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System of Fuzzy Relation equations: Criteria of solvability

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Abstract

In this paper, we observe and compare results concerning solvability of systems of fuzzy relation equations with different types of composition. In fact, two types of composition are considered $sup - *$ and $inf \rightarrow$ - composition where $*$ is a continuous t-norm and $\rightarrow$ is residuation operation with respect to the t-norm $*$.

Keywords: System of fuzzy relation equations, $sup - *$-composition and inf $\rightarrow$ composition, solvability of fuzzy relation equation system, maximal solution, minimal solution.

1 Introduction

The pioneering work in this topic was done by Sanchez (1976) [12], but many other researchers dealt with this problem, for instance De Baets [1], Di Nola, Sessa, Pedrycz [2], Gottwald [3], Klir, Yuan [7], Mamdani, Assilian [8], Perfilieva [9, 10], etc.

I fix on work by Perfilieva and Tonis [11] here. The authors in this article presented criterion of solvability for system of fuzzy relation equations with $sup - *$ - composition and I am trying to transfer the criterion for the system of fuzzy relation equations with $inf \rightarrow$ - composition.

1.1 Residuated lattice

We choose a complete residuated lattice as the basic algebra of operations.

Definition 1
A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$. with four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are the least and the greatest elements;*
- $\langle L, *, \mathbf{1} \rangle$ *is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * \mathbf{1} = a$;*
- *the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.*

$$a * b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

1.2 t-norm and other operations

I present some basic notions from this topic, more information is for instance in [5].

Definition 2
A Triangular norm (t-norm) is a binary operation $*$ on unit interval $[0, 1]$, i.e., a function $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$, such that for all $a, b, c \in [0, 1]$ the following four axioms are satisfied:

- $a * b = b * a$ *(commutativity);*
- $(a * b) * c = a * (b * c)$ *(associativity);*
- $a \leq b$ *implies* $a * c \leq b * c$ *(monotonicity);*
- $a * 1 = 1 * a = a$ *(boundary condition).*

The most important t-norms are:

- Gödel t-norm $* = \wedge$: $a \wedge b = \min(a, b)$.

- Łukasiewicz t-norm $* = \otimes$: $a \otimes b = \max(0, a + b - 1)$.

- Product t-norm $* = \cdot$: $a \cdot b$.

These t-norms are continuous.

Definition 3
Let $*$ be a t-norm. The Residuation operation is again binary operation $\rightarrow_*$: $[0,1] \times [0,1] \rightarrow [0,1]$ adjoined to $*$, is defined by

$$a \rightarrow_* b = \sup\{x \in [0,1] : a * x \leq b\}.$$

The following are the residuation operations corresponding to the basic t-norms:

- for Gödel t-norm:

$$a \rightarrow_\wedge b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

- for Łukasiewicz t-norm:

$$a \rightarrow_\otimes b = \min(1 - a + b, 1).$$

- for Product t-norm:

$$a \rightarrow_. b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{if } a > b. \end{cases}$$

If t-norm $*$ is continuous then residuation operation $\rightarrow_*$ is lower-semicontinuous. Every residuation operation $\rightarrow_*$ fulfills the adjunction condition $a * b \leq c$ iff $a \leq b \rightarrow_* c$.

Definition 4
Let $*$ be a continuous t-norm. Operation z is unary operation $z : [0,1] \rightarrow [0,1]$ so that

$$z(a) = \bigwedge_{k=1}^{\infty} a^k = \bigwedge_{k=1}^{\infty} a * a * \ldots * a.$$

Special cases of operation z

$$z(a) = \begin{cases} a & \text{if } * = \wedge, \\ 0 & \text{if } * = \otimes \text{ or } * = \cdot \text{ and } a < 1, \\ 1 & \text{if } * = \otimes \text{ or } * = \cdot \text{ and } a = 1. \end{cases}$$

Let $*$ be a continuous t-norm. Then the algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow_*, \mathbf{0}, \mathbf{1} \rangle$ is a BL-algebra.

1.3 Fuzzy sets and fuzzy relations

We accept here a mathematical definition of a fuzzy set. In the rest of this paper we suppose that a complete residuated lattice $\mathcal{L}$ with a support L is fixed, and $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary non-empty sets. Then a *fuzzy set* or better, a fuzzy subset A of $\mathbf{X}$, is identified with a function $A : \mathbf{X} \longrightarrow L$. This function is known as *membership function* of the fuzzy set A. The set of all fuzzy subsets of $\mathbf{X}$ is denoted by $\mathcal{F}(\mathbf{X})$, so that

$$\mathcal{F}(\mathbf{X}) = \{A \mid A : \mathbf{X} \longrightarrow L\} = L^{\mathbf{X}}.$$

For two fuzzy subsets A and B of $\mathbf{X}$ we write $A = B$ or $A \leq B$ if $A(x) = B(x)$ or $A(x) \leq B(x)$, holds for all $x \in \mathbf{X}$, respectively. A fuzzy set $A \in \mathcal{F}(\mathbf{X})$ is called *normal* if $A(x_0) = \mathbf{1}$ for some $x_0 \in \mathbf{X}$.

The algebra of operations over fuzzy subsets of $\mathbf{X}$ is introduced as an induced residuated lattice on $L^{\mathbf{X}}$. This means that each operation from $\mathcal{L}$ induces the corresponding operation on $L^{\mathbf{X}}$ taken pointwise. We demonstrate this on the example of $*$-operation between fuzzy sets A and B:

$$(A * B)(x) = A(x) * B(x).$$

Obviously, operations over fuzzy subsets fulfill the same properties as operations in the respective residuated lattice.

A (binary) *fuzzy relation* on $\mathbf{X} \times \mathbf{Y}$ is a fuzzy subset of the Cartesian product. $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$ denotes the set of all binary fuzzy relations on $\mathbf{X} \times \mathbf{Y}$. Analogously, an n-ary fuzzy relation can be introduced.

Let $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and $S \in \mathcal{F}(\mathbf{Y} \times \mathbf{Z})$, then the fuzzy relation $T = R \circ S$ on $\mathbf{X} \times \mathbf{Z}$

$$T(x,z) = \bigvee_{y \in \mathbf{Y}} (R(x,y) * S(y,z))$$

is called a $sup - *$ - composition of R and S, and the fuzzy relation $Q = R \looparrowright S$ on $\mathbf{X} \times \mathbf{Z}$

$$Q(x,z) = \bigwedge_{y \in \mathbf{Y}} (R(x,y) \to S(y,z))$$

is called an $inf \to$ - composition of R and S.

In particular, if A is a unary fuzzy relation on $\mathbf{X}$ or simply a fuzzy subset of $\mathbf{X}$ then the $sup - *$ ($inf \to$) - composition between A and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is the fuzzy subset of $\mathbf{Y}$ defined by

$$(A \circ R)(y) = \bigvee_{x \in \mathbf{X}} (A(x) * R(x,y)),$$

and respectively,

$$(A \looparrowright R)(y) = \bigwedge_{x \in \mathbf{X}} (A(x) \to R(x,y)).$$

1.4 Systems of fuzzy relation equations

Let $N \geq 1$. A $sup - *$ - system of fuzzy relation equations

$$A_i \circ R = B_i, \quad 1 \leq i \leq N, \tag{1}$$

or

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * R(x,y)) = B_i(y), \quad 1 \leq i \leq N,$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$, is considered with respect to unknown fuzzy relation R.

Analogously, we will consider an $inf \to$ - system of fuzzy relation equations

$$A_i \looparrowright R = B_i, \quad 1 \leq i \leq N, \tag{2}$$

or

$$\bigwedge_{x \in \mathbf{X}} (A_i(x) \to_* R(x,y)) = B_i(y), \quad 1 \leq i \leq n,$$

with the same description of parameters and also with respect to unknown fuzzy relation R. Operation $\to_*$ is residuated operation with respect t-norm $*$.

Because solutions of (1) and (2) may not exist, in general, the problem to investigate necessary and sufficient, or only sufficient conditions for solvability arises. This problem has been widely studied in the literature with respect to the system (1), and some nice theoretical results have been obtained. Let us recall [11, 9, 12] with necessary and sufficient conditions, or [3, 4] with sufficient conditions. Moreover, the approximate solvability of (1) has been also investigated in [6, 10]. Unfortunately, the intensive investigation of the solvability of (2) has not been attempted, and only basic results cited in the Introduction are known.

Two types of fuzzy relations have been always assumed with respect to the problem of solvability of (1) and (2), namely

$$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)) \tag{3}$$

considered in Mamdani & Assilian [8], and

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \tag{4}$$

first considered in Sanchez [12].

2 Criteria of solvability

In this section we will investigate whether the relation $\check{R}$ is a solution of (2). The following statements are common conditions of solvability.

Theorem 1
The system (1) is solvable if and only if the fuzzy relation $\hat{R}$ is its solution. If the system (1) is solvable then $\hat{R}$ is its greatest solution.

Theorem 2
The system (2) is solvable if and only if the fuzzy relation $\check{R}$ is its solution. If the system (2) is solvable then $\check{R}$ is its smallest solution.

The proofs of these Theorems were mentioned in many articles (e.g. [7]).

In [11] is written criterion of solvability for systems where X is a compact set and all the membership functions $A_1(x), \dots, A_n(x)$ are continuous. Let

$$X_i(y) = \{x \in X : A_i(x) \geq B_i(y)\},\ y \in Y.$$

Theorem 3
The system (1) is solvable if and only if for each i, $1 \leq i \leq N$, and every $y \in Y$ there exists $x \in X_i(y)$ such that $z(B_j(y)) < B_i(y)$ implies

$$A_i(x) \rightarrow_* B_i(y) \leq A_j(x) \rightarrow_* B_j(y) \tag{5}$$

for each $j \neq i$ and $1 \leq j \leq N$.

The proof of this criterion is based on following lemma and all proof is shown in the article [11] mentioned above.

Lemma 1
Let $a, b, c, d \in [0, 1]$ and $a \geq b$. Then $b \leq a * (c \rightarrow_* d)$ if and only if $a \rightarrow_* b \leq c \rightarrow_* d$ or $b \leq z(d)$.

In some sense the lemma represents some inverse adjunction condition. Analogous criterion of solvability for system (2) is not so easy. We must separate up two cases. The first case is for strict t-norms and the second is special for Łukasiewicz t-norm. The separation results from lemmas, which are made according to Lemma (1).

Lemma 2
Let $*$ be a strict t-norm, further let $a, b, c, d \in [0, 1]$ and $a \geq c * d$, then

$$b \geq a \rightarrow_* (c * d) \text{ if and only if } a * b \geq c * d.$$

PROOF: Let $*$ be a strict t-norm, $a, b, c, d \in [0, 1]$ and $a \geq c * d$, further let $a * b \geq c * d$

$$a \rightarrow_* (c * d) = \sup\{x \in [0, 1] | a * x \leq c * d\} = X$$

X is the biggest element, which satisfies $a * X \leq c * d$. We want to show $b \geq X$ so let $b < X$ than $a * b < a * X \leq c * d$, and this is contradictory with assumption.

Let $*$ be a strict t-norm, $a, b, c, d \in [0, 1]$ and $a \geq c * d$, further let $b \geq a \rightarrow_* (c * d)$

$$b \geq a \rightarrow_* (c * d) = \sup\{x \in [0, 1] | a * x \leq c * d\} = X$$

$*$ is continuous and $a \geq c * d$ hence there is X such that $a * X = c * d$ ($a * 0 \leq c * d \leq a * 1$). Necessary inequality results from assumption $b \geq X$ ($a * b \geq a * X = c * d$).

□

Lemma 3
Let $*$ be Łukasiewicz t-norm $\otimes$, further let $a, b, c, d \in [0,1]$ and $a \geq c \otimes d$, and $a + b \geq 1$, then

$$b \geq a \rightarrow_{\otimes} (c \otimes d) \text{ if and only if } a \otimes b \geq c \otimes d.$$

PROOF: Let $\otimes$ be Łukasiewicz t-norm, $a, b, c, d \in [0,1]$ and $a \geq c \otimes d$, $a + b \geq 1$, further let $a \otimes b \geq c \otimes d$

$$a \rightarrow_{\otimes} (c \otimes d) = \sup\{x \in [0,1] | a \otimes x \leq c \otimes d\} = X$$

X is the biggest element, which satisfies $a \otimes X \leq c \otimes d$.

$$a \otimes b \geq c \otimes d \geq a \otimes X$$

Now there are two cases.

1. $a \otimes b > 0$ then $0 < a \otimes b \geq a \otimes X$ hence $b \geq X = a \rightarrow_{\otimes} (c \otimes d)$.
2. $a \otimes b = 0$ then $b = a \rightarrow_{\otimes} 0$, because $a + b \geq 1$. $0 = a \otimes b \geq c \otimes d \geq a \otimes X$ hence $a \rightarrow_{\otimes} (c \otimes d) = a \rightarrow_{\otimes} 0 = b$

On the other side the proof is same as in Lemma (2). □

Further we are going to need following lemma.

Lemma 4
Let $a, b \in [0,1]$, further let $*$ satisfy the condition cancelation law (CCL) and $a + b > 0$ or $*$ satisfy the cancelation law (CL). Then $a \rightarrow_* (a * b) = b$.

PROOF: In every residuated lattice the inequality $b \leq a \rightarrow_* (a * b)$ holds.

$$a \rightarrow_* (a * b) = \sup\{x \in [0,1] | a * x \leq a * b\} = X$$

$*$ satisfy CL or CCL and $a * b > 0$ hence

$$a * X \leq a * b \text{ iff } X \leq b.$$

Therefore $a \rightarrow_* (a * b) = b$. □

With the help of these lemmas we derive following criteria. Let X be a compact set and all the membership functions $A_1(x), \ldots, A_n(x)$ are continuous.

2.1 For system with respect to strict t-norm

The first is for the system (2) with residuated operation with respect a strict t-norm. Let

$$X_i(y) = \{x \in X : A_i(x) \geq A_j(x) * B_j(y), \ \forall j \neq i\}, \ y \in Y.$$

Then we formulate Theorem (4) for strict t-norms.

Theorem 4
The system (2) is solvable if and only if for each i, $1 \leq i \leq N$, and every $y \in Y$ there exists $x \in X_i(y)$ such that

$$A_i(x) * B_i(y) \geq A_j(x) * B_j(y) \tag{6}$$

for each $j \neq i$ and $1 \leq j \leq N$.

PROOF: Let the system (2) be solvable so the fuzzy relation $\check{R}$ is solution of this system.

$$\begin{aligned} B_i(y) &= \bigwedge_{x\in X} \left(A_i(x) \to_* \check{R}(x,y)\right) = \bigwedge_{x\in X} \left(A_i(x) \to_* \bigvee_{j=1}^{N} A_j(x) * B_j(y)\right) \geq \\ &\geq \bigwedge_{x\in X} \left(A_i(x) \to_* A_i(x) * B_i(y)\right) \vee \left(A_i(x) \to_* \bigvee_{j\neq i} A_j(x) * B_j(y)\right) = \\ &= B_i(y) \vee \bigwedge_{x\in X} A_i(x) \to_* \bigvee_{j\neq i} A_j(x) * B_j(y) \geq B_i(y) \end{aligned}$$

Hence

$$B_i(y) \geq \bigwedge_{x\in X} \left(A_i(x) \to_* \bigvee_{j\neq i} \left(A_j(x) * B_j(y)\right)\right),$$

therefore there are $x_0 \in X_i(y)$ so

$$B_i(y) \geq A_i(x_0) \to_* \left(A_j(x_0) * B_j(y)\right),$$

for all $j \neq i$. From Lemma (2)

$$A_i(x_0) * B_i(y) \geq A_j(x_0) * B_j(y),$$

for all $j \neq i$.

Now for each i, $1 \leq i \leq N$, and every $y \in Y$ there exists $x \in X_i(y)$ such that

$$A_i(x) * B_i(y) \geq A_j(x) * B_j(y)$$

for each $j \neq i$ and $1 \leq j \leq N$. Hence (Lemma (2))

$$\begin{aligned} B_i(y) &\geq A_i(x) \to_* \left(A_j(x) * B_j(y)\right), \ \forall j \neq i \\ B_i(y) &\geq A_i(x) \to_* \bigvee_{j\neq i} \left(A_j(x) * B_j(y)\right), \end{aligned}$$

There is x satisfies this inequality, so

$$\begin{aligned} B_i(y) &\geq \bigwedge_{x\in X} A_i(x) \to_* \bigvee_{j\neq i} \left(A_j(x) * B_j(y)\right), \\ B_i(y) &= B_i(y) \vee \bigwedge_{x\in X} A_i(x) \to_* \bigvee_{j\neq i} A_j(x) * B_j(y). \end{aligned}$$

From Lemma (4)

$$B_i(y) = \bigwedge_{x\in X} \left(\left(A_i(x) \to_* A_i(x) * B_i(y)\right) \vee \left(A_i(x) \to_* \bigvee_{j\neq i} A_j(x) * B_j(y)\right)\right),$$

$*$ is a strict t-norm so

$$B_i(y) = \bigwedge_{x\in X} \left(A_i(x) \to_* \bigvee_{j=1}^{N} \left(A_j(x) * B_j(y)\right)\right) = \bigwedge_{x\in X} \left(A_i(x) \to_* \check{R}(x,y)\right)$$

□

2.2 For system with respect to Łukasiewicz t-norm

If the residuated operation in the system (2) is with respect Łukasiewicz t-norm, then we must formulate criterion of solvability other way. Now let

$$X_i(y) = \{x \in X : A_i(x) \geq A_j(x) \otimes B_j(y),\ \forall j \neq i \ \&\ A_i(x) + B_i(y) \geq 1\}.$$

Theorem 5
The system (2) is solvable if and only if for each i, $1 \leq i \leq N$, and every $y \in \mathrm{Supp}(B_i)$ there exists $x \in X_i(y)$ such that

$$A_i(x) \otimes B_i(y) \geq A_j(x) \otimes B_j(y) \tag{7}$$

for each $j \neq i$ and $1 \leq j \leq N$.

PROOF: The proof is similar to previous proof. At first let the system (2) be solvable so the fuzzy relation $\check{R}$ is solution of this system.

$$\begin{aligned} B_i(y) &= \bigwedge_{x \in X} \left(A_i(x) \to_\otimes \check{R}(x,y)\right) = \bigwedge_{x \in X} \left(A_i(x) \to_\otimes \bigvee_{j=1}^{N} A_j(x) \otimes B_j(y)\right) \geq \\ &\geq \bigwedge_{x \in X} (A_i(x) \to_\otimes A_i(x) \otimes B_i(y)) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right). \end{aligned}$$

Now there are three possibilities:
1. $x \in X_i(y)$, then

$$\begin{aligned} &(A_i(x) \to_\otimes A_i(x) \otimes B_i(y)) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right) = \\ &= B_i(y) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right) \geq B_i(y). \end{aligned}$$

2. $x \neq X_i(y)$ but $A_i(x) \geq A_j(x) \otimes B_j(y)$ (so that $A_i(x) + B_i(y) < 1$) then

$$\begin{aligned} &(A_i(x) \to_\otimes (A_i(x) \otimes B_i(y))) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right) = \\ &= (A_i(x) \to_\otimes 0) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right) \geq \\ &\geq A_i(x) \to_\otimes 0 = 1 - A_i(x). \end{aligned}$$

This is contradiction with assumption that $B_i(y) < 1 - A_i(x)$, so this case is out of the question.
3. $x \neq X_i(y)$ (so that $A_i(x) < A_j(x) \otimes B_j(y)$) then

$$\begin{aligned} &(A_i(x) \to_\otimes (A_i(x) \otimes B_i(y))) \vee \left(A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y)\right) = \\ &= (A_i(x) \to_\otimes (A_i(x) \otimes B_i(y))) \vee 1 = 1. \end{aligned}$$

From previous section follow

$$B_i(y) = B_i(y) \vee \bigwedge_{x \in X} (A_i(x) \to_\otimes \bigvee_{j \neq i} A_j(x) \otimes B_j(y))$$

so

$$B_i(y) \geq \bigwedge_{x\in X}\left(A_i(x) \rightarrow_{\otimes} \bigvee_{j\neq i} A_j(x)\otimes B_j(y)\right).$$

Hence there is an element $x_0 \in X_i(y)$ so that

$$\begin{aligned} B_i(y) &\geq A_i(x_0) \rightarrow_{\otimes} \bigvee_{j\neq i} A_j(x_0)\otimes B_j(y) \\ B_i(y) &\geq A_i(x_0) \rightarrow_{\otimes} A_j(x_0)\otimes B_j(y) \end{aligned}$$

for every $j \neq i$. From Lemma (3)

$$A_i(x_0)\otimes B_i(y) \geq A_j(x_0)\otimes B_j(y)$$

for every $j \neq i$.

Now we assume that for each i, $1 \leq i \leq N$, and every $y \in \mathrm{Supp}(B_i)$ there exists $x_0 \in X_i(y)$ such that

$$A_i(x_0)\otimes B_i(y) \geq A_j(x_0)\otimes B_j(y)$$

for each $j \neq i$ and $1 \leq j \leq N$. From Lemma (3):

$$B_i(y) \geq A_i(x_0) \rightarrow_{\otimes} A_j(x_0)\otimes B_j(y)$$

for every $j \neq i$, so that

$$B_i(y) \geq A_i(x_0) \rightarrow_{\otimes} \bigvee_{j\neq i} A_j(x_0)\otimes B_j(y)$$

Hence (from Lemma (4) and (3))

$$\begin{aligned} B_i(y) &= B_i(y) \vee \left(A_i(x_0) \rightarrow_{\otimes} \bigvee_{j\neq i} A_j(x_0)\otimes B_j(y)\right) = \\ &= (A_i(x_0) \rightarrow_{\otimes} A_i(x_0)\otimes B_i(y)) \vee \left(A_i(x_0) \rightarrow_{\otimes} \bigvee_{j\neq i} A_j(x_0)\otimes B_j(y)\right) = \\ &= A_i(x_0) \rightarrow_{\otimes} \bigvee_{j=1}^{N} (A_j(x_0)\otimes B_j(y)) = \bigwedge_{x\in X} A_i(x) \rightarrow_{\otimes} \bigvee_{i=1}^{N} (A_j(x)\otimes B_j(y)) = \\ &= \bigwedge_{x\in X} \left(A_i(x) \rightarrow_{\otimes} \check{R}(x,y)\right). \end{aligned}$$

The last step is result the following investigation:

- $x \in X$ such that $A_i(x) < A_j(x)\otimes B_j(y) \leq \bigvee_{j\neq i}(A_j(x)\otimes B_j(y))$ than

$$A_i(x) \rightarrow_{\otimes} \bigvee_{j=1}^{N} (A_j(x)\otimes B_j(y)) = 1 \geq B_i(y) = A_i(x_0) \rightarrow_{\otimes} \bigvee_{i=1}^{N} A_j(x_0)\otimes B_j(y)$$

- $x \in X$ such that $A_i(x) + B_i(y) < 1$

$$A_i(x) \rightarrow_{\otimes} \bigvee_{j=1}^{N} (A_j(x)\otimes B_j(y)) \geq A_i(x) \rightarrow_{\otimes} A_i(x)\otimes B_i(y) =$$

$$= A_i(x) \rightarrow_{\otimes} 0 > B_i(y) = A_i(x_0) \rightarrow_{\otimes} \bigvee_{i=1}^{N} A_j(x_0)\otimes B_j(y)$$

- $x \in X_i(y)$

$$A_i(x) \rightarrow_{\otimes} \bigvee_{j=1}^{N} (A_j(x) \otimes B_j(y)) \geq A_i(x) \rightarrow_{\otimes} (A_i(x) \otimes B_i(y)) =$$

$$= B_i(y) = A_i(x_0) \rightarrow_{\otimes} \bigvee_{i=1}^{N} A_j(x_0) \otimes B_j(y).$$

□

3 Sets of solutions

It is easy to see that if system (1) is solvable then the set of solutions forms a $\vee$-semi-lattice, i.e. if fuzzy relations R_1 and R_2 are its solutions then a fuzzy relation $R_1 \vee R_2$ is a solution to (1) too. This semi-lattice has the greatest element and in the case of finite universes $\mathbf{X}$ and $\mathbf{Y}$, it has minimal elements (see [1]).

Set of solution of the system (1), $\mathcal{R}_{sup}$, frames a Root system with stem $\hat{R}$ and set of offshoots $\hat{O}$, where $\hat{R}$ is the greatest solution and $\hat{O}$ is a set of the minimal solutions.

$$\mathcal{R}_{sup} = \bigcup_{R \in \hat{O}} [R, \hat{R}]$$

Analogous if system (2) is solvable then the set of solutions forms a $\wedge$-semi-lattice, i.e. a fuzzy relation $R_1 \wedge R_2$ is a solution to (2) whenever R_1 and R_2 are its solutions. Moreover, this semi-lattice has the least element and in the case of finite universes $\mathbf{X}$ and $\mathbf{Y}$, it has maximal elements (see [1]).

Set of solution of the system (2), $\mathcal{R}_{inf}$, frames a Crown system with stem $\check{R}$ and set of offshoots $\check{O}$, where $\check{R}$ is the least solution and $\check{O}$ is a set of the maximal solutions.

$$\mathcal{R}_{inf} = \bigcup_{R \in \check{O}} [\check{R}, R]$$

4 Conclusion

In this contribution I introduced some criterion of solvability for the system of fuzzy relation equations with dual composition. It is an analogy of criterion of solvability from [11] however situation is more difficult in case of systems with respect to $inf \rightarrow$ - composition.

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