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Pareto-optimality of compromise decisions*

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Abstract

This paper deals with the problem of existence of compromise decisions and conditions for a compromise decision to be weakly Pareto-optimal, i.e., to be a weak Pareto-maximizer, Pareto-maximizer, or strong Pareto-maximizer. The concept of compromise decision is generalized by adopting triangular norms. Further, a concept of fuzzy interval is introduced. Sufficient conditions for the existence of a max-t decision and for max-t decision to be a (weak, strong) Pareto-maximizer are derived. Finally, a concept called explicit quasiconcavity is defined, which is sufficient for a max-t decision to be a strong Pareto-maximizer.

Keywords: Pareto-optimality, multi-criteria decisions, triangular norms.

1 Introduction

In many decision problems, elements of a given set X of possible alternatives (variants, feasible solutions, etc.) are usually evaluated by several criteria each of which may be expressed by a fuzzy set on X, i.e. by a membership function $\mu_i : X \longrightarrow [0,1]$, $i \in I$, where I is an index set for the criteria. Bellman and Zadeh in [1] proposed a concept of compromise decision $x^* \in X$ maximizing the $\min\{\mu(x); i \in I\}$. On the other hand, nondominated decisions — a traditional concept — play an important role in multi-criteria decision analysis. In this paper we deal with the problem of existence of compromise decisions and with conditions for compromise decisions to be Pareto-optimal, i.e., to be weak Pareto-maximizers, Pareto-maximizers, or strong Pareto-maximizers. This problem is important not only in multi-criteria decision analysis and game theory but also in fuzzy logic and fuzzy control, see [3, 4]. For the sake of simplicity we restrict our consideration only to the simplest case of the real line $X = \mathbb{R}^1$ (or, simply $\mathbb{R}$), and two criteria μ_1, μ_2, i.e. $I = \{1, 2\}$. An extension to $\mathbb{R}^n, n > 1$, and also to more than two criteria is possible. Moreover, we generalize the concept of the compromise decision by adopting triangular norms, see [5].

2 Triangular norms and some of their properties

Definition 1 A *triangular norm* t (briefly t-norm) is a function

$$\mathrm{t} : [0,1] \times [0,1] \longrightarrow [0,1],$$

which satisfies the following conditions:

(i) $\mathrm{t}(a, 1) = a$ for every $a \in [0, 1]$, (boundary condition) (1)

(ii) $\mathrm{t}(a, b) \leq \mathrm{t}(c, d)$ whenever $a \leq c, b \leq d$, $a, b, c, d \in [0, 1]$, (monotonicity) (2)

(iii) $\mathrm{t}(a, b) = \mathrm{t}(b, a)$ for every $a, b \in [0, 1]$, (commutativity) (3)

(iv) $\mathrm{t}(\mathrm{t}(a, b), c) = \mathrm{t}(a, \mathrm{t}(b, c))$ for every $a, b, c \in [0, 1]$. (associativity) (4)

A t-norm is called *strictly monotone* if

$$\mathrm{t}(a, b) < \mathrm{t}(c, d) \quad \text{whenever} \quad a < c \quad \text{and} \quad b < d. \tag{5}$$

Examples of t-norms.

Let $a, b \in [0, 1]$.

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(i) $\mathrm{t}_1(a,b) = \min\{a,b\}$. (6)

This t-norm is continuous and strictly monotone.

(ii) $\mathrm{t}_2(a,b) = a.b$. (7)

This t-norm is continuous and strictly monotone.

(iii) $\mathrm{t}_3(a,b) = \max\{0, a+b-1\}$. (8)

This t-norm is continuous, but not strictly monotone.

(iv)

$$\mathrm{t}_4(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

This t-norm is neither continuous, nor strictly monotone.

Proposition 1 *Every triangular norm t satisfies the inequality*

$$\mathrm{t}_4(a,b) \le \mathrm{t}(a,b) \le \mathrm{t}_1(a,b) \quad \textit{for every} \quad a,b \in [0,1]. \quad (10)$$

PROOF: By monotonicity (2) and by (1), for any $a,b \in [0,1]$ we obtain

$$\mathrm{t}(a,b) \le \mathrm{t}(a,1) = a \quad \text{and}$$
$$\mathrm{t}(a,b) \le \mathrm{t}(1,b) = b$$

and, consequently,

$$\mathrm{t}(a,b) \le \min\{a,b\} = \mathrm{t}_1(a,b).$$

On the other hand, for $a,b \in]0,1[$ we have $\mathrm{t}(a,b) \ge 0 = \mathrm{t}_4(a,b)$ and by (1), (2) and (3), all triangular norms coincide on the boundary of the unit square $[0,1]^2$. Consequently,

$$\mathrm{t}_4(a,b) \le \mathrm{t}(a,b).$$

□

3 Fuzzy intervals, Pareto-optimal decisions and max-t decisions

Definition 2 A real function $m : \mathbb{R} \longrightarrow \mathbb{R}$ is called *quasiconcave* if

$$\mu(z) \ge \min\{\mu(x), \mu(y)\}, \quad (11)$$

whenever $x \le z \le y$, see [2].

We refer to [2] for properties and usefulness of this class of functions.

Definition 3 A fuzzy set $m : \mathbb{R} \longrightarrow [0,1]$ is called a *fuzzy interval* if

(i) m is quasiconcave on $\mathbb{R}$,

(ii) m is *normalized*, i.e. there exists $u \in \mathbb{R}$ with $\mu(u) = 1$,

(iii) $\lim \mu(x)$ is 0 or 1 for $x \to -\infty$, and $\lim \mu(x)$ is 0 or 1 for $x \to +\infty$.

A fuzzy interval μ is *trivial* if $\mu(x) = 1$ for all $x \in \mathbb{R}$, otherwise, it is *non-trivial.*

Remark 1 The last condition (iii) requires that "in plus and minus infinity μ is equal either to 0 or 1". The limits in question exist, see Proposition 2 below.

Proposition 2 *Let $\mu : \mathbb{R} \longrightarrow [0,1]$ be a fuzzy interval. Then there exist $a, b, c, d \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, $a \le b \le c \le d$, such that*

$$\mu(x) = \begin{cases} 0 \quad \textit{for} \quad x < a, \\ \textit{is non-decreasing for } a \le x \le b, \\ 1 \quad \textit{for} \quad b < x < c, \\ \textit{is non-increasing for } c \le x \le d, \\ 0 \quad \textit{for} \quad d < x. \end{cases}$$

PROOF: By (ii) in Definition 3 there exists $u \in \mathbb{R}$ with $\mu(u) = 1$. Set

$$b = \inf\{x \in \mathbb{R}; \mu(x) = 1\}, \tag{12}$$
$$c = sup\{x \in \mathbb{R}; \mu(x) = 1\}, \tag{13}$$

then $b \le u \le c$.

Let $x_1 < x_2 \le u$, then by quasiconcavity we obtain

$$\mu(x_2) \ge \min\{\mu(x_1), \mu(u)\} = \min\{\mu(x_1), 1\} = \mu(x_1),$$

i.e. $\mu(x)$ is non-decreasing in $]-\infty, b]$.

Using similar arguments, we obtain that $\mu(x)$ is non-increasing in $[c, +\infty[$. By monotonicity, there exist limits $\lim \mu(x)$ for $x \to -\infty$ and $\lim \mu(x)$ for $x \to +\infty$, and by (iii) in Definition 3, the values of the limits are either 0 or 1. Setting

$$a = \sup\{x \in \mathbb{R}; \mu(x) = 0, x \le b\},$$
$$d = \inf\{x \in \mathbb{R}; \mu(x) = 0, x \ge c\},$$

we obtain the required result. □

Definition 4 An *α-level set* $[\mu]_\alpha$, $\alpha \in [0,1]$, of a fuzzy set $\mu : \mathbb{R} \longrightarrow [0,1]$ is defined as

$$[\mu]_\alpha = \{x \in \mathbb{R}; \mu(x) \ge \alpha\}.$$

A fuzzy set $\mu : \mathbb{R} \longrightarrow [0,1]$ is bounded if there exists $\alpha \in [0,1]$ such that $[\mu]_\alpha$ is non-empty and bounded.

Now, we consider a *decision situation* with the set of decisions $X = \mathbb{R}$ and two criteria μ_1, μ_2, where μ_1 and μ_2 are membership functions of the corresponding fuzzy sets on X.

Definition 5 A decision $x' \in X$ is said to be:

(i) a weak Pareto-maximizer if there is no $x'' \in X$ such that

$$\mu_1(x') < \mu_1(x'') \quad \text{and} \quad \mu_2(x') < \mu_2(x''),$$

(ii) a Pareto-maximizer if there is no $x'' \in X$ such that

$$\mu_1(x') \le \mu_1(x'') \quad \text{and} \quad \mu_2(x') \le \mu_2(x''),$$
$$\mu_1(x') < \mu_1(x'') \quad \text{or} \quad \mu_2(x') < \mu_2(x''),$$

(iii) a strong Pareto-maximizer if there is no $x'' \in X$ such that

$$\mu_1(x') \le \mu_1(x'') \quad \text{and} \quad \mu_2(x') \le \mu_2(x''),$$

$x' \ne x''$.

Weak Pareto-maximizers, Pareto-maximizers, and strong Pareto-maximizers are called *Pareto-optimal decisions.*

Remark 2 Note that any strong Pareto-maximizer is a Pareto-maximizer and any Pareto-maximizer is a weak Pareto-maximizer.

Remark 3 By the above definition, if $x^* \in X$ is a (strong) Pareto-maximizer, $x' \in X$, and $\mu_1(x^*) < \mu_1(x')$, then

$$\mu_2(x^*) > \mu_2(x'). \tag{14}$$

Definition 6 Let μ_1, μ_2 membership functions of be fuzzy sets, t be a triangular norm. A decision $x^+ \in X$ is called a *max-t decision*, if

$$\mathrm{t}(\mu_1(x^+), \mu_2(x^+)) = \max\{\mathrm{t}(\mu_1(x), \mu_2(x)); x \in X\}. \tag{15}$$

Remark 4 Setting $\mathrm{t} = \min$, we obtain in Definition 6 the popular max-min decision. By Proposition 4 (see below), for upper semicontinuous fuzzy intervals defined by the membership functions μ_1, μ_2 and a continuous triangular norm t, max-t decisions exist.

Proposition 3 *An upper semi-continuous bounded fuzzy set given by*

$$\mu : \mathbb{R} \longrightarrow [0,1]$$

attains its maximum on $\mathbb{R}$.

Proof: By the assumption of boundedness there exists $\alpha \in [0,1]$ such that $[\mu]_\alpha$ is nonempty and bounded and by the assumption of upper semicontinuity, the set

$$[\mu]_\alpha = \{x \in \mathbb{R}; \mu(x) \geq \alpha\}$$

is closed, i.e. compact. Consequently, μ attains its maximum on $[\mu]_\alpha$, and also on $\mathbb{R}$. □

Proposition 4 *Let μ_1, μ_2 be the membership functions of upper semicontinuous fuzzy intervals,* t *be a continuous triangular norm. Then max-t decisions exist.*

Proof: As μ_1, μ_2 are membership functions of fuzzy intervals, then by (ii) in Definition 3 there exist v_1 and v_2 such that $\mu_1(v_1) = \mu_2(v_2) = 1$. Without loss of generality we may suppose $v_1 \leq v_2$.

By Proposition 2 both μ_1 and μ_2 are non-decreasing in $]-\infty, v_1]$ and non-increasing in $[v_2, +\infty[$. Using monotonicity (2) of the t-norm, we obtain that $\varphi = \mathrm{t}(\mu_1, \mu_2)$ is also non-decreasing in $]-\infty, v_1]$ and non-increasing in $[v_2, +\infty[$.

As μ_1, μ_2 are upper semicontinuous and t is a continuous triangular norm, then $\varphi = \mathrm{t}(\mu_1, \mu_2)$ is upper semicontinuous on $\mathbb{R}$, particularly on the compact interval $[v_1, v_2]$. Hence $\varphi = \mathrm{t}(\mu_1, \mu_2)$ attains its maximum on $[v_1, v_2]$, which is also a global maximum on $\mathbb{R}$. This maximum is a max-t decision. □

The following example shows that the assumption of upper semicontinuity in Proposition 4 cannot be removed.

Example 1 Set

$$\mu_1(x) = \begin{cases} e^x & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Here, μ_1, μ_2 are the membership functions of fuzzy intervals, μ_1 is continuous, μ_2 is not upper semicontinuous on $\mathbb{R}$. It is easy to see that for the continuous triangular norm $\mathrm{t} = \min$ we have that $\psi(x) = \min\{\mu_1(x), \mu_2(x)\}$ does not attain its maximum on $\mathbb{R}$, see Fig. 1.

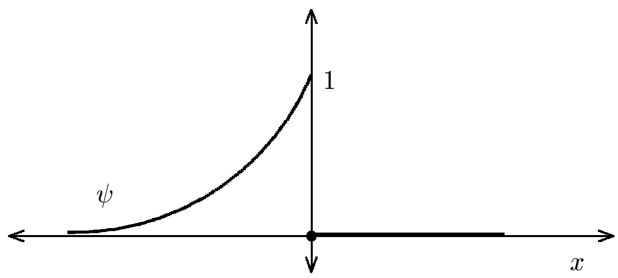

Figure 1: Membership function of $\psi(x) = \min\{\mu_1(x), \mu_2(x)\}$.

Proposition 5 *Let μ_1, μ_2 be the membership functions of fuzzy intervals,* t *be a strictly monotone t-norm, x^* be a max-t decision. Then x^* is a weak Pareto-maximizer.*

PROOF: Suppose that x^* is not a weak Pareto-maximizer, then by Definition 5 there exists x' such that $\mu_1(x^*) < \mu_1(x')$ and $\mu_2(x^*) < \mu_2(x')$. By strict monotonicity (5) we obtain

$$\mathrm{t}(\mu_1(x^*), \mu_2(x^*)) < \mathrm{t}(\mu_1(x'), \mu_2(x')),$$

i.e. x^* is not a max-t decision. □

Remark 5 Strict monotonicity of t is essential in the above proposition, as is evident from the following example.

Example 2 Let μ_1, μ_2 be as in Fig. 2 and let $\mathrm{t}(a,b) = \mathrm{t}_3(a,b) = max\{0, a+b-1\}$, see (8). Then t is not strictly monotone. Clearly, $\mathrm{t}_3(\mu_1(x), \mu_2(x)) = 0$, for all $x \in \mathbb{R}$, i.e. every $x \in \mathbb{R}$ is a max-t decision. However, for $x_1 < y < x_2$, y is not a weak Pareto-maximizer.

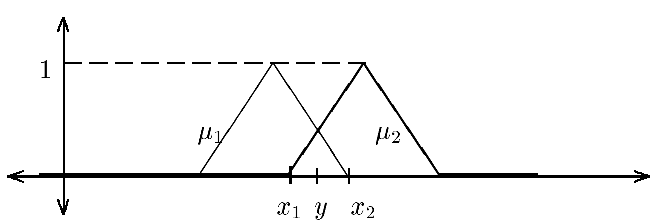

Figure 2: y is not a weak Pareto-maximizer

Proposition 6 *Let μ_1, μ_2 be upper semicontinuous membership functions of fuzzy intervals,* t *be a t-norm and let the set of all max-t decisions X^* be nonempty and closed. Then there exists a Pareto-maximizer $x^* \in X^*$.*

PROOF: By Definition 3 there exist u_1, u_2 with $u_1 \le u_2$ and $\mu_1(u_1) = \mu_2(u_2) = 1$.

Set $\varphi(x) = \mathrm{t}(\mu_1(x), \mu_2(x))$, then by Proposition 2 and monotonicity (2) we have that φ is nondecreasing on $]-\infty, u_1]$ and nonincreasing in $[u_2, +\infty[$.

Let $Y = [u_1, u_2] \cap X^*$, i.e. Y be the set of all max-t decisions from the closed interval $[u_1, u_2]$. Now we prove that Y is nonempty. Suppose the opposite, i.e. suppose that $Y = \emptyset$. Since X^* is nonempty, there exists $x^* \in X^*$, $x^* \notin [u_1, u_2]$.

(a) Let $x^* < u_1$, then by the above statement that is nondecreasing in $]-\infty, u_1]$, we have $\varphi(x^*) \le \varphi(u_1)$, which means that $u_1 \in X^*$ — a contradiction.

(b) Let $x^* > u_2$, then as φ is nonincreasing on $[u_2, +\infty[$, we obtain $\varphi(x^*) \leq \varphi(u_2)$, i.e. $u_2 \in X^*$, again a contradiction.

Consequently, Y is nonempty and as X^* is closed, then Y is compact. Now, set

$$x_1 = \inf Y \quad \text{and} \quad x_2 = \sup Y, \tag{16}$$

then $[x_1, x_2] \subseteq [u_1, u_2]$ and μ_1 is nonincreasing in $[x_1, x_2]$, μ_2 is nondecreasing in $[x_1, x_2]$. Define

$$x_1^* = \max\{x \in [x_1, x_2]; \mu_1(x) = \mu_1(x_1)\}, \tag{17}$$

$$x_2^* = \min\{x \in [x_1, x_2]; \mu_2(x) = \mu_2(x_2)\}. \tag{18}$$

As μ_1, μ_2 are upper semicontinuous, the above maximum and minimum (17), (18) exist. In the rest of this proof we prove that either x_1^*, or x_2^* is a Pareto-maximizer. Now, consider the following three cases:

1. $\mu_1(x_1) > \mu_2(x_1)$.
2. $\mu_1(x_2) < \mu_2(x_2)$.
3. $\mu_1(x_1) \leq \mu_2(x_1)$ and $\mu_1(x_2) \geq \mu_2(x_2)$.

Ad 1. Supposing that x_1^* is not a Pareto-maximizer, there exists x' with either

$$\mu_1(x') > \mu_1(x_1^*), \qquad \mu_2(x') \geq \mu_2(x_1^*),$$

or

$$\mu_1(x') \geq \mu_1(x_1^*), \qquad \mu_2(x') > \mu_2(x_1^*).$$

In the former case, by monotonicity (2) of t-norm we obtain

$$\varphi(x') = \mathrm{t}(\mu_1(x'), \mu_2(x')) \geq \mathrm{t}(\mu_1(x_1^*), \mu_2(x_1^*)) = \varphi(x_1^*). \tag{19}$$

However, by the strict inequality $\mu_1(x_1) > \mu_2(x_1)$, we obtain $x' \notin [x_1, x_2]$, which is a contradiction with the definition of $[x_1, x_2]$.

In the latter case, as μ_2 is nondecreasing then by the strict inequality $\mu_2(x') > \mu_2(x_1^*)$ we obtain $x' \geq x_1^*$ and since $x' \neq x_1^*$, we have $x' > x_1^*$ — a contradiction with the definition of x_1^*.

Ad 2. This case can be proved by similar arguments as in the above case, now applied "symmetrically" to x_2^*.

Ad 3. As μ_1 is nonincreasing and μ_2 is nondecreasing on $[x_1, x_2]$, we obtain

$$\mu_1(x) = \mu_2(x) = \mu_1(x_1^*) = \mu_2(x_2^*)$$

for every $x \in [x_1, x_2]$. Apparently, both x_1^* and x_2^* are Pareto-optimal. □

Remark 6 Let μ_1, μ_2 be upper semicontinuous membership functions of fuzzy intervals and t be a continuous triangular norm. Then the set of all max-t decisions X^* is nonempty and closed, and there exists a Pareto-maximizer $x^* \in X^*$. The first part of this assertion follows from Proposition 4, the second part follows from Proposition 4. The following example shows that upper semicontinuity in the above assertion cannot be removed, see also Example 1. The problem whether upper semicontinuity can be removed from the assumptions of Proposition 6 remains, however, open.

Example 3 Let μ_1, μ_2 be defined as in Fig. 3, μ_1 and μ_2 be quasiconcave but μ_1 is not upper semicontinuous, t = min, i.e. a continuous t-norm. Here, $X^* =]x_1, x_2]$ is the set of all max-min decisions. Evidently, if $x \in X^*$ then it is a weak Pareto-maximizer but it is not a Pareto-maximizer. Note, that X^* is not closed.

Remark 7 The Pareto-maximizer x^* in Proposition 6 need not be strong. Consider, e.g., trivial fuzzy intervals, i.e. $\mu_1(x) = \mu_2(x) = 1$ for all $x \in \mathbb{R}$, and t = min. Evidently, all $x \in \mathbb{R}$ are Pareto-maximizers, however, not strong Pareto-maximizers.

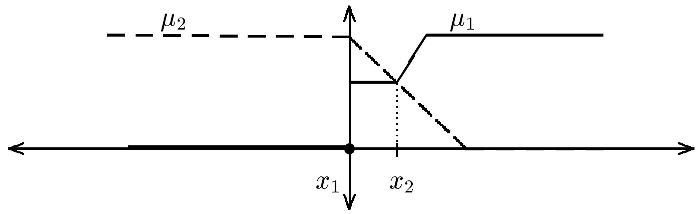

Figure 3: A max-min decision in $]x_1, x_2[$ is a (strong) Pareto-maximizer

4 Explicitly quasiconcave functions

Apparently, the minimum $\min\{\mu_1, \mu_2\}$ of two quasiconcave functions μ_1 and μ_2 is also quasiconcave. However, for some t-norms and quasiconcave μ_1, μ_2, the $\mathrm{t}(\mu_1, \mu_2)$ is not necessarily quasiconcave as is evident in the following example.

Example 4 Set

$$\mu_1(x) = \begin{cases} 0 & \text{if } x < -0.5, \\ \min\{x + 0.5, 1\} & \text{if } x \geq -0.5, \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } x < 0, \\ \max\{1 - x, 0\} & \text{if } x \geq 0. \end{cases}$$

Here, μ_1, μ_2 are continuous membership functions of fuzzy intervals, i.e., quasiconcave functions. Using t-norm (9), we define

$$\psi(x) = \mathrm{t}_4(\mu_1(x), \mu_2(x)),$$

then by (9) we obtain (see Fig. 2)

$$\psi(x) = \begin{cases} 0 & \text{if } x < -0.5, \quad \text{or} \quad x > 1, \\ x + 0.5 & \text{if } x \geq -0.5, \quad \text{and} \quad x \leq 0, \\ 1 - x & \text{if } x \geq 0.5, \quad \text{and} \quad x \leq 1, \\ 0 & \text{if } x > 0, \quad \text{and} \quad x < 0.5. \end{cases}$$

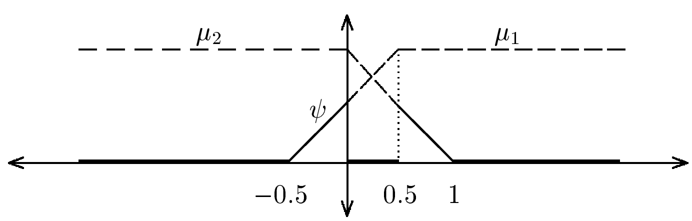

Figure 4: Membership function of $\psi = \mathrm{t}_4(\mu_1, \mu_2)$.

It is evident that $\psi = \mathrm{t}_4(\mu_1, \mu_2)$ is not quasiconcave. Notice that t_4 is not a continuous t-norm.

Definition 7 A real function $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be explicitly quasiconcave on $\mathbb{R}$, if

(i) μ is quasiconcave on $\mathbb{R}$,

(ii) $\mu(z) > \min\{\mu(x), \mu(y)\}$ (20)

whenever $x, y, z \in \mathbb{R}$, $x < z < y$, $\mu(x) \neq \mu(y)$ and

$$\min\{\mu(x), \mu(y)\} > \inf\{\mu(u); u \in \mathbb{R}\}. \tag{21}$$

The following proposition gives a characterization of explicitly quasiconcave fuzzy intervals.

Proposition 7 *Let $\mu : \mathbb{R} \longrightarrow [0,1]$ be an explicitly quasiconcave membership function of fuzzy interval. Then there exist $a, b, c, d \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, $a \leq b \leq c \leq d$, such that*

$$\mu(x) = \begin{cases} 0 \quad \textit{for} \quad x < a, \\ \textit{is increasing for } a \leq x \leq b, \\ 1 \quad \textit{for} \quad b < x < c, \\ \textit{is decreasing for } c \leq x \leq d, \\ 0 \quad \textit{for} \quad x > d. \end{cases}$$

PROOF: As μ is explicitly quasiconcave, it is quasiconcave and we define a, b, c, d by the same way as in the proof of Proposition 2.

Suppose that e.g. $-\infty < a < b$ and let $a < x < y \leq b$. Since

$$a = \sup\{x \in \mathbb{R}; \mu(x) = 0, x \leq b\}, \qquad b = \inf\{x \in \mathbb{R}; \mu(x) = 1\},$$

and using (20), (21), we obtain

$$0 < \mu(x) < \mu(y),$$

which means that μ is increasing in the interval $[a, b]$.

The rest of the proposition can be proved in a similar way. □

Proposition 8 *Let μ_1, μ_2 be the membership functions of fuzzy intervals and* t *be a t-norm. If $x^* \in X$ is a unique max-t decision then x^* is a strong Pareto-maximizer.*

PROOF: Suppose that x^* is a unique max-t decision and suppose that x^* is not a strong Pareto-maximizer. Then there exists $x^+ \in X$, $x^* \neq x^+$, $\mu_1(x^*) \leq \mu_1(x^+)$ and $\mu_2(x^*) \leq \mu_2(x^+)$. By monotonicity (2) we obtain

$$\mathrm{t}(\mu_1(x^*), \mu_2(x^*)) \leq \mathrm{t}(\mu_1(x^+), \mu_2(x^+)),$$

which is a contradiction with the uniqueness of x^*. □

Proposition 9 *Let μ_1, μ_2 be non-trivial explicitly quasiconcave membership functions of fuzzy intervals, let* t *be strictly monotone t-norm and let $x^* \in X$ be a max-t decision, such that*

$$0 < \mathrm{t}(\mu_1(x^*), \mu_2(x^*)) < 1. \tag{22}$$

Then x^* is a strong Pareto-maximizer.

PROOF: As μ_1, μ_2 are non-trivial, then by Definition 3(iii) we have

$$\inf \mu_1(z) = \inf \mu_2(z) = 0.$$

Suppose that a max-t decision x^* is not a strong Pareto-maximizer, i.e. there exist $x^+ \in X$, $x^* \neq x^+$ such that

$$\mu_1(x^*) \leq \mu_1(x^+) \quad \text{and} \quad \mu_2(x^*) \leq \mu_2(x^+). \tag{23}$$

Set $\varphi(x) = \mathrm{t}(\mu_1(x), \mu_2(x))$. Since x^* is a max-t decision

$$\psi(x^*) = \psi(x^+) \tag{24}$$

holds. As $\psi(x^*) > 0$, by (10) we have

$$\mu_1(x^*) \geq \min\{\mu_1(x^*), \mu_2(x^*)\} \geq \varphi(x^*) > 0,$$
$$\mu_2(x^*) \geq \min\{\mu_1(x^*), \mu_2(x^*)\} \geq \varphi(x^*) > 0$$

and by (25) we also have

$$\mu_1(x^+) > 0 \quad \text{and} \quad \mu_2(x^+) > 0.$$

Find $u, v \in \mathbb{R}$ with $\mu_1(u) = \mu_2(v) = 1$, by (22) we have $u \neq v$. Without loss of generality, suppose $x^+ < x^*$. Consider the following four cases: (a) – (d).

(a) If $\mu_1(x^*) < \mu_1(x^+)$ and $\mu_2(x^*) < \mu_2(x^+)$ then we obtain $\varphi(x^*) < \varphi(x^+)$ by (5), which is a contradiction with (24).

(b) Let

$$\mu_1(x^*) < \mu_1(x^+) \quad \text{and} \quad \mu_2(x^*) = \mu_2(x^+). \tag{25}$$

First, suppose that $\mu_2(x^*) = \mu_2(x^+) = 1$. Then by the first inequality in (25) and (1), we obtain $\varphi(x^*) < \varphi(x^+)$, a contradiction with (24).

Second, suppose

$$\mu_2(x^*) = \mu_2(x^+) < 1. \tag{26}$$

We investigate the following three cases:

1. $x^+ < x^* < v$,
2. $v < x^+ < x^*$,
3. $x^+ \leq v \leq x^*$.

Ad 1. By (28) we have

$$\mu_2(x^+) < 1 = \mu_2(v),$$

then by the explicit quasiconcavity of μ_2 we obtain

$$\mu_2(x^*) > \mu_2(x^+),$$

a contradiction with (25).

Ad 2. Again by (26) we have

$$\mu_2(x^*) < 1 = \mu_2(v),$$

then by the explicit quasiconcavity of μ_2 we obtain

$$\mu_2(x^*) < \mu_2(x^+),$$

which is again a contradiction with (25). Consequently, $v \notin [x^+, x^*]$ leads to the contradiction. This fact will be used again later on.

Ad 3. If $x^+ < v < x^*$ then by the explicit quasiconcavity of μ_1 we obtain

$$\mu_1(v) > \mu_1(x^*)$$

and by (26) we get

$$\mu_2(v) = 1 > \mu_2(x^*).$$

Consequently, by the strict monotonicity of t (see (5)) we obtain

$$\varphi(v) > \varphi(x^*),$$

which is a contradiction with the fact that x^* is a max-t decision.

(c) Let

$$\mu_1(x^*) = \mu_1(x^+) \quad \text{and} \quad \mu_2(x^*) < \mu_2(x^+). \tag{27}$$

This case may be carried out similarly to (b) when interchanging u with v and μ_1 with μ_2. Consequently, (27) leads again to the contradiction.

(d) Let

$$\mu_1(x^*) = \mu_1(x^+) \quad \text{and} \quad \mu_2(x^*) = \mu_2(x^+).$$

By (22) we may suppose that $\mu_1(x^*) = \mu_1(x^+) < 1$. Knowing that $u \notin [x^+, x^*]$ leads to the contradiction with explicit quasiconcavity of μ_1, we may suppose that $x^+ < u < x^*$ and there is also v' with $\mu_2(v') = 1$ and $x^+ < v' < x^*$. By (22) we have $u \neq v'$.

Setting $w = (u+v')/2$ and repeatedly using explicit quasiconcavity we obtain both $\mu_1(w) > \mu_1(x^*)$ and $\mu_2(w) > \mu_2(x^*)$. Then by strict monotonicity (5) we conclude that $\varphi(w) > \varphi(x^*)$, again a contradiction with the fact that x^* is a max-t decision. In all cases (a) – (d) we obtained contradictions. This completes the proof. □

Remark 8 If the restriction $\mathrm{t}(\mu_1(x^*), \mu_2(x^*)) < 1$ in Proposition 9 is not satisfied, i.e. $\mathrm{t}(\mu_1(x^*), \mu_2(x^*)) = 1$ then by (10) we obtain $\mu_1(x^*) = \mu_2(x^*) = 1$. This fact means that x^* is a Pareto-maximizer, however, not necessarily a strong Pareto-maximizer.

Remark 9 Explicit quasiconcavity in Proposition 9 cannot be replaced only by quasiconcavity as the following example demonstrates.

Example 5 Let μ_1, μ_2 be defined as in Fig. 5, then μ_1 and μ_2 are quasiconcave but μ_1 is not explicitly quasiconcave. Here, x^* is a max-min decision, however, not a Pareto-maximizer, x^+ is a Pareto-maximizer, however, not a strong Pareto-maximizer.

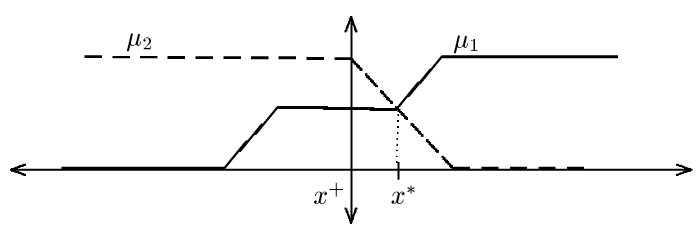

Figure 5: *Max-min decision x^* is not a Pareto-maximizer*

5 Conclusion

In this paper we have dealt with the problem of existence of a compromise decision and conditions for a compromise decision to be a weak Pareto-maximizer, Pareto-maximizer and a strong Pareto-maximizer. We have generalized the concept of compromise decision, e.g. max-min decision, by adopting triangular norms. Further, we have introduced the concept of fuzzy interval generalizing the characteristic function of crisp interval. We have derived sufficient conditions for the existence of a max-t decision (Proposition 4), weak Pareto-maximizer and Pareto-maximizer (Proposition 5 and 6). Here, quasiconcavity plays an essential role. Finally, we have defined a concept of explicit quasiconcavity. We have proved that, under the condition of the explicit quasiconcavity of fuzzy intervals, any max-t decision is a strong Pareto-maximizer (Proposition 9).

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