiv B) &= e(A) \sim e(B) \end{aligned}$$

for all formulas $A, B \in F_J$.

Definition 9 (Tautology)
A formula $A \in F_J$ *is a tautology if* $e(A) = \mathbf{1}$ *for each truth evaluation* $e : F_J \longrightarrow E$.

Remark 3
Despite the non-commutativity of $\otimes$, *our logic has only one implication because it is derived from equivalence (i.e. its interpretation is not residual operation adjoint with multiplication). We are convinced that this is an advantage.*

3.1.3 Logical axioms and inference rules

Now, we will present logical axioms and inference rules of the basic EQ-logic.

Definition 10 (Logical axioms)
The following formulas are logical axioms of the basic EQ-logic:

(EQ1) $(A \equiv \top) \equiv A$

(EQ2) $A \wedge B \equiv B \wedge A$

(EQ3) $(A \bigcirc B) \bigcirc C \equiv A \bigcirc (B \bigcirc C), \quad \bigcirc \in \{\wedge, \&\}$

(EQ4) $A \wedge A \equiv A$

(EQ5) $A \wedge \top \equiv A$

(EQ6) $A \& \top \equiv A$

(EQ7) $\top \& A \equiv A$

(EQ8a) $((A \wedge B) \& C) \Rightarrow (B \& C)$

(EQ8b) $(C \& (A \wedge B)) \Rightarrow (C \& B)$

(EQ9) $((A \wedge B) \equiv C) \& (D \equiv A) \Rightarrow (C \equiv (D \wedge B))$

(EQ10) $(A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$

(EQ11) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$

Note that logical axioms of the basic EQ-logic correspond to the axioms of the EQ-algebra. We only added (EQ1) to be the EQ-logic strong enough. The presence of this so called "goodness" axiom, besides others, makes axioms (E3) and (E7) (after rewriting to the formulas) provable (see the following subsection).

Remark 4

As we already mentioned, logical axioms reflect the algebraic axioms. The latter are originally formulated by the inequality. Thus, it would be possible to set a special short

$$A \preceq B := A \wedge B \equiv A, \tag{3.1.2}$$

rewrite logical axioms using (3.1.2) and then (because EQ-algebra is good), add the following lemma:

Lemma 11

EQ-logic proves $\vdash A \preceq B \equiv A \Rightarrow B$.

Since the notation using implication is relatively natural and the inequality is in EQ-algebra identified with implication, we consider this procedure as unnecessary.

Thus the above axioms (and also axioms in other kinds of EQ-logics presented below in this thesis) are written using $\Rightarrow$*. However, as we will see below, we apply equational-style proofs in the sense of [36], which are sequences of formulas of the form* $A_1 \equiv A_2, \ldots, A_{n-1} \equiv A_n$ *such that each of the individual theorems* $A_i \equiv A_{i+1}$ *has an independent individual proof. Therefore, a fussy presentation would require to rewrite all the axioms using* $\equiv$ *and either definition (3.1.1) or (3.1.2) only.*

Definition 11 (Inference rules)

(EA) *Let* $A, B \in F_J$*. Then, from* A *and* $A \equiv B$ *infer* B.

(Leib) *Let* $A \equiv B$*. Then infer* $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$ *where the expression of the form* $C[\mathbf{p} := X]$ *for* $X := A$ *or* $X := B$ *denotes a formula resulting from* C *by replacing all occurrences of the variable* $\mathbf{p}$ *in* C *by the formula* X.

The rule (EA) is the *equanimity rule* and (Leib) is the *Leibniz rule* for formulas (cf. [36] and elsewhere). Note that formula C in Leibniz rule will be also called as a "C-part" in the proofs below.

Remark 5

Let us stress that the inference rule of modus ponens, denoted by (MP) is derived rule in EQ-logics (see Lemma 14(c)).

The concept of provability and proof are defined in the same way as in classical logic. A formal *theory* is any subset $T \subseteq F_J$. As usual, we suppose that T is defined by a set of special axioms. If $A \in F_J$ is a formula then it is provable in T if there is a proof of A. Then we write $T \vdash A$.

3.1.4 Main properties

As already indicated, we prefer equational-style of proofs to Hilbert style which seems to be more natural for EQ-logics. Note, that equational-style of proof is a sequence of formulas of the form

$$A_1 \equiv A_2, A_2 \equiv A_3, \ldots, A_{n-1} \equiv A_n, A_n \equiv A_{n+1}. \tag{3.1.3}$$

Each of the formulas $A_i \equiv A_{i+1}$ must be either a logical axiom, or an assumption, or it must be proved earlier, or derived using the Leibniz rule (this is often used in the proofs below). The proof consisting of a series of applications of the Leibniz inference rule is linked implicitly by the transitivity. Each step of the proof is completed by an informative annotation which explains how we arrived at the formula $A_i \equiv A_{i+1}$. Observe that it is usual to write the sequence of formulas (3.1.3) by the following form

$$\begin{aligned}
& A_1 \\
\Leftrightarrow & \langle annotation \rangle \\
& A_2 \\
\Leftrightarrow & \langle annotation \rangle \\
& A_3 \\
& \vdots \\
& A_{n-1} \\
\Leftrightarrow & \langle annotation \rangle \\
& A_n \\
\Leftrightarrow & \langle annotation \rangle \\
& A_{n+1}
\end{aligned}$$

where $A_1 \Leftrightarrow A_2 \Leftrightarrow A_3$ means only $A_1 \equiv A_2$ and $A_2 \equiv A_3$.

An equational proof says that its first formula is equivalent to the last one (or vice-versa). We can say it due to transitivity of equivalence which is often used in the equational proofs (see Lemma 12). Thus, the equational proof needs not be built up to the final formula as in the case of Hilbert-style proof; whenever convenient, it can start with it and end up with some known formula. Moreover, none of the inference rules needs to be mentioned explicitly in an equational proof, which reduces the amount of writing when presenting the proof. Consequently, the proofs are more concise and thus, easy to read and remember (for more details see [18] or [36]).

The following lemmas illustrate the main properties of the basic EQ-logic. First of all, we prove a special derived rule (Trans), called transitivity (of equivalence). We prove it in both Hilbert style and equational style. The reader can compare both styles of proofs. However, the subsequent proofs will be equational. We will try to make the explanation concise and not overburdened with too many technical details. Therefore we will shorten the proofs wherever possible, or outline the main ideas only.

Lemma 12
$A \equiv B, B \equiv C \vdash A \equiv C$ *(Trans)*

PROOF: [Hilbert style]

(L.1) $\vdash A \equiv B$ (assumption)

(L.2) $\vdash B \equiv C$ (assumption)

(L.3) $\vdash (A \equiv B) \equiv (A \equiv C)$ (L.2 and (Leib) "C-part": $A \equiv \mathbf{p}$)

(L.4) $\vdash A \equiv C$ (L.1, L.3, (EA))

□

PROOF: [Equational style]

$A \equiv B$ (assumption)

$\Leftrightarrow$ ⟨(Leib) + assumption: $B \equiv C$; "C-part": $A \equiv \mathbf{p}$⟩

$A \equiv C$

This proof in equational style uses Leibniz rule. We apply it on the assumption $B \equiv C$ and obtain $(A \equiv \mathbf{p})[\mathbf{p} := B] \equiv (A \equiv \mathbf{p})[\mathbf{p} := C]$, and hence $(A \equiv B) \equiv (A \equiv C)$ which is put above. Since $A \equiv B$ is assumption (it is consider to be true), formula $A \equiv C$ is thus also true. □

Remark 6
The variable $\mathbf{p}$ *is called fresh in the proofs above. It means that* $\mathbf{p}$ *does not occur in any of* A, B, C. *We will also keep it in mind when* $\mathbf{p}$ *arises in the proofs below.*

Lemma 13
The following are special derived rules:

(a) $A \equiv \top \vdash A$. *(rule (T1))*

(b) $A \vdash A \equiv \top$. *(rule (T2))*

(c) $A \wedge D \equiv C, A \equiv B \vdash B \wedge D \equiv C$. *(rule (C))*

(d) $(A \equiv D) \equiv C, A \equiv B \vdash (B \equiv D) \equiv C$. *(rule (E))*

(e) $A \,\&\, D \equiv C, A \equiv B \vdash B \,\&\, D \equiv C$. *(rule (F1))*

(f) $D \,\&\, A \equiv C, A \equiv B \vdash D \,\&\, B \equiv C$. *(rule (F2))*

PROOF: (a) By the assumption and (EQ1) using equanimity rule.

(b)

$$
\begin{aligned}
& A \equiv \top \\
\Leftrightarrow & \langle \text{axiom (EQ1)} \rangle \\
& A
\end{aligned}
$$

Notice that this equational-style proof starts with formula, which we want to prove. Applying axiom (EQ1) we immediately obtain formula A, i.e. assumption. Thus, we are done.

(c)

$$
\begin{aligned}
& A \wedge D \equiv C \\
\Leftrightarrow & \langle \text{(Leib) + assumption } A \equiv B; \text{ “C-part”: } \mathbf{p} \wedge D \equiv C \rangle \\
& B \wedge D \equiv C
\end{aligned}
$$

(d), (e), (f) are proved in the same way as above. □

Lemma 14
(a) $\vdash A \equiv A$,

(b) $A \equiv B \vdash B \equiv A$,

(c) $A, A \Rightarrow B \vdash B,$ *(Modus Ponens)*

(d) $\vdash (\top \Rightarrow A) \equiv A,$

(e) $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C,$

(f) $A, B \vdash A \,\&\, B,$

(g) $\vdash (A \equiv B) \equiv (B \equiv A),$

(h) $\vdash (A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B),$

(i) $A \Rightarrow (B \equiv C), B \vdash A \Rightarrow C,$

(j) $\vdash (A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B)),$

(k) $A \Rightarrow (B \equiv C), B \equiv D \vdash A \Rightarrow (D \equiv C),$

(l) $A \Rightarrow (B \equiv C), C \equiv D \vdash A \Rightarrow (B \equiv D),$

(m) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B),$

(n) $\vdash (A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D),$

(o) $\vdash (A \Rightarrow B) \& (B \Rightarrow A) \Rightarrow (A \equiv B),$

(p) $A \Rightarrow B, C \Rightarrow D \vdash (A \,\&\, C) \Rightarrow (B \,\&\, D),$

(q) $A \equiv B, C \equiv D \vdash (A \,\&\, C) \equiv (B \,\&\, D),$

(r) $\vdash ((A \equiv B) \equiv C) \& (A \equiv D) \Rightarrow ((D \equiv B) \equiv C),$

(s) $\vdash B \Rightarrow (A \Rightarrow B).$

PROOF: (a)

$A \wedge A \equiv A$ (EQ4)

$\Leftrightarrow \langle$(Leib) + (EQ4); "C-part": $\mathbf{p} \equiv A\rangle$

$A \equiv A$

(b)

$(A \,\&\, \top) \equiv A$ (EQ6)

$\Leftrightarrow \langle$(Leib) + assumption: $A \equiv B$; "C-part": $\mathbf{p} \,\&\, \top \equiv A\rangle$

$(B \,\&\, \top) \equiv A$

$\Leftrightarrow \langle$(Leib) + (EQ6): $B \,\&\, \top \equiv B$; "C-part": $\mathbf{p} \equiv A\rangle$

$B \equiv A$

(c)

A

$\Leftrightarrow \langle$(EQ1) + Lemma 14(b)$\rangle$

$A \equiv \top$

$\Leftrightarrow \langle$(Leib) + assumption: $A \wedge B \equiv A$ + Lemma 14(b); "C-part": $\mathbf{p} \equiv \top\rangle$

$A \wedge B \equiv \top$

$\Leftrightarrow \langle$(Leib) + assumption: A + rule (T2); "C-part": $\mathbf{p} \wedge B \equiv \top\rangle$

$\top \wedge B \equiv \top$

$\Leftrightarrow \langle$(Leib) + (EQ2): $\top \wedge B \equiv B \wedge \top$; "C-part": $\mathbf{p} \equiv \top\rangle$

$B \wedge \top \equiv \top$

$\Leftrightarrow \langle$(Leib) + (EQ5): $B \wedge \top \equiv B$; "C-part": $\mathbf{p} \equiv \top\rangle$

$B \equiv \top$

$\Leftrightarrow \langle$(EQ1)$\rangle$

B

(d)

$\top \wedge A \equiv \top$ (i.e. $\top \Rightarrow A$)

$\Leftrightarrow \langle$(Leib) + (EQ2) ; "C-part": $\mathbf{p} \equiv \top\rangle$

$A \wedge \top \equiv \top$

$\Leftrightarrow \langle$(Leib) + (EQ5); "C-part": $\mathbf{p} \equiv \top\rangle$

$A \equiv \top$

$\Leftrightarrow \langle$(EQ1)$\rangle$

A

(e)

$A \wedge B \equiv A$ (assumption)

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $\mathbf{p} \equiv A\rangle$

$B \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + assumption: $B \wedge C \equiv B$ + Lemma 14(b); "C-part": $\mathbf{p} \wedge A \equiv A\rangle$

$(B \wedge C) \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + (EQ2): $B \wedge C \equiv C \wedge B$; "C-part": $\mathbf{p} \wedge A \equiv A\rangle$

$(C \wedge B) \wedge A \equiv A$

$\Leftrightarrow \langle$(Leib) + (EQ2): $(C \wedge B) \wedge A \equiv A \wedge (C \wedge B)$; "C-part": $\mathbf{p} \equiv A\rangle$

$A \wedge (C \wedge B) \equiv A$

$\Leftrightarrow \langle$(EQ11): $((A \wedge (C \wedge B)) \equiv A) \Rightarrow ((A \wedge C) \equiv A)$ + Lemma 14(c)$\rangle$

$A \wedge C \equiv A$

(f)

A (assumption)

$\Leftrightarrow \langle$(EQ6) + Lemma 14(b)$\rangle$

$A \,\&\, \top$

$\Leftrightarrow \langle$(Leib)+ assumption: B + rule (T2) + Lemma 14(b); "C-part": $A \,\&\, \mathbf{p}\rangle$

$A \,\&\, B$

(g)

$(A \equiv A) \,\&\, (B \equiv B)$ (2x Lemma 14(a) + Lemma 14(f))

$\Leftrightarrow \langle$(EQ10): $(A \equiv A)\,\&(B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv A)$ + Lemma 14(c)$\rangle$

$(A \equiv B) \equiv (B \equiv A)$

(h)

$$((A \wedge C) \equiv (A \wedge C))\,\&((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C)) \qquad \text{(EQ9)}$$

$\Leftrightarrow \langle$(Leib) + Lemma 14(a): $(A \wedge C) \equiv (A \wedge C)$ + rule (T2);
"C-part": $\mathbf{p}\,\&((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))\rangle$

$\top\,\&((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))$

$\Leftrightarrow \langle$(Leib) + (EQ7); "C-part": $\mathbf{p} \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))\rangle$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge B) \wedge C))$

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv \mathbf{p})\rangle$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv (C \wedge (A \wedge B)))$

$\Leftrightarrow \langle$(Leib) + (EQ3); "C-part": $((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv \mathbf{p})\rangle$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((C \wedge A) \wedge B))$

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv (\mathbf{p} \wedge B))\rangle$

$((A \wedge B) \equiv A) \Rightarrow ((A \wedge C) \equiv ((A \wedge C) \wedge B))$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g): $((A \wedge C) \equiv ((A \wedge C) \wedge B)) \equiv ((A \wedge C) \wedge B)) \equiv ((A \wedge C)$;
"C-part": $((A \wedge B) \equiv A) \Rightarrow \mathbf{p}\rangle$

$((A \wedge B) \equiv A) \Rightarrow (((A \wedge C) \wedge B) \equiv (A \wedge C))$
(i.e. $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$)

(i)

$$(A \wedge (B \equiv C)) \equiv A \qquad \text{(assumption)}$$

$\Leftrightarrow \langle$(Leib) + assumption: B + rule (T2); "C-part": $A \wedge (\mathbf{p} \equiv C) \equiv A\rangle$

$A \wedge (\top \equiv C) \equiv A$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g): $(\top \equiv C) \equiv (C \equiv \top)$; "C-part": $A \wedge \mathbf{p} \equiv A\rangle$

$A \land (C \equiv \top) \equiv A$

$\Leftrightarrow \langle$(Leib) + (EQ1): $(C \equiv \top) \equiv C$; "C-part": $A \land \mathbf{p} \equiv A\rangle$

$A \land C \equiv A$

(j)

$$(A \equiv D) \,\&\, (B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv D) \qquad \text{(EQ10)}$$

$\Leftrightarrow \langle$(Leib) + Lemma 14(a): $B \equiv B$ + rule (T2);
"C-part": $(A \equiv D) \,\&\, \mathbf{p} \Rightarrow ((A \equiv B) \equiv (B \equiv D))\rangle$

$(A \equiv D) \,\&\, \top \Rightarrow ((A \equiv B) \equiv (B \equiv D))$

$\Leftrightarrow \langle$(Leib) + (EQ6): $(A \equiv D) \,\&\, \top \equiv (A \equiv D)$;
"C-part": $\mathbf{p} \Rightarrow ((A \equiv B) \equiv (B \equiv D))\rangle$

$(A \equiv D) \Rightarrow ((A \equiv B) \equiv (B \equiv D))$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g): $(B \equiv D) \equiv (D \equiv B)$;
"C-part": $(A \equiv D) \Rightarrow ((A \equiv B) \equiv \mathbf{p})\rangle$

$(A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B))$

(k)

$$(A \land (B \equiv C)) \equiv A \qquad \text{(assumption)}$$

$\Leftrightarrow \langle$(Leib) + assumption: $B \equiv D$; "C-part": $(A \land (\mathbf{p} \equiv C)) \equiv A\rangle$

$(A \land (D \equiv C)) \equiv A$

(l) Can be proven in a similar way as above.

(m)

$$(((A \land A) \equiv A) \,\&\, (B \equiv A)) \land (A \equiv (B \land A)) \equiv ((A \land A) \equiv A) \,\&\, (B \equiv A) \qquad \text{(EQ9)}$$

$\Leftrightarrow \langle$(Leib) + (EQ4) + rule (T2);
"C-part": $(\mathbf{p} \,\&\, (B \equiv A)) \land (A \equiv (B \land A)) \equiv (\mathbf{p} \,\&\, (B \equiv A))\rangle$

$(\top \,\&\, (B \equiv A)) \land (A \equiv (B \land A)) \equiv (\top \,\&\, (B \equiv A))$

$\Leftrightarrow \langle$(Leib) + (EQ7); "C-part": $\mathbf{p} \wedge (A \equiv (B \wedge A)) \equiv \mathbf{p}\rangle$

$(B \equiv A) \wedge (A \equiv (B \wedge A)) \equiv (B \equiv A)$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g); "C-part": $\mathbf{p} \wedge (A \equiv (B \wedge A)) \equiv \mathbf{p}\rangle$

$(A \equiv B) \wedge (A \equiv (B \wedge A)) \equiv (A \equiv B)$

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $(A \equiv B) \wedge (A \equiv \mathbf{p}) \equiv (A \equiv B)\rangle$

$(A \equiv B) \wedge (A \equiv (A \wedge B)) \equiv (A \equiv B)$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g); "C-part": $(A \equiv B) \wedge \mathbf{p} \equiv (A \equiv B)\rangle$

$(A \equiv B) \wedge ((A \wedge B) \equiv A) \equiv (A \equiv B)$ (i.e. $(A \equiv B) \Rightarrow (A \Rightarrow B)$)

(n)

$(A \equiv B) \,\&\, (C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$ (EQ10)

$\Leftrightarrow \langle$(Leib) + Lemma 14(g): $(D \equiv B) \equiv (B \equiv D)$;
"C-part": $(A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \equiv C) \equiv \mathbf{p})\rangle$

$(A \equiv B) \,\&\, (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D)$

(o) Using Lemma 14(n) in the form

$$\vdash ((A \wedge B) \equiv A) \,\&\, ((B \wedge A) \equiv B) \Rightarrow ((A \wedge B) \equiv (B \wedge A)) \equiv (A \equiv B)$$

and (EQ2) as assumptions in Lemma 14(i) we obtain

$$\vdash ((A \wedge B) \equiv A) \,\&\, ((B \wedge A) \equiv B) \Rightarrow (A \equiv B)$$

and thus

$$\vdash (A \Rightarrow B) \,\&\, (B \Rightarrow A) \Rightarrow (A \equiv B).$$

(p) First, we use (EQ8a) $\vdash ((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$ and the Leibniz rule with the assumption $\vdash (A \wedge B) \equiv A$ to obtain $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, C)$. Then we use (EQ8b) $\vdash (B \,\&\, (C \wedge D)) \Rightarrow (B \,\&\, D)$ and the Leibniz rule again with the second assumption $\vdash (C \wedge D) \equiv C$ to obtain $\vdash (B \,\&\, C) \Rightarrow (B \,\&\, D)$. Finally, we use these results as the assumptions in Lemma 14(e) to obtain $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, D)$.

(q) From the assumptions, Lemma 14(m) and (c) we obtain $\vdash A \Rightarrow B$ and $\vdash C \Rightarrow D$ and thus, from Lemma 14(p) we find $\vdash (A \,\&\, C) \Rightarrow (B \,\&\, D)$. In the same way and, moreover, using Lemma 14(g) we obtain $\vdash B \Rightarrow A$ and $\vdash D \Rightarrow C$ and thus, from Lemma 14(p) we get $\vdash (B \mathbin{\&} D) \Rightarrow (A \mathbin{\&} C)$. Finally, form the previous formulas and Lemma 14(f), (o) and (c) we have $\vdash (A \mathbin{\&} C) \equiv (B \mathbin{\&} D)$.

(r) Using the assumptions $\vdash ((A \equiv B) \equiv C) \Rightarrow ((A \equiv B) \equiv C)$ (EQ4) and $\vdash (A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B))$ (Lemma 14(j)) in Lemma 14(p) we obtain

$$\vdash (((A \equiv B) \equiv C) \mathbin{\&} (A \equiv D)) \Rightarrow ((((A \equiv B) \equiv C) \mathbin{\&} ((A \equiv B) \equiv (D \equiv B))),$$

which together with axiom (EQ10) in the form

$$\vdash (((A \equiv B) \equiv C) \mathbin{\&} ((A \equiv B) \equiv (D \equiv B))) \\ \Rightarrow (((A \equiv B) \equiv (A \equiv B)) \equiv ((D \equiv B) \equiv C))$$

yields (by Lemma 14(e)) the formula

$$\vdash (((A \equiv B) \equiv C) \mathbin{\&} (A \equiv D)) \Rightarrow (((A \equiv B) \equiv (A \equiv B)) \equiv ((D \equiv B) \equiv C)).$$

Now we use Lemma 14(i) with the assumption of Lemma 14(a) and formula above to get

$$\vdash ((A \equiv B) \equiv C) \mathbin{\&} (A \equiv D) \Rightarrow ((D \equiv B) \equiv C).$$

(s)

$(\top \Rightarrow B) \Rightarrow ((\top \wedge A) \Rightarrow B)$ (Lemma 14(h))

$\Leftrightarrow \langle$(Leib) + Lemma 14(d); "C-part": $\mathbf{p} \Rightarrow ((\top \wedge A) \Rightarrow B)\rangle$

$B \Rightarrow ((\top \wedge A) \Rightarrow B)$

$\Leftrightarrow \langle$(Leib) + (EQ2); "C-part": $B \Rightarrow (\mathbf{p} \Rightarrow B)\rangle$

$B \Rightarrow ((A \wedge \top) \Rightarrow B)$

$\Leftrightarrow \langle$(Leib) + (EQ5); "C-part": $B \Rightarrow (\mathbf{p} \Rightarrow B)\rangle$

$B \Rightarrow (A \Rightarrow B)$

□

Lemma 15

(a) $\vdash (A \equiv B) \mathbin{\&} (B \equiv C) \Rightarrow (A \equiv C)$,

(b) $\vdash (A\,\&(A \equiv B)) \Rightarrow B$,

(c) $\vdash (A \Rightarrow B)\,\&(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$,

(d) $\vdash (A\,\&(A \Rightarrow B)) \Rightarrow B$,

(e) $\vdash (A\,\&\,B) \Rightarrow A$,

(f) $\vdash (A\,\&\,B) \Rightarrow B$,

(g) $\vdash B \Rightarrow (B \equiv \top)$,

(h) $A \Rightarrow (B \Rightarrow C) \vdash (A\,\&\,B) \Rightarrow C$,

(i) $\vdash A\,\&\,B \Rightarrow A \equiv B$,

(j) $\vdash (A \wedge B) \Rightarrow A$,

(k) $\vdash (C \Rightarrow A)\,\&(C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B))$,

(l) $\vdash (A\,\&\,B) \Rightarrow (A \wedge B)$,

(m) $\vdash (A \equiv B) \Rightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$,

(n) $\vdash A \Rightarrow ((A \equiv B) \equiv B)$,

(o) $\vdash A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$.

PROOF: (a) Follows from (EQ9) in the form

$$\vdash ((B \wedge \top) \equiv A)\,\&(C \equiv B) \Rightarrow (A \equiv (C \wedge \top))$$

and by clear modifications using the Leibniz rule.

(b) Using Lemma 15(a) in the form

$$\vdash (\top \equiv A)\,\&(A \equiv B) \Rightarrow (\top \equiv B),$$

and twice use of Leibniz rule with (EQ1) as assumptions.

(c) We start with (EQ9) in the form

$$\vdash ((B \wedge A) \equiv A)\,\&((B \wedge C) \equiv B) \Rightarrow (A \equiv ((B \wedge C) \wedge A))$$

and by easy sequence of Leibniz rule we get formula

$$\vdash (A \Rightarrow B)\,\&(B \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C)),$$

which together with (EQ11) and Lemma 14(e) yields (c).

(d) Immediately from Lemma 15(c), $\vdash (\top \Rightarrow A) \,\&\, (A \Rightarrow B) \Rightarrow (\top \Rightarrow B)$ and the Leibniz rule used twice with Lemma 14(d) as the assumption.

(e) Follows from (EQ8b) in a form $\vdash (A \,\&\, (B \wedge \top)) \Rightarrow (A \,\&\, \top)$, using Leibniz rule twice with (EQ5) and (EQ6).

(f) In the same way as above using (EQ8a).

(g)

$$B \wedge B \equiv B \qquad \text{(EQ4)}$$

$\Leftrightarrow \langle$(Leib)+ (EQ1): $(B \equiv \top) \equiv B$ + Lemma 14(b); "C-part": $(B \wedge \mathbf{p}) \equiv B\rangle$

$$(B \wedge (B \equiv \top)) \equiv B \qquad \text{(i.e. } B \Rightarrow (B \equiv \top))$$

(h) Using the assumption and Lemmas 15(g), 14(p) we derive

$$\vdash A \,\&\, B \Rightarrow ((B \wedge C) \equiv B) \,\&\, (B \equiv \top),$$

from here and by Lemma 15(a) in the form

$$\vdash ((B \wedge C) \equiv B) \,\&\, (B \equiv \top) \Rightarrow ((B \wedge C) \equiv \top)$$

we get

$$\vdash A \,\&\, B \Rightarrow ((B \wedge C) \equiv \top).$$

by Lemma 14(e). Now, we derive in an obvious way

$$\vdash A \,\&\, B \Rightarrow (((\top \wedge C) \wedge B) \equiv \top).$$

By (EQ11) we have

$$\vdash (((\top \wedge C) \wedge B) \equiv \top) \Rightarrow ((\top \wedge C) \equiv \top)$$

and thus

$$\vdash A \,\&\, B \Rightarrow ((\top \wedge C) \equiv \top)$$

or

$$\vdash A \,\&\, B \Rightarrow (\top \Rightarrow C).$$

Finally

$$A \,\&\, B \Rightarrow (\top \Rightarrow C)$$

$\Leftrightarrow \langle$(Leib)+ Lemma 14(d): $(\top \Rightarrow C) \equiv C$; "C-part": $A \,\&\, B \Rightarrow \mathbf{p})\rangle$

$A \,\&\, B \Rightarrow C$

(i) Obviously follows from (EQ10):

$$\vdash (A \equiv \top) \,\&\, (B \equiv \top) \Rightarrow (A \equiv B) \equiv (\top \equiv \top)$$

(j) From Lemma 14(a): $\vdash (A \wedge B) \equiv (A \wedge B)$ using (EQ4) and rule (C) we get $\vdash ((A \wedge A) \wedge B) \equiv (A \wedge B)$. We finish this proof using the Leibniz rule twice with assumptions (EQ3) and (EQ2) and also using (3.1.1).

(k) Follows from (EQ9) in the form

$$\vdash ((C \wedge A) \equiv C) \,\&\, ((C \wedge B) \equiv C) \Rightarrow (C \equiv ((C \wedge B) \wedge A))$$

and easy application of the Leibniz rule.

(l) First of all we use Lemma 15(e), (f) and Lemma 14(f) and then we finish this proof using Lemma 15(k) and 14(c).

(m) Since $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$ by Lemma 14(m) and $\vdash (A \equiv B) \Rightarrow (B \Rightarrow A)$ by Lemma 14(m), (g) using Leibniz rule, we get $\vdash ((A \equiv B) \Rightarrow (A \Rightarrow B)) \,\&\, ((A \equiv B) \Rightarrow (B \Rightarrow A))$ (see Lemma 14(f)) and thus $\vdash (A \equiv B) \Rightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$ by Lemma 15(k) and 14(c).

(n) We start with (EQ10) in the form $\vdash (A \equiv \top) \,\&\, (B \equiv B) \Rightarrow (A \equiv B) \equiv (B \equiv \top)$. By easy using Leibniz rule and (EQ1), Lemma 14(a), rule (T2) and (EQ6) we complete the proof.

(o) Using Lemma 15(n) in the form $\vdash A \Rightarrow ((A \equiv (A \wedge B)) \equiv (A \wedge B))$, definition of implication, Lemma 14(m) and (e) we obtain $\vdash A \Rightarrow ((A \Rightarrow B) \Rightarrow (A \wedge B))$. (EQ11) and Lemma 14(e) complete the proof. □

Lemma 16

(a) $A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$,

(b) $A \Rightarrow B \vdash (C \Rightarrow A) \Rightarrow (C \Rightarrow B)$,

(c) $\vdash (A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$,

(d) $\vdash (A \equiv D) \Rightarrow (((A \wedge B) \equiv C) \equiv ((D \wedge B) \equiv C))$,

(e) $\vdash (A \equiv D) \Rightarrow ((B \Rightarrow A) \equiv (B \Rightarrow D))$,

(f) $\vdash (A \Rightarrow D) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow D))$,

(g) $\vdash ((A \equiv B) \mathbin{\&} (C \equiv D)) \Rightarrow ((A \wedge C) \equiv (B \wedge D))$,

(h) $\vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$,

(i) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$.

PROOF: (a)

$(B \Rightarrow C) \Rightarrow ((B \wedge A) \Rightarrow C)$ (Lemma 14(h))

$\Leftrightarrow \langle$(Leib) +(EQ2); "C-part": $(B \Rightarrow C) \Rightarrow (\mathbf{p} \Rightarrow C)\rangle$

$(B \Rightarrow C) \Rightarrow ((A \wedge B) \Rightarrow C)$

$\Leftrightarrow \langle$(Leib) + assumption $\vdash (A \wedge B) \equiv A$; "C-part": $(B \Rightarrow C) \Rightarrow (\mathbf{p} \Rightarrow C)\rangle$

$(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(b) In a similar way as (a), only use (EQ11).

(c)

$((A \wedge C) \equiv (A \wedge C)) \mathbin{\&} (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$ (EQ9)

$\Leftrightarrow \langle$(Leib) + Lemma 14(a) + rule (T2); "C-part":
$\mathbf{p} \mathbin{\&} (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))\rangle$

$\top \mathbin{\&} (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$

$\Leftrightarrow \langle$(Leib) + (EQ7); "C-part": $\mathbf{p} \Rightarrow ((A \wedge C) \equiv (B \wedge C))\rangle$

$(B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$

$\Leftrightarrow \langle$(Leib) + Lemma 14(g); "C-part": $\mathbf{p} \Rightarrow ((A \wedge C) \equiv (B \wedge C))\rangle$

$(A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$

(d)

$(A \equiv D) \Rightarrow ((A \wedge B) \equiv (D \wedge B))$ (Lemma 16(c))

$\Leftrightarrow \langle$(Leib) +(EQ6) + rule (T2); "C-part": $(A \equiv D) \Rightarrow \mathbf{p}\rangle$

$(A \equiv D) \Rightarrow ((A \wedge B) \equiv (D \wedge B)) \,\&\, \top$

$\Leftrightarrow$ $\langle$(Leib) + Lemma 14(a)+ rule (T2); "C-part": $(A \equiv D) \Rightarrow ((A \wedge B) \equiv (D \wedge B)) \,\&\, \mathbf{p}\rangle$

$(A \equiv D) \Rightarrow ((A \wedge B) \equiv (D \wedge B)) \,\&(C \equiv C)$

Since Lemma 14(n) yields

$$\vdash ((A \wedge B) \equiv (D \wedge B)) \,\&(C \equiv C) \Rightarrow (((A \wedge B) \equiv C) \equiv ((D \wedge B) \equiv C)),$$

then using Lemma 14(e) we have

$$\vdash (A \equiv D) \Rightarrow (((A \wedge B) \equiv C) \equiv ((D \wedge B) \equiv C)).$$

(e) Immediately from Lemma 16(d)

$$\vdash (A \equiv D) \Rightarrow (((A \wedge B) \equiv B) \equiv ((D \wedge B) \equiv B))$$

using (Leib), (EQ2) and definition of $\Rightarrow$.

(f)

(L.1) $\vdash (A \equiv (A \wedge D)) \Rightarrow ((B \Rightarrow A) \equiv (B \Rightarrow (A \wedge D)))$ (Lemma 16(e))

(L.2) $\vdash ((B \Rightarrow A) \equiv (B \Rightarrow (A \wedge D))) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow (A \wedge D)))$ (Lemma 14(m))

(L.3) $\vdash (A \Rightarrow D) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow (A \wedge D)))$ (Definition of $\Rightarrow$; L.1, L.2, Lemma 14(e))

(L.4) $\vdash ((A \Rightarrow D) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow (A \wedge D))) \Rightarrow ((A \Rightarrow D) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow D)))$ (Lemma 15(j), triple Lemma 16(b))

(L.5) $\vdash (A \Rightarrow D) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow D))$ (L.3, L.4, Lemma 14(c))

(g) First we use Lemma 16(c) $\vdash (A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$. Then we use Lemma 16(c) again and also (EQ2) and the Leibniz rule to obtain $\vdash (C \equiv D) \Rightarrow ((B \wedge C) \equiv (B \wedge D))$. We use both formulas as assumptions in Lemma 14(p) to get $\vdash ((A \equiv B) \,\&(C \equiv D)) \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \,\&((B \wedge C) \equiv (B \wedge D))$. Finally using Lemma 15(a) and 14(e) we obtain Lemma 16(g).

(h) From Lemma 14(j) we have

$$\vdash ((A \wedge B) \equiv A) \Rightarrow (((A \wedge B) \equiv ((A \wedge B) \wedge C)) \equiv (A \equiv ((A \wedge B) \wedge C))).$$

By definition of $\Rightarrow$, (Leib), Lemma 14(g), (EQ2), (EQ3) we get

$$\vdash (A \Rightarrow B) \Rightarrow (((A \wedge B) \Rightarrow C) \equiv (A \Rightarrow (B \wedge C))).$$

Thus, this and Lemma 14(m) using Lemma 14(e) we obtain

$$\vdash (A \Rightarrow B) \Rightarrow (((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))). \tag{3.1.4}$$

By Lemma 15(j) + (Leib) + (EQ2) and twice Lemma 16(a) we get

$$\vdash (((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))). \tag{3.1.5}$$

Similarly, using twice Lemma 16(b) we deduce

$$\vdash ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)). \tag{3.1.6}$$

Thus, (3.1.5), (3.1.6) and Lemma 14(e) yield

$$\vdash (((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)),$$

Lemma 16(b) yield

$$\vdash ((A \Rightarrow B) \Rightarrow (((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C)))) \Rightarrow \\ ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$$

and finally, this and (3.1.4) using Lemma 14(c) complete the proof.

(i)

(L.1) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (((B \Rightarrow C) \Rightarrow C) \Rightarrow (A \Rightarrow C))$ (Lemma 16(h))

(L.2) $\vdash B \Rightarrow ((B \Rightarrow C) \Rightarrow C)$ (Lemma 15(o))

(L.3) $\vdash (B \Rightarrow ((B \Rightarrow C) \Rightarrow C)) \Rightarrow ((((B \Rightarrow C) \Rightarrow C) \Rightarrow (A \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$ (Lemma 16(h))

(L.4) $\vdash (((B \Rightarrow C) \Rightarrow C) \Rightarrow (A \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (L.2, L.3, Lemma 14(c))

(L.5) $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (L.1, L.4, Lemma 14(e))

□

It should be noted that the commutativity of **&** in general does not hold. On the other hand, however, its commutativity holds in a weaker form as the following lemma shows.

Lemma 17
$A \equiv B \vdash A \,\&\, B \equiv B \,\&\, A$

PROOF: This follows from the assumption, Lemma 14(b) and (q). □

3.1.5 Completeness

In this subsection we show that the basic EQ-logic is rich enough to enjoy the completeness property. Its proof is based on using of the well known Lindenbaum-Tarski technique.

Lemma 18
All axioms of the basic EQ-logic are tautologies.

PROOF: (EQ1)–(EQ7) are obviously tautologies. To verify (EQ8a) and (EQ8b) we use the property $a \wedge b \leq b$, axiom (E2) and Lemma 2(a). To verify (EQ9)–(EQ11) observe (E4)–(E6), Lemma 2(a) and also Theorem 1(a) in the proof of (EQ10). □

Lemma 19
The inference rules of basic EQ-logic are sound in the following sense: Let $e : F_J \longrightarrow E$ be a truth evaluation where E is a support of a good non-commutative EQ-algebra:

(a) If $e(A) = \mathbf{1}$ and $e(A \equiv B) = \mathbf{1}$ then $e(B) = \mathbf{1}$.

(b) If $e(B \equiv C) = \mathbf{1}$ then $e(A[\mathbf{p} := B] \equiv A[\mathbf{p} := C]) = \mathbf{1}$ for any formula A.

PROOF: (a) If $a = 1$ and $a \sim b = 1$ then necessarily $b = 1$ (from E9).

(b) By induction on the complexity of the formula A. If A is either $\top$ or $\mathbf{q}$ (other than $\mathbf{p}$) then $e(A[\mathbf{p} := B] \equiv A[\mathbf{p} := C]) = e(A \equiv A) = \mathbf{1}$. If on the other hand A is $\mathbf{p}$ then $e(A[\mathbf{p} := B] \equiv A[\mathbf{p} := C]) = e(B \equiv C) = \mathbf{1}$.

For induction step, we choose an arbitrary nonatomic A and prove

$$e(A[\mathbf{p} := B] \equiv A[\mathbf{p} := C]) = \mathbf{1}$$

that is

$$e(A[\mathbf{p} := B]) \sim e(A[\mathbf{p} := C]) = \mathbf{1}$$

and thus, on the basis of property (2.1.6) and the fact that every good algebra is separated we conclude

$$e(A[\mathbf{p} := B]) = e(A[\mathbf{p} := C]) \tag{3.1.7}$$

using the induction hypothesis (I.H.) that the claim $e(D[\mathbf{p} := B]) = e(D[\mathbf{p} := C])$ is true for all formulas with complexity smaller than A.

Let A be $E \wedge F$ and I.H. applies to E and F. Now, $A[\mathbf{p} := B] \equiv A[\mathbf{p} := C])$ implies $e(E[\mathbf{p} := B] \wedge F[\mathbf{p} := B] \equiv E[\mathbf{p} := C] \wedge F[p := C])$ and thus, we get (3.1.7) as follows:

$$\begin{aligned} e(E[\mathbf{p} := B] \wedge F[\mathbf{p} := B]) &= e(E[\mathbf{p} := B]) \wedge e(F[\mathbf{p} := B]) \\ &= e(E[\mathbf{p} := C]) \wedge e(F[\mathbf{p} := C]) \text{ (by I.H.)} \\ &= e(E[\mathbf{p} := C] \wedge F[\mathbf{p} := C]) \end{aligned}$$

The cases where A is $E \,\&\, F$ or $E \equiv F$ are proved analogously. □

The following is the standard Lindenbaum-Tarski technique. We use it to show that classes of provably equivalent formulas form a good non-commutative EQ-algebra (Lindenbaum algebra).

Definition 12
Put

$$A \approx B \quad \text{iff} \quad \vdash A \equiv B, \quad A, B \in F_J.$$

It follows from Lemmas 14(a), (g) and 15(a) that $\approx$ *is an equivalence relation on* F_J*. Let us denote by* $[A]$ *an equivalence class of* A *and put* $\bar{E} = \{[A] \mid A \in F_J\}$*. Finally we define operations on the set* $\bar{E}$

$$\begin{aligned} \mathbf{1} &= [\top], \\ [A] \wedge [B] &= [A \wedge B], \\ [A] \otimes [B] &= [A \,\&\, B], \\ [A] \sim [B] &= [A \equiv B]. \end{aligned}$$

Lemma 20
The algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$ is a good non-commutative EQ-algebra.

PROOF: The "goodness property" follows from (EQ1). For the properties of $\wedge$ see axioms (EQ2)–(EQ5). Note that we have

$$[A] \leq [B] \quad \text{iff} \quad [A] \wedge [B] = [A] \quad \text{iff} \quad \vdash (A \boldsymbol{\wedge} B) \equiv A$$
$$\text{iff} \quad \vdash A \Rightarrow B \quad \text{iff} \quad \vdash (A \Rightarrow B) \equiv \top \quad \text{iff} \quad [A] \to [B] = [\top].$$

For the associativity and isotonicity of $\otimes$, and existence of the neutral element see axioms (EQ3) and (EQ6)–(EQ8b). Axiom (E3) follows from Lemma 14(a), (E4) from (EQ9), (E5) from Lemma 14(n), (E6) from (EQ11) and (E7) from Lemma 15(i). □

Theorem 6 (Soundness)
The basic EQ-logic is sound.

PROOF: This is a consequence of Lemmas 18 and 19. □

Theorem 7 (Completeness)
The following is equivalent for every formula A:

(a) $\vdash A$,

(b) $e(A) = \mathbf{1}$ *for every good non-commutative EQ-algebra* $\mathcal{E}$ *and a truth evaluation* $e : F_J \longrightarrow E$.

PROOF: The implication (a) to (b) is soundness.

(b) to (a): By Lemma 20 the algebra $\bar{\mathcal{E}}$ of equivalence classes of formulas is a good non-commutative EQ-algebra. Thus, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). □

3.2 Extensions of the basic EQ-logic

This section will discuss some more special propositional EQ-logics, namely involutive EQ-logic (IEQ-logic), prelinear logic and finally residuated EQ-logic, which is equivalent to MTL-logic.

3.2.1 Involutive EQ-logic

This logic is characteristic by keeping the law of double negation and thus, the contraposition property. Therefore, we modify the language J of the basic EQ-logic by replacing the logical constant $\top$ by $\bot$ (falsum). Furthermore, we introduce the following shorts of formulas:

$$\top := \bot \equiv \bot, \tag{3.2.1}$$

$$\neg A := A \equiv \bot, \tag{3.2.2}$$

$$A \vee B := \neg(\neg A \wedge \neg B). \tag{3.2.3}$$

Formula (3.2.2) is the definition of *negation* and (3.2.3) the definition of *disjunction* in this logic.

Logical axioms of IEQ-logic are the same as in basic EQ-logic plus the following ones:

(EQ12) $(A \wedge \bot) \equiv \bot$ (ex falso quodlibet)

(EQ13) $\neg\neg A \equiv A$ (double negation)

Axiom (EQ12) characterizes the basic property of $\bot$ and we can also write it as $\bot \Rightarrow A$.

The fact that we introduced the double negation property in this logic makes it richer. For example, the disjunction is naturally introduced by (3.2.3) and all its expected properties can be easily proved. Let us also remark that the contraposition and double negation are often used by people when reasoning.

Lemma 21

(a) $\vdash (A \equiv B) \equiv (\neg A \equiv \neg B)$,

(b) $\vdash ((A \vee B) \equiv C) \,\&\, (D \equiv A) \Rightarrow (D \vee B) \equiv C$.

PROOF: (a) Follows from twice using of Lemma 14(n) in the forms

$$\vdash (A \equiv B) \,\&\, (\bot \equiv \bot) \Rightarrow (A \equiv \bot) \equiv (B \equiv \bot)$$

and

$$\vdash ((A \equiv \bot) \equiv (B \equiv \bot)) \,\&\, (\bot \equiv \bot) \Rightarrow ((A \equiv \bot) \equiv \bot) \equiv ((B \equiv \bot) \equiv \bot),$$

few simple modifications ((Leib), (3.2.1), (EQ6), (3.2.2) and (EQ13)) and finally using Lemma 14(f), (o) and (c).

(b) From axiom (EQ9) in the form

$$\vdash ((\neg A \wedge \neg B) \equiv \neg C) \,\&\, (\neg D \equiv \neg A) \Rightarrow (\neg C \equiv (\neg D \wedge \neg B))$$

multiple using the Leibniz rule and Lemma 21(a) we get

$$\vdash (\neg(\neg A \wedge \neg B) \equiv \neg\neg C) \,\&\, (D \equiv A) \Rightarrow (\neg\neg C \equiv \neg(\neg D \wedge \neg B)).$$

We complete the proof using Leibniz rule, (3.2.3), (EQ13) and Lemma 14(g). □

A contradiction is the formula $A \,\&\, \neg A$. Then we say that a theory T is *contradictory* if $T \vdash A \,\&\, \neg A$ for some formula $A \in F_J$. The following theorem demonstrates that contradictory theories in EQ-logic behave classically.

Theorem 8
A theory T is contradictory iff $T \vdash A$ for all $A \in F_{J(T)}$.

PROOF: If T is contradictory then $T \vdash A \,\&\, \neg A$, i.e. $T \vdash A \,\&\, (A \equiv \bot)$. Since $T \vdash (A \,\&\, (A \equiv \bot)) \Rightarrow \bot$ (Lemma 15(b)), using (Lemma 14(c)) we obtain $T \vdash \bot$. Let $B \in F_{J(T)}$ is an arbitrary formula. Then from $T \vdash \bot$ using (EQ12) and (Lemma 14(c)) we get $T \vdash B$.

Conversely, assume $T \vdash A$ and $T \vdash \neg A$. Then by Lemma 14(f) it follows that $T \vdash A \,\&\, \neg A$, thus the theory T is contradictory. □

Semantics of IEQ-logic is formed by non-commutative IEQ-algebras.

Theorem 9 (Completeness)
The following is equivalent for every formula A:

(a) $\vdash A$,

(b) $e(A) = \mathbf{1}$ *for every IEQ-algebra $\mathcal{E}$ and a truth evaluation $e : F_J \longrightarrow E$.*

PROOF: The implication (a) to (b): It is easy to verify that axioms (EQ12)–(EQ13) are tautologies. The remaining part of proof follows from soundness of basic EQ-logic.

(b) to (a): Form algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$ of equivalence classes of formulas. It is obvious $\bar{\mathcal{E}}$ is IEQ-algebra. Now, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). □

3.2.2 Prelinear EQ-logic

This logic seems to be closest to the residuated fuzzy logics in the sense that a stronger variant of the completeness theorem holds in it. The language of this logic is the same as that of basic EQ-logic extended by a short

$$A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A). \tag{3.2.4}$$

Formula (3.2.4) defines *disjunction* in this logic.

The axioms are the same as in the basic EQ-logic plus the following one:

(EQ14) $(A \Rightarrow B) \vee (D \Rightarrow (D \,\&(C \Rightarrow ((B \Rightarrow A) \,\& \, C))))$.

Remark 7
Axiom (EQ14) stands for prelinearity. Indeed, just put $C := D := \top$ and use clear modifications.

Semantics of this logic is formed by good non-commutative EQ-algebras in which condition (2.1.9) is satisfied.

Theorem 10 (Completeness)
For every formula $A \in F_J$ and for every theory T the following is equivalent:

(a) $T \vdash A$.

(b) $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every linearly ordered good non-commutative EQ-algebra* $\mathcal{E}$.

(c) $e(A) = \mathbf{1}$ *for every truth evaluation* $e : F_J \longrightarrow E$ *and every good non-commutative EQ-algebra* $\mathcal{E}$ *satisfying condition (2.1.9).*

PROOF: (a) $\Rightarrow$ (c): All axioms of T are true in all models of T (axiom (EQ14) is a tautology because of the property (2.1.9)).

(c) $\Rightarrow$ (a): If (c) holds then it holds also for $e : F_J \longrightarrow \bar{E}$, where $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1}\rangle$ is a good non-commutative EQ-algebra satisfying (2.1.9) (the Lindenbaum algebra). If $e(A) = \mathbf{1}$ then $[A] = [\top]$, i.e. $T \vdash A \equiv \top$ and so, $T \vdash A$ by rule (T1).

(c) $\Rightarrow$ (b): According to Definition 12 we use the Lindenbaum-Tarski technique to construct the algebra $\bar{\mathcal{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1}\rangle$. This algebra satisfies (2.1.9) (see (EQ14)), hence it is representable by Theorem 3. Thus we also have (b) $\Rightarrow$ (c). □

3.2.3 EQ(MTL)-logic

In this subsection we demonstrate how EQ-logic can be extended to make it equivalent with MTL-logic. The language J of EQ(MTL)-logic is the same as that of basic EQ-logic with the exception that $\top$ is replaced by $\bot$. The truth is defined by (3.2.1) and disjunction by (3.2.4). Its axioms are (EQ1)–(EQ11) plus the following:

(EQ15) $((A \mathbin{\&} B) \Rightarrow C) \equiv (A \Rightarrow (B \Rightarrow C))$

(EQ16) $(A \wedge \bot) \equiv \bot$

(EQ17) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

In the following lemma, we demonstrate that **&** is commutative in the EQ(MTL)-logic.

Lemma 22
$\vdash (A \mathbin{\&} B) \equiv (B \mathbin{\&} A)$.

PROOF: The equivalence will be obtained when proving the implications left to right and its opposite. Let's prove the first one:

(L.1) $\vdash (A \mathbin{\&} B) \Rightarrow (A \mathbin{\&} B)$ ((EQ4), (3.1.1))

(L.2) $\vdash A \Rightarrow (B \Rightarrow (A \mathbin{\&} B))$ (L.1, (EQ15), (EA))

(L.3) $\vdash B \Rightarrow (A \Rightarrow (A \mathbin{\&} B))$ (Lemma 16(i), L.2, (MP))

(L.4) $\vdash (B \Rightarrow (A \Rightarrow (A \mathbin{\&} B))) \equiv ((B \mathbin{\&} A) \Rightarrow (A \mathbin{\&} B))$
((EQ15), Lemma 14(g), (EA))

(L.5) $\vdash (B \mathbin{\&} A) \Rightarrow (A \mathbin{\&} B))$ (L.3, L.4, (EA))

In the same way we prove the converse implication. Both implications, Lemma 14(f), (o) and (c) complete the proof. □

Theorem 11
EQ(MTL)-logic is equivalent with MTL-logic.

PROOF: We will first show that axioms and rules of MTL-logic are provable from axioms and rules of EQ(MTL)-logic. Recall that implication is defined by (3.1.1).

Axiom (A1) is Lemma 16(h). We do not have to prove axiom (A2), because it is provable from the other ones (see [5]). (A3) follows from Lemma 22 (see also [7]). (A4) is Lemma 15(j). (A5) follows from (EQ2), Lemma 14(m) and Lemma 14(c), (A6) from Lemma 15(e), 15(d) and 14(f) and next Lemma 15(k) and 14(c), (A7a) and (A7b) from (EQ15), Lemma 14(m) and 14(c). Axiom (A8) is just (EQ17) and (A9) is (EQ16). Of course, the modus ponens (MP) is the derived rule in EQ(MTL)-logic.

Second, we will show, that axioms and inference rules of EQ(MTL)-logic are provable in MTL-logic. In the proof, we will use the properties of MTL-logic mentioned in Section 1.2.

Recall that the role of equivalence is in MTL taken by the bi-implication $A \Leftrightarrow B := (A \Rightarrow B) \mathbin{\&} (B \Rightarrow A)$. Hence, the EQ-axioms proved below hold with respect to the latter and we take $A \equiv B := A \Leftrightarrow B$. To prove the required equivalences in the EQ-axioms, we prove the corresponding two implications and then put them together by (T5) and twice use of (MP).

(EQ1):

(L.1) $\vdash (A \equiv \top) \Rightarrow (\top \Rightarrow A)$ ((T18), (T20), (A1), 2x (MP))

(L.2) $\vdash ((\top \Rightarrow A) \Rightarrow (\top \Rightarrow A)) \Rightarrow (\top \Rightarrow ((\top \Rightarrow A) \Rightarrow A))$ (T2)

(L.3) $\vdash (\top \Rightarrow A) \Rightarrow A$ ((T3), L.2, (MP); (T13), (MP))

(L.4) $\vdash (A \equiv \top) \Rightarrow A$ (L.1, L.3, (A1), 2x (MP))

For the proof of the converse implication we first need to prove the following property:

$$\vdash (A \mathbin{\&} B) \Rightarrow (A \equiv B). \qquad (3.2.5)$$

Thus

(L.1) $\vdash (A \Rightarrow (B \Rightarrow A)) \mathbin{\&} (B \Rightarrow (A \Rightarrow B))$ (2x (T1), (T5), 2x (MP))

(L.2) $\vdash (A \mathbin{\&} B) \Rightarrow (B \Rightarrow A) \mathbin{\&} (A \Rightarrow B)$ (L.1, (T7), (MP))

The proof of (3.2.5) is completed using definition of equivalence. Thus, the second implication of (EQ1) can be proved as follows:

(L.1) $\vdash A \Rightarrow (A \mathbin{\&} \top)$ ((T14), (A3), (A1), 2x (MP))

(L.2) $\vdash (A \,\&\, \top) \Rightarrow (A \equiv \top)$ (3.2.5)

(L.3) $\vdash A \Rightarrow (A \equiv \top)$ (L.1, L.2, (A1), 2x (MP))

(EQ2): This follows from twice use of (A5), (T5) and twice (MP).

(EQ3): Associativity of $\wedge$ and $\&$ follow from (T15) and (T16), respectively.

(EQ4): It follows from (T8), (T12), (T5) and twice (MP).

(EQ5): We immediately get one implication from (A5),(T8),(A1) and twice (MP). For the second implication:

(L.1) $\vdash A \Rightarrow (A \,\&\, \top)$ ((T14), (A3), (A1), 2x (MP))

(L.2) $\vdash A \Rightarrow (A \wedge \top)$ (L.1, (T9), (A1), 2x (MP))

(EQ6): One implication is just (A2). The converse implication follows from (T14), (A3), (A1) and twice (MP).

(EQ7): One implication is just (T14). The converse implication follows from (A3), (A2) and (A1) applying (MP).

(EQ8a):

(L.1) $\vdash (A \wedge B) \Rightarrow B$ (T8)

(L.2) $\vdash ((A \wedge B) \Rightarrow B) \Rightarrow (((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C))$ (T6)

(L.3) $\vdash ((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$ (L.1, L.2, (MP))

(EQ8b): This follows from (EQ8a) by using twice (A3), (A1) and (MP).

(EQ9): It is easy to see that using properties (T5), (T7) and (MP) this axiom follows from the following implications:

$$\vdash ((A \wedge B) \Rightarrow C) \,\&\, (D \Rightarrow A) \Rightarrow ((D \wedge B) \Rightarrow C),$$
$$\vdash (C \Rightarrow (A \wedge B)) \,\&\, (A \Rightarrow D) \Rightarrow (C \Rightarrow (D \wedge B)).$$

The first implication:

(L.1) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow B$ ((A2), (T8), (A1), 2x(MP))

(L.2) $\vdash ((D \wedge B) \Rightarrow D) \Rightarrow ((D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (D \,\&\, (D \Rightarrow A)))$ (T6)

(L.3) $\vdash (D \wedge B) \,\&\, (D \Rightarrow A) \Rightarrow (D \,\&\, (D \Rightarrow A))$ ((A4), L.2, (MP))

(L.4) $\vdash (D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow A$ (L.3, (T4), (A1), 2x(MP))

(L.5) $\vdash ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow A) \mathbin{\&} ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow B)$
(L.4, L.1, (T5), 2x(MP))

(L.6) $\vdash ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow A) \wedge ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow B)$ (L.5, (T9), (MP))

(L.7) $\vdash ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow A) \wedge ((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow B) \Rightarrow$
$((D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow (A \wedge B))$ (T11)

(L.8) $\vdash (D \wedge B) \mathbin{\&} (D \Rightarrow A) \Rightarrow (A \wedge B)$ (L.6, L.7, (MP))

(L.9) $\vdash ((D \Rightarrow A) \mathbin{\&} (D \wedge B)) \mathbin{\&} ((A \wedge B) \Rightarrow C) \Rightarrow (A \wedge B) \mathbin{\&} ((A \wedge B) \Rightarrow C)$
((A3), L.8, (A1), 2x (MP); (T6), (MP))

(L.10) $\vdash (((A \wedge B) \Rightarrow C) \mathbin{\&} (D \Rightarrow A)) \mathbin{\&} (D \wedge B) \Rightarrow ((D \Rightarrow A) \mathbin{\&} (D \wedge B)) \mathbin{\&}$
$((A \wedge B) \Rightarrow C)$ ((T16), (T20), (MP); (A3), (A1), 2x(MP))

(L.11) $\vdash (((A \wedge B) \Rightarrow C) \mathbin{\&} (D \Rightarrow A)) \mathbin{\&} (D \wedge B) \Rightarrow C$
(L.10, L.9, (A1), 2x(MP); (T4), (A1), 2x(MP))

(L.12) $\vdash ((A \wedge B) \Rightarrow C) \mathbin{\&} (D \Rightarrow A) \Rightarrow ((D \wedge B) \Rightarrow C)$ (L.11, (A7b), (MP))

Now, we prove the second implication:

(L.1) $\vdash (A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow B$ ((A2), (T8), (A1), 2x(MP))

(L.2) $\vdash ((A \wedge B) \Rightarrow A) \Rightarrow ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow (A \mathbin{\&} (A \Rightarrow D)))$ (T6)

(L.3) $\vdash (A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow (A \mathbin{\&} (A \Rightarrow D))$ ((A4), L.2, (MP))

(L.4) $\vdash (A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow D$ (L.3, (T4), (A1), 2x(MP))

(L.5) $\vdash ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow D) \mathbin{\&} ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow B)$
(L.4, L.1, (T5), 2x(MP))

(L.6) $\vdash ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow D) \wedge ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow B)$ (L.5, (T9), (MP))

(L.7) $\vdash ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow D) \wedge ((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow B) \Rightarrow$
$((A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow (D \wedge B))$ ((T11))

(L.8) $\vdash (A \wedge B) \mathbin{\&} (A \Rightarrow D) \Rightarrow (D \wedge B)$ (L.6, L.7, (MP))

(L.9) $\vdash (A \Rightarrow D) \Rightarrow ((A \wedge B) \Rightarrow (D \wedge B))$
((A3), L.8, (A1), 2x(MP); (A7b), (MP))

(L.10) $\vdash ((A \wedge B) \Rightarrow (D \wedge B)) \Rightarrow ((C \Rightarrow (A \wedge B)) \Rightarrow (C \Rightarrow (D \wedge B)))$ (T10)

(L.11) $\vdash (A \Rightarrow D) \Rightarrow ((C \Rightarrow (A \wedge B)) \Rightarrow (C \Rightarrow (D \wedge B)))$
(L.9, L.10, (A1), 2x (MP))

(L.12) $\vdash (C \Rightarrow (A \wedge B)) \,\&\, (A \Rightarrow D) \Rightarrow (C \Rightarrow (D \wedge B))$
(L.11, (A7a), (MP), (A3), (A1), 2x (MP))

(EQ10): Similarly as above, we prove the following two implications:

$$\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B)),$$
$$\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((D \equiv B) \Rightarrow (A \equiv C)).$$

(L.1) $\vdash (B \equiv A) \,\&\, (A \equiv C) \Rightarrow (B \equiv C)$ (T19)

(L.2) $\vdash ((B \equiv A) \,\&\, (A \equiv C)) \,\&\, (C \equiv D) \Rightarrow (B \equiv C) \,\&\, (C \equiv D)$ (L.1, (T6), (MP))

(L.3) $\vdash (B \equiv C) \,\&\, (C \equiv D) \Rightarrow (B \equiv D)$ (T19)

(L.4) $\vdash ((B \equiv A) \,\&\, (A \equiv C)) \,\&\, (C \equiv D) \Rightarrow (B \equiv D)$ (L.2, L.3, (A1), 2x(MP))

(L.5) $\vdash (B \equiv A) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow ((B \equiv A) \,\&\, (A \equiv C)) \,\&\, (C \equiv D)$
((T16), (T18), (MP); (T20), (MP))

(L.6) $\vdash (B \equiv A) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv D)$ (L.5, L.4, (A1), 2x(MP))

(L.7) $\vdash (A \equiv B) \Rightarrow (B \equiv A)$ (T18)

(L.8) $\vdash (A \equiv B) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv A) \,\&\, ((A \equiv C) \,\&\, (C \equiv D))$
(L.7, (T6), (MP))

(L.9) $\vdash (A \equiv B) \,\&\, ((A \equiv C) \,\&\, (C \equiv D)) \Rightarrow (B \equiv D)$ (L.8, L.6, (A1), 2x(MP))

(L.10) $\vdash (A \equiv B) \,\&\, ((C \equiv D) \,\&\, (A \equiv C)) \Rightarrow (A \equiv B) \,\&\, ((A \equiv C) \,\&\, (C \equiv D))$
((A3), (T6), (MP); 2x((A3), (A1), 2x (MP)))

(L.11) $\vdash (A \equiv B) \,\&\, ((C \equiv D) \,\&\, (A \equiv C)) \Rightarrow (B \equiv D)$ (L.10, L.9, (A1), 2x(MP))

(L.12) $\vdash ((A \equiv B) \,\&\, (C \equiv D)) \,\&\, (A \equiv C) \Rightarrow (B \equiv D)$
((T16), (T20), (MP); L.11, (A1), 2x(MP))

(L.13) $\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (B \equiv D))$ (L.12, (A7b), (MP))

(L.14) $\vdash ((A \equiv C) \Rightarrow (B \equiv D)) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B))$ ((T18), (T10), (MP))

(L.15) $\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \equiv C) \Rightarrow (D \equiv B))$ (L.13, L.14, (A1), 2x(MP))

The second implication is analogous. Using (T5), both already proved implications, (T9), (T11) and (MP) we get

$$\vdash (A \equiv B) \,\&\, (C \equiv D) \Rightarrow (((A \equiv C) \Rightarrow (D \equiv B)) \wedge ((D \equiv B) \Rightarrow (A \equiv C))).$$

Finally, using (T24), (T18), (T25), (A1) and (MP) we obtain axiom (EQ10).

(EQ11):

(L.1) $\vdash (B \wedge C) \Rightarrow B$ (A4)

(L.2) $\vdash ((B \wedge C) \Rightarrow B) \Rightarrow ((A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B))$ (T10)

(L.3) $\vdash (A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$ (L.1, L.2, (MP))

(EQ15): This follows immediately from (A7a), (A7b), (T5) using (MP).

(EQ16): This is just (A9).

(EQ17): This is just (A8).

Now we derive both rules of EQ(MTL)-logic. The equanimity rule immediately follows from (T20) and the assumptions applying (MP). The Leibniz rule requires the proof by induction on the complexity of the formula C. Assume that $\vdash A \equiv B$. Then

1. If C is either $\top$ or $\mathbf{q}$ (other than $\mathbf{p}$) then $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$ is $C \equiv C$ which is (T17).

2. If C is $\mathbf{p}$ then we immediately get the assumption.

3. Let C be $\mathbf{p} \,\&\, D$. Then, from the assumption and (T21) applying (MP) we have $\vdash (A \,\&\, D) \equiv (B \,\&\, D)$.

4. Let C be $\mathbf{p} \wedge D$. We prove two implications.

(L.1) $\vdash (A \Rightarrow B) \Rightarrow ((A \wedge D) \Rightarrow B)$ ((A4), (A1), (MP))

(L.2) $\vdash (A \wedge D) \Rightarrow B$ (assumption, (T20), (MP); L.1, (MP))

(L.3) $\vdash ((A \wedge D) \Rightarrow B) \,\&((A \wedge D) \Rightarrow D)$ (L.2, (T8), (T5), 2x(MP))

(L.4) $\vdash ((A \wedge D) \Rightarrow B) \wedge ((A \wedge D) \Rightarrow D)$ (L.3, (T9), (MP))

(L.5) $\vdash (A \wedge D) \Rightarrow (B \wedge D)$ (L.4, (T11), (MP))

In the same way we obtain the converse implication.

5. Let C be $\mathbf{p} \Rightarrow D$. Then from the assumption, (T22) and using (MP) we get $\vdash (A \Rightarrow D) \equiv (B \Rightarrow D)$.

6. Let C be $D \Rightarrow p$. Then from the assumption, (T23) and using (MP) we find $\vdash (D \Rightarrow A) \equiv (D \Rightarrow B)$.

□

This chapter introduced four kinds of propositional EQ-logics which we believe to be interesting. We developed basic EQ-logic, continued with involutive and prelinear EQ-logic and finished this chapter with EQ(MTL)-logic. However, deduction theorem which is necessary for evolving predicate EQ-logic does not hold in any of these EQ-logics. Thus, we solve this trouble, in the upcoming chapter, by adding delta connective to EQ-logic.

Chapter 4

EQ$_\Delta$-logic

In this chapter we introduce a propositional EQ-logic enriching it by Δ connective. First of all, we present basic EQ$_\Delta$-logic and then we extend the latter logic to a prelinear one.

4.1 Basic EQ$_\Delta$-logic

The basic EQ-logic can be extended to the basic EQ-logic with the delta connective. We can also say, that the basic EQ$_\Delta$-logic is the simplest logic definable on the basis of EQ$_\Delta$-algebra. This logic will be interesting for us especially due to the deduction theorem which we are able to prove here.

4.1.1 Syntax and semantics

Definition 13 (Language)
The language of basic EQ$_\Delta$-logic is the language of the basic EQ-logic expanded by the unary connective $\boldsymbol{\Delta}$*.*

Semantics of this logic is formed by a good non-commutative EQ$_\Delta$-algebra

$$\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \Delta, \mathbf{1}\rangle.$$

Definition 14 (Truth evaluation)
A truth evaluation of formulas $e : F_J \longrightarrow E$ *is defined in the same way as in Definition 8, we only add the following:*

$$e(\boldsymbol{\Delta} A) = \Delta e(A)$$

for all formulas $A, B \in F_J$.

4.1.2 Logical axioms and inference rules

Definition 15
Axioms of basic $EQ_{\boldsymbol{\Delta}}$-logic are those of the basic EQ-logic extended by the following ones:

(EQ$\boldsymbol{\Delta}$1) $\boldsymbol{\Delta} A \Rightarrow \boldsymbol{\Delta}\boldsymbol{\Delta} A$

(EQ$\boldsymbol{\Delta}$2) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta} A \equiv \boldsymbol{\Delta} B)$

(EQ$\boldsymbol{\Delta}$3) $\boldsymbol{\Delta}(A \wedge B) \equiv (\boldsymbol{\Delta} A \wedge \boldsymbol{\Delta} B)$

(EQ$\boldsymbol{\Delta}$4) $\boldsymbol{\Delta} A \equiv (\boldsymbol{\Delta} A \,\&\, \boldsymbol{\Delta} A)$

(EQ$\boldsymbol{\Delta}$5) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow ((A \,\&\, C) \equiv (B \,\&\, C))$

(EQ$\boldsymbol{\Delta}$6) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow ((C \,\&\, A) \equiv (C \,\&\, B))$

Note that logical axioms of the basic $EQ_{\boldsymbol{\Delta}}$-logic correspond to the axioms of the EQ_{Δ}-algebra.

Deduction rules of $EQ_{\boldsymbol{\Delta}}$-logic are *equanimity rule*, *Leibniz rule* and the *necessitation rule*

$$(N) \qquad \frac{A}{\boldsymbol{\Delta} A}.$$

4.1.3 Main properties

Lemma 23
All axioms of $EQ_{\boldsymbol{\Delta}}$-logic are tautologies.

PROOF: This is straightforward using the axioms and properties of EQ_{Δ}-algebra. □

Lemma 24
The deductive rules of $EQ_{\boldsymbol{\Delta}}$-logic are sound in the following sense. Let $e : F_J \longrightarrow E$ be a truth evaluation:

(a) If $e(A) = \mathbf{1}$ and $e(A \equiv B) = \mathbf{1}$ then $e(B) = \mathbf{1}$.

(b) If $e(B \equiv C) = \mathbf{1}$ then $e(A[p := B] \equiv A[p := C]) = \mathbf{1}$ for any formula A.

(c) If $e(A) = \mathbf{1}$ *then* $e(\mathbf{\Delta} A) = \mathbf{1}$.

PROOF: For (a) and (b) see Lemma 19. We only add to (b) the proof of the case where formula A is $\mathbf{\Delta} E$ (see proof of Lemma 19). Then $A[p := B] \equiv A[p := C])$ implies $e(\mathbf{\Delta} E[p := B] \equiv \mathbf{\Delta} E[p := C])$ and thus, we deduce (3.1.7) as follows:

$$e(\mathbf{\Delta} E[p := B]) = \Delta e(E[p := B]) = \Delta e(E[p := C]) = e(\mathbf{\Delta} E[p := C])$$

(c) follows from (EΔ1). □

The following lemma introduces several properties of $\mathbf{\Delta}$ connective provable in $\mathrm{EQ}_{\mathbf{\Delta}}$-logic.

Lemma 25

(a) $\vdash \mathbf{\Delta} A \Rightarrow A$,

(b) $\vdash \mathbf{\Delta} \top \equiv \top$,

(c) $\vdash \mathbf{\Delta}(A \Rightarrow B) \Rightarrow (\mathbf{\Delta} A \Rightarrow \mathbf{\Delta} B)$,

(d) $\vdash \mathbf{\Delta}(A \mathbin{\&} B) \Rightarrow (\mathbf{\Delta} A \mathbin{\&} \mathbf{\Delta} B)$,

(e) $\vdash \mathbf{\Delta} A \equiv \mathbf{\Delta}(A \mathbin{\&} A)$,

(f) $\vdash \mathbf{\Delta}((A \equiv B) \mathbin{\&} (C \equiv D)) \Rightarrow ((A \mathbin{\&} C) \equiv (B \mathbin{\&} D))$.

PROOF: (a)

$\mathbf{\Delta}(A \equiv \top) \Rightarrow ((A \mathbin{\&} \top) \equiv (\top \mathbin{\&} \top))$ (EQ$\mathbf{\Delta}$5)

$\Leftrightarrow \langle$(Leib) + (EQ6); "C-part": $\mathbf{\Delta}(A \equiv \top) \Rightarrow (\mathbf{p} \equiv (\top \mathbin{\&} \top))\rangle$

$\mathbf{\Delta}(A \equiv \top) \Rightarrow (A \equiv (\top \mathbin{\&} \top))$

$\Leftrightarrow \langle$(Leib) + (EQ6); "C-part": $\mathbf{\Delta}(A \equiv \top) \Rightarrow (A \equiv \mathbf{p})\rangle$

$\mathbf{\Delta}(A \equiv \top) \Rightarrow (A \equiv \top)$

$\Leftrightarrow \langle$(Leib) + (EQ1); "C-part": $\mathbf{\Delta p} \Rightarrow \mathbf{p}\rangle$

$\mathbf{\Delta} A \Rightarrow A$

(b) Follows from (EQ1) in the form $\vdash (\mathbf{\Delta} \top \equiv \top) \equiv \mathbf{\Delta} \top$ and from $\vdash \mathbf{\Delta} \top$ using (EA).

(c)

$$\mathbf{\Delta}((A \wedge B) \equiv A) \Rightarrow (\mathbf{\Delta}(A \wedge B) \equiv \mathbf{\Delta} A) \qquad (\mathrm{EQ}\mathbf{\Delta}2)$$

$$\Leftrightarrow \langle (\text{Leib}) + (\mathrm{EQ}\mathbf{\Delta}3);\ \text{“C-part”}:\ \mathbf{\Delta}((A \wedge B) \equiv A) \Rightarrow (\mathbf{p} \equiv \mathbf{\Delta} A)\rangle$$

$$\mathbf{\Delta}((A \wedge B) \equiv A) \Rightarrow ((\mathbf{\Delta} A \wedge \mathbf{\Delta} B) \equiv \mathbf{\Delta} A) \qquad (\text{i.e. (c)})$$

(d) By Lemma 15(e) and 15(f) using rule (N), Lemma 25(c) and Lemma 14(c) we get $\vdash \mathbf{\Delta}(A \,\&\, B) \Rightarrow \mathbf{\Delta} A$ and $\vdash \mathbf{\Delta}(A \,\&\, B) \Rightarrow \mathbf{\Delta} B$, respectively. Next, we use these formulas as assumptions in Lemma 14(p) to obtain $\vdash (\mathbf{\Delta}(A \,\&\, B) \,\&\, \mathbf{\Delta}(A \,\&\, B)) \Rightarrow (\mathbf{\Delta} A \,\&\, \mathbf{\Delta} B)$ and therefore $\vdash \mathbf{\Delta}(A \,\&\, B) \Rightarrow (\mathbf{\Delta} A \,\&\, \mathbf{\Delta} B)$ (see (EQ$\mathbf{\Delta}$4)).

(e) From (EQ$\mathbf{\Delta}$5) $\vdash \mathbf{\Delta}(\top \equiv A) \Rightarrow ((\top \,\&\, A) \equiv (A \,\&\, A))$ and thus $\vdash \mathbf{\Delta} A \Rightarrow (A \equiv (A \,\&\, A))$. Using Lemma 14(m) and applying transitivity of implication $\vdash \mathbf{\Delta} A \Rightarrow (A \Rightarrow (A \,\&\, A))$. By rule (N), twice use of Lemma 25(c), Lemma 14(c) and transitivity of implication observe $\vdash \mathbf{\Delta\Delta} A \Rightarrow (\mathbf{\Delta} A \Rightarrow \mathbf{\Delta}(A \,\&\, A))$, thus $\vdash \mathbf{\Delta} A \Rightarrow (\mathbf{\Delta} A \Rightarrow \mathbf{\Delta}(A \,\&\, A))$ (use (EQ$\mathbf{\Delta}$1) and transitivity of implication again). Then, from Lemma 15(h) $\vdash (\mathbf{\Delta} A \,\&\, \mathbf{\Delta} A) \Rightarrow \mathbf{\Delta}(A \,\&\, A)$ and finally, (EQ$\mathbf{\Delta}$4) gives $\vdash \mathbf{\Delta} A \Rightarrow \mathbf{\Delta}(A \,\&\, A)$.

We prove converse implication as follows:

$$\mathbf{\Delta}(A \,\&\, A) \Rightarrow (\mathbf{\Delta} A \,\&\, \mathbf{\Delta} A) \qquad (\text{Lemma 25(d)})$$

$$\Leftrightarrow \langle (\text{Leib}) + (\mathrm{EQ}\mathbf{\Delta}4);\ \text{“C-part”}:\ \mathbf{\Delta}(A \,\&\, A) \Rightarrow \mathbf{p}\rangle$$

$$\mathbf{\Delta}(A \,\&\, A) \Rightarrow \mathbf{\Delta} A$$

Both implications put together by Lemma 14(f) and then using Lemma 14(o) and 14(c) get the desired formula.

(f) First we use Lemma 25(d)

$$\vdash \mathbf{\Delta}((A \equiv B) \,\&\, (C \equiv D)) \Rightarrow (\mathbf{\Delta}(A \equiv B) \,\&\, \mathbf{\Delta}(C \equiv D)).$$

Using assumptions (EQ$\mathbf{\Delta}$5) and $\vdash \mathbf{\Delta}(C \equiv D) \Rightarrow \mathbf{\Delta}(C \equiv D)$ in Lemma 14(p) we obtain

$$\vdash (\mathbf{\Delta}(A \equiv B) \,\&\, \mathbf{\Delta}(C \equiv D)) \Rightarrow (((A \,\&\, C) \equiv (B \,\&\, C)) \,\&\, \mathbf{\Delta}(C \equiv D)).$$

In the same way (but using (EQ$\mathbf{\Delta}$6)) we get

$$\vdash (((A \,\&\, C) \equiv (B \,\&\, C)) \,\&\, \mathbf{\Delta}(C \equiv D)) \\ \Rightarrow (((A \,\&\, C) \equiv (B \,\&\, C)) \,\&\, ((B \,\&\, C) \equiv (B \,\&\, D))).$$

Finally from the three previous formulas and Lemma 15(a) in the form

$$\vdash (((A \,\&\, C) \equiv (B \,\&\, C)) \,\&\, ((B \,\&\, C) \equiv (B \,\&\, D))) \Rightarrow ((A \,\&\, C) \equiv (B \,\&\, D))$$

using Lemma 14(e) we find desired formula. □

4.1.4 Deduction theorem

Now, we present an important theorem – deduction theorem whose a slightly modified form copes well with the character of EQ-logic for which formulations using equivalences is more natural. It is notable that this generalized deduction theorem does not hold in basic EQ-logic but after enriching it by the $\boldsymbol{\Delta}$ connective we are able to prove it.

Theorem 12 (Deduction theorem)
For each theory T and formulas A, B, C it holds that

$$T \cup \{A \equiv B\} \vdash C \quad \textit{iff} \quad T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow C.$$

PROOF: [Hilbert style] Let $T \cup \{A \equiv B\} \vdash C$. The proof proceeds by induction on the length of the proof of C.

(a) If $C := (A \equiv B)$ then $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (A \equiv B)$ by Lemma 25(a).

(b) C is an axiom of T then $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow C$ using Lemma 14(s) and Lemma 14(c).

(c) Let C have been obtained using rule (EA) by the proof

$$\ldots, D, D \equiv C, C.$$

Then

(L.1) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}(A \equiv B) \,\&\, \boldsymbol{\Delta}(A \equiv B))$
((EQ$\boldsymbol{\Delta}$4), Lemma 14(m), (c))

(L.2) $T \vdash (\boldsymbol{\Delta}(A \equiv B) \,\&\, \boldsymbol{\Delta}(A \equiv B)) \Rightarrow (D \,\&\, (D \equiv C))$
(2x inductive assumption, Lemma 14(p))

(L.3) $T \vdash (D \,\&\, (D \equiv C)) \Rightarrow C$ (Lemma 15(b))

(L.4) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow C$ (L.1, L.2, L.3 and Lemma 14(e))

(d) Let $C := D[\mathbf{p} := E] \equiv D[\mathbf{p} := F]$ have been obtained using rule (Leib) by the proof

$$\ldots, E \equiv F, D[\mathbf{p} := E] \equiv D[\mathbf{p} := F].$$

Then the proof proceeds by induction on the complexity of the formula C.

(i) If D is either $\top$ or $\mathbf{q}$ (other than $\mathbf{p}$) then $D[\mathbf{p} := E] \equiv D[\mathbf{p} := F] = D \equiv D$ and $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (D \equiv D)$ follows from Lemma 14(a), Lemma 14(s) and Lemma 14(c).

(ii) If D is $\mathbf{p}$ then it follows immediately from the inductive assumption.

(iii) Let D be $G \bigcirc H$, where $\bigcirc \in \{\wedge, \&, \equiv\}$. Then we must prove

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G \bigcirc H)[\mathbf{p} := E] \equiv (G \bigcirc H)[\mathbf{p} := F]),$$

that is

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow \\ ((G[\mathbf{p} := E] \bigcirc H[\mathbf{p} := E]) \equiv (G[\mathbf{p} := F] \bigcirc H[\mathbf{p} := F]))$$

and thus

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G' \bigcirc H') \equiv (G'' \bigcirc H'')$$

where $G' := G[\mathbf{p} := E], H' := H[\mathbf{p} := E], G'' := G[\mathbf{p} := F]$ and $H'' := H[\mathbf{p} := F]$.

First, let D be $G \wedge H$. Then

(L.1) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}(A \equiv B) \,\&\, \boldsymbol{\Delta}(A \equiv B))$
((EQ$\boldsymbol{\Delta}$4), Lemma 14(m), (c))

(L.2) $T \vdash (\boldsymbol{\Delta}(A \equiv B) \,\&\, \boldsymbol{\Delta}(A \equiv B)) \Rightarrow ((G' \equiv G'') \,\&\, (H' \equiv H''))$
(2x inductive assumption, Lemma 14(p))

(L.3) $T \vdash ((G' \equiv G'') \,\&\, (H' \equiv H'')) \Rightarrow ((G' \wedge H') \equiv (G'' \wedge H''))$
(Lemma 16(g))

(L.4) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G' \wedge H') \equiv (G'' \wedge H''))$
(L.1, L.2, L.3 and Lemma 14(e))

(iv) Let D be $G \equiv H$. Then

(L.1) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}(A \equiv B) \,\&\, \boldsymbol{\Delta}(A \equiv B))$
((EQ$\boldsymbol{\Delta}$4), Lemma 14(m), (c))

(L.2) $T \vdash (\boldsymbol{\Delta}(A \equiv B) \mathbin{\&} \boldsymbol{\Delta}(A \equiv B)) \Rightarrow ((G' \equiv G'') \mathbin{\&} (H' \equiv H''))$
(2x inductive assumption, Lemma 14(p))

(L.3) $T \vdash ((G' \equiv G'') \mathbin{\&} (H' \equiv H'')) \Rightarrow ((G' \equiv H') \equiv (G'' \equiv H''))$
(Lemma 14(n))

(L.4) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G' \equiv H') \equiv (G'' \equiv H''))$
(L.1, L.2, L.3 and Lemma 14(e))

(v) Let D be $G \mathbin{\&} H$.

(L.1) $T \vdash (\boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G' \equiv G'') \mathbin{\&} (H' \equiv H''))$ (see above)

(L.2) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow \boldsymbol{\Delta\Delta}(A \equiv B)$ ((EQ$\boldsymbol{\Delta}$1))

(L.3) $T \vdash \boldsymbol{\Delta\Delta}(A \equiv B) \Rightarrow \boldsymbol{\Delta}((G' \equiv G'') \mathbin{\&} (H' \equiv H''))$
(L.1, rule (N), Lemma 25(c), Lemma 14(c))

(L.4) $T \vdash \boldsymbol{\Delta}((G' \equiv G'') \mathbin{\&} (H' \equiv H'')) \Rightarrow ((G' \mathbin{\&} H') \equiv (G'' \mathbin{\&} H''))$
(Lemma 25(f))

(L.5) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((G' \mathbin{\&} H') \equiv (G'' \mathbin{\&} H''))$
(L.2, L.3, L.4 and Lemma 14(e))

(vi) Let D be $\boldsymbol{\Delta}G$. Then we must show

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow ((\boldsymbol{\Delta}G)[\mathbf{p} := E] \equiv (\boldsymbol{\Delta}G)[\mathbf{p} := F]),$$

that is

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}(G[\mathbf{p} := E]) \equiv (\boldsymbol{\Delta}(G[\mathbf{p} := F])))$$

and thus (according to the marks above)

$$T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}G' \equiv \boldsymbol{\Delta}G'').$$

Then observe

(L.1) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (G' \equiv G'')$ (inductive assumption)

(L.2) $T \vdash \boldsymbol{\Delta\Delta}(A \equiv B) \Rightarrow \boldsymbol{\Delta}(G' \equiv G'')$
(L.1, rule (N), Lemma 25(c), Lemma 14(c))

(L.3) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow \boldsymbol{\Delta}(G' \equiv G'')$ ((EQ$\boldsymbol{\Delta}$1), L.2 and Lemma 14(e))

(L.4) $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta}G' \equiv \boldsymbol{\Delta}G'')$ (L.3, (EQ$\boldsymbol{\Delta}$2) and Lemma 14(e))

(e) Let $C := \boldsymbol{\Delta}D$ have been obtained using rule (N) by the proof

$$\ldots, D, \boldsymbol{\Delta}D.$$

Then $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow \boldsymbol{\Delta}D$ follows from inductive assumption using rule (N), Lemma 25(c), 14(c), (EQ$\boldsymbol{\Delta}$1) and Lemma 14(e).

The converse implication follows from rule (N) and Lemma 14(c). $\square$

If we put $B := \top$ in the deduction theorem above then we get the"standard" form of the delta deduction theorem.

Corollary 1
Let T be a theory and let A, C be formulas. Then it holds that

$$T \cup \{A\} \vdash C \quad \textit{iff} \quad T \vdash \boldsymbol{\Delta} A \Rightarrow C.$$

4.1.5 Completeness

We define Lindenbaum algebra of the equivalence classes of formulas $\bar{\mathcal{E}}_\Delta = \langle \bar{E} = \{[A] \mid A \in F_J\}, \wedge, \otimes, \sim, \Delta, \mathbf{1}\rangle$ in the same way as in Definition 12. Moreover, we define the Δ operation on the set $\bar{E}$ as follows:

$$\Delta[A] = [\boldsymbol{\Delta} A]. \tag{4.1.1}$$

Lemma 26
The algebra $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \otimes, \sim, \Delta, \mathbf{1}\rangle$ is a good non-commutative EQ$_\Delta$-algebra.

PROOF: We have to verify only axioms of the EQ$_\Delta$-algebra: for axioms (E1)–(E7) and the "goodness property" see the proof of Lemma 20 in Chapter 3. For axiom (EΔ1) see Lemma 25(b). For axioms (EΔ2)–(EΔ7) see (EQ$\boldsymbol{\Delta}$1)–(EQ$\boldsymbol{\Delta}$6). $\square$

Theorem 13 (Soundness)
The basic EQ$_{\boldsymbol{\Delta}}$-fuzzy logic is sound.

PROOF: This follows from Lemmas 23 and 24. $\square$

Theorem 14 (Completeness)
The following is equivalent for every formula A:

(a) $\vdash A$,

(b) $e(A) = \mathbf{1}$ *for every good non-commutative EQ$_\Delta$-algebra $\mathcal{E}_\Delta$ and a truth evaluation $e : F_J \longrightarrow E$.*

PROOF: The implication (a) to (b) is soundness.

(b) to (a): By Lemma 26 the algebra $\bar{\mathcal{E}}_\Delta$ of equivalence classes of formulas is a good non-commutative EQ$_\Delta$-algebra. Thus, if (b) holds then it holds also for

$e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). □

4.2 Prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic

Prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic is interesting because a stronger form of the completeness theorem holds in it. Moreover, this logic will become the basis for the development of the predicate EQ-logic.

4.2.1 Syntax and semantics

The language of this logic is that of the basic $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic extended by the logical connective $\vee$ and logical constant $\bot$ (falsum). We also extend the language by the short

$$\neg A := A \equiv \bot. \tag{4.2.1}$$

Formula (4.2.1) defines *negation* in this logic.

Semantics of this logic is formed by a non-commutative $\ell\mathrm{EQ}_{\Delta}$-algebra $\mathcal{E}_{\Delta} = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1}\rangle$ in which condition (2.1.9) is fulfilled.

4.2.2 Logical axioms and inference rules

The complete list of logical axioms of the prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic is the following:

(EQ1) $(A \equiv \top) \equiv A$

(EQ2) $A \wedge B \equiv B \wedge A$

(EQ3) $(A \bigcirc B) \bigcirc C \equiv A \bigcirc (B \bigcirc C), \quad \bigcirc \in \{\wedge, \&\}$

(EQ4) $A \wedge A \equiv A$

(EQ5) $A \wedge \top \equiv A$

(EQ6) $A \,\&\, \top \equiv A$

(EQ7) $\top \,\&\, A \equiv A$

(EQ8a) $((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$

(EQ8b) $(C \,\&(A \wedge B)) \Rightarrow (C \,\&\, B)$

(EQ9) $((((A \wedge B) \vee C) \equiv D) \,\&(F \equiv C)) \,\&(E \equiv A) \Rightarrow (D \equiv (F \vee (B \wedge E)))$

(EQ10) $(A \equiv B) \,\&(C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$

(EQ11) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$

(EQ12) $(A \vee B) \vee C \equiv A \vee (B \vee C)$

(EQ13) $A \vee (A \wedge B) \equiv A$

(EQ14) $(A \wedge \bot) \equiv \bot$

(EQ15) $(A \Rightarrow B) \vee (D \Rightarrow (D \,\&(C \Rightarrow ((B \Rightarrow A) \,\&\, C))))$

(EQ$\boldsymbol{\Delta}$1) $\boldsymbol{\Delta} A \Rightarrow \boldsymbol{\Delta}\boldsymbol{\Delta} A$

(EQ$\boldsymbol{\Delta}$2) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow (\boldsymbol{\Delta} A \equiv \boldsymbol{\Delta} B)$

(EQ$\boldsymbol{\Delta}$3) $\boldsymbol{\Delta}(A \wedge B) \equiv (\boldsymbol{\Delta} A \wedge \boldsymbol{\Delta} B)$

(EQ$\boldsymbol{\Delta}$4) $\boldsymbol{\Delta} A \equiv (\boldsymbol{\Delta} A \,\&\, \boldsymbol{\Delta} A)$

(EQ$\boldsymbol{\Delta}$5) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow ((A \,\&\, C) \equiv (B \,\&\, C))$

(EQ$\boldsymbol{\Delta}$6) $\boldsymbol{\Delta}(A \equiv B) \Rightarrow ((C \,\&\, A) \equiv (C \,\&\, B))$

(EQ$\boldsymbol{\Delta}$7) $\boldsymbol{\Delta}(A \vee B) \Rightarrow (\boldsymbol{\Delta} A \vee \boldsymbol{\Delta} B)$

(EQ$\boldsymbol{\Delta}$8) $\boldsymbol{\Delta} A \vee \neg \boldsymbol{\Delta} A$

Remark 8
Axioms of the basic EQ$_{\boldsymbol{\Delta}}$-logic are extended by axioms (EQ12)–(EQ14) and (EQ$\boldsymbol{\Delta}$7)–(EQ$\boldsymbol{\Delta}$8) which reflect the join-semilattice structure. Moreover, axiom (EQ15) stands for the prelinearity. Note also that axiom (EQ9) expresses a common substitution axiom both for $\wedge$ and for $\vee$ and thus it replaces the original substitution axiom in EQ-logics.

Deduction rules are the same as those of the basic EQ$_{\boldsymbol{\Delta}}$-logic.

4.2.3 Main properties

The main properties of the prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic, with emphasize to the disjunction connective are introduced in the following lemma. We will also give a proof of the substitution of $\wedge$ (Lemma 27(d)). It should be noted that no property used in its proof was proved using the same substitution property, i.e. there is no hidden vicious circle in the proof.

Lemma 27

(a) $\vdash A \vee B \equiv B \vee A$,

(b) $\vdash ((A \vee B) \equiv C) \,\&\, (D \equiv A) \Rightarrow ((D \vee B) \equiv C)$,

(c) $\vdash A \vee \bot \equiv A$,

(d) $\vdash ((A \wedge B) \equiv C) \,\&\, (D \equiv A) \Rightarrow (C \equiv (D \wedge B))$,

(e) $\vdash (A \Rightarrow B) \vee (B \Rightarrow A)$,

(f) $\vdash \top \vee A$,

(g) $\vdash A \Rightarrow (A \vee B)$,

(h) $\vdash A \wedge (A \vee B) \equiv A$,

(i) $((A \vee B) \equiv B) \equiv (A \Rightarrow B)$,

(j) $(A \Rightarrow C) \,\&\, (B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C)$,

(k) $((A \vee B) \Rightarrow C) \Rightarrow (A \Rightarrow C)$,

(l) $(A \Rightarrow B) \Rightarrow (A \Rightarrow (B \vee C))$,

(m) $(A \vee B) \equiv (((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A))$,

(n) $(A \equiv B) \Rightarrow ((A \vee C) \equiv (B \vee C))$,

(o) $(A \Rightarrow B) \Rightarrow ((A \vee C) \Rightarrow (B \vee C))$,

(p) $(A \Rightarrow B) \,\&\, (C \Rightarrow D) \Rightarrow ((A \vee C) \Rightarrow (B \vee D))$,

(q) $(A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \vee C) \equiv (B \vee D))$.

PROOF: (a) By (EQ9) we get $((((A \wedge \top) \vee B) \equiv \top) \,\&\, (B \equiv B)) \,\&\, (A \equiv A) \Rightarrow (\top \equiv (B \vee (\top \wedge A)))$. Using (Leib), (EQ5), (EQ1), Lemma 14(a) + rule (T2), (EQ6), (EQ2) and Lemma 14(g) we get $\vdash (A \vee B) \Rightarrow (B \vee A)$. In the same way we obtain $\vdash (B \vee A) \Rightarrow (A \vee B)$. Finally using Lemma 14(f), (o) and (c) we have $\vdash (A \vee B) \equiv (B \vee A)$.

(b) From (EQ9) we have $((((A \wedge \top) \vee B) \equiv C) \,\&\, (B \equiv B)) \,\&\, (D \equiv A) \Rightarrow (C \equiv (B \vee (\top \wedge D)))$. Using (Leib), (EQ5), Lemma 14(a) + rule (T2), (EQ6), (EQ2), Lemma 27(a) and Lemma 14(g) we obtain (b).

(c)

$A \vee \bot$

$\Leftrightarrow \langle$(Leib) + (EQ14) + Lemma 14(g); "C-part": $A \vee \mathbf{p}\rangle$

$A \vee (A \wedge \bot)$

$\Leftrightarrow \langle$(EQ13)$\rangle$

A

(d) From (EQ9) we have $((((A \wedge B) \vee \bot) \equiv C) \,\&\, (\bot \equiv \bot)) \,\&\, (D \equiv A) \Rightarrow (C \equiv (\bot \vee (B \wedge D)))$. Using (Leib), Lemma 27(c), Lemma 14(a) + rule (T2), (EQ6), Lemma 27(a) and (EQ2) we get (d).

(e) Directly from (EQ15) in the form $(A \Rightarrow B) \vee (\top \Rightarrow (\top \,\&\, (\top \Rightarrow ((B \Rightarrow A) \,\&\, \top))))$ using (Leib), (EQ6), (EQ7) and Lemma 14(d).

(f)

$(A \Rightarrow \top) \vee (\top \Rightarrow A)$ (Lemma 27(e))

$\Leftrightarrow \langle$(Leib) + Lemma 14(d); "C-part": $(A \Rightarrow \top) \vee \mathbf{p}\rangle$

$(A \Rightarrow \top) \vee A$

$\Leftrightarrow \langle$(Leib) + (EQ5): $(A \wedge \top) \equiv A$ + rule (T2); "C-part": $\mathbf{p} \vee A\rangle$

$\top \vee A$

(g) Just from Lemma 27(b) $\vdash ((\top \vee B) \equiv \top) \,\&\, (A \equiv \top) \Rightarrow ((A \vee B) \equiv \top)$ using Lemma 27(f) + rule (T2), Lemma 14(a) + rule (T2), (EQ7) and (EQ1).

(h) Immediately by the definition of implication in Lemma 27(g).

(i) The equivalence will be obtained when proving the implications left to right and its opposite. From Lemma 27(d) in the form

$$\vdash ((A \vee B) \wedge A) \equiv A) \,\&\, (B \equiv (A \vee B)) \Rightarrow (A \equiv (B \wedge A))$$

and by clear modifications using the Leibniz rule we have $\vdash ((A \vee B) \equiv B) \Rightarrow (A \Rightarrow B)$. The converse implication follows in the same way from Lemma 27(b) in the form

$$\vdash (((A \wedge B) \vee B) \equiv B) \,\&\, (A \equiv (A \wedge B)) \Rightarrow ((A \vee B) \equiv B).$$

Both implications, Lemma 14(f), (o) and (c) complete the proof.

(j) We start with Lemma 27(b) in the form

$$\vdash ((C \vee A) \equiv C) \,\&\, ((B \vee C) \equiv C) \Rightarrow ((B \vee C) \vee A) \equiv C).$$

Using Leibniz rule and Lemma 27(a), (EQ12) and Lemma 27(i) we finish the proof.

(k) Follows immediately from Lemma 14(h) in the form

$$\vdash ((A \vee B) \Rightarrow C) \Rightarrow (((A \vee B) \wedge A) \Rightarrow C)$$

using (Leib), (EQ2) and Lemma 27(h).

(l) Immediately from (EQ11) in the form

$$\vdash (A \Rightarrow ((B \vee C) \wedge B)) \Rightarrow (A \Rightarrow (B \vee C))$$

using (Leib), (EQ2) and Lemma 27(h).

(m) We prove two implications which yield equivalence using standard technique. Denote $C := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A)$. From Lemma 15(o) and also 14(s) in the form $\vdash A \Rightarrow ((B \Rightarrow A) \Rightarrow A)$ using Lemma 14(f), 15(k) and 14(c) we have $\vdash A \Rightarrow C$. By the similar way we get $\vdash B \Rightarrow C$. Thus, using Lemma 14(f), 27(j) and 14(c) we obtain

$$(A \vee B) \Rightarrow (((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A)).$$

Now, we show the converse implication. Let us consider the following formulas:

$$\vdash (A \Rightarrow B) \Rightarrow (((A \Rightarrow B) \Rightarrow B) \Rightarrow B) \qquad \text{(Lemma 15(o))}$$

$$\vdash ((((A \Rightarrow B) \Rightarrow B) \Rightarrow B) \Rightarrow (C \Rightarrow B) \qquad \text{(Lemma 14(h))}$$

$\vdash (C \Rightarrow B) \Rightarrow (C \Rightarrow (A \vee B))$ (Lemma 27(l))

The transitivity of $\Rightarrow$ results in $\vdash (A \Rightarrow B) \Rightarrow (C \Rightarrow (A \vee B))$. Similarly, we get $\vdash (B \Rightarrow A) \Rightarrow (C \Rightarrow (A \vee B))$. Using Lemma 14(f), 27(j) and 14(c) we have

$$\vdash ((A \Rightarrow B) \vee (B \Rightarrow A)) \Rightarrow (C \Rightarrow (A \vee B))$$

and therefore, using Lemma 27(e) and 14(c),

$$\vdash (((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A)) \Rightarrow (A \vee B).$$

(n) Let us consider the following variant of Lemma 27(b):

$$\vdash ((A \vee C) \equiv (A \vee C)) \,\&\, (B \equiv A) \Rightarrow ((B \vee C) \equiv (A \vee C)).$$

Using (Leib), Lemma 14(a) + rule (T2), (EQ7) Lemma 14(g) we obtain (n).

(o) From Lemma 27(n) we have $\vdash ((A \vee B) \equiv B) \Rightarrow (((A \vee B) \vee C) \equiv (B \vee C))$. By (Leib), Lemma 27(i), (a) and (EQ12) we get $\vdash (A \Rightarrow B) \Rightarrow (((A \vee C) \vee B) \equiv (B \vee C))$ and thus by Lemma 14(m) and transitivity of $\Rightarrow$ $\vdash (A \Rightarrow B) \Rightarrow (((A \vee C) \vee B) \Rightarrow (B \vee C))$. From this and from $\vdash (((A \vee C) \vee B) \Rightarrow (B \vee C)) \Rightarrow ((A \vee C) \Rightarrow (B \vee C))$ (i.e. Lemma 27(k)) using transitivity of $\Rightarrow$ again we conclude (o).

(p) Using Lemma 27(o) and (a) we obtain $\vdash (A \Rightarrow B) \Rightarrow ((A \vee C) \Rightarrow (B \vee C))$ and $\vdash (C \Rightarrow D) \Rightarrow ((B \vee C) \Rightarrow (B \vee D))$. Now, Lemma 14(p) yields

$$\vdash (A \Rightarrow B) \,\&\, (C \Rightarrow D) \Rightarrow ((A \vee C) \Rightarrow (B \vee C)) \,\&\, ((B \vee C) \Rightarrow (B \vee D)).$$

Finally, using Lemma 15(c) and transitivity of $\Rightarrow$ we complete the proof.

(q) If we use Lemma 27(n) twice as assumptions in Lemma 14(p) then we get

$$(A \equiv B) \,\&\, (C \equiv D) \Rightarrow ((A \vee C) \equiv (B \vee C)) \,\&\, ((B \vee C) \equiv (B \vee D)).$$

Hence, Lemma 15(a) and Lemma 14(e) yield

$$(A \equiv B) \,\&\, (C \equiv D) \Rightarrow (A \vee C) \equiv (B \vee D).$$

□

The following lemma shows properties of the delta connective in the prelinear $\mathrm{EQ}_{\mathbf{\Delta}}$-logic.

Lemma 28

(a) $\vdash \mathbf{\Delta}\bot \equiv \bot$,

(b) $\vdash \mathbf{\Delta}(\neg A) \Rightarrow \neg(\mathbf{\Delta} A)$,

(c) $\vdash \mathbf{\Delta}(A \vee B) \equiv (\mathbf{\Delta} A \vee \mathbf{\Delta} B)$,

(d) $\vdash \mathbf{\Delta}(A \Rightarrow B) \vee \mathbf{\Delta}(B \Rightarrow A)$,

(e) $\vdash (\mathbf{\Delta} A \,\&\, \mathbf{\Delta}(A \Rightarrow B)) \Rightarrow \mathbf{\Delta} B$.

PROOF: (a) $\vdash \mathbf{\Delta}\bot \Rightarrow \bot$ follows from Lemma 25(a). If we use (EQ14) and rewrite it in the implication form, we obtain the converse implication. Both implications together with Lemma 14(f), 14(o) and 14(c) yield the equivalence.

(b)

$$\mathbf{\Delta}(A \equiv \bot) \Rightarrow (\mathbf{\Delta} A \equiv \mathbf{\Delta}\bot) \qquad (\text{EQ}\mathbf{\Delta}2)$$

$$\Leftrightarrow \langle (\text{Leib}) + \text{Lemma 28(a); ``C-part'': } \mathbf{\Delta}(A \equiv \bot) \Rightarrow (\mathbf{\Delta} A \equiv \mathbf{p}) \rangle$$

$$\mathbf{\Delta}(A \equiv \bot) \Rightarrow (\mathbf{\Delta} A \equiv \bot)$$

$$\Leftrightarrow \langle (4.2.1) \rangle$$

$$\mathbf{\Delta}(\neg A) \Rightarrow \neg(\mathbf{\Delta} A)$$

(c) One implication is just (EQ$\mathbf{\Delta}$7). The proof of the second implication proceeds as follows: From Lemma 27(g), rule (N), Lemma 25(c) and 14(c) we find $\vdash \mathbf{\Delta} A \Rightarrow \mathbf{\Delta}(A \vee B)$ and $\vdash \mathbf{\Delta} B \Rightarrow \mathbf{\Delta}(A \vee B)$. Both formulas connect by $\&$ using Lemma 14(f) and finally, we complete it using Lemma 27(j) and 14(c).

(d) Follows from Lemma 27(e) using rule (N), (EQ$\mathbf{\Delta}$7) and Lemma 14(c).

(e) This can be obtained from Lemma 25(c) using Lemma 16(f) and 15(h). □

Remark 9
Deduction theorem mentioned in basic EQ_{Δ}-logic (Theorem 12) also holds in prelinear EQ_{Δ}-logic. However, the proof above must be completed, since this logic is enriched with disjunction connective. Namely (see proof above!), in case (d), if D is $G \vee H$. Then using Lemma 27(q), it easy to show that $T \vdash \mathbf{\Delta}(A \equiv B) \Rightarrow ((G' \vee H') \equiv (G'' \vee H''))$.

4.2.4 Completeness

Similarly as in Definition 12, we again construct Lindenbaum algebra of equivalence classes of formulas $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1}\rangle$. In addition to it and (4.1.1), we define

$$\mathbf{0} = [\bot],$$
$$[A] \vee [B] = [A \boldsymbol{\vee} B].$$

Lemma 29
The algebra $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1}\rangle$ is an ℓEQ_Δ-algebra satisfying condition (2.1.9).

PROOF: Axiom (E14) which substitutes axioms (E4) and (E13), follows from (EQ9). For the least element of $\bar{E}$ see (EQ14). For join-semilattice structure see (EQ12)–(EQ13) and Lemma 27(a). For axioms (EΔ8) and (EΔ9) see (EQ$\boldsymbol{\Delta}$7), (EQ$\boldsymbol{\Delta}$8) and rule (T2). For condition (2.1.9) see axiom (EQ15). The remaining axioms have been already proven in Lemma 26. □

Theorem 15 (Soundness)
The prelinear EQ$_{\boldsymbol{\Delta}}$-logic is sound.

PROOF: We have to show that all axioms of prelinear EQ$_{\boldsymbol{\Delta}}$-logic are tautologies. For axioms (EQ1)–(EQ8b), (EQ10)–(EQ11) and (EQ$\boldsymbol{\Delta}$1)–(EQ$\boldsymbol{\Delta}$6) see Lemma 23. For axiom (EQ9) see Lemma 7; axioms (EQ12)–(EQ14) follow from properties of lattice; axiom (EQ15) is a tautology because of the property (2.1.9); finally, to verify (EQ$\boldsymbol{\Delta}$7) and (EQ$\boldsymbol{\Delta}$8) observe (EΔ8) and (EΔ9). For soundness of the deductive rules see Lemma 24. It is easy to check whether Leibniz rule is soundness in the case that formula A is $\bot$ or $E \boldsymbol{\vee} F$ (see Lemma 19). □

Theorem 16 (Completeness)
For every formula $A \in F_J$ and for every theory T the following is equivalent:

(a) $T \vdash A$.

(b) $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every linearly ordered, ℓEQ_Δ-algebra $\mathcal{E}_\Delta$.

(c) $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every ℓEQ_Δ-algebra $\mathcal{E}_\Delta$ satisfying condition (2.1.9).

PROOF: (a) $\Rightarrow$ (c): All axioms of T are true in all models of T (see Theorem 15).

(c) $\Rightarrow$ (a): If (c) holds then it holds also for $e : F_J \longrightarrow \bar{E}$, where $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$ is an ℓEQ$_\Delta$-algebra satisfying (2.1.9) (the Lindenbaum algebra). If $e(A) = \mathbf{1}$ then $[A] = [\top]$, i.e. $T \vdash A \equiv \top$ and so, $T \vdash A$ by rule (T1).

(c) $\Rightarrow$ (b): We use the Lindenbaum-Tarski technique to construct the algebra $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$. This algebra satisfies (2.1.9) (see (EQ15)), hence it is representable by Theorem 5. Thus we have also deduced (b) $\Rightarrow$ (c). □

This chapter presented two kinds of EQ-logics which are extended by delta connective. First, we formed the basic EQ$_{\boldsymbol{\Delta}}$-logic which is considered to be the simplest EQ-logic with delta connective. Further, we extended it by disjunction connective and arose the prelinear EQ$_{\boldsymbol{\Delta}}$-logic. This logic is strong enough (also due to the holding of the deduction theorem) for the development of the predicate EQ-logic outlined in the following chapter.

Chapter 5

Outline of the predicate EQ-logic

After we have established the propositional EQ-logic, we are ready to start our investigation of the predicate EQ-logic. Since the research of predicate EQ-logic is still in progress, we did not include it in this thesis. However, in this chapter, we, at least, outline our vision how we intend to develop it. We build the predicate EQ-logic on propositional prelinear EQ_Δ-logic, which is the strongest propositional EQ-logic.

The language of the predicate EQ-logic consist of object variables $x, y, \dots$, object constants $\mathbf{u}, \mathbf{v}, \dots$, predicate symbols $P, Q, \dots$, logical constants $\top$ and $\bot$, connectives $\wedge, \vee, \&, \equiv, \boldsymbol{\Delta}$ and quantifiers (universal and existential) $\forall$, $\exists$.

Logical axioms of the predicate EQ-logic should determine the behavior of connectives and constants, and they should also tell us how both quantifiers behave. Thus, the axiom set of the predicate EQ-logic naturally includes all the axioms of propositional prelinear EQ_Δ-logic (i.e. (EQ1)–(EQ15) and (EQ$\boldsymbol{\Delta}$1)–(EQ$\boldsymbol{\Delta}$8), completed by five well known axioms on quantifiers used in many other kinds predicate fuzzy logics:

- $(\forall x)A(x) \Rightarrow A(t)$ (t substituable for x in $A(x)$),
- $A(t) \Rightarrow (\exists x)A(x)$ (t substituable for x in $A(x)$),
- $(\forall x)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x)B)$ (x not free in A),
- $(\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow B)$ (x not free in B),
- $(\forall x)(A \vee B) \Rightarrow ((\forall x)A \vee B)$ (x not free in B).

The deduction rules are Equanimity rule, the rule of necessitation and generalization (from A infer $\forall A$). The Leibniz rule will be the same as in propositional EQ-logic, only completed by requirement: From $A \equiv B$ infer $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$ provided that $\mathbf{p}$ is not in the scope of a quantifier in C.

Classically, semantics of the predicate EQ-logics is set by a structure, which gives a meaning to symbols from the language. Notice only that a structure of truth values is formed by a non-commutative linearly ordered $\ell\mathrm{EQ}_\Delta$-algebra $\mathcal{E} = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1}\rangle$.

The following items will play an important role in proving completeness:

1. Deduction theorem in the form: $T \cup \{A \equiv B\} \vdash C$ iff $T \vdash \boldsymbol{\Delta}(A \equiv B) \Rightarrow C$ holds for each theory T and closed formulas A, B and arbitrary formula C.

2. Linear (complete) theory: A theory T is linear (complete) if for every two formulas A, B, either $T \vdash A \Rightarrow B$ or $T \vdash B \Rightarrow A$.

3. Henkin theory (in our case extensionally complete theory): A theory T is extensionally complete if for every closed formula $(\forall x)(A(x) \equiv B(x))$, $T \not\vdash (\forall x)(A(x) \equiv B(x))$ there is a constant $\mathbf{u}$ such that $T \not\vdash (A_x[\mathbf{u}] \equiv B_x[\mathbf{u}])$.

We conclude this chapter by the observation that the delta connective must be included in the predicate EQ-logic, because otherwise we cannot prove the deduction theorem which, however, is indispensable for the proof of the completeness theorem.

Chapter 6

Conclusion

The main goal of this thesis was to show a new approach to the development of fuzzy logic. Namely, to develop fuzzy logic on the basis of fuzzy equality. Such class of logics called EQ-logics. Unlike other kinds of fuzzy logics [8, 13, 15, 16, 29], EQ-logics have equivalence as the basic connective instead of implication. Moreover, they are not equivalent with residuated fuzzy logics as in the case of classical logic developed in this way (cf. [36]).

First, in Chapter 2, we gave the survey of three special kinds of algebras — EQ-algebras which have been proposed as the structures of truth values for EQ-logics. We have remembered the most important properties of them, we have proved a representation theorem of the strongest EQ-algebra and we have found examples of all kinds of algebras presented in this Chapter. Note that for this purpose, a special software for checking axioms of EQ-algebras on finite sets has been developed in the Institute for Research and Applications of Fuzzy Modeling at University of Ostrava.

The reason why we have chosen this class of algebras is that the main connective of equivalence in EQ-logics is interpreted by a primary algebraic operation. Moreover, implication is not as closely tied to multiplication as in residuated lattices. Consequently, EQ-algebras are not residuated. This enables us to develop the non-commutative and non-residuated EQ-logics with only one implication. We think that one implication is more natural than two implications which have to be considered in non-commutative implication-based fuzzy logics (cf. [21]).

In Chapter 3, we established several kinds of the propositional EQ-logics. First, we have developed in detail syntax and semantics of the basic EQ-logic. We call it "basic", because it is the simplest logic developed on the basis of EQ-algebras. It lies even deeper than the logics based on MTL-algebras. Another reason is that it can be extended to stronger logics. If we introduce the double negation axiom, then we get an involutive EQ-logic (IEQ-logic). Furthermore, if we add the prelinearity axiom (i.e. (EQ14)), then we obtain prelinear EQ-logic in which the stronger form of the completeness theorem holds. Many properties of these three logics were described and their completeness theorems were proved. Finally, by adding further axioms we obtain EQ(MTL)-logic which is proved to be equivalent to the residuated MTL-logic. EQ(MTL)-logic underlines the difference between EQ-logics and the residuated core logics (these are expansions of the MTL-one). It should be emphasized that formal proofs in this work proceed mostly in equational style, which is for EQ-logics more natural than the Hilbert style.

In Chapter 4, we introduced a class of EQ-logics which are extended by additional unary connective of Δ. This connective is very useful in fuzzy logics because it makes it possible to work with crisp formulas. For EQ-logics it also enables to prove the generalized deduction theorem (namely, the Δ-deduction one) which does not hold in EQ-logics because they are weaker than MTL-logic. We have proved this theorem and demonstrated that EQ_Δ-logics have reasonable properties including completeness. By introducing prelinear EQ_Δ-logic, we sowed the seeds of the development of the predicate EQ-logic, in which the deduction theorem is indispensable. This is indicated in Chapter 5.

There are several interesting questions and tasks to be solved in the future:

1. Is it possible to substitute the inference rules of EQ-logic (Equanimity and Leibniz rule) for the modus ponens only? We guess that the answer is negative. Probably a few axioms or rules would be necessary to add.
2. To develop the first-order EQ-logic in detail.
3. To investigate whether it is possible to use for our EQ-logic the tools presented in the recently published handbook [9] for proving general results in fuzzy logics (e.g. completeness).

Author's contribution

Here I summarize my contribution in this thesis.

1. Chapter 2: Examples of all types of EQ-algebras as well as their properties proved in this Chapter.

2. Chapter 3: All results concerning properties of the basic EQ-logic (Subsection 3.1.4, 3.1.5). Results concerning extensions of the basic EQ-logic presented in Section 3.2.

3. All definitions and results in Chapter 4.

List of Author's Publications

- Dyba, M., Novák, V., EQ-logics with delta connective. *Fuzzy Sets and Systems*, submitted.
- Dyba, M., Novák, V., EQ-logics: Non-commutative fuzzy logics based on fuzzy equality. *Fuzzy Sets and Systems*, 2011, Vol. 172, pp. 13–32.
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- Novák, V., Dyba, M., Non-commutative EQ-logics and their extensions. *Proceedings of IFSA World Congress/EUSFLAT Conference.* Technical University of Lisbon, Lisbon, 2009, pp. 1422–1427.

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