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Organized by the Institute for Research and Applications of Fuzzy Modeling, University of Ostrava

Invited lectures

Demonstrative Uses

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In my talk I shall discuss cases of non-standard uses of indexicals such as ‘now’ and ‘I’. I shall offer an analysis of the phenomena that is conservative with respect to the Kaplanian account of indexicality presented in his Logic of Demonstratives. The point of departure of the paper is the observation that some proper indexicals have demonstrative uses. It is argued that treating some occurrences of ‘now’ and ‘I’ as cases of such uses results in an intuitive and simple analysis of the puzzling phenomena. In my talk I shall also address the question regarding the demarcation line between indexical and demonstrative uses of some expressions.

What do nice proof systems look like?

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(Joint work with Amir Tabatabai.)

A proof system is a system with a set of basic rules indicating how to produce proofs. “Nice” proof systems are useful to study various properties of a logic, such as admissibility of the rules. However, it is not clear what a nice proof system is. In this talk, we focus on this question by introducing a connection between the existence of a nice sequent calculus and the Craig and uniform interpolation properties of the corresponding logic that the calculus captures.

As positive applications, we provide a uniform method to prove the Craig and uniform interpolation properties for the logics FL_e, FL_{ew}, CFL_e, CFL_{ew}, IPC, CPC and their K and KD-type modal extensions. However, on the negative side the relationship finds its more interesting application to show that many substructural logics including $Ł_n$, G_n, BL, R and RM^e, almost all superintutionistic logics (except at most seven of them) and almost all extensions of S4 (except thirty seven of them) do not have a nice calculus.

Edge colourings of complete graphs and representations of certain non-associative relation algebras

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(Joint work with B. Al-Juaid, M. Jackson, and J. Koussas.)

In an edge n-colouring of a complete graph, each triangle of edges is either monochromatic, dichromatic or trichromatic. We explore edge-colourings determined by disallowed triangle colour combinations, but also requiring others. Thus, disallowing monochromatic triangles restricts to edge-coloured complete graphs within the Ramsey bound $R(3, 3, \ldots, 3)$. It is however natural to impose also the dual constraint requiring that all remaining colour combinations (trichromatic and dichromatic) are present. Is it then possible to find such a colouring? Apart from being a natural combinatorial question, there is an additional motivation by way of the algebraic foundations of qualitative reasoning. The constraint language underlying a typical qualitative reasoning system determines a kind of non-associative relation algebra, which is attracting considerable attention from a theoretical computer science perspective, as well as practical applications, for example in scheduling, navigation and geospatial positioning. Satisfiability of such a constraint system is equivalent to a weak form of representability. In the cases we consider, it is also equivalent to existence of a colouring. A few cases we consider have nontrivial solutions, and moreover provide some novel extensions of classically understood connections between certain associative relation algebras and combinatorial geometries.

Partial Type Theory TT^* with Extensions

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Simple type theory (STT), originated in Church 1940, is λ-*calculus* with terms that all are (strictly) *typed* (i.e. associated with type symbols τ). Since it allows anonymous calls of *functions* via λ-*operator*, it's present to some extent in most of modern i. *programming languages* (since McCarthy's Lisp), and because of its vast expressive power, it's also involved in various approaches to ii. *natural language processing* (NLP) (since Montague's intensional logic).

Because of STT's *higher-order quantification* (over functions of various 'degrees'), it's *higher-order logic* (HOL). It allows encoding of all classical logical operators using the equality $=^\tau$ only (as shown by Church and Henkin), and also numbers and thus arithmetic (as shown by Church). Hence it's incomplete (Gödel), but Henkin 1950 showed its *Henkin completeness* w.r.t. *general models*. For canonical development of STT, see esp. Andrews 1986, Benzmüller et al. 2008.

The Andrews-style STT was modified by Farmer 1990 by adopting selected *partial functions* (i.e. mappings undefined for at least one its argument). These are necessary for modelling numerous partiality phenomena such as: programs getting stuck, unsuccessful database searches, mathematical and natural-language *empty* ('invalid') *expressions* (e.g. "$3 \div 0$", "$\lim_{x \mapsto a} f(x)$" for some values, "the King of France", "the greatest prime"). Many similar projects, but based on three-valued, four-valued and even fuzzy logics were proposed in literature. In the Czech Republic, the members of IRAFM team – Novák, Běhounek, Daňková, Dvořák – developed *partial fuzzy type theory* (see e.g. Běhounek and Novák 2015, 2019, Běhounek 2016). Most of the approaches straightforwardly extend STT, while they utilise so-called *dummy values* (the values 'undefined', $\bot_\tau$) to represent that a function/expression has no functional/denotational value. Overall, they usually contribute to HOL as such.

The talk presents a variant of *partial type theory* TT^* recently developed by Raclavský and his close collaborator Kuchyňka who elaborated a detailed proof of its Henkin completeness, see Kuchyňka and Raclavský 2023. TT^* can be best seen as a certain Andrews-style STT enriched by some proposals by Tichý (esp. his pioneer paper 1982). Tichý's mature project was rather NLP, but his partial STT presents a self-sustaining (though 'pre-Henkin') HOL; we supplement it by modification of some his late ideas (Tichý 1988). Here are few distinctive features of the resulting system TT^*:

1. it handles all total and partial functions-as-mappings (not only some of them);

2. it also handles *algorithmic computations* of these (so a certain ramification of typing steps in);

3. the *semantics* is not denotational, but *procedural* (hence e.g. dummy values are not needed, abortive algorithmic computations suffice);

4. its *sequent-style natural deduction with signed formulas* retains monotonicity of logical consequence relation.

In the talk, we present some main features of TT^*, but we focus on its extensions using *evaluation terms*. Evaluation terms correspond to Lisp's evaluation terms; see Farmer 2016 for a discussion of their need in computer science and also a set of problems brought by their adoption. The current version of TT^* successfully resists the problems and allows thus deduction not only with higher-order functions-as-mappings, but also with (total or partial) functions-as-algorithms w.r.t. (total or partial) functions-as-mappings.

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Contributed talks

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β-conversion, syntactic vs. semantic computing, and dual procedures

Abstract Transparent Intensional Logic (TIL) is a higher-order, hyperintensional λ-calculus with procedural semantics. The terms of the ideography of TIL denote algorithmically structured *procedures* producing mappings rather than the mappings themselves. Mappings are semantically secondary, while the procedures are semantically primary. Procedures are *sui generis* objects of the ontology of TIL. They include a pair of dual procedures operating on lower-order procedures, namely Trivialisation (^0C) and Double Execution (^2C). While ^0C *presents* procedure C as an object to operate on by other super-procedures, ^2C cancels the effect of Trivialization; ^2C produces what is produced (if anything) by the procedure produced by C. It means that there are two basic modes in which procedure C can occur. While in ^0C, procedure C occurs *displayed*, in ^2C (as well as in C), it occurs in the *execution* mode. The context of a displayed procedure is *hyperintensional.*

Hence, hyperintensions in TIL are not 'black boxes'; rather, their procedural structure is visible, and we specify the extensional logic of hyperintensions. We have developed the substitution method to operate in a hyperintensional context, and the definition of *correct substitution* is fundamental. The correctness of λ-conversions, in particular β-conversion) and of the substitution method, is based on this definition. The basic principle of substitution is that one can substitute only for variables occurring freely. Hence, the definition of the free occurrence of a variable is also fundamental. Tichý, the founder of TIL, defined correct substitution in a restrictive way in (1988). He was careful not to substitute into a context within the scope of a Trivialization, which makes the context hyperintensional. For Tichý, as soon as a variable occurs within the scope of a Trivialization, it is Trivialization-bound, and thus not free for substitution. Tichý proved the basic theorem covering the validity of substitution and thus also the validity of λ-conversions, namely the *compensation principle*, thereby proving consistency. Yet, Tichý proved this principle only for procedures of order 1, thus ignoring the hyperintensional levels of TIL. Therefore, he did not get around to considering the fact that Double Execution cancels the effect of Trivialisation.

For this reason, Duží et al. (2010) aimed to extend the definitions of free variable and correct substitution to also include those occurrences of variables that, apparently, become free due to the duality of Double Execution and Trivialisation. A variable occurring within the scope of a Trivialization can become free for substitution if it also occurs within the scope of Double Execution. Since (2010), the definition of substitution, as well as its correctness, seemed to be a completed task. TIL has been applied not least to natural-language processing, and when analysing ordinary natural-language sentences, everything seemed to be all right.

However, Kosterec (2020), (2021) demonstrated inconsistencies in TIL that are due to too liberal a definition of a variable occurring free for substitution, which in turn goes hand in hand with too liberal a definition of the execution mode for occurrences of procedures. Some conversions that were apparently proved to be equivalent transformations turned out not to be equivalent, as Kosterec discovered a similar mistake in the proof of the compensation principle in Duží et al. (2010). There are limited cases where the *redex* procedure does not *v*-produce (i.e., produce relative to a valuation function) the same entity as does the *contractum* procedure. However, Kosterec did not propose a solution. In (2020) he proposed a new version of the basic definitions, but it turned out that new inconsistencies cropped up, so a proof of consistency is still outstanding. The goal of the present paper is to fill this gap. We are going to propose a new version of the fundamental definitions together with a proof of consistency.

Our diagnosis is that the quintessence of the problem stems from mixing up two levels of abstraction: the '*syntactic*' level (i.e., the very structure of a procedure) and the *semantic* level of evaluation of procedures (i.e., the level of computing what is produced by a given procedure with respect to a valuation of free variables occurring in the procedure).[1] Pezlar in (2019) speaks about two

[1] In TIL, we usually do not talk about the syntax and semantics of the TIL language of procedures. It is because this language comes with an interpreted syntax (our ideography). The terms are isomorphic with the procedures

notions of computation in TIL. One is *syntactic* computation, which corresponds to *term rewriting* in ordinary λ-calculi and the other is *semantic* computation, which might be compared to *term interpretation*. Syntactic computation is specified by the rules of λ-conversion. These are α-, β- and η-conversion. The most problematic of them is β-conversion, because in TIL we work with *partial* functions, and it has been proved that *β-conversion by name* is not an equivalent transformation in the logic of partial functions.[2] In a simple and general form, this conversion can be specified like this: $[\lambda x\, F(x)\, A] \rightarrow F(A/x)$, where $F(A/x)$ arises from F by a *correct substitution* of A for the variable x. Hence, the fundamental issue is to correctly define the occurrence of a *variable free for substitution*. Since this definition is closely connected to the distinction between a procedure occurring either displayed or executed, this definition also calls for repair. The new versions of these definitions must be fully *syntactic*; we must consider only the structure of a procedure and disregard its execution.

All of Kosterec's examples of inconsistency are rooted in (Double) Execution. A Trivialization-*bound* variable can become *free* due to the application of Double Execution; new variables that are apparently free for substitution can emerge by executing Double Execution. This is something we do not want to happen. Whether a variable occurs free or bound must be determined merely syntactically. The form of the solution is that one can substitute only for those variables that are 'syntactically' free, which excludes Trivialisation-bound variables. This is the way Tichý chose, and we are going to adhere to Tichy's original definition with one important extension. There is a ^{20}-elimination rule, i.e., $^{20}C \approx C$, which is valid for any procedure C, and this rule should be accounted for. But ^{20}-elimination concerns only those cases where the rule can be applied correctly: it must not 'propagate' Double Execution inside a given procedure, thus making variables free during execution. This needs to be blocked.

The goal of this paper is to define correct collision-less substitution based on an updated definition of a variable free for substitution that would account for the correct application of the ^{20}-elimination rule. Another no less important goal is to prove the compensation principle, hence the consistency of this version of TIL. We are solving a problem germane to TIL. Still, the problem and its solution are relevant to any kind of procedural semantics that comes with a pair of dual procedures (operations) where one cancels out the other and which allows more than one kind of variable-binding (in TIL, both λ-binding and Trivialization-binding), such that it may be syntactically indeterminate whether an occurrence of a variable is free or bound.

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they denote. Hence, we speak directly about the *structure* of those procedures, which corresponds to the 'syntactic' level. However, in this paper, we will stick to the terms 'syntactic' and 'semantic' to make the exposition easier to read for those who are acquainted with traditional model-theoretic λ-calculi rather than TIL.

[2] See, for instance, Duží and Kosterec (2017) or Duží (2019).

Graded Peterson's Square of Opposition as immediate inference

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Abstract

We will deal with graded Peterson's square of opposition in fuzzy natural logic. The main focus will be on its properties, especially how to use its properties to infer new information.

The main topic of this presentation will be inference, which generally serves to acquire new information or knowledge. We will deal with the theory of quantifiers, where we can distinguish mediate inferences which use two or more quantifiers for inference (such as syllogisms) or immediate inferences which use for inference only one quantifier (see [2]). Our focus will be on immediate inferences.

Let me note that we will focus on quantifiers of type $\langle 1, 1 \rangle$ which we can express for example as follows:

All children like chocolate.
Most dogs like to sleep.

Our motivation comes from classical Aristotle's square of opposition ([6]) which consist of classical quantifiers **A, E, I, O**. Aristotle's square of opposition describes properties as *contrary, subcontrary, contradictory, subaltern, superaltern* between quantifiers. We can use these properties as immediate inferences as we can see in the following example.

Example. Let us assume that we know that the following statement is true:

All children like chocolate.

Then we can infer the following information:

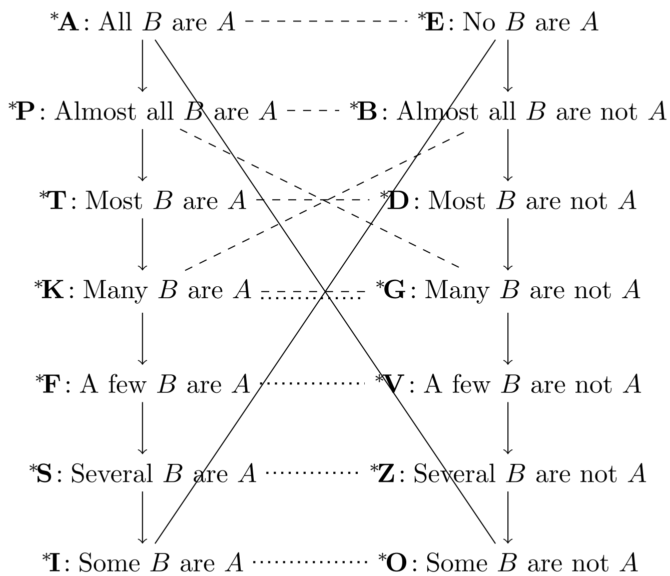
The statement *No children like chocolate.* is not true.
The statement *Some children like chocolate.* is true.
The statement *Some children do not like chocolate.* is not true.

Aristotle's square of opposition can be extended by adding some quantifiers (see [7]), also can be extended to the cube of opposition (see [3]) or hexagon of opposition (see [1, 8]).

In our presentation, we will focus on graded Peterson's square of opposition ([5, 4]) that we can see in Table 1. Our goal will be to use properties (*contrary, subcontrary, contradiction, subaltern, superaltern*) of graded Peterson's square of opposition similarly, as we presented in **Example**.

As we can see in Table 1, this structure obtains more quantifiers than Aristotle's square of opposition. Another extension comes from the fact that Graded Peterson's square of opposition was introduced in fuzzy natural logic, so we will distinguish truth values from interval $[0, 1]$.

Table 1: Graded Peterson's square of opposition



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Łukasiewicz logic, Rational Pavelka logic, and logics with graded syntax

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Łukasiewicz infinite-valued logic was first considered by Łukasiewicz and Tarski in their paper [13]. That work then developed into one of several grand avenues toward formal many-valued and fuzzy logic (among other ones provided by Gödel, Heyting, or Post; see [10, Section 10.1] for an account) and also into advanced topics in Łukasiewicz logic and its semantics, the class of MV-algebras [2, 3, 4, 14, 5].

Łukasiewicz logic can be expanded with constants for rational elements in the interval $[0, 1]$ [10, 11]: nullary connectives are added to the language and their desired behaviour fixed by suitable axioms. Such expansions have quite some pedigree [6, 7, 19, 15, 10, 18]. In particular, the system called Rational Pavelka logic (RPL) [10] can be viewed as the fruit of several subsequent simplifications [8, 9, 10] of the systems originally proposed by Pavelka [19].

Logics with graded syntax [15, 18, 16, 17], also originating with Pavelka's work, are distinguished by taking the use of graded formulas, rather than just a presence of propositional constants for the real or the rational numbers in the interval $[0, 1]$, to be the true insignia of fuzzy logic. In particular, it has been suggested [1, 17] that there is a significant difference between the logic (we call here) Graded Rational Pavelka Logic (GRPL), as in [8], and RPL, as in [9, 10]: namely, "genuine notions of abstract fuzzy logic, such as that of degree of provability, are 'simulated' by the ordinary notions", available in RPL; see [1].

In the talk, I will introduce the three propositional logics Ł, RPL, and GRPL and go briefly over their evolution. Implicit definability of rational elements of the MV-algebra on the interval $[0, 1]$ with natural order, and its relevance for the study of RPL [10], will also be mentioned. A simple, faithful embedding of the graded system GRPL into itself will be presented, in order to show that the provability relation in GRPL can be captured using only graded formulas whose grade is 1. This observation, which presumably was made in the course of events, leads directly to the introduction of RPL, it substantiates the claim on 'simulation', and it upsets the issue of RPL being an inauthentic rendering of GRPL.

The talk is based on the survey paper [12].

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On the complexity of addition

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Integer addition is one of the most basic computational problems. What is its time complexity? (Here we assume the standard multi-tape Turing machine model of computation, and we measure the number of steps taken by the machine as a function of the length of the input, denoted n. Integers are written in binary or decimal, hence the length of X is roughly $\log X$.)

For addition of two integers $X + Y$, the answer is obvious: the algorithm we all know from elementary school takes linear time $O(n)$, and this is optimal, as any Turing machine requires time n even to read the input.

The question becomes more interesting when we want to compute the sum of *multiple* integers. That is, we consider the following problem SeqSum:

> **Input:** A sequence of non-negative integers $\langle X_i : i < k \rangle$ written in binary and separated with commas.
>
> **Output:** $\sum_{i<k} X_i$.

The length of the input is $n = k + \sum_{i<k} n_i$, where n_i is the length of X_i. Can we still compute the problem in linear time?

The simplest algorithm for SeqSum uses one tape as an accumulator Y, adding the X_i's to Y one by one as we encounter them. Each addition takes time linear in the length of X_i and Y, which is at most n, thus the whole algorithm works in time $O(nk) \subseteq O(n^2)$. Considering an input consisting of a large number followed by a sequence of small numbers, we see that Y may indeed have length essentially n for almost the whole time, and even though in many cases adding a small number to a large one only needs time proportional to the length of the smaller operand, it may require time proportional to the size of the *larger* operand in the worst case due to carry propagation. Thus, it is not clear how to improve the $O(n^2)$ bound.

A well known example in amortized complexity [2] is a *binary counter*: we count from 0 to n (in binary) by a sequence of n increments. Since each increment takes time $O(\log n)$ in the worst case, it would seem that the whole algorithm requires time $O(n \log n)$. But in fact, it only takes time $O(n)$: $n/2$ of the increments only change the last bit, $n/4$ two bits, $n/8$ three bits, etc., hence the *average* time of an increment is only $O(1)$; see e.g. [1]. Observe that the binary counter is essentially the accumulator algorithm for SeqSum in the special case when $X_i = 1$ for all i.

Surprisingly, it does not seem to be widely known that a similar amortized analysis applies to *arbitrary* instances of the accumulator algorithm: it is possible to show that the total amount of carries is at most n, thus despite first appearances, the algorithm actually computes SeqSum in time $O(n)$ in all cases. This will be the topic of this talk.

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Porphyrian Containment Logic

Abstract

Containment logics have received substantial attention and elaboration in recent years (*e.g.,* Ferguson, 2017). However, a dissonance exists between at least two distinct families of containment logics: one based on G. W. Leibniz's understanding of containment, and the other based on Parry's systems. I suggest a third option of *porphyrian containment logic* that combines elements of both approaches.

Unlike Parry's containment logic, *porphyrian containment logic* makes use of Aristotelian logic and the scholastic tradition, which can also be traced in the logical works of G. W. Leibniz. However, unlike Leibniz's *algebra of concepts* and his containment theory of truth, it addresses the nature of the links between those concepts that are said to be contained in one another, and thus, it captures the *relevance* of the containment relation.

The notion of *porphyrian containment* is based on Porphyry's interpretation of Aristotelian definition, which states that *definitio fit per genus proximum et differentiam specificam.* For example, by analyzing the sentence "Man is a rational animal", we say that "animal" is the genus, and "man" is differentiated from other species of this genus by a single distinguishing property—that of being rational. This projection is sometimes called a *porphyrian tree*, which is presented here in an interpreted form as a *labeled* binary tree:

Definition 1. *A* porphyrian tree $\mathfrak{T}$ *is a labeled binary tree data structure such that:*

- *its elements are tuples* $\langle a,b,l\rangle$ *where a is a direct descendant of b decorated with label l*
- *if* $\langle a,b,l\rangle$ *and* $\langle a,b',l\rangle$ *appear in* $\mathfrak{T}$ *then* $b'=b$

From a mathematical perspective, we can treat the tree as a set of tuples. Additionally, I use the symbol $\preccurlyeq$ to indicate the *subtree* relation. From an ontological perspective, the nodes represent concepts. The leaf (*external*) nodes represent *individual concepts*, and the *internal* nodes (*i.e.*, nodes having at least one pair of descendant nodes) represent *universal concepts*. It follows that whether a concept is a *genus* or a *differentia* (species) depends on the perspective. In a hypothetical complete tree, any genus, except for the root node, is also a species. Siblings, or complements, are represented as two direct descendants of their common ancestor. The label c represents a property common to two nodes connected by a branch.

Definition 2. *A* porphyrian model $\mathfrak{M}$ *is a tuple* $\langle \mathscr{T},\mathscr{C},\mathscr{R},f_d,f_a\rangle$ *such that*

- $\mathscr{T}$ *is a class of porphyrian trees whose nodes and labels belong to* $\mathscr{C}$
- $\mathscr{R}\subseteq\mathscr{C}\times\mathscr{C}\times\mathscr{C}$
- $\mathscr{T}$ *is closed under subtrees*
- f_d *and* f_a *are functions from* $\mathbf{At}\rightarrow\mathscr{C}$ *such that* $\langle f_d(p),f_a(p),l\rangle\in\mathscr{R}$ *for some* $l\in\mathscr{C}$

Definition 3. $g_a^{\mathfrak{T}}(b)$*—the* immediate ancestor *of a node b on a porphyrian tree* $\mathfrak{T}$*—is the node n such that* $\langle b,n,l\rangle$ *is on* $\mathfrak{T}$ *for some* $l\in\mathscr{C}$.

Definition 4. $g_d^{\mathfrak{T}}(b)$*—the* pair of immediate descendants *of a node b on a porphyrian tree* $\mathscr{T}$*—is the set of nodes n such that* $\langle b,n,l\rangle$ *is on* $\mathfrak{T}$ *for some* $l\in\mathscr{C}$.

Definition 5. $c^{\mathfrak{T}}(b)$*—the* complement *of a node b on a porphyrian tree* $\mathfrak{T}$*—is the immediate descendant of the immediate ancestor of b on* $\mathfrak{T}$*,* i.e.*, the element* b' *of* $g_d^{\mathfrak{T}}(g_a^{\mathfrak{T}}(b))$ *distinct from b.*

Importantly, the complement function is preserved under supertrees:

Lemma 1. *If* $\mathfrak{T}\preccurlyeq\mathfrak{T}'$ *then* $c^{\mathfrak{T}}\subseteq c^{\mathfrak{T}'}$

In the notation of truth conditions, I use Nelson's truth and false-making notation (as seen in Nelson, 1949).

Definition 6. *The truth conditions and falsity conditions for formulae on a tree are defined in tandem:*

$$\begin{array}{lll}
\mathfrak{T} \Vdash^{+} p & \textit{iff} & \exists l \in \mathscr{C} \textit{ such that } \langle f_d(p), f_a(p), l \rangle \in \mathscr{R} \textit{ and } \langle f_d(p), f_a(p), l \rangle \preccurlyeq \mathfrak{T} \\
\mathfrak{T} \Vdash^{+} \neg\varphi & \textit{iff} & \mathfrak{T} \Vdash^{-} \varphi \\
\mathfrak{T} \Vdash^{+} \varphi \wedge \psi & \textit{iff} & \exists \mathfrak{T}', \mathfrak{T}'' \preccurlyeq \mathfrak{T} \textit{ such that } \mathfrak{T}' \Vdash^{+} \varphi \textit{ and } \mathfrak{T}'' \Vdash^{+} \psi \\
\mathfrak{T} \Vdash^{+} \varphi \vee \psi & \textit{iff} & \exists \mathfrak{T}' \preccurlyeq \mathfrak{T} \textit{ such that either } \mathfrak{T}' \Vdash^{+} \varphi \textit{ or } \mathfrak{T}' \Vdash^{+} \psi \\
\mathfrak{T} \Vdash^{+} \varphi \to \psi & \textit{iff} & \exists \mathfrak{T}' \preccurlyeq \mathfrak{T}'' \preccurlyeq \mathfrak{T} \textit{ such that } \mathfrak{T}' \Vdash^{+} \varphi \textit{ and } \mathfrak{T}'' \Vdash^{+} \psi \\
 & & \\
\mathfrak{T} \Vdash^{-} p & \textit{iff} & \exists l \in \mathscr{C} \textit{ s.t. } \langle c^{\mathfrak{T}}(f_d(p)), f_a(p), l \rangle \in \mathscr{R} \textit{ and } \langle c^{\mathfrak{T}}(f_d(p)), f_a(p), l \rangle \preccurlyeq \mathfrak{T} \\
\mathfrak{T} \Vdash^{-} \neg\varphi & \textit{iff} & \mathfrak{T} \Vdash^{+} \varphi \\
\mathfrak{T} \Vdash^{-} \varphi \wedge \psi & \textit{iff} & \exists \mathfrak{T}' \preccurlyeq \mathfrak{T} \textit{ such that either } \mathfrak{T}' \Vdash^{-} \varphi \textit{ or } \mathfrak{T}' \Vdash^{-} \psi \\
\mathfrak{T} \Vdash^{+} \varphi \vee \psi & \textit{iff} & \exists \mathfrak{T}', \mathfrak{T}'' \preccurlyeq \mathfrak{T} \textit{ such that } \mathfrak{T}' \Vdash^{-} \varphi \textit{ and } \mathfrak{T}'' \Vdash^{-} \psi \\
\mathfrak{T} \Vdash^{-} \varphi \to \psi & \textit{iff} & \exists \mathfrak{T}', \mathfrak{T}'' \preccurlyeq \mathfrak{T} \textit{ such that } \mathfrak{T}' \Vdash^{+} \varphi \textit{ and } \mathfrak{T}'' \Vdash^{-} \psi
\end{array}$$

Another important preservation result is that truth and falsity are preserved under supertrees.

Lemma 2. *Truth and falsity are preserved under supertrees,* i.e.*, if* $\mathfrak{T} \preccurlyeq \mathfrak{T}'$ *then:*

- *if* $\mathfrak{T} \Vdash^{+} \varphi$ *then* $\mathfrak{T}' \Vdash^{+} \varphi$
- *if* $\mathfrak{T} \Vdash^{-} \varphi$ *then* $\mathfrak{T}' \Vdash^{-} \varphi$

While this axiomatic system is a work in progress, it might provide some philosophically interesting results. For example, it follows from Definition 6 that if implication holds, it entails that conjunction holds as well. I will attempt to show that this result is philosophically well motivated and it is in accordance with our natural language use of conditionals.

To sum up, *porphyrian containment logic* can be provided with a reasonable semantic theory which is philosophically motivated. In my talk, I will compare this approach to alternative accounts of containment, such as Parry's analytic implication, especially compared to the semantics provided by Kit Fine. Eventually, I will discuss future work, philosophical directions, and axiomatization.

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Many-Valued Predicate Lifting and Nabla Modalities

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Many-valued and modal logics present two prominent examples of non-classical logics. Their instances and applications have been a focus of research since at least the last century. While the former provides a formalism for reasoning about graded concepts, vagueness, and under uncertainty, the latter presents a framework for deductions with necessity, possibility, and many other epistemic or dynamic modalities. Since the publication of [7], there has been increasing interest in systems integrating these two types of logics to understand the behavior of graded modalities. At the same time, many applications of the many-valued modal logic have been found in philosophy, computer science, and AI [9].

On the other hand, it is desirable to have a general framework at hand to encompass a variety of semantics and various examples of modal logic. Coalgebraic modal logic provides such an abstract framework [4]. Analogously to classical coalgebraic logic, many-valued coalgebraic modal logic can provide a uniform framework for many-valued modal logics. Technically, there are two approaches towards coalgebraic modal logic – one based on the concept of relation lifting and a single cover modality [6], and the other on the more usual syntax with modalities defined via predicate liftings [8]. Bílková and Dostál studied both directions in the many-valued setting in [2, 3]. Based on their work, **a natural question is: Can we find a mutual translation between these two approaches?** As R.A. Leal indicates in [5], it can be done with some further restrictions in the Boolean case. In this contribution, we demonstrate how to generalize this within the many-valued setting.

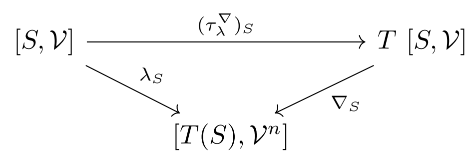
To start with, we consider T to be a Kripke polynomial endofunctor in **Set**. For the simplicity, we let $\mathcal{V}$ be a complete Gödel chain, and Λ be a set of n-ary many-valued predicate liftings $\lambda : [-, \mathcal{V}^n] \to [T(-), \mathcal{V}]$ for all $n \in \omega$. The predicate lifting language $\mathcal{L}_P(\Lambda)$ is constructed using $\wedge$, $\vee$ $\to$, $\bot$,$\top$ and the modal operator $\Box_\lambda$ for $\lambda \in \Lambda$ as logical symbols, and Var as a fixed countable set of propositional variables. The cover modality language $\mathcal{L}_M(T)$ with respect to a functor T consists of the same logical symbols as in $\mathcal{L}_P$ except the single modal operator being defined as $\nabla\alpha$ with $\alpha \subseteq T\mathcal{L}_M$. The many-valued coalgebraic semantics [2, 3] for $\mathcal{L}_P$ is denoted as $\|\cdot\|_\sigma$, and for $\mathcal{L}_P$ as $\Vdash_\sigma$ with respect to a coalgebra $\mathbb{S} = \langle S, \sigma \rangle$.

Intuitively, translating the formulas must not change their meaning, and we follow this intuition to define translations formally.

Definition 1. *A translation from $\mathcal{L}_P(\Lambda)$ to $\mathcal{L}_M(T)$ is a function $Tr : \mathcal{L}_P(\Lambda) \to \mathcal{L}_M(T)$ such that Tr is a non-modal homomorphism and $\|\Box_\lambda\varphi\|_\sigma(s) = s \Vdash_\sigma Tr(\Box_\lambda\varphi)$. A translation from $\mathcal{L}_M(T)$ to $\mathcal{L}_P(\Lambda)$ can be defined similarly with $\|Tr(\nabla\alpha)\|_\sigma(s) = s \Vdash_\sigma \nabla\alpha$.*

We can also formulate the definition of translations in a categorical setting under the restriction that the truth value of modal formulas should be preserved through mappings.

Definition 2. *A translator τ is a natural transformation $(-)^n \to T(-)$. A translator for a predicate lifting λ and a cover modality ∇ is a natural transformation $\tau_\lambda^\nabla : [-, \mathcal{V}^n] \to T[-, \mathcal{V}]$ such that the following diagram commutes*



From [5], we know that $\Box_\lambda\bot$ and $\Box_\lambda\top$ cannot be translated from $\mathcal{L}_P(\Lambda)$ to $\mathcal{L}_M(T)$ if we only allow finitary formulas on the language. Therefore, we have to expand our language by allowing infinitary formulas over a regular cardinal κ. We use $\mathcal{L}^\kappa_P(\Lambda)$ and $\mathcal{L}^\kappa_M(T)$ to denote the corresponding predicate lifting languages and cover modality languages. The definitions above can be generalized from n to any regular cardinal ζ. With the techniques above, we can give a translation from $\mathcal{L}^\kappa_P(\Lambda')$ to $\mathcal{L}^\kappa_M(T)$ as follows:

Theorem 1. *Let T be an accessible and weakly pullback preserving functor, let Λ' be a set of predicate liftings such that each members has a translator. Then we can find a translation from $\mathcal{L}^\kappa_P(\Lambda') \to \mathcal{L}^\kappa_M(T)$.*

We use the naturality of τ to define the translation Tr for the modal formula $\Box_\lambda\varphi$ as $\tau_{\mathcal{L}^\kappa_M(T)}(Tr\varphi)$. Non-modal formulas can be matched directly between the two different languages. One can then use the theorem above to study translation between $\mathcal{L}^\kappa_P(\Lambda')$ and $\mathcal{L}^\kappa_M(T)$ for T to be a $\mathcal{V}$-valued Kripke polynomial functor.

For the converse direction, we can prove a better translation theorem by restricting to the finitary case. Using the fact from [1], we know every finitary endofunctor T of **Set** is presentable by $H : \amalg_{n\in\omega} id^n \times T(n)$ with an epimorphism $\epsilon : H \to T$. Using this, one can define a translator ϵ_p between a single predicate lifting and cover modality with fixing $p \in T(n)$.

Theorem 2. *Let T be a finitary and weakly pullback preserving functor. Then there is a translation from $\mathcal{L}_M(T)$ to $\mathcal{L}_P(\Lambda)$.*

Similarly as above, non-modal formulas can be matched directly between the two different languages. The translation Tr of the cover modality formula is defined as $Tr(\nabla\alpha) = \Box_{\lambda^p}(Tr(\overrightarrow{\varphi}))$ where $Tr(\overrightarrow{\varphi}) = (Tr(\varphi_i))_{i\leq m}$ with $m \in \omega$ and $\epsilon_{\mathcal{L}_M(T)}(p, \overrightarrow{\varphi}) = \alpha$.

The future study is to extend the results towards fully abstract many-valued coalgebraic logic and study their corresponding equational systems in the two languages.

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Two types of structures in mathematics

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Contemporary mathematics is heavily dependent on the ('hilbertean', 'structural') axiomatic method which provides means to organise the presentation, communication and justification of mathematical knowledge.[10][4][7][21][24] Axiomatic method does not only provide foundations for modern mathematics in the sense of 'the accepted norms of interpersonal and intergenerational transfer and justification of mathematical knowledge'[14, 1213], it is also instrumental in significantly economising them.[4]

Axiomatic mathematics being the rule, there seems to be quite a general agreement among mathematicians, logicians and the philosophers of mathematics that axiomatically given mathematical theories 'define' [4][19][7], 'describe' [12][5], or 'determine' [11] a class (kind, type) of structures.[21][24] Indeed, some approaches even identify structures with theories[18], while others claim that 'mathematical structures become, properly speaking, the only 'objects' of mathematics'[4, 225–6, ftn].

Why and in what sense are axiomatic descriptions structural? The key is in the way the usual axiomatic presentations work: at their heart they are relational. This property translates into metamathematics (mathematical logic) in guise of all propositions being build around certain relational properties – constructed upon (applied) predicates. The relational (and thus structural) nature of (contemporary) mathematics has been acknowledged by the recent (and swiftly growing) approach of structuralism within the philosophy of mathematics. [23][8][20][25][26][22][13][5][6][9]

Although most of the mathematical structuralists discussions seem to be consistent with the view of mathematical structure being a property of a mathematical system introduced via its axiomatic presentation, there appears to exist an important exception in the form of category theory structuralism. While being sympathetic to the general notion that it is structural properties that are fundamental for mathematics, the categorial structuralists claim to endorse a substantially different notion of structure. For them, it is always a certain category which provides (arrowy) means to 'determine', 'define', 'characterise' or 'describe' 'mathematical structure of a given kind.' [15][16][17][1][2][3]

While seemingly easily reconcilable by pointing out that one group speaks of structural properties of whole mathematical systems whereas the other one of structural properties of objects within a mathematical system (a category), this 'misunderstanding' still indicates certain fundamental difference worthy of further considerations. It concerns the distinction between the mathematical and the foundational. In the present case, the contrast stands between explicitly treating structure mathematically and implicitly 'presupposing' it at the foundational level. The situation is somewhat similar to the one where a mathematical treatment of foundations (i.e. metamathematics) leads to explicit mathematical specification of semantic interpretations (models) of a theory and thus a mathematical treatment of structure (e.g. such as in standard FOL) – even a system of mathematical logic is a specimen of a mathematical theory with its own foundations and its own overall (implicit) structure.

In the paper we shall explain how these two types of structure (mathematical and foundational) carry with itself two different types of isomorphism (or satisfaction, in the case of formal logic). We shall argue that the utilisation of both types of structure is justified, even necessary, and that a sort of trade-off between flexibility vs. exactness as well as (a possible) trade-off between complexity and exactness obtains there. Specifically, at the example of **Sets**, the complexity of a category of structures will be compared with that of the structure it purports to define, as well as the structural complexity of the FOL assumptions which are necessary mathematically to treat the ZFC universe of sets as an explicit model of its own theory. An interesting connection between the categorial structuralists preoccupation with 'pure structures' and the philosophical problem of universals (types) will be briefly mentioned.

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On the ultraproduct theorem in fuzzy type theory

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In this paper we extend the concept of ultraproduct to models of fuzzy type theory (higher-order fuzzy logic; FTT). The latter was introduced in [7, 8] and it is a generalization of the classical type theory [1]. A few results in the model theory of FTT have been obtained in [9]. However, generalization of the ultraproduct theorem has not yet been suggested (as fas as the author knows).

The ultraproduct theorem in fuzzy logic was proved in [5]. Note that this is straightforward but not trivial generalization of the classical ultraproduct theorem presented, e.g. in [4]. Our theorem is based on these results. We form it w.r.t. general EQ-algebra-based FTT (cf. [8]).

Let J be a language of FTT and $(M_\alpha)_{\alpha\in \mathit{Types}}$ be a system of sets called *basic frame* such that M_o, M_ϵ are sets and for each $\alpha,\beta \in$ *Types*, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$ is a set of weakly extensional functions $M_\alpha \longrightarrow M_\beta$. The *general frame* is a tuple

$$\mathscr{M}^{\mathscr{E}} = \langle (M_\alpha, \doteq_\alpha)_{\alpha\in \mathit{Types}}, \mathscr{E}\rangle \tag{1}$$

such that the following holds:

(i) The $\mathscr{E}$ is an algebra of truth values with the support $M_o = E$. We assume that each of the sets $M_{oo}, M_{(oo)o}$ contains all the unary and binary operations from the algebra $\mathscr{E}$, respectively. The fuzzy equivalence (equality on truth values) is $\doteq_o := \sim$.

(ii) The set M_ϵ is the set of *individuals* with the fuzzy equality $\doteq_\epsilon$ on M_ϵ.

(iii) If $\alpha \neq o, \epsilon$ then $\doteq_\alpha$ is a fuzzy equality

$$[h \doteq h'] = \bigwedge_{m\in M_\alpha} [h(m) \doteq_\beta h'(m)], \qquad h, h' \in M \tag{2}$$

where $\doteq_\beta$ is a fuzzy equality on a set M_β.

We call $\mathscr{M}^{\mathscr{E}}$ a *general model* if $\mathscr{E}$ is a complete linearly ordered EQ-algebra and for any formula A_α, $\alpha \in$ *Types*

$$\mathscr{M}_p^{\mathscr{E}}(A_\alpha) \in M_\alpha.^{*)}$$

Let I be an index set and consider a set of general models

$$\mathscr{M}_i^{\mathscr{E}_i} = \langle (M_{\alpha,i}, \doteq_{\alpha,i})_{\alpha\in \mathit{Types}}, \mathscr{E}_i\rangle, \qquad i \in I. \tag{3}$$

Let $\alpha \in \{o, \epsilon\}$. Then for all $\overline{m}_\alpha, \overline{m}'_\alpha \in \prod_{i\in I} M_{\alpha,i}$ we define a binary relation

$$\overline{m}_\alpha\, \theta_\alpha\, \overline{m}'_\alpha \quad \text{iff} \quad \{i \in I \mid [\overline{m}_\alpha(i) \doteq_{\alpha,i} \overline{m}'_\alpha(i)] = \mathbf{1}\} \in G.^{\dagger)} \tag{4}$$

It can be verified that (4) is an equivalence which induces equivalence θ_α on $\prod_{i\in I} M_{\alpha,i}$ for all $\alpha \in$ *Types*. We will denote the corresponding equivalence classes by $[\overline{m}_\alpha]^G$.

*) Note that the definition of general model includes also the concept of *safe structure* introduced in [6] for the first-order fuzzy logics.

†) Due to the assumption that $\doteq_\alpha$ is separated for all $\alpha \in$ *Types*, the fuzzy equality in (4) reduces to the classical equality.

Assumption 1. *Let us consider a set of general models (3) and their direct product (**??**). Let κ be a regular cardinal. We assume that for each $\alpha \in Types$, the cardinality $|\prod_{i\in I} M_{\alpha,i}| < \kappa$. Furthermore, we assume that $G \subseteq P(I)$ is a κ-complete filter.*

By [3], there exist $\aleph_1$-complete ultrafilters on ω.

Theorem 1 (Ultraproduct theorem)**.** *Under Assumption 1, let*

$$\mathscr{M}_i^{\mathscr{E}_i} = \langle (M_{\alpha,i}, \stackrel{\circ}{=}_{\alpha,i})_{\alpha\in Types}, \mathscr{E}_i\rangle, \qquad i \in I$$

be a set of general models, $\prod_G \mathscr{M}_i^{\mathscr{E}_i}$ be their ultraproduct over an ultrafilter G and $\bar{p} \in \mathrm{Asg}(\prod_G \mathscr{M}_i^{\mathscr{E}_i})$ be an assignment. Then for every formula $A_\alpha \in Form_\alpha$ and $\alpha \in Types$

$$(\prod_G \mathscr{M}_i^{\mathscr{E}_i})_{\bar{p}}(A_\alpha) = [\overline{m}_\alpha] \quad \textit{iff} \quad \{i \mid (\mathscr{M}_i^{\mathscr{E}_i})_{p_i}(A_\alpha) = \overline{m}_\alpha(i)\} \in G. \tag{5}$$

In the paper, we also introduce the concept of *saturated model* and prove that saturated models can be obtained in FTT using ultraproduct construction.

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A new justification for the law of Pseudo-Scotus

In this talk, we consider the law of Pseudo-Scotus (also known as the ex falso quodlibet rule (EFQ), the explosion principle, or simply the falsity or absurdity rule)

$$\frac{\bot}{A}\ \text{EFQ}$$

as a derived rule and propose a new justification for it based on a rule we call the collapse rule (we assume $\lor$ is commutative)

$$\frac{A \lor \bot}{A}\ \text{collapse}$$

The collapse rule is a mix between EFQ and disjunctive syllogism (DS) and, informally, it says that a choice between a proposition A and $\bot$, which is understood as nullary disjunction, is no choice at all and it defaults to A ("the implosion principle"). Furthermore, we show that the collapse rule can be also used to justify DS and that all these three rules have the same deductive strength: they are all interderivable. Thus, the discussions about the acceptability of EFQ or DS can be reduced to a discussion about the acceptability of the collapse rule. Finally, we consider the computational meaning of the collapse rule with the help of the Curry-Howard correspondence (Curry (1958), Howard (1980)).

More specifically, using the collapse rule and the standard disjunction introduction rule, we can derive EFQ as follows

$$\dfrac{\dfrac{\bot}{A \lor \bot}\ \lor\text{I}_r}{A}\ \text{collapse}$$

And to derive the corresponding ex falso formula $\bot \to A$, all we need to do is apply the implication introduction rule to the last step of the above derivation.

The DS can be derived as follows (in addition to the collapse rule, we also need negation elimination, and disjunction introduction and elimination)

$$\dfrac{A \lor B \qquad [A]^1 \qquad \dfrac{\dfrac{\dfrac{\neg B \qquad [B]^2}{\bot}\ \neg\text{E}}{A \lor \bot}\ \lor\text{I}_r}{A}\ \text{collapse}}{A}\ \lor\text{E}_{1,2}$$

And to capture the computational meaning of the collapse rule, we introduce a new non-canonical eliminatory operator collapse that behaves similarly to EFQ's abort

$$\frac{c : A \lor \bot}{\mathsf{collapse}(c) : A}$$

but while the abort function has no instructions for computation (since $\bot$ can never be true), our function collapse has instructions (since $A \lor \bot$ can be true).

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Schematic validity and completeness in Prawitz's semantics

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Prawitz's *semantics of valid argument* [5, SVA] stems from Prawitz's normalisation theorems for Gentzen's natural deduction [4]. These theorems state that natural deduction derivations reduce to *normal* forms, i.e. to derivations without *detours*. A detour obtains when a formula occurs both as conclusion of an introduction and as major premise of an elimination, and it can be eliminated through suitable reduction functions, such as

$$\dfrac{\dfrac{\begin{array}{c}[A]\\ \mathscr{D}_1\\ B\end{array}}{A \to B} \qquad \begin{array}{c}\mathscr{D}_2\\ A\end{array}}{B} \qquad \Longrightarrow \qquad \begin{array}{c}\mathscr{D}_2\\ [A]\\ \mathscr{D}_1\\ B\end{array}$$

Some corollaries of Prawitz's theorems hint at a semantic interpretation of normalisation theory where normalisability plays the role of validity criterion. However, this requires that derivations in specific systems be broadened towards *argument structures* where arbitrary inferences may occur, and that reduction functions be accordingly broadened towards *justification functions* for arbitrary inferences in non-introductory form. An argument, namely, a pair $\langle \mathscr{D}, \mathfrak{J} \rangle$ where $\mathscr{D}$ is an argument structure and $\mathfrak{J}$ is a set of justification functions, is then said to be valid when $\mathscr{D}$ reduces through $\mathfrak{J}$ to a structure whose last step is an introduction inference. Argumental validity is firstly relative to *atomic bases* $\mathfrak{B}$, i.e. to set of rules for atomic derivability; logical validity amounts instead to validity relative to universal quantification on every $\mathfrak{B}$.

In a more recent variant, which I shall call *base semantics* [BS], proof-theoretic semantics in Prawitz's style are however dealt with with no reference to argument structures or justification functions. The proof-burden is limited to atomic bases, whereas argumental validity is replaced by a base-dependent consequence relation over (sets of) formulas, written $\Gamma \models_{\mathfrak{B}} A$. Works in the BS tradition have led to fundamental completeness and incompleteness results for intuitionistic logic and other super-intuitionistic frameworks—for an overview see [2], while more recent works are [3, 7, 8].

My talk has a twofold aim. First of all, I argue that the incompleteness results achieved in the BS tradition can be extended to an approach where argument structures and justification functions are not disregarded. This is done following the natural idea—also suggested by [3]—of interpreting $\Gamma \models_{\mathfrak{B}} A$ constructively as existence of a $\langle \mathscr{D}, \mathfrak{J} \rangle$ from assumptions Γ to conclusion A which be valid relative to $\mathfrak{B}$. Secondly, I argue that this extension of the BS-incompleteness results forces a quite deep modification of the original SVA framework, i.e.:

- logical consequence of A from Γ means that, for every $\mathfrak{B}$, there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ from Γ to A which is valid relative to $\mathfrak{B}$, whereas SVA seems to work with inverted quantifiers, i.e. there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ from Γ to A which is valid relative to every $\mathfrak{B}$;
- the quantifiers inversion can be dealt with by adopting a broader notion of justification function than the one at play in SVA, i.e.:

– justification functions are allowed to range on atomic bases, so that one can define *choice*-functions picking the right arguments on every base;
– justification functions are allowed to be non-schematic in nature, namely they may not give rise to "recursive" rewriting systems, so that the role of the choice-functions above be played by union-sets of justification functions for "locally" valid arguments, or of "local" *reduction sequences* along the lines of [6].

However, in SVA justification functions are always defined on argument structures, and are seemingly required to be schematic in nature.

I conclude by discussing the notion of *schematicity*. In particular, I argue that some SVA-significant results may come from Pezlar's recent work on the *Split rule* [1], or else from some "linearity" constraints to be put on justification functions, so that the latter be closed under substitution of the argument structures which they apply to, i.e., if we indicate a justification function with ϕ and the result of its computation with `comp`,

$$\phi(\mathscr{D})[\mathscr{D}^*/\mathscr{D}] = \texttt{comp}(\phi(\mathscr{D}))[\mathscr{D}^*/\mathscr{D}].$$

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Equational definability of logical filters

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A central concern of algebraic logic, particularly of its subfield known as abstract algebraic logic (AAL), is to systematically relate the consequence relation of a propositional logic L with the equational consequence relation of some class of algebras, typically the *algebraic counterpart* of L (a class of algebras Alg L canonically associated to any logic). For example, the consequence relation IL of intuitionistic logic is linked to the equational consequence relation of the class HA of Heyting algebras by the following equivalence:

$$\gamma_1, \dots, \gamma_n \vdash_{\mathrm{IL}} \varphi \iff \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \vDash_{\mathsf{HA}} \varphi \approx 1.$$

Here the right-hand side of the equivalence states that the corresponding quasi-equation holds in all Heyting algebras, i.e. instead we may equally well write

$$\mathsf{HA} \vDash (\gamma_1 \approx 1 \ \& \ \dots \ \& \ \gamma_n \approx 1 \implies \varphi \approx 1).$$

Such an equivalence is called an *equational completeness theorem* [1]. Indeed, it is a *standard* equational completeness theorem: the class HA is the algebraic counterpart of IL.

However, this not the only way to provide the consequence relation of intuitionistic logic with an algebraic reading. We can also translate intuitionistic consequence into the equational theory of Heyting algebras as follows:

$$\gamma_1, \dots, \gamma_n \vdash_{\mathrm{IL}} \varphi \iff \mathsf{HA} \vDash \gamma_1 \wedge \dots \wedge \gamma_n \leq \varphi.$$

Rather than individual formulas getting translated into equations, here the entire consequence $\gamma_1, \dots, \gamma_n \vdash \varphi$ is translated into an equation. Accordingly, we shall call this kind of equivalence an *equational definition of consequence* (EDC). In particular, the above equivalence is a *standard* equational definition of consequence: again, the class HA is the algebraic counterpart of IL.

Both of these algebraic readings of logical consequence are available for intuitionistic logic, where it is in fact easy to derive one from the other. However, other logics might only admit one of these readings. Consider for instance the local and global variants of basic modal logic, K^ℓ and K^{g}. These are two distinct consequence relations: $x \vdash_{\mathrm{K}^{\mathrm{g}}} \Box x$ but $x \nvdash_{\mathrm{K}^\ell} \Box x$. The algebraic counterpart of both of these logics is the class BAO of Boolean algebras with an operator, i.e. with a unary operation $\Box$ preserving finite meets. The logic K^ℓ has a standard EDC:

$$\gamma_1, \dots, \gamma_n \vdash_{\mathrm{K}^\ell} \varphi \iff \mathsf{BAO} \vDash \gamma_1 \wedge \dots \wedge \gamma_n \leq \varphi.$$

However, it does not admit any standard equational completeness theorem [1, Corollary 9.7]. The opposite situation holds for K^{g}. This logic has a standard equational completeness theorem of the following form:

$$\gamma_1, \dots, \gamma_n \vdash_{\mathrm{K}^{\mathrm{g}}} \varphi \iff \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \vDash_{\mathsf{BAO}} \varphi \approx 1,$$

but we show that it does not have a standard global EDC.

Although the logic K^g does not admit a standard EDC, it does admit the following more general kind of definition of consequence:

$$\gamma_1, \ldots, \gamma_n \vdash_{K^g} \varphi \iff \mathsf{BAO} \models \Box_k(\gamma_1 \wedge \cdots \wedge \gamma_n) \leq \varphi \text{ for some } k \in \omega,$$

where $\Box_0 x := x$ and $\Box_{i+1} x := x \wedge \Box\Box_i x$. Similarly, we can show that the infinite-valued Lukasiewicz logic Ł (whose algebraic counterpart is the class MV of MV-algebras) does not have an EDC, but it again admits a more general kind of definition of consequence:

$$\gamma_1, \ldots, \gamma_n \vdash_{Ł} \varphi \iff \mathsf{MV} \models (\gamma_1 \wedge \cdots \wedge \gamma_n)^k \leq \varphi \text{ for some } k \in \omega,$$

where $\varphi^0 := 1$ and $\varphi^{i+1} := \varphi \odot \varphi^i$. Such an equivalence, where the right-hand side states that at least one equation in a certain family of equations holds, will be called a *local* equational definition of consequence. Generalizing in an orthogonal direction, one can also consider *parametrized* EDCs, where instead of equations we consider existentially quantified conjunctions of equations. Parametrized local EDCs generalize EDCs in both of these directions.

Unfortunately, we do not know how to characterize which logics admit an EDC. (To be fair, the same can be said about equational completeness theorems.) However, most logics with an EDC in fact admit an *equational definition of compact filters (EDCF)*: for each $n \in \omega$ there is a set of formulas $\mathbf{\Theta}(x_1, \ldots, x_n, y)$ such that for each $\mathbf{A} \in \operatorname{Alg} \mathrm{L}$ and each $a_1, \ldots, a_n \in \mathbf{A}$

$$b \in \operatorname{Fg}^{\mathbf{A}}_{\mathrm{L}}(a_1, \ldots, a_n) \iff \mathbf{A} \models \mathbf{\Theta}(a_1, \ldots, a_n, b).$$

Here $\operatorname{Fg}^{\mathbf{A}}_{\mathrm{L}}(a_1, \ldots, a_n)$ is the L-filter generated on $\mathbf{A}$ by $a_1, \ldots, a_n$, i.e. the smallest subset of $\mathbf{A}$ extending $\{a_1, \ldots, a_n\}$ which is, informally speaking, closed under all the inference rules of L. EDCFs again have local and parametrized variants, much like EDCs.

In this contribution, we give an intrinsic characterization of logics which admit an EDCF of a given kind, and showcase how it can easily be used to prove that particular logics do not admit an EDCF of a given kind. These characterizations are similar to, but in interesting ways distinct from, the known characterization of logics with a so-called Deduction–Detachment Theorem (DDT). For the specialist in AAL, we remark that what we are doing here can roughly be seen as extending the existing hierarchy of DDTs beyond the case of protoalgebraic logics.

Theorem. *Let* L *be a finitary logic such that* $\operatorname{Alg} \mathrm{L}$ *is closed under subalgebras. Then:*

(i) L *has a parametrized local EDCF.*

(ii) L *has a parametrized EDCF if and only if for each family* $\mathbf{A}_i \in \operatorname{Alg} \mathrm{L}$ *with* $i \in I$ *and each* $a_1, \ldots, a_n \in \mathbf{A} := \prod_{i \in I} \mathbf{A}_i$ *(with projections* $\pi_i \colon \mathbf{A} \to \mathbf{A}_i$*)*

$$\operatorname{Fg}^{\mathbf{A}}_{\mathrm{L}}(a_1, \ldots, a_n) = \prod_{i \in I} \operatorname{Fg}^{\mathbf{A}_i}_{\mathrm{L}}(\pi_i(a_1), \ldots, \pi_i(a_n)).$$

(iii) L *has a local EDCF if and only if for all algebras* $\mathbf{A} \leq \mathbf{B}$ *in* $\operatorname{Alg} \mathrm{L}$ *and all* $a_1, \ldots, a_n \in \mathbf{A}$

$$\operatorname{Fg}^{\mathbf{A}}_{\mathrm{L}}(a_1, \ldots, a_n) = A \cap \operatorname{Fg}^{\mathbf{B}}_{\mathrm{L}}(a_1, \ldots, a_n).$$

(iv) L *has an EDCF if and only if it has local EDCF and for each* $\mathbf{A} \in \operatorname{Alg} \mathrm{L}$ *and each* $a_1, \ldots, a_n, b \in \mathbf{A}$ *there is a smallest congruence* $\theta \in \operatorname{Con} \mathbf{A}$ *such that* $\mathbf{A}/\theta \in \operatorname{Alg} \mathrm{L}$ *and*

$$b/\theta \in \operatorname{Fg}^{\mathbf{A}/\theta}_{\mathrm{L}}(a_1/\theta, \ldots, a_n/\theta).$$

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Embedded Analytic Containment

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In (1977; 1989) Richard B. Angell introduced the logic AC of analytic containment. It is a very unusual logic that aims at capturing a notion of entailment satisfying the following condition: α entails β only if the meaning of β is contained in the meaning of α. This notion of entailment is formalized as an *analytic implication*. Synonymity can then be regarded as *analytic equivalence*, definable in terms of this implication: α is synonymous with β if and only if α analytically implies β and β analytically implies α. Angell actually proceeded in the reversed order and took synonymity as primitive and defined analytic implication as follows: α analytically implies β if and only if α is synonymous with $\alpha \wedge \beta$. We will use the symbol $\Rightarrow$ for analytic implication and $\Leftrightarrow$ for analytic equivalence.

In Angell's system analytic implication and equivalence can occur only as the main connectives and cannot be embedded under other operators. In fact, the logic AC is a subsystem of the logic known as First Degree Entailment (FDE). Thus far, no extensions of AC have been introduced that are defined over a language allowing for higher degree formulas (i.e. formulas including nested and embedded conditionals).

AC can be viewed as a relevant logic. As such, it has the variable sharing property. It even possesses a stronger version of the variable sharing property which more common relevant logics (including FDE) lack: $\alpha \Rightarrow \beta$ is a theorem of AC only if α contains all the propositional variables occurring in β.[1] As a consequence, AC does not validate such principles as $p \Rightarrow (p \vee q)$. This makes perfect sense given the intended interpretation of implication ($p \vee q$ includes some extra content that is not contained in p).

In my paper I will connect Angell's approach to relevant implication with a more common approach to relevant logic that led to the introduction of the paradigmatic relevant logic R (Belnap, 1967). In particular, I will introduce a framework that combines in a natural way both AC and the implicational fragment of R. This framework allows us to define analytic implication $\Rightarrow$ via a combination of the R-implication $\rightarrow$ and a new connective $\rightharpoonup$ that is obtained by a peculiar semantic symmetry from a semantic characterization of the R-implication. So, in the framework we can define:

$$\varphi \Rightarrow \psi =_{def} (\varphi \rightarrow \psi) \wedge (\varphi \rightharpoonup \psi) \text{ and } \varphi \Leftrightarrow \psi =_{def} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi).$$

This definition of analytic implication allows us to consider also nested and embedded occurrences of this connective, which is something that cannot be done in Angell's original system. This is possible by a combination of various frameworks that are interconnected in surprising ways and the main aim of this paper is to reveal these connections.

Both AC and R were originally introduced only as syntactic systems. Much later, Kit Fine (2016) formulated a *truthmaker* semantics for AC which can be viewed as a particular version of situation semantics (Barwise & Perry, 1983) and which reflects nicely the original Angell's informal interpretation. Fine (2016) also introduced truthmaker semantics for intuitionistic logic which was recently generalized to some substructural logics including the implicational

[1] This fact was proven independently in (Ferguson, 2016) and (Fine, 2016).

fragment of R (Majer, Punčochář & Sedlár, to appear).[2] At the same time, truthmaker semantics is intimately connected to inquisitive logic.[3] Inquisitive logic is a logic of questions but we will not rely on this particular interpretation of the system.[4] However, we will show that the disjunction of the logic AC is best viewed as the disjunction of inquisitive logic, and it can be viewed as the disjunction of the inquisitive version of the logic R introduced in (Punčochář, 2020).

All these strands are put together in the semantic framework that I will introduce in my talk. In the presentation, I will focus on the conceptual matters, I will be mainly concerned with the philosophical motivations behind the framework.

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[3] As far as we know this connection was originally discovered and described by Ivano Ciardelli in an unpublished manuscript (Ciardelli, unpublished).

[4] For more details on the standard framework of inquisitive logic including its interpretation in terms of questions see (Ciardelli, Groenendijk & Roelofsen, 2019; Ciardelli, 2022).

Mathematical Logic in the History of Logic: Łukasiewicz's Contribution and Its Reception

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1 Abstract

Although Łukasiewicz was not the first one who used mathematical logic in his research in the history of logic, he is considered to be the founder of this approach. In particular, he (along with Heinrich Scholz) was the first to insist on and promote the use of mathematical logic in historical research (see Bocheński 1961, 9–10). The careful and explicit formulation of the requirement was due to the incorrect presentation of Stoic logic that appeared in Carl Prantl's book *Geschichte der Logik im Abendlande*. While Prantl (1855, 469–470) criticised Stoic's syllogism as dull and unoriginal compared to Aristotle's syllogistic, Łukasiewicz ([1935-36] 1970, 197–198) demonstrated that Stoic logic differs, in fact, from Aristotle's syllogistic. While the former system is propositional logic, the latter is calculus of names. Łukasiewicz ([1935-36] 1970, 198) argued that Prantl conflated these two systems due to his insufficient knowledge of mathematical logic and, consequently, that the knowledge is essential for research in the history of the subject.

However, Łukasiewicz had methodological reasons for this approach. Firstly, mathematical logic was a suitable method for scientific philosophy in general, according to Łukasiewicz ([1910] 1987, 180–183). Secondly, Łukasiewicz was convinced that there is a unity between systems of logic from ancient times to current modern systems of logic (see Woleński 1987, xxi). He ([1910] 1987, 7) pointed out that Aristotle's syllogistic is a system of formal logic, even though it is not formalised. Therefore, he did not find the use of modern mathematical logic inappropriate.

Łukasiewicz focused his research on the history of logic in two areas, in particular: Stoic logic and Aristotle's syllogistic. It was the first area that led to the formulation of Łukasiewicz's approach and the requirement for any future historian of logic. The second area, however, made Łukasiewicz more famous as a historian of logic, for the result of Łukasiewicz's research (1957) was the book *Aristotle's Syllogistics from the Point of View of Modern Formal Logic*, published by Clarendon Press in 1951 and as a second and expanded edition in 1957.

Concerning the current knowledge of Łukasiewicz's contribution, it seems to be more widespread among historians of logic that focus on Aristotle. Firstly, it is discussed in the entries on *SEP* and *IEP* on Aristotle's syllogistic but not in those on Stoic logic (see Baltzly 2019; Groarke 2023; Pigliucci 2023; Smith 2020). Secondly, there is a difference between the citations of Łukasiewicz's work on Stoic

logic and Aristotle's syllogistic. Łukasiewicz's book on Aristotle's syllogistic was cited considerably more times his work on Stoic logic.

However, the papers that appeared in the last five years on *Scopus* and the *Web of Science* demonstrate a different picture. Although Łukasiewicz's contribution is mentioned more often among scholars writing about Aristotle's syllogistic, historians of Stoic logic also mention it. In addition, the difference between the number of mentions is not conclusive.

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Adding Weights to Kleene Algebra

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Imperative programs can be formalized using a language that extends the language of classical propositional logic with a set of program variables, and adds operators corresponding to the skip statement, sequential composition, if-then-else conditionals and while loops. One type of operational-style semantics for this language maps programs into certain regular languages of computation sequences over an alphabet consisting of program variables and Boolean formulas. This makes it possible to use the algebra of regular languages (Kleene algebra [4]) to reason about the behavior of programs.

Weighted programs [1] are a recent extension of standard imperative programs, motivated as a general framework for specifying mathematical models such as optimization problems or probability distributions. Similarly to weighted automata [2, 6, 7], the basic idea behind weighted programs is that possible computation paths of a program carry *weights*, typically taken from some semiring. On the operation-style semantic perspective described above, programs with weighted computation sequences correspond to (rational) *formal power series*, that is, (a certain class of) functions from the free monoid over the program alphabet to the given semiring of weights. Weighted programs add two operators that are not present in the basic case, namely, nondeterministic choice and an operator representing addition of weight to the current computation path. Both correspond to standard (rational) operations on formal power series. We are interested in the following question: Is there a version of Kleene algebra that can be used to reason about weighted programs?

We show, first, that ordinary Kleene algebra can be used to reason about the most abstract aspects of weighted programs using an arbitrary idempotent and complete semiring of weights. If the semiring of weights is idempotent and complete, then the given class of formal power series forms a Kleene algebra; moreover, every algebra of regular languages can be seen as a Kleene algebra of formal power series where the semiring of weights is the two-element Boolean semiring. Hence, by Kozen's completeness result [4], an equation $p \approx q$ is provable in Kleene algebra iff it is valid in every Kleene algebra of (rational) formal power series.

Second, to handle more interesting cases, we generalize Kleene algebras to Kleene algebras with weights. A Kleene algebra with weights is a Kleene algebra K with a distinguished subsemiring S. Hence, we obtain a two-sorted structure, similarly to Kleene algebra with tests [5], but here S is not assumed to be a Boolean algebra representing meanings of Boolean formulas, but a semiring representing weights. This generalization is related to graded Kleene algebras with tests [3], but there are important differences between the two approaches which we will discuss.

We conjecture a number of generalizations of Kozen's completeness result connecting various classes of Kleene algebras with weights and classes of Kleene algebras of (rational) formal power series based on the specific kind of semiring of weights used.

This work extends the conference paper [8].

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Bolzano’s Measurable Numbers

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The section VII of Bernard Bolzano’s *Reine Zahlelehre*, which is called *Infinite Quantity Concepts* [5], is an important but problematic text. It was written during the early 1830’s as the last part of a foundational account of numbers and their properties. It is important because it is one of the first attempts of the arithmetization of continuum several decades before the works on real numbers of Cantor, Weierstrass, Méray, and Dedekind. However, it is problematic for several reasons. It survived only as an unfinished manuscript that was not intended for publication. It contains some ambiguities, inconsistent terminology and potential mistakes. It was not known until 1962 when Karel Rychlík published the manuscript but not in full, a completed transcription of the final version was published by Jan Berg in 1976 [3].

This has motivated a large discussion about meaning and consistency of the theory [4], [7], possible interpretations [2] and the significance of Bolzano’s basic concepts [3]. Most scholars stay with this and evaluate Bolzano’s theory from this perspective. It escapes attention that Bolzano established and *cum grano salis* proved for his newly constructed *measurable* numbers all the properties of the contemporary real numbers. [6].

In my talk, I will present the main ideas of Bolzano’s theory, mention reasons for the misunderstanding and show its consistent interpretation in contemporary mathematics.

The basic notion is an infinite number expression, which is a generalization of the notion of rational numbers. While rational numbers can be considered *number expressions*, which are formed by using a finite number of arithmetic operations (addition, subtraction, multiplication and division) with integers, *infinite number expressions* are formed by using an infinite number of arithmetic operations. Bolzano introduces several examples.

1. $1+2+3+4+\ldots$ in inf.
2. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$ in inf.
3. $(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{8})(1-\frac{1}{16})\ldots$ in inf.
4. $a+\frac{b}{1+1+1+\ldots\text{in inf.}}$ where a,b is a pair of integers.

A *measurable number* is an infinite number expression S that we determine by *measuring as precisely as we please* which means that for every positive integer q we determine the integer p such that

$$S=\frac{p}{q}+P_1=\frac{p+1}{q}-P_2.$$

where P_1 and P_2 denote a pair of strictly positive number expressions (the former possibly being zero).

This very definition is the source of problems of Bolzano’s theory. There are several ways to get around it. Perhaps the simplest is to modify the definition as follows. S is measurable if for every positive integer q we determine the greatest integer p such that

$$S=\frac{p-1}{q}+P_1=\frac{p+1}{q}-P_2.$$

where P_1 and P_2 denote a pair of strictly positive number expressions.[1]

Infinite number concept is *infinitely small* if its absolute value is less than $\frac{1}{q}$ for any natural number q, and it is *infinitely great* if it is greater than any q. Two measurable numbers S,P are *equal (equivalent)*, if *their difference is infinitely small*. In contemporary terms, he makes a factorization by this relation. Nevertheless, he still calls them measurable numbers. The *ordering* of measurable numbers is consistently defined: $P > S$, if their difference is positive and not infinitely small.

Bolzano formulates and proves properties of measurable numbers: the ordering is transitive, dense, unbounded, and Archimedean. Measurable numbers are closed to addition and multiplication, commutativity, associativity and distributivity apply. The basic propreties of fractions are designated. Finally, completeness and consequently the supremum property, defined by Bolzano as soon as 1817, is proved. One may have some objections against particular details in proofs but in principle the line of reasoning is right.

Bolzano's concept of infinity provides good reasons for interpreting infinite number expressions as sequences of partial results, and most scholars indeed do so. If we accept the modified definition, then measurable numbers correspond to Bolzano-Cauchy sequences, infinitely small numbers to sequences converging to zero and infinitely great numbers to divergent sequences. Addition and multiplication are defined componentwise in agreement with the Bolzano way of counting.

Measurable numbers can be interpreted in usual Cantorian way of builiding real numbers from rationals. However, Bolzano's construction is also somewhat similar to the construction of real numbers from non-standard rational numbers [1]. Of course, Bolzano does not have ultrafilters. His measurable numbers without equality are sufficient for this purpose. It is a non-Archimedean, commutative ring. The set of all infinitely small quantities forms its maximal ideal. The introduction of the equality of measurable numbers entails the factorization by the equality relation, i.e. the factorization of the ring by its maximal ideal. The result is a linearly ordered field, in this case a field of real numbers.

Then it is easy to comprehend why Bolzano did not build analysis on infinitely small numbers although he defined them here and defended their existence in his last work *Paradoxes of the Infinite*. The reason is the same as for hyperrational numbers - an important transfer principle fails. Bolzano was obviously aware of this fact. In *Theory of Functions*, the work following *Reine Zahlelehre*, he returned to his primordial idea of building the calculus on the basis of *quantities which can become smaller than any given quantities*, an idea similar to the later Weierstassian "$\varepsilon - \delta$ analysis".

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[1] Bolzano suggests a similar solution in his concluding remarks, where he says that the theory could perhaps be simplified if we formulated the equation $\frac{p}{q} + P = S = \frac{p+n}{q} - P$ for the identical n and indefinitely increased q, which has the same effect.