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Extensional subobjects in categories of Ω-fuzzy sets

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1 Introduction

In a fuzzy set theory there are several categories which play important role in fuzzy logic interpretation. Two of these categories of fuzzy sets over MV-algebra $\Omega = (L, \otimes, \rightarrow)$ will be investigated in the paper. The first one is a category $\mathbf{Set}(\Omega)$ with objects (A, δ) where A is a set and $\delta : A \times A \rightarrow \Omega$ is a similarity relation such that

(a) $(\forall x \in A) \quad \delta(x, x) = 1$,

(b) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,

(c) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$.

A morphism $f : (A, \delta) \rightarrow (B, \gamma)$ in $\mathbf{Set}(\Omega)$ is a map $f : A \times B \rightarrow \Omega$ satisfying the following conditions .

(a) $(\forall x, z \in A)(\forall y \in B) \quad \delta(x, z) \otimes f(x, y) \leq f(z, y)$,

(b) $(\forall x \in A)(\forall y, z \in B) \quad \gamma(y, z) \otimes f(x, y) \leq f(x, z)$,

(c) $(\forall x \in A)(\forall y, z \in B) \quad f(x, y) \otimes f(x, z) \leq \gamma(y, z)$,

(d) $(\forall x \in A) \quad 1 = \bigvee\{f(x, y) : y \in B\}$.

If $f : \mathbf{A} \rightarrow \mathbf{B}$ and $g : \mathbf{B} \rightarrow \mathbf{C}$ are two morphisms then their composition is a function $g \circ f : A \times C \rightarrow \Omega$ such that

$$g \circ f(x, z) = \bigvee_{y \in B} (f(x, y) \otimes g(y, z)).$$

The other category $\mathbf{SetF}(\Omega)$ will have the same objects as the category $\mathbf{Set}(\Omega)$. A morphism $f : (A, \delta) \rightarrow (B, \gamma)$ in $\mathbf{SetF}(\Omega)$ is a map $f : A \rightarrow B$ such that $(\forall x, y \in A) \quad \gamma(f(x), f(y)) \geq \delta(x, y)$.

In [9],[10] we investigated some principal properties of a category $\mathbf{SetF}(\Omega)$ and we proved that the *extensional subobjects* of objects (A, δ) in this category $\mathbf{SetF}(\Omega)$ can be identified with some *characteristic morphism* $(A, \delta) \rightarrow (\Omega^*, \mu)$. Namely we proved that if $S : \mathbf{SetF}(\Omega) \rightarrow \mathbf{Set}$ is a functor such that $S(A, \delta) = \{s : s \text{ is an extensional subobject of } (A, \delta)\}$ then there exists a natural isomorphism

$$\zeta : S(-) \rightarrow \mathrm{Hom}_{\mathbf{SetF}(\Omega)}(-, \Omega^*).$$

This classification property, which is one of the most important properties of a topos category are frequently used for interpretation of formulas of fuzzy logic in a category $\mathbf{SetF}(\Omega)$ in such a way that interpretation of a fuzzy logic formula is defined as a special extensional subobject of some object (A, δ).

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Hence, it seems natural that extensional subobjects of objects in a category $\mathbf{SetF}(\Omega)$ play important role for further investigation of that category.

In this paper we are interested in the following problem related to extensional subobjects. For any object (A,δ) of a category $\mathbf{SetF}(\Omega)$ we can define a set $\Omega^{(A,\delta)}$ of all (or some special, respectively) extensional subobjects of (A,δ). This set can be transformed (in several different ways) into an object of a category $\mathbf{SetF}(\Omega)$. In that way we obtain a full subcategory $\Omega^{\mathbf{SetF}(\Omega)}$ of a category $\mathbf{SetF}(\Omega)$ consisting of these special objects. We will be interested in conditions under which that subcategory $\Omega^{\mathbf{SetF}(\Omega)}$ is a quasi-reflective subcategory in $\mathbf{SetF}(\Omega)$. Recall that a subcategory $\mathbf{L}$ of a category $\mathbf{K}$ is a *quasi-reflective* subcategory in $\mathbf{K}$ if there exists a functor $G:\mathbf{K}\to\mathbf{L}$ such that for any object $a\in\mathbf{K}$ there should exist a morphism $a@>u_a>>G(a)$ such that for any object $b\in\mathbf{L}$ and any morphism $f:a\to b$ (in $\mathbf{K}$) there exists a morphism (in general non unique) $\widehat{f}:G(a)\to b$ such that the diagram commutes:

$$\begin{array}{ccc} a & \xrightarrow{u_a} & G(a) \\ f\big\downarrow & & \big\downarrow\widehat{f} \\ b & = & b. \end{array}$$

A functor G is then called a *quasi-reflector*. We will be interested in several subcategories of a category $\mathbf{SetF}(\Omega)$ consisting of various objects $(sub(A,\delta),\sigma)$, where $sub(A,\delta)$ will be a subset of the set of all extensional subobjects of (A,δ) and σ will be a similarity relation defined on that subset. We prove that all these subcategories are quasi-reflective subcategories in a category $\mathbf{SetF}(\Omega)$. We also introduce a notion of a weak singleton extensional subobject of (A,δ) in a category $\mathbf{Set}(\Omega)$ and we prove that a subcategory consisting of these subobjects is also quasi-reflective subcategory of a category $\mathbf{Set}(\Omega)$.

2 Subcategories of extensional subobjects

We show firstly a simple result which states the existence of a functor between categories $\mathbf{SetF}(\Omega)$ and $\mathbf{Set}(\Omega)$.

Lemma 2.1 *There exists a functor* $F:\mathbf{SetF}(\Omega)\to\mathbf{Set}(\Omega)$.

For $(A,\delta)\in\mathbf{SetF}(\Omega)$ we set $F(A,\delta)=(A,\delta)$ and for a morphism $f:(A,\delta)\to(B,\gamma)$ in $\mathbf{SetF}(\Omega)$ we define a map $F(f):A\times B\to\Omega$ such that $F(f)(a,b)=\gamma(f(a),b)$ for any $a\in A, b\in B$. Then $F(f)$ is a morphism in $\mathbf{Set}(\Omega)$. In fact, we have for example

$$F(f)(a,b)\otimes\delta(a,a')=\gamma(f(a),b)\otimes\delta(a,a')\leq$$
$$\leq\gamma(f(a),b)\otimes\gamma(f(a),f(a'))\leq\gamma(f(a'),b)=F(f)(a',b).$$

Recall that an *extensional subobject* of (A,δ) in a category $\mathbf{SetF}(\Omega)$ is a map $s:A\to\Omega$ such that

$$s(x)\otimes\delta(x,y)\leq s(y).$$

An extensional subobject can be defined in a category $\mathbf{Set}(\Omega)$ as well. In fact, it is clear that $(\Omega,\leftrightarrow)$ is an object in $\mathbf{SetF}(\Omega)$, where $\alpha\leftrightarrow\beta=(\alpha\to\beta)\wedge(\beta\to\alpha)$. Then s is an extensional subobject of (A,δ) in a category $\mathbf{SetF}(\Omega)$ if $s:(A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{SetF}(\Omega)$. Analogously s will be called an *extensional subobject* of (A,δ) in $\mathbf{Set}(\Omega)$ if $s:(A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{Set}(\Omega)$, i.e.

(a) $(\forall a,a'\in A)(\alpha\in\Omega)\quad s(a,\alpha)\otimes\delta(a,a')\leq s(a',\alpha)$,

(b) $(\forall a\in A)(\forall\beta,\alpha\in\Omega)\quad s(a,\alpha)\otimes(\alpha\leftrightarrow\beta)\leq s(a,\beta)$,

(c) $(\forall a\in A)(\alpha,\beta\in\Omega)\quad s(a,\alpha)\otimes s(a,\beta)\leq\alpha\leftrightarrow\beta$,

(d) $(\forall x\in A)\quad 1=\bigvee_{\alpha\in\Omega}s(x,\alpha)$.

An extensional subobject s of (A,δ) in $\mathbf{SetF}(\Omega)$ is called *normal* if $\bigvee_{a\in A}s(a)=1$. Then an extensional subobject t of (A,δ) in $\mathbf{Set}(\Omega)$ is called *normal* if $\bigvee_{a\in A}t(a,1)=1$. Moreover, we say that t is a *weak singleton* of (A,δ) in $\mathbf{Set}(\Omega)$ if t is a normal extensional subobject of (A,δ) in $\mathbf{Set}(\Omega)$ and $t(a,1)\otimes t(b,1)\leq$

$\delta(a,b)$ for all $a,b\in A$. A special normal extensional subobject in $\mathbf{SetF}(\Omega)$ is called singleton. Recall that an extensional subobject $s:A\to\Omega$ of (A,δ) (in a category $\mathbf{SetF}(\Omega)$)is a *singleton* if it satisfies the condition

$$(\forall x,y\in A)\quad s(x)\otimes s(y)\le\delta(x,y).$$

It is clear that a map $\{a\}=\delta(a,-):A\to\Omega$ is an example of a singleton for any $a\in A$. On the other hand an object (A,δ) is called *complete* if for any singleton s of (A,δ) there exists $a\in A$ such that $s=\delta(a,-)$.

Let (A,δ) be an object of $\mathbf{Set}(\Omega)$ (or $\mathbf{SetF}(\Omega)$).We introduce the following notation.

$$\Omega^{(A,\delta)}=\{s: s \text{ is an extensional subobject of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\},$$
$$\Omega_1^{(A,\delta)}=\{s: s \text{ is an extensional and normal subobject of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\},$$
$$\text{w-singl}(A,\delta)=\{s: s \text{ is a weak singleton of } (A,\delta) \text{ in } \mathbf{Set}(\Omega)\},$$
$$\text{singl}(A,\delta)=\{s\in\Omega_1^{(A,\delta)}: s \text{ is singleton of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\}.$$

All the previous sets can be transformed into objects of categories $\mathbf{Set}(\Omega)$ and $\mathbf{SetF}(\Omega)$, respectively. In fact, for any object (A,δ) and for any $s,t\in\Omega^{(A,\delta)}$, $p,q\in$ w-singl(A,δ) we set

$$\sigma(s,t)=\sigma_{(A,\delta)}(s,t)=\bigwedge_{x\in A}s(x)\leftrightarrow t(x),$$
$$\tau(s,t)=\tau_{(A,\delta)}(s,t)=\begin{cases}\bigvee_{x\in A}s(x)\otimes t(x), & \text{if } s\ne t\\ 1, & \text{if } s=t\end{cases},$$
$$\rho(p,q)=\rho_{(A,\delta)}(p,q)=\bigwedge_{x\in A}p(x,1)\leftrightarrow q(x,1).$$

Lemma 2.2 *For any object (A,δ) there exists a morphism $\widehat{\ }:(singl(A,\delta),\sigma_{(A,\delta)})\to(w\text{-}singl(A,\delta),\rho_{(A,\delta)})$.*

Proof. For $s\in$ singl(A,δ) we set $\widehat{s}(a,\alpha)=s(a)\leftrightarrow\alpha$. It is then clear that $\widehat{s}=F(s)$ (see Lemma 1), since $s:(A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{SetF}(\Omega)$. It follows that $\widehat{s}$ is an extensional (and clearly normal) subobject in $\mathbf{Set}(\Omega)$. Since s is a singleton, $\widehat{s}$ is a weak singleton. Moreover according to Lemma 5 for any $s,t\in\Omega_1^{(A,\delta)}$ we have

$$\rho_{A,\delta)}(\widehat{s},\widehat{t})=\bigwedge_{a\in A}(s(a)\leftrightarrow 1)\leftrightarrow(t(a)\leftrightarrow 1)=$$
$$\bigwedge_{a\in A}s(a)\leftrightarrow t(a)=\sigma_{(A,\delta)}(s,t).$$

Lemma 2.3 *For any object (A,δ) the pairs $(\Omega^{(A,\delta)},\sigma_{(A,\delta)})$, $(\Omega^{(A,\delta)},\tau_{(A,\delta)})$, $(w\text{-}singl(A,\delta),\rho_{(A,\delta)})$ and $(singl(A,\delta),\tau_{(A,\delta)})$, respectively are objects of a category $\mathbf{Set}(\Omega)$ (and $\mathbf{SetF}(\Omega)$, simultaneously).*

The proof of this lemma can be done by simple computation.

Lemma 2.4 *Let Ω be an MV-algebra. Let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be two sets of elements of Ω and let $p:I\to I$ be a bijection map.*

(a) $\bigvee_{i\in I}a_i\leftrightarrow\bigvee_{i\in I}b_{i\in I}\ge\bigwedge_{j\in I}(a_j\leftrightarrow b_{p(j)})$,

(b) $\bigwedge_{i\in I}a_i\leftrightarrow\bigwedge_{i\in I}b_i\ge\bigwedge_{j\in I}(a_j\leftrightarrow b_{p(j)})$.

The proof can be done by simple computation and will be omitted.

Lemma 2.5 *Let Ω be an MV-algebra.*

(a) $(\forall a,b,x\in\Omega)\quad(x\leftrightarrow a)\leftrightarrow(x\leftrightarrow b)\ge a\leftrightarrow b$.

(b) $(\forall a,b,c,d \in \Omega) \quad (a \leftrightarrow c) \leftrightarrow (b \leftrightarrow d) \geq (a \leftrightarrow b) \otimes (c \leftrightarrow d)$.

(1) We have

$$(x \to a) \leftrightarrow (x \to b) \geq (x \leftrightarrow x) \otimes (a \leftrightarrow b) = a \leftrightarrow b,$$
$$(a \to x) \leftrightarrow (b \to x) \geq (a \leftrightarrow b) \otimes (x \leftrightarrow x) = a \leftrightarrow b,$$

and it follows

$$(x \leftrightarrow a) \leftrightarrow (x \leftrightarrow b) = ((x \to a) \wedge (a \to x)) \leftrightarrow ((x \to b) \wedge (b \to x)) \geq$$
$$((x \to a) \leftrightarrow (x \to b)) \wedge ((a \to x) \leftrightarrow (b \to x)) \geq a \leftrightarrow b.$$

(2) According to Lemma 4 we have

$$(a \leftrightarrow c) \leftrightarrow (b \leftrightarrow d) = ((a \to c) \wedge (c \to a)) \leftrightarrow ((b \to d) \wedge (d \to b)) \geq$$
$$((a \to c) \leftrightarrow (b \to d)) \wedge ((c \to a) \leftrightarrow (d \to b)) \geq$$
$$((a \leftrightarrow b) \otimes (c \leftrightarrow d)) \wedge ((c \leftrightarrow d) \otimes (a \leftrightarrow b) = (a \leftrightarrow b) \otimes (c \leftrightarrow d).$$

We will consider the following subcategories of categories $\mathbf{Set}(\Omega)$ and $\mathbf{SetF}(\Omega)$, respectively.

(a) A full subcategory $\mathbf{SetF}(\Omega)_{comp} \hookrightarrow \mathbf{SetF}(\Omega)$ consisting of complete objects of a category $\mathbf{SetF}(\Omega)$,

(b) A full subcategory $\Omega^{\mathbf{SetF}(\Omega)}_{\leftrightarrow} \hookrightarrow \mathbf{SetF}(\Omega)$ with objects $(\Omega^{(A,\delta)}, \sigma_{(A,\delta)})$ for any object (A,δ),

(c) A full subcategory $\Omega^{\mathbf{SetF}(\Omega)}_{\otimes} \hookrightarrow \mathbf{SetF}(\Omega)$ with objects $(\Omega^{(A,\delta)}, \tau_{(A,\delta)})$ for any object (A,δ),

(d) A full subcategory $\Omega^{\mathbf{Set}(\Omega)}_{1,\otimes} \hookrightarrow \mathbf{Set}(\Omega)$ with objects $(\Omega^{(A,\tau)}_1, \sigma_{(A,\tau)})$ for any object (A,δ),

(e) A full subcategory $\Omega^{\mathbf{Set}(\Omega)} \hookrightarrow \mathbf{Set}(\Omega)$ with objects $(\text{w-singl}(A,\delta), \rho_{(A,\delta)})$ for any object (A,δ).

Theorem 2.1 *There is a functor $C : \mathbf{SetF}(\Omega) \to \mathbf{SetF}(\Omega)_{comp}$ which is a quasi-reflector.*

Let (A,δ) be an object in $\mathbf{SetF}(\Omega)$. We show firstly that (singl(A,δ),τ) is a complete object. Let S be a singleton in $(\text{singl}(A,\delta), \tau_{(A,\delta)})$. Then we define a map $e_S : A \to \Omega$ such that

$$e_S(x) = \bigvee_{t \in \text{singl}(A,\delta)} t(x) \otimes S(t).$$

We show that $e_S \in \text{singl}(A,\delta)$. It can be easy to see that e_S is a normal extensional subobject. Moreover, we have

$$e_S(x) \otimes e_s(y) = \bigvee_{t,p \in \text{singl}(A,\delta)} t(x) \otimes p(y) \otimes S(t) \otimes S(p) \leq$$
$$\bigvee_{t,p \in \text{singl}(A,\delta)} t(x) \otimes p(y) \otimes \tau(t,p) = \bigvee_{t,p \in \text{singl}(A,\delta)} t(x) \otimes (\bigvee_{a \in A} t(a) \otimes p(a) \otimes p(y)) \leq$$
$$\bigvee_{t,p \in \text{singl}(A,\delta)} t(x) \otimes (\bigvee_{a \in A} t(a) \otimes \delta(a,y)) \leq \bigvee_{t \in \text{singl}(A,\delta)} t(x) \otimes t(y) \leq \delta(x,y).$$

Then $S = \{e_S\}$. In fact, let $s \in \text{singl}(A,\delta)$, then we have

$$\{e_S\}(s) = \tau_{(A,\delta)}(e_S, s) = \bigvee_{x \in A} \bigvee_{t \in \text{singl}(A,\delta)} t(x) \otimes S(t) \otimes s(x) =$$
$$\bigvee_{t \in \text{singl}(A,\delta)} (\bigvee_{x \in A} t(x) \otimes s(x)) \otimes S(t) =$$
$$\bigvee_{t \in \text{singl}(A,\delta)} \tau_{(A,\delta)}(s,t) \otimes S(t) \geq \tau_{(A,\delta)}(s,s) \otimes S(s) = S(s),$$

and on the other hand since $\tau(s,t)\otimes S(t)\leq S(s)$, we obtain that $\{e_S\}(s)=S(s)$. We define a functor

$$C:\mathbf{SetF}(\Omega)\to\mathbf{SetF}(\Omega)_{comp}$$

such that $C(A,\delta)=(\mathrm{singl}(A,\delta),\tau_{(A,\delta)})$ and for a morphism $f:(A,\delta)\to(B,\gamma)$ in $\mathbf{SetF}(\Omega)$ we set $C(f)=\overline{f}$, where $\overline{f}(s)(b)=\bigvee_{a\in A}s(a)\otimes\gamma(f(a),b)$. Then $\overline{f}$ is a morphism in $\mathbf{SetF}(\Omega)_{comp}$. In fact it is clear that $\overline{f}(s)$ is a normal extensional subobject in (B,γ). For any $b,c\in B$ we have

$$\begin{aligned}\overline{f}(s)(b)\otimes\overline{f}(s)(c)=\bigvee_{x,y\in A}s(x)\otimes s(y)\otimes\gamma(f(x),b)\otimes\gamma(f(y),c)\leq\\ \bigvee_{x,y\in A}\delta(x,y)\otimes\gamma(f(x),b)\otimes\gamma(f(x),c)\leq\\ \bigvee_{x,y\in A}\gamma(f(x),f(y))\otimes\gamma(f(x),b)\otimes\gamma(f(y),c)\leq\\ \bigvee_{x,y\in A}\gamma(b,f(y))\otimes\gamma(f(y),c)\leq\gamma(b,c).\end{aligned}$$

Hence $\overline{f}(s)$ is a singleton in (B,γ). We show further that $\overline{f}:(\mathrm{singl}(A,\delta),\tau_{(A,\delta)})\to(\mathrm{singl}(B,\gamma),\tau_{(B,\gamma)})$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, we have

$$\begin{aligned}\tau_{(B,\gamma)}(\overline{f}(s),\overline{f}(t))=\bigvee_{b\in B}\overline{f}(s)(b)\otimes\overline{f}(t)(b)=\\ \bigvee_{b\in B}(\bigvee_{x\in A}s(x)\otimes\gamma(f(x),b))\otimes(\bigvee_{y\in A}t(y)\otimes\gamma(f(y),b))\geq\\ \bigvee_{b\in B}\bigvee_{x\in A}s(x)\otimes\gamma(f(x),b)\otimes t(x)\otimes\gamma(f(x),b)\geq\\ \bigvee_{x\in A}s(x)\otimes\gamma(f(x),f(x))\otimes t(x)\otimes\gamma(f(x),f(x))=\\ \bigvee_{x\in A}s(x)\otimes t(x)=\tau_{(A,\delta)}(s,t).\end{aligned}$$

Let us now consider the singleton map

$$(A,\delta)\to C(A,\delta)=(\mathrm{singl}(A,\delta),\tau_{(A,\delta)}).$$

We show that C is a quasi-reflector. Since $\tau(\{x\},\{y\})=\delta(x,y)$, it is clear that $\{-\}$ is a morphism in $\mathbf{SetF}(\Omega)$. Moreover let (B,γ) be a complete object and let $f:(A,\delta)\to(B,\gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\widetilde{f}:(\mathrm{singl}(A,\delta),\tau)\to(B,\gamma)$ such that the diagram commutes:

$$\begin{array}{ccc}(A,\delta) & \xrightarrow{\{-\}} & (\mathrm{singl}(A,\delta),\tau_{(A,\delta)})\\ f\downarrow & & \downarrow\widetilde{f}\\ (B,\gamma) & = & (B,\gamma)\end{array}$$

The map $\widetilde{f}$ is defined as follows. Let $s\in\mathrm{singl}(A,\delta)$. Then $C(f)(s)=\overline{f}(s):B\to\Omega$ is a singleton in (B,γ). Since (B,γ) is complete, there exists the unique element $b\in B$ such that $\overline{f}(s)=\{b\}$. We set $\widetilde{f}(s)=b$. We show that $\widetilde{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, let $\widetilde{f}(s)=b,\widetilde{f}(t)=c$. Then we have $\gamma(\widetilde{f}(s),\widetilde{f}(t))=\gamma(b,c)=\tau_{(B,\gamma)}(\{b\},\{c\})=\tau_{(B,\gamma)}(\overline{f}(s),\overline{f}(t))\geq\tau_{(A,\delta)}(s,t)$ since $\overline{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. We show that the above mentioned diagram commutes. In fact, let $a\in A$, then we have $\widetilde{f}(\{a\})=b$, where $\overline{f}(\{a\})=\{b\}$. But we have $\overline{f}(\{a\})(y)=\bigvee_{x\in A}\{a\}(x)\otimes\gamma(f(x),y)=\bigvee_{x\in A}\delta(a,x)\otimes\gamma(f(x),y)\geq\delta(a,a)\otimes\gamma(f(a),y)=\gamma(f(a),y)=\{f(a)\}(y)$. On the other hand we have $\overline{f}(\{a\})(y)\leq\bigvee_{x\in A}\gamma(f(a),f(x))\otimes\gamma(f(x),y)\leq\bigvee_{x\in A}\gamma(f(a),y)=\gamma(f(a),y)=\{f(a)\}(y)$. Hence we have $\{b\}=\overline{f}(\{a\})=\{f(a)\}$ and it follows that $b=f(a)$.

Theorem 2.2 *There exists a functor $D : \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ which is a quasi-reflector.*

Let (A,δ) be an object of $\mathbf{SetF}(\Omega)$ and let $f : (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. We define a functor $D : \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ such that

$$D(A,\delta) = (\Omega^{(A,\delta)}, \sigma_{(A,\delta)}), \quad D(f) : D(A,\delta) \to D(B,\gamma),$$
$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad D(f)(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(b, f(x)).$$

It is clear that this definition is correct.

Now let (A,δ) be an object in $\mathbf{SetF}(\Omega)$. We consider a map

$$(A,\delta) \to D(A,\delta) = (\Omega^{(A,\delta)}, \sigma_{(A,\delta)}).$$

We show that this map is a morphism in $\mathbf{SetF}(\Omega)$. In fact, for $x, y \in A$ we have

$$\sigma_{(A,\delta)}(\{x\},\{y\}) = \bigwedge_{a\in A} \delta(a,x) \leftrightarrow \delta(a,y) \geq$$
$$\bigwedge_{a\in A} (\delta(a,x) \to \delta(x,y) \otimes \delta(a,x)) \wedge (\delta(a,y) \to \delta(a,y) \otimes \delta(x,y)) \geq \delta(x,y).$$

On the other hand we have $\sigma(\{x\},\{y\}) \leq \delta(x,y)$ and it follows that $\sigma(\{x\},\{y\}) = \delta(x,y)$.

Finally, let $f : (A,\delta) \to (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\widehat{f}$ such that the following diagram commutes.

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{\{-\}} & D(A,\delta) \\ f\big\downarrow & & \big\downarrow\widehat{f} \\ (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)}) & = & (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)}) \end{array}$$

The morphism $\widehat{f} : \Omega^{(A,\delta)} \to \Omega^{(B,\gamma)}$ is defined as follows.

$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad \widehat{f}(s)(b) = \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s).$$

This definition is correct. In fact, we show firstly that $\widehat{f}(s) \in \Omega^{(B,\gamma)}$. Let $b, c \in B$, then we have

$$\widehat{f}(s)(b) \otimes \gamma(b,c) = \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s) \otimes \gamma(b,c) \leq$$
$$\bigvee_{x\in A} f(x)(c) \otimes \sigma_{(A,\delta)}(\{x\}, s) = \widehat{f}(s)(c).$$

Further, $\widehat{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, we have

$$\sigma_{(B,\gamma)}(\widehat{f}(s), \widehat{f}(t)) = \bigwedge_{b\in B} \widehat{f}(s)(b) \leftrightarrow \widehat{f}(t)(b).$$

According to Lemma 4 and Lemma 5 we have

$$\widehat{f}(s)(b) \leftrightarrow \widehat{f}(t)(b) =$$
$$(\bigvee_{a\in A} f(a)(b) \otimes \sigma_{(A,\delta)}(\{a\}, s)) \leftrightarrow (\bigvee_{c\in A} f(c)(b) \otimes \sigma_{(A,\delta)}(\{c\}, t)) \geq$$
$$\bigwedge_{x\in A} (f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s) \leftrightarrow f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, t)) \geq$$
$$\bigwedge_{a\in A} \sigma_{(A,\delta)}(\{a\}, s) \leftrightarrow \sigma_{(A,\delta)}(\{a\}, t) =$$
$$\bigwedge_{a\in A} ((\bigwedge_{x\in A} \delta(a, x) \leftrightarrow s(x)) \leftrightarrow (\bigwedge_{y\in A} \delta(a, y) \leftrightarrow t(y))) \geq$$
$$\bigwedge_{a\in A} \bigwedge_{x\in A} (\delta(a, x) \leftrightarrow s(x)) \leftrightarrow (\delta(a, x) \leftrightarrow t(x)) \geq$$
$$\bigwedge_{a\in A} \bigwedge_{x\in A} (\delta(a, x) \leftrightarrow \delta(a, x)) \otimes (s(x) \leftrightarrow t(x)) =$$
$$\bigwedge_{x\in A} s(x) \leftrightarrow t(x) = \sigma_{(A,\delta)}(s, t).$$

It follows that $\sigma_{(B,\gamma)}(\widehat{f}(s), \widehat{f}(t)) \geq \sigma_{(A,\delta)}(s, t)$. We show that the diagram commutes. In fact, let $a \in A, b \in B$. Then

$$\widehat{f}(\{a\})(b) = \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, \{a\}) =$$
$$\bigvee_{x\in A} f(x)(b) \otimes \delta(x, a) \leq \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(B,\gamma)}(f(x), f(a)) =$$
$$\bigvee_{x\in A} f(x)(b) \otimes (\bigwedge_{y\in B} f(x)(y) \leftrightarrow f(a)(y)) =$$
$$\bigvee_{x\in A} \bigwedge_{y\in B} f(x)(b) \otimes (f(x)(y) \leftrightarrow f(a)(y)) \leq \bigvee_{x\in A} f(x)(b) \otimes (f(x)(b) \leftrightarrow f(a)(b)) \leq$$
$$\bigvee_{x\in A} f(x)(b) \otimes (f(x)(b) \to f(a)(b)) \leq f(a)(b)$$

On the other hand we have

$$\widehat{f}(\{a\})(b) = \bigvee_{x\in A} f(x)(b) \otimes \delta(x, a) \geq f(a)(b).$$

Hence the diagram commutes and $D : \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ is a quasi-reflector.

It should be observed that $\widehat{f}$ is the smallest morphism for which the above diagram commutes. In fact, let $g : D(A, \delta) \to (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$ such that the diagram commutes. Then for any $s \in \Omega^{(A,\delta)}, b \in B$ we have

$$\widehat{f}(s)(b) = \bigvee_{a\in A} f(a)(b) \otimes \sigma_{(A,\delta)}(\{a\}, s) \leq$$
$$\bigvee_{a\in A} f(a)(b) \otimes \sigma_{(B,\gamma)}(g(\{a\}), g(s)) = \bigvee_{a\in A} f(a)(b) \otimes \sigma_{(B,\gamma)}(f(a), g(s)) \leq$$
$$\bigvee_{a\in A} f(a)(b) \otimes \bigwedge_{x\in B} f(a)(x) \to g(s)(x) \leq \bigvee_{a\in A} f(a)(b) \otimes (f(a)(b) \to g(s)(b)) \leq$$
$$g(s)(b).$$

Theorem 2.3 *There exists a functor $E : \mathbf{SetF}(\Omega) \to \Omega_{\otimes}^{\mathbf{SetF}(\Omega)}$ which is a quasi-reflector.*

Let (A, δ) be an object in $\mathbf{SetF}(\Omega)$ and let $f : (A, \delta) \to (B, \gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. We define a functor E such that

$$E(A, \delta) = (\Omega^{(A,\delta)}, \tau_{(A,\delta)}), \quad E(f) : E(A, \delta) \to E(B, \gamma),$$
$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad D(f)(s)(b) = \bigvee_{x \in A} s(x) \otimes \gamma(b, f(x)).$$

It is clear that $\tau_{(A,\delta)}(s, t) \leq \tau_{(B,\gamma)}(E(f)(s), E(f)(t))$ for any $s, t \in \Omega^{(A,\delta)}$ and it follows that E is defined correctly. We consider a map

$$(A, \delta) \to (\Omega^{(A,\delta)}, \tau_{(A,\delta)})$$

such that $\{-\}(a)(b) = \tau_{(A,\delta)}(\{a\}, \{b\})$. Since $\tau_{(A,\delta)}(\{a\}, \{b\}) = \delta(a, b)$, the definition is correct.

Finally, let $f : (A, \delta) \to (\Omega^{(B,\gamma)}, \tau_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\widehat{f}$ such that the following diagram commutes.

$$\begin{array}{ccc} (A, \delta) & \xrightarrow{\{-\}} & E(A, \delta) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ (\Omega^{(B,\gamma)}, \tau_{(B,\gamma)}) & = & (\Omega^{(B,\gamma)}, \tau_{(B,\gamma)}) \end{array}$$

The morphism $\widehat{f} : E(A, \delta) \to \Omega^{(B,\gamma)}$ is defined as follows.

$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad \widehat{f}(s)(b) = \bigvee_{x \in A} f(x)(b) \otimes \tau_{(A,\delta)}(\{x\}, s) = \bigvee_{x \in A} f(x)(b) \otimes s(x).$$

We show that $\widehat{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, let $s, t \in \Omega^{(A,\delta)}$. Then we have

$$\begin{aligned} \tau_{(B,\gamma)}(\widehat{f}(s), \widehat{f}(t)) &= \bigvee_{y \in B} \bigvee_{x,z \in A} f(x)(y) \otimes f(z)(y) \otimes s(x) \otimes t(z) = \\ &\bigvee_{x,z \in A} (\bigvee_{y \in B} f(x)(y) \otimes f(z)(y)) \otimes s(x) \otimes t(z) = \\ \bigvee_{x,z \in A} \tau_{(B,\gamma)}(f(x), f(z)) \otimes s(x) \otimes t(z) &\geq \bigvee_{x,z \in A} \delta(x, z) \otimes s(x) \otimes t(z) \\ &\geq \bigvee_{x \in A} s(x) \otimes t(x) = \tau_{(A,\delta)}(s, t). \end{aligned}$$

The above mentioned diagram commutes. In fact, let $x \in A, b \in B$. Then we have

$$\widehat{f}(\{a\})(b) = \bigvee_{x \in A} f(x)(b) \otimes \{a\}(x) \geq f(a)(b),$$
$$\widehat{f}(\{a\})(b) = \bigvee_{x \in A} f(x)(b) \otimes \delta(a, x) \leq$$
$$\bigvee_{x \in A} f(x)(b) \otimes \tau_{(B,\gamma)}(f(x), f(a)) \leq \bigvee_{x \in A} f(x)(b) \otimes \sigma_{(B,\gamma)}(f(x), f(a)) \leq f(a)(b).$$

Hence, $E : \mathbf{SetF}(\Omega) \to \Omega_{\otimes}^{\mathbf{SetF}(\Omega)}$ is a quasi-reflector.

Theorem 2.4 *There exists a functor $G : \mathbf{Set}(\Omega) \to \Omega_{1,\otimes}^{\mathbf{Set}(\Omega)}$ which is a quasi-reflector.*

The functor G is defined such that $G(A,\delta)=(\Omega_1^{(A,\delta)},\tau_{(A,\delta)})$ which is considered as an object of a category $\mathbf{Set}(\Omega)$. Let $f:(A,\delta)\to(B,\gamma)$ be a morphism in $\mathbf{Set}(\Omega)$. The morphism $G(f)$ will be defined later. Firstly let us define a morphism $v_{(A,\delta)}=v:(A,\delta)\to G(A,\delta)$ in $\mathbf{Set}(\Omega)$ such that $v=F(\{-\})$, where $\{-\}:(A,\delta)\to(\Omega_1^{(A,\delta)},\tau_{(A,\delta)})$ is a singleton morphism in $\mathbf{SetF}(\Omega)$ and F is a functor from Lemma 1. This definition is correct since for any $a,b\in A$ we have $\tau_{(A,\delta)}(\{a\},\{b\})=\delta(a,b)$ as it can be proved simply. Then for any object (B,γ) and any morphism $f:(A,\delta)\to(\Omega_1^{(B,\gamma)},\tau_{(B,\gamma)})$ in $\mathbf{Set}(\Omega)$ there exists a morphism $\widetilde{f}$ in $\mathbf{Set}(\Omega)$ such that the diagram commutes.

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{v_{(A,\delta)}} & G(A,\delta) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ (\Omega_1^{(B,\gamma)},\tau_{(B,\gamma)}) & = & (\Omega_1^{(B,\gamma)},\tau_{(B,\gamma)}) \end{array}$$

In fact, we set

$$(\forall s\in\Omega_1^{(A,\delta)})(\forall t\in\Omega_1^{(B,\gamma)})\quad \widetilde{f}(s,t)=\bigvee_{x\in A} f(x,t)\otimes v_{(A,\delta)}(x,s).$$

Then we have

$$\bigvee_{t\in\Omega_1^{(B,\gamma)}}\widetilde{f}(s,t)=\bigvee_{t\in\Omega_1^{(B,\gamma)}}\bigvee_{x\in A} f(x,t)\otimes v(x,s)=$$
$$\bigvee_{x\in A}\Big(\bigvee_{t\in\Omega_1^{(B,\gamma)}} f(x,t)\Big)\otimes v(x,s)=\bigvee_{x\in A} v(x,s)=\bigvee_{x\in A}\tau_{(A,\delta)}(\{x\},s)\geq$$
$$\bigvee_{x\in A} s(x)=1$$

Further,

$$\widetilde{f}(s,t)\otimes\widetilde{f}(s,t')=\bigvee_{a,b\in A} f(a,t)\otimes f(b,t')\otimes v(a,s)\otimes v(b,s)\leq$$
$$\bigvee_{a,b\in A} f(a,t)\otimes f(b,t')\otimes\delta(a,b)\leq\bigvee_{b\in A} f(b,t)\otimes f(b,t')\leq\tau_{(A,\delta)}(t,t').$$

The other properties of a morphism in $\mathbf{Set}(\Omega)$ can be proved analogously.
We show that the diagram commutes. In fact, let $a\in A, t\in\Omega_1^{(B,\gamma)}$. Then we have

$$(\widetilde{f}\circ v)(a,t)=\bigvee_{s\in\Omega_1^{(A,\delta)}} v(a,s)\otimes\widetilde{f}(s,t)=$$
$$\bigvee_{s\in\Omega_1^{(A,\delta)}} v(a,s)\otimes\Big(\bigvee_{x\in A} f(x,t)\otimes v(x,s)\Big)\geq$$
$$v(a,\{a\})\otimes\bigvee_{x\in A} f(x,t)\otimes v(x,\{a\})=\bigvee_{x\in A} f(x,t)\otimes v(x,\{a\})\geq$$
$$f(a,t),$$

and on the other hand we have

$$\widetilde{f}\circ v(a,t)=\bigvee_{s\in\Omega_1^{(A,\delta)}}\bigvee_{x\in A}\tau_{(A,\delta)}(\{a\},s)\otimes\tau_{(A,\delta)}(\{x\},s)\otimes f(x,t)\leq$$
$$\bigvee_{s\in\Omega_1^{(A,\delta)}}\bigvee_{x\in A}\tau_{(A,\delta)}(\{a\},\{x\})\otimes f(x,t)=\bigvee_{x\in A}\delta(a,x)\otimes f(x,t)\leq f(a,t).$$

The morphism $G(f) : G(A,\delta) \to G(B,\gamma)$ will be defined such that $G(f) = \widetilde{v_{(B\gamma)}} \circ f$. Hence, more explicitly, we have

$$G(f)(s,t) = \bigvee_{x\in A} \bigvee_{y\in B} f(x,y) \otimes v_{(B,\gamma)}(y,t) \otimes v_{(A,\delta)}(x,s).$$

Theorem 2.5 *There exists a functor $H : \mathbf{Set}(\Omega) \to \Omega^{\mathbf{Set}(\Omega)}$ which is a quasi-reflector.*

Let (A,δ) be na object in $\mathbf{Set}(\Omega)$. We set $H(A,\delta) = (\text{w-singl}(A,\delta), \rho_{(A,\delta)})$. For an element $a \in A$ we define a morphism $[a] : (A,\delta) \to (\Omega, \leftrightarrow)$ in a category $\mathbf{Set}(\Omega)$ such that $[a](x,\alpha) = F(\{a\})(x,\alpha)$, for any $x \in A, \alpha \in \Omega$, where F is a functor from Lemma 1 and $\{a\} : (A,\delta) \to (\Omega,\leftrightarrow)$ is a morphism in a category $\mathbf{SetF}(\Omega)$ such that $\{a\}(x) = \delta(a,x)$. Then $[a]$ is a normal extensional subobject of (A,δ) in a category $\mathbf{Set}(\Omega)$. Since $[a](x,1) \otimes [a](y,1) \le \delta(x,y)$ we obtain that $[a]$ is a weak singleton. Moreover, since $\rho_{(A,\delta)}([a],[b]) = \delta(a,b)$ we obtain that $[-] : (A,\delta) \to (\text{w-singl}(A,\delta), \rho_{(A,\delta)})$ is a morphism in $\mathbf{SetF}(\Omega)$.

We define a morphism $u = u_{(A,\delta)} : (A,\delta) \to H(A,\delta) = (\text{w-singl}(A,\delta), \rho_{(A,\delta)})$ in $\mathbf{Set}(\Omega)$ such that $u(a,s) = F([-])(a,s)$ for all $a \in A, s \in \text{w-singl}(A,\delta)$, where F is a functor from Lemma 1.

Let $f : (A,\delta) \to (\text{w-singl}(B,\gamma), \rho_{(B,\gamma)})$ be a morphism in $\mathbf{Set}(\Omega)$. Then we define a morphism $\widetilde{f} : (\text{w-singl}(A,\delta), \rho_{(A,\delta)}) \to (\text{w-singl}(B,\gamma), \rho_{(B,\gamma)})$ such that

$$\widetilde{f}(s,t) = \bigvee_{x\in A} f(x,t) \otimes u(x,s),$$

for all $s \in \text{w-singl}(A,\delta), t \in \text{w-singl}(B,\gamma)$. Then $\widetilde{f}$ is a morphism in $\mathbf{Set}(\Omega)$. In fact, we have

$$\bigvee_{t\in \text{w-singl}(B,\gamma)} \widetilde{f}(s,t) = \bigvee_{x\in A} \Big(\bigvee_{t\in \text{w-singl}(B,\gamma)} f(x,t)\Big) \otimes u(x,s) =$$
$$\bigvee_{x\in A} u(x,s) = \bigvee_{x\in A} \bigwedge_{y\in A} \delta(x,y) \leftrightarrow s(y,1) \ge \bigvee_{x\in A} s(x,1) = 1.$$

Moreover, the following diagram commutes.

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{\ u\ } & (\text{w-singl}(A,\delta), \rho_{(A,\delta)}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ (\text{w-singl}(B,\gamma), \rho_{(B,\gamma)}) & = & (\text{w-singl}(B,\gamma), \rho_{(B,\gamma)}) \end{array}$$

In fact, we have

$$\widetilde{f} \circ u(a,t) = \bigvee_{s\in \text{w-singl}(A,\delta)} u(a,s) \otimes \widetilde{f}(s,t) =$$
$$\bigvee_{s\in \text{w-singl}(A,\delta)} u(a,s) \otimes \bigvee_{x\in A} u(x,s) \otimes f(x,t) \ge u(a,[a]) \otimes \bigvee_{x\in A} u(x,[a]) \otimes f(x,t) =$$
$$\bigvee_{x\in A} u(x,[a]) \otimes f(x,t) \ge f(a,t),$$

and on the other hand

$$\widetilde{f} \circ u(a,t) \le \bigvee_{x\in A} \rho_{(A,\delta)}([a],[x]) \otimes f(x,t) = \bigvee_{x\in A} \delta(x,a) \otimes f(x,t) \le f(a,t).$$

Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{Set}(\Omega)$. The morphism $H(f)$ will be defined as $\widetilde{u_{(B,\gamma)}} \circ f$.

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