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Fuzzy and Non-deterministic Automata*

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Abstract

An existence of an isomorphism between a category of fuzzy automata and a category of chains of non-deterministic automata is proved and some relationships between output fuzzy sets of these systems are investigated.

1 Introduction

A notion of a fuzzy automaton was introduced by several authors, see e.g. [1], [2], [3], [4]. Analogously as in a theory of classical automata there are several definitions of a fuzzy automaton and, hence, there are several categories of these fuzzy automata with possible different properties. Analogously a notion of non-deterministic automaton was introduced which seems to be similar (in some aspect) to the notion of a fuzzy automaton, although a definition of this notion is very far from a fuzzy set theory.

In this paper we will be dealing with two categories - a category of fuzzy automata with fuzzy morphisms and with a category of chains of non-deterministic automata with some specific morphims. The principal result will be a theorem which states that these two categories are isomorphic. Moreover, we will be dealing with output fuzzy sets of these two kinds of automata and we prove that there are very specific relationships between these fuzzy sets. Hence, a conclusion of this paper is a fact that instead of a fuzzy automaton we can deal equivalently with a chain of non-deterministic automata. This idea is then an analogy of a well known relation between fuzzy set A and a chain of its α-level sets $\{A_\alpha : \alpha \in (0,1]\}$.

2 Types of fuzzy automata

To unify various existing definitions of fuzzy automata, the following elementary form of a fuzzy automaton can be identified as a common generic form in a fuzzy automaton theory.

Definition 2.1 *Let $(M,*)$ be a semigroup (called* **sequences of input alpha- bet***). A fuzzy automaton over M is then a system $\mathbf{A} = (S,F)$, where S is a set (called a* **set of states***) and F is a fuzzy set in $S \times M \times S$ (called a* **fuzzy transition function***) which has to satisfy the following conditions*

1. $F(s,1_M,t) = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{if } s \neq t \end{cases}$
2. $F(s,m*n,t) = \bigvee_{z\in S} \left(F(s,m,z) \wedge F(z,n,t)\right)$

for all $s,t \in S$.

This definition is correct since for $m \in M$ we have

$$(\forall s,t \in S)F(s,1_M * m,t) = \bigvee_{u\in S} \left(F(s,1_M,u) \wedge F(u,m,t)\right) =$$
$$= F(s,1_M,s) \wedge F(s,m,t) = F(s,m,t)$$

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and analogously for $F(s, m * 1_M, t)$. We will now recall a notion of a fuzzy commutativity of a diagram which is a fuzzy analogy of a classical commutativity in a category of fuzzy sets. This definition was firstly introduced in [4].

Definition 2.2 *Let A, B be sets and let $f \underset{\sim}{\subset} A, g \underset{\sim}{\subset} B$. Then a diagram*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (0,1] & = & (0,1] \end{array}$$

is called to **fuzzy commute**, *if the following condition holds.*

$$\big(\forall b \in u(A)\big)\, g(b) = \bigvee_{\substack{a \in A \\ u(a)=b}} f(a)$$

Sometimes it is useful to use instead of this condition the following system of two conditions which are equivalent to the original one:

1. $(\forall a \in A) f(a) \leq gu(a)$
2. $\big(\forall b \in u(A)\big) g(b) \leq \bigvee_{u(a)=b} f(a)$

Then the following definition recalls a principal category of fuzzy automata we will be dealing with which was introduced in [4]. Let $(M, *)$ be a semigroup. By $\mathcal{F}_M$ we denote a category of fuzzy automata the objects of which are fuzzy automata over M defined in Definition 2.1 and morphisms between fuzzy automata $\mathbf{A} = (S, F), \mathbf{B} = (T, G)$ are maps $\alpha : S \longrightarrow T$ such that the following diagram fuzzy commutes.

$$\begin{array}{ccc} S \times M \times S & \xrightarrow{\alpha \times 1_M \times \alpha} & T \times M \times T \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ (0,1] & = & (0,1] \end{array} \tag{1}$$

The following simple lemma then states that a "global" notion of a morphism in a category $\mathcal{F}_M$ can be substituted by a system of "local" conditions.

Lemma 2.3 *The diagram* (1) *fuzzy commutes if and only if for any $m \in M$ the following diagram fuzzy commutes:*

$$\begin{array}{ccc} S \times S & \xrightarrow{\alpha \times \alpha} & T \times T \\ {\scriptstyle F_m}\downarrow & & \downarrow{\scriptstyle G_m} \\ (0,1] & = & (0,1] \end{array}$$

where $F_m(s, s') = F(s, m, s')$ and $G_m(t, t') = G(t, m, t')$ for all $s, s' \in S, t, t' \in T$.

The proof of this lemma is trivial and will be omitted.

Sometimes fuzzy automata were defined over a set Λ of input signals only instead over a semigroup (see e.g. [4], [5]). In this case a fuzzy automaton is only a system $\mathbf{A} = (S, F)$, where F is a fuzzy set in $S \times \Lambda \times S$ (without any additional conditions required). Even in this case we can extend this simplified version of a fuzzy automaton onto the previous one, but defined over a free semigroup $(M, *)$ over Λ. In fact, a function F can be extended onto a fuzzy set F^* in $S \times M \times S$ such that for $m = \lambda_1 * \cdots * \lambda_k \in M$ we put

$$(\forall s, t \in S) F^*(s, m, t) = (F_{\lambda_1} \circ \cdots \circ F_{\lambda_k})(s, t),$$

where for any $\lambda \in \Lambda$, a fuzzy relation $F_\lambda \underset{\sim}{\subset} S \times S$ is defined such that $F_\lambda(s, s') = F(s, \lambda, s')$ for any $s, s' \in S$ and "$\circ$" is a composition of fuzzy relations in $S \times S$.

In this case (S, F^*) is a fuzzy automaton over M in an above sense. In fact, let $m = \lambda_1 * \cdots * \lambda_g, n = \rho_1 * \cdots * \rho_h \in M$. Then for any $s, t \in S$ we have

$$F^*(s, m*n, t) = (F_{\lambda_1} \circ \cdots \circ F_{\lambda_g} \circ F_{\rho_1} \circ \cdots \circ F_{\rho_h})(s,t) =$$
$$= \bigvee_{r\in S} \big((F_{\lambda_1} \circ \cdots \circ F_{\lambda_g})(s,r) \wedge (F_{\rho_1} \circ \cdots \circ F_{\rho_h})(r,t)\big) =$$
$$= \bigvee_{r\in S} \big(F^*(s,m,r) \wedge F^*(r,n,t)\big).$$

Recall that a classical automaton over a semigroup $(M, *)$ is a system $\mathbf{B} = (S, d)$ such that $d : M \times S \longrightarrow S$ is a map (called a **transition function**) satisfying the conditions

1. $(\forall s \in S) d(1_M, s) = s$
2. $(\forall m, n \in M, s \in S) d(m*n, s) = d(n, d(m,s))$

Then it is clear that any classical automaton (S, d) over M is a fuzzy automaton (S, F) as well, where we set

$$F(s,m,t) = \begin{cases} 1, & \text{if } t = d(m,s) \\ 0, & \text{otherwise} \end{cases}$$

Further, we will introduce a notion of a general non-deterministic automaton over a semigroup $(M, *)$. Recall that by 2^S we understand the set of all subsets of S.

Definition 2.4 *A system (S, d) is called a non-deterministic automaton over a semigroup $(M, *)$ (in abbreviation :* **ND-automaton***), if S is a set (of* **states***) and $d : S \times M \longrightarrow 2^S$ is a (***non-deterministic transition***) function such that*

1. $(\forall s \in S) d(s, 1_M) = \{s\}$,
2. $(\forall m, n \in M, s \in S) d(s, m*n) = \bigcup_{t\in d(s,m)} d(t, n)$.

As in a case of fuzzy automata, even ND-automata are sometimes defined only for an abstract set Λ instead of a semigroup. Hence, instead of a ND-automaton over a semigroup M we can consider a ND-automaton (S, d) over a set Λ, where $d : S \times \Lambda \longrightarrow 2^S$ is a map. Then the following extension principle holds for these ND-automata.

Proposition 2.5 *Let (S, d) be a ND-automaton over a set Λ. Then a map d can be extended onto a map $\bar{d} : S \times M \longrightarrow 2^S$, where $(M, *)$ is a free semigroup over Λ in such a way that $(S, \bar{d})$ is a ND-automaton over a semigroup $(M, *)$ defined in 2.4.*

Proof. An extension $\bar{d}$ will be defined by induction principle applied on the length $|m|$ of elements $m \in M$ in the following way.

1. Let $m \in M$ be such that $|m| = 1$ (hence, $m \in \Lambda$). Then we set $\bar{d}(s, m) = d(s, m)$ for any $s \in S$.
2. Let $\bar{d}$ be defined for any $m \in M$ such that $|m| \le a \in \mathbb{N}$. Let $m \in M, \lambda \in \Lambda$ be such that $|m| \le a$. Then we set
$$(\forall s \in S)\bar{d}(s, m*\lambda) = \bigcup_{t\in \bar{d}(s,m)} d(t, \lambda).$$
3. $(\forall s \in S)\bar{d}(s, 1_M) = \{s\}$.

We show that then $\bar{d}$ satisfies the condition

$$(\forall s \in S)(\forall m, n \in M)\bar{d}(s, m*n) = \bigcup_{t\in \bar{d}(s,m)} \bar{d}(t, n).$$

The proof will be done by induction principle applied on the length $a = |m * n|$. For $a = 1$ we have either $m * n = m * 1_M$ or $m * n = 1_M * n$, where $m, n \in \Lambda$. Then in the first case we have

$$\bigcup_{t \in \bar{d}(s,m)} \bar{d}(t, 1_M) = \bigcup_{t \in d(s,m)} \{t\} = d(s,m) = \bar{d}(s, m * 1_M)$$

and in the other case we have

$$\bigcup_{t \in \bar{d}(s,1_M)} \bar{d}(t,m) = \bigcup_{t=s} d(t,m) = \bar{d}(s,m) = \bar{d}(s, 1_M * m).$$

Let us assume that this proposition holds for any $m, n \in M$ such that $|m * n| \leq a$. Let $m, n \in M$ be such that $|m * n| = a + 1$, i.e. there exist $\lambda \in \Lambda, n' \in M$ such that $m * n = m * n' * \lambda$. Then according the induction assumption we have

$$\bar{d}(s, m * n) = \bar{d}(s, (m * n') * \lambda) =$$

$$= \bigcup_{t \in \bar{d}(s, m*n')} d(t, \lambda) = \bigcup_{t \in \cup_{v \in \bar{d}(s,m)} \bar{d}(v,n')} d(t, \lambda) \tag{2}$$

Let $x \in \bar{d}(s, m * n)$. According to (2) there exist $v \in \bar{d}(s,m)$ and $t \in \bar{d}(v, n')$ such that $x \in d(t, \lambda)$. Then according to the induction assumption we have

$$x \in \bigcup_{t \in \bar{d}(v,n')} d(t,\lambda) = \bar{d}(v, n' * \lambda) = \bar{d}(v,n) \subseteq \bigcup_{r \in \bar{d}(s,m)} \bar{d}(r,n).$$

Conversely, let $x \in \bigcup_{t \in \bar{d}(s,m)} \bar{d}(t,n)$. Then there exists $t \in \bar{d}(s,m)$ such that according to the induction assumption we have

$$x \in \bar{d}(t,n) = \bar{d}(t, n' * \lambda) = \bigcup_{r \in \bar{d}(t,n')} \bar{d}(r, \lambda).$$

Then according to (2) we have $x \in \bar{d}(s, m * n)$. □

We now introduce a category $\mathcal{ND}_M$ of ND-automaton the object of which are ND-automata over a semigroup M and $\alpha : (S,d) \longrightarrow (T,f)$ is morphism in $\mathcal{ND}_M$ if $\alpha : S \longrightarrow T$ is a map such that

$$(\forall s \in S)(\forall m \in M)\alpha\Big(\bigcup_{\substack{s' \in S \\ \alpha(s')=\alpha(s)}} d(s',m)\Big) = f\big(\alpha(s), m\big) \cap \alpha(S).$$

Proposition 2.6 *There exist a functor $\mathbb{A} : \mathcal{ND}_M \longrightarrow \mathcal{F}_M$.*

Proof. Let $\mathbf{A} = (S,d) \in \mathcal{ND}_M$. We set $\mathbb{A}(\mathbf{A}) = (S,F)$, where $F \subseteq S \times M \times S$ be defined such that

$$(\forall s,t \in S)(\forall m \in M) F(s,m,t) = \begin{cases} 1, & \text{if } t \in d(s,m) \\ 0, & \text{if } t \notin d(s,m) \end{cases}$$

Then $\mathbb{A}(\mathbf{A}) \in \mathcal{F}_M$. In fact, a condition (1) from a definition 2.1 is clearly satisfied. By a simple computation can be verified that also the condition (2) is satisfied. Now, let $\alpha : \mathbf{A} = (S,d) \longrightarrow \mathbf{B} = (T,f)$ be a morphism in $\mathcal{ND}_M$. Then for $\mathbb{A}(\mathbf{A}) = (S,F), \mathbb{A}(\mathbf{B}) = (T,G)$ also α is a morphism from (S,F) into (T,G) in $\mathcal{F}_M$. To prove it, we have to verify that for any $t, t' \in \alpha(S)$,

$$G_m(t,t') = \bigvee_{\substack{(s,s') \in S \times S \\ \alpha(s)=t, \alpha(s')=t'}} F_m(s,s'). \tag{3}$$

Let $G_m(t,t') = 1$. Then $t' \in f(t,m) \cap \alpha(S)$, $t = \alpha(s)$, and there exists $s'' \in d(r,m)$, $\alpha(r) = \alpha(s)$, such that $\alpha(s'') = t'$. Then $F_m(r, s'') = 1$, $\alpha(r) = \alpha(s) = t$,$\alpha(s'') = t'$ and it follows that (3) holds. For the case $G_m(t,t') = 0$ the proof can be done analogously. □

Using a generic form of a fuzzy automaton which was introduced in 2.1, we can define a fuzzy automaton with initial and final fuzzy states.

Definition 2.7 *A fuzzy automaton over a semigroup M with initial and final fuzzy states is a system (S, F, G, P) such that*

1. *(S, F) is a fuzzy automaton over M,*
2. *G and P are fuzzy sets in S called an* **initial** *and* **final fuzzy states***, respectively.*

By $\mathcal{F}_M^0$ we denote the category of fuzzy automata over a semigroup M with initial and final fuzzy states (sometimes we call these systems shortly fuzzy automata as well) and with α as a morphism between automaton (S_1, F_1, G_1, P_1) and automaton (S_2, F_2, G_2, P_2), if

1. α is a morphism between (S_1, F_1) and (S_2, F_2) in a category $\mathcal{F}_M$,
2. the following diagramms fuzzy commute

$$\begin{array}{ccccccc} S_1 & \xrightarrow{\alpha} & S_2 & S_1 & \xrightarrow{\alpha} & S_2 \\ {\scriptstyle G_1}\downarrow & & {\scriptstyle G_2}\downarrow & \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ (0,1] & = & (0,1] & (0,1] & = & (0,1] \end{array}$$

An analogical form of these automata we can introduce for ND-automata as well.

Definition 2.8 *A non-deterministic automaton over a semigroup M with initial and final sets of states is a system (S, d, P, G) such that*

1. *(S, d) is a ND-automaton over M,*
2. *$P, G \subseteq S$.*

By $\mathcal{ND}_M^0$ we denote a category of these ND-automata with initial and final sets of states where $\alpha : (S_1, d_1, P_1, G_1) \longrightarrow (S_2, d_2, P_2, G_2)$ is a morphism if

1. $\alpha : (S_1, d_1) \longrightarrow (S_2, d_2)$ is a morphism in $\mathcal{ND}_M$,
2. $\alpha(P_1) = P_2 \cap \alpha(S_1), \alpha(G_1) = G_2 \cap \alpha(S_2)$.

Recall that any fuzzy automaton $\mathbf{A} = (S, F, P, G) \in \mathcal{F}_M^0$ defines an output fuzzy set $\Omega(\mathbf{A}) \underset{\sim}{\subset} M$ such that

$$(\forall m \in M)\Omega(\mathbf{A})(m) = P \circ F_m \circ G.$$

Analogously, any ND-automaton $\mathbf{B} = (S, d, P, G) \in \mathcal{ND}_M^0$ defines an output set $\Psi(\mathbf{B}) \subseteq M$ such that

$$m \in \Psi(\mathbf{B}) \Longleftrightarrow (\exists s \in P) d(s, m) \cap G \neq \emptyset.$$

Proposition 2.9 *There exist a functor $\mathbb{B} : \mathcal{ND}_M^0 \longrightarrow \mathcal{F}_M^0$ such that if $\mathbf{A} \in \mathcal{ND}_M^0$, then $\Psi(\mathbf{A}) = \Omega(\mathbb{B}(\mathbf{A}))$.*

The proof follows directly from a Proposition 2.6 and from definitions by simple computation and it will be omitted.

3 Fuzzy automata and nets of ND-automata

Although it is not necessary for proofs of all propositions in this section, we will consider all automata in this section to be finite, i.e. their sets of states are finite. We begin this section with an existence of a functor which is a partial converse of a functor $\mathbb{B}$ from proposition 2.9. The following lemma is then a non-deterministic analogy of a [4]; Theorem 2.4.

Lemma 3.1 *Let $\epsilon \in (0,1]$.*

1. *There exists a functor $\mathbb{C}_\epsilon : \mathcal{F}_M \longrightarrow \mathcal{ND}_M$.*
2. *There exists a functor $\mathbb{D}_\epsilon : \mathcal{F}^0_M \longrightarrow \mathcal{ND}^0_M$ such that*

$$(\forall \mathbf{A} \in \mathcal{F}^0_M)(\forall m \in M)\Omega(\mathbf{A})(m) \geq \epsilon \Longleftrightarrow m \in \Psi(\mathbb{D}_\epsilon(\mathbf{A})).$$

Proof. (1) For $\mathbf{A} = (S, F) \in \mathcal{F}_M$ and $\epsilon > 0$ we set $\mathbb{C}_\epsilon(\mathbf{A}) = (S, d_\epsilon)$, where $d_\epsilon(s,m) = \{t \in S : F(s,m,t) \geq \epsilon\}$. Then $\mathbb{C}_\epsilon(\mathbf{A}) \in \mathcal{ND}^0_M$. In fact, we have

$$(\forall s \in S) d_\epsilon(s, 1_M) = \{t \in S : F(s, 1_M, t) \geq \epsilon > 0\} = \{s\}.$$

By using only a simple computation and a fact that S is finite we can prove that

$$(\forall m, n \in M)(\forall s \in S) d_\epsilon(s, m * n) = \bigcup_{t \in d_\epsilon(s,m)} d_\epsilon(t, n).$$

We show that $\mathbb{C}_\epsilon$ is a functor. Let α be a morphism (in $\mathcal{F}_M$) from $\mathbf{A}^1 = (S_1, F^1)$ into $\mathbf{A}^2 = (S_2, F^2)$ and let $\mathbb{C}_\epsilon(\mathbf{A}^i) = (S_i, d^i_\epsilon)$. Then we have

$$(\forall s, s' \in S_1)(\forall m \in M) F^2\big(\alpha(s), m, \alpha(s')\big) = \bigvee_{\substack{\alpha(t)=\alpha(s) \\ \alpha(t')=\alpha(s')}} F^1(t, m, t').$$

Using this equation and a fact that S_1 is finite, by simple computation we can prove that $(\forall s \in S_1)(\forall m \in M)\alpha\big(\bigcup_{\alpha(s')=\alpha(s)} d^1_\epsilon(s', m)\big) = d^2_\epsilon(\alpha(s), m) \cap \alpha(S_1)$. Hence, α is a morphism in $\mathcal{ND}_M$.

(2) For $\mathbf{A} = (S, F, P, G) \in \mathcal{F}^0_M$ and $\epsilon > 0$ we set $\mathbb{D}_\epsilon(\mathbf{A}) = (S, d_\epsilon, P_\epsilon, G_\epsilon)$, where $(S, d_\epsilon) = \mathbb{C}_\epsilon(S, F)$ and P_ϵ and G_ϵ are ϵ-level sets of corresponding fuzzy sets. Then $\mathbb{D}_\epsilon(\mathbf{A}) \in \mathcal{ND}^0_M$ as follows from a part (1).

The required relationship between $\Omega(\mathbf{A})$ and $\Psi(\mathbb{D}_\epsilon(\mathbf{A}))$ follows directly from the relation

$$(\forall m \in M) P \circ F_m \circ G \geq \epsilon \Leftrightarrow (\exists s \in P_\epsilon)(\exists t \in G_\epsilon) t \in G_\epsilon \cap d_\epsilon(s, m).$$

Finally, we show that $\mathbb{D}_\epsilon$ is a functor. Let α be a morphism (in $\mathcal{F}^0_M$) from $\mathbf{A}^1 = (S_1, F^1, P^1, G^1)$ into $\mathbf{A}^2 = (S_2, F^2, P^2, G^2)$ and let $\mathbb{D}_\epsilon(\mathbf{A}^i) = (S_i, d^i_\epsilon, P^i_\epsilon, G^i_\epsilon)$. Then from a part (1) it follows that α is a morphism from (S_1, d^1_ϵ) to (S_2, d^2_ϵ) and we have

$$G^2(\alpha(s)) = \bigvee_{\alpha(t)=\alpha(s)} G^1(t), \quad P^2(\alpha(s)) = \bigvee_{\alpha(t)=\alpha(s)} P^1(t).$$

Using these equations and a fact that S_1 is finite, by simple computation we can prove that

1. $\alpha(P^1_\epsilon) = P^2 \cap \alpha(S_1)$,
2. $\alpha(G^1_\epsilon) = G^2 \cap \alpha(S_1)$.

Hence, α is a morphism in $\mathcal{ND}^0_M$. □

A functor $\mathbb{D}_\epsilon$ is only a partial inverse of a functor $\mathbb{B}$ since from $\mathbb{D}_\epsilon(\mathbf{A})$ we are not able to reconstruct completely an original fuzzy automaton $\mathbf{A}$. On the other hand, it seem hopefull that we would be able to reconstruct $\mathbf{A}$ from a set $\{\mathbb{D}_\epsilon(\mathbf{A}) : \epsilon \in (0,1]\}$ as we are able to reconstruct a fuzzy set A from its ϵ-level sets A_ϵ. In fact, the following simple lemma holds.

Lemma 3.2 *Let S be (in general nonfinite) set and let $A \subsetsim S$. Let $s \in S$ and let $\alpha = \bigvee_{s \in A_\epsilon} \epsilon$, where $A_\epsilon = \{x \in S : A(x) \geq \epsilon\}$. Then $s \in A_\alpha$ and $A(s) = \alpha$.*

The proof is straightforward and it will be omitted.

Hence, we introduce another categories $\mathcal{CND}_M$ and $\mathcal{CND}^0_M$ the objects of which will be some chains of ND-automata with some special morphisms between such chains.

Definition 3.3 *Let $\mathcal{CND}_M$ be a category with objects all chains $\mathbf{C} = \{(S, d_\epsilon) : \epsilon \in (0,1]\}$, where*

1. $(S, d_\epsilon) \in \mathcal{ND}_M$ *for any* $\epsilon \in (0,1]$,
2. $(\forall \epsilon, \tau \in (0,1])(\forall s \in S)(\forall m \in M)\epsilon \le \tau \implies d_\tau(s,m) \subseteq d_\epsilon(s,m)$,
3. $(\forall s, t \in S)(\forall m \in M)\gamma = \bigvee_{t \in d_\epsilon(s,m)} \epsilon \implies t \in d_\gamma(s,m)$.

If $\mathbf{C}^i = \{(S_i, d^i_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}_M$, $i = 1,2$, then $\alpha : \mathbf{C}^1 \longrightarrow \mathbf{C}^2$ is a morphism if $\alpha : S_1 \longrightarrow S_2$ is a map such that it is a morphim from (S_1, d^1_ϵ) into (S_2, d^2_ϵ) in $\mathcal{ND}_M$ for any ϵ.

As we have mentioned above, the category $\mathcal{CND}_M$ could then serve as a reconstruction category for fuzzy automata. To prove this assertion we construct another two mutually inverse functors between categories $\mathcal{F}_M$ and $\mathcal{CND}_M$ which then enable us to construct any fuzzy automaton from a sequence of ND-automata.

Theorem 3.4 *There exist functors $\mathbb{J} : \mathcal{F}_M \longrightarrow \mathcal{CND}_M$ and $\mathbb{K} : \mathcal{CND}_M \longrightarrow \mathcal{F}_M$ such that $\mathbb{J}$ and $\mathbb{K}$ are mutually inverse isomorphisms. Hence, categories $\mathcal{F}_M$ and $\mathcal{CND}_M$ are isomorphic.*

Proof. Let $\mathbf{A} = (S, F) \in \mathcal{F}_M$. We set

$$\mathbb{J}(\mathbf{A}) = \big\{\mathbb{C}_\epsilon(\mathbf{A}) : \epsilon \in (0,1]\big\}.$$

If for $\mathbf{A} \in \mathcal{F}_M$ and $\epsilon > 0$ we have $\mathbb{C}_\epsilon(\mathbf{A}) = (S, d_\epsilon)$, then it is clear that $d_\tau \subseteq d_\epsilon$ for $\epsilon \le \tau$ and the condition (2) from 3.3 holds. Moreover, since $A = F(s,m,-) \underset{\sim}{\subseteq} S$ is a fuzzy set such that $A_\epsilon = d_\epsilon(s,m)$, then the condition (3) from 3.3 follows directly from a lemma 3.2. Hence, $\mathbb{J}(\mathbf{A}) \in \mathcal{CND}_M$.

Since $\mathbb{C}_\epsilon$ is a functor, it is clear that if α is a morphism in $\mathcal{F}_M$ then α is a morphism in $\mathcal{CND}_M$ as well. Hence, $\mathbb{J}$ is a functor.

Conversely, for $\mathbf{R} = \{(S, d_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}_M$ we put $\mathbb{K}(\mathbf{R}) = (S, F)$, where

$$(\forall s, t \in S)(\forall m \in M)F(s,m,t) = \bigvee_{t \in d_\epsilon(s,m)} \epsilon.$$

Then $(S, F) \in \mathcal{F}_M$. In fact, we have

$$F(s, 1_M, t) = \bigvee_{t \in d_\epsilon(s,1_M)} \epsilon = \bigvee_{t \in \{s\}} \epsilon = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{if } s \ne t \end{cases}$$

Moreover, we prove that for any $s, t \in S, m \in M$,

$$(a =) \quad F(s, m*n, t) = \bigvee_{r \in S} \big(F(s,m,r) \wedge F(r,n,t)\big) \quad (= b).$$

In fact, let ϵ be such that $t \in d_\epsilon(s, m*n)$. According to 2.4(2), there exists $r \in d_\epsilon(s,m)$ such that $t \in d_\epsilon(r,n)$. Then $\epsilon \le F(s,m,r) \wedge F(r,n,t) \le b$ and it follows that $a \le b$. Conversely, let $r \in S$ and let for example $\alpha = F(s,m,r) \le F(r,n,t) = \beta$. Then according to 3.3(3), we have $r \in d_\alpha(s,m)$ and $t \in d_\beta(r,n)$. Since $\alpha \le \beta$, according to 3.3(2), we have $t \in d_\beta(r,n) \subseteq d_\alpha(r,n)$ and it follows that $t \in \bigcup_{r \in d_\alpha(s,m)} d_\alpha(r,n) = d_\alpha(s, m*n)$ according to 2.4(2). Hence, $F(s, m*n, t) = \bigvee_{t \in d_\epsilon(s,m*n)} \epsilon \ge \alpha = F(s,m,r) \wedge F(r,n,t)$ and it follows that $a \ge b$. Therefore, $(S, F) \in \mathcal{F}_M$.

Further, we show that $\mathbb{K}$ is a functor. In fact, let $\mathbf{R}_i = \{(S_i, d^i_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}_M$, $i = 1,2$ and let α be a morphism $\mathbf{R}_1 \longrightarrow \mathbf{R}_2$ in $\mathcal{CND}_M$. Then for $\mathbb{K}(\mathbf{R}_i) = (S_i, F^i)$, $\alpha : (S_1, F^1) \longrightarrow (S_2, F^2)$ is a morphism in $\mathcal{F}_M$, as well. In fact, according to the definition of a category $\mathcal{F}_M$ we have to prove that for any $m \in M, s, t \in S$, $F^2(\alpha(s), m, \alpha(t)) = \bigvee_{\substack{\alpha(s') = \alpha(s) \\ \alpha(t') = \alpha(t)}} F^1(s', m, t')$ holds. Let us denote

$$A = \big\{\tau : \alpha(t) \in d^2_\tau\big(\alpha(s), m\big)\big\},$$
$$B = \big\{\epsilon : (\exists t', s' \in S_1)\alpha(t') = \alpha(t), \alpha(s') = \alpha(s), t' \in d^1_\epsilon(s', m)\big\}.$$

Let $\tau \in A$. Since α is a morphism in $\mathcal{CND}_M$, we have

$$\alpha(t) \in d^2_\tau(\alpha(s), m) \cap \alpha(S_1) = \alpha\Big(\bigcup_{\alpha(s')=\alpha(s)} d^1_\tau(s', m)\Big)$$

and it follows that $\tau \in B$. Conversely, let $\epsilon \in B$, then for some $s', t' \in S_1$ such that $\alpha(s') = \alpha(s), \alpha(t') = \alpha(t)$ we have $\alpha(t) \in \alpha(d^1_\epsilon(s', m)) \subseteq d^2_\epsilon(\alpha(s), m)$ and it follows that $\epsilon \in A$. Therefore, the required equality holds. Hence, $\mathbb{K}$ is a functor.

Finally, we show that $\mathbb{J}$ and $\mathbb{K}$ are mutually inverse. Let $\mathbf{R} = \{(S, d_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}_M$ and let $\mathbb{K}(\mathbf{R}) = (S, F)$, $\mathbb{J}((S,F)) = \{(S, h_\epsilon) : \epsilon \in (0,1]\}$. Then $h_\epsilon = d_\epsilon$. In fact, let $t \in h_\epsilon(s,m)$. Then $\epsilon \le \alpha = F(s,m,t) = \bigvee_{t \in d_\tau(s,m)} \tau$ and according to 3.3, we have $t \in d_\alpha(s,m) \subseteq d_\epsilon(s,m)$. Conversely, for $t \in d_\epsilon(s,m)$ we have $F(s,m,t) = \bigvee_{t\in d_\tau(s,m)} \tau \ge \epsilon$ and it follows that $t \in h_\epsilon(s,m)$. Hence $\mathbb{J}(\mathbb{K}(\mathbf{R})) = \mathbf{R}$.

Conversely, let $\mathbf{A} = (S, F) \in \mathcal{F}_M$ and let $\mathbb{J}(\mathbf{A}) = \{(S, d_\epsilon) : \epsilon \in (0,1]\}$, $\mathbb{K}(\mathbb{J}(\mathbf{A})) = (S, G)$. Then $G = F$. In fact, let $\alpha = G(s,m,t)$, then according to 3.3, $t \in d_\alpha(s,m)$ and it follows that $F(s,m,t) \ge \alpha$. Conversely, let $\beta = F(s,m,t)$. Then $t \in d_\beta(s,m)$ and it follows that $G(s,m,t) \ge \beta$. Hence, $\mathbb{K}(\mathbb{J}(\mathbf{A})) = \mathbf{A}$. □

A category $\mathcal{CND}^0_M$ will be now defined as follows.

Definition 3.5 *By $\mathcal{CND}^0_M$ we denote a category the object of which are chains $\{(S, d_\epsilon, P_\epsilon, G_\epsilon) : \epsilon \in (0,1]\}$ such that*

1. $\{(S, d_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}_M$,
2. $(\forall \epsilon > 0)(S, d_\epsilon, P_\epsilon, G_\epsilon) \in \mathcal{ND}^0_M$,
3. $(\forall \epsilon, \tau > 0)(\forall s \in S)\epsilon \le \tau \implies G_\tau \subseteq G_\epsilon, P_\tau \subseteq P_\epsilon$,
4. $(\forall s \in S)\alpha = \bigvee_{s \in P_\epsilon} \epsilon \implies s \in P_\alpha$
5. $(\forall s \in S)\beta = \bigvee_{s\in G_\tau} \tau \implies s \in G_\beta$.

If $\mathbf{C}^i = \{(S_i, d^i_\epsilon, P^i_\epsilon, G^i_\epsilon) : \epsilon > 0\} \in \mathcal{CND}^0_M, i = 1,2$, then $\alpha : \mathbf{C}^1 \longrightarrow \mathbf{C}^2$ is a morphism if

1. $\alpha : \{(S_1, d^1_\epsilon) : \epsilon > 0\} \longrightarrow \{(S_2, d^2_\epsilon) : \epsilon > 0\}$ *is a morphism in* $\mathcal{CND}_M$,
2. $(\forall \epsilon > 0)\alpha(P^1_\epsilon) = P^2_\epsilon \cap \alpha(S_1), \alpha(G^1_\epsilon) = G^2_\epsilon \cap \alpha(S_1)$.

Analogously as we did for ND-automata with initial and final states, we can define an output fuzzy set for automata from a category $\mathcal{CND}^0_M$. In fact, let $\mathbf{R} = \{\mathbf{R}_\epsilon = (S, d_\epsilon, P_\epsilon, G_\epsilon) : \epsilon \in (0,1]\} \in \mathcal{CND}^0_M$. Then $\mathbf{R}$ defines an output fuzzy set $\Phi(\mathbf{R}) \underset{\sim}{\subset} M$ such that

$$(\forall m \in M)\Phi(\mathbf{R})(m) = \bigvee\{\epsilon : m \in \Psi(\mathbf{R}_\epsilon)\}.$$

Hence a "global" output fuzzy set $\Phi(\mathbf{R})$ is defined by "local" output sets $\Psi(\mathbf{R}_\epsilon)$, $\epsilon \ge 0$.

Then the following theorem is an analogy of Theorem 3.4 and Proposition 2.9 for automata with with initial and final states.

Theorem 3.6 *There exist functors $\mathbb{U} : \mathcal{F}^0_M \longrightarrow \mathcal{CND}^0_M$ and $\mathbb{V} : \mathcal{CND}^0_M \longrightarrow \mathcal{F}^0_M$ such that*

1. *Functors $\mathbb{U}$ and $\mathbb{V}$ are mutually inverse isomorphisms. Hence, categories $\mathcal{F}^0_M$ and $\mathcal{CND}^0_M$ are isomorphic.*
2. *For any $\mathbf{R} \in \mathcal{CND}^0_M$ we have $\Phi(\mathbf{R}) = \Omega(\mathbb{V}(\mathbf{R}))$.*

Proof. (1) A functor $\mathbb{U}$ is defined as follows. Let $\mathbf{A} = (S,F,P,G) \in \mathcal{F}^0_M$. Then we set $\mathbb{U}(\mathbf{A}) = \{\mathbb{D}_\epsilon(\mathbf{A}) : \epsilon > 0\}$. Then according to the lemma 3.2 and lemma 3.1, $\mathbb{U}(\mathbf{A}) \in \mathcal{CND}^0_M$. Let $\alpha : \mathbf{A}_1 = (S_1,F^1,P^1,G^1) \longrightarrow \mathbf{A}_2 = (S_2,F^2,P^2,G^2)$ be a morphism in $\mathcal{F}^0_M$. Then for any $\tau > 0$ we have $\alpha(P^1_\tau) = P^2_\tau \cap \alpha(S_1), \alpha(G^1_\tau) = G^2_\tau \cap \alpha(S_1)$ as follows directly from the fact that diagramms

$$\begin{array}{ccccccc} S_1 & \xrightarrow{\alpha} & S_2 & S_1 & \xrightarrow{\alpha} & S_2 \\ {\scriptstyle P^1}\downarrow & & {\scriptstyle P^2}\downarrow & \downarrow{\scriptstyle G^1} & & \downarrow{\scriptstyle G^2} \\ (0,1] & = & (0,1] & (0,1] & = & (0,1] \end{array}$$

fuzzy commute and from a fact that S_1 is a finite set. Hence, according to 3.1, $\alpha : \mathbb{U}(\mathbf{A}_1) \longrightarrow \mathbb{U}(\mathbf{A}_2)$ is a morphism in a category $\mathcal{CND}^0_M$ and $\mathbb{U}$ is a functor.

Further, let $\mathbf{R} = \{(S, D_\epsilon, P_\epsilon, G_\epsilon) : \epsilon > 0\} \in \mathcal{CND}^0_M$ and let $(S,F) = \mathbb{K}(\{(S,d_\epsilon) : \epsilon > 0\})$. Let fuzzy sets P and $G \underset{\sim}{\subset} S$ be defined such that $P(s) = \bigvee\{\tau : s \in P_\tau\}, G(s) = \bigvee\{\tau : s \in G_\tau\}$ for all $s \in S$. Then we put $\mathbb{V}(\mathbf{R}) = (S,F,P,G)$. From 3.5, it follows that $\mathbb{V}(\mathbf{R}) \in \mathcal{F}^0_M$. Further, let $\mathbf{R}_i = \{(S_i, d^i_\epsilon, P^i_\epsilon, G^i_\epsilon) : \epsilon > 0\}, i = 1,2$, and let $\alpha : \mathbf{R}_1 \longrightarrow \mathbf{R}_2$ be a morphism in $\mathcal{CND}^0_M$. Then $\alpha : (S_1,F^1) \longrightarrow (S_2,F^2)$ is a morphism in $\mathcal{F}^0_M$, where $(S_i,F^i) = \mathbb{K}(\{(S_i,d^i_\epsilon) : \epsilon > 0\})$. Moreover, if $\mathbb{V}(\mathbf{R}_i) = (S_i,F^i,P^i,G^i)$, then

$$(\forall s \in S_1) P^2(\alpha(s)) = \bigvee_{\alpha(t)=\alpha(s)} P^1(t), \quad G^2(\alpha(s)) = \bigvee_{\alpha(t)=\alpha(s)} G^1(t)$$

as it follows from a fact that α is a morphism in $\mathcal{CND}^0_M$. Hence, $\mathbb{V}$ is a functor. From 3.5, and by a simple computation we can also obtain that $\mathbb{V}$ and $\mathbb{U}$ are mutually inverse functors.

(2) Let $\mathbf{R} = \{(S,d_\epsilon,P_\epsilon,G_\epsilon) : \epsilon > 0\} \in \mathcal{CND}^0_M$ and let $\mathbb{V}(\mathbf{R}) = (S,F,P,G)$, where according to the part (1), we have

$$(\forall m \in M)(\forall s,t \in S) F(s,m,t) = \bigvee_{t \in d_\epsilon(s,m)} \epsilon, \quad P(s) = \bigvee_{s \in P_\tau} \tau, \quad G(t) = \bigvee_{t \in G_\tau} \tau.$$

Let us denote for any $m \in M$

$$\begin{aligned} A_0(m) &= \{\tau \geq 0 : (\exists s,t \in S)\tau = P(s) \wedge F(s,m,t) \wedge G(t)\} \\ A(m) &= \{\tau \geq 0 : (\exists s,t \in S)\tau \leq P(s) \wedge F(s,m,t) \wedge G(t)\} \\ B(m) &= \{\epsilon \geq 0 : (\exists s \in P_\epsilon) d_\epsilon(s,m) \cap G_\epsilon \neq \emptyset\}. \end{aligned}$$

Then from definitions of Ω and Φ it follows that for any $m \in M$ we have

$$\Phi(\mathbf{R})(m) = \bigvee_{\epsilon \in B(m)} \epsilon, \quad \Omega(\mathbb{V}(\mathbf{R}))(m) = \bigvee_{\tau \in A_0(m)} \tau = \bigvee_{\tau \in A(m)} \tau.$$

Let $\epsilon \in B(m)$. Then there exist $s \in P_\epsilon$ and $t \in d_\epsilon(s,m) \cap G_\epsilon$ and it follows that $P(s) \wedge F(s,m,t) \wedge G(t) \geq \epsilon$. Hence $\epsilon \in A(m)$.

Conversely, let $\tau \in A_0(m)$, i.e $\tau = P(s) \wedge F(s,m,t) \wedge G(t)$ for some $s,t \in S$. Then $\tau \leq P(s) = \bigvee_{s \in P_\epsilon} \epsilon$. If $\tau = \bigvee_{s \in P_\epsilon} \epsilon$, then according to 3.5, we have $s \in P_\tau$. If $\tau < \bigvee_{s \in P_\epsilon} \epsilon$, then there exists $\epsilon > \tau$ such that $s \in P_\epsilon \subseteq P_\tau$. Hence, $s \in P_\tau$. Analogously we can prove that $t \in G_\tau$. Since $\alpha = F(s,m,t) = \bigvee_{t \in d_\epsilon(s,m)} \epsilon \geq \tau$, we have $t \in d_\alpha(s,m) \subseteq d_\tau(s,m)$ and it follows that $t \in d_\tau(s,m) \cap G_\tau$. Hence, $\tau \in B$. Therefore, the part (2) is proved. □

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