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Covariant functors in categories of fuzzy sets over MV-algebras

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1 Introduction

In general fuzzy set theory several categories are important as a natural generalization of classical $[0,1]$-fuzzy sets. A special position among these categories has the category $\mathbf{SetF}(\Omega)$ of Ω-fuzzy sets $\mathbf{A} = (A, \delta)$, where $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete MV-algebra, A is a set and $\delta : A \times A \rightarrow \Omega$ is a similarity relation. A morphism $f : (A, \delta) \rightarrow (B, \beta)$ in this category is a map $f : A \rightarrow B$ such that

(a) $(\forall x, y \in A) \quad \beta(f(x), f(y)) \geq \delta(x, y)$,

(b) $(\forall x \in A) \quad \beta(f(x), f(x)) = \delta(x, x)$.

This category was (in a little more general form) introduce by U.Höhle [5]-[9]. He observed that this category (although it is not a topos) can be used for interpretation of some part of fuzzy logic. This property of the category $\mathbf{SetF}(\Omega)$ is connected with some special fuzzy sets and subobjects of $(A, \delta) \in \mathbf{SetF}(\Omega)$. Recall ([13],[14]) that a map $s : A \rightarrow \Omega$ is called *extensional set* ($s \subseteq (A, \delta)$, in symbol) if

(a) $(\forall x \in A) \quad s(x) \leq \delta(x, x)$,

(b) $(\forall x, y \in A) \quad s(x) \otimes (\delta(x, x) \rightarrow \delta(x, y)) \leq s(y)$.

Moreover a subset $S \subseteq A$ is called *complete* (see [13]) in (A, δ) if

$$S = \{a \in A : \bigvee_{x \in S} \delta(a, x) = \delta(a, a)\}.$$

Complete subsets in (A, δ) then define some closure system $X \mapsto \overline{X}$. Principal properties of the category $\mathbf{SetF}(\Omega)$ can be then described by some special *contravariant* functors

$$\text{Sub},\ \text{Sub}^c, \mathcal{S},\ \text{Hom}(-, \Omega^*) : \mathbf{SetF}(\Omega)^{op} \rightarrow \mathbf{Set},$$

where for a morphism $f : (A, \delta) \rightarrow (B, \beta)$ in the category $\mathbf{SetF}(\Omega)$ we have

(a)

$$\begin{aligned} \text{Sub}(A, \delta) &= \{(S, \delta) : S \subseteq A\}, \\ \text{Sub}(f) :\ \text{Sub}(B, \beta) &\rightarrow\ \text{Sub}(A, \delta), \\ \text{Sub}(f)(T, \beta) &= (f^{-1}(T), \delta),\ \text{for } (T, \beta) \in\ \text{Sub}(B, \beta), \end{aligned}$$

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(b)

$$\begin{aligned} \mathrm{Sub}^c(A,\delta) &= \{(S,\delta) : S \subseteq A \text{ is complete in } (A,\delta)\}, \\ \mathrm{Sub}^c(f) &: \ \mathrm{Sub}^c(B,\beta) \to \ \mathrm{Sub}^c(A,\delta), \\ \mathrm{Sub}^c(f)(T,\beta) &= (f^{-1}(T),\delta), \text{ for } (T,\beta) \in \ \mathrm{Sub}^c(B,\beta), \end{aligned}$$

(c)

$$\begin{aligned} \mathcal{S}(A,\delta) &= \{s : s \subseteq (A,\delta) \text{ is an extensional set}\}, \\ \mathcal{S}(f) &: \mathcal{S}(B,\beta) \to \mathcal{S}(A,\delta), \\ \mathcal{S}(f)(t) &= t.f, \text{ for } t \in \mathcal{S}(B,\beta), \end{aligned}$$

(d)

$$\begin{aligned} \mathrm{Hom}((A,\delta),\Omega^*) &= \{g : (A,\delta) \xrightarrow{\ g\ } (\Omega^*,\mu) \text{ is a morphism in } \mathbf{SetF}(\Omega)\}, \\ \mathrm{Hom}(f) &: \ \mathrm{Hom}((B,\beta),\Omega^*) \to \ \mathrm{Hom}((A,\delta),\Omega^*), \\ \mathrm{Hom}(f)(h) &= h.f, \text{ where } h \in \ \mathrm{Hom}((B,\beta),\Omega^*). \end{aligned}$$

Here we have

$$\begin{aligned} \Omega^* &= (\{(\alpha,\beta) \in L \times L \mid \alpha \geq \beta\},\mu), \\ \mu((\alpha_1,\beta_1),(\alpha_2,\beta_2)) &= \alpha_1 \otimes (\beta_1 \to \beta_2) \wedge \alpha_2 \otimes (\beta_2 \to \beta_1). \end{aligned}$$

In [13] we introduced several natural transformations between pairs of these contravariant functors. The fact that all these functors are *contravariant* could be interpreted as some disadvantage of these functors. In fact, one of the most important tools in classical fuzzy set theory is *Zadeh's extension principle*. It enables to extend any map $f : A \to B$ into a map $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ where, for every set X, $\mathcal{F}(X)$ denotes the class of fuzzy subsets of X, i.e. maps $s : X \to \Omega$ (in case we consider Ω-fuzzy sets instead of $[0,1]$-fuzzy sets only). Namely,

$$\mathcal{F}(f)(b) = \bigvee_{a \in A, f(a)=b} s(a),$$

for $s \in \mathcal{F}(A)$ and $b \in f(A) \subseteq B$. Hence, by using this principle we can obtain a *covariant* functor

$$\mathcal{F} : \Omega - \text{ fuzzy sets } \to \mathbf{Set},$$

(here we do not specify a category of corresponding Ω-fuzzy sets).

Unfortunately the same approach cannot be used in the category $\mathbf{SetF}(\Omega)$, where the extensional sets $s \subseteq (A,\delta)$ play the role of Ω-fuzzy sets. In fact, if $f : (A,\delta) \to (B,\beta)$ is a morphism in the category $\mathbf{SetF}(\Omega)$ and $s \subseteq (A,\delta)$ is an extensional set then $\mathcal{F}(f)(s)$ is not an extensional set in (B,β), in general, as the following example shows.

Example 1.1

Let Ω be a complete MV-algebra such that there exists $\alpha \in \Omega$ such that $\alpha^3 = \alpha \otimes \alpha \otimes \alpha < \alpha \otimes \alpha = \alpha^2$. Let $A = \{a_1, a_2\}$ and define $\delta, \beta : A \times A \to \Omega$ such that

$$\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise,} \end{cases} \quad \beta(x,y) = \begin{cases} 1, & \text{if } x = y \\ \alpha, & \text{otherwise.} \end{cases}$$

It is clear that $(A,\delta),(A,\beta) \in \mathbf{SetF}(\Omega)$ and that $f = \mathbf{id} : (A,\delta) \to (A,\beta)$ is a morphism in this category. Let $s : A \to \Omega$ be defined such that

$$s(a_1) = \alpha, \quad s(a_2) = \alpha^3.$$

Then $s \subseteq (A, \delta)$ and $\mathcal{F}(f)(s)(a_i) = s(a_i); \quad i = 1, 2$. On the other hand we have

$$\alpha^2 = \mathcal{F}(f)(s)(a_1) \otimes (\beta(a_1, a_1) \to \beta(a_1, a_2)) \not\leq \mathcal{F}(f)(s)(a_2) = \alpha^3,$$

and it follows that $\mathcal{F}(f)(s)$ is not extensional set in (A, β).

The principal aim of this paper is to introduce some generalization of Zadeh's extension principle for objects which are of some interest from fuzzy set theory point of view and which could be derived from objects of the category $\mathbf{SetF}(\Omega)$. For any object $(A, \delta) \in \mathbf{SetF}(\Omega)$ there are three principal sets of such objects:

(a) a set $\mathcal{C}(\mathcal{A}, \delta)$ of all extensional sets $s \subseteq (A, \delta)$,

(b) a set $\mathrm{sub}(A, \delta)$ of all subobjects of (A, δ) which are of the form (S, δ), where $S \subseteq A$, and

(c) a set $\hom(A, \delta)$ of all morphisms from (A, δ) to Ω^*.

Elements of all these sets then represent some generalized fuzzy sets in a category $\mathbf{SetF}(\Omega)$. It means that generalizations of Zadeh's extension principle to the category $\mathbf{SetF}(\Omega)$ are based on those covariant functors $\mathcal{C}$, sub, hom : $\mathbf{SetF}(\otimes) \to \mathbf{Set}$, respectively. In this paper we introduce these functors and we investigate some natural transformations between pairs of these functors. Finally we derive a generalizations of Zadeh's extension principle based on these covariant functors.

2 Complete sets and extensional sets in $\mathbf{SetF}(\Omega)$

Let $\Omega = (L, \wedge, \vee, \otimes, \to, 1_\Omega, 0_\Omega)$ be a complete MV- algebra, i.e. a complete residuated lattice, where $(a \to b) \to b = a \vee b$ holds for every $a, b \in L$. By an *Ω-fuzzy set* we mean (A, δ), where A is a set and $\delta : A \times A \to \Omega$ is a map such that

(a) $(\forall x, y \in A) \quad \delta(x, y) \leq \delta(x, x) \wedge \delta(y, y)$,

(b) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,

(c) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes (\delta(y, y) \to \delta(y, z)) \leq \delta(x, z)$.

Moreover, a Ω-fuzzy set (A, α) is called *separated* if it satisfies the axiom

$$\alpha(x, x) \vee \alpha(y, y) \leq \alpha(x, y) \Rightarrow x = y.$$

The category $\mathbf{SetF}(\Omega)$ of Ω-fuzzy sets then consists of separated Ω-fuzzy sets as objects and morphisms between objects $(A, \alpha), (B, \beta)$, which are maps $f : A \to B$ such that

(a) $(\forall x, y \in A) \quad \beta(f(x), f(y)) \geq \alpha(x, y)$,

(b) $(\forall x \in A) \quad \alpha(x, x) = \beta(f(x), f(x))$.

The composition of morphisms is the usual composition of maps. In this category $\mathbf{SetF}(\Omega)$ a special object (Ω^*, μ) exists (see the definition from the introduction) which is an analogy of a subobject classifier in a topos but only for some special subobjects. These subobjects are connected with *extensional sets* of an Ω-fuzzy set $\mathbf{A} = (A, \delta)$ ($s \subseteq (A, \delta)$, in symbol), i.e. with maps $s : A \to \Omega$ such that

(a) $(\forall x \in A) \quad s(x) \leq \delta(x, x)$,

(b) $(\forall x, y \in A) \quad s(x) \otimes (\delta(x, x) \to \delta(x, y)) \leq s(y)$.

In [13] we introduced a *contravariant* functor $\mathcal{S} : \mathbf{SetF}(\Omega)^{op} \to \mathbf{Set}$ such that $\mathcal{S}(A, \delta) = \{s : s \subseteq (A, \delta)\}$. This functor then represents a *contravariant* generalization of classical Zadeh's extension principle which could be defined as a covariant functor $\mathcal{F} : \mathbf{SetF}(\Omega) \to \mathbf{Set}$, where $\mathcal{F}(A, \delta) = \{s : s : A \to \Omega \text{ is a map }\}$. As we observed this covariant functor does not respect the fact that s is extensional set in (A, δ), in general. In the following definition we do the first step to derive a generalized Zadeh's extension principle for extensional sets in a category $\mathbf{SetF}(\Omega)$, i.e. we will define a map $\mathcal{S}(\mathcal{A}, \delta) \to \mathcal{S}(\mathcal{B}, \beta)$.

Definition 2.1 *Let $f : (A, \delta) \to (B, \beta)$ be a morphism in a category* $\mathbf{SetF}(\Omega)$*. Then the map $\widehat{f} : \mathcal{S}(A, \delta) \to \mathcal{S}(B, \beta)$ is defined such that for $s \in \mathcal{S}(A, \delta)$,*

$$b \in B, \quad \widehat{f}(s)(b) = \bigvee_{x \in A} s(x) \otimes \beta(b, f(x)).$$

This definition is correct, i.e. $\widehat{f}(s) \in \mathcal{S}(\mathcal{B}, \beta)$. In fact, we have $\widehat{f}(s)(s) \leq \beta(b, b)$ and for $b, c \in B$,

$$\widehat{f}(s)(b) \otimes (\beta(b, b) \to \beta(b, c)) =$$
$$= \bigvee_{x \in A} s(x) \otimes \beta(f(x), b) \otimes (\beta(b, b) \to \beta(b, c)) \leq$$
$$\leq \bigvee_{x \in A} s(x) \otimes \beta(f(x), c) = \widehat{f}(s)(c).$$

Recall that a subset $S \subseteq A$ is called *complete* in (A, δ) if

$$S = \overline{S} = \{a \in A : \delta(a, a) = \bigvee_{x \in S} \delta(a, x)\}.$$

Then $X \mapsto \overline{X}$ is a closure system (see [13]) which is used for some functors construction. It could be observed that any extensional set $s \subseteq (A, \delta)$ is then (in some sense) continuous with respect to the closure system in the full subcategory $\mathbf{SetF}(\Omega)_0$ of $\mathbf{SetF}(\Omega)$ which consists of Ω-fuzzy set (A, δ) such that $\delta(x, x) = 1_\Omega$ for all $x \in A$. In fact, on the value MV-algebra Ω a similarity relation ω can be defined such that $\omega(\alpha, \beta) = \alpha \leftrightarrow \beta$. Then the following simple lemma holds.

Lemma 2.1 *Let $s \subseteq (A, \delta)$ be an extensional set and let $X \subseteq A$. Then $s : (A, \delta) \to (\Omega, \omega)$ is a morphisms in $\mathbf{SetF}(\Omega)_0$ and $s(\overline{X}) \subseteq \overline{s(X)}$.*

Proof. Since $s(x) \otimes \delta(x, y) \leq s(y))$, we have $\delta(x, y) \leq \omega(s(x), s(y))$. Further, for $y \in \overline{X}$ we have $\bigvee_{s(x) \in s(X)} \omega(s(y), s(x)) \geq \bigvee_{x \in X} \delta(y, x) = 1$.

In $\mathbf{SetF}(\Omega)_0$ for $s \subseteq (A, \delta)$ any α-cut of s is closed, i.e. for $s_\alpha = \{x \in A : s(x) \geq \alpha\}$ we have $\overline{s_\alpha} = s_\alpha$. In fact, let $a \in \overline{s_\alpha}$; i.e. $\bigvee_{x \in A, s(x) \geq \alpha} \delta(a, x) = \delta(a, a) = 1$. Since $s(a) \geq s(x) \otimes \delta(a, x)$ for any $x \in A$, we have

$$s(a) \geq \bigvee_{x \in s_\alpha} s(x) \otimes \delta(a, x) \geq \alpha \otimes \bigvee_{x \in s_\alpha} \delta(a, x) = \alpha \otimes \delta(a, a) = \alpha.$$

Hence, $a \in s_\alpha$.

It could be observed that this statement does not hold true in $\mathbf{SetF}(\Omega)$, in general. Let us consider the following example.

Example 2.1

Let Ω be such that there exists α with the property $\alpha^2 < \alpha$. Let $A = \{a_1, a_2\}$, $\delta(a_1, a_1) = \alpha, \delta(a_2, a_2) = \delta(a_1, a_2) = \alpha^2$. Then (A, δ) is Ω-fuzzy set and for $s : A \to \Omega$ such that $s(a_1) = \alpha, s(a_2) = \alpha^2$ we have $s_\alpha = \{a_1\}$, $\overline{s_\alpha} = \{a_1, a_2\}$.

The following lemma describes some analogy of this property of α-cuts of the map $\widehat{f}$.

Lemma 2.2 *Let $f : (A, \delta) \to (B, \beta)$ be a morphism in $\mathbf{SetF}(\Omega)$ such that $\beta(b, b) = 1$ for all $b \in B$ and let $s \in \mathcal{S}(\mathcal{A}, \delta)$. Then for any $\alpha \in \Omega$ we have*

$$\overline{f(s_\alpha)} \subseteq (\widehat{f}(s))_\alpha.$$

Proof. Let $b \in \overline{f(s_\alpha)}$, then we have $\beta(b,b) = \bigvee_{x\in s_\alpha} \beta(b, f(x))$ and it follows that

$$\widehat{f}(s)(b) = \bigvee_{x\in A} s(x) \otimes \beta(f(x), b) \geq \bigvee_{x\in s_\alpha} s(x) \otimes \beta(f(x), b) \geq$$
$$\alpha \otimes \bigvee_{x\in s_\alpha} \beta(f(x), b) = \alpha \otimes \beta(b,b) = \alpha.$$

It could be observed that the opposite inclusion in 2.2 does not hold, in general. In fact, let (A,δ), (A,β) and $s \subseteq (A,\delta)$ be the same as in Example 1.1. Then for $f = \mathbf{id} : (A,\delta) \to (A,\beta)$ we have

$$\widehat{f}(s)(a_1) = (\alpha \otimes \alpha) \vee (\alpha^3 \otimes \alpha) = \alpha,$$
$$\widehat{f}(s)(a_2) = (\alpha^3 \otimes 1) \vee (\alpha \otimes \alpha) = \alpha^2.$$

Hence, we have $(\widehat{f}(s))_{\alpha^2} = \{a_1, a_2\}$ and on the other hand $\overline{f(s_{\alpha^2})} = \overline{\{a_1\}} = \{a_1\}$.

Proposition 2.1 *Let $f : (A,\delta) \to (B,\beta)$ be a morphism in $\mathbf{SetF}(\Omega)$.*

(a) If $s \leq t$ in $\mathcal{S}(\mathcal{A},\delta)$ then $\widehat{f}(s) \leq \widehat{f}(t)$.

(b) For any $s,t \in \mathcal{S}(\mathcal{A},\delta)$, $\widehat{f}(s \vee t) = \widehat{f}(s) \vee \widehat{f}(t)$, where $(s \vee t)(a) = s(a) \vee t(a)$.

Proof. It is clear that $s \vee t \in \mathcal{S}(\mathcal{A},\delta)$. Moreover, for $b \in B$ we have

$$\widehat{f}(s \vee t)(b) = \bigvee_{x\in A} (s \vee t)(x) \otimes \beta(f(x), b) =$$
$$= \bigvee_{x\in A} (s(x) \otimes \beta(f(x), b)) \vee (t(x) \otimes \beta(f(x), b)) =$$
$$= \widehat{f}(s)(b) \vee \widehat{f}(t)(b) = (\widehat{f}(s) \vee \widehat{f}(t))(b).$$

Lemma 2.3 *Let $f : (A,\delta) \to (B,\beta)$ be a morphism in $\mathbf{SetF}(\Omega)$ and let $C \subseteq A$. Then f is continuous with respect to the closure systems in (A,δ) and (B,β), i.e.*

$$f(\overline{C}) \subseteq \overline{f(C)}.$$

Proof. Let $a \in \overline{C}$. Then we have

$$\beta(f(a), f(a)) \geq \bigvee_{x\in C} \beta(f(a), f(x)) \geq \bigvee_{x\in C} \delta(a, x) = \delta(a,a) = \beta(f(a), f(a))$$

and it follows that $f(a) \in \overline{f(C)}$.

The closure operator $X \mapsto \overline{X}$ in (A,δ) has some importance for functors and their natural transformations in the category $\mathbf{SetF}(\Omega)$. The following proposition shows when this closure space is discrete.

Proposition 2.2 *The closure operator in (A,δ) is discrete if and only if the following condition is satisfied in (A,δ):*

$$(\forall a \in A)(\exists \alpha_a < \delta(a,a))(\forall x \in A, x \neq a) \quad \delta(a,x) \leq \alpha_a. \tag{1}$$

Proof. Let the closure operator be discrete and let us assume by contradiction that there exists $a \in A$ such that for any $\alpha < \delta(a,a)$ there exists $x_\alpha \neq a$ in A with $\delta(a, x_\alpha) > \alpha$. We put $X = \{x_\alpha : \alpha \in \Omega, \alpha < \delta(a,a)\}$. Then $a \notin X$ and we have $\delta(a,a) = \bigvee_{x\in X} \delta(a,x)$. In fact, if $\bigvee_{x\in X} \delta(a,x) = \alpha_0 < \delta(a,a)$, we have $x_{\alpha_0} \in X$ and it follows that $\alpha_0 < \delta(a,a) \leq \alpha_0$, a contradiction. Hence, $a \in \overline{X}$, $a \notin X$ and the operator is not discrete. Conversely, let (A,δ) satisfy the condition (1) and let $X \subseteq A$. For $a \in \overline{X}$ we have $\delta(a,a) = \bigvee_{x\in X} \delta(a,x)$. If $a \notin X$ then according to (1) we have $\delta(a,x) < \alpha_a$ for all $x \in A$. Therefore, $\bigvee_{x\in X} \delta(a,x) \leq \alpha_a < \delta(a,a)$, a contradiction.

Proposition 2.3 *Let the closure operator in (A,δ) be discrete. Then there exists an extensional set $s \subseteq (A,\delta)$ which is sub-normal, i.e. $s(x) < \delta(x,x)$ for all $x \in A$.*

Proof. According to 2.2, for any $a \in A$ there exists $\alpha_a < \delta(a,a)$ satisfying the condition (1). Then the map $s : A \to \Omega$ can be defined such that $s(a) = \alpha_a$ and for any $x, y \in A, x \neq y$ we have

$$s(x) \otimes (\delta(a,a) \to \delta(x,y)) \leq s(x) \otimes (s(x) \to \delta(x,y)) =$$
$$= s(x) \wedge \delta(x,y) \leq s(x) \wedge \alpha_x \wedge \alpha_y \leq s(y).$$

Hence, $s \subseteq (A,\delta)$.

Lemma 2.4 *Let $(A,\delta) \in \mathbf{SetF}(\Omega)$ and $a \in A, X \subseteq A$. Then*

$$\bigvee_{x \in X} \delta(a,x) = \bigvee_{x \in \overline{X}} \delta(a,x).$$

Proof. We have

$$\bigvee_{y \in \overline{X}} \delta(a,y) = \bigvee_{y \in \overline{X}} \delta(a,y) \wedge \delta(y,y) =$$
$$= \bigvee_{y \in \overline{X}} \delta(a,y) \wedge (\bigvee_{z \in X} \delta(y,z)) = \bigvee_{y \in \overline{X}} \bigvee_{z \in X} \delta(a,y) \wedge \delta(y,z) \leq$$
$$\leq \bigvee_{y \in \overline{X}} \bigvee_{y \in X} \delta(a,y) \otimes (\delta(y,y) \to \delta(y,z)) \leq \bigvee_{z \in X} \delta(a,z)$$

3 Covariant functors in $\mathbf{SetF}(\Omega)$

In [13] we introduced several contravariant functors

$$\mathcal{S},\ \mathrm{Sub},\ \mathrm{Sub}^c,\ \mathrm{Hom}(-,\Omega^*) : \mathbf{SetF}(\Omega)^{op} \to \mathbf{Set}.$$

In this part we want to introduce covariant analogy of these functors which have the same object functions as the original functors. Nevertheless covariant versions of these functors will be defined for some full subcategory $\mathbf{SetF}(\Omega)_0$ only, which consists of Ω-fuzzy set (A,δ) which are *normal*, i.e. $\delta(a,a) = 1_\Omega$ for all $a \in A$. These covariant analogy of functors $\mathcal{S}$, Sub and Hom are introduced in the following definition.

Definition 3.1 *Let $f : (A,\delta) \to (B,\beta)$ be a morphism in $\mathbf{SetF}(\Omega)_0$. Then we set*

(a)

$$\mathcal{C}(A,\delta) := \mathcal{S}(A,\delta),$$
$$\mathcal{C}(f) : \mathcal{C}(A,\delta) \to \mathcal{C}(B,\beta), \quad \mathcal{C}(f) := \widehat{f}.$$

(b)

$$\mathrm{sub}(A,\delta) := \mathrm{Sub}(A,\delta),$$
$$\mathrm{sub}(f) : \mathrm{sub}(A,\delta) \to \mathrm{sub}(B,\beta),$$
$$(\forall (S,\delta) \in \mathrm{sub}(A,\delta)) \quad \mathrm{sub}(f)(S,\delta) := (f(S),\beta).$$

(c)

$$\mathrm{sub}^c(A,\delta) := \mathrm{Sub}^c(A,\delta),$$
$$\mathrm{sub}^c(f) : \mathrm{sub}^c(A,\delta) \to \mathrm{sub}^c(B,\beta),$$
$$(\forall (S,\delta) \in \mathrm{sub}^c(A,\delta)) \quad \mathrm{sub}^c(f)(S,\delta) := (\overline{f(S)},\beta).$$

(d)

$$\begin{aligned}\text{hom}(A,\delta) &:= \text{Hom}((A,\delta),\Omega^*),\\ \text{hom}(f):\ \text{hom}(A,\delta) &\to \text{hom}(B,\beta),\\ (\forall u\in \text{hom}(A,\delta), b\in B)\quad \text{hom}(f)(u)(b) &:= (1_\Omega, \widehat{f}(pr_2.u)(b)),\end{aligned}$$

where $pr_2:\Omega^*\to\Omega$ is the second projection map.

From the next lemma it follows that the definition of hom is correct.

Lemma 3.1 *Let $(A,\delta)\in \mathbf{SetF}(\Omega)$ and let $u:(A,\delta)\to(\Omega^*,\mu)$ be a morphism. Then $pr_2.u$ is an extensional set in (A,δ).*

Proof. Let $s=pr_2.u$ and $a\in A$. We set $u(a)=(u_1,u_2)\in\Omega^*$. Then we have $1_\Omega=\delta(a,a)=u_1$ and $s(a)\le\delta(a,a)$. Moreover, for $a,b\in A$ we have

$$\delta(a,b)\le\mu(u(a),u(b))\le\delta(a,a)\otimes(s(a)\to s(b)),$$

and it then follows that

$$\begin{aligned}&s(a)\otimes(\delta(a,a)\to\delta(a,b))\le\\ &\le s(a)\otimes(\delta(a,a)\to(\delta(a,a)\otimes(s(a)\to s(b))))=\\ &=s(a)\otimes((s(a)\to s(b))\vee\neg\delta(a,a))=\\ &=(s(a)\otimes(s(a)\to s(b)))\vee(s(a)\otimes\neg\delta(a,a))\le\\ &\le s(b)\vee(\delta(a,a)\otimes\neg\delta(a,a))=s(b).\end{aligned}$$

Hence, $s\subseteq(A,\delta)$.

Theorem 3.1 *$\mathcal{C}$, sub, sub^c and hom are covariant functors $\mathbf{SetF}(\Omega)_0\to\mathbf{Set}$.*

Proof. Let $f:\mathbf{A}=(A,\delta)\to(B,\beta)$, $g:(B,\gamma)\to(C,\gamma)$ be morphisms in $\mathbf{SetF}(\Omega)_0$.

(1) To prove that $\mathcal{C}$ is a functor we have to show that $\mathcal{C}(1_\mathbf{A})=1_{\mathcal{C}(\mathbf{A})}$ and $\mathcal{C}(g.f)=\mathcal{C}(g).\mathcal{C}(f)$. Let $s\in\mathcal{C}(A,\delta)$, $a\in A$. Then

$$s(a)=s(a)\otimes\delta(a,a)\le\bigvee_{x\in A}s(x)\otimes\delta(a,x)\le s(a).$$

Let $c\in C$. Then we have

$$\begin{aligned}\mathcal{C}(g).\mathcal{C}(f)(s)(c)&=\bigvee_{b\in B}\bigvee_{a\in A}s(a)\otimes\beta(f(a),b)\otimes\gamma(g(b),c)\le\\ &\bigvee_{b\in B}\bigvee_{a\in A}s(a)\otimes\gamma(gf(a),g(b))\otimes\gamma(g(b),c)\le\\ &\le\bigvee_{a\in A}s(a)\otimes\gamma(gf(a),c)=\mathcal{C}(g.f)(s)(c).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathcal{C}(g).\mathcal{C}(f)(s)(c)&\ge\bigvee_{a'\in A}\bigvee_{a\in A}s(a)\otimes\beta(f(a),f(a'))\otimes\gamma(gf(a'),c)\ge\\ &\ge\bigvee_{a'\in A}s(a')\otimes\gamma(gf(a'),c)=\mathcal{C}(g.f)(s)(c).\end{aligned}$$

Hence $\mathcal{C}$ is a functor.

(3) Let $(C,\delta)\in\text{sub}^c(A,\delta)$. Then according to Lemma 2.4, we have

$$\text{sub}^c(g.f)(C,\delta)=(\overline{g.f(C)},\gamma)=(\overline{g(\overline{f(C)})},\gamma)=\text{sub}^c(g).\,\text{sub}^c(f)(C,\delta).$$

(4) Let $u : (A, \delta) \to \Omega^*$ be a morphism. For simplicity we set $\tilde{u} := \hom(f)(u) : B \to \Omega^*$. We have to prove that $\tilde{u} : (B, \beta) \to (\Omega^*, \mu)$ is a morphism. Let $b, c \in B$. Then

$$\mu(\tilde{u}(b), \tilde{u}(c)) = \bigvee_{x\in A} pr_2.u(x) \otimes \beta(f(x), b) \leftrightarrow \bigvee_{z\in X} pr_2.u(z) \otimes \beta(f(z), c).$$

We have

$$\beta(b, c) \otimes \bigvee_{x\in A} (pr_2.u(x) \otimes \beta(f(x), b)) \le \bigvee_{x\in A} pr_2.u(x) \otimes \beta(f(x), c)$$

and it follows that

$$\beta(b, c) \le \bigvee_{x\in A} pr_2.u(x) \otimes \beta(f(x), b) \to \bigvee_{z\in A} pr_2.u(z) \otimes \beta(f(z), c).$$

By using the symmetry of β we obtain $\beta(b, c) \le \mu(\tilde{u}(b), \tilde{u}(c))$. Moreover, by simple computation we can show that $\hom(g.f) = \hom(g). \hom(f)$. The rest of the proof is trivial.

In [13] we introduced several natural transformations between pairs of contravariant functors $\mathcal{S}$, Sub, Hom. The following are principal of these transformations.

$$\mathcal{S} \xrightarrow{\ \zeta\ } \mathrm{Hom}(-, \Omega^*) \xrightarrow{\ \zeta^{-1}\ } \mathcal{S} \xrightarrow{\ \sigma\ } \mathrm{Sub},$$

where for $\mathbf{A} = (A, \delta), s \in \mathcal{S}(A, \delta), u \in \mathrm{Hom}(\mathbf{A}, \Omega^*)$ we have

$$\zeta_{\mathbf{A}}(s)(x) = (\delta(x, x), s(x)), \quad x \in A \tag{2}$$

$$\zeta_{\mathbf{A}}^{-1}(u) = pr_2.u, \tag{3}$$

$$\sigma_{\mathbf{A}}(s) = (\{a \in A : s(a) = \delta(a, a)\}, \delta). \tag{4}$$

Moreover, if $\mathbf{SetF}(\Omega)_1$ is a subcategory of $\mathbf{SetF}(\Omega)$ with the same objects and with morphisms $f : (A, \delta) \to (B, \beta)$ such that f is surjective and $\beta(f(x), f(y)) = \delta(x, y)$ for all $x, y \in A$ then according to [13]; 3.3, there exists a natural transformation

$$\psi : \mathrm{Sub}_1 \to \mathcal{S}_1$$

$$\psi_{\mathbf{A}}(S, \delta)(x) = \bigvee_{y\in S} \delta(x, y); \quad x \in A, \tag{5}$$

where Sub_1 and $\mathcal{S}_1$ are restrictions of $\mathrm{Sub}, \mathcal{S}$ onto $\mathbf{SetF}(\Omega)_1$, respectively.

In the next proposition we show the existence of some natural transformations also between covariant extension functors.

Proposition 3.1 *The object $\psi = \{\psi_{\mathbf{A}} : \mathbf{A} \in \mathbf{SetF}(\Omega)_0\}$ is a natural transformation $sub \to \mathcal{C}$, where $\psi_{\mathbf{A}}$ are defined by (5).*

Proof. According to [13];3.3, $\psi_{\mathbf{A}}(S, \delta)$ is a extensional set in (A, δ) for any $S \subseteq A$. We have to show only that for any morphism $(A, \delta) \xrightarrow{\ f\ } (B, \beta)$ in $\mathbf{SetF}(\Omega)_0$ the following diagram commutes.

$$\begin{array}{ccc} \mathrm{sub}(A, \delta) & \xrightarrow{\ \psi_{\mathbf{A}}\ } & \mathcal{C}(\mathcal{A}, \delta) \\ {\scriptstyle \mathrm{sub}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\{)} \\ \mathrm{sub}(B, \beta) & \xrightarrow{\ \psi_{\mathbf{B}}\ } & \mathcal{C}(\mathcal{B}, \beta). \end{array}$$

Let $(S, \delta) \in \mathrm{sub}(A, \delta)$ and $b \in B$. Then we have

$$\mathcal{C}(f).\psi_{\mathbf{A}}(S, \delta)(b) = \bigvee_{x\in A} \bigvee_{z\in S} \delta(z, x) \otimes \beta(f(x), b),$$

$$\psi_{\mathbf{B}}. \mathrm{sub}(f)(S, \delta)(b) = \bigvee_{y\in S} \beta(f(y), b).$$

On the other hand we have

$$\bigvee_{x\in A}\bigvee_{z\in S}\delta(z,x)\otimes\beta(f(x),b)\le\bigvee_{x\in A}\bigvee_{z\in S}\beta(f(z),f(x))\otimes\beta(f(x),b)\le$$
$$\bigvee_{y\in S}\beta(f(y),b)=\bigvee_{x\in S}\delta(x,x)\otimes\beta(f(x),b)\le$$
$$\le\bigvee_{x\in S}\bigvee_{z\in S}\delta(z,x)\otimes\beta(f(x),b)\le$$
$$\le\bigvee_{x\in A}\bigvee_{z\in S}\delta(z,x)\otimes\beta(f(x),b).$$

The following proposition is then an analogy of subobject classification theorem [13]; Proposition 3.1, for generalized Zadeh's extension principle $\mathcal{C}$.

Proposition 3.2 *The objects $\zeta=\{\zeta_{\mathbf{A}}:\mathbf{A}\in\mathbf{SetF}(\Omega)_0\}:\mathcal{C}\to$ hom and $\zeta^{-1}=\{\zeta_{\mathbf{A}}:\mathbf{A}\in\mathbf{SetF}(\Omega)_0\}:$ hom $\to\mathcal{C}$ are mutually inverse natural transformations, where $\zeta_{\mathbf{A}}$ and $\zeta^{-1}_{\mathbf{A}}$ are defined by (2) and (3), respectively.*

Proof. In [13] it was proved that $\zeta_{\mathbf{A}}$ and $\zeta^{-1}_{\mathbf{A}}$ are defined correctly. Hence, we have only to show that the corresponding diagrams commute for any morphism $f:(A,\delta)\to(B,\beta)$ in $\mathbf{SetF}(\Omega)_0$. Let us consider the following diagram.

$$\begin{array}{ccc}\mathcal{C}(A,\delta) & \xrightarrow{\zeta_{\mathbf{A}}} & \hom(A,\delta)\\ {\scriptstyle\mathcal{C}(f)}\downarrow & & \downarrow{\scriptstyle\hom(f)}\\ \mathcal{C}(B,\beta) & \xrightarrow{\zeta_{\mathbf{B}}} & \hom(B,\beta).\end{array}$$

Let $s\in\mathcal{C}(A,\delta)$ and $b\in B$. Then we have

$$\hom(f).\zeta_{\mathbf{A}}(s)(b)=(1_\Omega,\widehat{pr_2.\zeta_{\mathbf{A}}(s)}(b))=$$
$$(1_\Omega,\bigvee_{x\in A}s(x)\otimes\beta(f(x),b))=\zeta_{\mathbf{B}}.\mathcal{C}(f)(s)(b).$$

It can be shown analogously that the diagram for ζ^{-1} commutes.

Proposition 3.3 *The object $\psi=\{\psi_{\mathbf{A}}:\mathbf{A}\in\mathbf{SetF}(\Omega)_0\}$ is a natural transformation $sub^c\to\mathcal{C}$.*

Proof. Let $f:(A,\delta)\to(B,\beta)$ be a morphism in $\mathbf{SetF}(\Omega)$ and let $(S,\delta)\in\ \mathrm{sub}^c(A,\delta)$. Then the following diagram commutes.

$$\begin{array}{ccc}\mathrm{sub}^c(A,\delta) & \xrightarrow{\psi_{\mathbf{A}}} & \mathcal{C}(A,\delta)\\ {\scriptstyle\mathrm{sub}^c(f)}\downarrow & & \downarrow{\scriptstyle\mathcal{C}(f)}\\ \mathrm{sub}^c(B,\beta) & \xrightarrow{\psi_{\mathbf{B}}} & \mathcal{C}(B,\beta).\end{array}$$

In fact, according to 2.4, we have

$$\psi_{\mathbf{A}}.\mathcal{C}(f)(S,\delta)(b)=\bigvee_{x\in A}\bigvee_{z\in S}\delta(z,x)\otimes\beta(f(x),b)=$$
$$\bigvee_{y\in f(S)}\beta(y,b)=\bigvee_{y\in\overline{f(S))}}\beta(y,b)=\psi_{\mathbf{B}}.\ \mathrm{sub}^c(f)(S,\delta)(b).$$

Let $\mathbf{SetF}(\Omega)_{0,1}$ be the full subcategory of $\mathbf{SetF}(\Omega)_1$ with objects from the category $\mathbf{SetF}(\Omega)_0$ and let $\mathcal{C}_{0,1}$ and $\mathrm{sub}_{0,1}$ be restrictions of $\mathcal{C}$ and sub, respectively, on the category $\mathbf{SetF}(\Omega)_{0,1}$.

Proposition 3.4 *The object $\sigma=\{\sigma_{\mathbf{A}}:\mathbf{A}\in\mathbf{SetF}(\Omega)_{0,1}\}$ is a natural transformation $\mathcal{C}_{0,1}\to$ $sub_{0,1}$.*

Proof. Let $f : (A, \delta) \to (B, \beta)$ be a morphism in $\mathbf{SetF}(\Omega)_{0,1}$. Then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_{0,1}(A,\delta) & \xrightarrow{\sigma_{\mathbf{A}}} & \mathrm{sub}_{0,1}(A,\delta) \\ {\scriptstyle \mathcal{C}_{0,1}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{sub}_{0,1}} \\ \mathcal{C}_{0,1}(B,\beta) & \xrightarrow{\sigma_{\mathbf{B}}} & \mathrm{sub}_{0,1}(B,\beta). \end{array}$$

In fact, let $s \subseteq (A, \delta)$. Then we have

$$\sigma_{\mathbf{B}}.\mathcal{C}_{0,1}(f)(s) = (\{b \in B : \bigvee_{x\in A} s(x) \otimes \beta(f(x), b) = 1_\Omega\}, \beta),$$
$$\mathrm{sub}_{0,1}(f).\sigma_{\mathbf{A}}(s) = (\{f(a) : a \in A, s(a) = 1_\Omega\}, \beta).$$

Let $a \in A$ be such that $s(a) = 1_\Omega$. Then $1_\Omega \geq \bigvee_{x\in A} s(x) \otimes \beta(f(a), f(x)) \geq s(a) = 1_\Omega$. Conversely, for $b = f(a) \in B$ we have

$$1_\Omega = \bigvee_{x\in A} s(x) \otimes \beta(f(a), f(x)) = \bigvee_{x\in A} s(x) \otimes \delta(a, x) \leq s(a) \leq 1_\Omega.$$

We present now some relationships between contravariant functors $\mathcal{S}$, Hom, Sub, Sub^c on one hand and their covariant versions $\mathcal{C}$, hom, sub, sub^c on the other hand. Let $f : (A, \delta) \to (B, \beta)$ be a morphism in $\mathbf{SetF}(\Omega)_0$, $\mathcal{G}$ be one of the above contravariant functor and let $\mathcal{G}_{cov}$ be its covariant version. Then we obtain two maps

$$\mathcal{F}(A, \delta) \xrightarrow{\mathcal{F}_{cov}(f)} \mathcal{F}(B, \beta) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(A, \delta)\cdot$$

In the following proposition we present some properties of their compositions.

Proposition 3.5 *Let $f : (A, \delta) \to (B, \beta)$ be a morphism in $\mathbf{SetF}(\Omega)_0$.*

(a) For any $s \subseteq (A, \delta)$, we have $s \subseteq \mathcal{S}(f).\mathcal{C}(f)(s)$.

(b) If f is surjective, then $\mathcal{C}(f).\mathcal{S}(f) = \mathbf{id}$.

(c) $Hom(f).\ hom(f) \geq \mathbf{id}$.

(d) If f is surjective, then $hom(f).\ Hom(f) = \mathbf{id}$.

(e) If f is surjective, then $sub^c(f).\ Sub^c(f) = \mathbf{id}$.

(f) $\mathbf{id} \subseteq\ Sub^c(f).\ sub^c(f)$.

Proof. We will prove (d) and (e) only since the rest is straightforward.

(d) Let $v \in \mathrm{Hom}((B, \beta), \Omega^*), b \in B$. Then we have

$$\mathrm{hom}(f).\ \mathrm{Hom}(f)(v)(b) = \mathrm{hom}(f)(v.f)(b) =$$
$$(1_\Omega, \bigvee_{x\in A} pr_2.v(f(x)) \otimes \beta(f(x), b)) \leq (1_\Omega, \bigvee_{x\in A} pr_2.v(b)) = v(b),$$

as follows from the fact

$$pr_2.v(b) \otimes \beta(b, c) \leq pr_2.v(c)$$

for any $b, c \in B$. On the other hand, we have

$$\bigvee_{x\in A} pr_2.v(f(x)) \otimes \beta(f(x), b) \geq pr_2.v(b) \otimes \beta(b, b) = pr_2.v(b).$$

(e) Let $(T, \beta) \in \mathrm{Sub}^c(B, \beta)$. Then

$$\mathrm{sub}^c(f).\ \mathrm{Sub}^c(f)(T, \beta) = (\overline{f(f^{-1}(T))}, \beta) = (\overline{T}, \beta) = (T, \beta).$$

4 Zadeh's extension principle in $\mathbf{SetF}(\Omega)$

Zadeh's extension principle can be used to extend any map $h : \prod_{i\in I} A_i \to A$, where A_i and A are sets, into a map $\mathcal{F}(h) : \prod_{i\in I} \mathcal{F}(A_i) \to \mathcal{F}(A)$. This property is the most important for applications. In this section we want to present similar properties of our generalized extension principles $\mathcal{C}$, sub, sub^c and hom.

Let $(A_i, \delta_i), i \in I$, be objects in the category $\mathbf{SetF}(\Omega)_0$ and let $(\prod_{i\in I} A_i, \delta) = \prod_{i\in I}(A_i, \delta_i)$ be a product in this complete category, i.e. $\delta(\mathbf{a}, \mathbf{b}) = \bigwedge_{i\in I} \delta(a_i, b_i)$, where $\mathbf{a} = (a_i)_i, \mathbf{b} = (b_i)_i$. Generalized Zadeh's extension principles $\mathcal{G} =$ sub, sub^c, $\mathcal{C}$, hom, introduced in Theorem 3.1, enable then to extend any morphism $f : \prod_{i\in I}(A_i, \delta_i) \to (B, \beta)$ in $\mathbf{SetF}(\Omega)_0$ to a map $\mathcal{G}(f) : \mathcal{G}(\prod_{i\in I} A_i, \delta) \to \mathcal{G}(B, \beta)$. Nevertheless to be able to use this principle in applications we need (analogously as we do for classical Zadeh's extension principle) to extend a morphism f to a map

$$\mathcal{G}_{\prod}(f) : \prod_{i\in I} \mathcal{G}(A_i, \delta_i) \to \mathcal{G}(B, \beta).$$

It is clear that $\prod_{i\in I} \mathcal{G}(A_i, \delta_i) \neq \mathcal{G}(\prod_{i\in I} A_i, \delta)$, in general, and it follows that this map $\mathcal{G}_{\prod}(f)$ has to be defined newly. In this part we will show how to define this map and we investigate some properties of this map.

For definition of $\mathcal{G}_{\prod}$ we need to introduce in a specific way a map

$$u_{\mathcal{G}} : \prod_{i\in I} \mathcal{G}(A_i, \delta_i) \to \mathcal{G}(\prod_{i\in I}(A_i, \delta_i)).$$

If such a map is defined than the map $\mathcal{G}_{\prod}(f)$ can be defined as follows:

$$\forall (X_i)_i \in \prod_{i\in I} \mathcal{G}(A_i, \delta_i), \quad \mathcal{G}_{\prod}(f)((X_i)_i) = \mathcal{G}(f)(u_{\mathcal{G}}((X_i)_i).$$

In the following definition we introduce some examples of a map $u_{\mathcal{G}}$ for our generalized extension functors.

Definition 4.1 *Let $(A_i, \delta_i) \in \mathbf{SetF}(\Omega)_0$ for $i \in I$.*

(a) Let $\mathcal{G} = \mathcal{C}$ and let $s_i \in \mathcal{C}(A_i, \delta_i)$ for $i \in I$. Then $u_{\mathcal{C}}((s_i)_i)$ is defined such that for all $\mathbf{a} = (a_i)_i \in \prod_{i\in I} A_i$, we have

$$u_{\mathcal{C}}((s_i)_i)(\mathbf{a}) = \bigwedge_{i\in I} s_i(a_i).$$

(b) Let $\mathcal{G} =$ sub and let $(C_i, \delta_i) \in$ sub(A_i, δ_i) for $i \in I$. Then

$$u_{\text{sub}}((C_i, \delta_i)_i) = (\prod_{i\in I} C_i, \delta).$$

(c) Let $\mathcal{G} =$ sub^c. The definition of u_{sub^c} is then formally the same as for sub.

(d) Let $\mathcal{G} =$ hom and let $g_i \in$ hom(A_i, δ_i) for $i \in I$. Then for any $\mathbf{a} = ((a_i)_i) \in \prod_{i\in I} A_i$ we put

$$u_{\text{hom}}((g_i)_i)(\mathbf{a}) = \bigwedge_{i\in I} g_i(a_i).$$

We show that this definition is correct. We have to prove firstly that $u_{\mathcal{C}}((s_i)_i) \subseteq (\prod_{i\in I} A_i, \delta)$. In fact,

$$u_{\mathcal{C}}((s_i)_i)(\mathbf{a}) \otimes \delta(\mathbf{a}, \mathbf{b}) = (\bigwedge_{i\in I} s_i(a_i)) \otimes (\bigwedge_{j\in I} \delta_j(a_j, b_j) \leq$$

$$\leq \bigwedge_{i\in I} s_i(a_i) \otimes \delta_i(a_i, b_i) \leq \bigwedge_{i\in I} s_i(b_i) = u_{\mathcal{C}}((s_i)_i)(\mathbf{b}).$$

Further we have to prove that if C_i is complete in (A_i, δ_i) for any $i \in I$, then $\prod_{i\in I} C_i$ is complete in $(\prod_{i\in I} A_i, \delta)$. Let $\mathbf{a} = (a_i)_i \in \overline{\prod_{i\in I} C_i}$. Then we have

$$\begin{aligned}1_\Omega = \bigvee_{\mathbf{x}\in\prod C_i} \delta(\mathbf{a},\mathbf{x}) &= \bigvee_{\mathbf{x}\in\prod C_i} \bigwedge_{i\in I} \delta_i(a_i, x_i) \le \\ \le \bigwedge_{j\in I} \bigvee_{\mathbf{x}\in\prod C_i} \delta_j(a_j, x_j) &= \bigwedge_{j\in I} \bigvee_{x\in C_j} \delta_j(a_j, x) \le \\ &\le \bigvee_{x\in C_j} \delta_j(a_j, x) \le 1_\Omega,\end{aligned}$$

for all $j \in I$. Hence, $a_j \in \overline{C_j} = C_j$ and $\mathbf{a} \in \prod_{i\in I} C_i$.

Finally we show that for $((g_i)_i) \in \prod_{i\in I} \ \mathrm{hom}(A_i, \delta_i)$, $u_{\ \mathrm{hom}}((g_i)_i) : (\prod_i A_i, \delta) \to (\Omega^*, \mu)$ is a morphism. Let $\mathbf{a}, \mathbf{b} \in \prod_i A_i$. For $g_i(a_i) = (\alpha', \alpha)$ we have $1_\Omega = \delta_i(a_i, a_i) = \mu(g_i(a_i), g_i(a_i)) = \alpha'$ and it follows that $g_i(a_i) = (1_\Omega, \alpha_i), g_i(b_i) = (1_\Omega, \beta_i)$. Then we have

$$\begin{aligned}\delta(\mathbf{a},\mathbf{b}) = \bigwedge_{i\in I} \delta_i(a_i, b_i) \le \bigwedge_{i\in I} \mu(g_i(a_i), g_i(b_i)) = \\ = \bigwedge_{i\in I} \mu((1_\Omega, \alpha_i), (1_\Omega, \beta_i)) = \bigwedge_{i\in I} (\alpha_i \leftrightarrow \beta_i) \le (\bigwedge_i \alpha_i) \leftrightarrow (\bigwedge_i \beta_i) = \\ = \mu(\bigwedge_i g_i(a_i), \bigwedge_i g_i(b_i)) = \mu(u_{\ \mathrm{hom}}((g_i)_i)(\mathbf{a}), u_{\ \mathrm{hom}}((g_i)_i)(\mathbf{b})).\end{aligned}$$

For any morphism $f : \prod_i (A_i, \delta_i) \to (B, \beta)$ in $\mathbf{SetF}(\Omega)$ we obtain maps

$$\begin{aligned}\mathcal{C}_{\prod}(f) &: \prod_i \mathcal{C}(A_i, \delta_i) \to \mathcal{C}(B, \beta), \\ \mathrm{sub}_{\prod}(f) &: \prod_i \ \mathrm{sub}(A_i, \delta_i) \to \ \mathrm{sub}(B, \beta), \\ \mathrm{sub}^c_{\prod}(f) &: \prod_i \ \mathrm{sub}^c(A_i, \delta_i) \to \ \mathrm{sub}^c(B, \beta), \\ \mathrm{hom}_{\prod}(f) &: \prod_i \ \mathrm{hom}(A_i, \delta_i) \to \ \mathrm{hom}(B, \beta).\end{aligned}$$

Then $\mathcal{C}_{\prod}(f)$ is clearly a generalization of Zadeh's extension principle since for any sets $A_i, i \in I$ and B we can define trivial similarity relations δ_i on A_i and δ on B such that $\delta_i(x, x) = 1_\Omega$ and $\delta_i(x, y) = 0_\Omega$, otherwise (and similarly for δ). Then $\mathcal{C}(A_i, \delta_i) = \mathcal{F}(A_i)$ and $\mathcal{C}_{\prod}(f)$ coincides with classical Zadeh's extension of a map $f : \prod_i A_i \to B$.

In the previous section we presented several natural transformations between pairs of functors $\mathcal{C}$, sub, sub^c and hom. In this last section we show that similar relations exist also between maps $\mathcal{C}_{\prod}(f)$, $\mathrm{sub}_{\prod}(f)$, $\mathrm{sub}^c_{\prod}(f)$ and $\mathrm{hom}_{\prod}(f)$.

Proposition 4.1 *Let $\mathbf{A}_i = (A_i, \delta_i)$ be objects in $\mathbf{SetF}(\Omega)$ for $i \in I$ and let $f : \prod_i \mathbf{A}_i \to \mathbf{B} = (B, \beta)$ be a morphism in this category.*

(a) If $\mathbf{A}_i, i \in I, \mathbf{B}$ and f are objects of the category $\mathbf{SetF}(\Omega)_{0,1}$, then the following diagram commutes.

$$\begin{array}{ccc} \prod_i \ sub(\mathbf{A}_i) & \xrightarrow{sub_{\prod}(f)} & sub(\mathbf{B}) \\ \prod_i \psi_{\mathbf{A}_i} \downarrow & & \downarrow \psi_{\mathbf{B}} \\ \prod_i \mathcal{C}(\mathbf{A}_i) & \xrightarrow{\mathcal{C}_{\prod}(f)} & \mathcal{C}(\mathbf{B}), \end{array}$$

where ψ was introduced in 3.3.

(b) The following diagram commutes.

$$\begin{array}{ccc}
\prod_i \mathcal{C}(\mathbf{A}_i) & \xrightarrow{\mathcal{C}_{\prod}(f)} & \mathcal{C}(\mathbf{B}) \\
{\scriptstyle \prod_i \zeta_{\mathbf{A}_i}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{\mathbf{B}}} \\
\prod_i \; hom(\mathbf{A}_i) & \xrightarrow{hom_{\prod}(f)} & hom(\mathbf{B}),
\end{array}$$

where ζ was introduced in 3.2.

(c) If $\mathbf{A}_i, i \in I, \mathbf{B}$ and f are objects of the category $\mathbf{SetF}(\Omega)_{0,1}$, then the following diagram commutes.

$$\begin{array}{ccc}
\prod_i \; sub^c(\mathbf{A}_i) & \xrightarrow{sub^c_{\prod}(f)} & sub^c(\mathbf{B}) \\
{\scriptstyle \prod_i \psi_{\mathbf{A}_i}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{\mathbf{B}}} \\
\prod_i \mathcal{C}(\mathbf{A}_i) & \xrightarrow{\mathcal{C}_{\prod}(f)} & \mathcal{C}(\mathbf{B}).
\end{array}$$

(d) If $\mathbf{A}_i, i \in I, \mathbf{B}$ and f are objects of the category $\mathbf{SetF}(\Omega)_{0,1}$, then the following diagram commutes.

$$\begin{array}{ccc}
\prod_i \mathcal{C}(\mathbf{A}_i) & \xrightarrow{\mathcal{C}_{\prod}(f)} & \mathcal{C}(\mathbf{B}) \\
{\scriptstyle \prod_i \sigma_{\mathbf{A}_i}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{\mathbf{B}}} \\
\prod_i \; sub(\mathbf{A}_i) & \xrightarrow{sub_{\prod}(f)} & sub(\mathbf{B}),
\end{array}$$

where σ was introduced in 3.4.

Proof. (a) Let $\mathbf{A}_i, \mathbf{B}$ and f are objects of the category $\mathbf{SetF}(\Omega)_{0,1}$ for any $i \in I$ and let $((C_i, \delta_i)_i) \in \prod_i \text{ sub}(\mathbf{A}_i)$. Then for any $b \in B$ we have

$$\psi_{\mathbf{B}}.\ \text{sub}_{\prod}(f)((C_i,\delta_i)_i)(b) = \psi_{\mathbf{B}}(\ \text{sub}(f)(u_{\ \text{sub}}((C_i,\delta_i)_i)))(b) =$$
$$\psi_{\mathbf{B}}(f(\prod_i C_i),\beta)(b) = \bigvee_{\mathbf{z}\in\prod C_i} \beta(b, f(\mathbf{z})).$$

Let $b = f(\mathbf{a})$, then we have

$$(\mathcal{C}_{\prod}(f).(\prod_i \psi_{\mathbf{A}_i})((C_i,\delta_i)_i)(b) = \mathcal{C}(f)(u_{\mathcal{C}}((\psi_{\mathbf{A}_i}(C_i,\delta_i)))(b) =$$
$$\bigvee_{\mathbf{x}\in\prod A_i} (\bigwedge_i \psi_{\mathbf{A}_i}(C_i,\delta_i))(\mathbf{x}) \otimes \beta(b, f(\mathbf{x})) =$$
$$= \bigvee_{\mathbf{x}\in\prod A_i} \bigwedge_i \psi_{\mathbf{A}_i}(C_i,\delta_i)(\mathbf{x}) \otimes \delta(\mathbf{a},\mathbf{x}) \leq \bigvee_{\mathbf{x}\in\prod A_i} (\bigwedge_i \psi_{\mathbf{A}_i}(C_i,\delta_i))(\mathbf{a}) =$$
$$= \bigwedge_i (\psi_{\mathbf{A}_i}(C_i,\delta_i)(a_i)) = \bigwedge_i (\bigvee_{z_i\in C_i} \delta_i(z_i,a_i)) =$$
$$= \bigvee_{\mathbf{z}\in\prod C_i} (\bigwedge_i \delta_i(z_i,a_i)) = \bigvee_{\mathbf{z}\in\prod C_i} \delta(\mathbf{a},\mathbf{z}) =$$
$$= \bigvee_{\mathbf{z}\in\prod C_i} \beta(b, f(\mathbf{z})).$$

On the other hand we have

$$(\mathcal{C}_{\prod}(f).(\prod_i \psi_{\mathbf{A}_i})((C_i,\delta_i)_i))(b) =$$
$$= \bigvee_{\mathbf{x}\in\prod A_i} (\bigwedge_i (\bigvee_{z\in C_i} \delta_i(x_i,z))) \otimes \beta(b, f(\mathbf{x})) \geq$$
$$\geq \bigvee_{\mathbf{x}\in\prod C_i} (\bigwedge_i \delta_i(x_i,x_i)) \otimes \beta(b, f(\mathbf{x})) = \bigvee_{\mathbf{x}\in\prod C_i} \beta(b, f(\mathbf{x})).$$

(b) Let $(s_i)_i \in \prod_i \mathcal{C}(\mathbf{A}_i)$, $b \in B$. Then we have

$$\zeta_{\mathbf{B}}.\mathcal{C}_{\prod}(f)((s_i)_i)(b) = (1_\Omega, \mathcal{C}_{\prod}(f)((s_i)_i)(b)) =$$
$$= (1_\Omega, \bigvee_{\mathbf{x}\in\prod A_i} (\bigwedge_i s_i(x_i)) \otimes \beta(b, f(\mathbf{x}))).$$

On the other hand we have

$$\hom_{\prod}(f).\prod_i \zeta_{\mathbf{A}_i}((s_i)_i)(b) = \hom(f)(u_{\hom}((\zeta_{\mathbf{A}_i}(s_i))_i))(b) =$$
$$= (1_\Omega, \bigvee_{\mathbf{x}\in\prod A_i} (pr_2.u_{\hom}((\zeta_{\mathbf{A}_i}(s_i))_i))(\mathbf{x}) \otimes \beta(b, f(\mathbf{x}))) =$$
$$= (1_\Omega, \bigvee_{\mathbf{x}\in\prod A_i} pr_2(\bigwedge_i (1_\Omega, s_i(x_i))) \otimes \beta(b, f(\mathbf{x}))) =$$
$$= (1_\Omega, \bigvee_{\mathbf{x}\in\prod A_i} (\bigwedge_i (s_i(x_i)) \otimes \beta(b, f(\mathbf{x})))).$$

Hence, the diagram commutes.

(c) It follows from (a) and Lemma 2.4.

(d) Let $\mathbf{A}_i, \mathbf{B}$ and f be objects from the category $\mathbf{SetF}(\Omega)_{0,1}$ for all $i \in I$. Let $(s_i)_i \in \prod_i \mathcal{C}(\mathbf{A}_i)$. Let $(X, \beta) = \sigma_{\mathbf{B}}.\mathcal{C}_{\prod}(f)((s_i)_i)$, $(Y, \beta) = \mathrm{sub}_{\prod}(f).\prod_i \sigma_{\mathbf{A}_i}((s_i)_i)$. Then we have

$$X = \{b \in B : \bigvee_{\mathbf{x}\in\prod A_i} u_{\mathcal{C}}((s_i)_i)(\mathbf{x}) \otimes \beta(b, f(\mathbf{x})) = 1_\Omega\},$$
$$Y = f(\prod_i \{x \in A_i : s_i(x) = 1_\Omega\}).$$

Let $b \in Y$, then $b = f(\mathbf{a})$, where $s_i(a_i) = 1_\Omega$ for all $i \in I$ and we obtain

$$1_\Omega \geq \bigvee_{\mathbf{x}\in\prod A_i} u_{\mathcal{C}}((s_i)_i)(\mathbf{x}) \otimes \beta(b, f(\mathbf{x})) \geq$$
$$\geq u_{\mathcal{C}}((s_i)_i)(\mathbf{a}) = \bigwedge_i s_i(a_i) = 1.$$

Hence, $b \in X$. Conversely, let $b = f(\mathbf{a}) \in X$, then

$$1_\Omega = \bigvee_{\mathbf{x}\in\prod A_i} u_{\mathcal{C}}((s_i)_i)(\mathbf{x}) \otimes \beta(b, f(\mathbf{x})) = \bigvee_{\mathbf{x}\in\prod A_i} u_{\mathcal{C}}((s_i)_i)(\mathbf{x}) \otimes \delta(\mathbf{a}, \mathbf{x}) \leq$$
$$\leq \bigvee_{\mathbf{x}\in\prod A_i} u_{\mathcal{C}}((s_i)_i)(\mathbf{a}) = \bigwedge_i s_i(a_i) \leq 1_\Omega.$$

Hence, $s_i(a_i) = 1_\Omega$ for all $i \in I$ and $b \in Y$.

References

[1] Eytan, M., Fuzzy sets: a topos-logical point of view. *Fuzzy Sets and Systems* **5**(1981), 47–67.

[2] Goguen, J.A., L-fuzzy sets. *J.Math. Anal. Appli. 18*(1967), 145-174.

[3] Goldblatt, R., *TOPOI, The categorical analysis of logic*, North-Holland Publ. Co., Amsterdam, New York, Oxford(1979)

[4] Higgs, D., *A category approach to Boolean-valued set theory.* University of Waterloo (1973).

[5] Höhle, U., Presheaves over GL-monoids. *Non-Classical Logic and Their Applications to Fuzzy Subsets*, Kluwer Academic Publ., Dordrecht, New York(1995), 127-157.

[6] Höhle, U., M-Valued sets and sheaves over integral, commutative cl-monoids. *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publ., Dordrecht, Boston (1992), 33-72.

[7] Höhle, U., Classification of Subsheaves over GL-algebras. *Proceedings of Logic Colloquium 98 Prague*, Springer Verlag (1999).

[8] Höhle, U., Commutative, residuated l-monoids. *Non-Classical Logic and Their Applications to Fuzzy Subsets*, Kluwer Academic Publ. Dordrecht, New York (1995), 53–106.

[9] Höhle, U., Monoidal closed categories, weak topoi and generalized logics. *Fuzzy Sets and Systems* **42**(1991), 15–35.

[10] Johnstone, P.T., *Topos Theory*, Academic Press, London, New York, San Francisco(1977)

[11] Lawvere, F.W., (editor)*Toposes, Algebraic Geometry and Logic*, Springer Verlag, Berlin, Heidelberg, New York(1971)

[12] Makkai, M., Reyes, E.G. *Firts Order Categorical Logic*, Springer Verlag, Berlin, Heidelberg, New York(1977)

[13] Močkoř, J., Complete subobjects of fuzzy sets over MV-algebras. *Czech.Math.J.* **129(54)**(2004), 379–392.

[14] Novák,V., Perfilijeva, I., Močkoř,J., *Mathematical principles of fuzzy logic* Kluwer Academic Publishers,Boston,Dordrecht, London (1999).

[15] Pitts, A.M., Fuzzy sets do not form a topos, *Fuzzy Sets and Systems* **8**(1982),101–104

[16] Ponasse, D., Categorical studies of fuzzy sets, *Fuzzy Sets and Systems*, **28**(1988), 235-244