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Model-Theoretic Constructions in Fuzzy Logic with Evaluated Syntax

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Abstract

This paper studies the model theory as generalization of classical model theory in predicate fuzzy logic with evaluated syntax ($\mathrm{Ev_Ł}$). There exist a lot of possibilities how to find the models of fuzzy theories. In this paper we will describe the theoretic constructions of the models in $\mathrm{Ev_Ł}$ where these models are built by using elementary extensions and downward Löwenheim-Skolem-Tarski theorem which will be discussed.

Keywords: Model theory, Łukasiewicz MV-algebra, predicate fuzzy logic with evaluated syntax.

1 Introduction

This paper is a contribution to the model theory of the predicate fuzzy logic with evaluated syntax which is in detail presented in [6] where also its model theory has been founded. It is specific for $\mathrm{Ev_Ł}$ that the set of truth values must be the Łukasiewicz MV-algebra whose support set is the interval of reals $[0, 1]$. Model theory of fuzzy logic with evaluated syntax has been initiated by V. Novák in [6, 5] and further this theory has been elaborated by Murinová and Novák in [2, 3, 4].

The paper is organized as follows: We introduce with a preliminaries-section which is divided in two subsections. The first subsection recounted a syntax of $\mathrm{Ev_Ł}$ with all logical constants and the second subsection contain a semantics for this logic. In the next section we introduce the definitions of submodel, strong submodel, elementary submodel and elementary diagram in $\mathrm{Ev_Ł}$. The last section contains downward Löwenheim-Skolem-Tarski theorem which is constructed as generalization of classical theorem which can be found in [1].

2 Preliminaries

The set of truth values is supposed to form *Łukasiewicz MV-algebra*

$$\mathcal{L}_Ł = \langle [0, 1], \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where $\otimes$ is the operation of Łukasiewicz conjunction defined by $a \otimes b = 0 \vee (a + b - 1)$, $\oplus$ is the operation of Łukasiewicz disjunction defined by $a \oplus b = 1 \wedge (a + b)$ and $\neg$ is the negation operation defined by $\neg a = 1 - a$ for all $a, b \in [0, 1]$. We may introduce lattice operations by

$$a \vee b = (a \otimes \neg b) \oplus b, \quad a \wedge b = (a \oplus \neg b) \otimes b.$$

The following two subsections contains a brief overview of the main concepts and notation of $\mathrm{Ev_Ł}$.

2.1 Syntax

The *language* of $\mathrm{Ev_Ł}$ is denoted by J and it consists of a set of *object variables* $x, y, \ldots$, a set of *object constants* $\mathbf{u_1}, \ldots$, a set of n-ary *functional symbols* $f, g \ldots$, a set of n-ary *predicate symbols* $P, Q \ldots$[1], *implication connective* $\Rightarrow$, *logical constants* $\mathbf{a}$ being names of all the truth values $a \in L$ and the *general quantifier* $\forall$. We write $\top, \bot$ instead of the logical constant $\mathbf{1}, \mathbf{0}$, respectively. Other connectives ($\wedge$, $\vee$, $\neg$, $\Leftrightarrow$,$\&$,∇) are defined in [6]. If L is uncountable then J is uncountable as well.

Remark 1 *By* $\|J\|$ *we mean a* cardinality *of the language* J*.*

The set of all the well-formed formulas for the language J is denoted by F_J. A couple $(^a/A)$ where $a \in L$ and $A \in F_J$ is called an *evaluated formula*.

A fuzzy theory T is a fuzzy set of formulas $T \underset{\sim}{\subset} F_J$ given by the triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle$$

[1]The arity n of each functional symbol as well as of predicate symbol, of course, may vary depending on the given symbol. If unnecessary, we will not explicitly stress this in the sequel.

where $\mathrm{LAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subset} F_J$ is a fuzzy set of special axioms, and R is a set of sound inference rules which contains the rules of modus ponens (r_{MP}), generalization (r_G) and the rule of logical constant introduction (r_{LC}). For the precise definitions see [6].

Given a fuzzy theory T and a formula A, a proof (denoted by w_A) of a formula A is a finite sequence of evaluated formulas which are axioms, or are derived using the inference rules. If w_A is a proof with the value $\mathrm{Val}_T(w_A)$ then $T \vdash_a A$ means that A is *provable in the fuzzy theory T in the degree*

$$a = \bigvee \{\mathrm{Val}(w) \mid w \text{ is a proof of } A \text{ in } T\}.$$

If there exists a proof w_A such that $\mathrm{Val}_T(w_A) = a$ then we say that A is effectively provable in T in the degree a (note that this may not always be the case).

Definition 1 *A fuzzy theory T is* contradictory *if there is a formula A and proofs w_A and $w_{\neg A}$ of A and $\neg A$, respectively, such that*

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0.$$

It is consistent *in the opposite case.*

2.2 Semantics

The semantics is defined by generalization of the classical semantics of predicate logic. A *model* for the language J is

$$\mathcal{V} = \langle V, P_V, \ldots, f_V, \ldots, u_1, \ldots \rangle$$

where V is a set, $P_V \underset{\sim}{\subset} V^n$ are n-ary fuzzy relations assigned to each n-ary predicate symbol P (n depends on P), ..., f_V are ordinary n-ary functions on V assigned to each n-ary functional symbol f, and $u_1, \ldots \in V$ are designated elements which are assigned to each object constant $\mathbf{u_1} \ldots \in J$.

Let $\mathcal{V}$ be a model for the language J. A *cardinal* of the model $\mathcal{V}$ is the cardinal $|V|$. $\mathcal{V}$ is said to be finite, countable or uncountable if $|V|$ is finite, countable or uncountable.

Let $\mathcal{V}$ be a model for the language J. We extend J to the language J_V by new constants being names for all the elements from V, i.e. constants will be denoted by the corresponding bold-face letter, namely

$$J_V = J \cup \{\mathbf{v} \mid v \in V\}. \tag{1}$$

For interpretation of closed terms, formulas and the derived connectives (see [6]).

Let $\mathrm{Var}(J)$ be a set of all variables of the language J and $\mathcal{V}$ be a model for J. Interpretation of a general formula A is using an evaluation

$$e : Var(J) \longrightarrow V.$$

Definition 2 *A formula $A(x_1, \ldots, x_n)$ is* satisfied *in $\mathcal{V}$ by the evaluation e, $e(x_1) = v_1, \ldots e(x_n) = v_n$ in the degree a if $\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v_1}, \ldots, \mathbf{v_n}]) = a$.*

Definition 3 *A formula $A(x_1, \ldots, x_n)$ is* true *in $\mathcal{V}$ in the degree a if*

$$a = \mathcal{V}(A) = \bigwedge \{\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v_1}, \ldots, \mathbf{v_n}]) \mid \text{ for all evaluation } e\}.$$

Definition 4 *Let T be a fuzzy theory over Ev_L and $\mathcal{V}$ be a model for J. We say that the model $\mathcal{V}$ is a* model *of the fuzzy theory T and write $\mathcal{V} \models T$, if $\mathrm{SAx}(A) \leq \mathcal{V}(A)$ holds for all formulas $A \in F_J$.*

3 Elementary extension

Definition 5 *Let $\mathcal{V}$ and $\mathcal{W}$ be models for J. Then we say that $\mathcal{V}$ is a* submodel *of $\mathcal{W}$, in symbols $\mathcal{V} \subset \mathcal{W}$, if $V \subseteq W$ and for every atomic formula $A \in F_J$*

$$\mathcal{V}(A_{x_1,\ldots,x_n}[v_1, \ldots, v_n]) = \mathcal{W}(A_{x_1,\ldots,x_n}[v_1, \ldots, v_n]) \tag{2}$$

holds where $v_1, \ldots, v_n \in V$.

Definition 6 *Let $\mathcal{V}$ and $\mathcal{W}$ be models for J. We say that $\mathcal{V}$ is a* strong submodel *of $\mathcal{W}$, in symbols $\mathcal{V} \leq \mathcal{W}$, if $V \subseteq W$ and for every formula $A \in F_J$*

$$\mathcal{V}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n]) \leq \mathcal{W}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n]) \tag{3}$$

holds where $v_1,\ldots,v_n \in V$.

Remark 2 *If (3) holds only for every atomic formula $A \in F_J$ then we say that $\mathcal{V}$ is a* weak submodel *of $\mathcal{W}$, in symbols $\mathcal{V} \subseteq \mathcal{W}$.*

Definition 7 *Let $\mathcal{V}$ and $\mathcal{W}$ be models for J. Then we say, that $\mathcal{V}$ is an* elementary submodel *of $\mathcal{W}$, in symbols $\mathcal{V} \prec \mathcal{W}$, if $V \subseteq W$, $\mathcal{V} \subset \mathcal{W}$ and for every formula $A \in F_J$*

$$\mathcal{V}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n]) = \mathcal{W}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n]) \tag{4}$$

holds where $v_1,\ldots,v_n \in V$.

When $\mathcal{V}$ is an elementary submodel of $\mathcal{W}$ then we also say that $\mathcal{W}$ is an *elementary extension*.

Lemma 1 *Let $\mathcal{V}$ and $\mathcal{W}$ be models for the same language J. Let $\mathcal{V} \subset W$ and for every $w \in W - V$ there is $v \in V$ such that $\mathcal{W}(A[w]) \leq \mathcal{W}(A[v])$ holds for every formula from F_J. Then the following are equivalent:*

a) $\mathcal{V} \prec \mathcal{W}$

b) For every formula $(\exists x)A(x, x_1,\ldots,x_n)$ and every $v_1,\ldots,v_n \in V$ the following holds: If $\mathcal{W}((\exists x)A[v_1,\ldots,v_n]) = a$ then there is an $v \in V$ such that $\mathcal{W}(A[v, v_1,\ldots,v_n]) = a$.

Proof. First we prove the direction from left to the right. Let $\mathcal{V}$ and $\mathcal{W}$ be models for the same language J, $\mathcal{V} \subset W$ and for every $w \in W - V$ there is $v \in V$ such that $\mathcal{W}(A[w]) \leq \mathcal{W}(A[v])$ holds for every formula from F_J. Whenever A is a formula which has all its free variables among $x_1,\ldots,x_n$ and $v_1,\ldots,v_n \in V$, then we show that

$$\mathcal{V}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n]) = \mathcal{W}(A_{x_1,\ldots,x_n}[v_1,\ldots,v_n])$$

holds for every formula $A \in F_J$ and every n-tuple of elements $v_1,\ldots,v_n \in V$. The induction from the assumption $\mathcal{V} \subset \mathcal{W}$ is easy for atomic formulas, and for going through sentential connectives. In the crucial step of going from $A(x_1,\ldots,x_n)$ to $(\exists x_1)A(x_2,\ldots,x_n)$ we note the following: Given $v_2,\ldots,v_n \in V$, if

$$\mathcal{V}((\exists x_1)A[v_2,\ldots,v_n]) = a$$

then there is an $v_1 \in V$ such that

$$\mathcal{V}(A[v_1, v_2,\ldots,v_n]) = a.$$

By induction, $\mathcal{W}(A[v_1, v_2,\ldots,v_n]) = a$, then

$$\mathcal{W}((\exists x_1)A[v_2,\ldots,v_n]) = \bigvee_{w\in W} (\mathcal{W}(A[w, v_2,\ldots,v_n])) =$$

$$= \bigvee_{v\in V} (\mathcal{V}(A[v, v_2,\ldots,v_n])) \wedge \bigvee_{w\in W-V} (\mathcal{W}(A[w, v_2,\ldots,v_n])).$$

Because $\mathcal{W}(A[w]) \leq \mathcal{W}(A[v])$ holds for every $A \in F_J$ then we can see that $\mathcal{W}((\exists x_1)A[v_2,\ldots,v_n]) = a$ which means that $\mathcal{V} \prec \mathcal{W}$.

The opposite implication is constructed analogously. Suppose that $\mathcal{V} \prec \mathcal{W}$. Then

$$a = \mathcal{W}(((\exists x)A[v_1,\ldots,v_n])) = \bigvee_{w\in W} (\mathcal{W}(A[w, v_1,\ldots,v_n]))$$

$$= \bigvee_{v\in V} (\mathcal{V}(A[v, v_1,\ldots,v_n])) \wedge \bigvee_{w\in W-V} (\mathcal{W}(A[w, v_1,\ldots,v_n])).$$

But from the assumption of Lemma we know that for every $w \in W - V$ there is $v \in V$ such that $\mathcal{W}(A[w]) \leq \mathcal{W}(A[v])$ holds for every formula from F_J. This means that

$$a = \bigvee_{v\in V} (\mathcal{V}(A[v, v_1,\ldots,v_n])) = \bigvee_{v\in V} (\mathcal{W}(A[v, v_1,\ldots,v_n]))$$

which gives the second direction of the lemma. □

3.1 Elementary diagram

Let $\mathcal{V}$ be a model for J. Let J_V be a language which was defined in (1) as extension of J by new constants. We may then expand $\mathcal{V}$ to the model

$$\mathcal{V}_V = (\mathcal{V}, v)_{v \in V}$$

for J_V by interpreting each new constat $\mathbf{v}$ by the element v.

Definition 8 *The* elementary diagram *of $\mathcal{V}$ is the fuzzy theory $Th(\mathcal{V}_V)$ such that*

$$Th(\mathcal{V}_V) = \{{}^{a}/A(x_1, \dots, x_n) \mid \mathcal{V}_V(A(x_1, \dots, x_n)) = a\}$$

holds for every closed formula $A(x_1, \dots, x_n)$ of J_V.

Sometimes instead of $\mathrm{Th}(\mathcal{V}_V)$ we will write Γ_V.

Lemma 2 *Let Γ_V be the elementary diagram of $\mathcal{V}$. If $\mathcal{V} \subset \mathcal{W}$, then $\mathcal{V} \prec \mathcal{W}$ if and only if $(\mathcal{W}, v)_{v \in V} \models \Gamma_V$.*

Proof. Let Γ_V be the elementary diagram of $\mathcal{V}$. Then $(\mathcal{V}, v)_{v \in V} \models \Gamma_V$. The results follows from $(\mathcal{V}, v)_{v \in V} \prec (\mathcal{W}, v)_{v \in V}$. □

4 Downward Löwenheim-Skolem-Tarski theorem

Theorem 1 *Let $\mathcal{V}$ be a model of the cardinality α and let $\|J\| \leq \beta \leq \alpha$. Then $\mathcal{V}$ has an elementary submodel of the cardinality β. Furthermore, given any set $X \subset V$ of the cardinality $\leq \beta$, $\mathcal{V}$ has an elementary submodel of the cardinality β which contains X.*

Proof. We may assume that X has the cardinality β. For each formula $A(x, x_1, \dots, x_n)$ and each n-tuple $v_1, \dots, v_n \in X$ of elements such that

$$\mathcal{V}((\exists x)A[v_1, \dots, v_n]) = a$$

choose an element $w \in V$ such that

$$\mathcal{V}(A[w, v_1, \dots, v_n]) = a.$$

Let X_1 be the set X plus all the w's so chosen. Since $\mid X \mid = \beta$ and $\|J\| \leq \beta$, X_1 has the cardinality β. Now repeat the process countably many times, forming a chain

$$X \subset X_1 \subset X_2 \subset \cdots .$$

Let $W = \bigcup_{n<\omega} X_n$. Each X_n has the cardinality β, so W has the cardinality β. Let $\mathcal{W}$ be the submodel of $\mathcal{V}$ with the universe W. Consider a formula $A(x, x_1, \dots, x_n)$ and an n-tuple $w_1, \dots, w_n \in W$ such that

$$\mathcal{V}((\exists x)A[w_1, \dots, w_n]) = c \geq a.$$

For some $m < \omega$, we have $w_1, \dots, w_n \in X_m$. Then there exists $w \in X_{m+1}$ such that

$$\mathcal{V}(A[w, w_1, \dots, w_n]) = c \geq a.$$

Thus $w \in W$. From this construction we can see that for every $v \in V - W$ there is $w \in W$ such that $\mathcal{V}(A[v]) \leq \mathcal{V}(A[w])$ then by Lemma 1 we have $\mathcal{W} \prec \mathcal{V}$. □

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