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Abstract. The technique of direct and inverse fuzzy (F-)transforms of three different types is introduced and approximating properties of the inverse F-transforms are described. A method of lossy image compression and reconstruction on the basis of the F-transform is presented.

1 Introduction

In this paper, we put a bridge between the well known classical transforms (Fourier, Laplace, integral, wavelet etc.) and the fuzzy transform (or, shortly, *F-transform*) introduced in [5] and further elaborated in [7]. Classical transforms are used besides others as powerful methods for construction of approximation models. The same purpose is pursued by the fuzzy transform. The main idea of all these transforms consists in transforming the original model into a special space where the computation is simpler. The transform back to the original space produces an approximate model or an approximate solution. The fuzzy transform also encompasses approximation methods based on elaboration of fuzzy IF-THEN rules.

Originally, the fuzzy transform based on arithmetic operations [5] (we call it ordinary) has been introduced. In [6], we investigated numerical methods (specially for solution of ordinary differential equations) based on this type of transform. Then we have generalized the ordinary F-transform to the case when operations are taken from a residuated lattice. This gave us an opportunity to realize that the lattice based F-transforms can be used in methods of data compression and decompression as proposed in [2]. We have extended these methods to the ordinary F-transform, and it turned out that the latter has certain advantages over its predecessors.

In this paper, we will explain a construction of approximation models on the basis of three different fuzzy transforms (Sections 2,3,4 and 5) and show, how they can be applied to data compression and decompression (Sections 6). A method of lossy image compression and reconstruction on the basis of F-transforms is illustrated on examples of pictures.

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2 Fuzzy Partition of the Universe and the Direct F-transform

The core idea of the technique proposed in this paper is a fuzzy partition of an interval as a universe. We claim that for a sufficient (approximate) representation of a function, defined on the interval, we may consider its average values within subintervals which constitute a partition of that interval. Then, an arbitrary function can be associated with a mapping from thus obtained set of subintervals to the set of average values of this function. Moreover, the set of the above mentioned average values gives an approximate (compressed) representation of the considered function. Let us give the necessary details (see [7]).

Definition 1. *Let $x_1 < \ldots < x_n$ be fixed nodes within $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ defined on $[a, b]$, form a* fuzzy partition *of $[a, b]$ if they fulfil the following conditions for $k = 1, \ldots, n$:*

1. *$A_k : [a, b] \longrightarrow [0, 1]$, $A_k(x_k) = 1$;*
2. *$A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$;*
3. *$A_k(x)$ is continuous;*
4. *$A_k(x)$, $k = 2, \ldots, n$, monotonically increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1, \ldots, n-1$, monotonically decreases on $[x_k, x_{k+1}]$;*
5. *for all $x \in [a, b]$*

$$\sum_{k=1}^{n} A_k(x) = 1. \tag{1}$$

The membership functions $A_1(x), \ldots, A_n(x)$ are called basic functions.

Let us remark that basic functions are specified by a set of nodes $x_1 < \ldots < x_n$ and the properties 1–5. The shape of basic functions is not predetermined and therefore, it can be chosen additionally.

We say that a fuzzy partition $A_1(x), \ldots, A_n(x)$, $n > 2$, is *uniform* if the nodes $x_1, \ldots, x_n$ are equidistant, i.e. $x_k = a + h(k-1)$, $k = 1, \ldots, n$, where $h = (b-a)/(n-1)$, and two more properties are fulfilled for $k = 2, \ldots, n-1$:

6. $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0, h]$,
7. $A_k(x) = A_{k-1}(x - h)$, for all $x \in [x_k, x_{k+1}]$ and $A_{k+1}(x) = A_k(x - h)$, for all $x \in [x_k, x_{k+1}]$

Figure 1 shows a uniform partition by sinusoidal shaped basic functions. Their formal expressions are given below.

$$A_1(x) = \begin{cases} 0.5(\cos\frac{\pi}{h}(x - x_1) + 1), & x \in [x_1, x_2], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k(x) = \begin{cases} 0.5(\cos\frac{\pi}{h}(x - x_k) + 1), & x \in [x_{k-1}, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}$$

where $k = 2, \dots n-1$, and

$$A_n(x) = \begin{cases} 0.5(\cos \frac{\pi}{h}(x - x_n) + 1), & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise.} \end{cases}$$

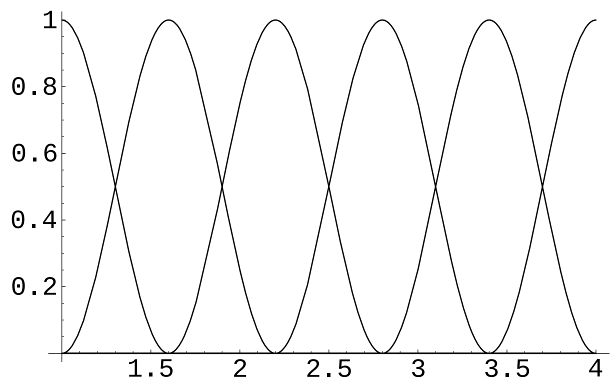

Fig. 1. An example of a uniform fuzzy partition of $[1, 4]$ by sinusoidal membership functions

The following lemma [6, 7] shows that, in the case of a uniform partition, the definite integral of a basic function does not depend on its concrete shape. This property will be further used to simplify the expression of F-transform components.

Lemma 1. *Let the uniform partition of* $[a, b]$ *be given by basic functions* A_1, $\dots, A_n$, $n \geq 3$. *Then*

$$\int_{x_1}^{x_2} A_1(x)dx = \int_{x_{n-1}}^{x_n} A_n(x)dx = \frac{h}{2}, \tag{2}$$

and for $k = 2, \dots, n-1$

$$\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx = h \tag{3}$$

where h *is the distance between each two neighboring nodes.*

Let $C([a, b])$ be the set of continuous functions on the interval $[a, b]$. The following definition (see also [6, 7]) introduces the fuzzy transform of a function $f \in C([a, b])$.

Definition 2. *Let* $A_1, \dots, A_n$ *be basic functions which form a fuzzy partition of* $[a, b]$ *and* f *be any function from* $C([a, b])$. *We say that the* n*-tuple of real* *numbers* $[F_1, \ldots, F_n]$ *given by*

$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \qquad k = 1, \ldots, n, \tag{4}$$

is the (integral) F-transform of f *with respect to* $A_1, \ldots, A_n$.

Denote the F-transform of a function $f \in C([a,b])$ with respect to $A_1, \ldots, A_n$ by $\mathbf{F}_n[f]$. Then according to Definitions 2, we can write

$$\mathbf{F}_n[f] = [F_1, \ldots, F_n]. \tag{5}$$

The elements $F_1, \ldots, F_n$ are called *components of the F-transform.*

It is easy to see that if the fuzzy partition of $[a, b]$ (and therefore, basic functions) is fixed then the F-transform establishes a linear mapping from $C([a, b])$ to $\mathbb{R}^n$ so that

$$\mathbf{F}_n[\alpha f + \beta g] = \alpha \mathbf{F}_n[f] + \beta \mathbf{F}_n[g]$$

for $\alpha, \beta \in \mathbb{R}$ and functions $f, g \in C([a, b])$. This linear mapping is denoted by $\mathbf{F}_n$ where n is the dimension of the image space.

At this point we will refer to [7] for some useful properties of the F-transform components. The most important one we are going to present below. This property concerns the following problem: how well is the original function f represented by its F-transform? We will show that under certain assumptions on the original function, the components of its F-transform are the *weighted mean values* of the given function where the weights are given by the basic functions.

Theorem 1. *Let* f *be a continuous function on* $[a, b]$ *and* $A_1, \ldots, A_n$ *be basic functions which form a fuzzy partition of* $[a, b]$. *Then the* k*-th component of the integral F-transform gives minimum to the function*

$$\Phi(y) = \int_a^b (f(x) - y)^2 A_k(x)dx \tag{6}$$

defined on $[f(a), f(b)]$.

Proof. By the assumptions, the function $(f(x) - y)^2 A_k(x)$ is continuously differentiable with respect to y in $(f(a), f(b))$, and we may write

$$\Phi'(y) = -2 \int_a^b (f(x) - y)A_k(x)dx.$$

Moreover, it is easy to see that the function $\Phi(y)$ reaches its minimum at the point which gives a solution to the equation $\Phi'(y) = 0$, i.e.

$$y = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}.$$

This is the exact expression of the k-th F-transform component (cf. 4).

Let us specially consider a discrete case, when the original function f is known (may be computed) only at some nodes $p_1, \ldots, p_l \in [a, b]$. We assume that the set P of these nodes is *sufficiently dense with respect to the fixed partition*, i.e.

$$(\forall k)(\exists j) A_k(p_j) > 0. \tag{7}$$

Then the (discrete) F-transform of f is introduced as follows.

Definition 3. *Let a function f be given at nodes $p_1, \ldots, p_l \in [a, b]$ and A_1, $\ldots, A_n$, $n < l$, be basic functions which form a fuzzy partition of $[a, b]$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$ is the discrete F-transform of f with respect to $A_1, \ldots, A_n$ if*

$$F_k = \frac{\sum_{j=1}^{l} f(p_j) A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)}. \tag{8}$$

Similarly to the integral F-transform, we may show that the components of the discrete F-transform are the *weighted mean values* of the given function where the weights are given by the basic functions.

Lemma 2. *Let function f be given at nodes $p_1, \ldots, p_l \in [a, b]$ and $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$. Then the k-th component of the discrete F-transform gives minimum of the function*

$$\Phi(y) = \sum_{j=1}^{l} (f(p_j) - y)^2 A_k(p_j) \tag{9}$$

defined on $[f(a), f(b)]$.

Proof. The proof is similar to the proof of Theorem 1 and therefore, it is omitted.

2.1 Inverse F-transform

A reasonable question is the following: can we reconstruct the function from its F-transform? The answer is clear: not precisely in general because we are loosing information when passing to the F-transform. However, the function that can be reconstructed (by the inversion formula) approximates the original one in such a way that a universal convergence can be established. Moreover, the inverse F-transform fulfils the best approximation criterion which can be called the piecewise integral least square criterion.

Definition 4. *Let $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$ and f be a function from $C([a, b])$. Let $\mathbf{F}_n[f] = [F_1, \ldots, F_n]$ be the integral F-transform of f with respect to $A_1, \ldots, A_n$. Then the function*

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x) \tag{10}$$

is called the inverse F-transform.

The theorem below shows that the inverse F-transform $f_{F,n}$ can approximate the original continuous function f with an arbitrary precision.

Theorem 2. *Let f be a continuous function on $[a,b]$. Then for any $\varepsilon > 0$ there exist n_ε and a fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$ of $[a,b]$ such that for all $x \in [a,b]$*

$$|f(x) - f_{F,n_\varepsilon}(x)| \leq \varepsilon \tag{11}$$

where f_{F,n_ε} is the inverse F-transform of f with respect to the fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$.

Proof. Note that the function f is uniformly continuous on $[a,b]$, i.e. for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x', x'' \in [a,b]$, $|x' - x''| < \delta$ implies $|f(x') - f(x'')| < \varepsilon$. To prove our theorem we choose some $\varepsilon > 0$ and find the nodes $x_1, \ldots, x_n \in [a,b]$ such that $a = x_1 < \ldots < x_n = b$ and $|f(x') - f(x'')| < \varepsilon$ whenever $x', x'' \in [x_{k-1}, x_{k+1}]$, $k = 2, \ldots, n-1$. Let us put $n = n_\varepsilon$ and take a fuzzy partition determined by the chosen nodes and constituted by basic functions $A_1, \ldots, A_n$. To complete the proof it remains to verify (11).

Let $F_1, \ldots, F_n$ be the components of the F-transform of f w.r.t. basic functions $A_1, \ldots, A_n$. Then for all $t \in [x_k, x_{k+1}]$, $k = 1, \ldots, n-1$, we evaluate

$$|f(t) - F_k| = |f(t) - \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx}| \leq \frac{\int_{x_{k-1}}^{x_{k+1}} |f(t) - f(x)|A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx} \leq \varepsilon$$

and analogously,

$$|f(t) - F_{k+1}| \leq \varepsilon$$

where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$. Therefore, having in mind (1), we obtain

$$|f(t) - \sum_{i=1}^{n} F_i A_i(t)| = |f(t)\sum_{i=1}^{n} A_i(t) - \sum_{i=1}^{n} F_i A_i(t)| \leq$$

$$\leq \sum_{i=1}^{n} A_i(t)|f(t) - F_i| = \sum_{i=k}^{k+1} A_i(t)|f(t) - F_i| \leq \varepsilon \sum_{i=k}^{k+1} A_i(t) = \varepsilon \sum_{i=1}^{n} A_i(t) = \varepsilon.$$

Because the argument t has been chosen arbitrary within the interval $[a,b]$, this proves the inequality (11).

In the proof of Theorem 2 we have constructed the non-uniform partition of $[a,b]$. We can reformulate the result of Theorem 2 for the case of uniform fuzzy partitions of $[a,b]$ having in mind the fact that the number of nodes n determines the uniform fuzzy partition up to the shape of membership functions.

Corollary 1. *Let f be a continuous function on $[a,b]$ and let $\{(A_1^{(n)}, \ldots, A_n^{(n)})_n\}$ be a sequence of uniform fuzzy partitions of $[a,b]$, one for each n. Let $\{f_{F,n}(x)\}$*

be the sequence of inverse F-transforms, each with respect to the given n-tuple $A_1^{(n)}, \dots, A_n^{(n)}$. *Then for any* $\varepsilon > 0$ *there exists* n_ε *such that for each* $n > n_\varepsilon$ *and for all* $x \in [a, b]$

$$|f(x) - f_{F,n}(x)| \leq \varepsilon. \tag{12}$$

Proof. The proof easily follows from the fact that for a chosen $\varepsilon > 0$ we can always find the respective value $n_\varepsilon > 2$ such that the corresponding value of $h = (b - a)/(n_\varepsilon - 1)$ guarantees that

$$|f(x') - f(x'')| < \varepsilon \quad \text{whenever} \quad |x' - x''| < h.$$

Corollary 2. *Let the assumptions of Corollary 1 be fulfilled. Then the sequence of inverse F-transforms* $\{f_{F,n}\}$ *uniformly converges to* f.

In the discrete case, we define the inverse F-transform only at nodes where the original function is given.

Definition 5. *Let function* $f(x)$ *be given at nodes* $p_1, \dots, p_l \in [a, b]$ *and* $\mathbf{F}_n[f] = [F_1, \dots, F_n]$ *be the discrete F-transform of* f *w.r.t.* $A_1, \dots, A_n$. *Then the function*

$$f_{F,n}(p_j) = \sum_{k=1}^{n} F_k A_k(p_j),$$

defined at the same nodes, is the inverse discrete F-transform.

Analogously to Theorem 2, we may show that the inverse discrete F-transform $f_{F,n}$ can approximate the original function f at common nodes with an arbitrary precision (see [7]) for the proof.

Theorem 3. *Let a function* f *be given at nodes* $p_1, \dots, p_l$ *constituting the set* $P \subset [a, b]$. *Then, for any* $\varepsilon > 0$, *there exist* n_ε *and a fuzzy partition* $A_1, \dots, A_{n_\varepsilon}$ *of* $[a, b]$ *such that* P *is sufficiently dense with respect to* $A_1, \dots, A_{n_\varepsilon}$ *and for all* $p \in \{p_1, \dots, p_l\}$

$$|f(p) - f_{F,n_\varepsilon}(p)| < \varepsilon \tag{13}$$

holds true.

3 F-Transforms of Functions of Two and More Variables

The direct and inverse F-transforms of a function of two and more variables can be introduced as a direct generalization of the case of one variable. We will do it briefly and refer to [8] for more details.

Suppose that the universe is a rectangle $[a, b] \times [c, d]$ and $x_1 < \ldots < x_n$ are fixed nodes from $[a, b]$ and $y_1 < \ldots < y_m$ are fixed nodes from $[c, d]$, such that $x_1 = a$, $x_n = b$, $y_1 = c$, $x_m = d$ and $n, m \geq 2$. Let us formally extend the set of nodes by $x_0 = a$, $y_0 = c$ and $x_{n+1} = b$, $y_{m+1} = d$. Assume that $A_1, \dots, A_n$ are basic functions which form a fuzzy partition of $[a, b]$ and $B_1, \dots, B_m$ are basic functions which form a fuzzy partition of $[c, d]$. Let $C([a, b] \times [c, d])$ be the set of continuous functions of two variables $f(x, y)$.

Definition 6. *Let $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$ and $B_1, \ldots, B_m$ be basic functions which form a fuzzy partition of $[c, d]$. Let $f(x, y)$ be any function from $C([a, b] \times [c, d])$. We say that the $n \times m$-matrix of real numbers $\mathbf{F}_{nm}[f] = (F_{kl})$ is the (integral) F-transform of f with respect to $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ if for each $k = 1, \ldots, n$, $l = 1, \ldots, m$,*

$$F_{kl} = \frac{\int_c^d \int_a^b f(x,y) A_k(x) B_l(y) dx dy}{\int_c^d \int_a^b A_k(x) B_l(y) dx dy}. \tag{14}$$

In the discrete case, when an original function $f(x, y)$ is known only at some nodes $(p_i, q_j) \in [a, b] \times [c, d]$, $i = 1, \ldots, N$, $j = 1, \ldots, M$, the (discrete) F-transform of f can be introduced analogously to the case of a function of one variable. We assume additionally that sets $P = \{p_1, \ldots, p_N\}$ and $Q = \{q_1, \ldots, q_M\}$ of these nodes are *sufficiently dense with respect to the chosen partitions*, i.e.

$$(\forall k)(\exists j) A_k(p_j) > 0,$$
$$(\forall k)(\exists j) B_k(q_j) > 0.$$

Definition 7. *Let a function f be given at nodes $(p_i, q_j) \in [a, b] \times [c, d]$, $i = 1, \ldots, N$, $j = 1, \ldots, M$, and $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ where $n < N$, $m < M$, be basic functions which form fuzzy partitions of $[a, b]$ and $[c, d]$ respectively. Suppose that sets P and Q of these nodes are sufficiently dense with respect to the chosen partitions. We say that the $n \times m$-matrix of real numbers $\mathbf{F}_{nm}[f] = (F_{kl})$ is the discrete F-transform of f with respect to $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ if*

$$F_{kl} = \frac{\sum_{j=1}^{M} \sum_{i=1}^{N} f(p_i, q_j) A_k(p_i) B_l(q_j)}{\sum_{j=1}^{M} \sum_{i=1}^{N} A_k(p_i) B_l(q_j)}. \tag{15}$$

holds for all $k = 1, \ldots, n$, $l = 1, \ldots, m$.

As in the case of functions of one variable, the elements F_{kl}, $k = 1, \ldots, n$, $l = 1, \ldots, m$, are called *components of the F-transform.*

If the partitions of $[a, b]$ and $[c, d]$ by $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ are uniform then the expression (14) for the components of the F-transform may be simplified on the basis of expressions which can be easily obtained from Lemma 1:

$$\begin{aligned}
F_{11} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_1(y) dx dy, \\
F_{1m} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_m(y) dx dy, \\
F_{n1} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_1(y) dx dy, \\
F_{nm} =& \frac{4}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_m(y) dx dy,
\end{aligned}$$

and for $k = 2, ..., n-1$ and $l = 2, ..., m-1$

$$F_{k1} = \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_1(y) dx dy,$$
$$F_{km} = \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_m(y) dx dy,$$
$$F_{1l} = \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_1(x) B_l(y) dx dy,$$
$$F_{nl} = \frac{2}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_n(x) B_l(y) dx dy,$$
$$F_{kl} = \frac{1}{h_1 h_2} \int_c^d \int_a^b f(x,y) A_k(x) B_l(y) dx dy.$$

Remark 1. The expression (14) can be rewritten with the help of a repeated integral

$$F_{kl} = \frac{\int_c^d (\int_a^b f(x,y) A_k(x) dx) B_l(y) dy}{\int_c^d B_l(y) dy \int_a^b A_k(x) dx}.$$

On the basis of this expression, all the properties (linearity etc.) proved for the F-transform of a function of one variable can be easily generalized and proved for the considered case too.

Definition 8. *Let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be basic functions which form fuzzy partitions of $[a,b]$ and $[c,d]$ respectively. Let f be a function from $C([a,b] \times [c,d])$ and $\mathbf{F}_{nm}[f]$ be the F-transform of f with respect to $A_1, \dots, A_n$ and $B_1, \dots, B_m$. Then the function*

$$f^F_{nm}(x,y) = \sum_{k=1}^{n} \sum_{l=1}^{m} F_{kl} A_k(x) B_l(y) \tag{16}$$

is called the the inverse F-transform.

Similarly to the case of a function of one variable we can prove that the inverse F-transform $f^F_{n,m}$ can approximate the original continuous function f with an arbitrary precision.

Theorem 4. *Let f be any continuous function on $[a,b] \times [c,d]$. Then for any $\varepsilon > 0$ there exist n_ε, m_ε and fuzzy partitions $A_1, \dots, A_{n_\varepsilon}$ and $B_1, \dots, B_{m_\varepsilon}$ of $[a,b]$ and $[c,d]$ respectively, such that for all $(x,y) \in [a,b] \times [c,d]$*

$$|f(x,y) - f^F_{n_\varepsilon m_\varepsilon}(x,y)| \le \varepsilon. \tag{17}$$

The function $f^F_{n_\varepsilon m_\varepsilon}$ in (17) is the inverse F-transform of f with respect to $A_1, \dots, A_{n_\varepsilon}$ and $B_1, \dots, B_{m_\varepsilon}$.

4 F-Transforms Expressed By Residuated Lattice Operations

Our purpose here is to introduce two new fuzzy transforms which are based on operations of a residuated lattice on $[0,1]$. These transforms lead to new approximation models which are formally represented using weaker operations than the arithmetic ones used above in the case of the (ordinary) F-transform. However, these operations are successfully used in modeling of dependencies characterized by words of natural language (e.g. fuzzy IF–THEN rules) and also, in modeling of continuous functions. Therefore, two new F-transforms that we are going to introduce in this section extend and generalize the F-transform considered above.

There is another important application of fuzzy transforms based on residuated lattice operations – an application to image processing. By this, we mean an application to image compression and reconstruction. We will discuss this later in Section 6.

Let us briefly introduce the concept of residuated lattice [1] which will be a basic algebra of operations in the sequel.

Definition 9. *A residuated lattice is an algebra*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle.$$

with four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice where the ordering* $\leq$ *defined using operations* $\vee, \wedge$ *as usual, and* $\mathbf{0}, \mathbf{1}$ *are the least and the greatest elements, respectively;*
- $\langle L, *, \mathbf{1} \rangle$ *is a commutative monoid, that is,* $*$ *is a commutative and associative operation with the identity* $a * \mathbf{1} = a$*;*
- *the operation* $\rightarrow$ *is a residuation operation with respect to* $*$*, i.e.*

$$a * b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The well known examples of residuated lattices are boolean algebras, Gödel, Łukasiewicz and product algebras. In the particular case $L = [0,1]$, the multiplication $*$ is called t-norm. In the foregoing text we will operate with some fixed residuated lattice $\mathcal{L}$ on $[0,1]$.

4.1 Direct $\mathbf{F}^{\uparrow}$ and $\mathbf{F}^{\downarrow}$-transforms

Let the universe be the interval $[0,1]$. We redefine here the notion of *fuzzy partition* of $[0,1]$ assuming that it is given by fuzzy sets $A_1, \ldots, A_n$, $n \geq 2$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ fulfilling the following (only one!) *covering property*

$$(\forall x)(\exists i) \quad A_i(x) > 0. \tag{18}$$

As above, the membership functions $A_1(x), \ldots, A_n(x)$ are called the *basic functions*. In the sequel, we fix the value of $n \geq 2$ and some fuzzy partition of $[0,1]$ by basic functions $A_1, \ldots, A_n$.

We assume that a finite subset $P = \{p_1, \ldots, p_l\}$ of $[0,1]$ is fixed. Moreover, we assume that P is sufficiently dense with respect to the fixed partition, i.e. (7) holds.

Definition 10. *Let a function f be defined at nodes $p_1, \ldots, p_l \in [0,1]$ and $A_1, \ldots, A_n$, $n < l$, be basic functions which form a fuzzy partition of $[a,b]$. We say that the n-tuple of real numbers $[F_1^{\uparrow}, \ldots, F_n^{\uparrow}]$ is a (discrete) $F^{\uparrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ if*

$$F_k^{\uparrow} = \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)) \tag{19}$$

and the n-tuple of real numbers $[F_1^{\downarrow}, \ldots, F_n^{\downarrow}]$ is the (discrete) $F^{\downarrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ if

$$F_k^{\downarrow} = \bigwedge_{j=1}^{l} (A_k(p_j) \to f(p_j)). \tag{20}$$

Denote the $\mathrm{F}^{\uparrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ by $\mathbf{F}_n^{\uparrow}[f]$ and the $\mathrm{F}^{\downarrow}$-transform of f w.r.t. $A_1, \ldots, A_n$ by $\mathbf{F}_n^{\downarrow}[f]$. Then we may write:

$$\mathbf{F}_n^{\uparrow}[f] = [F_1^{\uparrow}, \ldots, F_n^{\uparrow}], \qquad \mathbf{F}_n^{\downarrow}[f] = [F_1^{\downarrow}, \ldots, F_n^{\downarrow}].$$

Analogously to Theorem 1 we will show that components of the lattice based F-transforms are *lower mean values* (respectively, *upper mean values*) of an original function which give least (greatest) elements to certain sets.

Lemma 3. *Let a function f be defined at nodes $p_1, \ldots, p_l \in [0,1]$ and $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[0,1]$. Then the k-th component of the $F^{\uparrow}$-transform is the least element of the set*

$$S_k = \{a \in [0,1] | \quad A_k(p_j) \leq (f(p_j) \to a) \textit{ for all } j = 1, \ldots, l\} \tag{21}$$

and the k-th component of the $F^{\downarrow}$-transform is the greatest element of the set

$$T_k = \{a \in [0,1] | \quad A_k(p_j) \leq a \to f(p_j) \textit{ for all } j = 1, \ldots, l\} \tag{22}$$

where $k = 1, \ldots, n$.

Proof. The proof will be given for the first statement and can be obtained for the second one by the adjunction property.

It is easy to see that $F_k^{\uparrow} \in S_k$. We will show that $a \in S_k$ implies $F_k^{\uparrow} \leq a$. Indeed, from (21) we have

$$A_k(p_j) \leq (f(p_j) \to a) \text{ for all } j = 1, \ldots, l$$

which implies (with help of the adjunction property)

$$a \geq A_k(p_j) * f(p_j) \text{ for all } j = 1, \dots, l$$

and therefore,

$$a \geq \bigvee_{j=1}^{l} (A_k(p_j) * f(p_j)) = F_k^{\uparrow}.$$

5 Inverse $\mathbf{F}^{\uparrow}$ ($\mathbf{F}^{\downarrow}$)-Transforms

All F-transforms (the ordinary one and those based on the lattice operations) convert the respective space of functions into the space of n-dimensional real vectors. We have defined the inverse F-transform in Subsection 2.1. In this section, we will define inverse $\mathrm{F}^{\uparrow}$ and inverse $\mathrm{F}^{\downarrow}$-transforms and prove their approximation properties.

In the construction of the inverse $\mathrm{F}^{\uparrow}$- and $\mathrm{F}^{\downarrow}$-transforms we use the fact that the operations $*$ and $\rightarrow$ are mutually adjoint in a residuated lattice.

Definition 11. *Let function f be defined at nodes $p_1, \dots, p_l \in [a, b]$ and let $\mathbf{F}_n^{\uparrow}[f] = [F_1^{\uparrow}, \dots, F_n^{\uparrow}]$ be the $F^{\uparrow}$-transform of f and $\mathbf{F}_n^{\downarrow}[f] = [F_1^{\downarrow}, \dots, F_n^{\downarrow}]$ be the $F^{\downarrow}$-transform of f w.r.t. basic functions $A_1, \dots, A_n$. Then the following functions, defined at the same nodes as f, are called the* inverse $\mathrm{F}^{\uparrow}$-transform

$$f_{F,n}^{\uparrow}(p_j) = \bigwedge_{k=1}^{n} (A_k(p_j) \rightarrow F_k^{\uparrow}), \tag{23}$$

and the inverse $\mathrm{F}^{\downarrow}$-transform

$$f_{F,n}^{\downarrow}(p_j) = \bigvee_{k=1}^{n} (A_k(p_j) * F_k^{\downarrow}), \tag{24}$$

The following theorem shows that the inverse $\mathrm{F}^{\uparrow}$- and $\mathrm{F}^{\downarrow}$-transforms approximate the original function from above and from below.

Theorem 5. *Let function f be defined at nodes $p_1, \dots, p_l \in [0, 1]$. Then for all $j = 1, \dots, l$*

$$f_{F,n}^{\downarrow}(p_j) \leq f(p_j) \leq f_{F,n}^{\uparrow}(p_j). \tag{25}$$

Remark 2. Let us remark that similarly to Definition 7, the direct and inverse lattice based F-transforms of a function of two and more variables can be introduced as a direct generalization of the case of one variable.

6 Application of the F-transform to Image Compression and Reconstruction

A method of lossy image compression and reconstruction on the basis of fuzzy relations has been proposed in a number of papers (see e.g. [2, 3]). When analyzing these methods, we have realized that they can be expressed using F-transforms based on lattice operations. In this section, we will explain how the general technique of F-transform can be successfully applied to the compression and reconstruction of images and compare the effectiveness of all three types of the F-transform with respect to this problem. We will see that all considered examples witnessed the advantage of the ordinary F-transform (4) over the lattice based F-transforms (cf. (19) and (20)).

Let an image I of the size $N \times M$ pixels be represented by a function of two variables (a fuzzy relation) $f_I : \mathbb{N} \times \mathbb{N} \longrightarrow [0,1]$ partially defined at nodes $(i,j) \in [1,N] \times [1,M]$. The value $f_I(i,j)$ represents an intensity range of each pixel. We propose to compress this image with the help of the one of the three discrete F-transforms of a function of two variables by the $n \times m$-matrix of real numbers

$$\mathbf{F}_{nm}[f_I] = \begin{pmatrix} F_{11} & \dots & F_{1m} \\ \vdots & \vdots & \vdots \\ F_{n1} & \dots & F_{nm} \end{pmatrix}.$$

The following expression reminds the components F_{kl} obtained by the ordinary F-transform of a function of two variables (14):

$$F_{kl} = \frac{\sum_{j=1}^{M} \sum_{i=1}^{N} f_I(i,j) A_k(i) B_l(j)}{\sum_{j=1}^{M} \sum_{i=1}^{N} A_k(i) B_l(j)}$$

where $A_1, \dots, A_n$, $B_1, \dots, B_m$, are basic functions which form fuzzy partitions of $[1,N]$ and $[1,M]$, respectively and $n < N$, $m < M$. We refer to Remark 2 for the case of lattice based F-transforms. The value $\rho = nm/NM$ is called a *compression ratio.*

A reconstruction of the image f_I, being compressed by $\mathbf{F}_{nm}[f_I] = (F_{kl})$ with respect to $A_1, \dots, A_n$ and $B_1, \dots, B_m$, is given by the inverse F-transform (16) adapted to the domain $[1,N] \times [1,M]$:

$$f_{nm}^{F}(i,j) = \sum_{k=1}^{n} \sum_{l=1}^{m} F_{kl} A_k(i) B_l(j).$$

On the basis of Theorem 4 we know that the reconstructed image is close to the original one and moreover, that it can be obtained with a prescribed level of accuracy. On Fig. 2 and Fig. 3 (the figures are taken from [4]), we illustrate the proposed compression method and reconstruction based on F-transforms of all three types. Let us show the advantage of the ordinary F-transform over the lattice based F-transforms in what concerns images “Bird” and “Bridge” compression/reconstruction. We have evaluated the quality of the reconstructed images using PSNR value given by

$$PSNR = 20 \log_{10} \frac{255}{\varepsilon}$$

where ε is the root mean square error (RMSE) given by

$$\varepsilon = \sqrt{\frac{\sum_{i=1}^{N} \sum_{j=M}^{n} (f_I(i,j) - f_{nm}^F(i,j))}{MN}}.$$

The table below [4] contains values of PSNR for images “Bird” and “Bridge”.

“Bird”			“Bridge”		
ρ	F-transform	$F^{\uparrow}$-transform	ρ	F-transform	$F^{\uparrow}$-transform
0.56	38.3182	35.9521	0.56	38.3182	35.9521
0.39	36.4782	28.4277	0.39	36.4782	28.4277

On the basis of Lemma 2, we may proclaim that the PSNR-quality is the best for compression/reconstruction methods based on the ordinary F-transform in comparison with the lattice based F-transforms.

7 Conclusion

We have introduced a new technique of direct and inverse fuzzy transforms (F-transforms for short) which enables us to construct various approximating models depending on the choice of basic functions. The best approximation property of the inverse F-transform is established within the respective approximating space.

A method of lossy compression and reconstruction data on the basis of fuzzy transforms has been proposed and its advantage over the similar method based on a lattice based F-transform is discussed. The proposed method can be applied not only to pictures, but also to music and other kinds of digital data.

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a)

b)

c)

d)

e)

Fig. 2. The original image "Bird" a) is compressed and reconstructed by the ordinary F-transform method (pictures b) and d)) and by the lattice based $\mathrm{F}^{\uparrow}$-transform method (pictures c) and e)). The compression ratio $\rho = 0.56$ has been used for images on pictures b) and c) and the compression ratio $\rho = 0.39$ has been used for images on pictures d) and e).

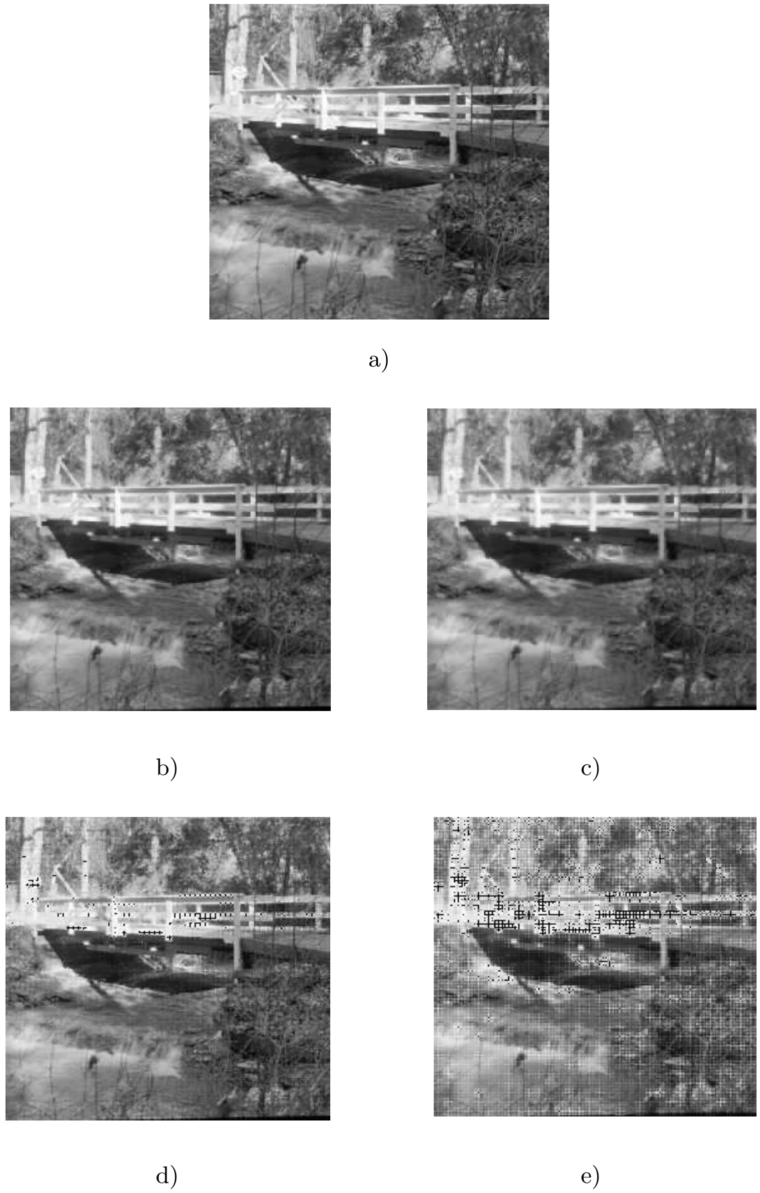

Fig. 3. The original image "Bridge" a) is compressed and reconstructed by the ordinary F-transform method (pictures b) and d)) and by the lattice based $F^{\uparrow}$-transform method (pictures c) and e)). The compression ratio $\rho = 0.56$ has been used for images on pictures b) and c) and the compression ratio $\rho = 0.39$ has been used for images on pictures d) and e).