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FUZZY LOGIC DEDUCTION IN THE FRAME OF FUZZY LOGIC IN A BROADER SENSE

Ph.D. THESIS

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FUZZY LOGICKÁ DEDUKCE V RÁMCI FUZZY LOGIKY V ŠIRŠÍM SMYSLU

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Summary

An important role in fuzzy logic and fuzzy control is played by *linguistic descriptions*, i.e. finite sets of IF-THEN rules. These rules often include so-called evaluating linguistic expressions – natural language expressions which characterize a position on an ordered scale, usually on a real interval. Examples of evaluating linguistic expressions are *small*, *more or less medium*, *approximately 20* etc. Given a linguistic description of a process, situation, environment etc., and an *observation*, i.e. a value measured in some concrete situation, the task is to determine the *conclusion* by some plausible method.

This thesis proposes a methodology for dealing with the above-described situation and studies its properties. The basis for it is fuzzy logic in a narrow sense with evaluated syntax [27]. IF-THEN rules are understood as linguistically expressed logical implications. We consistently distinguished three levels of study – linguistic, syntactic and semantic. The meaning of evaluating linguistic expression is characterized on syntactic level by its *intension* and on semantic level by a class of its *extensions*.

First we investigate properties that a formal theory aimed at the characterization of evaluating linguistic expressions should have. We call the theory which fulfills these properties the *theory of evaluating expressions*. Further we use theories of evaluating expressions for the determination of the formal theory which characterizes the meaning of a linguistic description, called the *theory of linguistic description*. The central part of our thesis is a study of fuzzy logic deduction. It uses the theory of linguistic description and another theory describing an observation for determination of conclusion by means of formal proving in clearly defined logic. First we prove some general properties of fuzzy logic deduction and then we study one important aspect of the work with linguistic descriptions, so-called inconsistencies. Last part of our thesis is devoted to a situation in which an observation is given as a crisp number.

Keywords: Logical deduction, intension, linguistic description, IF-THEN rules, fuzzy logic.

Anotace

Důležitou roli ve fuzzy logice a fuzzy regulaci hrají *jazykové popisy*, tj. konečné množiny JESTLIŽE-PAK pravidel. V mnoha případech tyto pravidla obsahují tzv. evaluační jazykové výrazy – výrazy přirozeného jazyka charakterizující pozici na uspořádané škále, obvykle na intervalu reálných čísel. Příklady evaluačních jazykových výrazů jsou *malý*, *více méně střední*, *přibližně* 20 apod. Máme-li zadán jazykový popis nějakého procesu, situace, prostředí apod. a *pozorování*, tedy hodnotu naměřenou v nějaké konkrétní situaci, je naším úkolem určit *závěr* pomocí nějaké vědecky podložené metody.

Tato disertace navrhuje metodologii popisující tuto situaci a studuje její vlastnosti. Základem pro tuto metodologii je fuzzy logika v užším smyslu s ohodnocenou syntaxí [27]. JESTLIŽE-PAK pravidla jsou chápána jako jazykově vyjádřené logické implikace. Důsledně rozlišujeme tři úrovně práce s jazykovými výrazy – jazykovou, syntaktickou a sémantickou. Význam evaluačního jazykového výrazu je vyjádřen jeho *intenzí* na syntaktické úrovni a třídou jeho extenzí na úrovni sémantické.

Nejprve jsou zkoumány podmínky, které musí splňovat formální teorie zaměřená na charakterizaci evaluačních jazykových výrazů. Teorii splňující tyto podmínky nazýváme *teorií evaluačních výrazů*. Teorie evaluačních výrazů jsou dále použity k sestrojení formální teorie charakterizující význam jazykového popisu, tzv. *teorie jazykového popisu*. Ústřední částí této disertace je studium fuzzy logické dedukce. Zde je použita teorie jazykového popisu spolu s další teorií popisující pozorování k určení závěru pomocí formálního dokazování v jednoznačně definované logice. Nejprve jsou ukázány obecné vlastnosti fuzzy logické dedukce, dále pak studujeme jeden význačný aspekt práce s jazykovými popisy, tzv. nekonzistence v jazykových popisech. Závěrečná část disertace je věnována fuzzy logické dedukci s pozorováním ve formě reálného čísla.

Klíčová slova: Logická dedukce, intenze, jazykový popis, JESTLIŽE-PAK pravidla, fuzzy logika.

Preface

Fuzzy logic is an important tool for the modeling of *vagueness*. The importance of vagueness and its transmission and propagation in human thinking and everyday human communication is indisputable. Fuzzy logic offers powerful and transparent methodology which allows to describe and model vague phenomena. The applications of fuzzy logic in control, decision making and other areas are numerous and successful. However, from the emergence of fuzzy logic and fuzzy set theory in the 1960's up to the 1990's objections often occurred reproaching that fuzzy logic lacked solid mathematical foundations. These objections lost ground at the end of the 1990's when the important books [13], [27] and some others appeared. However, there are still areas in mathematical foundations of fuzzy logic that have to be investigated.

The aim of this thesis is to propose and investigate mathematical model of the meaning of *evaluating linguistic expressions*, i.e. linguistic expressions which characterize a position on an ordered scale. Further, the structure and meaning of *linguistic descriptions*, i.e. finite sets of IF-THEN rules are investigated. These meanings are described separately on syntactic and semantic levels. The concepts introduced and results obtained are then used in the central part of this thesis, namely in the investigation of *fuzzy logic deduction*. The deduction is performed on syntactic level as a formal proving in some formal theory. An important concept of *inconsistency* of linguistic description is discussed and its two definitions are proposed. Finally, fuzzy logic deduction with crisp observations, which is important in applications of fuzzy logic, is also investigated.

The methodology and results obtained in this thesis are, or will be, used in the software system LFLC 2000 developed in our Institute for Research and Applications of Fuzzy Modeling. LFLC 2000 is a software system which allows to design, test and apply linguistic descriptions. It proved itself to be useful also in practical applications.

I want to thank my supervisor Prof. Vilém Novák for his support, permanent encouragement and working conditions he created in Institute for Research and Applications of Fuzzy Modeling. I would also like to thank colleagues from our Institute and my parents.

Ostrava, April 2003 Antonín Dvořák

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Chapter 1

Introduction

This thesis is a study of one approach to approximate reasoning over a set of IF-THEN rules called *fuzzy logic deduction*. The novelty of our approach lies in the fact that the inference is performed on syntactic level as a formal proving in a formal logical system. This is a substantial difference in comparison with the classical method, namely the *compositional rule of inference* (see [35, 13]), which is defined on the semantic level.

Fuzzy logic has been, from its boom in 1980's till now, one of the most widely used methodologies for dealing with the phenomenon of indeterminacy. Its applications are numerous, mainly in the fields of control and optimization, but also in other areas, such as clustering, pattern recognition, expert systems, economy, psychology etc.

The phenomenon which could be most successfully modeled by means of fuzzy logic is called *vagueness*. It is one of the facets of the above-mentioned indeterminacy (another is *uncertainty*, that emerges due to lack of knowledge about the occurrence of some event). The vagueness occurs in the process of grouping together objects which have some property ϕ (e.g. tallness). Quite often is such a grouping imprecise, there exist borderline elements, i.e. elements which have the property ϕ only to some extent. There are various approaches trying to model this phenomenon, inside or outside of classical logic (see e.g. [15]). One of the important properties of vagueness, which we will often use in the following, is its *continuity*. It means that for similar objects should also the extent in which they have the property ϕ be similar.

In a great part of applications of fuzzy logic, so called IF-THEN rules are used. They have the form IF X is $\mathcal{A}$ THEN Y is $\mathcal{B}$ where X, Y are *linguistic variables* and

$\mathcal{A}, \mathcal{B}$ are linguistic expressions. Most widely used are so-called *evaluating linguistic expressions*, which express the position on some ordered scale, most often on an interval of real numbers. However, in practice there are usually more than one linguistic variables on the left side of IF-THEN rule. IF-THEN rules are then grouped into (finite) set, called rulebase (we will call it *linguistic description*). This set of IF-THEN rules is a natural way to catch human knowledge about some process, situation etc. Then, given some *observation* (in the form of real number, interval of real numbers, linguistic expression, fuzzy set etc.), we need to determine a *conclusion*. If the rulebase is comprehensible to humans, then the conclusion should correspond with human intuition - should be similar to the conclusion carried out by humans.

There was a lot of discussions in the literature about the suitability of fuzzy logic for modeling of the phenomenon of vagueness and of the theoretical background of fuzzy logic and, namely, of fuzzy control. In recent years a lot of progress in this field has been done (as the most important contributions let us name books [14, 27, 11], see also Chapter 2). However, there is still a lot of open problems and not-fully-understood areas. Our thesis tries to contribute to this area of research.

We can distinguish two approaches in the general theory of fuzzy IF-THEN rules. The first one, called *fuzzy approximation* (see [30, 31]) uses linguistic form of the IF-THEN rules as a motivation for finding special formulas called *disjunctive* (DNF) and *conjunctive* (CNF) *normal forms*. The goal is to approximate a function described by IF-THEN rules in some model with prescribed accuracy.

The second possibility is to take the set of IF-THEN rules as the set of genuine linguistic expressions, find their logical interpretation and work with this interpretation inside some formal logical theory. This approach is based on *fuzzy logic in narrow sense with evaluated syntax*. This thesis focuses on the second approach. We will show the possible interpretation of linguistic description and the way how logical deduction based on it can be performed.

To be able to perform this task, we should first study the modeling of the meaning of linguistic expressions occurring in these IF-THEN rules. We restrict ourselves to so-called simple evaluating linguistic expressions (such as *medium*, *more or less small*, *extremely big* etc.). This class of expressions is most widely used in practice. We then use the same methodology for modeling of the meaning of one IF-THEN rule and sets of them.

We will always carefully distinguish three levels on which we are working, namely linguistic (e.g linguistic expressions such as *small*), syntactic (e.g. intensions of these expressions, formal theories etc.) and semantic (extensions of linguistic expressions in some possible world). We will focus our attention mainly on the syntactic level, because in the inclusion of this level lies the novelty of our approach. Moreover, all results obtained in syntactic level are automatically (due to the correctness of fuzzy predicate logic) valid in all possible worlds.

First, in Chapter 2, we review some approaches to the study of IF-THEN rules and deduction over sets of them. We briefly describe the approaches of Hájek, Novák and Vojtáš.

In Chapter 3 we introduce necessary preliminaries. Besides the material concerning predicate first-order fuzzy logic, we will also add linguistic preliminaries and review some other notions used in further chapters (namely *tolerance relations* and *real unimodal functions of one variable*).

In Chapter 4 we start to study formal theories for the modeling of so-called evaluating linguistic expressions. We call such theories *theories of evaluating expressions*. We will not try to determine one ("unique", "best") theory, we rather postulate several axioms every such a theory should obey. The basis for our study is first-order fuzzy logic in a narrow sense with evaluated syntax [27].

On the syntactic level, the meaning of an evaluating expression is characterized by its *intension*. We assign to evaluating expression $\mathcal{A}$ a formula $A(x)$ with one free variable. The intension of $\mathcal{A}$ is then the set of evaluated formulas

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A}_{\langle x \rangle} = \left\{ \tilde{\alpha}_A(t)/A_x[t] \,\middle|\, t \in M, \tilde{\alpha}_A(t) \in L \right\} \tag{1.1}$$

where M is a set of closed terms, L is a set of truth values and $\tilde{\alpha}_A(t)$ are syntactic evaluations of instances of formula $A(x)$. Syntactic evaluations could be obtained as provability degrees of the respective instances of $A(x)$ in some theory [21]. To every considered simple evaluating expression $\mathcal{A}_i$ we add a predicate symbol G_i to our formal language. Instances of atomic formulas $G_{i,x}[t]$ together with their syntactic evaluations $\tilde{\alpha}_{G_i}(t)$ form an essential part of the theory of evaluating expressions denoted by T^{ev} (the theory should also include some technical axioms which assure correct behavior of equality predicate etc.). All simple evaluating linguistic expressions are treated uniformly, i.e. we are not considering any structure over the set of them (e.g. linguistic hedges). But, if there is some theory where hedges are taken into account (like in e.g. [21]), it can be translated into our system and we can study its properties. This means that our level of study is lower, but at the same time more general than in [21].

We will prove that theories of evaluating expressions indeed exist (Lemma 4.2). On the semantic level, the meaning of an evaluating expression is modeled by its *extension* – satisfaction set of formula $A(x)$ in a model $\mathcal{V}$ of a theory T^{ev}:

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ \mathcal{V}(A(v/x))\big/ v \,\middle|\, v \in V \right\}$$

where V is a support of the structure $\mathcal{V}$. Models of T^{ev} are also called *possible worlds*. Then we will study special class of models of such theories, so-called *canonical possible worlds* (Section 4.2). These structures have a property that truth values of instances of formulas $A_x[t]$ from (1.1) coincide with syntactic evaluations of these instances. If such a structure does not exist for some theory of evaluating expressions, then the effort of its forming would be futile, because we never attain a proper correspondence between syntactic and semantic levels. Theorem 4.7 will show that under some conditions, which are usually fulfilled, such structures exist.

A well-known property of the relationship between syntax and semantics in (not only) fuzzy logic is that the truth value of a formula A in some model $\mathcal{D}$ denoted by $\mathcal{D}(A)$ is, in general, greater than or equal to its provability degree. The equality of these two quantities is attained for complete theories. Therefore, it is possible that the theory T^{ev} mentioned above can have a model $\mathcal{D}^\star$ in which all truth degrees for all instances of atomic formulas $G_{i,x}[t]$ have value 1 and on that account they are in $\mathcal{D}^\star$ undistinguishable. We propose (in Section 4.3) to restrict the range of these truth values by means of an addition of axioms of the form $\{\tilde{\beta}_{G_i}(t)\big/\neg G_{i,x}[t]\}$, where it should hold that $\tilde{\alpha}_{G_i}(t) \otimes \tilde{\beta}_{G_i}(t) = 0$. We call such theories *extended theories of evaluating expressions* T^{evx}. We will again define a structure (called *extended possible world*) which plays similar role as the canonical possible world above. We will prove that such structures exist (Theorem 4.13) and show some properties of these models. At the end of Chapter 4 there is a short section concerning theories of evaluating expressions with fuzzy equality.

In the following Chapter 5 we study linguistic descriptions, i.e. sets of IF-THEN rules. We again start on linguistic level, then we proceed to syntactic (Section 5.2) and semantic (Section 5.3) levels. On syntactic level we form an intension of one IF-THEN rule $\mathbf{R}_{\langle x,y\rangle}$. From these intensions we form a *theory of linguistic description* T_I. On semantic level, we define a canonical possible world for the theory T_I and show that it always exists (Theorem 5.6).

Chapter 6 "Fuzzy logic deduction" is the central part of our thesis. We will introduce (in Section 6.1) its so-called *basic schema*. The deduction is performed on syntactic level and the conclusion is obtained in the form of closed instances of atomic formulas $B_y[s]$ and their respective provability degrees $\tilde{\alpha}'_B(s)$ in the theory T_D. This theory is formed as a union of the theory T_I introduced in the previous chapter, and of a theory T' representing an observation. Theorem 6.5 (proved already in [27]), Theorem 6.1) shows how these provability degrees can be computed. We then show some general properties of our fuzzy logic deduction (Theorems 6.6, 6.9 and Corollary 6.10). It is interesting that these results show that our fuzzy logic deduction satisfies the conditions imposed by C. Cornelis and E. E. Kerre in [4] on general inference procedures.

When we postulate our basic schema of fuzzy logic deduction, we are able to investigate its properties from several viewpoints. In our thesis, we focus on the description of so-called *inconsistencies in linguistic descriptions*. By inconsistency we understand the situation when there are identical (or similar) antecedents and different consequents present in linguistic description. First, in Section 6.2 we take a negation of the standard formula defining the condition which a relation should fulfil to be a function and rewrite it into our formalism, using fuzzy equality. This notion of inconsistency is called $\approx$-inconsistency.

Then, in Section 6.3 we present another approach. We use a notion of unimodal function and say that some theory T_D is u-inconsistent if the function $\tilde{\alpha}'_B(s)$ mentioned above is not unimodal (more precisely, we should consider the real function γ'_B associated to $\tilde{\alpha}'_B$). In this way we can describe this type of inconsistency for one observation. Then we generalize this notion to be able to describe inconsistency for all sensible observations. This more general notion is called $u^\star$-inconsistency. The dual notion is $u^\star$-consistency. We will show necessary and sufficient conditions for this type of consistency of linguistic description in Theorem 6.22.

Finally, in Chapter 7 we apply the methodology and results of previous chapter to the situation, when a crisp number measured in some possible world is taken as an observation. In particular, we study a case when one IF-THEN rule is taken and the deduction is performed by means of this most suitable rule. To this purpose we introduce an operation Suit, which selects the most suitable linguistic expression, and, consequently, also the most suitable IF-THEN rule for the given observation. We also introduce special defuzzification method *DEEs* (*defuzzification of evaluating expressions*) and study some of its properties, namely its (dis)continuity.

Then we can investigate a real function *LD* which describes the overall behavior of the algorithm of fuzzy logic deduction with crisp observations. We conclude that this function is piece-wise continuous. This result is then explained and discussed.

Author's contribution

Here I summarize my contribution in this thesis.

1. In Chapter 3: the definition of unimodal function and its properties (Section 3.4.2).

2. All definitions and results in Chapter 4.

3. The definition of theory T_I in Section 5.2 and the definition of canonical possible world there and relevant results.

4. In Chapter 6: a formalization of a basic schema of fuzzy logic deduction and theorems in Section 6.1 (with exception of Theorem 6.5), concepts and results concerning two approaches to inconsistencies of linguistic descriptions (Sections 6.2 and 6.3).

5. In Chapter 7: a formalization of the operation Suit (Section 7.2), a formalization and results concerning the defuzzification operation $DEEs$ and the overall behavior of fuzzy logic deduction with crisp observations (Sections 7.3 and 7.4).

Results included in this thesis have not been published yet. Several definitions and results from Chapters 4, 5 and 6 were included in submitted paper [10] and accepted paper [9] (where a part of the content of Chapter 7 in a preliminary form is included).

Chapter 2

Current State of Research

In this section we summarize several approaches to the approximate inference and analysis of IF-THEN rules in logical setting. From our point of view, the most important sources are books [27], Chapter 6 of which is most inspiring for our approach, further, book [13], where in Chapter 7 "On Approximate Inference" there is described approach and properties of approximate inference in Łukasiewicz predicate logic with classical (non-evaluated) syntax. Another sources are papers [3, 34] and book [11].

Let us first review the traditional formulation of Zadeh's *compositional rule of inference* [35, 13], which served as a basis of the study and applications of fuzzy IF-THEN rules. Let D_X and D_Y be nonempty sets. Then:

From 'X is A' *and* '(X, Y) is R' *infer* 'Y is B' if for all $v \in D_Y$,

$$r_B(v) = \sup_{u \in D_X} (r_A(u) \star r_R(u, v)) \tag{2.1}$$

where $\star$ is a continuous t-norm, $r_A \underset{\sim}{\subseteq} D_X$ and $r_B \underset{\sim}{\subseteq} D_Y$ are fuzzy sets and

$$r_R \underset{\sim}{\subseteq} D_X \times D_Y$$

is a binary fuzzy relation.

2.1 Novák's approach

Here we present only main ideas, a more detailed treatment will be included in the foregoing sections. The novelty of his approach lies in the fact that, unlikely as in the classical compositional rule of inference, the main portion of inference process is performed on syntactic level. The inference is, in his approach, the formal proving in exactly defined formal system. Also, the stress is put on the analysis of the meaning of linguistic expressions. He introduced the notions of *intension*, *extension* and *possible world*, well-known from the field of philosophical logic and introduced originally by Rudolf Carnap.

Here, intension is defined on syntactic level. A formal model of intension is so-called multiformula, i.e. a fuzzy set of instances of evaluated formulas. Extension is an interpretation of intension in some specific structure (possible world). A linguistic expression has one intension and a class of extensions, one in every possible world.

This approach supposes the existence of a sufficiently rich set of closed terms in the formal language. These terms are understood as some prototypical formal objects and they are substituted to formulas assigned to linguistic expressions. Intensions can be also considered as some *prototypical interpretations*, which restrict (from below) truth functions of extensions.

2.2 Hájek's approach

Hájek studies IF-THEN rules and approximate inference in BL∀, i.e. basic predicate logic. The main difference between Novák and Hájek approach lies in the fact that Hájek incorporates in the language $\mathcal{I}$ only one object constant for each sort, which is in a model interpreted as an actual value of the variables X and Y (called *variates* there). The expression 'X is A' is then translated into predicate logic as an atomic formula $A(X)$. An IF-THEN rule 'IF X is A THEN Y is B' may be interpreted as $A(X) \Rightarrow B(Y)$. Hence, the language $\mathcal{I}$ further contains unary predicate symbols A and B, and a binary predicate symbol R. Let $\mathcal{D}$ be a structure for the language $\mathcal{I}$. The compositional rule of inference is expressed by the condition that formula

$$Comp := (\forall y)(B(y) \Leftrightarrow (\exists x)(A(x) \,\&\, R(x,y))$$

is 1-true in the model $\mathcal{D}$. Then two particular cases of the formula $Comp$ are studied. First, for Zadeh's Generalized modus ponens (where the relation R from (2.1) is defined as $r_R(u,v) = r_A(u) \to r_B(v)$, and $\to$ is the residuum of t-norm $\star$)

$$Comp_{MP} := (\forall y)(B^\star(y) \Leftrightarrow (\exists x)(A^\star(x) \,\&(A(x) \Rightarrow B(y))))$$

and it is showed that BL∀ proves

$$(Comp_{MP} \,\&\, A^\star(X) \,\&(A(X) \Rightarrow B(Y))) \Rightarrow B^\star(Y). \tag{2.2}$$

Second, generalized conjunctive rule, where the relation R from (2.1) is defined as $r_R(u,v) = r_A(u) \& r_B(v)$:

$$Comp_{CR} := (\forall y)(B^\star(y) \Leftrightarrow (\exists x)(A^\star(x) \,\&\, A(x) \,\&\, B(y)))$$

and it is showed that BL∀ proves

$$(Comp_{CR} \,\&\, A^\star(X) \,\&(A(X) \,\&\, B(Y))) \Rightarrow B^\star(Y) \tag{2.3}$$

In the second part (Chapter 7.3) are X and Y also unary predicates and the assertion 'X is A_i' is in the syntactic level translated into formula

$$(\forall x)(X(x) \Rightarrow A(x))$$

(or briefly $X \subseteq A$). This corresponds to a situation when actual values of variables are given not as crisp numbers, but only vaguely (as fuzzy sets). Then, an alternative composition for generalized modus ponens is the formula

$$Comp_{MPA} := (\forall y)(B^{\star\star}(y) \Leftrightarrow ((\forall x)(A^\star(x) \Rightarrow A(x)) \Rightarrow B(y))).$$

Then BL∀ proves

$$Comp_{MPA} \,\&((X \subseteq A^\star) \,\&((X \subseteq A) \Rightarrow (Y \subseteq B))) \Rightarrow (Y \subseteq B^{\star\star}). \tag{2.4}$$

Formulas (2.2), (2.3) and (2.4) can be visualized as deduction rules, for example formula (2.2) as

$$\frac{Comp_{MP}, A^\star(X), A(X) \Rightarrow B(Y)}{B^\star(Y)}$$

which can be read as: If $Comp_{MP}$, $A^\star(X)$ and $A(X) \Rightarrow B(Y)$ are 1-true in a given structure $\mathcal{D}$, then $B^\star(Y)$ is 1-true in $\mathcal{D}$.

This approach is the extensional one, because the meanings of linguistic expressions are modeled only on the semantic level. However, it also shows some general properties of inference valid in all models.

2.3 IF-THEN rules as fuzzy logic programs

Vojtáš [34] treats IF-THEN rules in the framework of fuzzy logic programming without negation. In his system he uses various truth functions for conjunction, disjunction and implication; and also aggregation operations. He introduces a notion of *fuzzy theory* as a partial mapping assigning to formulas rational numbers from $(0, 1]$. Then, *many valued modus ponens* is introduced by a schema

$$\frac{(B, x), (B \Rightarrow A, y)}{A, mp_{\Rightarrow}(x, y)}$$

where x, y are axiomatically assigned truth values of formulas B and $B \Rightarrow A$, respectively, and $mp_{\Rightarrow}$ is a function calculating truth value of the answer A. This function has to be a truth function of a conjunction (but it need not to be a truth function of conjunctions present in the language). Then the semantics is defined and the completeness of this system is showed.

Gerla in his book [11] studies fuzzy logic mainly from the point of view of fuzzy closure operators. In Chapter 10 "On Approximate Inference" he transforms a set of IF-THEN rules to a fuzzy program.

Chapter 3

Preliminaries

3.1 Logical preliminaries

The formal system we are working in is *fuzzy logic in a narrow sense with evaluated syntax* (FLn). Main source here is the book [27], Chapter 4. We recall here the main points of this formal system, for the details we refer to the above-mentioned book.

3.1.1 The structure of truth values

The two most important algebraic structures for our system of fuzzy logic, used as algebras of truth values, are residuated lattices and MV-algebras.

Definition 3.1
A residuated lattice is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

such that

(i) $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with the ordering defined using the operations $\vee, \wedge$, and $\mathbf{0}, \mathbf{1}$ are its least and greatest elements, respectively,

(ii) $\langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid, i.e. $\otimes$ is commutative and associative operation with the identity $a \otimes \mathbf{1} = a$,

(iii) it holds that

$$a \otimes b \leq c \text{ iff } a \leq b \rightarrow c$$

i.e. the operation $\rightarrow$ is a residuation operation with respect to $\otimes$.

It holds that the operation $\rightarrow$ is antitonic in the first and isotonic in the second variable.

Definition 3.2
An MV-algebra is an algebra

$$\mathcal{L} = \langle L, \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

in which the following identities are valid:

$$\begin{aligned} a \oplus b &= b \oplus a, & a \otimes b &= b \otimes a, \\ a \oplus (b \oplus c) &= (a \oplus b) \oplus c, & a \otimes (b \otimes c) &= (a \otimes b) \otimes c, \\ a \oplus \mathbf{0} &= a, & a \otimes \mathbf{1} &= a, \\ a \oplus \mathbf{1} &= \mathbf{1}, & a \otimes \mathbf{0} &= \mathbf{0}, \\ a \oplus \neg a &= \mathbf{1}, & a \otimes \neg a &= \mathbf{0}, \\ \neg(a \oplus b) &= \neg a \otimes \neg b, & \neg(a \otimes b) &= \neg a \oplus \neg b, \\ a &= \neg\neg a, & \neg\mathbf{0} &= \mathbf{1}, \\ \neg(\neg a \oplus b) \oplus b &= \neg(\neg b \oplus a) \oplus a. \end{aligned}$$

Lattice operations in an MV-algebra can be introduced as follows:

$$a \vee b = \neg(\neg a \oplus b) \oplus b, \quad a \wedge b = \neg(\neg a \vee \neg b).$$

Residuation in an MV-algebra can be defined as $a \rightarrow b = \neg a \oplus b$. It holds that every MV-algebra is a residuated lattice, and that a residuated lattice is an MV-algebra iff

$$(a \rightarrow b) \rightarrow b = a \vee b.$$

The fuzzy logic in a narrow sense with evaluated syntax is based on Łukasiewicz MV-algebra $\mathcal{L}$ of truth values:

$$\mathcal{L} = \langle L, \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where the set of truth values is $L = [0, 1]$, with standard ordering and Łukasiewicz operations $\otimes$, $\oplus$ and $\neg$, defined as follows:

$$a \otimes b = \max(a + b - 1, 0), \tag{3.1}$$

$$a \oplus b = \min(a+b, 1), \tag{3.2}$$

and

$$\neg a = 1 - a \tag{3.3}$$

for $a, b \in L$. Residuation in Łukasiewicz MV-algebra is given by

$$a \rightarrow b = \max(1 - a + b, 0). \tag{3.4}$$

The following simple lemma will be used in Chapter 6.

Lemma 3.3
Let $\mathcal{L}$ be Łukasiewicz MV-algebra, $a, b \in L$. If $b < a$ then $\neg b \otimes a > 0$.

PROOF: $\neg b \otimes a = (1 - b + a - 1) \vee 0 = (a - b) \vee 0$. If $b < a$ then $b - a > 0$. □

3.1.2 Language and syntax of FLn

The language J of FLn consists of the following:

(i) Countable set of object variables $x_1, x_2, \ldots$.

(ii) Set of object constants $\mathbf{u}_i$, $i \in I$.

(iii) Finite or countable set of function symbols $f, g, \ldots$ together with their arities.

(iv) Nonempty finite or countable set of predicate symbols $P, Q, \ldots$ together with their arities.

(v) Logical constants $\{\mathfrak{a} \mid a \in L\}$.

(vi) Symbols $\Rightarrow$ and $\forall$ for implication and universal quantifier, respectively.

(vii) Auxiliary symbols (brackets etc.).

Other standard (fuzzy) logical connectives $\neg$, $\wedge$, &, $\vee$, ∇ and $\Leftrightarrow$ and existential quantifier $\exists$ are introduced in usual way. For definitions of terms and formulas see [27], Section 4.3. The set of all well-formed formulas of language J is denoted by F_J. The set of inference rules and fuzzy set of axioms is also assumed to be identical with that one of [27], Section 4.3.

Let $A(x_1, \ldots, x_n)$ be a formula and $t_1, \ldots, t_n$ be terms substitutable into A for the variables $x_1, \ldots, x_n$, respectively. By $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ we denote an instance of A in which all the free occurrences of the variables $x_1, \ldots, x_n$ are replaced by the respective terms $t_1, \ldots, t_n$.

3.1.3 Semantics of FLn

The semantics is defined by generalization of the classical semantics of predicate logic. A *structure* for a language J is

$$\mathcal{V} = \langle V, f_{\mathcal{V}}, \ldots, P_{\mathcal{V}}, \ldots, u_{\mathcal{V}}, \ldots \rangle \tag{3.5}$$

where V is a nonempty set, $f_{\mathcal{V}} : V^n \longrightarrow V$ are n-ary functions on V assigned to function symbols $f \in J$ of arity n, $P_{\mathcal{V}} \underset{\sim}{\subset} V^m$ are m-ary fuzzy relations on V assigned to predicate symbols $P \in J$ of arity m and $u_{\mathcal{V}} \in V$ are designated elements assigned to object constants $\mathbf{u} \in J$. If the concrete symbols $f_{\mathcal{V}}, P_{\mathcal{V}}, u_{\mathcal{V}}, \ldots$ are unimportant for the explanation then we will simplify (3.5) only to $\mathcal{V} = \langle V, \ldots \rangle$. When we are dealing with a particular structure $\mathcal{D}$, we extend the language J to language $J(\mathcal{D}) = J \cup \{\mathbf{d} \mid d \in D\}$ in such a way that if $d \in D$ then $\mathbf{d} \in M$ is a constant denoting it.

Interpretation of a closed term $t \in M$ in a structure $\mathcal{V}$ is understood as a mapping denoted by $\mathcal{V}$

$$\begin{aligned} \mathcal{V} :\ & M \longrightarrow V \\ & t \longmapsto \mathcal{V}(t) \end{aligned}$$

which is defined as follows:

$$\begin{aligned} \mathcal{V}(\mathbf{u}_i) &= u_i, \quad \mathbf{u}_i \in J,\ u_i \in V, \\ \mathcal{V}(\mathbf{d}) &= d, \quad d \in V, \\ \mathcal{V}(f(t_1, \ldots, t_n)) &= f_{\mathcal{V}}(\mathcal{V}(t_1), \ldots, \mathcal{V}(t_n)). \end{aligned}$$

Interpretation of closed formulas is a (partial) mapping denoted also by $\mathcal{V}$:

$$\begin{aligned} \mathcal{V} :\ & F_J \longrightarrow \mathcal{L} \\ & A \longmapsto \mathcal{V}(A). \end{aligned}$$

Let $t_1, \ldots, t_n$ be closed terms. Then

$$\begin{aligned} \mathcal{V}(\mathbf{\mathfrak{a}}) &= a, \quad a \in L, \\ \mathcal{V}(P(t_1, \ldots, t_n)) &= P_{\mathcal{V}}(\mathcal{V}(t_1), \ldots, \mathcal{V}(t_n)), \\ \mathcal{V}(A \Rightarrow B) &= \mathcal{V}(A) \rightarrow \mathcal{V}(B), \\ \mathcal{V}((\forall x)A) &= \bigwedge \{\mathcal{V}(A_x[\mathbf{v}]) \mid v \in V\}. \end{aligned}$$

Interpretation of derived connectives $\neg$, $\wedge$, &, $\vee$, ∇ and $\Leftrightarrow$ and of existential quantifier $\exists$ is as follows:

$$\begin{aligned}
\mathcal{V}(\neg A) &= 1 - \mathcal{V}(A),\\
\mathcal{V}(A \wedge B) &= \mathcal{V}(A) \wedge \mathcal{V}(B),\\
\mathcal{V}(A \,\&\, B) &= \mathcal{V}(A) \otimes \mathcal{V}(B),\\
\mathcal{V}(A \vee B) &= \mathcal{V}(A) \vee \mathcal{V}(B),\\
\mathcal{V}(A \nabla B) &= \mathcal{V}(A) \oplus \mathcal{V}(B),\\
\mathcal{V}(A \Leftrightarrow B) &= \mathcal{V}(A) \leftrightarrow \mathcal{V}(B),\\
\mathcal{V}((\exists x)A) &= \bigvee\{\mathcal{V}(A_x[\mathbf{v}]) \mid v \in V\}
\end{aligned}$$

where the operation $\leftrightarrow$ is defined by $a \leftrightarrow b = 1 - |a - b|$.

To complete the definition of the mapping $\mathcal{V}$, we have to define it for general (open) formulas. Let us denote by $FV(A)$ the set of variables from J which includes all free variables of the formula A. For interpretation of a general formula A in the structure $\mathcal{V}$ we need an evaluation of its free variables $e : FV(A) \longrightarrow V$. A formula $A(x_1, \ldots, x_n)$ is satisfied in $\mathcal{V}$ in the degree a by the evaluation e, $e(x_1) = v_1$, $e(x_2) = v_2, \ldots, e(x_n) = v_n$, (denoted by $\mathcal{V}^e(A) = a$), if

$$\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n]) = a. \tag{3.6}$$

A formula $A(x_1, \ldots, x_n)$ is true in $\mathcal{V}$ in the degree a (denoted by $\mathcal{V}(A) = a$) if

$$a = \bigwedge\{\mathcal{V}^e(A) \mid e \text{ is an evaluation}\}.$$

In the sequel we will usually consider the following class of theories: with uncountable languages, where the set of closed terms M_J has the cardinality of continuum. Structures $\mathcal{V}$ in which we interpret these theories will be defined in such a way that its support has the same cardinality, usually the real interval $[a, b]$. Consequently, there is a bijection between M_J and the support of $\mathcal{V}$ and, if we equip the set M_J with appropriate operations, the before-mentioned bijection is an isomorphism (in the sense of isomorphisms of ordered abelian fields). We denote this isomorphism by $s_{\mathcal{V}} : M_J \longrightarrow V$. Then it is not necessary to extend the language J into $J(\mathcal{V})$, because for every element $v \in V$ there already exists (unique) term $\mathbf{v}$ (its abstract name).

With the help of the mapping $s_{\mathcal{V}}$ we can define the interpretation of a general formula $A(x_1, \ldots, x_n)$ in a structure $\mathcal{V}$ with an evaluation e, $e(x_1) = v_1$, $e(x_2) =$

$= v_2, \ldots, e(x_n) = v_n$, denoted by $\mathcal{V}^e(A) = a$, by formula formally identical to (3.6)

$$\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n]) = a \tag{3.7}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n \in J$ and it holds that $\mathbf{v}_i = s_{\mathcal{V}}^{-1}(v_i)$, $i = 1, \ldots, n$.

3.1.4 Many-sorted fuzzy predicate logic

To distinguish objects of various kinds in the analysis of fuzzy logic deduction, we employ many-sorted language. The main purpose is to divide objects which belong to antecedent and consequent parts of fuzzy IF-THEN rules. Definition of the language J from Subsection 3.1.2 is modified in the following way: there is a nonempty finite set $\mathcal{Y}$ of sorts, for every sort ι there is an infinite set of variables of the sort ι. Each nonempty finite sequence of sorts $\langle \iota_1, \ldots, \iota_n \rangle$ is a *type*. Then there is a nonempty set of predicate symbols P, Q, each having a type, set of function symbols f, g also having types, and sets of constant symbols $\mathbf{u}_\iota$. The other symbols are the same as in Subsection 3.1.2. Terms and formulas are defined in the same way as in fuzzy predicate logic with the exception that we have terms of various sorts.

Structures for many-sorted language are defined as follows:

Definition 3.4
A structure for the language J of many-sorted fuzzy predicate logic is

$$\mathcal{V} = \langle \{V_\iota \mid \iota \in \mathcal{Y}\}, P_V, \ldots, f_V, \ldots, u_V, \ldots \rangle$$

where V_ι are nonempty sets. If P is a predicate symbol of type $\langle \iota_1, \ldots, \iota_n \rangle$ then it is assigned a fuzzy relation $P_V \subsetapprox V_{\iota_1} \times \cdots V_{\iota_n}$. If f is a function symbol of type $\langle \iota_1, \ldots, \iota_n, \iota_{n+1} \rangle$ then it is assigned a (standard) function $f_V : V_{\iota_1} \times \cdots V_{\iota_n} \longrightarrow V_{\iota_{n+1}}$. Finally, if $\mathbf{u}_\iota$ is a constant symbol then it is assigned an element $u_{\mathcal{V}} \in V_\iota$.

3.1.5 Fuzzy theories, provability degrees, completeness

A fuzzy theory T is a fuzzy set of formulas $T \subsetapprox F_J$ given by the triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle$$

where $\mathrm{LAx} \subsetapprox F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \subsetapprox F_J$ is a fuzzy set of special axioms and R is a set of inference rules which includes the rules modus ponens

(r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}). In the following we suppose that SAx and R are the same as in [27], Section 4.3.1. If T is a fuzzy theory then its language is denoted by $J(T)$.

We will usually define a fuzzy theory only by the *fuzzy set of its special axioms*, i.e. we write

$$T = \{{}^{a}/A \mid \ldots\} \tag{3.8}$$

understanding that $a > 0$ in (3.8) and A is a special axiom of T.

We say that a structure $\mathcal{V}$ is a *model* of the fuzzy theory T and write $\mathcal{V} \models T$ if $\mathrm{SAx}(A) \leq \mathcal{V}(A)$ holds for every formula $A \in F_{J(T)}$.

By generalization of the classical definition of proof, it is possible to define *evaluated proof*, see [27], Definition 4.4, page 99. Given a fuzzy theory T and a formula A, we denote (some) evaluated proof of A by w_A and its value by $\mathrm{Val}(w_A)$. The concept of provability degree of a formula A is expressed by means of notation $T \vdash_a A$, which means that A is provable in the theory T in the degree a,

$$a = \bigvee \{\mathrm{Val}(w_A) \mid w_A \text{ is a proof of A}\}\,.$$

If $a = 1$ then we write $T \vdash A$ instead of $T \vdash_1 A$.

A fuzzy theory T is *inconsistent* if there is a formula A and proofs w_A and $w_{\neg A}$ of A and $\neg A$, respectively, such that

$$\mathrm{Val}(w_A) \otimes \mathrm{Val}(w_{\neg A}) > 0.$$

It is *consistent* in the opposite case.

Truth degree of a formula A in a theory T is expressed by means of notation $T \models_a A$. It means that A is true in T in the degree a,

$$a = \bigwedge \{\mathcal{V}(A) \mid \mathcal{V} \models T\}\,.$$

Let us stress that the provability degree coincides with the truth due to the completeness theorem.

Theorem 3.5 (Completeness)

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_{J(T)}$ and every consistent fuzzy theory T.

3.1.6 Independence of formulas

The notion of a set of independent formulas formalizes the intuitive idea of a collection of "different" formulas, e.g. atomic formulas with different predicate symbols.

Definition 3.6

(i) Two formulas A and B are independent if no variant or instance of one is a subformula of the other one.

(ii) Let $F_0 \subset F_J$ be a set of evaluated formulas, which fulfills the following conditions:

1. If ${}^{a}\!/A$, ${}^{b}\!/B \in F_0$ then A, B are independent.
2. To each A there is at most one $a > 0$ such that ${}^{a}\!/A \in F_0$.
3. If A is a logical axiom of FLn then ${}^{a}\!/A \in F_0$ implies $a = \mathrm{LAx}(A)$.

We will call F_0 the set of independent evaluated formulas. If formulas $A(x)$, $B(y)$ are independent, then also their respective instances are independent.

Lemma 3.7

Let F_0 be a set of independent evaluated formulas. Let $T = \{{}^{a}\!/A \mid {}^{a}\!/A \in F_0\}$ be consistent. Then there is a model $\mathcal{V} \models T$ such that

$$\mathcal{V}(A) = a \tag{3.9}$$

holds for all $\{{}^{a}\!/A\} \in F_0$.

Remark 3.8 (i) For the proof see [27], page 243.

(ii) The requirement of consistency of the theory T is necessary. For example, $\{{}^{1}\!/A \,\&\, \neg A\}$ is a set of independent formulas (with the cardinality equal to 1), whose corresponding theory T is evidently inconsistent.

(iii) This lemma plays an important role in the study of fuzzy logic deduction (Section 6). In general, models with property (3.9) may not exist. Nevertheless, a model with similar property can exist also in the case when the set of special axioms of the theory T is not a set of independent evaluated formulas, as is shown by the following Lemma, proved in [24]. Note that by *truth valuation* we mean a mapping which assigns truth degrees to formulas in a structure in the sense of [27], Definitions 4.16 and 4.17, page 118.

Lemma 3.9

Let T be a consistent fuzzy theory and S a set of closed atomic formulas such that $T \vdash_a A$, $a > 0$, for every $A \in S$. Let the fuzzy set of special axioms of the theory T fulfil for every formula B and every truth valuation $\mathcal{W} : F_{J(T)} \longrightarrow L$ the following condition: if A is an atomic subformula of B and $\mathrm{SAx}(A) \leq \mathcal{W}(A)$ then $\mathrm{SAx}(B) \leq \mathcal{W}(B)$. Then there is a model $\mathcal{W} \models T$ such that

$$\mathcal{W}(A) = a$$

for every $A \in S$.

Remark 3.10 Consider for example a theory $T = \{{}^{a}/A, {}^{b}/B, {}^{\mathbf{1}}/A \vee B\}$, $\mathbf{0} < a$, $b < \mathbf{1}$. Then, as it is shown in [24], the structure $\mathcal{V}$, in which $\mathcal{V}(A) = a$, $\mathcal{V}(B) = b$ holds, is not a model of T.

3.1.7 Isomorphism of models in fuzzy logic

The following definition is a generalization of Definition 4.32 from [27], page 169. We will employ it in Section 4.3.

Definition 3.11

Let $\mathcal{H}$ be a subset of a set of predicate symbols $\mathcal{P}$ of the language J. We say that two structures $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\mathcal{H}$-isomorphic in the degree c, $\mathcal{V}_1 \cong_c^{\mathcal{H}} \mathcal{V}_2$ if there is a bijection $g : V_1 \longrightarrow V_2$ such that the following hold for all $v_1, \ldots, v_n \in V_1$:

(i) for each couple of functions $f_{\mathcal{V}_1}$ in $\mathcal{V}_1$ and $f_{\mathcal{V}_2}$ in $\mathcal{V}_2$ assigned to function symbol $f \in J$,

$$g(f_{\mathcal{V}_1}(v_1, \ldots, v_n)) = f_{\mathcal{V}_2}(g(v_1), \ldots, g(v_n)).$$

(ii) It holds that

$$c = \bigwedge_{P \in \mathcal{H}} \bigwedge_{v_1, \ldots, v_n \in \mathcal{V}_1} (P_{\mathcal{V}_1}(v_1, \ldots, v_n) \leftrightarrow P_{\mathcal{V}_2}(g(v_1), \ldots, g(v_n))).$$

(iii) For each couple of constants u in $\mathcal{V}_1$ and v in $\mathcal{V}_2$ assigned to a constant symbol $\mathbf{u}$ in J,

$$g(u) = v.$$

If $\mathcal{H} = \mathcal{P}$, we say that structures $\mathcal{V}_1$ and $\mathcal{V}_2$ are *isomorphic* in the degree c. If $c = 1$, then $\mathcal{V}_1$ and $\mathcal{V}_2$ are *isomorphic*.

3.1.8 Theories with fuzzy equality

Fuzzy theory with fuzzy equality T contains in its language binary predicate $\approx$ which should fulfil the following axioms (cf. [23]):

(E1) ${}^{1}/(x \approx x)$.

(E2) There are $m_1, \dots m_n \geq 1$ such that for every n-ary function symbol f

$$
\begin{aligned}
{}^{1}/((x_1 \approx y_1)^{m_1} \Rightarrow (\dots \Rightarrow ((x_n \approx y_n)^{m_n} \Rightarrow \\
\Rightarrow (f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n)) \dots).
\end{aligned}
$$

(E3) There are $m_1, \dots m_n \geq 1$ such that for every n-ary predicate symbol P

$$
\begin{aligned}
{}^{1}/((x_1 \approx y_1)^{m_1} \Rightarrow (\dots \Rightarrow ((x_n \approx y_n)^{m_n} \Rightarrow \\
\Rightarrow (P(x_1, \dots, x_n) \Rightarrow P(y_1, \dots, y_n)) \dots).
\end{aligned}
$$

Usually the numbers $m_1, \dots, m_n$ are supposed to be equal to 1. It is also possible to restrict the validity of axioms (E2) and (E3) to only some of function and predicate symbols of the language in concern. It is also possible to have several different fuzzy equalities present in our language at the same time. It can be shown that fuzzy equality has the usually required properties of symmetry and transitivity. A special case of fuzzy equality is the crisp one, denoted by $=$, which also fulfills the crispness property $T \vdash (x = y) \vee \neg(x = y)$. It follows that

$$T \vdash (\forall x)(\forall y)((x = y) \Rightarrow (x \approx y)). \qquad (3.10)$$

3.2 Linguistic preliminaries

3.2.1 Linguistic predications

Throughout this thesis, we denote linguistic expressions by the *slanted typeface*. A general surface structure of fuzzy IF-THEN rule is

$$\mathsf{IF}\ \langle \text{noun} \rangle_1\ \mathsf{is}\ \mathcal{A}\ \mathsf{THEN}\ \langle \text{noun} \rangle_2\ \mathsf{is}\ \mathcal{B}. \qquad (3.11)$$

It is a conditional statement characterizing relation between linguistic expressions of the form

$$\langle \text{noun} \rangle\ \mathsf{is}\ \mathcal{A}. \qquad (3.12)$$

We will call expressions (3.12) *linguistic predications*.

Examples of such predications are *temperature is quite high*, *angle of the wheel is negative very big*, *(braking) force is more or less small*, etc. The structure of (3.12) is quite general since it is possible to transform a great deal of more complicated expressions into it. For example, the former predications can be obtained from *turn the wheel very much to the left*, or *brake but not too much*. Note that the latter possibility — transformation of commands into predications — has opened the door to the applications of fuzzy logic in control.

For the purpose of modeling using fuzzy logic, we are not usually interested in the objects denoted by nouns occurring in the linguistic predications. In practice, they are replaced by numbers. Therefore, we replace ⟨noun⟩ in (3.11) by some variable $X, Y, \ldots$, etc. Consequently, the general surface structure of fuzzy IF-THEN rule considered further is

$$\mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}. \tag{3.13}$$

3.2.2 Evaluating linguistic expressions

A special class of the expressions occurring in the linguistic predications, which deserves our attention, are *evaluating linguistic expressions* (cf. [27, 21]). These are special natural language expressions, which characterize sizes, distances, etc. In general, they characterize a position on an ordered scale. Among them, we distinguish *atomic evaluating expressions* which include any of the adjectives *small*, *medium*, or *big*(and possibly other adjectives of the same kind, such as *cold*, *hot*, etc.), or fuzzy quantity *approximately* z. The latter is a linguistic expression characterizing some quantity z from an ordered set.

Atomic evaluating expressions usually form pairs of antonyms, i.e. the pairs

⟨nominal adjective⟩ — ⟨antonym⟩.

Of course, there are a lot of pairs of antonyms, for example *young* — *old*, *ugly* — *nice*, *stupid* — *clever*, etc. When completed by the middle term, such as *medium*, *average*, etc., they form the so-called *basic linguistic trichotomy*. Let us stress that the basic linguistic trichotomy "*small*, *medium*, *big*" should be taken as canonical, which represents a lot of other corresponding trichotomies, such as "*short*, *average*, *long*", "*deep*, *medium deep*, *shallow*", etc.

Simple evaluating expressions are expressions of the form

$$\langle\text{linguistic hedge}\rangle\langle\text{atomic evaluating expression}\rangle.$$

Examples of simple evaluating expressions are *very small*, *more or less medium*, *roughly big*, *about twenty five*, *approximately* z, etc. Linguistic hedges are special adjectives modifying the meaning of adjectives before which they stand. In general, we speak about linguistic hedges with *narrowing effect* (*very*, *significantly*, etc.) and those with *widening effect* (*more or less*, *roughly*, etc.).

If $\mathcal{A}$ in (3.12) is an evaluating expression then (3.12) is called *evaluating linguistic predication.* If $\mathcal{A}$ is simple then (3.12) is called *simple evaluating predication.* Examples of simple evaluating predications are, e.g., *temperature is very high* (here *high* is taken instead of *big*), *pressure is roughly small*, *income is roughly three million*, etc.

Various linguistic expressions can be connected by the connectives *AND* and *OR*, thus forming *compound linguistic expressions*

$$\mathcal{C} := \mathcal{A}\ \langle\text{connective}\rangle\ \mathcal{B}. \tag{3.14}$$

If $\mathcal{A}$, $\mathcal{B}$ are evaluating linguistic expressions (or predications) then (3.14) is correspondingly called *compound evaluating linguistic expression (or predication).*

All the above discussed expressions of natural language (evaluating expressions, predications, etc.) form a part $\tilde{\mathcal{S}}$ of natural language, which is formalized using the means of fuzzy logic. Besides others, we must introduce the concepts of intension, extension and possible world. Let us stress here that we do not pretend that the formalism used in the subsequent sections enables us to model natural language semantics as a whole. We better say that our formalism is powerful enough to model such its part, which covers at least evaluating expressions, evaluating predications and simple conditional sentences formed from them. Hence, when speaking about a linguistic expression, we will usually have in mind some of the expressions from the above-considered set $\tilde{\mathcal{S}}$ in the sequel.

3.3 Meaning of linguistic expressions

3.3.1 Intensions and extensions

A linguistic expression may in general be understood as a name of some property. In the linguistic theory, we speak about its *intension* (instead of property named by it). Furthermore, we have to consider a *possible world* (cf. [20, 33]), which can be informally understood as "a particular state of affairs". For us, the possible world is a set of objects, which may carry the properties in concern. Hence, the intension of the linguistic expression determines in each possible world its *extension*, i.e. a grouping of objects having the given property. Since there can exist infinite number of possible worlds, one intension may lead to a class of extensions.

We will formalize these concepts using the means of predicate FLn with evaluated syntax. The level of formal syntax is identified with the syntax of FLn and the semantic level is identified with the semantics of FLn. In the sequel, we suppose some fixed predicate language J. By F_J we denote the set of well-formed formulas and by M the set of all closed terms of J (we will further suppose that M contains at least two elements).

To define the mathematical model of the *intension* of a linguistic expression $\mathcal{A} \in \tilde{\mathcal{S}}$, we start by assigning some formula $A(x) \in F_J$ to $\mathcal{A}$. However, this is not sufficient since this does not grasp inherent vagueness of the property represented by $\mathcal{A}$. This can be accomplished in FLn using the concept of *evaluated formula*. Namely, if $A(x)$ is a formula with one free variable then the evaluated formula $a\big/A_x[t]$ means that some object represented by the term t has the property A in the degree at least $a \in L$. This renders the hint for formalization of intensions of linguistic expressions given in Definition 3.12 below.

3.3.2 Possible worlds

The *extension* is characterized on the semantic level, which is identified with the semantics of FLn. Hence, the concept of *possible world* is understood as a special structure $\mathcal{V}$ for J

$$\mathcal{V} = \langle V, P_{\mathcal{V}}, \ldots \rangle.$$

We will usually suppose that all fuzzy relations assigned to predicate symbols of J in the possible world $\mathcal{V}$ have continuous membership function. Moreover, some further assumptions on $\mathcal{V}$ can be made, for example the unimodality of some membership functions, specific topological structure defined on the support V, etc. When dealing with evaluating linguistic expressions, V is assumed to be a *linearly ordered interval* $V = [{}^l v, {}^r v]$.

Therefore, in the sequel we make the following convention: for a given theory T, we say that $\mathcal{V}$ is a *model* of T if $\mathcal{V} \models T$. We say that $\mathcal{V}$ is a *possible world* for T if

$\mathcal{V} \models T$ and the support of $\mathcal{V}$ is a real interval.

3.3.3 Meaning of evaluating expressions

Definition 3.12
Let $\mathcal{A} \in \tilde{\mathcal{S}}$ be a natural language expression and let it be assigned a formula $A(x)$.

(i) The intension of $\mathcal{A}$ is a set of evaluated formulas (also called *multiformula*)

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A}_{\langle x \rangle} = \left\{ {}^{a_t}\!/A_x[t] \,\middle|\, t \in M, a_t \in L \right\}. \tag{3.15}$$

(ii) The extension of $\mathcal{A}$ in the possible world $\mathcal{V}$ is the satisfaction fuzzy set

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ {}^{\mathcal{V}(A_x[\mathbf{v}])}\!/v \,\middle|\, v \in V \right\}. \tag{3.16}$$

(iii) The meaning of $\mathcal{A}$ is the couple

$$\mathrm{Mean}(\mathcal{A}) = \langle \mathrm{Int}(\mathcal{A}), \mathbf{Ext}(\mathcal{A}) \rangle$$

where $\mathbf{Ext}(\mathcal{A}) = \{\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) \mid \mathcal{V} \text{ is a possible world}\}$ is a class of all its extensions.

3.4 Other preliminaries

3.4.1 Tolerance relations

Definition 3.13

(i) A binary relation in a set is called *tolerance* if it is reflexive and symmetric.

(ii) Let R be a tolerance relation in the set X. A set $A \subseteq X$ is called a *block* of R if $\langle x, y \rangle \in R$ holds for all $x, y \in A$. A block A of R is called a *maximal block* if it is maximal wrt. set inclusion, i.e. if there is no block B of R such that $A \subset B$. The set of all maximal blocks of a tolerance relation R in X denoted by X/R is called *factor set* of X by R.

The following lemma will be used in the proof of Theorem 6.22, Section 6.3.

Lemma 3.14
Let A and B be sets and $f : A \longrightarrow B$ a function. Let R_A and R_B be tolerance relations on A and B, respectively, such that for all $x, y \in A$

$$\textit{if } \langle x, y \rangle \in R_A \textit{ then } \langle f(x), f(y) \rangle \in R_B \tag{3.17}$$

holds true. Then the following holds:

$$\text{If } A \text{ is a block of } R_A \text{ then } f(A) \text{ is a block of } R_B.$$

PROOF: We have to show that for any $b_1, b_2 \in f(A)$ it holds that $\langle b_1, b_2\rangle \in R_B$. For any $b_1, b_2 \in f(A)$ there are $a_1, a_2 \in A$ such that $f(a_1) = b_1$, $f(a_2) = b_2$. Because A is a block of R_A, $\langle a_1, a_2\rangle \in R_A$. By (3.17), $\langle f(a_1), f(a_2)\rangle = \langle b_1, b_2\rangle \in R_B$. □

3.4.2 Unimodality

The notion of unimodal real function will be important for our study of modeling of meaning of evaluating linguistic expressions, and especially in the investigation of inconsistencies in linguistic descriptions in Section 6.3.

Definition 3.15 (Unimodality)
Let $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$ be real intervals, f a continuous function from $[a_1, b_1]$ to $[a_2, b_2]$, let $M = \sup_{y\in[a_1,b_1]} f(y)$. The function f is called *unimodal*, if there are $c_1, c_2, d_1, d_2 \in [a_1, b_1]$, $d_1 \le c_1 \le c_2 \le d_2$ such that $f(x) = M$ for $x \in [c_1, c_2]$, restriction of f to $[d_1, c_1]$ is strictly increasing and to $[c_2, d_2]$ is strictly decreasing, and restrictions of f to $[a_1, d_1]$ and $[d_2, b_1]$ are constant functions (see Figure 3.1). The set of all unimodal functions from $[a_1, b_1]$ to $[a_2, b_2]$ is denoted by $\mathcal{C}_u$.

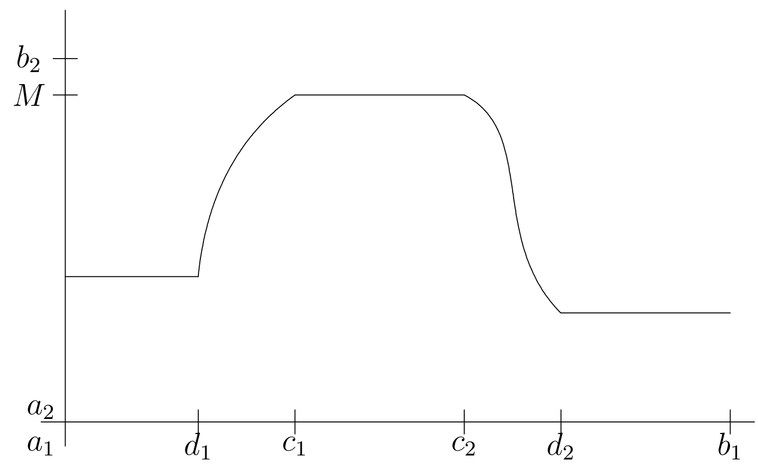

Figure 3.1: Example of unimodal function.

Lemma 3.16
Let $\{f^i \mid i \in J\}$ be a finite family of unimodal functions from $[a_1, b_1]$ to $[a_2, b_2]$. If $\bigcap_{i\in J}[c_1^i, c_2^i] \neq \emptyset$ holds, then the function $f(x) = \bigvee_{i\in J} f^i(x)$ is also unimodal.

PROOF: Let $M^j = \sup_{x\in[a_1,b_1]} f^j(x)$. The assumptions imply that there exists $w \in [a_1, a_2]$ such that $f(w) = M^j$ for all $j \in J$. It follows that the restrictions of f^j to $[a_1, w]$ and to $[w, a_2]$ are non-decreasing and non-increasing, respectively. It holds that if $\{g^i \mid i \in I\}$ is a family of non-decreasing (non-increasing) functions, then the function $g(x) = \bigvee_{i\in I} g^i(x)$ is non-decreasing (non-increasing). Hence, the restrictions of f to $[a_1, w]$ and $[w, a_2]$ are non-decreasing and non-increasing, respectively. It follows that $f = \bigvee_{i\in J} f^j$ is unimodal. □

Remark 3.17 (i) We can distinguish three types of unimodal functions, which in the literature are usually denoted by Z, S and Π (see Figure 3.2 and Definition 3.18) where S denotes non-decreasing unimodal functions, Z non-increasing unimodal functions, and Π functions which could be divided to non-decreasing and non-increasing parts. We can introduce a relation uni on the set $\mathcal{C}_u$ of all unimodal functions. Two functions f_1 and f_2 are in relation uni if they possess the same type of unimodality (one of Z, S, Π). It is easy to see that the relation uni is an equivalence.

(ii) There is also another classification of unimodal functions, which will be used in Chapter 6. Due to this classification, two unimodal functions are in the same class if the intervals in which they attain its maximal values have nonempty intersection.

Definition 3.18

(i) Unimodal function f is of type

1. Z if $a_1 = c_1 = d_1$ and $c_2 \neq b_1$.
2. S if $b_1 = c_2 = d_2$ and $c_1 \neq a_1$.
3. Π otherwise.

(ii) Two unimodal functions f^1 and f^2 are in relation uni iff they both possess the same type of unimodality (one of Z, S or Π).

(iii) Two unimodal functions $f^1, f^2 : [a_1, b_1] \longrightarrow [a_2, b_2]$ are in relation uni$_m$ iff there exists $x_0 \in [a_1, b_1]$ such that $f^1(x_0) = \bigvee_{x\in[a_1,b_1]} f^1(x)$ and $f^2(x_0) = \bigvee_{x\in[a_1,b_1]} f^2(x)$.

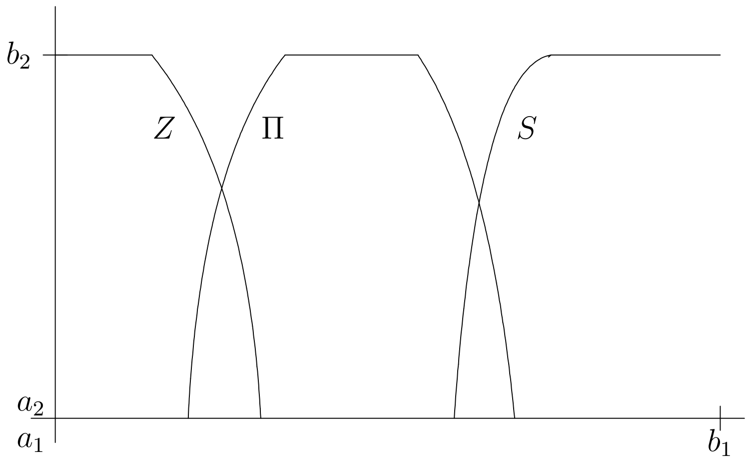

Figure 3.2: Three types of unimodal functions.

Lemma 3.19

(i) Any unimodal function f has just one type of unimodality.

(ii) The relation uni *is an equivalence relation.*

(iii) The relation uni_m *is reflexive and symmetric.*

PROOF: Immediate. □

Remark 3.20 It is easy to see that the relation uni_m is not, in general, a transitive one. Hence, it is a *tolerance relation* (see Section 3.4.1 and [32]). It follows that the maximal blocks of the relation uni_m (i.e. maximal subsets U of $\mathcal{C}_u$ such that for all $f_i, f_j \in U$ $f_i \,\mathrm{uni}_m\, f_j$) induced by this relation are not disjoint, in general. Note that if there is a family of unimodal functions $\{f_i \mid i \in I\}$, where I is some index set, and for all $i, j \in I$ it holds that $f_i \,\mathrm{uni}_m\, f_j$, then Lemma 3.16 applies to this family and the function $\bigvee_{i \in J} f_i$ is an unimodal one. We will use this fact in Chapter 6. The next lemma shows a condition under which both relations uni and uni_m coincide.

Lemma 3.21

Let $\mathcal{U} = \{f^i \mid i \in I\}$, $\mathcal{U} \subseteq \mathcal{C}_u$ be a family of unimodal functions,

$$f^i : [a_1, b_1] \longrightarrow [a_2, b_2]$$

for all $i \in I$. Denote by M^{f_i} the value $\sup_{x \in [a_1, b_1]} f^i(x)$. Let the following conditions hold:

(i) There exist $x_0 \in [a_1, b_1]$ and $M \in [a_2, b_2]$ such that for all functions $\{f^j \mid j \in J\}$ of type Π from $\mathcal{U}$, $J \subseteq I$, it holds that $f^j(x_0) = M^{f_j} = M$.

(ii)

$$(f^i \,\mathrm{uni}_m\, f^j) \textit{ implies } (f^i \,\mathrm{uni}\, f^j)$$

for all $i, j \in I$.

Then the relations uni *and* uni_m *coincide.*

PROOF: We have to show that

$$(f^i \,\mathrm{uni}\, f^j) \text{ implies } (f^i \,\mathrm{uni}_m\, f^j) \tag{3.18}$$

For two unimodal functions f_1 and f_2 of type Z or S, (3.18) follows from the fact that there is always $x_0 = a_1$ (or $x_0 = b_1$) such that $f_1(x_0) = f_2(x_0) = M_1 = M_2$. For two unimodal functions of type Π, the existence of such x_0 follows directly from assumption (i). □

The following lemma will be used in the proof of Theorem 6.22, Section 6.3.

Lemma 3.22
Let f^1 and f^2 be two unimodal functions from $[a_1, b_1]$ to $[a_2, b_2]$ satisfying

(i) $\bigvee_{x \in [a_1, b_1]} f^1(x) = \bigvee_{x \in [a_1, b_1]} f^2(x)$,

(ii) f^1 and f^2 are not in the relation uni_m.

Then a function $f(x) = f^1(x) \vee f^2(x)$ is not unimodal.

PROOF: Denote by M the value $M = \bigvee_{x \in [a_1, b_1]} f^1(x) = \bigvee_{x \in [a_1, b_1]} f^2(x)$. Suppose that $f^1(x_1) = f^2(x_2) = M$. Because f^1 and f^2 are not in the relation uni_m, we know that either $x_1 < x_2$ or $x_1 > x_2$. Suppose $x_1 < x_2$. Then it follows from the continuity of functions f^1 and f^2 that there exists x_0 such that $x_1 < x_0 < x_2$ and $f^1(x_0) < M$ as well as $f^2(x_0) < M$. It follows that $f(x_0) = f^1(x_0) \vee f^2(x_0) < M$. Because $f(x_1) = f(x_2) = M$, it follows that the function $f(x) = f^1(x) \vee f^2(x)$ is not unimodal. The case $x_1 > x_2$ can be proved similarly. □

Remark 3.23 In the following text we will need to extend the notion of unimodal functions in such a way that it includes also (non-continuous) intervals, i.e. functions

$$f(x) = \begin{cases} e_1, & x \in [c_1, c_2], \\ e_2 & \text{otherwise} \end{cases}$$

where $e_1 > e_2$. If $c_1 = c_2$, then this interval degenerates to a single point. Definitions of the relations uni and uni_m are identical for this wider class of unimodal functions and it can be demonstrated that all lemmas in this section also hold, so we will not explicitly distinguish between these two classes in further explanation.

Chapter 4

Theories for Modeling of the Meaning of Evaluating Linguistic Expressions

Let us remind that in the following, we say that a structure $\mathcal{V}$ is a *model* of the theory T if it holds that $\mathrm{SAx}_T(A) \leq \mathcal{V}(A)$ for all $A \in F_{J(T)}$. We call a structure $\mathcal{V}$ a *possible world* for the theory T if it is a model of T and its support is a real interval $[v^l, v^r]$ (or, in case of many-sorted languages, intervals $[v^{l_1}, v^{r_1}], \ldots, [v^{l_n}, v^{r_n}]$). Moreover, membership functions – interpretations of predicate symbols from $J(T)$ are continuous with respect to the standard metric on reals.

4.1 Definition of theory T^{ev}

In this section we state several natural requirements which a logical theory aimed at a characterization of the meaning of linguistic expressions should fulfil. We are not going to define one ("unique", "best") theory, but postulate several axioms which all such theories should obey. We work on a lower level of abstraction than in [21]. We suppose that the members of the set $\mathcal{S}$ of simple evaluating expressions are translated into our formal system by means of unary predicate symbols, i.e. we have no counterpart of linguistic hedges in our language. However, even if we consider also some formalization of hedges (by unary connectives), it is possible to transform it into the system presented here, and to study its properties (e.g. if the proposed interpretation of linguistic hedges on syntactic level is done in such a way that it guarantees correct behavior of the overall formal system with respect to deduction) and to compare it with other ones.

We denote a fuzzy theory aimed at a characterization of simple evaluating expressions by T^{ev} and its first-order language by $J(T^{ev})$. The (finite) nonempty set of simple evaluating linguistic expressions is denoted by $\mathcal{S}$. The corresponding set of unary predicate symbols in the language $J(T^{ev})$ is denoted by $\mathcal{G}$. We denote by $m : \mathcal{S} \longrightarrow \mathcal{G}$ a bijection between $\mathcal{S}$ and $\mathcal{G}$ which assigns to every simple evaluating expression $\mathcal{A} \in \mathcal{S}$ the unary predicate $m(\mathcal{A}) \in \mathcal{G}$. The set of all closed terms of $J(T^{ev})$ is denoted by M. In Definition 3.12, an intension of linguistic expression has been generally defined as a set of (instances of) evaluated formulas. In the following, we use provability degrees in certain formal theory T as the evaluations of instances of formulas.

The intension of simple evaluating expression $\mathcal{A} \in \mathcal{S}$ in a theory T is then

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A} = \left\{ \tilde{\alpha}_G(t) / G_x[t] \mid G = m(\mathcal{A}), t \in M, T \vdash_{\tilde{\alpha}_G(t)} G_x[t] \right\} \tag{4.1}$$

where $\tilde{\alpha}_G : M \longrightarrow L$ is a function called an *intensional mapping*. Note that $\tilde{\alpha}_G(t)$ depends also on the theory T we are working in. Because $\mathcal{G}$ is a set of unary predicate symbols, for every nonempty subset $\{G_i \mid i \in I\} \subseteq \mathcal{G}$ holds that the set $\left\{ \tilde{\alpha}_{G_i}(t) / G_{i,x}[t] \mid i \in I, t \in M \right\}$ is a set of independent evaluated formulas in the sense of Definition 3.6.

Definition 4.1
The fuzzy theory T^{ev} with the set of unary predicate symbols $\mathcal{G}$, $\mathcal{G} \subset J(T^{ev})$ is called a *theory of evaluating expressions* if it fulfills the following conditions:

1. The language $J(T^{ev})$ contains a set of constants large enough for the representation of real numbers, e.g. let the set of constant symbols M (which is simultaneously also the set of closed terms, provided that there are no function symbols in $J(T^{ev})$ be $M = \left\{ t^{(z)} \mid z \in [0,1] \right\}$.

2. For every formula $G(x)$, $G \in \mathcal{G}$, there exist closed terms $t_1, t_2 \in M$ such that $T^{ev} \vdash G_x[t_1]$ and $T^{ev} \vdash_0 G_x[t_2]$.

3. Let $\gamma_G : [0,1] \longrightarrow [0,1]$ be functions adjoined to functions $\tilde{\alpha}_G$ by putting

$$\gamma_G(z) = v \quad \text{iff} \quad \tilde{\alpha}_G\left(t^{(z)}\right) = v \quad \text{iff} \quad T^{ev} \vdash_v G_x\left[t^{(z)}\right]. \tag{4.2}$$

   The functions γ_G are continuous with respect to standard topology on $\mathbb{R}$, differentiable (with the exception of at most finitely many points) and unimodal (see Definition 3.15).

4. For all $G, G' \in \mathcal{G}$, if $\gamma_G = \gamma_{G'}$ then $G = G'$.

5. For every $t \in M$ there is at least one $G \in \mathcal{G}$ such that $T^{ev} \vdash_c G_x[t]$ with $c > 0$.

6. $$T^{ev} \vdash t^{(z_1)} \neq t^{(z_2)} \text{ iff } z_1 \neq z_2.$$

7. $$T^{ev} \vdash t^{(z_1)} \leq t^{(z_2)} \text{ iff } z_1 \leq z_2$$

   in the standard ordering on $[0,1]$. Further, the following two axioms which ensure the existence of endpoints of linear ordering must be introduced:

   (i) $(\exists x)(\forall y)(x \leq y)$,

   (ii) $(\exists x')(\forall y)(y \leq x')$

   and crispness of $\leq$:

   $$(\forall x)(\forall y)(x \leq y) \vee \neg(x \leq y) \tag{4.3}$$

   and $=$:

   $$(\forall x)(\forall y)(x = y) \vee \neg(x = y). \tag{4.4}$$

Lemma 4.2
There exists a consistent theory $T^\star$ which fulfills the requirements of Definition 4.1.

PROOF: Let the language of $T^\star$ be $J(T^\star) = \langle \leq, G_1, G_2, \{t^{(z)} \mid z \in [0,1]\}\rangle$ where $\leq$ is binary predicate symbol, G_1 and G_2 are unary predicate symbols. Let the set of special axioms of the theory $T^\star$ be

$$\begin{aligned} \mathrm{SAx}_{T^\star} = &\left\{\tilde{\alpha}_{G_1}(t)/G_{1,x}[t] \mid t \in M\right\} \cup \left\{\tilde{\alpha}_{G_2}(t)/G_{2,x}[t] \mid t \in M\right\} \cup \\ &\cup \left\{1/t^{(z_1)} < t^{(z_2)} \mid z_1 < z_2, z_1, z_2 \in [0,1]\right\} \cup \left\{1/(\exists x)(\forall y)(x \leq y)\right\} \\ &\cup \left\{1/(\exists x')(\forall y)(y \leq x')\right\} \cup \left\{1/(\forall x)(\forall y)(x \leq y) \vee \neg(x \leq y)\right\} \\ &\cup \left\{1/(\forall x)(\forall y)(x = y) \vee \neg(x = y)\right\} \end{aligned} \tag{4.5}$$

where $t^{(z_1)} < t^{(z_2)}$ is an abbreviation for $\left(t^{(z_1)} \leq t^{(z_2)}\right)$ & $\left(t^{(z_1)} \neq t^{(z_2)}\right)$. Let the functions $\tilde{\alpha}_{G_1}$ and $\tilde{\alpha}_{G_2}$ be

$$\tilde{\alpha}_{G_1}\left(t^{(z)}\right) = (1 - \frac{4}{3}z) \vee 0, \quad \tilde{\alpha}_{G_2}\left(t^{(z)}\right) = (-\frac{1}{3} + \frac{4}{3}z) \vee 0.$$

Let us define a structure $\mathcal{M} = \langle [0,1], \leq, \hat{G}_1, \hat{G}_2\rangle$ where $\leq$ is the standard ordering of real numbers, $\hat{G}_1$, $\hat{G}_2$ are unary fuzzy relations defined by $\hat{G}_1(x) = \tilde{\alpha}_{G_1}\left(t^{(x)}\right)$ and

$\hat{G}_2(x) = \tilde{\alpha}_{G_2}\left(t^{(x)}\right)$. The interpretation of constant symbols from $J(T^\star)$ is defined by $\mathcal{M}\left(t^{(z)}\right) = z$, $z \in [0,1]$. It is easy to see that $\mathcal{M}$ is a model of $T^\star$, hence $T^\star$ is consistent.

We show that $T^\star$ fulfills the requirements of Definition 4.1. Item 1 of Definition 4.1 is fulfilled due to the definition of $J(T^\star)$, for item 2 consider terms $t^{(0)}$ and $t^{(1)}$ for G_1: $T^\star \vdash G_{1,x}\left[t^{(0)}\right]$ because there is a special axiom of the form $1/G_{1,x}\left[t^{(0)}\right]$ in $\mathrm{SAx}_{T^\star}$. Because $\mathcal{M}\left(G_{1,x}\left[t^{(1)}\right]\right) = 0$, it follows that $T^\star \vdash_0 G_{1,x}\left[t^{(1)}\right]$. Analogously, $T^\star \vdash G_{2,x}\left[t^{(1)}\right]$ and $T^\star \vdash_0 G_{2,x}\left[t^{(0)}\right]$. Items 3, 4, 6 and 7 are straightforward, for item 5 consider the set $M_1 = \left\{t^{(z)} \mid z \in [0, 0.6]\right\}$, for which it holds that $T^\star \vdash_{c_t} G_{1,x}[t]$, $t \in M_1$, $c_t > 0$, and analogously $M_2 = \left\{t^{(z)} \mid z \in [0.4, 1]\right\}$, for which it holds that $T^\star \vdash_{c_t} G_{2,x}[t]$, $t \in M_2$, $c_t > 0$. It follows that for all $t \in M$, $T^\star \vdash_{c_t} G_x[t]$ for some $G \in \mathcal{G}$ and with $c_t > 0$. □

Remark 4.3 (i) In the previous definition we index the members of the set M of closed terms either by subscripts (e.g. t_1, t_2) where subscripts mean the usual indexing via some fixed index set, or by superscripts (e.g. $t^{(z)}, z \in [0,1]$), by which we denote (unique) closed term corresponding to the real number z. We will keep this notation throughout the rest of the thesis.

(ii) Item 2 of the previous definition articulates the requirement that the properties expressed by simple evaluating expressions should be completely valid for some objects and completely invalid for the other ones. Item 3 expresses the fact that vagueness cannot change abruptly (it is continuous). Item 4 requires that the adjoined functions γ_G should be pairwise different. The reason is that the predicate symbols G are formalizations of *different* evaluating linguistic expressions and it is counterintuitive to assign the same meaning to different expressions. Item 5 states that every abstract object $t^{(z)}$ corresponding to the real number z from $[0,1]$ has some property in a non-zero degree, i.e. that the set $\mathcal{S}$ of simple evaluating expressions is rich enough in the sense that there are no "gaps" for which no properties are described by the theory T^{ev}. Finally, Item 7 ensures us that the set M of closed terms is linearly ordered in the same way as interval $[0,1]$ and this ordering has endpoints.

(iii) The intensional mappings of the atomic formulas $G(x) \in \mathcal{G}$ can be computed inside some other theory, e.g. the *theory of evaluating syntagms* T^{EV} described in [21]. Formulas which correspond to simple evaluating expressions are there composed from predicate symbol for horizon L, members of a set of unary connectives $\lhd_\alpha$ representing linguistic hedges, and function symbol η for order--reversing automorphism. It means that these formulas are not independent, in general.

(iv) The intended semantics of the theory T^{ev} is as follows: the support of such a structure is a real interval, predicate symbols $G \in G$ are interpreted as unary fuzzy relations (fuzzy subsets) of D. Object constants $t^{(z)}$ are interpreted by real numbers $z \in D$ in such a way that if $z_1 < z_2$ then $\mathcal{D}\left(t^{(z_1)}\right) < \mathcal{D}\left(t^{(z_2)}\right)$.

(v) We require in Item 3 that functions $\gamma_G(z)$, $z \in [0,1]$ have to be unimodal. Because unimodal functions could be divided to several categories by means of the relations uni and uni_m (see Remark 3.17), we can define relations $\mathrm{uni}^{T^{ev}}$ and $\mathrm{uni}_m^{T^{ev}}$ on the set $\mathbf{S}_{\langle x\rangle}$ of intensions of simple evaluating expressions (for formal definitions see Definition 4.4). Two intensions $\mathbf{A}_{1,\langle x\rangle}$ and $\mathbf{A}_{2,\langle x\rangle}$ are in relation $\mathrm{uni}^{T^{ev}}$ iff for the functions γ_{A_i} associated to their intensional mappings $\tilde{\alpha}_{A_i}$ hold that $\gamma_{A_1}\,\mathrm{uni}\,\gamma_{A_2}$, and similarly is defined the relation $\mathrm{uni}_m^{T^{ev}}$. Note that these relations are dependent on the assignment of intentions $\mathbf{A}_{i,\langle x\rangle}$ to evaluating linguistic expressions $\mathcal{A}_i \in \mathcal{S}$ and consequently on theory T^{ev}. It is easy to see that the relations $\mathrm{uni}^{T^{ev}}$ and $\mathrm{uni}_m^{T^{ev}}$ are equivalence and tolerance relations, respectively. Therefore, the relation $\mathrm{uni}^{T^{ev}}$ induce a partition on the set $\mathbf{S}_{\langle x\rangle}$. In accordance with the discussion presented in Section 3.2, we prefer to use three atomic evaluating expressions *small*, *medium* and *big* in such a way that their intensional mappings belong to classes Z, Π and S, respectively. In this case the intensions $A_{\langle x\rangle} \in \mathbf{S}_{\langle x\rangle}$ of simple evaluating expressions are divided by relations $\mathrm{uni}^{T^{ev}}$ and $\mathrm{uni}_m^{T^{ev}}$ into three disjoint subsets and both relations coincide (due to Lemma 3.21). We can also include into the definition of the theory T^{ev} the requirement, that if two linguistic expressions $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same atomic expression, then their respective intensions $\mathbf{A}_{1,\langle x\rangle}$ and $\mathbf{A}_{2,\langle x\rangle}$ should be in the relation $\mathrm{uni}_m^{T^{ev}}$.

Definition 4.4

Denote by $\mathbf{S}_{\langle x\rangle}$ a set of intensions of simple evaluating linguistic expressions (with respect to a given theory T^{ev}), i.e.

$$\mathbf{S}_{\langle x\rangle} = \left\{\mathbf{A}_{\langle x\rangle} \mid A = m(\mathcal{A}) \text{ and } \mathcal{A} \in \mathcal{S}\right\}$$

where $\mathbf{A}_{\langle x\rangle} = \{\tilde{\alpha}_A(t)/A_x[t] \mid t \in M\}$. A relation $\mathrm{uni}^{T^{ev}} \subseteq \mathbf{S}_{\langle x\rangle} \times \mathbf{S}_{\langle x\rangle}$ is defined by means of the intensional mappings $\tilde{\alpha}_A$:

$$(\mathbf{A}_{1,\langle x\rangle}\,\mathrm{uni}^{T^{ev}}\,\mathbf{A}_{2,\langle x\rangle}) \;\text{ iff }\; (\gamma_{A_1}\,\mathrm{uni}\,\gamma_{A_2})$$

where the functions γ_{A_i}, $i = 1, 2$ are defined by (4.2) and the relation uni is defined in Definition 3.18. Analogously is defined a relation $\mathrm{uni}_m^{T^{ev}} \subseteq \mathbf{S}_{\langle x\rangle} \times \mathbf{S}_{\langle x\rangle}$:

$$(\mathbf{A}_{1,\langle x\rangle}\ \mathrm{uni}_m^{T^{ev}}\ \mathbf{A}_{2,\langle x\rangle}) \ \text{ iff } \ (\gamma_{A_1}\ \mathrm{uni}_m\ \gamma_{A_2}).$$

Lemma 4.5
For any theory of evaluating expressions T^{ev} it holds that:

(i) The relation $\mathrm{uni}^{T^{ev}}$ *is an equivalence relation.*

(ii) The relation $\mathrm{uni}_m^{T^{ev}}$ *is a tolerance relation.*

PROOF: Item 3 of Definition 4.1 requires that the functions $\tilde{\alpha}_A$ should be unimodal for any theory of linguistic description T^{ev}. The claims then follow from Definition 3.18 of relations uni and uni_m. □

4.2 Canonical possible worlds

Definition 4.6
A *canonical possible world* $\mathcal{V}_C$ of the theory of evaluating expressions T^{ev} is a model $\mathcal{V}_C = \langle [0,1], \ldots\rangle$ of T^{ev} such that $\mathcal{V}_C\left(t^{(z)}\right) = z$ for $z \in [0,1]$, for which the following property holds for all $G \in \mathcal{G}$ and $t \in M$:

$$\mathcal{V}_C(G_x[t]) = \tilde{\alpha}_G(t). \tag{4.6}$$

The question of existence of such a canonical possible world is addressed by the following theorem.

Theorem 4.7
Let a theory T fulfil the conditions of Definition 4.1. Suppose that the fuzzy set of special axioms of the theory T has the following property (C): For every formula B, every atomic formula $G_x[t]$, $G \in \mathcal{G}$, $t \in M$ and every truth valuation $\mathcal{W} : F_{J(T)} \longrightarrow L$ it holds that: If $G_x[t]$ is a subformula of B and

$$\mathrm{SAx}(G_x[t]) \leq \mathcal{W}(G_x[t])$$

then $\mathrm{SAx}(B) \leq \mathcal{W}(B)$. *Then there exists canonical possible world $\mathcal{V}_C \models T$ in the sense of Definition 4.6.*

PROOF: Due to item 2 of Definition 4.1, the theory T is consistent (because there exists some formula with provability degree equal to 0), hence it has a model. From item 6 of Definition 4.1 and Löwenheim-Skolem Theorem (cf. Theorem 4.35, [27]) it follows that there exists a model $\mathcal{U}$ of T of cardinality of continuum. From Item 7 of Definition 4.1 follows that the support of $\mathcal{U}$ is linearly ordered and this ordering has endpoints.

Consider a structure $\mathcal{V}$ with the support $V = [0, 1]$ and an order-preserving bijection $g : U \longrightarrow V$. Let the interpretation of constants in $\mathcal{V}$ be defined by $\mathcal{V}(t) = g(\mathcal{U}(t))$, of function symbols by $\mathcal{V}(f(t_1, \ldots, t_n)) = g(\mathcal{U}(f(t_1, \ldots, t_n)))$ and interpretation of predicate symbols by $\mathcal{V}(P(t_1, \ldots, t_n)) = \mathcal{U}(P(t_1, \ldots, t_n))$. It follows that $\mathcal{V}$ is a model of T.

Now we can construct a structure $\mathcal{V}_C$ from $\mathcal{V}$ which differs only in the evaluations of $G_x[t]$ defined by $\mathcal{V}_C(G_x[t]) = \tilde{\alpha}_G(t)$. It is possible because $G_x[t]$ are closed instances of atomic predicate formulas. We show that $\mathcal{V}_C$ is a model of T. If B contains no subformula of the form $G_x[t]$ then $\mathcal{V}_C(B) = \mathcal{V}(B) \geq \mathrm{SAx}(B)$. Otherwise, $\mathrm{SAx}(B) \leq \mathcal{V}_C(B)$ by assumption (C). It follows that $\mathcal{V}_C$ is a canonical possible world for T. □

Remark 4.8 A theory T which fulfills property (C) will be called *simple*. For simple theories of evaluating expressions, canonical possible worlds always exist. The existence of canonical possible worlds is important, because it guarantees the existence of possible worlds where membership functions of fuzzy sets, which are interpretations of predicate symbols $G \in \mathcal{G}$, have the properties required in Definition 4.1.

4.3 Extended theories

Canonical possible worlds from Definition 4.6 can be understood as models in which truth functions mimic the behavior of intensional mappings of predicate symbols $G \in \mathcal{G}$ from the theory T^{ev}. In the following we define a wider class of models in which the interpretations of predicate symbols $G \in \mathcal{G}$ behave "similarly" as in the canonical possible world $\mathcal{V}$, but these interpretations are restricted also from above. To this purpose we extend (some fixed) fuzzy theory T^{ev} by axioms on negations of (instances of) predicate symbols $G \in \mathcal{G}$ and we also require continuity and unimodality of membership functions of their interpretations in appropriate models.

Definition 4.9

An *extended theory of evaluating expressions* T^{evx} arises from the theory T^{ev} by expanding it by axioms

$$\left\{\tilde{\beta}_G(t)/\neg G_x[t] \mid G \in \mathcal{G}, t \in M\right\} \tag{4.7}$$

where $\tilde{\beta}_G(t) : M \longrightarrow L$ and

$$\tilde{\alpha}_G(t) \otimes \tilde{\beta}_G(t) = 0 \tag{4.8}$$

holds for all $t \in M$.

Remark 4.10 (i) The requirement (4.8) is equivalent to $\tilde{\beta}_G(t) \leq \neg\tilde{\alpha}_G(t)$ for all $t \in M$ and also to $\tilde{\alpha}_G(t) \leq \neg\tilde{\beta}_G(t)$ for all $t \in M$.

(ii) We can unify the notation by denoting both sets of evaluated formulas (i.e. instances of formulas $A_x[t]$ and $\neg A_x[t]$) by one symbol $\mathbf{A}^e$, i.e.

$$\mathbf{A}^e := \left\{\tilde{\alpha}_A(t)/A_x[t] \mid t \in M\right\} \cup \left\{\tilde{\beta}_A(t)/\neg A_x[t] \mid t \in M\right\}, \tag{4.9}$$

provided that (4.8) holds for all $t \in M$.

(iii) The values $\neg\tilde{\alpha}_G(t)$ are *consistency thresholds* for the formulas $G_x[t]$ in the sense of Definition 1 from [26]. There is also some relation to the concept of *intuitionistic fuzzy sets* introduced by K. Atanasov.

Lemma 4.11

If T^{ev} is simple (see Remark 4.8), then T^{evx} is consistent and $T^{evx} \vdash_d \neg G_x[t]$ where $d = \tilde{\beta}_G(t)$, for all $G \in \mathcal{G}$ and $t \in T$.

PROOF: The canonical possible world $\mathcal{V}_C$ of T^{ev} from Definition 4.6 is also a model of T^{evx}, because $\mathcal{V}_C(\neg G_x[t]) = \neg\mathcal{V}_C(G_x[t]) = \neg\tilde{\alpha}_G(t) \geq \tilde{\beta}_G(t) = \mathrm{SAx}(\neg G_x[t])$ holds. Therefore, T^{evx} is consistent. For the second part of lemma, it is sufficient to construct such a model $\mathcal{U}$ for which $\mathcal{U}(\neg G_x[t]) = \tilde{\beta}_G(t)$, which can be done by the same technique as in the proof of Theorem 4.7. The claim then follows from the Completeness Theorem. □

Definition 4.12

An *extended possible world* of T^{evx} is a model $\mathcal{V}_E$ of the extended theory of evaluating expressions such that its support is a closed interval $[a, b] \subset \mathbb{R}$, interpretation of closed terms is $\mathcal{V}_E\left(t^{(z)}\right) = a + (b - a)z$, $z \in [0, 1]$, $t^{(z)} \in M$ and interpretations of predicate symbols $G \in \mathcal{G}$ are continuous and unimodal functions $G_{\mathcal{V}_E} : [a, b] \longrightarrow L$.

Theorem 4.13
If there exists the canonical possible world $\mathcal{V}_C$ of T^{ev} then there exists also the extended possible world $\mathcal{V}_E$ of T^{evx} and

$$\tilde{\alpha}_G(t) \leq \mathcal{V}_E(G_x[t]) \leq \neg\tilde{\beta}_G(t) \tag{4.10}$$

holds for all $G \in \mathcal{G}$ and all $t \in M$.

PROOF: The canonical possible world $\mathcal{V}_C$ of T^{ev} is also a model of the extended theory T^{evx} (see the proof of Lemma 4.11).

For any structure $\mathcal{U}$ such that $\mathcal{U} \models T^{evx}$, $\mathcal{U}(G_x[t]) \geq \mathrm{SAx}(G_x[t]) = \tilde{\alpha}_G(t)$. Let us prove the inequality $\mathcal{V}_E(G_x[t]) \leq \neg\tilde{\beta}_G(t)$. Suppose that $\mathcal{U}(G_x[t]) > \neg\tilde{\beta}_G(t)$. Then $\mathcal{U}(\neg G_x[t]) = \neg\mathcal{U}(G_x[t]) < \tilde{\beta}_G(t) = \mathrm{SAx}(\neg G_x[t])$, hence $\mathcal{U}$ is not a model of T^{evx} – a contradiction. □

Lemma 4.14
Let T^{evx} be an extended theory of evaluating expressions such that for all $G \in \mathcal{G}$ and all $t \in M$, $\tilde{\beta}_G(t) = \neg\tilde{\alpha}_G(t)$. Then all extended possible worlds $\mathcal{V}_E$ of T^{evx} are $\mathcal{G}$-isomorphic (see Definition 3.11) with the canonical possible world $\mathcal{V}_C$.

PROOF: It is easy to see that

$$\left(\tilde{\beta}_G(t) = \neg\tilde{\alpha}_G(t)\right) \text{ implies } \left(\tilde{\alpha}_G(t) = \neg\tilde{\beta}_G(t)\right). \tag{4.11}$$

From (4.6) we obtain that $\mathcal{V}_C(G_x[t]) = \tilde{\alpha}_G(t)$ and from (4.10) and (4.11) it follows that $\tilde{\alpha}_G(t) = \mathcal{V}_E(G_x[t])$, i.e. $\mathcal{V}_C(G_x[t]) = \mathcal{V}_E(G_x[t])$. Let $g : V_C \longrightarrow V_E$ be defined by $g(z) = a + (b-a)z$. This is a bijection between $V_C = [0,1]$ and $V_E = [a,b] \subset \mathbb{R}$. Now we show that $G_{\mathcal{V}_C}(z) = G_{\mathcal{V}_E}(g(z))$.

$$G_{\mathcal{V}_C}(z) = \mathcal{V}_C\left(G_x\left[t^{(z)}\right]\right) = \tilde{\alpha}_G\left(t^{(z)}\right) = \mathcal{V}_E\left(G_x\left[t^{(z)}\right]\right) = G_{\mathcal{V}_E}(a+(b-a)z) = G_{\mathcal{V}_E}(g(z)).$$

Consequently, $\mathcal{V}_C$ and $\mathcal{V}_E$ are $\mathcal{G}$-isomorphic. □

As a generalization of this result for general extended theories we obtain the following

Theorem 4.15
Let T^{evx} be an extended theory of evaluating expressions. Then two extended possible worlds $\mathcal{V}_{E1}$, $\mathcal{V}_{E2}$ of T^{evx} are $\mathcal{G}$-isomorphic in the degree greater than or equal to

$$c = 1 - \bigvee_{G\in\mathcal{G}} \bigvee_{t\in M} \left(\neg\tilde{\beta}_G\left(t^{(z)}\right) - \tilde{\alpha}_G\left(t^{(z)}\right)\right). \tag{4.12}$$

PROOF: Let $V_{Ei} = [a_i, b_i]$, $i = 1, 2$. We have to show that there exists a function g, $g : V_{E1} \longrightarrow V_{E2}$ such that (i) g is a bijection between V_{E1} and V_{E2}, (ii) $g\left(\mathcal{V}_{E1}\left(t^{(z)}\right)\right) = \mathcal{V}_{E2}\left(t^{(z)}\right)$ for all $z \in [0, 1]$ and (iii) $(G_i^1(y) \leftrightarrow G_i^2(g(y))) \geq c$, where $G_i^j = \mathcal{V}_{Ej}(G_i)$, $G_i \in \mathcal{G}$, $j = 1, 2$ and $y \in V_{E1}$.

Let g be defined as the linear transformation of V_{E1} to V_{E2}, i.e. for $y \in V_{E1}$ let

$$g(y) = a_2 + \frac{b_2 - a_2}{b_1 - a_1}(y - a_1).$$

Then (i) is fulfilled, (ii)

$$g\left(\mathcal{V}_{E1}\left(t^{(z)}\right)\right) = g(a_1 + (b_1 - a_1)z) = a_2 + (b_2 - a_2)z = \mathcal{V}_{E2}\left(t^{(z)}\right).$$

(iii) Let $y \in V_{E1}$, $y = \mathcal{V}_{E1}(t)$ for some $t \in M$ and let $G_i \in \mathcal{G}$. Then from Theorem 4.13 it follows that

$$(G_i^1(y) \leftrightarrow G_i^2(g(y))) = 1 - |G_i^1(y) - G_i^2(g(y))| \geq 1 - (\neg\tilde{\beta}_{G_i}(t) - \tilde{\alpha}_{G_i}(t)),$$

because $g(y) = \mathcal{V}_{E2}(t)$. The claim follows from the fact that we considered general $y \in V_{E1}$ and $G_i \in \mathcal{G}$, and that there is a bijection between the set of closed terms M and the set V_{E1}. □

Hence, if we put $\tilde{\beta}_G(t) = \neg\tilde{\alpha}_G(t)$ for all $G \in \mathcal{G}$ and $t \in M$, then all possible worlds with the support $[a, b]$ and natural interpretation of object constants $t^{(z)}$, $z \in [0, 1]$ are $\mathcal{G}$-isomorphic with the canonical possible world $\mathcal{V}_C$. It means that possible worlds of such a theory behave in the same way as models of complete theories with respect to atomic formulas of the form $G_x[t]$ – their provability degrees copy the structure of truth values, i.e. if $T^{evx} \vdash_{\tilde{\alpha}_{G_1}(t)} G_{1,x}[t]$ and $T^{evx} \vdash_{\tilde{\alpha}_{G_2}(t)} G_{2,x}[t]$ then

$$T^{evx} \vdash_{c(t)} G_{1,x}[t] \mathbin{\&_\odot} G_{2,x}[t]$$

where $c(t) = \tilde{\alpha}_{G_1}(t) \odot \tilde{\alpha}_{G_2}(t)$. Here $\&_\odot$ is a propositional connective and $\odot$ is its semantic interpretation. This situation is convenient. However, it also seems to be too restrictive – all possible worlds are isomorphic and there is no freedom in choice of the interpretations of extensions of simple evaluating expressions.

Now we prove (with a use of the Completeness Theorem) a lemma about lower bounds for provability degrees of propositional formulas in extended theories. In the sequel we use $(A \not\Leftrightarrow B)$ as an abbreviation for $\neg(A \Leftrightarrow B)$.

Lemma 4.16
Let T be a consistent fuzzy theory, A and B closed formulas, $T \vdash_{d_1} A$, $T \vdash_{d_2} B$, $T \vdash_{e_1} \neg A$ and $T \vdash_{e_2} \neg B$, and it holds that $e_1 \leq \neg d_1$ and $e_2 \leq \neg d_2$, then

(i)
$$T \vdash_d A \Rightarrow B, \ \textit{where } d \geq \neg e1 \rightarrow d_2, \tag{4.13}$$

(ii)
$$T \vdash_d A \Leftrightarrow B, \ \textit{where } d \geq (\neg e_1 \rightarrow d_2) \wedge (\neg e_2 \rightarrow d_1), \tag{4.14}$$

(iii)
$$T \vdash_d A \not\Leftrightarrow B, \ \textit{where } d \geq ((d_1 \otimes e_2) \vee (d_2 \otimes e1)), \tag{4.15}$$

(iv)
$$T \vdash_d A \,\&\, B, \ \textit{where } d_1 \otimes d_2 \leq d \leq \neg e_1 \otimes \neg e_2, \tag{4.16}$$

(v)
$$T \vdash_d A \wedge B, \ \textit{where } d = d_1 \wedge d_2, \tag{4.17}$$

(vi)
$$T \vdash_d A \vee B, \ \textit{where } d_1 \vee d_2 \leq d \leq \neg e_1 \vee \neg e_2. \tag{4.18}$$

PROOF:

(i) We use Completeness Theorem: $T \vdash_d A \Rightarrow B$ implies

$$d = \bigwedge_{\mathcal{D}} \{\mathcal{D}(A \Rightarrow B) \mid \mathcal{D} \models T\} = \bigwedge_{\mathcal{D}} \{\mathcal{D}(A) \rightarrow \mathcal{D}(B) \mid \mathcal{D} \models T\} \geq$$
$$\geq \bigvee_{\mathcal{D}} \{\mathcal{D}(A) \mid \mathcal{D} \models T\} \rightarrow \bigwedge_{\mathcal{D}} \{\mathcal{D}(B) \mid \mathcal{D} \models T\} = \neg e_1 \rightarrow d_2$$

where we have used the fact that the operation $\rightarrow$ is antitonic in the first and isotonic in the second variable.

(ii) $A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A)$. By the same consideration as in the previous item, $T \vdash_d A \Leftrightarrow B$ implies

$$d = \bigwedge_{\mathcal{D}} \{\mathcal{D}(A \Leftrightarrow B) \mid \mathcal{D} \models T\} = \bigwedge_{\mathcal{D}} \{\mathcal{D}(A) \leftrightarrow \mathcal{D}(B) \mid \mathcal{D} \models T\} =$$
$$= \bigwedge_{\mathcal{D}} \{(\mathcal{D}(A) \rightarrow \mathcal{D}(B)) \wedge (\mathcal{D}(B) \rightarrow \mathcal{D}(A)) \mid \mathcal{D} \models T\} =$$
$$= \bigwedge_{\mathcal{D}} \{(\mathcal{D}(A) \rightarrow \mathcal{D}(B)) \mid \mathcal{D} \models T\} \wedge \bigwedge_{\mathcal{D}} \{(\mathcal{D}(B) \rightarrow \mathcal{D}(A)) \mid \mathcal{D} \models T\} \geq$$
$$\geq (\neg e_1 \rightarrow d_2) \wedge (\neg e_2 \rightarrow d_1).$$

(iii)

$$d = \bigwedge_{\mathcal{D}} \{\mathcal{D}(A \not\Leftrightarrow B) \mid \mathcal{D} \models T\} = \bigwedge_{\mathcal{D}} \{(\mathcal{D}(\neg(A \Leftrightarrow B)) \mid \mathcal{D} \models T\} =$$
$$= \bigwedge_{\mathcal{D}} \{1 - \mathcal{D}(A \Leftrightarrow B) \mid \mathcal{D} \models T\} = \bigwedge_{\mathcal{D}} \{1 - (\mathcal{D}(A) \leftrightarrow \mathcal{D}(B)) \mid \mathcal{D} \models T\} =$$
$$= 1 - \bigvee_{\mathcal{D}} \{\mathcal{D}(A) \leftrightarrow \mathcal{D}(B) \mid \mathcal{D} \models T\} = 1 - \bigvee_{\mathcal{D}} \{1 - |\mathcal{D}(A) - \mathcal{D}(B)| \mid \mathcal{D} \models T\} =$$
$$= \bigwedge_{\mathcal{D}} \{|\mathcal{D}(A) - \mathcal{D}(B)| \mid \mathcal{D} \models T\}.$$

It can be easily seen that the expression on the last line attains minimal value at the point

$$\left(\bigwedge_{\mathcal{D}} \{\mathcal{D}(A) \mid \mathcal{D} \models T\} - \bigvee_{\mathcal{D}} \{\mathcal{D}(B) \mid \mathcal{D} \models T\}\right) \vee$$
$$\vee \left(\bigwedge_{\mathcal{D}} \{\mathcal{D}(B) \mid \mathcal{D} \models T\} - \bigvee_{\mathcal{D}} \{\mathcal{D}(A) \mid \mathcal{D} \models T\}\right) \vee 0.$$

And this expression is equal to

$$((d_1 - \neg e_2) \vee (d_2 - \neg e1)) \vee 0$$

which could be written, using the definition of Łukasiewicz conjunction, as

$$((d_1 \otimes e_2) \vee (d_2 \otimes e1)).$$

(iv) The fact that $d_1 \otimes d_2 \leq d$ has been proved in the item (b) of Theorem 10 of [27]. Suppose that $d > \neg e_1 \otimes \neg e_2$. By the completeness, $d = \bigwedge_{\mathcal{D}} (\mathcal{D}(A \,\&\, B)) = \bigwedge_{\mathcal{D}} (\mathcal{D}(A) \otimes \mathcal{D}(B))$. But $\mathcal{D}(A) \leq \neg e_1$ and $\mathcal{D}(B) \leq \neg e_2$, so $\mathcal{D}(A) \otimes \mathcal{D}(B) \leq \neg e_1 \otimes \neg e_2$, hence $\bigwedge_{\mathcal{D}} (\mathcal{D}(A) \otimes \mathcal{D}(B)) \leq \neg e_1 \otimes \neg e_2$, which contradicts $d > \neg e_1 \otimes \neg e_2$.

(v) Coincides with item (d) of Theorem 10 of [27].

(vi) Analogous to the proof of item (iii).

□

Remark 4.17 Lemma 4.16 shows that in extended theories we can use the additional information provided by axioms of the form $\left\{\tilde{\beta}_G(t)/\neg G_x[t] \mid G \in \mathcal{G}, t \in M\right\}$

for sharper determination of the bounds of provability degrees for formulas of type $G_{1,x}[t]\&_{\odot}G_{2,x}[t]$. For example, the lower bound for the provability degree d of formula $A \Rightarrow B$ in a theory T, provided that we have $T \vdash_{d_1} A$ and $T \vdash_{d_2} B$, is $d \geq d_2$ (compare with formula (4.13)).

4.4 Theories T^{ev} and T^{evx} with fuzzy equality

When we use theories T^{ev} or extended theories T^{evx} it is advantageous to allow also the possibility to express similarity among objects. Therefore we extend the language $J(T^{ev})$ (or $J(T^{evx})$) by fuzzy equality predicate $\approx$ (see Section 3.1.8).

Definition 4.18
A theory of evaluating expressions $T^{ev}_{\approx}$ is called a *theory of evaluating expressions with fuzzy equality* if its language contains the fuzzy equality predicate $\approx$ and the set of its special axioms $\mathrm{SAx}(T^{ev}_{\approx})$ includes the axiom schemata (E1), (E2) and (E3) from Section 3.1.8 with the exponents $m_1, \ldots, m_n$ equal to 1.

Lemma 4.19
Let $t_1, t_2 \in M$. Denote by c the value

$$c = \bigvee_{\mathcal{D}} \{\mathcal{D}(t_1 \approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\}.$$

Then

$$T^{ev}_{\approx} \vdash_{\neg c} t_1 \not\approx t_2.$$

PROOF: From the Completeness Theorem (cf. Theorem 3.5) it follows that

$$T^{ev}_{\approx} \vdash_{d} t_1 \not\approx t_2,$$

where

$$d = \bigwedge_{\mathcal{D}} \{\mathcal{D}(t_1 \not\approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\} = \bigwedge_{\mathcal{D}} \{\neg\mathcal{D}(t_1 \approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\} =$$
$$= \bigwedge_{\mathcal{D}} \{1 - \mathcal{D}(t_1 \approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\} = 1 - \bigvee_{\mathcal{D}} \{\mathcal{D}(t_1 \approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\} =$$
$$= \neg \bigvee_{\mathcal{D}} \{\mathcal{D}(t_1 \approx t_2) \mid \mathcal{D} \models T^{ev}_{\approx}\} = \neg c.$$

□

Chapter 5

Linguistic Descriptions

This section presents the treatment of linguistic descriptions on linguistic, syntactic and semantic levels.

5.1 Linguistic level

On the linguistic level, a linguistic description is a set of linguistic expressions of the form "IF $\mathcal{A}$ THEN $\mathcal{B}$", where $\mathcal{A}$ and $\mathcal{B}$ are evaluating linguistic predications (cf. Section 3.2.1).

Definition 5.1
A *linguistic description* in FLb is a finite set $\mathcal{LD}^I = \{\mathcal{R}_1^I, \mathcal{R}_2^I, \ldots, \mathcal{R}_r^I\}$ of conditional clauses

$$\mathcal{R}_i^I := \text{IF } \mathcal{A}_i \text{ THEN } \mathcal{B}_i, \quad i = 1, 2, \ldots, r \tag{5.1}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ are evaluating predications. If all evaluating predications $\mathcal{A}_i$, $\mathcal{B}_i$ are simple (cf. Section 3.2.2) then also the linguistic description $\mathcal{LD}^I$ is called simple.

Remark 5.2 (i) By $\mathrm{An}(\mathcal{R}_i)$ we denote the antecedent part of the rule $\mathcal{R}_i$, i.e. $\mathrm{An}(\mathcal{R}_i) = \mathcal{A}_i$, and, similarly for the succedent part, $\mathrm{Succ}(\mathcal{R}_i) = \mathcal{B}_i$. $\mathrm{An}(\mathcal{LD}^I)$ denotes the set of all antecedents and $\mathrm{Succ}(\mathcal{LD}^I)$ the set of all consequents of the linguistic description $\mathcal{LD}^I$, i.e. $\mathrm{An}(\mathcal{LD}^I) = \{\mathcal{A}_i \mid i = 1, 2, \ldots, r\}$ and $\mathrm{Succ}(\mathcal{LD}^I) = \{\mathcal{B}_i \mid i = 1, 2, \ldots, r\}$.

(ii) It is possible to consider also IF-THEN rules which include negations too, e.g.

$$\mathcal{R}_i^{In_1} := \text{IF } \mathcal{A}_i \text{ THEN NOT } \mathcal{B}_i, \tag{5.2}$$

$$\mathcal{R}_i^{In_2} := \text{IF NOT } \mathcal{A}_i \text{ THEN } \mathcal{B}_i, \tag{5.3}$$

$$\mathcal{R}_i^{In_3} := \text{IF NOT } \mathcal{A}_i \text{ THEN NOT } \mathcal{B}_i. \tag{5.4}$$

5.2 Level of syntax

The intensions of individual conditional clauses $\mathcal{R}_i^I$ are determined by the theories of evaluating expressions T_1^{ev} and T_2^{ev} for antecedent and consequent parts of IF-THEN rules, respectively. The theories T_1^{ev} and T_2^{ev} could be identical, but it is usually advantageous to use different theories for antecedents and for consequents. The language in which the intension of IF-THEN rule is written down is the two-sorted first-order language

$$J(T_I) = \langle \mathcal{G}_1, \mathcal{G}_2, \left\{t^{(z)} \mid z \in [0,1]\right\}, \left\{s^{(z)} \mid z \in [0,1]\right\}, \ldots\rangle$$

where $\mathcal{G}_i,\ i = 1,2$ is the set of atomic predicate symbols for sort i, discussed in the Section 4.1. The set of all closed terms of the sort i are denoted by M_i. The intension of one IF-THEN rule is constructed from the intensions of its antecedent part

$$\mathbf{A}_{i,\langle x\rangle} = \left\{\tilde{\alpha}_{A_i}(t)\big/ A_{i,x}[t] \mid t \in M_1\right\} \tag{5.5}$$

and its succedent part

$$\mathbf{B}_{i,\langle y\rangle} = \left\{\tilde{\alpha}_{B_i}(s)\big/ B_{i,y}[s] \mid s \in M_2\right\} \tag{5.6}$$

in the following way:

$$\begin{aligned}\mathbf{R}_{i,\langle x,y\rangle} &= \mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle} = \\ &= \left\{\tilde{\alpha}_{A_i \Rightarrow B_i}(t,s)\big/ A_{i,x}[t] \Rightarrow B_{i,y}[s] \mid t \in M_1, s \in M_2, A_i \in \mathcal{G}_1, B_i \in \mathcal{G}_2\right\}.\end{aligned} \tag{5.7}$$

In the sequent by A_i, B_i we denote the formulas from (5.5) and (5.6) respectively, i.e. the formulas the instances of which are used in construction of the intension of i-th IF-THEN rule in the linguistic description $\mathcal{LD}^I$. The syntactic evaluations $\tilde{\alpha}_{A_i \Rightarrow B_i}$ should by constructed by means of syntactic evaluations of A_i and B_i, respectively. This means that

$$\neg\tilde{\beta}_{A_i}(t) \to \tilde{\alpha}_{B_i}(s) \leq \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s) \leq \tilde{\alpha}_{A_i}(t) \to \neg\tilde{\beta}_{B_i}(s) \tag{5.8}$$

should hold (in case that T_1^{ev} or T_2^{ev} are not extended theories of evaluating expressions, $\tilde{\beta}_A(t)$ are equal to 0.) It can be easily shown that the inequality

$$\neg\tilde{\beta}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s) \leq \tilde{\alpha}_{A_i}(t) \rightarrow \neg\tilde{\beta}_{B_i}(s)$$

from (5.8) always holds, provided that $\tilde{\alpha}_{A_i}(t) \leq \neg\tilde{\beta}_{A_i}(t)$ and $\tilde{\alpha}_{B_i}(s) \leq \neg\tilde{\beta}_{B_i}(s)$, which follows from Definition 4.9.

Remark 5.3 The question which arises now is how to determine the intensional mapping $\tilde{\alpha}_{A_i \Rightarrow B_i}(t,s)$. The most natural possibility, which is also in accordance with Frege principle of compositionality (saying that the intension of a compound expression is a function of intensions of its parts), is to put

$$\tilde{\alpha}_{A_i \Rightarrow B_i}(t,s) = \tilde{\alpha}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s). \tag{5.9}$$

Intension of the conditional clause IF $\mathcal{A}$ THEN $\mathcal{B}$ is defined in this way in [27], Definition 6.10, p. 238. However, it is not necessary to suppose this in the proof of Theorem 6.5 of this thesis.

A linguistic description $\mathcal{LD}^I$ naturally leads to the *theory of linguistic description* T_I in the language $J(T_I)$ if we construct its fuzzy set of special axioms from the intensions $\mathbf{R}_{i,\langle x,y\rangle}$, i.e. we put

$$T_I = \left\{\mathbf{R}_{i,\langle x,y\rangle} \mid i = 1,2,\ldots,r\right\}. \tag{5.10}$$

Remark 5.4 If we work with extended theories of evaluating expressions T^{evx} (see Section 4.3), and there are IF-THEN rules (given by expert or extracted by some algorithm) which include negations, then it is possible (and advantageous) to extend also the theory T_I by corresponding axioms. Let us denote by $\bar{\mathbf{A}}_{\langle x\rangle}$, $\bar{\mathbf{B}}_{\langle y\rangle}$ the multiformulas

$$\bar{\mathbf{A}}_{\langle x\rangle} = \left\{\tilde{\beta}_A(t)/\neg A_x[t] \mid t \in M_1\right\}, \tag{5.11}$$

$$\bar{\mathbf{B}}_{\langle y\rangle} = \left\{\tilde{\beta}_B(s)/\neg B_y[s] \mid s \in M_2\right\}, \tag{5.12}$$

respectively, i.e. the intensional mapping of multiformula $\bar{\mathbf{A}}_{\langle x\rangle}$ is the mapping $\tilde{\beta}_A(t)$ from Definition 4.9. If there are IF-THEN rules

$$\mathcal{R}_j^I := \text{IF } \mathcal{A}_j \text{ THEN NOT } \mathcal{B}_j, \quad j \in 1,2,\ldots,r, \tag{5.13}$$

$$\mathcal{R}_k^I := \text{IF NOT } \mathcal{A}_k \text{ THEN NOT } \mathcal{B}_k, \quad k \in 1,2,\ldots,r, \tag{5.14}$$

$$\mathcal{R}_l^I := \text{IF NOT } \mathcal{A}_l \text{ THEN } \mathcal{B}_l, \quad l \in 1, 2, \ldots, r, \tag{5.15}$$

then we can construct the multiformulas $\mathbf{A}_{j,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{j,\langle y\rangle}$, $\bar{\mathbf{A}}_{k,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{k,\langle y\rangle}$ and $\bar{\mathbf{A}}_{l,\langle x\rangle} \Rightarrow \mathbf{B}_{l,\langle y\rangle}$, respectively, with their intensional mappings computed via the compositionality principle, i.e.

$$\mathbf{A}_{i,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{i,\langle y\rangle} = \left\{\tilde{\alpha}_{A_i}(t) \to \tilde{\beta}_{B_i}(s)\big/A_{i,x}[t] \Rightarrow \neg B_{i,y}[s] \mid t \in M_1, s \in M_2\right\}, \tag{5.16}$$

$$\bar{\mathbf{A}}_{i,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{i,\langle y\rangle} = \left\{\tilde{\beta}_{A_i}(t) \to \tilde{\beta}_{B_i}(s)\big/\neg A_{i,x}[t] \Rightarrow \neg B_{i,y}[s] \mid t \in M_1, s \in M_2\right\}, \tag{5.17}$$

$$\bar{\mathbf{A}}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle} = \left\{\tilde{\beta}_{A_i}(t) \to \tilde{\alpha}_{B_i}(s)\big/\neg A_{i,x}[t] \Rightarrow B_{i,y}[s] \mid t \in M_1, s \in M_2\right\}. \tag{5.18}$$

We will denote this theory by T_I^E; note that if the original theories T^{ev} are not extended ones, we put $\tilde{\beta}_{A_i}(t) = 0$ for all $i = 1, \ldots, r$ and $t \in M_1$ and, similarly, $\tilde{\beta}_{B_i}(s) = 0$ for all $i = 1, \ldots, r$ and $s \in M_2$.

Then, we put

$$T_I^E = \{\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}, \mathbf{A}_{i,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{i,\langle y\rangle}, \bar{\mathbf{A}}_{i,\langle x\rangle} \Rightarrow \bar{\mathbf{B}}_{i,\langle y\rangle}, \bar{\mathbf{A}}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}\}. \tag{5.19}$$

Remark 5.5 (i) Let us consider the following form of IF-THEN rule (5.2), namely

$$\mathcal{R}_j^I := \text{IF } \mathcal{A} \text{ THEN NOT } \mathcal{B}$$

where $\mathcal{B}$ is a simple evaluating predication. Then, the interval of admissible syntactic evaluations of a formula $A_x[t] \Rightarrow \neg B_y[s]$ is

$$\neg\tilde{\beta}_A(t) \to \tilde{\beta}_B(s) \le \tilde{\alpha}_{A \Rightarrow \neg B}(t, s) \le \tilde{\alpha}_A(t) \to \neg\tilde{\alpha}_B(s). \tag{5.20}$$

(ii) In the sequel we will need to work with antecedent and consequent parts of the linguistic description $\mathcal{LD}^I$ on the level of syntax. The following theories are introduced:

$$T_I^A = \{\text{Int}(\text{An}(\mathcal{R}_i)) \mid i = 1, 2, \ldots, r\} = \{\mathbf{A}_i \mid i = 1, 2, \ldots, r\} \tag{5.21}$$

and

$$T_I^S = \{\text{Int}(\text{Succ}(\mathcal{R}_i)) \mid i = 1, 2, \ldots, r\} = \{\mathbf{B}_i \mid i = 1, 2, \ldots, r\}. \tag{5.22}$$

For extended theories in the sense of Definition 4.9 the theories T_I^A and T_I^S are also extended by the axioms $\{\tilde{\beta}_{A_i}(t)/\neg A_{i,x}[t] \mid t \in M_1, i \in 1, 2, \ldots, r\}$ and $\{\tilde{\beta}_{B_i}(s)/\neg B_{i,y}[s] \mid s \in M_2\}$, respectively.

5.3 Level of semantics

On the *semantic level*, given a possible world

$$\mathcal{V} = \langle\langle V_1, V_2\rangle, \{\mathrm{A}_i \mid i \in \{1,\dots,r\}\}, \{\mathrm{B}_i \mid i \in \{1,\dots,r\}\}, \dots\rangle$$

where A_i, B_i are fuzzy sets – interpretations of atomic predicate symbols $A_i \in \mathcal{G}_1$, $B_i \in \mathcal{G}_2$, each fuzzy IF-THEN rule $\mathcal{R}_i \in \mathcal{LD}^I$ is assigned an *extension* in $\mathcal{V}$:

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{R}_i) = \left\{ \mathcal{V}(A_{i,x}[\mathbf{u}] \Rightarrow B_{i,y}[\mathbf{v}]) / \langle u, v\rangle \,\middle|\, \langle u, v\rangle \in V_1 \times V_2 \right\}. \tag{5.23}$$

Note that the extension of the rule $\mathcal{R}_i$ is a *fuzzy relation* $\mathrm{Ext}_{\mathcal{V}}(\mathcal{R}_i) = R \underset{\sim}{\subset} V_1 \times V_2$.

Theorem 5.6
Let $\mathcal{LD}^I$ be a linguistic description and (5.9) hold. Then the theory T_I is consistent and there exists possible world

$$\mathcal{W} = \langle\langle W_1, W_2\rangle, \{\mathrm{A}_i \mid i \in \{1,\dots,r\}\}, \{\mathrm{B}_i \mid i \in \{1,\dots,r\}\}, \dots\rangle$$

such that

$$\mathcal{W}(A_{i,x}[t] \Rightarrow B_{i,y}[s]) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t, s).$$

PROOF: Let us construct the model $\mathcal{W}$: define $W_j = [0,1]$, $j = 1, 2$ and $\mathrm{A}_i(x) = \tilde{\alpha}_{A_i}\left(t^{(x)}\right)$ for $x = \mathcal{W}\left(t^{(x)}\right)$, $t^{(x)} \in M_1$ and $x \in [0,1]$. Analogously, $\mathrm{B}_i(y) = \tilde{\alpha}_{B_i}\left(s^{(y)}\right)$ for $y = \mathcal{W}\left(s^{(y)}\right)$, $s^{(y)} \in M_2$ and $y \in [0,1]$. Then

$$\begin{aligned}\mathcal{W}(A_{i,x}[t] \Rightarrow B_{i,y}[s]) &= \mathcal{W}(A_{i,x}[t]) \rightarrow \mathcal{W}(B_{i,y}[s]) = \mathrm{A}_i(\mathcal{W}(t)) \rightarrow \mathrm{B}_i(\mathcal{W}(s)) = \\ &= \tilde{\alpha}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t, s).\end{aligned}$$

It follows that $\mathcal{W} \models T_I$ and, therefore, T_I is a consistent theory. □

Possible worlds isomorphic to the possible world $\mathcal{W}$ from the proof of Theorem 5.6 will be called *canonical possible worlds* for T_I.

Chapter 6

Fuzzy Logic Deduction

In this chapter we present basic scheme of fuzzy logic deduction and analyze the concept of *inconsistency of linguistic description*. We present two different definitions of inconsistency. Let us stress that this inconsistency is not considered in the strict logical sense, but rather from the point of view of applications of a linguistic description in concern.

6.1 Basic scheme

The basic scheme of fuzzy logic deduction is the following: We have at our disposal the fuzzy theory T_I (5.10) composed of implications and a fuzzy theory T' which represents an observation. From these theories we form a theory $T_D = T_I \cup T'$. The theory T_I expresses the relationship between antecedent and succedent variables. The problem (addressed in Chapter 7) is how an observation u' measured in some possible world $\mathcal{V}$ can be transformed into its logical counterpart T'. The general form of T' is

$$T' = \left\{ \mathbf{A}'_{i,\langle x \rangle} \mid i \in I \right\} \tag{6.1}$$

where $I \subseteq \{1, 2, \ldots, r\}$ and

$$\mathbf{A}'_{i,\langle x \rangle} = \left\{ \tilde{\alpha}'_{A_i}(t) / A_{i,x}[t] \mid A_i = m(\mathcal{A}_i) \right\}. \tag{6.2}$$

Definition 6.1
Let a theory $T_D = T_I \cup T'$ be consistent. Then the *conclusion* is defined by

$$\mathbf{B}' = \{ \mathbf{B}'_{i,\langle y \rangle} \mid i \in K \} \tag{6.3}$$

where

$$\mathbf{B}'_{i,\langle y\rangle} = \left\{\tilde{\alpha}'_{B_i}(s)/B_{i,y}[s] \mid s \in M_2\right\}, \tag{6.4}$$

and $\tilde{\alpha}'_{B_i}(s) = c$ iff $T_D \vdash_c B_{i,y}[s]$.

Generally, a theory T' can contain several intensions $\mathbf{A}'_i$, and then also the conclusion $\mathbf{B}'$ is composed of several parts.

Remark 6.2 (i) Note that formulas $A_{i,x}[t]$ appear in the intensions of antecedent part of individual IF-THEN rules in the theory T_I and also in the intensions (6.2). But, their syntactic evaluations, i.e. the functions $\tilde{\alpha}_{A_i}(t)$ and $\tilde{\alpha}'_{A_i}(t)$ can be (and usually are) different.

(ii) Index sets I and K are not identical, in general. They have the same cardinality in case there do not exist two IF-THEN rules with identical antecedent and different succedent parts in the linguistic description $\mathcal{LD}^I$. Otherwise, if the theory T_I contains intensions

$$\mathbf{R}_{i,\langle x,y\rangle} = \mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$$

and

$$\mathbf{R}_{j,\langle x,y\rangle} = \mathbf{A}_{j,\langle x\rangle} \Rightarrow \mathbf{B}_{j,\langle y\rangle}$$

where $A_i = A_j$ and $B_i \neq B_j$ and

$$T' = \left\{\mathbf{A}'_{i,\langle x\rangle}\right\} = \left\{\tilde{\alpha}'_{A_i}(t)/A_{i,x}[t] \mid t \in M_1\right\}$$

which means that $I = \{i\}$, then both intensions $\mathbf{R}_{i,\langle x,y\rangle}$ and $\mathbf{R}_{j,\langle x,y\rangle}$ are used in deduction and the conclusion $\mathbf{B}'$ will be

$$\mathbf{B}' = \{\mathbf{B}'_{i,\langle y\rangle} \mid i \in K\}$$

where $K = \{i, j\} \neq I$. Of course, linguistic descriptions with such rules are in some sense incorrect or "inconsistent", we will study such descriptions and problems connected with them in Section 6.2.

(iii) Formula (6.2) also means that we can use only intensions contained in some of antecedent parts of IF-THEN rules in a linguistic description $\mathcal{LD}^I$, when we are translating the observation (obtained on semantic level) into the level of formal syntax. Hence, even if there is some linguistic expression $\mathcal{A} \in \mathcal{S}$ and corresponding unary predicate $A = m(\mathcal{A}) \in \mathcal{G}$ which would be best for the description of the observation, it could not be used unless there is a rule $\mathcal{R}_i :=$ IF X IS $\mathcal{A}$ THEN Y IS $\mathcal{B}$ in $\mathcal{LD}^I$.

(iv) Note that our basic scheme implies that the following basic principle of (logical) deduction is kept: *The deduction is performed only when there is enough evidence for doing it.* This means in our case that the deduction gives some result only if there is the same formula A_i in the intension of some rule $\mathcal{R}_i$ from $\mathcal{LD}^I$, and in $\mathbf{A}'_{i,\langle x\rangle}$. If it is not the case (for example consider IF-THEN rules with antecedents $\mathcal{A}_1 = \textit{small}$ and $\mathcal{A}_2 = \textit{big}$, and the observation is e.g. *about 0.5*) then it happens that the conclusion $\mathbf{B}'$ is empty.

The theory T' usually represents a single observation. The observation can be a crisp number or linguistic expression, which expresses precisely or vaguely the position on an ordered scale. Therefore, the theory T' cannot be completely arbitrary and should fulfil some requirements. We can distinguish two situations — the index set I has cardinality equal to one, which means that the observation is translated into one multiformula. This is possible in situations when there is one rule $\mathcal{R}_i$ in $\mathcal{LD}^I$ whose antecedent fits the observation better than antecedents of other IF-THEN rules. If there is no such rule, then it is necessary to use several rules $\{\mathcal{R}_i \mid i \in I\}$, but the theory T' should still reflect the fact that it represents single observation. We will formulate this requirement in the following definition.

Definition 6.3
We say that a theory

$$T' = \left\{\mathbf{A}'_{i,\langle x\rangle} \mid i \in I\right\}$$

represents a single observation, if

1. all functions $\tilde{\alpha}'_{A_i}$ are identical, i.e. $\tilde{\alpha}'_{A_{i_1}} = \tilde{\alpha}'_{A_{i_2}} = \cdots = \tilde{\alpha}'_{A_{i_n}} = \tilde{\alpha}'$ where $i_1, i_2, \ldots, i_n \in I$.

2. The function γ adjoined to the function $\tilde{\alpha}'$ in the same way as in formula (4.2) is either

   (a) continuous, unimodal and there are $z_1, z_2 \in [0,1]$ such that $\gamma(z_1) = 1$ and $\gamma(z_2) = 0$, or

   (b) $\gamma(z) = 1$ for $z \in [z_l, z_r]$ and $\gamma(z) = 0$ otherwise.

Remark 6.4 Let us explain the intuition behind the previous definition. The degrees $\tilde{\alpha}'_{A_i}(t)$ express the truth of the proposition "Object named t characterizes the observed quantity." Therefore it should not depend on the (atomic) formula A_i (Item 1 of the definition.)

If we accept the principle saying that the vagueness can change only continuously, then it is natural to require that the function γ adjoined to functions $\tilde{\alpha}'_{A_i}$ should be continuous. If it would not be unimodal, then it can hardly represent single observation, having e.g. two local maxima. We should also exclude too non-specific representations like $\tilde{\alpha}'(t) = 1$ for all t, which are, strictly speaking, unimodal, but do not characterize any value. On the other hand, there is a natural non-continuous characterization of a single value, namely singleton or interval (Item 2b).

The fact that the following theorem holds is crucial for the applicability of fuzzy logic deduction. It enables us to compute intensional mappings $\tilde{\alpha}'_{B_i}$ of the conclusion. It is also in accordance with the classical formula obtained on the level of semantics by standard manipulations with fuzzy relations.

Theorem 6.5
Let $\mathcal{LD}^I$ be a simple linguistic description and the theory $T_D = T_I \cup T'$ be constructed as above. Then it is consistent and the conclusion is $\mathbf{B}' = \{\mathbf{B}'_{i,\langle y\rangle} \mid i \in K\}$ where the intensions $\mathbf{B}'_i$ are

$$\mathbf{B}'_{i,\langle y\rangle} = \left\{\tilde{\alpha}'_{B_i}(s)/B_{i,y}[s] \mid s \in M_2,\ B_i = m(\mathcal{B}_i),\ i \in K\right\} \tag{6.5}$$

where

$$\tilde{\alpha}'_{B_i}(s) = \bigvee_{t \in M_1} \left(\tilde{\alpha}'_{A_i}(t) \otimes \tilde{\alpha}_{A_i \Rightarrow B_i}(t,s)\right), \tag{6.6}$$

and all $\tilde{\alpha}'_{B_i}(s)$ in $\mathbf{B}'_i$, $i \in K$ are maximal.

PROOF: Analogous theorem, which differs only in notation, was proved as Theorem 6.1 in [27], page 249. □

Theorem 6.6
Let $\mathcal{LD}^I$ be a linguistic description. Let the theory T'_1 have the form

$$T'_1 = \{\tilde{\alpha}'_{1,A_j}(t)/A_{j,x}[t] \mid t \in M_1\}$$

and the theory T'_2 have the form $T'_2 = \{\tilde{\alpha}'_{2,A_j}(t)/A_{j,x}[t] \mid t \in M_1\}$ and let

$$\tilde{\alpha}'_{1,A_j}(t) \leq \tilde{\alpha}'_{2,A_j}(t)$$

hold for all $t \in M_1$. Then it holds for the conclusions $\mathbf{B}'_{1,\langle y\rangle}$ and $\mathbf{B}'_{2,\langle y\rangle}$, obtained from the theories $T_{D1} = T_I \cup T'_1$ and $T_{D2} = T_I \cup T'_2$, respectively, that $\tilde{\alpha}'_{1,B_j}(s) \leq \tilde{\alpha}'_{2,B_j}(s)$ for all $s \in M_2$.

PROOF: The intensional mapping of the conclusion $\tilde{\alpha}'_{i,B_j}(s)$, $i \in \{1,2\}$, is, due to Theorem 6.5, $\tilde{\alpha}'_{i,B_j}(s) = \bigvee_{t \in M_1} \left(\tilde{\alpha}'_{i,A_j}(t) \otimes \tilde{\alpha}_{A_j \Rightarrow B_j}(t,s) \right)$. If we denote the interior of round brackets by $a'_{i,j}(t,s)$, i.e. $a'_{i,j}(t,s) = \tilde{\alpha}'_{i,A_j}(t) \otimes \tilde{\alpha}_{A_j \Rightarrow B_j}(t,s)$, then it holds that $\tilde{\alpha}'_{i,B_j}(s) = \bigvee_{t \in M_1} a'_{i,j}(t,s)$. From the assumptions we can deduce that for all $t \in M_1$ and all $s \in M_2$, $a'_{1,j}(t,s) \leq a'_{2,j}(t,s)$ (because $\tilde{\alpha}_{A_j \Rightarrow B_j}(t,s)$ remains the same). Then the claim follows from the isotonicity of the operation $\otimes$ and properties of supremum.

□

Remark 6.7 Theorem 6.6 can be generalized for theories of the form

$$T'_1 = \left\{ \{ \tilde{\alpha}'_{1,A_i} / A_{i,x}[t] \mid t \in M_1 \} \mid i \in I \right\}$$

and

$$T'_2 = \left\{ \{ \tilde{\alpha}'_{2,A_i} / A_{i,x}[t] \mid t \in M_1 \} \mid i \in I \right\}$$

if it holds that $\tilde{\alpha}'_{1,A_i}(t) \leq \tilde{\alpha}'_{2,A_i}(t)$ for all $i \in I$ and all $t \in M_1$.

Lemma 6.8

Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x \rangle} \Rightarrow \mathbf{B}_{i,\langle y \rangle}$, $i = 1, 2, \ldots, r$ be constructed by means of (5.9). Let the theory T' have the form $T' = \{ \tilde{\alpha}'_{A_j}(t) / A_{j,x}[t] \mid t \in M_1 \}$ for some $j \in 1, \ldots, r$ and let there is some $t_0 \in M_1$ such that $\tilde{\alpha}'_{A_j}(t_0) = \tilde{\alpha}_{A_j}(t_0) = 1$ holds. Then for the conclusion $\mathbf{B}'_{\langle y \rangle}$ holds that $\mathbf{B}'_{\langle y \rangle} = \left\{ \mathbf{B}'_{j,\langle y \rangle} \right\}$ and $\tilde{\alpha}'_{B_j}(s) \geq \tilde{\alpha}_{B_j}(s)$ for all $s \in M_2$.

PROOF: In Theorem 6.5 it has been shown that

$$\tilde{\alpha}'_{B_j}(s) = \bigvee_{t \in M_1} \left(\tilde{\alpha}'_{A_j}(t) \otimes \tilde{\alpha}_{A_j \Rightarrow B_j}(t,s) \right),$$

and if the intensional mapping $\tilde{\alpha}_{A_j \Rightarrow B_j}$ is computed by means of (5.9), we can rewrite it to

$$\tilde{\alpha}'_{B_j}(s) = \bigvee_{t \in M_1} \left(\tilde{\alpha}'_{A_j}(t) \otimes \left(\tilde{\alpha}_{A_j}(t) \rightarrow \tilde{\alpha}_{B_j}(s) \right) \right).$$

For $t = t_0$ the expression inside the supremum has the value

$$\tilde{\alpha}'_{A_j}(t_0) \otimes \left(\tilde{\alpha}_{A_j}(t_0) \rightarrow \tilde{\alpha}_{B_j}(s) \right) = \tilde{\alpha}_{A_j}(t_0) \rightarrow \tilde{\alpha}_{B_j}(s)$$

and because $\tilde{\alpha}_{A_j}(t_0) = 1$, it holds that

$$\tilde{\alpha}_{A_j}(t_0) \rightarrow \tilde{\alpha}_{B_j}(s) \geq \tilde{\alpha}_{B_j}(s)$$

which follows from properties of the operation $\rightarrow$. The claim then follows from the fact that $\bigvee_{i \in I} a_i \geq a_j$ for any $j \in I$ and any index set I. □

Theorem 6.9
Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$, $i = 1, 2, \ldots, r$ be constructed by means of (5.9). Let the theory T' have the form $T' = \{\tilde{\alpha}'_{A_j}(t)/A_{j,x}[t] \mid t \in M_1\}$ for some $j \in 1, \ldots, r$, and let

(i) $\tilde{\alpha}'_{A_j}(t) \leq \tilde{\alpha}_{A_j}(t)$ *for all* $t \in M_1$,

(ii) *there is* $t_0 \in M_1$ *such that* $\tilde{\alpha}'_{A_j}(t_0) = \tilde{\alpha}_{A_j}(t_0) = 1$.

hold. Then it holds for the conclusion $\mathbf{B}'_{\langle y\rangle}$ that $\mathbf{B}'_{\langle y\rangle} = \left\{\mathbf{B}'_{j,\langle y\rangle}\right\}$ and $\tilde{\alpha}'_{B_j}(s) = \tilde{\alpha}_{B_j}(s)$ for all $s \in M_2$, and, consequently, $\mathbf{B}'_{j,\langle x\rangle} = \mathbf{B}_{j,\langle x\rangle}$.

PROOF: Due to Theorem 6.5 (see also the proof of Lemma 6.8),

$$\tilde{\alpha}'_{B_j}(s) = \bigvee_{t\in M_1} \left(\tilde{\alpha}'_{A_j}(t) \otimes \left(\tilde{\alpha}_{A_j}(t) \to \tilde{\alpha}_{B_j}(s)\right)\right).$$

If $t = t_0$, then the expression inside the supremum on the righthand side of previous equation is equal to

$$\tilde{\alpha}'_{A_j}(t_0) \otimes \left(\tilde{\alpha}_{A_j}(t_0) \to \tilde{\alpha}_{B_j}(s)\right), \tag{6.7}$$

and, because $\tilde{\alpha}'_{A_j}(t_0) = \tilde{\alpha}_{A_j}(t_0) = 1$, it is equal to $1 \to \tilde{\alpha}_{B_j}(s) = \tilde{\alpha}_{B_j}(s)$. Otherwise, for an arbitrary $t^\star \in M_1$, is the expression equal to

$$\tilde{\alpha}'_{A_j}(t^\star) \otimes \left(\tilde{\alpha}_{A_j}(t^\star) \to \tilde{\alpha}_{B_j}(s)\right).$$

From properties of operations $\otimes$ and $\to$ (see e.g. [27], Lemma 2.5 (c), page 26) we know that

$$\tilde{\alpha}_{A_j}(t^\star) \otimes \left(\tilde{\alpha}_{A_j}(t^\star) \to \tilde{\alpha}_{B_j}(s)\right) \leq \tilde{\alpha}_{B_j}(s).$$

Hence, due to assumption *(i)* (which assures that $\tilde{\alpha}'_{A_j}(t^\star) \leq \tilde{\alpha}_{A_j}(t^\star)$ in (6.7)), isotonicity of the operation $\otimes$ and properties of supremum, we deduce that

$$\tilde{\alpha}'_{B_j}(s) = \tilde{\alpha}_{B_j}(s)$$

for all $s \in M_2$, which is the same as $\mathbf{B}'_{j,\langle x\rangle} = \mathbf{B}_{j,\langle x\rangle}$. □

Corollary 6.10
Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$, $i = 1, 2, \ldots, r$ be constructed by means of (5.9). Let the theory T' have the form $T' = \mathbf{A}_{j,\langle x\rangle}$ for some $j \in 1, \ldots, r$. Then $\tilde{\alpha}'_{B_j}(s) = \tilde{\alpha}_{B_j}(s)$ holds for all $s \in M_2$, and, consequently,

$$\mathbf{B}'_{\langle y\rangle} = \left\{\mathbf{B}'_{j,\langle x\rangle}\right\} = \left\{\mathbf{B}_{j,\langle x\rangle}\right\}. \tag{6.8}$$

PROOF: The theory T' fulfills the assumptions of Theorem 6.9, hence (6.8) holds. □

Remark 6.11 (i) Previous Corollary 6.10 says that if the theory T' contains only one intension $\mathbf{A}_{j,\langle x\rangle}$ which is identical with the antecedent of one of IF-THEN rules, then the conclusion is identical with the succedent of this IF-THEN rule. Theorem 6.9 shows that the result of fuzzy logic deduction is identical for quite wide class of intensions for which the conditions (i) and (ii) hold. It means that fuzzy logic deduction is not sensitive to the changes of the intensional mapping $\tilde{\alpha}'_{B_j}$ provided that this mapping satisfies above-mentioned conditions.

(ii) The results in this section, i.e. Lemma 6.8, Theorem 6.6, Corollary 6.10 and Theorem 6.9 correspond to conditions A.1 to A.4, respectively, from [4]. These conditions are there imposed on general inference procedures.

6.2 Inconsistencies in linguistic description

We have shown in Theorem 5.6 that the theory T_I of a linguistic description $\mathcal{LD}^{\mathcal{I}}$ is consistent. It is a natural result, because it is not possible to derive a contradiction from implications only. However, there are some linguistic descriptions which behave "inconsistently." By this is meant that for some observation there are derived conclusions, whose interpretations are at the same time "small" and "big". A typical example of "inconsistent" linguistic description is

$$\mathcal{R}_1 := \text{IF } X \text{ is } \textit{medium} \text{ THEN } Y \text{ is } \textit{big},$$
$$\mathcal{R}_2 := \text{IF } X \text{ is } \textit{medium} \text{ THEN } Y \text{ is } \textit{small},$$
$$\ldots$$

Here, if the observation is $\mathcal{A}' = \textit{medium}$, we cannot decide between rules $\mathcal{R}_1$ and $\mathcal{R}_2$ and we should use both rules and, consequently, deduce that Y is *big* and Y is *small* at the same time. If we understand Y as the (precisely of imprecisely known) value of some variable, then there is a contradiction in the fact, that Y is at the same time "small" and "big". However, if we do not suppose some structure of the set $\mathcal{G}$ of predicate symbols, which serve as intensions of simple linguistic expressions, then there is no contradiction in the standard sense. The reason is that all predicate symbols $G \in \mathcal{G}$ are on the same level, there are no relationships such as '*small* is a negation (in a logical sense) of *big*." Moreover, we can also understand 'Y is *big*' and 'Y is *small*' as statements about the properties of individuals of the set M_2 of closed terms and then there is no contradiction in the fact that big numbers are big and small numbers are small.

To characterize this type of inconsistency of the theory $T_D = T' \cup T_I$, we propose to capture the fact that Y is a variable and therefore cannot have more than one value at a given time, (or, for one observation) by the provability of some formulas. We use fuzzy equality predicate $\approx$ to express this property. Let us denote by $T_D^{\approx}$ the fuzzy theory

$$T_D^{\approx} = T_D \cup T_2^{\approx} \tag{6.9}$$

where

$$T_2^{\approx} = \{{}^{d(s_1,s_2)}/s_1 \not\approx s_2 \mid s_1, s_2 \in M_2\} \tag{6.10}$$

where $s_1 \not\approx s_2$ is an abbreviation for $\neg(s_1 \approx s_2)$ and $d(s_1, s_2)$ is a provability degree of formula $s_1 \not\approx s_2$ in some theory T_2, i.e. $d(s_1, s_2) = d$ iff $T_2 \vdash_d s_1 \not\approx s_2$, and T_2 is some subtheory of the theory of evaluating expressions $T_{2\approx}^{evx}$, i.e. the extended theory of evaluating expressions with fuzzy equality (see Section 3.1.8 and also Remark 6.13).

Definition 6.12
The theory $T_D^{\approx} = T_I \cup T_2^{\approx}$ is *$\approx$-inconsistent in the degree κ* if for some theory T' which represents a single observation (see Definition 6.3) the theory $T_D^{\approx} \cup T'$ proves

$$T_D^{\approx} \cup T' \vdash_\kappa (\exists y_1)(\exists y_2)\ (y_1 \not\approx y_2)\ \&\ B_1(y_1)\ \&\ B_2(y_2) \tag{6.11}$$

for some $B_1, B_2 \in \mathcal{G}$. If $\kappa = 1$ then $T_D^{\approx}$ is called *$\approx$-inconsistent*.

Remark 6.13 (i) The previous definition is motivated by the classical requirement which a relation has to fulfil in order to be a function:

$$(\forall x)(\forall y)(\forall z)\ (y = f(x)\ \&\ z = f(x)) \Rightarrow (y = z).$$

Negation of this formula is logically equivalent to

$$(\exists x)(\exists y)(\exists z)\ (y = f(x)\ \&\ z = f(x)\ \&\ z \neq y). \tag{6.12}$$

Because $B_1(y_1)$ and $B_2(y_2)$ from (6.11) express a value of variable Y for some single observation characterized by the theory T', formula (6.11) expresses in our formalism the same property as formula (6.12) in the classical case.

(ii) Let us stress that the above-defined notion of $\approx$-inconsistency is dependent on the intensional mappings $\tilde{\alpha}_G$ of predicate symbols $G \in \mathcal{G}$. This is the reason why we defined the inconsistency of theory $T_D^{\approx}$ and not the inconsistency of the linguistic description $\mathcal{LD}^I$. If we include into the theory $T_2^{\approx}$ axioms of the form ${}^{d(s_1,s_2)}\!/s_1 \not\approx s_2$ with degrees $d(s_1, s_2)$ computed from the whole theory T^{evx}, i.e. the theory which describes the intensions of the whole set $\mathcal{S}$ of simple evaluating expressions, then the definition of $\approx$-inconsistency can become too sensitive. It means that $T_D^{\approx}$ can prove that $x \not\approx y$ for x and y which are "close" to each other (in standard metric on [0,1]) but are distinguished by some predicate $B_i \in \mathcal{G}$ in the sense that

$$T^{evx} \vdash B_i(x) \not\Leftrightarrow B_i(y) \tag{6.13}$$

and, consequently, $T_D^{\approx} \vdash x \not\approx y$. To be able to prove formulas such as (6.13), we have to work with the extended theory of evaluating expressions. Then, it is possible to use Lemma 4.16(iii), which allows us to estimate lower bounds of the provability degrees for formulas (6.13), $x \not\approx y$ and, consequently, (6.11). As a possible solution we can use some subtheory T_2 of T^{evx}, i.e. a theory which includes axioms of the form (4.1) and (4.7) for some subset $\{S_k \mid k \in K\}$ of the set $\mathcal{S}$. The structure of the set of evaluating linguistic expression (see Subsection 3.2.2) can give us a clue which predicate symbols should be chosen. We can choose e.g. predicates which model atomic evaluating expressions *small*, *medium* and *big* or predicates which model some wider evaluating expressions for every subset of evaluating expressions with the same atomic one, such as *roughly small*, *roughly medium* and *roughly big*.

Lemma 6.14
If there are closed terms s_1 and $s_2 \in M_2$ such that $T_D \vdash B_{i,y}[s_1]$ and $T_D \vdash B_{j,y}[s_2]$ and

$$T_2 \vdash B_{k,y}[s_1] \not\Leftrightarrow B_{k,y}[s_2] \tag{6.14}$$

for some $i, j \in 1, 2, \ldots, r$ and some $k \in K$, then the theory $T_D^{\approx}$ is $\approx$-inconsistent.

PROOF: Formula

$$(s_1 \approx s_2) \Rightarrow (B_{k,y}[s_1] \Leftrightarrow B_{k,y}[s_2])$$

is an instance of fuzzy equality axiom (E3), i.e.

$$T_2 \vdash (s_1 \approx s_2) \Rightarrow (B_{k,y}[s_1] \Leftrightarrow B_{k,y}[s_2]).$$

Consequently,

$$T_2 \vdash (B_{k,y}[s_1] \not\Leftrightarrow B_{k,y}[s_2]) \Rightarrow (s_1 \not\approx s_2).$$

Then from (6.14) and Modus Ponens follows that $T_2 \vdash (s_1 \not\approx s_2)$. From the definition of the theory $T_D^{\approx}$ (see (6.9) and (6.10)) it follows that also $T_D^{\approx} \vdash (s_1 \not\approx s_2)$. This and other assumptions imply that (6.11) holds. □

Theorem 6.15
Let $\mathcal{LD}^I$ be a simple linguistic description which includes rules $\mathcal{R}_j$ and $\mathcal{R}_k$ such that $\mathcal{R}_j :=$ IF $\mathcal{A}$ THEN $\mathcal{B}_j$ and $\mathcal{R}_k :=$ IF $\mathcal{A}$ THEN $\mathcal{B}_k$. Let the theories of evaluating expressions T_1^{evx} and T_2^{evx} are such that the theory $T_D^{\approx} = T_D \cup T_2$ (6.9), where T' has the form $T' = \{\tilde{\alpha}'_A(t) / A_x[t] \mid t \in M_1\}$ and it holds that there is some $t_0 \in M_1$ such that $\tilde{\alpha}'_A(t_0) = \tilde{\alpha}_A(t_0) = 1$, proves for some $s_1, s_2 \in M_2$ that $T_D^{\approx} \vdash B_{j,y}[s_1]$, $T_D^{\approx} \vdash B_{k,y}[s_2]$, and

$$T_2 \vdash (\exists y)(B_j(y) \not\Leftrightarrow B_k(y)) \tag{6.15}$$

where $T_2 \subsetapprox T_{2\approx}^{evx}$. Then the theory $T_D^{\approx}$ is $\approx$-inconsistent.

PROOF: We have to show that from (6.15) it follows that there exist closed terms s_a and s_b from M_2 such that $T_2 \vdash s_a \not\approx s_b$ (and, consequently, $T_D^{\approx} \vdash s_a \not\approx s_b$) and in the same time that $T_D^{\approx} \vdash B_{j,y}[s_a]$ and $T_D^{\approx} \vdash B_{k,y}[s_b]$.

From (6.15) and the Completeness Theorem it follows that for some $s \in M_2$ either a) $T_2 \vdash B_{j,y}[s]$ and $T_2 \vdash \neg B_{k,y}[s]$, or b) $T_2 \vdash \neg B_{j,y}[s]$ and $T_2 \vdash B_{k,y}[s]$. Suppose a). Then for some $s_a, s_b \in M_2$ either a$_1$) $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$ and $T_2 \vdash s_a \leq s$, $T_2 \vdash s < s_b$, or a$_2$) $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$ and $T_2 \vdash s \leq s_a$, $T_2 \vdash s_b < s$. Suppose a$_1$). From assumptions a) and a$_1$) it also follows that

$$T_2 \vdash B_{k,y}[s_b] \not\Leftrightarrow B_{k,y}[s],$$

and, due to provable inequalities $s_a \leq s$, $s < s_b$ and unimodality of functions $\tilde{\alpha}_{B_k}$ and $\tilde{\beta}_{B_k}$, also

$$T_2 \vdash B_{k,y}[s_a] \not\Leftrightarrow B_{k,y}[s_b]$$

(see Figure 6.1, for the sake of simplicity we depict the associated functions γ_{B_j} and γ_{B_k} to functions $\tilde{\alpha}_{B_j}$ and $\tilde{\alpha}_{B_k}$, respectively). By the same consideration as in the proof of Lemma 6.14 we show that $T_2 \vdash s_a \not\approx s_b$. From the definition of theory $T_D^{\approx}$ it follows that also $T_D^{\approx} \vdash (s_a \not\approx s_b)$. Because $T_2 \vdash B_{j,y}[s_a]$, $T_2 \vdash B_{k,y}[s_b]$, it holds that $\tilde{\alpha}_{B_j}(s_a) = \tilde{\alpha}_{B_k}(s_b) = 1$. The assumptions of Lemma 6.8 are fulfilled, hence we deduce that also $\tilde{\alpha}'_{B_j}(s_a) = \tilde{\alpha}'_{B_k}(s_b) = 1$. Hence, we have shown that $T_D^{\approx} \vdash B_{j,y}[s_a]$,

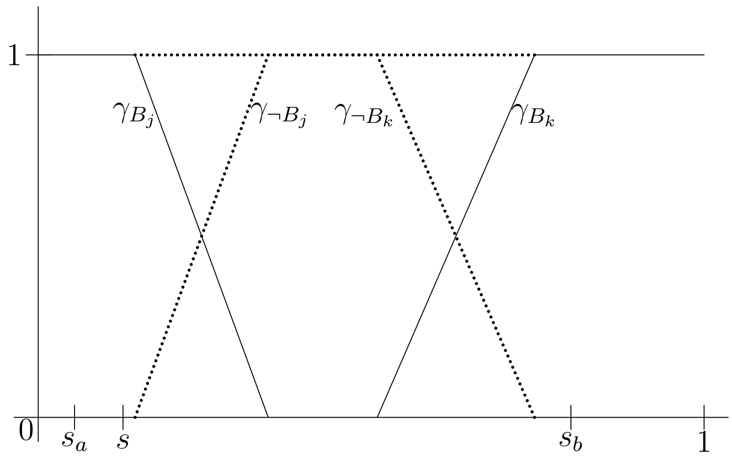

Figure 6.1: Situation a_1) from the proof of Theorem 6.15.

$T_D^{\approx} \vdash B_{k,y}[s_b]$ and $T_D^{\approx} \vdash s_a \not\approx s_b$, and it follows (using the fact that $A_x[t] \Rightarrow (\exists x)A(x)$ is provable in fuzzy predicate calculus and the Completeness Theorem) that

$$T_D^{\approx} \vdash (\exists y_1)(\exists y_2)\ (y_1 \not\approx y_2) \,\&\, B_j(y_1) \,\&\, B_k(y_2),$$

i.e. the theory $T_D^{\approx}$ is $\approx$-inconsistent. The cases b) and a$_2$) can be proved analogously.

□

6.3 Alternative approach to inconsistency of linguistic descriptions

In this section we present an alternative approach to a definition of inconsistency of linguistic descriptions. The main idea can be described as follows: The linguistic description $\mathcal{LD}^I$ is inconsistent if it is possible to deduce for two objects s_1 and s_3 from M_2 that they describe the result of inference in a high degree, and at the same time, to deduce for an object s_2 lying between s_1 and s_3 that it describes the result of inference in a low degree.

Definition 6.16
We say that the theory T_D (see (6.1)) is u-inconsistent in the degree τ if

$$T_D \vdash_a B_{i,y}[s_1],\ T_D \vdash_b B_{j,y}[s_2],\ T_D \vdash_c B_{k,y}[s_3], \tag{6.16}$$

$$T_2^{ev} \vdash (s_1 \leq s_2) \,\&(s_2 \leq s_3),$$

and

$$\tau = \bigvee \{a \otimes \neg b \otimes c \mid i, j, k \in K, s_1, s_2, s_3 \in M_2\}. \tag{6.17}$$

If $\tau = 0$, we call the theory T_D *u*-consistent and if $\tau = 1$, we call the theory T_D *u*-inconsistent.

Now we prove an auxiliary lemma.

Lemma 6.17
Let $f : \mathbb{R} \longrightarrow [0,1]$ be a continuous function such that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 1$. f is unimodal iff for all $a, b, c \in \mathbb{R}$ such that $a \leq b \leq c$,

$$f(a) \otimes \neg f(b) \otimes f(c) = 0. \tag{6.18}$$

holds.

PROOF:

(i) First we show that (6.18) holds for non-increasing and non-decreasing functions. Let f be non-decreasing. Then, $f(a) \leq f(b)$. It follows that

$$\neg f(b) \leq \neg f(a). \tag{6.19}$$

In every residuated lattice it holds that $f(a) \otimes \neg f(a) = 0$. The claim then follows from (6.19) and isotonicity of the operation $\otimes$. If f is non-increasing, then $f(c) \leq f(b)$. The rest is analogous.

For a general unimodal function f (see Definition 3.15) it always holds that if $a \leq b \leq c$, then either $f(b) \geq f(a)$ or $f(b) \geq f(c)$. The rest is obvious.

(ii) If f is not unimodal then there exist $a, b, c \in \mathbb{R}$ such that $a \leq b \leq c$, $f(b) < f(a)$, $f(b) < f(c)$. From the assumptions follows that we can choose a and c such that either $f(a) = 1$ or $f(c) = 1$ (or both). Suppose that $f(a) = 1$. Then we have to show that $\neg f(b) \otimes f(c) > 0$. But this follows from Lemma 3.3. If $f(c) = 1$ and $f(a) < 1$, then again above-mentioned Lemma implies that $\neg f(b) \otimes f(a) > 0$.

□

According to the discussion presented in Section 6.2, we interpret the value of the function $\tilde{\alpha}'_B(s)$ from intensions of conclusion $\mathbf{B}'$ not as a degree, in which the object $s \in M_2$ have the property B, but as a degree in which the object with the name s describes the vaguely characterized value of the conclusion. Therefore, it is possible to consider a function $\tilde{\alpha}^\star : M_2 \longrightarrow [0,1]$,

$$\tilde{\alpha}^\star(s) = \bigvee_{i \in K} \tilde{\alpha}'_{B_i}(s)$$

where $\tilde{\alpha}'_{B_i}(s)$ are defined by (6.6). The value of the function $\tilde{\alpha}^\star(s)$ expresses the maximal degree in which we can deduce (from the theory T_D) for the object named by s that it describes the result of the deduction. This is important in case that the index set K in the conclusion $\mathbf{B}' = \{B'_i \mid i \in K\}$ has cardinality greater than 1. A high value $\tilde{\alpha}^\star(s)$ then means that it was deduced from some IF-THEN rule $\mathcal{R}_i$ in $\mathcal{LD}^I$ that the object named by s describes the result of deduction with a high plausibility.

With respect to the previous discussion and ideas, we are able to formulate the following theorem.

Theorem 6.18
Let a linguistic description $\mathcal{LD}^I$ be simple, $T_D = T' \cup T_I$,

$$T' = \mathbf{A}'_{\langle x \rangle} = \{\tilde{\alpha}'_A(t)/A_x[t] \mid t \in M_1\} \tag{6.20}$$

and there is just one intension $\mathbf{R}_{i,\langle x,y \rangle} = \mathbf{A}_{i,\langle x \rangle} \Rightarrow \mathbf{B}_{i,\langle y \rangle}$ in T_I such that $A_i = A$ (formula A is from (6.20)). Moreover, this intension is constructed from the intensions of $\mathbf{A}_{i,\langle x \rangle}$ and $\mathbf{B}_{i,\langle y \rangle}$ by means of formula (5.9) and there is $t_0 \in M_1$ such that $\tilde{\alpha}_A(t_0) = \tilde{\alpha}'_A(t_0) = 1$. Then T_D is u-consistent.

PROOF: Denote by B the formula B_i. It follows from the assumptions that the conclusion will be

$$\mathbf{B}'_{\langle y \rangle} = \{\tilde{\alpha}'_B(s)/B_y[s] \mid s \in M_2\}.$$

We have to show that the function $\gamma'_B : [0,1] \longrightarrow [0,1]$ associated to $\tilde{\alpha}'_B$ by means of

$$\gamma'_B(z) = w \ \text{ iff } \ \tilde{\alpha}'_B\left(s^{(z)}\right) = w \ \text{ iff } \ T_D \vdash_w B_y\left[s^{(z)}\right], \ z \in [0,1] \tag{6.21}$$

is unimodal, and there is $z^\star \in [0,1]$ such that $\gamma'_B(z^\star) = 1$. Then, due to Lemma 6.17, T_D would be u-consistent.

From Theorem 6.5 and using the assumption that the intension of

$$\mathbf{A}_{i,\langle x \rangle} \Rightarrow \mathbf{B}_{i,\langle y \rangle}$$

is constructed by means of (5.9) it follows that the function $\gamma'_B(z)$ has the following form:

$$\gamma'_B(z) = \bigvee_{v\in[0,1]} (\gamma'_A(v) \otimes (\gamma_A(v) \to \gamma_B(z)))$$

where the functions γ'_A, γ_A and γ_B, all from $[0,1]$ to $[0,1]$, are associated to functions $\tilde{\alpha}'_A$, $\tilde{\alpha}_A$ and $\tilde{\alpha}_B$, respectively, in the same way as in (6.21).

Let us define a function $\gamma'_{B,v}(z) : [0,1] \longrightarrow [0,1]$ by

$$\gamma'_{B,v}(z) := \gamma'_A(v) \otimes (\gamma_A(v) \to \gamma_B(z)), \quad v \in [0,1]. \tag{6.22}$$

Then it holds that

$$\gamma'_B = \bigvee_{v\in[0,1]} \gamma'_{B,v}.$$

Now we have to show that the functions $\gamma'_{B,v}$ are unimodal and that there is $z^\star \in [0,1]$ such that $\gamma'_{B,v}(z^\star) = M^v$ for all $v \in [0,1]$, where $M^v = \sup_{z\in[0,1]} \gamma'_{B,v}(z)$. The functions γ_A and γ_B from formula (6.22) are continuous and unimodal due to Definition 4.1. For a fixed $v \in [0,1]$ we can rewrite (6.22) as

$$\gamma'_{B,v}(z) = C \otimes (D \to \gamma_B(z)) \tag{6.23}$$

where $C = \gamma'_A(v)$ and $D = \gamma_A(v)$. Due to Definition 4.1 there is $z_0 \in [0,1]$ such that $\gamma_B(z_0) = 1$. The function $D \to \gamma_B(z)$ is unimodal and has a value 1 in the point z_0. Hence, the function $\gamma'_{B,v}(z) = C \otimes (D \to \gamma_B(z))$ is also unimodal and has the value C in the point z_0, and this value is maximal. The unimodality of γ'_B then follows from Lemma 3.16. From the assumptions it follows that there is x_0 such that $\gamma_A(x_0) = \gamma'_A(x_0) = 1$. Hence, $\gamma'_{B,x_0}(z_0) = \gamma'_B(z_0) = 1$. From it and Lemma 6.17 it follows that

$$\{\gamma'_B(z_1) \otimes \neg\gamma'_B(z_2) \otimes \gamma'_B(z_3) \mid z_1 \le z_2 \le z_3,\ z_1, z_2, z_3 \in [0,1]\} = 0.$$

Because the values of the function $\gamma'_B(z)$ are provability degrees of formulas $B_y\left[s^{(z)}\right]$ in the theory T_D, it follows that the value τ from (6.17) is equal to 0, hence T_D is u-consistent. □

Remark 6.19 Now we can generalize the notion of u-inconsistent theory by considering some class $\mathcal{T}'$ of theories T' representing a single observation (see Definition 6.3) and arrive at the notion of u-consistent (or u-inconsistent) linguistic description. However, we have to consider the linguistic descriptions on syntactic, not linguistic level, because only on syntactic level there are the intensional mappings $\tilde{\alpha}_{A_i \Rightarrow B_i}$ assigned to linguistic expressions of the form 'IF $\mathcal{A}_i$ THEN $\mathcal{B}_i$.' Therefore, the fact whether the linguistic description $\mathcal{LD}^I$ is, or is not u-consistent, depends also on the theories T_1^{ev} and T_2^{ev} (or T_1^{evx} and T_2^{evx}) and on the way how intensional mappings $\tilde{\alpha}_{A_i \Rightarrow B_i}$ are constructed by means of them. Consequently, we will in the sequel speak about the consistency of the theory T_I.

Definition 6.20
We say that a theory of linguistic description T_I is $u^\star$-*inconsistent* in the degree $\tau^\star$ if

$$\tau^\star = \bigvee \{\tau_{T'} \mid T_D = T_I \cup T' \text{ is } u\text{-inconsistent in the degree } \tau_{T'},\ T' \in \mathcal{T}'\}. \quad (6.24)$$

If $\tau^\star = 0$, we say that theory T_I is $u^\star$-*consistent*.

Remark 6.21 We cannot consider here all possible theories T' representing a single observation, because in this case it is still possible to construct a theory T' in such a way that all rules in linguistic description $\mathcal{LD}^I$ would be used and, consequently, even reasonable theories T_I will be $u^\star$-inconsistent. As a reasonable class $\mathcal{T}'$ we use the class which includes all theories $\left\{\mathbf{A}'_{i,\langle x\rangle} \mid i \in I\right\}$ with intensions (6.2), for which the intensions $\mathbf{A}_{i,\langle x\rangle}$ have the same type of unimodality (described by the relation $\mathrm{uni}_m^{T^{ev}}$, see Definition 4.4) for all $i \in I$. Formally, $T' = \left\{\mathbf{A}'_{i,\langle x\rangle} \mid i \in I\right\} \in \mathcal{T}'$ iff

$$\left(\mathbf{A}_{i,\langle x\rangle}\, \mathrm{uni}_m^{T_1^{ev}}\, \mathbf{A}_{j,\langle x\rangle}\right) \text{ for all } i, j \in I \quad (6.25)$$

and

$$\left(\mathbf{A}_{i,\langle x\rangle}\, \mathrm{uni}_m^{T_1^{ev}}\, \mathbf{A}'_{i,\langle x\rangle}\right) \text{ for all } i \in I \quad (6.26)$$

where T_1^{ev} is the theory of evaluating expressions used for modeling of the meanings of evaluating expressions in the antecedent part of IF-THEN rules (see Section 5.2). Moreover, the functions $\tilde{\alpha}'_{A_i}$, $i \in I$ should be such that there is $t_i \in M_1$ such that $\tilde{\alpha}'_{A_i}(t_i) = 1$. Formula (6.25) means that the intensions $\left\{\mathbf{A}_{i,\langle x\rangle} \mid i \in I\right\}$ form a block of the set $\mathbf{S}_{\langle x\rangle}$ of intensions of simple evaluating expressions with respect to the tolerance relation $\mathrm{uni}_m^{T_1^{ev}}$. On the linguistic level, this requirement means that in the theory T' there could be intensions of simple evaluating expressions with the same atomic expression (e.g. *small*) and possibly various linguistic hedges. Formula (6.26) expresses the requirement that observations should have the same type of unimodality as the corresponding intensions on the antecedent part of IF-THEN rules. For example, if there is a rule IF X is $\mathcal{A}$ THEN Y is $\mathcal{B}$ and $\mathcal{A} = $ *small*, then the intensional mapping of $\mathbf{A}'$ should have the same type of unimodality as the intension of $\mathcal{A}$ = *small*. This is the articulation of a well known condition used elsewhere in the theory of approximate reasoning: the observation needs not be the same as the antecedent, but on the other hand, they are not supposed to be completely different.

Theorem 6.22

Let $\mathcal{LD}^I$ be a simple linguistic description, $T' \in \mathcal{T}'$ and the intensions of IF-THEN rules $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$ are constructed by means of (5.9). The theory T_I is $u^\star$-consistent iff the theories of evaluating expressions T_1^{ev} and T_2^{ev} are such that

$$\textit{if } (\mathbf{A}_{i,\langle x\rangle} \,\mathrm{uni}_m^{T_1^{ev}}\, \mathbf{A}_{j,\langle x\rangle}) \textit{ then } (\mathbf{B}_{i,\langle y\rangle} \,\mathrm{uni}_m^{T_2^{ev}}\, \mathbf{B}_{j,\langle y\rangle}) \tag{6.27}$$

holds for all $i, j \in 1, 2, \ldots, r$.

PROOF: ($\Leftarrow$) Let us denote by I the index set $I = \{j \mid \mathbf{A}'_{j,\langle x\rangle} \in T'\}$. From Lemma 3.14 and (6.27) it follows that the set $\{\mathbf{B}_{j,\langle y\rangle} \mid j \in I\}$ is a block of the set $\mathbf{S}_{\langle y\rangle}$ of intensions with respect to the tolerance relation $\mathrm{uni}_m^{T_2^{ev}}$. Hence, for all $i, j \in I$ it holds that $\mathbf{B}_{i,\langle y\rangle} \,\mathrm{uni}_m^{T_2^{ev}}\, \mathbf{B}_{j,\langle y\rangle}$. Consider again the functions $\gamma_A, \gamma'_A, \gamma_B, \gamma'_B$ associated to functions $\tilde{\alpha}_A, \tilde{\alpha}'_A, \tilde{\alpha}_B, \tilde{\alpha}'_B$, respectively, in the same way as in (6.21). Due to the definition of the relation $\mathrm{uni}_m^{T_2^{ev}}$, the set $\cup_{j\in I}\{y \in [0,1] \mid \gamma_{B_j}(y) = 1\}$ is nonempty, i.e. there exists $y' \in [0,1]$ such that $\gamma_{B_j}(y') = 1$ for all $j \in I$. Now we have to show that, if all assumptions are fulfilled, all functions γ'_{B_j}, $j \in I$, are unimodal and there exists $y_0 \in [0,1]$ such that $\gamma'_{B_j}(y_0) = 1$ for all $j \in I$. Because for all $j \in I$, $\mathbf{A}_{j,\langle x\rangle} \,\mathrm{uni}_m^{T_1^{ev}}\, \mathbf{A}'_{j,\langle x\rangle}$, and we know that there are $x_{j_0} \in [0,1]$, $j_0 \in I$ such that $\gamma'_{A_j}(x_{j_0}) = \gamma_{A_j}(x_{j_0}) = 1$ (this follows from the definition of $\mathcal{T}'$) we can use Lemma 6.8, and deduce that indeed $\gamma'_{B_j}(y') = 1$ for all $j \in I$. The unimodality of functions γ'_{B_j}, $j \in I$ is proved in the same way as in the proof of Theorem 6.18, and, because we considered arbitrary $T' \in \mathcal{T}'$, it follows that the theory T_I is $u^\star$-consistent (see the last sentence of the proof of Theorem 6.18).

($\Rightarrow$) Suppose that (6.27) does not hold. Then it holds for some $i_0, j_0 \in I$ that

$$\mathbf{A}_{i_0,\langle x\rangle} \,\mathrm{uni}_m^{T_1^{ev}}\, \mathbf{A}_{j_0,\langle x\rangle} \tag{6.28}$$

and in the same time

$$\neg \left(\mathbf{B}_{i_0,\langle y\rangle} \,\mathrm{uni}_m^{T_2^{ev}}\, \mathbf{B}_{j_0,\langle y\rangle}\right). \tag{6.29}$$

Now we form a theory $T'_0 = \mathbf{A}_{i_0,\langle x\rangle} \cup \mathbf{A}_{j_0,\langle x\rangle}$. Due to (6.28), $T'_0 \in \mathcal{T}'$. Due to Theorem 6.9 it holds that if $\mathbf{A}_{i,\langle x\rangle} = \mathbf{A}'_{i,\langle x\rangle}$ then $\mathbf{B}_{i,\langle y\rangle} = \mathbf{B}'_{i,\langle y\rangle}$. Hence,

$$\mathbf{B}'_{\langle y\rangle} = \{\mathbf{B}_{i_0,\langle y\rangle}\} \cup \{\mathbf{B}_{j_0,\langle y\rangle}\}.$$

From Lemma 3.22 it follows that a function $\bigvee_{i\in\{i_0,j_0\}} \tilde{\alpha}'_{B_i}$ is not a unimodal one, and therefore, the value $\tau^\star$ from (6.24) is greater than 0 and the theory T_I is not $u^\star$-consistent. □

As an easy consequence we add the following corrolary, saying that if (the intensions of) the antecedents have different types of unimodality, then the theory T_I assigned to a linguistic description $\mathcal{LD}^I$ is $u^\star$-consistent.

Corollary 6.23

Let the assumptions are the same as in Theorem 6.22. Let for all $i, j \in 1, 2, \ldots r$ it holds that

$$(\mathbf{A}_{i,\langle x\rangle} \, \mathrm{uni}_m^{T_1^{ev}} \, \mathbf{A}_{j,\langle x\rangle}) \text{ implies } i = j. \tag{6.30}$$

Then the theory T_I is $u^\star$-consistent.

PROOF: It follows from the assumption (6.30) that (6.27) holds. Then the claim follows from Theorem 6.22. □

Chapter 7

An Application — Deduction with Crisp Observations

In this chapter we apply results of the previous chapters, especially Chapter 6, to a particular, but very frequent situation, in which the observation (in some possible world) is a crisp number. Hence, the support of the theory T', understood as a fuzzy set of axioms, have cardinality equal to 1. The problem then does not lay so much in the computation of the conclusion $\mathbf{B}'_i$, but more in the determination of the theory $T' = \{{}^1/A_{i,x}[t_0]\}$ – which formula A_i should we choose? We assume in this chapter that there is given a simple linguistic description $\mathcal{LD}^I$ and the corresponding theory T_I. Further, a possible world $\mathcal{W} = \langle\langle W_1, W_2\rangle, \ldots\rangle$ is given, without a loss of generality we put $W_1 = W_2 = [0, 1]$. We also suppose that $\mathcal{W}$ is canonical possible world for T_I (see Section 5.3).

The theory T' now has the following form:

$$T' = \left\{{}^1/A_{i,x}[t_o] \mid A_i = m(\mathcal{A}_i),\ i \in \{1, \ldots, r\}\right\} \tag{7.1}$$

where $\mathcal{A}_i$ is the antecedent of the i-th IF-THEN rule $\mathcal{R}_i$ and $t_0 \in M_1$ is the term which corresponds to the crisp observation u_0 in a possible world $\mathcal{W}_1$. First, we investigate the properties of fuzzy logic deduction in the situation when the theory T' has the form (7.1). Then we turn our attention to the problem of the determination of formula A_i, given a crisp number $u_0 \in W_1$.

7.1 General properties

Lemma 7.1
Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$, $i = 1, 2, \ldots, r$ be constructed by means of (5.9). Let the theory T' have the form (7.1). Then it holds for the conclusion $\mathbf{B}'_{\langle y\rangle}$ that $\mathbf{B}'_{\langle y\rangle} = \left\{\mathbf{B}'_{i,\langle y\rangle}\right\}$ and $\tilde{\alpha}'_{B_i}(s) \geq \tilde{\alpha}_{B_i}(s)$ for all $s \in M_2$. If $\tilde{\alpha}_{A_i}(t_0) = 1$, then $\tilde{\alpha}'_{B_i}(s) = \tilde{\alpha}_{B_i}(s)$ for all $s \in M_2$.

PROOF: If $\tilde{\alpha}_{A_i}(t_0) = 1$ then the claim follows directly from Theorem 6.9. Otherwise, we can express $\tilde{\alpha}'_{B_i}(s)$ for the particular $s \in M_2$ as (see Theorem 6.5)

$$\tilde{\alpha}'_B(s) = \bigvee_{t\in M_1} \left(\tilde{\alpha}'_{A_i}(t) \otimes \tilde{\alpha}_{A_i\Rightarrow B_i}(t,s)\right) = \tilde{\alpha}_{A_i}(t_0) \rightarrow \tilde{\alpha}_{B_i}(s), \tag{7.2}$$

because $\tilde{\alpha}_{A_i\Rightarrow B_i}(t,s) = \tilde{\alpha}_{A_i}(t) \rightarrow \tilde{\alpha}_{B_i}(s)$ and $\tilde{\alpha}'_{A_i}(t) = 0$ for $t \neq t_0$. Consequently,

$$\tilde{\alpha}'_{B_i}(s) = (1 - \tilde{\alpha}_{A_i}(t_0) + \tilde{\alpha}_{B_i}(s)) \wedge 1 = 1 \wedge (C + \tilde{\alpha}_{B_i}(s))$$

where $C = 1 - \tilde{\alpha}_{A_i}(t_0)$. It follows that $\tilde{\alpha}'_{B_i}(s)$ is always greater than or equal to $\tilde{\alpha}_{B_i}(s)$. □

Remark 7.2 (i) It is also possible to consider theories of the form

$$T' = \left\{{}^{a_{t_0}}\!/A_{i,x}[t_o] \mid A_i = m(\mathcal{A}_i), t_0 \in M_1\right\}$$

with $a_{t_0} < 1$. We can use the value of a_{t_0} less than 1 in situations when our confidence that the term t_0 indeed represents the observation is not absolute. However, because the resulting function $\tilde{\alpha}'_{B_i}$ is

$$\tilde{\alpha}'_{B_i}(s) = a_{t_0} \otimes (\tilde{\alpha}_{A_i}(t_0) \rightarrow \tilde{\alpha}_{B_i}(s)),$$

i.e. it is the same as for $a_{t_0} = 1$, only multiplied by a_{t_0}, this technique is not suitable.

(ii) The theory T' can also have the form $T' = \left\{\mathbf{A}'_{i,\langle x\rangle} \mid i \in I\right\}$ with $\tilde{\alpha}'_{A_i}(t) = 1$ for $t = t_0$ and $\tilde{\alpha}'_{A_i}(t) = 0$ otherwise. This is in accordance with our definition of the theory representing a single observation (Definition 6.3). The conclusion has the form $\left\{\mathbf{B}'_{i,\langle x\rangle} \mid i \in K\right\}$ where $\mathbf{B}'_{i,\langle x\rangle} = \left\{{}^{\tilde{\alpha}'_{B_i}(s)}\!/B_{i,y}[s] \mid s \in M_2\right\}$. Again, by the same technique as in Lemma 7.1 we can prove that for all $i \in K$ it holds that $\tilde{\alpha}'_{B_i}(s) \geq \tilde{\alpha}_{B_i}(s)$. However, we prefer to represent an observation by the theory T' of the form (7.1), i.e. to choose only one simple evaluating expression $\mathcal{A}_i$ from the expressions occurring in the antecedents of the linguistic description $\mathcal{LD}^I$, because we usually need a crisp conclusion as a result of fuzzy logic deduction.

Lemma 7.3
Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$, $i = 1, 2, \dots, r$ be constructed by means of (5.9). Let the theory T' be of the form (7.1). Then the function $\gamma'_{B_i} : W_2 \longrightarrow [0,1]$ adjoined to function $\tilde{\alpha}'_{B_i}(s)$ by means of

$$\gamma'_{B_i}(z) = c \;\text{ iff }\; \tilde{\alpha}'_{B_i}\left(s^{(z)}\right) = c, \quad z \in [0,1]$$

where $\tilde{\alpha}'_{B_i}$ is defined by (7.2), is continuous and unimodal.

PROOF: The *continuity* follows from the fact that both $\gamma_{A_i}(x)$ and $\gamma_{B_i}(y)$ adjoined to functions $\tilde{\alpha}_{A_i}(t)$, $A_i = m(\mathcal{A}_i)$ and $\tilde{\alpha}_{B_i}(s)$, $B_i = m(\mathcal{B}_i)$, respectively, are continuous, and the operation $\rightarrow$ is also continuous.
Unimodality: due to the condition 3 from Definition 4.1 the function $\gamma_{B_i}(y)$ is unimodal. We can express $\gamma'_{B_i}(y)$ for particular $x \in W_1$ as

$$\gamma'_{B_i}(y) = \gamma_{A_i}(x) \rightarrow \gamma_{B_i}(y) = (1 - \gamma_{A_i}(x) + \gamma_{B_i}(y)) \wedge 1 = 1 \wedge (C + \gamma_{B_i}(y))$$

where $C = 1 - \gamma_{A_i}(x)$. This means that increasing parts of γ_{B_i} remain increasing (or constant) in γ'_{B_i}, decreasing parts of γ_{B_i} remain decreasing (or constant) in γ'_{B_i}, no new local extreme can occur and the maximum of γ_{B_i} is preserved in γ'_{B_i}. Hence, the function γ'_{B_i} is unimodal. □

Lemma 7.4
Let $\mathcal{LD}^I$ be a simple linguistic description and let the intensions $\mathbf{A}_{i,\langle x\rangle} \Rightarrow \mathbf{B}_{i,\langle y\rangle}$, $i = 1, 2, \dots, r$ be constructed by means of (5.9). Let the theory T' be of the form $T'_z = \left\{{}^1/A_{i,x}\left[t^{(z)}\right]\right\}$, where $z \in W_1$. Then the mapping $\Gamma : \langle W_1, |\,|\rangle \longrightarrow \langle [0,1]^{W_2}, d\rangle$ which assigns to every $z \in W_1$ a function γ'_{B_i} (by means of the theory T'_z, see also Lemma 7.3) is continuous (d is a metric $d(\gamma_1, \gamma_2) = \sup_{y \in W_2} |\gamma_1(y) - \gamma_2(y)|$).

PROOF: The claim is a straightforward consequence of the continuity of functions γ_{A_i} and γ_{B_i} as well as the operation $\rightarrow$ used in the computation of the function γ'_{B_i}.
□

7.2 The operation Suit

As it has been mentioned above, we have to choose the most suitable expression among evaluating expressions occurring in the antecedents of IF-THEN rules in the linguistic description $\mathcal{LD}^I$, which corresponds to the term t_0. Informally, this means that in every possible world the element assigned to t_0 intuitively corresponds to the meaning of this most suitable expression; for example, if t_0 is assigned the expression *big* then every interpretation v of t_0 in every possible world is intuitively indeed "big".

The algorithm which selects the predicate symbol $G \in \mathcal{G}$ (and, consequently, also the simple evaluating expression $\mathcal{A} \in \mathcal{S}$, $\mathcal{A} = m^{-1}(G)$) given an observation u_0 in some possible world $\mathcal{W}$ is described as follows.

We suppose some structure of the set $\mathcal{G}$ of atomic predicate symbols which model the meanings of linguistic evaluating expressions. This structure is motivated by a linguistic intuition (see Section 3.2) — that there are several atomic evaluating expressions (adjectives like *small*, *big* etc.). Their meaning can be modified by *linguistic hedges* (see e.g. [16]) – the adverbs *very*, *more or less* etc. Then the set of simple evaluating expressions is divided into several groups with the same atomic expression, and these groups are totally ordered by an order relation characterizing a *sharpness* of the hedges. We have already met this structure when we introduced the relation $\mathrm{uni}_m^{T^{ev}}$ on the set of intensions of evaluating linguistic expressions (see Section 4.1). If the intensions $\mathbf{A}_{1,\langle x\rangle}$ and $\mathbf{A}_{2,\langle x\rangle}$ of simple evaluating expressions $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}$ are in relation $\mathrm{uni}_m^{T^{ev}}$, we can suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same atomic expression and differ in linguistic hedge only. (We cannot be sure, but for *reasonable* theories T^{ev} it should hold).

The sharpness characterizes the degree of precision imposed by a hedge on an atomic expression, e.g. if we know that something is *very small*, we have more information than if we know that it is *small* (because the former implies the latter, but not vice versa – at least in the inclusive interpretation of hedges, see [16]). Similarly, to be *more or less small* is less specific than to be *small*. We express this by means of the sharpness relation: the linguistic expression $\mathcal{A}_1$ is sharper than $\mathcal{A}_2$ if it is more specific in the sense explained above.

Therefore we suppose the following structure of the set $\mathcal{G}$ of atomic predicate symbols: $\mathcal{G} = \bigcup_{i\in P} \mathcal{G}_i$ and it holds that if $i \neq j$ then $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$. The sets $\mathcal{G}_i$, $i \in P$ correspond to the sets of simple evaluating expressions with the same atomic evaluating expression. Further, $\mathcal{G}_i = \{G_{i,\sigma} \mid \sigma \in Q_i\}$ and there is a total order relation on the set Q_i: $\sigma_1 < \sigma_2$ iff $m^{-1}(G_{i,\sigma_1})$ is sharper than $m^{-1}(G_{i,\sigma_2})$, i.e. iff the simple evaluating expression $\mathcal{A}$ corresponding to G_{i,σ_1} is sharper than the simple evaluating expression $\mathcal{A}'$ corresponding to G_{i,σ_2}. It is possible that the cardinalities of the sets $Q_i, i \in P$ are not equal, i.e. that there is a various number of hedges used for the individual atomic expressions. This is the case e.g. in our software system LFLC, where we are using nine simple evaluating expressions with the atomic expressions *small* and *big*, but only five with the atomic expression *medium*.

We further suppose some fixed total order relation $\prec$, defined on the index set P, given to us apriori from outside. This relation will be used for a determination of a predicate symbol G to be chosen in a situation where there are several candidates with the same sharpness for the most suitable formula G and a given term t. This situation can occur e.g. when the interpretation of t lies between *small* and *medium*. Then it may happen that the most suitable expression could be e.g. *more or less small* or *more or less medium*. Hence we have to decide between two expressions with the identical hedge. Of course, it is possible to define only one total ordering (called also "sharpness") on the set $\mathcal{G}$ which unify these two order relations, but then the intuition behind the notion of sharpness can be lost.

Let us define an operation $p : F_{J(T)} \longrightarrow L$ for the given threshold $c_0 \in (0, 1]$ by

$$p_{T,c_0}(A) = \begin{cases} c, & T \vdash_c A \text{ and } c \leq c_0, \\ c_0, & T \vdash_c A \text{ and } c > c_0. \end{cases}$$

Definition 7.5
Let us denote the maximal value of the operation p on some subset $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ for the given term $t \in M$ by m_t, i.e.

$$m_t = \max_{G \in \tilde{\mathcal{G}}} \{p_{T,c_0}(G_x[t])\}.$$

The most suitable linguistic expression operation $\mathrm{Suit} : M \longrightarrow \tilde{\mathcal{G}}$ for the given theory T and the threshold $c_0 \in (0, 1]$ is defined by the following formula:

$$\mathrm{Suit}^{c_0}_{\tilde{\mathcal{G}}}(t_0) = \begin{cases} \min_{i \in P}\{G_{i,\sigma_0} \mid \sigma_0 = \min\{\sigma \mid p_{T,c_0}(G_{i,\sigma,x}[t_0]) = m_{t_0},\ G_{i,\sigma} \in \tilde{\mathcal{G}}\}\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad m_{t_0} > 0 & \\ \text{undefined} \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad m_{t_0} = 0. & \end{cases} \tag{7.3}$$

The first minimum in formula (7.3) is taken with respect to the ordering $\prec$ discussed above.

The next theorem shows that if the theory T is the theory of evaluating expressions in the sense of Definition 4.1, then the result of the operation Suit is always defined, provided that $\tilde{\mathcal{G}} = \mathcal{G}$.

Theorem 7.6
Let a theory T^{ev} fulfills the conditions of Definition 4.1 and $c_0 \in (0, 1]$. Then to every $t \in M$ there is $G \in \mathcal{G}$ such that

$$\mathrm{Suit}_{\mathcal{G}}^{c_0}(t) = G.$$

PROOF: Condition 5 from Definition 4.1 says that for every $t \in M$ there is at least one $G \in \mathcal{G}$ such that $T^{ev} \vdash_c G_x[t]$ with $c > 0$, which means that also $m_t > 0$. It follows that there always exists some $G_{i,\sigma}$ such that $p_{T^{ev},c_0}(G_{i,\sigma,x}[t]) = m_t$ and therefore the set

$$S_{\mathcal{G}}^{c_0}(t_0) = \{G_{i,\sigma_0} \mid \sigma_0 = \min\{\sigma \mid p_{T,c_0}(G_{i,\sigma,x}[t_0]) = m_{t_0},\ G_{i,\sigma} \in \mathcal{G}\}\} \tag{7.4}$$

is nonempty for every $t \in T$ and have the property that if $G_{i,\sigma} \in S_{\mathcal{G}}^{c_0}(t_0)$ and $G_{j,\beta} \in S_{\mathcal{G}}^{c_0}(t_0)$ then $i \neq j$. Because $\prec$ is total order, the set $\{i \mid i \in P,\ G_{i,\sigma} \in S_{\mathcal{G}}^{c_0}(t_0)\}$ has unique minimum with respect to $\prec$, and the claim follows. □

Remark 7.7 The meaning of the threshold c_0 appearing in the definition of the Suit operation is the following: all the provability degrees with respect to the theory T, which are greater than c_0, are regarded as maximal, and therefore they are considered as possible candidates for the result of the Suit operation. This is necessary e.g. in situations when there are for some $t \in M$ two (or more) predicate symbols $G_1 = G_{i,\sigma_j}$ and $G_2 = G_{i,\sigma_k}$ from $\mathcal{G}$, $i \in P$, $\sigma_j < \sigma_k$ with high values of $\tilde{\alpha}_{G_i}(t)$ and also

$$\tilde{\alpha}_{G_1}(t) \leq \tilde{\alpha}_{G_2}(t)$$

for all $t \in M$. This correspond to the so-called *inclusive interpretation* of linguistic hedges [16]. Then it is unsatisfactory to prefer *small* to *very small* in the situation when $\tilde{\alpha}_{G_2}(t)$ is 1 and $\tilde{\alpha}_{G_1}(t)$ is, say, 0.99, $G_1 = m(\textit{very small})$ and $G_2 = m(\textit{small})$. The threshold c_0 allows us to adjust the behavior of fuzzy logic deduction in such situations.

In the following we denote by An a set of all atomic predicate symbols $G_{i,\sigma}$ – meanings of simple evaluating expressions appearing in the antecedents of IF-THEN rules from a linguistic description $\mathcal{LD}^I$, i.e.

$$An = \{m(\mathrm{An}(\mathcal{R}_i)) \mid i = 1, \ldots, r\}. \tag{7.5}$$

Theorem 7.8
Let a simple linguistic description $\mathcal{LD}^I$ be given. Then $\mathrm{Suit}^{c_0}_{An}(t)$ is total function iff the set An defined by (7.5) has the following property (P): For every $t \in M_1$ there exists $G \in An$ such that $\tilde{\alpha}_G(t) > 0$.

PROOF: Suppose that P holds. Then the set $S^{c_0}_{An}(t)$ defined by (7.4) is nonempty and the set $\{i \mid i \in P,\ G_{i,\sigma} \in S^{c_0}_{An}(t_0)\}$ has unique minimum with respect to $\prec$ for every $t \in M_1$ and therefore, $\mathrm{Suit}^{c_0}_{An}(t)$ is total function. Vice-versa, if P does not hold, then there exists some $t_0 \in M_1$ such that $\tilde{\alpha}_G(t_0) = 0$ for all $G \in An$. It follows that $m_{t_0} = 0$ and, according to (7.3), $\mathrm{Suit}^{c_0}_{An}(t_0)$ is not defined for this t_0 and $\mathrm{Suit}^{c_0}_{An}(t)$ is not total function. □

7.3 The algorithm of fuzzy logic deduction with crisp observations

Now, the operation Suit is used in the algorithm of fuzzy logic deduction. In the sequel, we suppose that linguistic descriptions do not include IF-THEN rules with identical antecedents and different consequents. The reason is that the operation Suit then can force us to choose more than one rule.

For an observation $u' \in W_1$ we find the corresponding closed term $t_0 \in M_1$ such that $\mathcal{W}(t_0) = u'$. We assume that there are enough terms in M_1 so that t_0 always exists. We suppose that there are some theories of evaluating expressions T^{ev}_1 and T^{ev}_2 at our disposal. Last, we choose a threshold $c_0 \in (0, 1]$. Then we find atomic predicate symbol $G_i \in An$, $\mathrm{Suit}^{c_0}_{An}(t_0) = G_i$, if the result of $\mathrm{Suit}^{c_0}_{An}(t_0)$ is defined. This means that the i-th IF-THEN rule 'IF $\mathcal{A}_i$ THEN $\mathcal{B}_i$' is selected to perform the inference. The situation in which the result of $\mathrm{Suit}^{c_0}_{An}(t_0)$ is undefined corresponds to the situation in which we have no relevant information for the decision about the result of inference. Then also the result of fuzzy logic deduction for such a $u' \in W_1$ is left undefined.

Now we form the theory $T_D = T' \cup T_I$ where T' is defined by (7.1). Next, we derive the conclusion $\mathbf{B}'$ defined by formula (6.5). In this special case, when only one IF-THEN rule is used, (6.5) has the form

$$\mathbf{B}'_i = \left\{\tilde{\alpha}'_{B_i}(s) \big/ B_{i,y}[s] \mid s \in M_2,\ B_i = m(\mathcal{B}_i)\right\} \tag{7.6}$$

where

$$\tilde{\alpha}'_{B_i}(s) = \tilde{\alpha}_{A_i \Rightarrow B_i}(t_0, s) = \tilde{\alpha}_{A_i}(t_0) \rightarrow \tilde{\alpha}_{B_i}(s), \tag{7.7}$$

and $A_i = m(\mathcal{A}_i)$, provided that the intension of IF-THEN rule $\mathcal{R}_i$ is determined by (5.9).

Then the extension of linguistic expression $\mathcal{B}'_i$ with the intension $\mathbf{B}'_i$ in the possible world $\mathcal{W}$ is found,

$$\mathrm{Ext}_{\mathcal{W}}\, \mathcal{B}'_i = \left\{ \mathcal{W}(B'_{i,y}[\mathbf{w}])/w \middle| \, w \in W_2 \right\}.$$

We can assign linguistic expression $\mathcal{B}'_i$ to $\mathbf{B}'_i$ by means of some *linguistic approximation* algorithm [5]. The last step is to find $v' \in W_2$, $v' = \mathrm{DEF}(\mathrm{Ext}_{\mathcal{W}}\, \mathcal{B}'_i)$.

The defuzzification operation is defined on the semantic level: for a possible world $\mathcal{W}$, it is an operation DEF : $(L^W - \{\emptyset\}) \longrightarrow W$ which to any nonempty fuzzy set $A \underset{\sim}{\subset} W$ assigns element $\mathrm{DEF}(A) \in W$ such that $A(\mathrm{DEF}(A)) > 0$ ([27], p. 214, see also [19]). The problem of logically well-founded defuzzification method is one of directions of our further research. Currently we are using defuzzification method *DEEs* (Defuzzification of Evaluating Expressions) which is a combination of the well known defuzzification methods LOM (Last of Maxima), MOM (Mean of Maxima) and FOM (First of Maxima). It takes the right edge of the kernel, center of gravity, or the left edge of the kernel for unimodal functions of type Z, Π and S, respectively (see Definition 7.9).

Definition 7.9

Denote by $\mathcal{K}$ the set of all continuous functions from $[a, b]$ to $[0, 1]$. The *DEEs* defuzzification method is defined as follows: Let $A \in \mathcal{K}$ be unimodal function (cf. Definition 3.15, $M = \sup_{x \in [a,b]} A(x)$). Then

(i) if $A(a) = M$ and $A(b) < M$ then $DEEs(A) = c_2$,

(ii) if $A(b) = M$ and $A(a) < M$ then $DEEs(A) = c_1$,

(iii) if $A(b) < M$ and $A(a) < M$ then $DEEs(A) = \frac{c_1 + c_2}{2}$,

otherwise it is defined as a center of gravity of A, i.e. $DEEs(A) = \frac{\int_{[a,b]} A(x)x\,\mathrm{dx}}{\int_{[a,b]} A(x)\,\mathrm{dx}}$.

Lemma 7.10

The DEEs defuzzification method is not continuous wrt. topology induced by the metrics

$$d(A, B) = \sup_{x \in [a,b]} |A(x) - B(x)|$$

where $A, B \in \mathcal{K}$.

PROOF: It is well known [18] that a mapping $f : X_1 \longrightarrow X_2$ where (X_1, ρ_1), (X_2, ρ_2) are metric spaces, is continuous in $x_0 \in X_1$ iff for every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in X_1$, if $\rho_1(x_0, x) < \delta$ then $\rho_2(f(x_0), f(x)) < \epsilon$. Without a loss of generality, we put $[a, b] = [0, 1]$, i.e. let $\mathcal{K}$ be the set of all continuous functions from $[0, 1]$ to $[0, 1]$. We denote by $\langle a_1, c_2, d_2\rangle$ $(0 < a_1 < c_2 < d_2 \leq 1)$ a piecewise linear function given by (see Figure 7.1):

$$\langle a_1, c_2, d_2\rangle(x) = \begin{cases} 1 - a_1 + x & x \in [0, a_1], \\ 1 & x \in [a_1, c_2], \\ 1 - \frac{x - c_2}{d_2 - c_2} & x \in [c_2, d_2], \\ 0 & x \in [d_2, 1]. \end{cases}$$

We show that $DEEs$ is not continuous in a point A_0 defined as

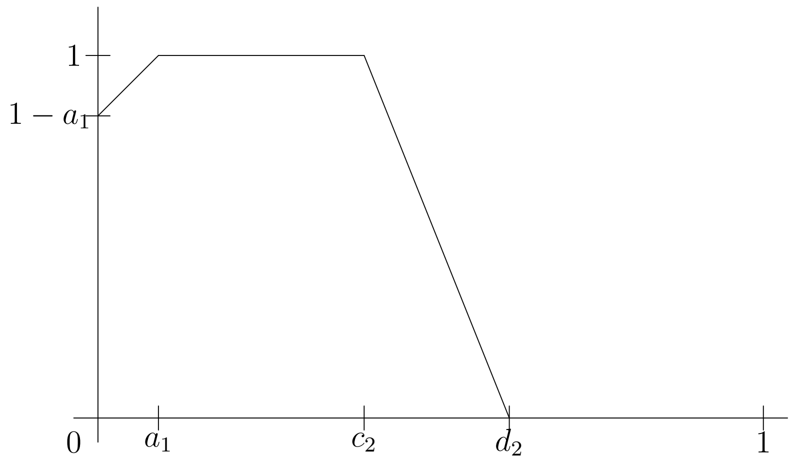

Figure 7.1: The function $\langle a_1, c_2, d_2\rangle$ from the proof of Lemma 7.10.

$$A_0(x) = \begin{cases} 1 & x \in [0, c_2], \\ 1 - \frac{x - c_2}{d_2 - c_2} & x \in [c_2, d_2], \\ 0 & x \in [d_2, 1]. \end{cases}$$

Let $0 < \epsilon < c_2/4$. It holds that $DEEs(A_0) = c_2$. Then there is no $\delta > 0$ such that for every $A \in \mathcal{K}$, if $d(A_0, A) < \delta$ then $|DEEs(A_0) - DEEs(A)| < \epsilon$. Indeed, for any $\delta > 0$ there is $0 < \zeta < \delta$, $\zeta < \frac{c_2}{2}$ and for the function $\langle \zeta, c_2, d_2\rangle$ it holds that $d(\langle \zeta, c_2, d_2\rangle, A_0) = \sup_{x \in [a,b]} |\langle \zeta, c_2, d_2\rangle(x) - A_0(x)| = \zeta < \delta$, but $DEEs(\langle \zeta, c_2, d_2\rangle) = \\ = \frac{c_2 + \zeta}{2}$ and $|DEEs(A_0) - \frac{c_2+\zeta}{2}| = |c_2 - \frac{c_2+\zeta}{2}| = \frac{c_2 - \zeta}{2} > \epsilon$ (because $\zeta < \frac{c_2}{2}$). □

Remark 7.11 Despite this result, the defuzzification method $DDEs$ behaves continuously, if we restrict the set of considered functions to such functions, which can occur as a result of fuzzy logic deduction with crisp observations.

Definition 7.12
A continuous unimodal function $f : [a, b] \longrightarrow [0, 1]$ is called *proper unimodal* if

(i) there is $x_1 \in [a, b]$ such that $f(x_1) = 1$.

(ii) If f is of type S or Z, then there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$.

(iii) If f is of type Π, then there exist $x_{01}, x_{02} \in [a, b]$ such that $x_{01} < x_1 < x_{02}$ and $f(x_{01}) = f(x_{02}) = 0$.

Theorem 7.13
Let $f_0 : [a, b] \longrightarrow [0, 1]$ is proper unimodal . Denote by $\mathcal{U}_{f_0} \subset \mathcal{K}$ the set

$$\mathcal{U}_{f_0} = \{f_0 \oplus f_C \mid C \in [0, 1)\} \tag{7.8}$$

where $f_C(x) = C$ for all $x \in [a, b]$ and $\oplus$ is Łukasiewicz t-conorm. (sum of functions is defined pointwise). Then the function $DDEs : \mathcal{U}_{f_0} \longrightarrow [a, b]$ is continuous wrt. metric d.

PROOF: It is easy to see that the mapping d is a metric on the set $\mathcal{U}_{f_0}$. Note that the definition of $\mathcal{U}_{f_0}$ and the requirement that $C < 1$ in (7.8) assure us that all functions in $\mathcal{U}_{f_0}$ are of the same type of unimodality and the set $\mathcal{U}_{f_0}$ is totally ordered by relation $\leq$ defined pointwise. Let us denote by $f + C$, $f \in \mathcal{U}_{f_0}$, $C \in [0, 1]$ a function $f \oplus f_C$ if $f \oplus f_C \in \mathcal{U}_{f_0}$, otherwise let $f + C$ be the function $(f_0 \oplus f_1)(x) = f_0(x) \oplus 1 = f_1$. Similarly, let $f - C$ be a function f' such that $f' \oplus f_C = f$. If such a function does not exist, let $f - C$ be f_0. It holds that $d(f, f + C) \leq C$ and $d(f, f - C) \leq C$.

Due to the definition of $DEEs$ method (Definition 7.9), which takes into account only right edge of the kernel, left edge of the kernel or both for unimodal functions of types Z, S, Π, respectively, it is sufficient to prove only the following: The functions $c_1, c_2 : \mathcal{U}_{f_0} \longrightarrow [a, b]$ which assign to functions from $\mathcal{U}_{f_0}$ the left and the right edge of the kernel, respectively, are continuous wrt. metric d. Note that if $f_1 \leq f_2$ then $c_1(f_1) \geq c_1(f_2)$ and $c_2(f_1) \leq c_2(f_2)$.

Let f_0 be of type Z or Π. Let us check the function c_2. We have to show that for all $f' \in \mathcal{U}_{f_0}$, for any $\epsilon > 0$ there is $\delta > 0$ such that for all $f \in \mathcal{U}_{f_0}$,

$$d(f, f') < \delta \text{ implies } |c_2(f) - c_2(f')| < \epsilon. \tag{7.9}$$

For $\epsilon > 0$, choose δ such that $\delta < M$, where

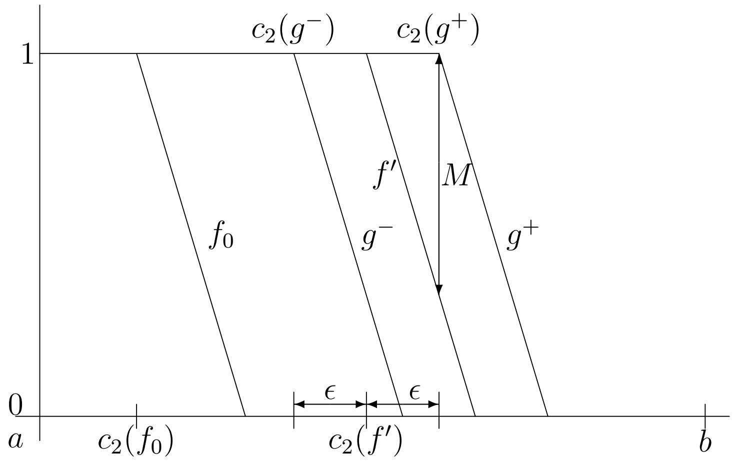

Figure 7.2: Continuity of DEEs defuzzification method.

(i) $M = 1 - f_0(c_2(f_0) + \epsilon)$ if $c_2(f') + \epsilon \leq b$,

(ii) $M = 1 - f'(b)$, if $c_2(f') + \epsilon > b$.

We show that (7.9) holds. We have $M > 0$, because $c_2(f_0)$ is right edge of kernel, hence (i) $f_0(c_2(f_0) + \epsilon) < 1$, or (ii) $1 - f'(b) < 1$.

Consider a situation when $c_2(f')+\epsilon \leq b$ and $\epsilon < c_2(f')-c_2(f_0)$. For a function g^+, for which it holds that $d(f', g^+) = M$ and $g^+ > f'$, it holds that $c_2(g^+) = c_2(f') + \epsilon$. Similarly, for a function g^-, for which it holds that $d(f', g^-) = M$ and $g^- < f'$, it holds that $c_2(g^-) = c_2(f') - \epsilon$ (see Figure 7.2, for the sake of simplicity, the function f_0 is piecewise linear, but the situation is the same for a general continuous and unimodal function). Hence, for all $\delta < M$ and for all functions $f \in \mathcal{U}_{f_0}$ such that $g^- < f < g^+$ condition (7.9) holds.

If $c_2(f') + \epsilon > b$ or $\epsilon > c_2(f') - c_2(f_0)$, then the range of functions, for which $d(f, f') < \delta$ holds, is restricted (because it is limited by the greatest and the smallest elements of the set $\mathcal{U}_{f_0}$, respectively), and (7.9) holds again. It follows that the function c_2 is continuous for functions f_0 of type Z or Π. The proof for the functions f_0 of types Π or S proceeds in the same way. Hence, both functions c_1 and c_2 are continuous with respect to the metric d. The continuity of $DEEs$ defuzzification method for functions f_0 of types Z and S is immediate. For functions f_0 of type Π, the continuity follows from the continuity of the function which defines the result of $DEEs$ method in this case (namely $DEEs(f) = (c_1(f) + c_2(f))/2$).

□

Remark 7.14 The continuity of DEEs defuzzification could be violated if f_0 is not proper unimodal or if we allow $C = 1$ in (7.8). In such situations it is possible that the type of unimodality changes for some functions from $\mathcal{U}_{f_0}$. Consequently, the formula for the computation of $DEEs(f)$ (see Definition 7.9) changes, too.

7.4 The behavior of fuzzy logic deduction with crisp observations

Generally speaking, the behavior of fuzzy logic deduction with crisp observations depends on several factors, namely:

1. concrete settings of functions $\tilde{\alpha}_G$,
2. the value of the threshold c_0,
3. the defuzzification method DEF.

We can study the behavior of fuzzy logic deduction by examining the function LD, $LD : W_1 \longrightarrow W_2$, which assigns the conclusion $v' \in W_2$ to a given observation $u' \in W_1$. It follows from Theorem 7.8 that if the linguistic description $\mathcal{LD}^I$ is such that it fulfills the property (P), $\text{Suit}^{c_0}_{An}(t)$ is total function, which means that for every observation there is one IF-THEN rule selected and, consequently, the conclusion $\mathbf{B}'$ is defined and therefore also LD is total function.

Because only one IF-THEN rule from the linguistic description is selected for the given observation $u' \in W_1$, the function LD is only piece-wise continuous.

Lemma 7.15
Let the linguistic description $\mathcal{LD}^I$ contain one IF-THEN rule, i.e.

$$\mathcal{LD}^I = \{\mathit{IF}\ \mathcal{A}\ \mathit{THEN}\ \mathcal{B}\}.$$

Let the function $\tilde{\alpha}_B$ is proper unimodal. Then the function $LD : W^\star \longrightarrow W_2$, where the set $W^\star \subseteq W_1$ is defined as $W^\star = \{x \mid x \in W_1, \tilde{\alpha}_A\left(t^{(x)}\right) > 0\}$, is continuous wrt. standard metric.

PROOF: From the definition of operation Suit (Definition 7.5) and of the algorithm of fuzzy logic deduction with crisp observations (Section 7.3) it follows that the function LD can be depicted by the following scheme:

$$x \in W_1 \overset{(1)}{\mapsto} t^{(x)} \in M_1 \overset{(2)}{\mapsto} \tilde{\alpha}'_B \in [0,1]^{M_2} \overset{(3)}{\mapsto} \mathrm{B}' \in \mathcal{U}_{f_0} \overset{(4)}{\mapsto} DEEs(x) \in W_2. \tag{7.10}$$

Hence, the mapping LD is continuous if the mappings (1), (2), (3), (4) from (7.10) are continuous (provided that the continuity is considered with respect to the same metrics "in" and "out" in the spaces M_1, $[0,1]^{M_2}$ and $\mathcal{U}_{f_0}$). We had not introduced M_1 and $[0,1]^{M_2}$ as metric spaces, but we can consider adjoined metric spaces $[0,1]$ and $[0,1]^{[0,1]}$ (the space of functions γ_B first introduced in Definition 4.1). The space $\mathcal{U}_{f_0}$ was defined by (7.8). The function f_0 is put equal to the function γ_B adjoined to $\tilde{\alpha}_B$. The continuity of mappings (1) and (3) is straightforward. The continuity of the mapping (2) was shown in Lemma 7.4. We can see from the proof of Lemma 7.3 that the space obtained after step (3) in (7.10) is indeed $\mathcal{U}_{\gamma_B}$. The continuity of the mapping (4) was proved in Theorem 7.13. □

Theorem 7.16
Let a linguistic description $\mathcal{LD}^I$ with the associated theory T_1^{ev} fulfilling the property (P) (see Theorem 7.8) be given, such that the functions γ_{B_i} adjoined to functions $\tilde{\alpha}_{B_i}$, $i = 1, \ldots, r$ are proper unimodal. Further, let a canonical possible world $\mathcal{W}$, $\mathcal{W} = \langle\langle W_1, W_2\rangle, \ldots\rangle$ be given. Then the function LD defined above is piecewise continuous (i.e. it has finitely many discontinuities of the first type).

PROOF: We showed in Theorem 7.8 that the function $\mathrm{Suit}_{\mathrm{An}}^{c_0} : M \longrightarrow \mathrm{An}$ is total function. It easily follows that also the function $LD : W_1 \longrightarrow W_2$ is total. We can divide the domain of the function LD in several parts in such a way that in every part is used only one IF-THEN rule $\mathcal{R}_i$ (selected by means of the operation $\mathrm{Suit}_{\mathrm{An}}^{c_0}$). As was shown in previous Lemma 7.15, for one IF-THEN rule $\mathcal{R}_i$, $i = 1, \ldots, r$ is the function LD a continuous one. Because every linguistic description can contain only finite number of IF-THEN rules, we can deduce that the overall function LD is piece-wise continuous, and the number of the discontinuities is finite. □

However, the previous result, i.e. the non-continuity of function LD is not necessarily a major drawback. In decision-type problems, for example, we are not looking for continuity, but for the well-defined behavior in every possible situation. Consider, e.g. the linguistic description

$$\mathcal{R}_1 := \text{IF } X \text{ is } \textit{small} \text{ THEN } Y \text{ is } \textit{big},$$
$$\mathcal{R}_2 := \text{IF } X \text{ is } \textit{very small} \text{ THEN } Y \text{ is } \textit{small},$$
$$\ldots$$

If we interpret X as "distance to an obstacle" and Y as "angle of a steering wheel", then it can be dangerous to use for small observations more than one rule, because it can result in “medium” conclusion and crash with the obstacle. This can be amended by a special defuzzification method adapted to such situations, or by a special shape of functions $\tilde{\alpha}_{B_i}$. However, we are convinced that such a well-defined behavior conforms with the way of human reasoning and so, we prefer to find a transparent inference mechanism.

The generalization of results in this section for the (in practice much more important) situation where there are several antecedent variables $X_1, \ldots, X_n$ and IF-THEN rules have the form

$$\mathcal{R}_i^I := \text{IF } \mathcal{A}_{i,1} \text{ AND } \ldots \text{ AND } \mathcal{A}_{i,n} \text{ THEN } \mathcal{B}_i, \quad i = 1, 2, \ldots, r$$

can be done in the straightforward way and is omitted in this thesis.

In the future research, we will study the behavior of fuzzy logic deduction in more details, we will concentrate on its modifications which allow the function LD to be continuous and study also approximation capabilities of it. Another field of study, which can be well formulated using our formalism, is fuzzy logic deduction with linguistically expressed observations.

Chapter 8

Conclusion

We presented in this thesis the methodology which describes the meaning of evaluating linguistic expressions on syntactic level (as an intension) by means of formal fuzzy theory. By means of such theories for antecedent and consequent parts of IF-THEN rules we constructed the intension of IF-THEN rule and treated a set of IF-THEN rules (i.e. a linguistic description) again as a formal theory. Then we used this in the central part of our thesis – the formalization of fuzzy logic deduction. We added another formal theory describing an observation and obtain the conclusion as a fuzzy set of instances of atomic formulas in first-order fuzzy logic. We proved some properties and discussed one important notion, namely inconsistencies in linguistic descriptions. Finally, as an application of the results of previous chapters, we studied fuzzy logic deduction with crisp observations.

It was not possible to cover in this thesis all directions of research which our methodology offers. Among open problems, which we will pursue in the future, we name the study of the properties of fuzzy logic deduction over linguistic descriptions, where the meanings of evaluating linguistic expressions are modeled by extended theories of evaluating expressions (Section 4.3). The properties of the deduction for linguistic descriptions with negated evaluating expressions (e.g. *IF* X is *small* THEN Y is NOT *big*) is also worth of study. We would like also study more carefully the notion of possible world and model-theoretic properties of models of theories of evaluating expressions. Finally, our main future interest will lie in the study of our formal system enriched by generalized quantifiers (e.g. *many*).

The methodology and results have their counterparts in algorithms implemented in software system LFLC (Linguistic Fuzzy Logic Controller) [6] developed in Institute for Research and Applications of Fuzzy Modeling at University of Ostrava. It uses the algorithm of fuzzy logic deduction with crisp observation from Chapter 7. The structure of the set of simple evaluating expressions is the following: there are three atomic evaluating expressions *small*, *medium* and *big* and linguistic hedges (with decreasing sharpness) *extremely*, *highly*, *very*, *more or less*, *roughly*, *quite roughly*, *very roughly* interpreted inclusively, i.e. if x is *very small* then x is *small* as well. However, users can also use their own linguistic expressions and fuzzy numbers in their linguistic descriptions. The defuzzification method $DEEs$ from Section 7.3 is also implemented. It proved itself the most suitable method for fuzzy logic deduction among the known defuzzification methods, provided that evaluating linguistic expressions are used. LFLC, of course, also implements fuzzy approximation methods mentioned in Introduction.

The LFLC system and fuzzy logic deduction implemented in it proved themselves useful in practical applications (see [25]). Fuzzy logic deduction has been also used in methods for learning linguistic descriptions from data [2, 7, 8].

List of Author's Publications

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- Dvořák, A.: On Linguistic Approximation in the Frame of Fuzzy Logic Deduction. *Soft Computing,* 1999, Vol. 3, pp. 111–115.
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