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On approximate reasoning with graded rules

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Research report No. 94

2006

Submitted/to appear:

Fuzzy Sets and Systems, special volume

Supported by:

Grant MSM6198898701 of the MŠMT ČR

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Abstract

This contribution is aimed at provide the reader with the comprehensive view on problems of approximating reasoning with imprecise knowledge in the form of a collection of fuzzy IF-THEN rules formalized by approximating formulas of the special type. Two alternatives that follow from the dual character of approximating formulas are developed in parallel. And the link to the theory of fuzzy control systems is explained.

Keywords: Approximate reasoning, Fuzzy approximation, Normal forms, Fuzzy control systems.

1 Introduction

This paper is aimed at providing systematic investigation of knowledge-based systems of the special types. The type of a system is determined by interpretation of a knowledge that is at disposal. We will distinguish the following types of the knowledge formalization

$$\underbrace{\begin{array}{l} A_1 \text{ and } B_1, \text{ or} \\ \dots \\ A_k \text{ and } B_k \end{array}}_{\text{DNF}} \quad \text{and} \quad \begin{array}{l} \text{if } A_1 \text{ then } \neg B_1, \text{ and} \\ \dots \\ \text{if } \underbrace{A_k \text{ then } \neg B_k}_{\text{CNF}} \end{array}$$

or even more simple

$$A_1 \text{ or } \dots \text{ or } A_k \quad \text{and} \quad \neg A_1 \text{ and } \dots \text{ and } \neg A_k.$$

Differentiation between formalizations follows naturally from the requirement of creating a description of the given fact as simple as possible. Indeed, some facts are much easier to be described by eliminating inadmissible cases.

For example, imagine decision-making situation in diagnostics, where we detect disease on the basis of symptoms. The set of symptoms $\{s_1, \dots, s_p\}$ enables to detect a disease D, while the rejection of the set of symptoms $\{s'_1, \dots, s'_r\}$ leads to the same result, whereas r may be rapidly smaller than p.

Since the graded approach (as stated in [10]) seems to be a general principle of the human mind, therefore we assume rules DNF and CNF to express the fact in consideration using vague predicates. Additionally, there may exist a certain amount of inaccuracy or dubiousness over the knowledge base, which is reflected in truth constants (assigned to this inexactness) that equip the respective formalization. In [8], the authors distinguish between gradual rules, certainty rules and their mixture. Additionally, they cope with a problem of missing information that relates to a problem of interpolation which will not be our case. Here, we assume some statement S (e.g. $S \equiv A(x)\&B(y)$) and its verity is supported by exemplary samples $D = \{(c_i, d_i)|, i \in I\}$. To each data from D is assigned a weight that impact the degree f in which the statement S is satisfied. Hence, the final degree f of S is produced on the basis of knowledge about the data set D relatively to the structure of S. Therefore, our graded rules may become one of the rules specified by the authors of [8].

Finally, approximate inferences that mimic approximate reasoning in the sense of [16] and cope with graded rules will be introduced. Inference with graded DNF is based on Zadeh's compositional rule of inference that allows to derive a consequence of what is known (deduction), i.e. A^* and DNF. Unlike in the case of inference with graded CNF, which is used to explain what is known (abduction), i.e. A^* and CNF. For the illustration, let us consider the following formulae

$$A\&B\&A^* \to B^*, \tag{1}$$

$$A^*\&B^{**}\&A \to B, \tag{2}$$

and we wish to find B^* and B^{**} satisfying (1) and (2), respectively. Then obviously, we come to $B^* \equiv [A^*\&(A\&B)]$ and $B^{**} \equiv [A^* \to (A \to B)]$ fulfil these requirements. The dual character of DNF and CNF together with this two kinds of reasoning lead to the inference rules specified in Section 2.3. The properties of these inferences formulated in Section 3.2 are strongly influenced by the ones obtained by Hájek in [9], where the case of DNF is investigated. In this contribution, we adapt Hájek's results to the case of graded DNF and we build the dual approach for graded CNF in parallel. Moreover, the whole methodology is in the spirit of [2].

2 Preliminaries

2.1 Many-sorted fuzzy predicate logic(BL∀)

The language J with sorts $s_1, \dots, s_n$ of BL∀ includes a non-empty set of predicates of any type, a set of object constants, object variables $x_1, y_1, z_1, \dots,\ x_2, y_2, z_2, \dots, x_n, y_n, z_n, \dots, x_1, y_1, z_1, \dots$ of the sort s_i, a set of connectives $\{\&, \rightarrow\}$, truth constant $\bar{0}$ ($\bar{1} \equiv_{df} \bar{0} \rightarrow \bar{0}$), quantifiers $\forall, \exists$ and does not include functional symbols.

Terms are object variables and object constants.

By a formula, we mean a formula of BL∀ in the language J, build in the usual way, i.e. each $P(t_1, \dots, t_n)$, where P is an n-ary predicate and $t_1, \dots, t_n$ are terms, and each truth constant is an atomic formula. Each formula results from atomic formulae by iterated use of the following rule: if φ, ψ are formulae and x is an object variable then $(\forall x)\varphi$, $(\exists x)\varphi$, $\varphi\&\psi$ and $\varphi \rightarrow \psi$ are formulae.

In BL∀ it is possible to define the following derived connectives:

$$
\begin{array}{ccl}
\neg\varphi & \text{as} & \varphi \rightarrow \bar{0} \\
\varphi \wedge \psi & \text{as} & \varphi\&(\varphi \rightarrow \psi) \\
\varphi \vee \psi & \text{as} & [(\varphi \rightarrow \psi) \rightarrow \psi] \wedge [(\psi \rightarrow \varphi) \rightarrow \varphi] \\
\varphi \leftrightarrow \psi & \text{as} & (\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)
\end{array}
$$

Before we summarize logical calculus of BL∀, let us briefly recall that an object constant is always substitutable into a formula and a variable y is substitutable into φ for x if the substitution does not change any free occurrence of x in φ into a bound occurrence of y.

BL∀ consists of the following axioms of BL for connectives:

$$(\varphi \rightarrow \psi) \rightarrow [(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)], \tag{3}$$
$$(\varphi\&\psi) \rightarrow \varphi, \tag{4}$$
$$(\varphi\&\psi) \rightarrow (\psi\&\varphi), \tag{5}$$
$$[\varphi\&(\varphi \rightarrow \psi)] \rightarrow [\psi\&(\psi \rightarrow \varphi)], \tag{6}$$
$$[\varphi \rightarrow (\psi \rightarrow \chi)] \leftrightarrow [(\varphi\&\psi) \rightarrow \chi], \tag{7}$$
$$[(\varphi \rightarrow \psi) \rightarrow \chi] \rightarrow [((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \varphi], \tag{8}$$
$$\bar{0} \rightarrow \varphi, \tag{9}$$

together with the axioms on quantifiers:

$$(\forall x)\varphi(x) \rightarrow \varphi(t) \quad (t \text{ is substitutable for } x \text{ in } \varphi), \tag{10}$$
$$\varphi(t) \rightarrow (\exists x)\varphi(x) \quad (t \text{ is substitutable for } x \text{ in } \varphi), \tag{11}$$
$$(\forall x)(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\forall x)\psi) \quad (x \text{ is not free in } \varphi), \tag{12}$$
$$(\forall x)(\varphi \rightarrow \psi) \rightarrow ((\exists x)\varphi \rightarrow \psi) \quad (x \text{ is not free in } \psi), \tag{13}$$
$$(\forall x)(\varphi \vee \psi) \rightarrow ((\forall x)\varphi \vee \psi) \quad (x \text{ is not free in } \psi), \tag{14}$$

and two deduction rules

$$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi} \quad \text{(modus ponens - MP)},$$
$$\frac{\varphi}{(\forall x)\varphi} \quad \text{(generalization - GR)}.$$

The notions of theory T, theory over BL∀, proof and provability in BL∀, proof and provability in a theory over BL∀ are defined in the same way as in classical logic.

Let the interpretation of the connectives forms a BL-algebra

$$\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow_*, 0, 1\rangle.$$

Any $\mathcal{L}$-structure of the form

$$\mathcal{M} = \langle (D_i)_{for\, s_i}, (r_P)_{P-predicate}, (m_c)_{c-constant}\rangle,$$

is understood as a fuzzy structure for the many-sorted language J. There $D_1, \dots, D_n$ are non-empty sets of objects, r_P is a L-fuzzy relation of the respective type and m_c belongs to D_i provided that c is of the type s_i.

Finally, we summarize useful theorems of BL∀.

Lemma 2.1
The following are theorems of BL∀:

$$[\varphi \& (\varphi \to \psi)] \to \psi, \tag{15}$$
$$\varphi \to (\varphi \vee \psi), \tag{16}$$
$$(\varphi \to \psi) \to [(\varphi \& \chi) \to (\psi \& \chi)], \tag{17}$$
$$(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi), \tag{18}$$
$$(\forall x)(\varphi \to \psi) \leftrightarrow (\varphi \to (\forall x)\psi) \quad (x \text{ is not free in } \varphi), \tag{19}$$
$$(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi), \tag{20}$$
$$(\forall x)(\varphi \to \psi) \leftrightarrow ((\exists x)\varphi \to \psi) \quad (x \text{ is not free in } \psi), \tag{21}$$
$$[(\forall x)\varphi \& (\exists x)\psi] \to (\exists x)(\varphi \& \psi), \tag{22}$$
$$(\exists x)(\varphi \& \psi) \leftrightarrow ((\exists x)\varphi \& \psi) \quad (x \text{ is not free in } \psi). \tag{23}$$

2.2 Formalization of graded fuzzy IF-THEN rules

Below, we recall special approximating formulae (known as normal forms introduced by Perfilieva in [13]) formalizing collection of fuzzy IF-THEN rules.

Conventions 2.2
For the sake of brevity, let us denote

$$\begin{array}{ccc} [x_1, \dots, x_n] & \text{by} & \bar{x} \\ R_1 \dots R_n & \text{by} & \bar{R} \\ R_1(x_1, y_1) \& \cdots \& R_n(x_n, y_n) & \text{by} & R(\bar{x}, \bar{y}) \end{array}$$

Analogous shortening we use for any other formulae, predicate or variable.

Definition 2.1 *Let $k \in \mathbb{N}$ and $I_k = \{1, \dots, k\}$. The language J_k extends $J(BL\forall)$ by*

- *a finite set of n-tuples of object constants $\{\bar{\mathbf{c}}_i = [\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}] \mid i \in I_k\}$, where each $\mathbf{c}_{i_j}$ is of the sort s_i for all $j = 1, \dots, n$,*
- *binary predicate symbols $R_1, \dots, R_n$, each R_i of the type $\langle s_i, s_i \rangle$,*
- *and an n-ary predicate F of the type $\langle s_1, \dots, s_n \rangle$.*

We define the following operations

$$\mathrm{DNF}_{F,k}(\bar{x}) \equiv_{df} \bigvee_{i \in I_k} (R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)), \tag{24}$$
$$\mathrm{CNF}_{F,k}(\bar{x}) \equiv_{df} \bigwedge_{i \in I_k} (R(\bar{x}, \bar{\mathbf{c}}_i) \to F(\bar{\mathbf{c}}_i)), \tag{25}$$
$$\mathrm{DNF}^{\exists}_{F}(\bar{x}) \equiv_{df} (\exists \bar{y})(R(\bar{y}, \bar{x}) \& F(\bar{y})), \tag{26}$$
$$\mathrm{CNF}^{\forall}_{F}(\bar{x}) \equiv_{df} (\forall \bar{y})(R(\bar{x}, \bar{y}) \to F(\bar{y})). \tag{27}$$

We can also define the usual properties of formulae and binary predicates:

$$
\begin{array}{rcll}
\mathrm{Ext}_{\bar{R}}(\varphi) & \equiv_{df} & (\forall \bar{x}, \bar{y})(R(\bar{x}, \bar{y}) \& \varphi(\bar{x}) \rightarrow \varphi(\bar{y})) & \bar{R}\textit{-extensionality} \\
\varphi(\bar{x}) \subseteq \psi(\bar{x}) & \equiv_{df} & (\forall \bar{x})(\varphi(\bar{x}) \rightarrow \psi(\bar{x})) & \textit{inclusion} \\
\varphi(\bar{x}) \approx \psi(\bar{x}) & \equiv_{df} & (\forall \bar{x})(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) & \textit{bi-inclusion} \\
\mathrm{Refl}(R) & \equiv_{df} & (\forall x) Rxx & \textit{reflexivity} \\
\mathrm{Sym}(R) & \equiv_{df} & (\forall x, y)(Rxy \rightarrow Ryx) & \textit{symmetry} \\
\mathrm{Trans}(R) & \equiv_{df} & (\forall x, y, z)(Rxy \& Ryz \rightarrow Rxz) & \textit{transitivity}
\end{array}
$$

Notice that $\mathrm{DNF}_F^{\exists}$ and $\mathrm{CNF}_F^{\forall}$ are known in literature as image and pre-image of F in relation R, see e.g. [14]. Here, these formulae represent limit elements of sequence of approximating formulae $\mathrm{DNF}_{F,k}$ and $\mathrm{CNF}_{F,k}$ for $k \rightarrow \infty$, respectively.

Conventions 2.3
For the sake of brevity, we will write only D(C)NF *and* I *instead of* $\mathrm{D(C)NF}_{F,k}$ *and* I_k*, respectively, whenever* F *and* k *will be clear from the context.*

A partial knowledge formalized by (24) or (25) can be interpreted in fuzzy control as a rule-base consisting of k rules in the following form

$$\text{IF } [(X \text{ is } \mathcal{R}_{i_1}) \text{ AND } (Y \text{ is } \mathcal{R}_{i_2})] \text{ THEN “it is valid in degree” } F_i, \tag{28}$$

or in the sense of [10], we can write

$$^{F_i}/(X \text{ is } \mathcal{R}_{i_1}) \text{ AND } (Y \text{ is } \mathcal{R}_{i_2}). \tag{29}$$

Notice that the concept of CNF as a formalization of the eliminative knowledge outlined at the beginning of the paper works only in the case of Łukasiewicz logic because of the involutivity of the negation. Generally, $\neg$ does not have such a good properties. To keep the dual concept of the descriptions, we propose to interpret a particular rule in CNF, e.g. $A \rightarrow p$ as “`nearly not` A” on the basis of $p \leftrightarrow 0$, i.e. $(p \leftrightarrow 0) \rightarrow (A \rightarrow p)$.

2.3 Inferring with graded rules

In fuzzy control, an approximate inference rule is considered as a basis for dealing with fuzzy IF-THEN rules and non-precise input knowledge. A generalized rule of modus ponens as a particular case of compositional rule of inference in the global concept of many-valued logics has been introduced by L. Zadeh in [15]. The analysis of logical aspects of Zadeh's compositional rules of inference was done by P. Hájek in [9] or V. Novák in [12, 10] (for evaluated syntax). Inference rules were also intensively studied from the algebraical point of view as special operations called compositions see e.g. [11, 8, 3].

In the sequel, we will assume $J_{FC} = J_k \cup \{A^*\}$, where A^* is a predicate of the type $\langle s_1, \ldots, s_p \rangle$.

Let us suppose $n > 1$, $p \in \mathbb{N}$ such that $1 \leq p \leq n-1$, and $\bar{x}_p = [x_1, \ldots, x_p]$, $\bar{y}_p = [x_{p+1}, \ldots, x_n]$. Then, we define

$$B^*_{\mathrm{DNF}}(\bar{y}_p) \equiv_{df} (\exists\, \bar{x}_p)(A^*(\bar{x}_p) \& \mathrm{DNF}_{F,k}(\bar{x})), \tag{30}$$

$$B^*_{\mathrm{CNF}}(\bar{y}_p) \equiv_{df} (\forall\, \bar{x}_p)(A^*(\bar{x}_p) \rightarrow \mathrm{CNF}_{F,k}(\bar{x})), \tag{31}$$

which define B^*_{DNF} and B^*_{CNF} from A^* using the Zadeh's compositional rule of inference and Bandler-Kouhout's product (BK-product [1]), respectively. Since BK-product can be viewed as a dual to Zadeh's composition, moreover $\mathrm{CNF}_{F,k}$ is dual to $\mathrm{DNF}_{F,k}$, hence also B^*_{CNF} is in some sense dual to B^*_{DNF}.

Remark 2.1 *Approximate inferences based on (30) and (31) can be visualized as inference rules of the following forms*

$$\frac{A^*, \mathrm{DNF}_{F,k}}{B^*_{\mathrm{DNF}}} \quad \textit{and} \quad \frac{A^*, \mathrm{CNF}_{F,k}}{B^*_{\mathrm{CNF}}}.$$

Later, we will investigate properties of fuzzy control systems without taking into account fuzzification and defuzzification methods.

3 Logical approximation based fuzzy control systems

3.1 Approximating formulae

In this subsection, we will mostly reformulate the results from [7] in accordance with a methodology manifested in [2]. These results are in the scope of the logical approximation, which is a theory aimed at studying properties of a class of formulae in an simplified form relating to some initial formula. There, the significant rule is played by formula of the form $\epsilon \to (\varphi \leftrightarrow \varphi_S)$ called conditional equivalence. We may interpret it as a lower bounded or graded equivalence between the given formula φ and its simplified version φ_S.

As it was pointed out in [13], considering standard model, conditional equivalence expresses a precision of approximation of "φ" by "φ_S". To prove a conditional equivalence for normal forms, we have to demand additionally the extensionality property from the initial formula. Remind that this property is natural to each propositional formula and in the case of formulae containing extensional predicates, we have the extensionality for the whole formula as well (see Lemma 5.6.8 in [9]).

Without considering the extensionality, we are able to prove only the following relationship between both normal forms.

Lemma 3.1
Let $m \in \mathbb{N}$, $m < k$ and $C_k(\bar{x})$ stands for $(\forall \bar{x}) \bigvee_{i \in I} R(\bar{x}, \bar{\mathbf{c}}_i) \& R(\bar{\mathbf{c}}_i, \bar{x})$. Moreover, let us define

$$D_i(x, y) \equiv_{df} \bigwedge_{j \in I} R_i(\mathbf{c}_{j_i}, x) \to R_i(\mathbf{c}_{j_i}, y), \tag{32}$$

$$E_i(x, y) \equiv_{df} \bigwedge_{j \in I} R_i(y, \mathbf{c}_{j_i}) \to R_i(x, \mathbf{c}_{j_i}). \tag{33}$$

Then

$$\mathrm{DNF}_{F,m}(\bar{x}) \subseteq \mathrm{DNF}_{F,m+1}(\bar{x}), \tag{34}$$
$$\mathrm{CNF}_{F,m+1}(\bar{x}) \subseteq \mathrm{CNF}_{F,m}(\bar{x}), \tag{35}$$
$$(\forall \bar{x}) C_k(\bar{x}) \to \mathrm{CNF}_{F,k}(\bar{x}) \subseteq \mathrm{DNF}_{F,k}(\bar{x}), \tag{36}$$
$$\mathrm{Ext}_{\bar{D}} \mathrm{DNF}_{F,k}, \tag{37}$$
$$\mathrm{Ext}_{\bar{E}} \mathrm{CNF}_{F,k}, \tag{38}$$
$$\mathrm{Trans}(\bar{R}) \to \mathrm{Ext}_{\bar{R}}\, \mathrm{DNF}_{F,k}, \tag{39}$$
$$\mathrm{Trans}(\bar{R}) \to \mathrm{Ext}_{\bar{R}}\, \mathrm{CNF}_{F,k}. \tag{40}$$

PROOF:
(34) – (35) Obvious.
(36) From (15), it follows

$$R(\bar{x}, \bar{\mathbf{c}}_i) \& (R(\bar{x}, \bar{\mathbf{c}}_i) \to F(\bar{\mathbf{c}}_i)) \to F(\bar{\mathbf{c}}_i)$$
$$R(\bar{x}, \bar{\mathbf{c}}_i) \& R(\bar{\mathbf{c}}_i, \bar{x}) \& (R(\bar{x}, \bar{\mathbf{c}}_i) \to F(\bar{\mathbf{c}}_i)) \to R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)$$
$$R(\bar{x}, \bar{\mathbf{c}}_i) \& R(\bar{\mathbf{c}}_i, \bar{x}) \& \bigwedge_{i \in I} (R(\bar{x}, \bar{\mathbf{c}}_i) \to F(\bar{\mathbf{c}}_i)) \to \bigvee_{i \in I} (R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i))$$
$$\bigvee_{i \in I} R(\bar{x}, \bar{\mathbf{c}}_i) \& R(\bar{\mathbf{c}}_i, \bar{x}) \& \bigwedge_{i \in I} (R(\bar{x}, \bar{\mathbf{c}}_i) \to F(\bar{\mathbf{c}}_i)) \to \bigvee_{i \in I} (R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)).$$

(37) By (17) and (7), it follows

$$R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i) \longrightarrow ((R(\bar{\mathbf{c}}_i, \bar{x}) \to R(\bar{\mathbf{c}}_i, \bar{y})) \to R(\bar{\mathbf{c}}_i, \bar{y}) \& F(\bar{\mathbf{c}}_i)) \longrightarrow$$
$$\bigvee_{i \in I} [(R(\bar{\mathbf{c}}_i, \bar{x}) \to R(\bar{\mathbf{c}}_i, \bar{y})) \to R(\bar{\mathbf{c}}_i, \bar{y}) \& F(\bar{\mathbf{c}}_i)] \longrightarrow$$
$$\bigwedge_{i \in I} (R(\bar{\mathbf{c}}_i, \bar{x}) \to R(\bar{\mathbf{c}}_i, \bar{y})) \to \bigvee_{i \in I} R(\bar{\mathbf{c}}_i, \bar{y}) \& F(\bar{\mathbf{c}}_i),$$

and $D_i(\bar{x},\bar{y}) \leftrightarrow \bigwedge_{i\in I}(R(\bar{\mathbf{c}}_i,\bar{x}) \to R(\bar{\mathbf{c}}_i,\bar{y}))$, hence we prove $\mathrm{Ext}_{\bar{D}}\,\mathrm{DNF}_{F,k}$.

(38) Analogous.

(39) – (40) Obviously follows by transitivity of $\to$. □

Remark 3.1 *From (37) and (38), it is clear that we approximate F by extensional formulae. Moreover, a degree in which* DNF *is lower than* CNF *is given as a composition of a domain covering qualities (partition of the input and output space) of both normal forms expressed by C_k, (see (36)). For the recent advances in formal theory of fuzzy partitions see [6].*

Involving the extensionality property allows us to prove the following formulae.

Lemma 3.2

$$\mathrm{Ext}_{\bar{R}}\,F \to \mathrm{DNF}_{F,k}(\bar{x}) \subseteq F(\bar{x}), \tag{41}$$

$$\mathrm{Ext}_{\bar{R}}\,F \to F(\bar{x}) \subseteq \mathrm{CNF}_{F,k}(\bar{x}), \tag{42}$$

$$\mathrm{Ext}_{\bar{R}}\,F \wedge (\forall\bar{x})C_k(\bar{x}) \to (\mathrm{DNF}_{F,k}(\bar{x}) \approx F(\bar{x})), \tag{43}$$

$$\mathrm{Ext}_{\bar{R}}\,F \wedge (\forall\bar{x})C_k(\bar{x}) \to (\mathrm{CNF}_{F,k}(\bar{x}) \approx F(\bar{x})), \tag{44}$$

$$\mathrm{Refl}\,\bar{R} \to (\mathrm{Ext}_{\bar{R}}\,F \leftrightarrow \mathrm{DNF}^{\exists}_{F}(\bar{x}) \approx F(\bar{x})), \tag{45}$$

$$\mathrm{Refl}\,\bar{R} \to (\mathrm{Ext}_{\bar{R}}\,F \leftrightarrow \mathrm{CNF}^{\forall}_{F}(\bar{x}) \approx F(\bar{x})). \tag{46}$$

PROOF:

(41)

$$\begin{aligned}
\mathrm{Ext}_{\bar{R}}\,F \to& (\forall\bar{x})(R(\bar{\mathbf{c}}_i,\bar{x})\&F(\bar{\mathbf{c}}_i) \to F(\bar{x})) \quad \text{for all } i\in I, \text{ hence}\\
\mathrm{Ext}_{\bar{R}}\,F \longrightarrow& \bigwedge_{i\in I}(\forall\bar{x})(R(\bar{\mathbf{c}}_i,\bar{x})\&F(\bar{\mathbf{c}}_i) \to F(\bar{x})) \longleftrightarrow\\
& (\forall\bar{x})\bigwedge_{i\in I}(R(\bar{\mathbf{c}}_i,\bar{x})\&F(\bar{\mathbf{c}}_i) \to F(\bar{x})) \longrightarrow\\
& (\forall\bar{x})[\bigvee_{i\in I} R(\bar{\mathbf{c}}_i,\bar{x})\&F(\bar{\mathbf{c}}_i) \to F(\bar{x})] \longleftrightarrow \mathrm{DNF}(\bar{x}) \subseteq F(\bar{x}).
\end{aligned}$$

(42) Analogously

$$\begin{aligned}
\mathrm{Ext}_{\bar{R}}\,F \longrightarrow& (\forall\bar{x})\bigwedge_{i\in I}(F(\bar{x}) \to (R(\bar{\mathbf{c}}_i,\bar{x}) \to F(\bar{\mathbf{c}}_i))) \longrightarrow\\
& (\forall\bar{x})(F(\bar{x}) \to \bigwedge_{i\in I}(R(\bar{\mathbf{c}}_i,\bar{x}) \to F(\bar{\mathbf{c}}_i))) \longleftrightarrow F(\bar{x}) \subseteq \mathrm{CNF}(\bar{x}).
\end{aligned}$$

(43) For e.g. we prove the formulae for $\mathrm{DNF}_{F,k}$. By (36) and (42)

$$\begin{aligned}
\mathrm{Ext}_{\bar{R}}\,F \wedge (\forall\bar{x})C_k(\bar{x}) \longrightarrow& (F(\bar{x}) \subseteq \mathrm{CNF}(\bar{x})) \wedge (\mathrm{CNF}(\bar{x}) \subseteq \mathrm{DNF}(\bar{x})) \longrightarrow\\
& F(\bar{x}) \subseteq \mathrm{DNF}(\bar{x}) \longrightarrow_{\text{by (41)}} F(\bar{x}) \approx \mathrm{DNF}(\bar{x}).
\end{aligned}$$

(44) Analogous to (43).

(45)

$$\begin{aligned}
&\mathrm{Ext}_{\bar{R}}\,F \longleftrightarrow (\forall\bar{x},\bar{y})(R(\bar{y},\bar{x})\&F(\bar{y}) \to F(\bar{x})) \longleftrightarrow\\
&\qquad (\forall\bar{x})[(\exists\bar{y})(R(\bar{y},\bar{x})\&F(\bar{y})) \to F(\bar{x})] \longleftrightarrow \mathrm{DNF}^{\exists}_{F}(\bar{x}) \subseteq F(\bar{x}),\\
&\mathrm{Refl}\,\bar{R} \longrightarrow (\bar{1} \to \mathrm{Refl}\,\bar{R}) \longrightarrow (\forall\bar{x})[F(\bar{x}) \to F(\bar{x})\&R(\bar{x},\bar{x})] \longrightarrow\\
&\qquad (\forall\bar{x})[F(\bar{x}) \to (\exists\bar{y})(R(\bar{y},\bar{x})\&F(\bar{y}))] \longleftrightarrow F(\bar{x}) \subseteq \mathrm{DNF}^{\exists}_{F}(\bar{x}),\\
&F(\bar{x}) \approx \mathrm{DNF}^{\exists}_{F}(\bar{x}) \longrightarrow \mathrm{DNF}^{\exists}_{F}(\bar{x}) \subseteq F(\bar{x}) \longleftrightarrow \mathrm{Ext}_{\bar{R}}\,F,\\
&\mathrm{Refl}\,\bar{R} \to [\mathrm{Ext}_{\bar{R}}\,F \leftrightarrow F(\bar{x}) \approx \mathrm{DNF}^{\exists}_{F}(\bar{x})].
\end{aligned}$$

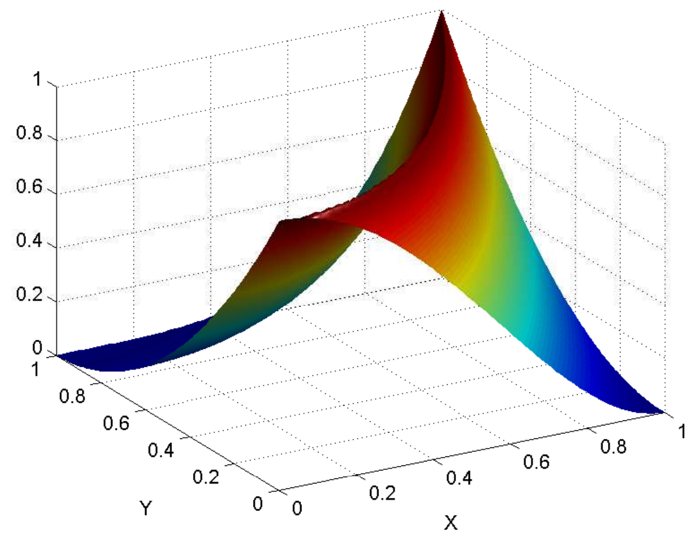


Figure 1: Relation to be approximated from Example 3.3.7.

(46) Analogous to (45). □

Remark 3.2 *Due to the extensionality property, we are able to show that*

$$\begin{aligned} \mathrm{DNF}_{F,k}(\bar{x}) &\subseteq F(\bar{x}), \quad (\text{“}\mathrm{DNF}_{F,k}\text{” creates a lower approximation of “}F\text{”}),\\ F(\bar{x}) &\subseteq \mathrm{CNF}_{F,k}(\bar{x}), \quad (\text{“}\mathrm{CNF}_{F,k}\text{” is an upper approximation of “}F\text{”}). \end{aligned}$$

And moreover, $F \approx \mathrm{D(C)NF}_{F,k}$ is determined by choice of $\bar{\mathbf{c}}_i$ and $\bar{R}$ in a particular model, as it can be seen from C_k.

By (34)+(41) and ((35)+(42)), we see that the sequence of approximating formulae DNF_F, k for increasing k approaches F and in the limit case, it coincide with F whenever $\bar{R}$ is reflexive (45), (46). Note that we can find semantical proofs (not in graded form and under additional conditions required from $\bar{R}$) of (45) and (46) in [4] or [3].

As it follows from (43) and (44), we can efficiently approximate only extensional formulae. Below, we show some examples.

Example 3.3
Let us assume examples $D = \{\bar{\mathbf{d}}_i | \, i \in J\}$

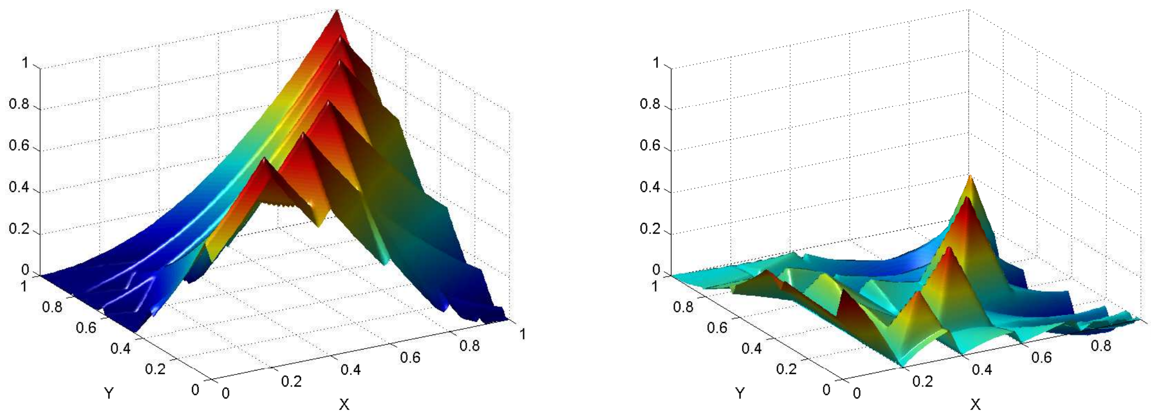
1. $F_{\mathrm{DNF}}(x) \longleftrightarrow \bar{0} \vee \bigvee_{i\in J} R(\bar{x}, \bar{\mathbf{d}}_i)$ *is extensional w.r.t. $\bar{Q}$, where each*

$$Q_i(x,y) \equiv_{df} \bigwedge_{j\in J} R(x, \mathbf{d}_{j_i}) \to R(y, \mathbf{d}_{j_i}).$$

2. $F_{\mathrm{CNF}}(x) \longleftrightarrow \bar{1} \wedge \bigwedge_{i\in J} R(\bar{\mathbf{d}}_i, \bar{x})$ *is extensional w.r.t. $\bar{Q}'$, where each*

$$Q'_i(x,y) \equiv_{df} \bigwedge_{j\in J} R(\mathbf{d}_{j_i}, x) \to R(\mathbf{d}_{j_i}, y).$$

3. $\mathrm{Trans}\, \bar{R} \to \mathrm{Ext}_R\, F_{\mathrm{DNF}}$ *and* $\mathrm{Trans}\, \bar{R} \to \mathrm{Ext}_R\, F_{\mathrm{CNF}}$.



(a) $\mathrm{DNF}_{F,6}(x,y)$ (b) Error of approximation by DNF

Figure 2: Approximation of the relation from Figure 1 by DNF.

4. *Let* $\mathbf{k}$ *be a constant and* $(\forall \bar{x}_p)[\bigvee_{i\in J} R(\bar{x}_p, \bar{\mathbf{d}}_{i_p}) \leftrightarrow \mathbf{k}]$ *then*

$$F_1(x) \longleftrightarrow \bar{0} \vee [\bigvee_{i\in J} R(\bar{x}_p, \bar{\mathbf{d}}_{i_p}) \rightarrow \bigvee_{i\in J} R(\bar{x}, \bar{\mathbf{d}}_i)]$$

is extensional w.r.t. $\bar{Q}$.

5. *Let* $\mathbf{k}$ *be a constant and* $(\forall \bar{x}_p)[\bigvee_{i\in J} R(\bar{\mathbf{d}}_{i_p}, \bar{x}_p) \leftrightarrow \mathbf{k}]$ *then*

$$F_2(x) \longleftrightarrow \bar{1} \wedge [\bigwedge_{i\in J} R(\bar{\mathbf{d}}_{i_p}, \bar{x}_p) \rightarrow \bigwedge_{i\in J} R(\bar{\mathbf{d}}_i, \bar{x})]$$

is extensional w.r.t. $\bar{Q}'$.

6. *Let* f *be a functional symbol. The following formula expresses a property of* f *that is called compatibility w.r.t.* $\approx_1$ *and* $\approx_2$ *(see [3])*

$$Comp_{\approx_{1,2}} f \equiv_{df} (\forall x, y)[x \approx_1 y \rightarrow f(x) \approx_2 f(y)], \tag{47}$$

where x, y *are of the sort* s, $f(x), f(y)$ *are of the sort* s' *and* $\approx_1$ $(\approx_2)$ *is of the type* $\langle s, s\rangle(\langle s', s'\rangle)$. *Define* F *as follows*

$$F_f(x, y) \equiv_{df} y \approx_2 f(x). \tag{48}$$

Then, it is easy to prove that

$$Comp_{\approx_{1,2}} f \,\&\, \mathrm{Sym}\,(\approx_2)\&(\mathrm{Trans}\,(\approx_2))^2 \rightarrow \mathrm{Ext}_{\approx_{1,2}} F_f, \tag{49}$$

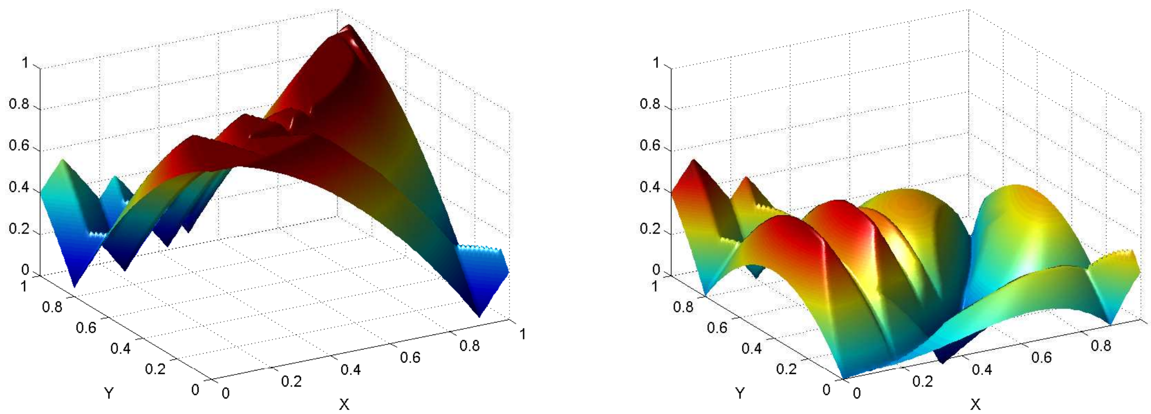
and additionally, we are able to show the property of functionality of F_f *(expressed by (50)) saying that any two images of the same input* x *are "close enough"*

$$\mathrm{Func}_{(\approx)} F \equiv_{df} (\forall x, y, y')[F(x, y)\&F(x, y') \rightarrow y \approx y'], \tag{50}$$

$$\mathrm{Trans}\,(\approx_2)\&\,\mathrm{Sym}\,(\approx_2) \rightarrow \mathrm{Func}_{(\approx_2)} F_f. \tag{51}$$

7. *For illustration of the above example, let us assume the standard Łukasiewicz algebra* $\mathcal{L} = \langle [0,1], \otimes, \Rightarrow, 0, 1\rangle$ *as an interpretation of the logical operations, let* $\{f, F_f, \approx_1, \approx_2\}$ *be interpreted as* $\{\tilde{f}, \tilde{F}_f, \cong, \cong\}$, *respectively, where* $\tilde{f} = x^2$, $x \cong y = (1 - |x - y|)^2$ *and* $M = [0, 1]$.

   In this case, $\cong$ *is similarity relation (reflexive, symmetric, transitive) and* $\tilde{F}_f$ *(see Figure 1) is a fuzzy function in Hajek's sense ([9]), i.e.* $\tilde{F}_f$ *is extensional w.r.t.* $\cong$ *(since for all* $x, y \in M$: $|x^2 - y^2| \leq$



(a) $\mathrm{CNF}_{F,6}(x,y)$ (b) Error of approximation by CNF

Figure 3: Approximation of the relation from Figure 1 by CNF.

$|x-y|$, hence $x \cong y \leq x^2 \cong y^2$) and maps the same inputs to indistinguishable outputs due to (50). Approximation given by DNF_{F_f} is shown on Figure 2 and similarly, approximation by CNF_{F_f} on Figure 3. The difference between these two approximations is evident form Figure 4. There, the nodes suitable for approximating formulae are depicted (circles for DNF and diamonds for CNF). It is clear that nodes in which we construct DNF (CNF) locate in local maxima (minima).

Remark 3.3 *In (§7, [9]), examples of F from $D = \{(c_i, d_i) | \, i \in I\}$ are assumed to be 1-true, i.e. the formula $\bigwedge_{i \in J} F(c_i, d_i)$ is taken as an axiom and approximating formulae for fuzzy function F (see Example 3.3.7) are defined as*

$$\mathrm{MAMD}(x,y) \equiv_{df} \bigvee_{i \in I} (x \approx_1 c_i) \& (y \approx_2 d_i),$$

$$\mathrm{RULES}(x,y) \equiv_{df} \bigwedge_{i \in I} (x \approx_1 c_i) \rightarrow (y \approx_2 d_i).$$

Notice that MAMD *and* DNF *constructed in the examples from D are identical, unlike* RULES *and* CNF *as it can be seen from Figure 5 and Figure 3.*

3.2 Properties of approximate inferences

The following results are analogous to the one given in [9] for Zadeh's compositional rule of inference and IF-THEN rules of the Mamdani type (formalized as MAMD see Remark 3.3). First, we will focus at relationship between outputs of approximate inferences.

Lemma 3.4
Under the present notation

$$[(\exists \bar{x}_p) A^*(\bar{x}_p) \rightarrow (\forall \bar{y}_p) B^*_{\mathrm{DNF}}(\bar{y}_p)] \leftarrow (\forall \bar{x})\, \mathrm{DNF}_{F,k}(\bar{x}), \tag{52}$$

$$(\forall \bar{x}_p) A^*(\bar{x}_p) \& (\forall \bar{y}_p) B^*_{\mathrm{CNF}}(\bar{y}_p) \rightarrow (\forall \bar{x})\, \mathrm{CNF}_{F,k}(\bar{x}), \tag{53}$$

$$(\mathrm{Ext}_{\bar{R}}\, F)^2 \& (\forall \bar{x}_1, \bar{x}_2) D(\bar{x}_1, \bar{x}_2) \rightarrow B^*_{\mathrm{DNF}}(\bar{y}_{1_p}) \subseteq B^*_{\mathrm{CNF}}(\bar{y}_{2_p}), \tag{54}$$

$$(\forall \bar{x}) C_k(\bar{x}) \& (\exists \bar{x}_p)(A^*(\bar{x}_p))^2 \rightarrow B^*_{\mathrm{CNF}}(\bar{y}_p) \subseteq B^*_{\mathrm{DNF}}(\bar{y}_p), \tag{55}$$

$$(\mathrm{Ext}_{\bar{R}}\, F)^2 \rightarrow B^*_{\mathrm{DNF}}(\bar{y}_p) \subseteq B^*_{\mathrm{CNF}}(\bar{y}_p), \tag{56}$$

$$(\forall \bar{x}) C_k(\bar{x}) \& (\exists \bar{x}_p)(A^*(\bar{x}_p))^2 \& (\mathrm{Ext}_{\bar{R}}\, F)^2 \rightarrow B^*_{\mathrm{DNF}}(\bar{y}_p) \approx B^*_{\mathrm{CNF}}(\bar{y}_p), \tag{57}$$

for arbitrary $p \in \mathbb{N}$, $1 \leq p < n$.

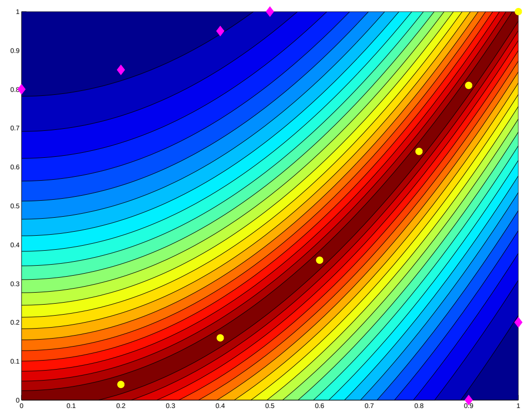
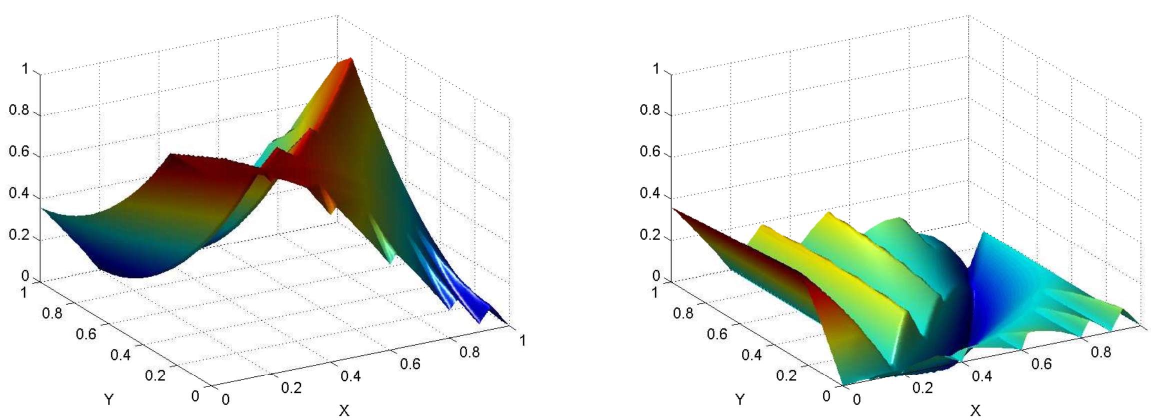


Figure 4: Nodes marked by circles (diamonds) in which approximating formula DNF(CNF) is constructed.



(a) RULES(x, y)

(b) Error of approximation by RULES

Figure 5: Approximation of the relation from Figure 1 by approximating formula RULES from Remark 3.3.

PROOF: (52) – (53) Obvious.

(54) By (41), (42) and (37), we have

$$
\begin{aligned}
&(\mathrm{Ext}_{\bar{R}}\, f)^2 \& (\forall \bar{x}_1,\bar{x}_2) D(\bar{x}_1,\bar{x}_2) \longrightarrow (\forall \bar{x}_1,\bar{x}_2)(\mathrm{DNF}(\bar{x}_1) \to \mathrm{CNF}(\bar{x}_2)) \longrightarrow \\
&\quad (\forall \bar{x}_1,\bar{x}_2)[A^*(\bar{x}_{1_p}) \& A^*(\bar{x}_{2_p}) \& \,\mathrm{DNF}(\bar{x}_1) \to \mathrm{CNF}(\bar{x}_2)] \longrightarrow \\
&\quad (\forall \bar{x}_1,\bar{x}_2)[A^*(\bar{x}_{1_p}) \& \,\mathrm{DNF}(\bar{x}_1) \to (A^*(\bar{x}_{2_p}) \to \mathrm{CNF}(\bar{x}_2))] \longrightarrow \\
&\quad (\forall \bar{y}_{1_p},\bar{y}_{2_p})[(\exists \bar{x}_{1_p}) A^*(\bar{x}_{1_p}) \& \,\mathrm{DNF}(\bar{x}_1) \to (\forall \bar{x}_{2_p})(A^*(\bar{x}_{2_p}) \to \mathrm{CNF}(\bar{x}_2))].
\end{aligned}
$$

(55) By (36) and transitivity axiom (3)

$$
\begin{aligned}
&(A^*(\bar{x}_p) \to \mathrm{CNF}(\bar{x})) \longrightarrow A^*(\bar{x}_p) \to (C_k(\bar{x}) \to \mathrm{DNF}(\bar{x})) \longrightarrow \\
&\quad A^*(\bar{x}_p) \to [C_k(\bar{x}) \to \mathrm{DNF}(\bar{x})] \& [A^*(\bar{x}_p) \to A^*(\bar{x}_p)] \longrightarrow \\
&\quad A^*(\bar{x}_p) \to [C_k(\bar{x}) \& A^*(\bar{x}_p) \to \mathrm{DNF}(\bar{x}) \& A^*(\bar{x}_p)] \longleftrightarrow \\
&\quad (A^*(\bar{x}_p))^2 \& C_k(\bar{x}) \to \mathrm{DNF}(\bar{x}) \& A^*(\bar{x}_p),
\end{aligned}
$$

$$
\begin{aligned}
&(\forall \bar{x}_p)(A^*(\bar{x}_p) \to \mathrm{CNF}(\bar{x})) \to \\
&\quad [(\exists \bar{x}_p)(A^*(\bar{x}_p))^2 \& C_k(\bar{x}) \to (\exists \bar{x}_p)\,\mathrm{DNF}(\bar{x}) \& A^*(\bar{x}_p)], \\
&(\forall \bar{y}_p)(\exists \bar{x}_p)(A^*(\bar{x}_p))^2 \& C_k(\bar{x}) \to (\forall \bar{y}_p)[B^*_{\mathrm{CNF}}(\bar{y}_p) \to B^*_{\mathrm{DNF}}(\bar{y}_p)].
\end{aligned}
$$

(56) From (54) and reflexivity of D.

(57) By (55) and (56). □

Remark 3.4 *As one would expect from the relationship between* D(C)NF *and F, it is not completely true that $B^*_{\mathrm{DNF}} \subseteq B^*_{\mathrm{CNF}}$ (54,56) nor $B^*_{\mathrm{CNF}} \subseteq B^*_{\mathrm{DNF}}$ (55). From (56), it follows that the extensionality is essential there, and the part $(\forall \bar{x}_1,\bar{x}_2)D(\bar{x}_1,\bar{x}_2)$ from (54) relates to the degree of inclusion, see (32).*

Now, we will investigate a relationship between the precise value of F and conclusion B^* of the inferences based on the approximate description.

Lemma 3.5

$$P_1 \to F(\bar{x}) \subseteq (A^*(\bar{x}_p) \to B^*_{\mathrm{DNF}}(\bar{y}_p)), \tag{58}$$

$$P_1 \to (A^*(\bar{x}_p) \& B^*_{\mathrm{CNF}}(\bar{y}_p)) \subseteq F(\bar{x}), \tag{59}$$

$$P_2 \to F(\bar{x}) \approx (A^*(\bar{x}_p) \to B^*_{\mathrm{DNF}}(\bar{y}_p)), \tag{60}$$

$$P_2 \to F(\bar{x}) \approx (A^*(\bar{x}_p) \& B^*_{\mathrm{CNF}}(\bar{y}_p)), \tag{61}$$

where

$$
\begin{aligned}
&P_1 \equiv_{df} \mathrm{Ext}_{\bar{R}}\, F \wedge (\forall \bar{x}) C_k(\bar{x}), \\
&P_2 \equiv_{df} P_1 \& (\mathrm{Ext}_{\bar{R}}\, F)^2.
\end{aligned}
$$

PROOF:

(58) From (43)

$$(\mathrm{Ext}_{\bar{R}}\, F \wedge C_k(\bar{x})) \& F(\bar{x}) \& A^*(\bar{x}_p) \to (\exists \bar{x}_p)(A^*(\bar{x}_p) \& \,\mathrm{DNF}(\bar{x})),$$

and using adjunction.

(59) Analogously.

(60) – (61) Obviously by (56). □

Remark 3.5 *Since F represents some real situation that is to be approximately described by approximating formulae, therefore we wish to know the relationship w.r.t. $B^*_{\mathrm{D(C)NF}}$ that has been investigated in the above lemma. From $P_{1,2}$ it follows that besides extensionality we need a appropriate partition, i.e. the distribution of input and output fuzzy sets associated to $\bar{R}(\bar{\mathbf{c}}, \bar{x}), \bar{R}(\bar{x}, \bar{\mathbf{c}})$, that lead to the indistinguishability between $B^*_{\mathrm{D(C)NF}}$ and F.*

Lemma 3.6
Let us denote $[\mathbf{c}_{i_{p+1}}, \ldots, \mathbf{c}_{i_n}]$ by $\bar{\mathbf{d}}_{i_p}$. Then

$$L_1 \to [R(\bar{\mathbf{d}}_{i_p}, \bar{y}_p) \& F(\bar{\mathbf{c}}_i)] \subseteq B^*_{\mathrm{DNF}}(\bar{y}_p), \tag{62}$$
$$L_2 \to B^*_{\mathrm{CNF}}(\bar{y}_p) \subseteq [R(\bar{y}_p, \bar{\mathbf{d}}_{i_p}) \to F(\bar{\mathbf{c}}_i)], \tag{63}$$
$$L_3 \to B^*_{\mathrm{DNF}}(\bar{y}_p) \subseteq [R(\bar{\mathbf{d}}_{i_p}, \bar{y}_p) \& F(\bar{\mathbf{c}}_i)], \tag{64}$$
$$L_4 \to [R(\bar{y}_p, \bar{\mathbf{d}}_{i_p}) \to F(\bar{\mathbf{c}}_i)] \subseteq B^*_{\mathrm{CNF}}(\bar{y}_p), \tag{65}$$
$$L_5 \to B^*_{\mathrm{DNF}}(\bar{y}_p) \approx [R(\bar{\mathbf{d}}_{i_p}, \bar{y}_p) \& F(\bar{\mathbf{c}}_i)], \tag{66}$$
$$L_6 \to B^*_{\mathrm{CNF}}(\bar{y}_p) \approx [R(\bar{y}_p, \bar{\mathbf{d}}_{i_p}) \to F(\bar{\mathbf{c}}_i)], \tag{67}$$

where

$$\begin{aligned}
L_1 &\equiv_{df} [R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \subseteq A^*(\bar{x}_p)] \& (\exists\, \bar{x}_p) R^2(\bar{\mathbf{c}}_{i_p}, \bar{x}_p), \\
L_2 &\equiv_{df} [R(\bar{x}_p, \bar{\mathbf{c}}_{i_p}) \subseteq A^*(\bar{x}_p)] \& (\exists\, \bar{x}_p) R^2(\bar{x}_p, \bar{\mathbf{c}}_{i_p}), \\
L_3 &\equiv_{df} [A^*(\bar{x}_p) \subseteq R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p)] \& \bigwedge_{i \neq j} \neg [R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& R(\bar{\mathbf{c}}_{j_p}, \bar{x}_p)], \\
L_4 &\equiv_{df} [A^*(\bar{x}_p) \subseteq R(\bar{x}_p, \bar{\mathbf{c}}_{i_p})] \& \bigwedge_{i \neq j} \neg [R(\bar{x}_p, \bar{\mathbf{c}}_{i_p}) \& R(\bar{x}_p, \bar{\mathbf{c}}_{j_p})], \\
L_5 &\equiv_{df} [A^*(\bar{x}_p) \approx R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p)] \& ((\exists\, \bar{x}_p) R^2(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \wedge \bigwedge_{i \neq j} \neg [R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& R(\bar{\mathbf{c}}_{j_p}, \bar{x}_p)]), \\
L_6 &\equiv_{df} [A^*(\bar{x}_p) \approx R(\bar{x}_p, \bar{\mathbf{c}}_{i_p})] \& ((\exists\, \bar{x}_p) R^2(\bar{x}_p, \bar{\mathbf{c}}_{i_p}) \wedge \bigwedge_{i \neq j} \neg [R(\bar{x}_p, \bar{\mathbf{c}}_{i_p}) \& R(\bar{x}_p, \bar{\mathbf{c}}_{j_p})]).
\end{aligned}$$

PROOF:

(62) Let us denote $R(\bar{\mathbf{d}}_{i_p}, \bar{y}_p) \& F(\bar{\mathbf{c}}_i)$ by $B(\bar{y}_p)$

$$\begin{aligned}
&(\exists\, \bar{x}_p) R^2(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& (\forall \bar{x}_p)(R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \to A^*(\bar{x}_p)) \& B(\bar{y}_p) \longrightarrow \\
&\quad (\exists\, \bar{x}_p)[R^2(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& (R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \to A^*(\bar{x}_p)) \& B(\bar{y}_p)] \longrightarrow_{\text{by (15)}} \\
&\quad (\exists\, \bar{x}_p)[A^*(\bar{x}_p) \& R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)] \longrightarrow \\
&\quad (\exists\, \bar{x}_p)[A^*(\bar{x}_p) \& \bigvee_{i \in I} R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)] \longrightarrow B^*_{\mathrm{DNF}}(\bar{y}_p).
\end{aligned}$$

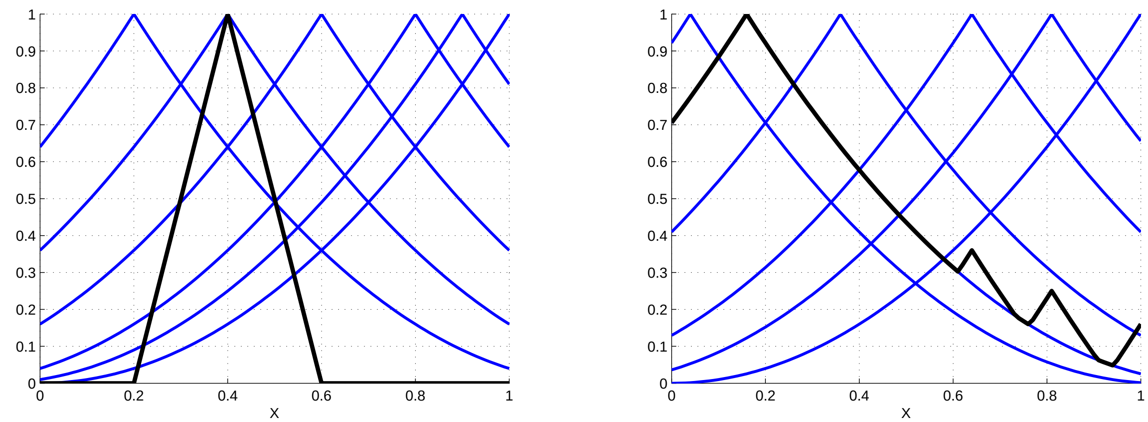
(63) Analogous to (62).

(64)

$$\begin{aligned}
&B^*_{\mathrm{DNF}}(\bar{y}_p) \& [A^*(\bar{x}_p) \subseteq R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p)] \longrightarrow \\
&\quad (\exists\, \bar{x}_p)[A^*(\bar{x}_p) \& \bigvee_{i \in I} R(\bar{\mathbf{c}}_i, \bar{x}_p) \& F(\bar{\mathbf{c}}_i)] \& [A^*(\bar{x}_p) \to R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p)] \longrightarrow \\
&\quad (\exists\, \bar{x}_p) R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& \bigvee_{i \in I} R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i) \longrightarrow \\
&\quad (\exists\, \bar{x}_p)[R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& R(\bar{\mathbf{c}}_i, \bar{x}) \& F(\bar{\mathbf{c}}_i)] \longrightarrow R(\bar{\mathbf{d}}_{i_p}, \bar{y}_p) \& F(\bar{\mathbf{c}}_i).
\end{aligned}$$

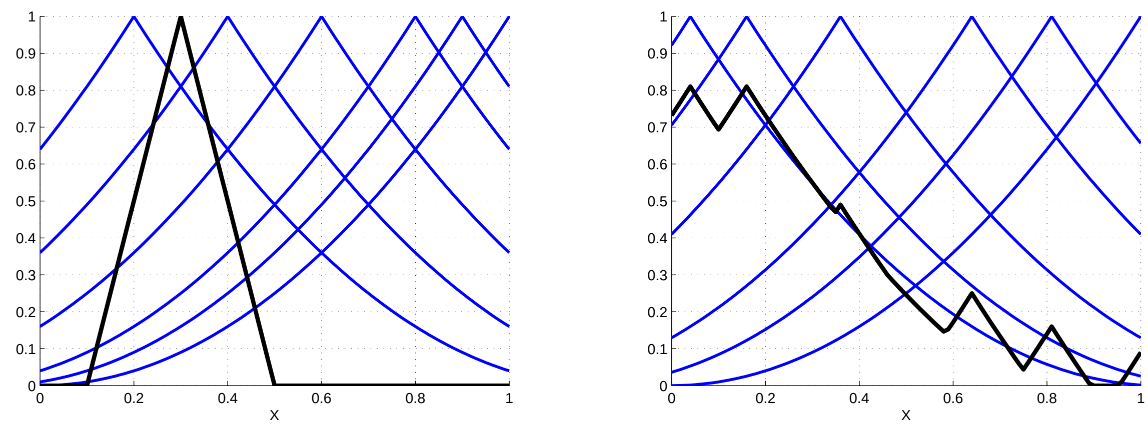
(65) Analogous to (64).

(66) – (67) Obviously by $L_{5(6)} \to L_{1(2)} \wedge L_{3(4)}$. □



(a) Input fuzzy number (black line) assigned to A^* and fuzzy sets assigned to $R_1(\mathbf{c}_{i_1}, x)$, $i \in I$

(b) Output fuzzy set determined by B^*_{DNF} (black line) based on A^* from (a) and fuzzy sets assigned to $R_2(\mathbf{c}_{i_2}, x)$, $i \in I$



(c) Analogous to (a) with the shifted fuzzy number (black line)

(d) Analogous to (b), where A^* is due to (c)

Figure 6: Inference with DNF and triangular fuzzy number.

Remark 3.6 *In order to understand causes of the proved relationships (62)–(67), it is worth to pay attention to the following formulae:*

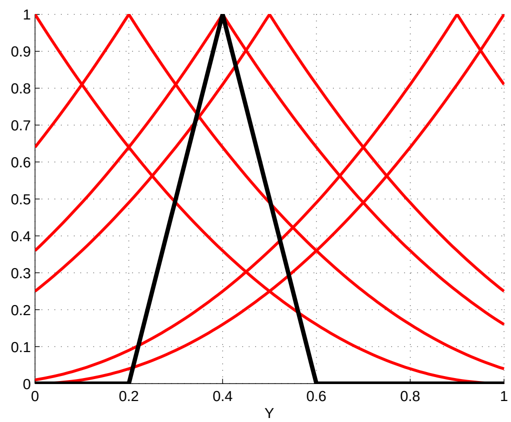
$$(\exists\, \bar{x}_p) R^2(\bar{\mathbf{c}}_{i_p}, \bar{x}_p), \quad (\exists\, \bar{x}_p) R^2(\bar{x}_p, \bar{\mathbf{c}}_{i_p}), \tag{68}$$

$$\bigwedge_{i \neq j} \neg[R(\bar{\mathbf{c}}_{i_p}, \bar{x}_p) \& R(\bar{\mathbf{c}}_{j_p}, \bar{x}_p)], \quad \bigwedge_{i \neq j} \neg[R(\bar{x}_p, \bar{\mathbf{c}}_{i_p}) \& R(\bar{x}_p, \bar{\mathbf{c}}_{j_p})]. \tag{69}$$

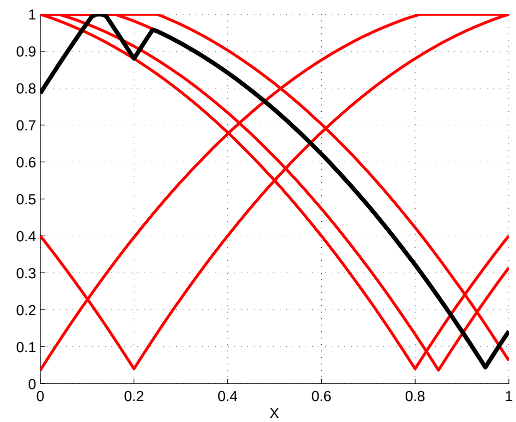
Let $\langle \{\tilde{R}_j\}_{j\in J}, \{c_i\}_{i\in I}\rangle$ be a model for $\langle \{R_j\}_{j\in J}, \{\bar{\mathbf{c}}_i\}_{i\in I}\rangle$, where $j \in \{1, \ldots, n\}$. Then each formula in (68) determines the height of a fuzzy set describing "very close" neighborhood of c_i being a subset of $\tilde{R}(c_i, x)$ or $\tilde{R}(x, c_i)$ that represents this neighborhood. And formula (69) says how much $\tilde{R}_i(c_i, x)$ is disjoint from all the others $\tilde{R}_j(c_j, x)$ (analogously for $\tilde{R}_i(x, c_i)$).

Example 3.7
We will continue with Example 3.3.7 and demonstrate how inferences with DNF *and* CNF *work in this setting.*

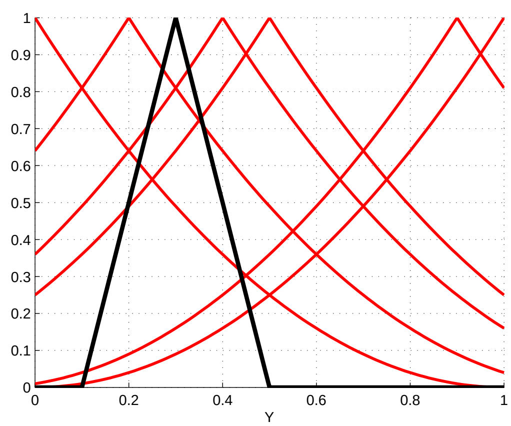
Figures 6 and 7 illustrates Lemma formulae from 3.6. From (a) + (b), we see that whenever A^* is contained in one of the left sides of the rules then the output nearly coincides with the right side of this



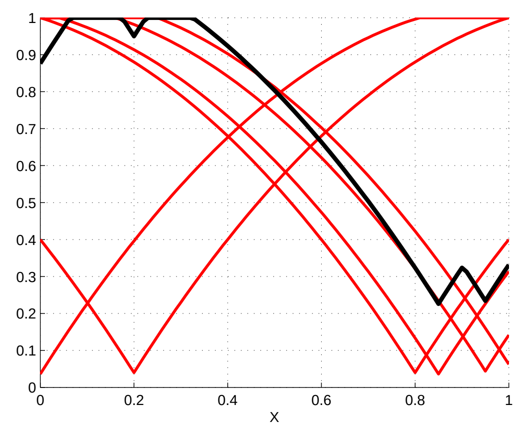
(a) Input fuzzy number (black line) assigned to A^* and fuzzy sets assigned to $R_1(x, \mathbf{c}_{i_1})$, $i \in I$



(b) Output fuzzy set determined by B^*_{CNF} (black line) based on A^* from (a) and fuzzy sets assigned to $R_2(x, \mathbf{c}_{i_2})$, $i \in I$



(c) Analogous to (a) with the shifted fuzzy number (black line)



(d) Analogous to (b), where A^* is due to (c)

Figure 7: Inference with CNF and triangular fuzzy number.

particular rule. While the cases (c) + (d), where A^* is not fully the subset of any of the left sides of the rules and hence also the output fuzzy set reflects this lack of knowledge included in D(C)NF *is transfer to B^*.*

4 Conclusions

All the results shad light on the problems of approximate reasoning based on an imprecise description of some real physical system. Therefore, this work contributes to the theory of fuzzy control systems. Especially the formulae from Section 3.2 provide a quality estimation of fuzzy control system consisting of rule-base interpreted as DNF or CNF and appropriate inference rule. The traditional approach with Mamdani's type of the rules and Zadeh's compositional rule of inference has been extended by graded rules of the type D(C)NF together with the compositional rule of inference $B^*_{\mathrm{D(C)NF}}$. And the dual systems (deductive and abductive) were build in parallel.

Acknowledgement The author would like to thank Libor Běhounek for his helpful comments. This investigation has been partially supported by project MSM6198898701 of the MŠMT ČR.

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