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Fuzzy Transform of a Function on the Basis of Triangulation

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Abstract

The technique of the direct and inverse fuzzy-transform of a function of two variables to the case of a circular area is generalize. The problem of an approximation of a continuous function of two variables which is defined on a circular domain is solved on the basis of a special triangulation of the given domain. The theorem about uniform convergence has been proved. The method of the fuzzy-transform is illustrated on the elementary example.
K e y w o r d s: fuzzy transform, basic functions, approximation, triangulation.

1 INTRODUCTION

Fuzzy transform (F-transform) is a known technique for an approximation of continuous functions. This technique belongs to the area of fuzzy approximations. This method has been developed by I. Perfilieva (see in [2]) for the case of functions of one variable and has been used for solving ordinary differential equations (see in [1]). A generalization of the F-transform for functions of two variables for the case of rectangular areas has been performed by M. Štěpnička and R. Valášek (see in [4]). In this contribution, this technique is generalized for the case of circular and elliptical areas.

2 F-TRANSFORM: TWO VARIABLES ON A CIRCULAR AREA

2.1 Triangulation

1. Let Ω be a polygon in the Euclidean space $\mathbb{R}^2$. We define a triangulation τ_h of Ω with the discretization parameter h by the following rules:

   - $\overline{\Omega} = \bigcup_{K\in\tau_h} K$, where $K \subset \overline{\Omega}$ is a triangle (closed), $int(K) \neq \emptyset$.
   - For all $K_1, K_2 \in \tau_h$ such that $K_1 \neq K_2$ and the following holds: $int(K_1) \cap int(K_2) = \emptyset$.
   - Let S be an edge of a triangle $K \in \tau_h$. Then either $S \subset \delta\Omega$ or there exists another triangle $K' \in \tau_h$ so that S is the common edge with this triangle.

   The discretization h measures the largest length of all edges in τ_h. Let $\mathcal{V}$ denote the set of all vertices of the triangulation τ_h of Ω.

2. Now, let Ω be a circular area. We approximate this area Ω by a polygonal area Ω_h, where $\Omega_h \subset \Omega$ and it holds:

   - Polygon Ω_h is determined by its vertices $V_1, \cdots, V_N$ which lie on the boundary $\delta\Omega$ and for these vertices

     $$\|V_i - V_{i+1}\| \leq h, \qquad i = 1, \cdots, N, \quad V_{N+1} \equiv V_1,$$

     holds where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$.

Moreover, for arbitrary two points P, T which belong to the sauce triangle we have $\| P - T \| \leq h$.

- we create a triangulation $\widetilde{\tau}_h$ of the polygon Ω_h such that each edge of the polygon Ω_h is the edge of a triangle $K \in \widetilde{\tau}_h$.

3. Example of triangulation
   On Figure 1 the triangulation has two parameters:
   n - number of nodes in the smallest circle (in our case of $n = 6$)
   m - number of concentric circles

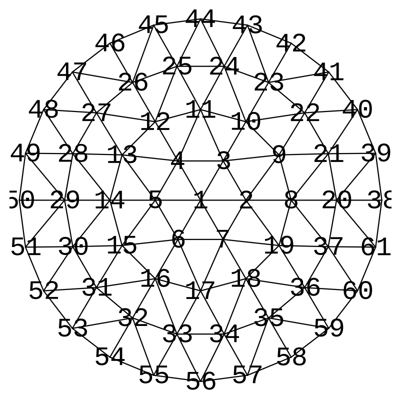

Figure 1: Example of triangulation

2.2 Basic functions

Let Ω be a polygon and τ_h be one of its triangulations. We will consider basic functions defined on Ω of linear type. A basic function A_V corresponds to the vertex V of the triangulation. For all basic function A_V the following hold:

- $A_{K,V}(x, y)$ is a linear function where $A_{K,V}$ is a restriction of the function A_V on the triangle K.
- $A_{K,V}(x, y) = 0$ if $V \notin K$.
- $A_V(V) = 1$
- $A_V(x, y)$ is a continuous function.

Let $V_1 = (x_1, y_1), V_2 = (x_2, y_2), V_3 = (x_3, y_3)$ be the vertices of a triangle K. From the above given requirement we can compute that

$$A_{K,V_1}(x, y) = 1 - a - b \qquad A_{K,V_2}(x, y) = a \qquad A_{K,V_3}(x, y) = b$$

Where

$$a = \frac{(x - x_1)(y_3 - y_1) - (x_3 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},$$

$$b = \frac{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}.$$

Therefore

$$\sum_{V \in \mathcal{V}} A_V(x, y) = \sum_{V \in K,\ (x,y) \in K} A_{K,V}(x, y) = 1.$$

2.3 F-transform

In the previous subsection, we described a triangulation on the circular area and defined basic functions. And now we will describe the method of the direct and inverse F-transform in this area. By the direct F-transform, we will create approximate values in the nodes of the triangulation as follows:

Definition 1
Let τ_h is a triangulation of a polygon Ω with vertices $V_1, \cdots, V_k$. Let $A_1(x, y), \cdots, A_k(x, y)$ be basic functions corresponding to the triangulation τ_h and $f(x, y)$ be any continuous function. We say that an n-tuple of real numbers $[F_1, \cdots, F_k]$ given by

$$F_V = \frac{\int_\Omega f(x, y) A_V(x, y) dxdy}{\int_\Omega A_V(x, y) dxdy} \tag{1}$$

is the F-transform of f with respect to $A_1, \cdots, A_k$.

By the inverse F-transform, we create the approximated function from values which we obtained by the direct F-transform.

Definition 2
Let $A_1, \cdots, A_k$ be basic functions corresponding to a triangulation τ_h and $f(x, y)$ be a continuous function. Let $F_k[f] = [F_1, ..., F_k]$ be the F-transform of f with respect to $A_1, ..., A_k$. Then the function

$$f_{\tau_h}(x, y) = \sum_{V \in \mathcal{V}} F_V A_V(x, y) \tag{2}$$

is called the inverse F-transform.

We will show in the following theorem that the method have good convergent properties.

Theorem 1
Let $f \in C(\overline{\Omega})$, where Ω is a circular area. Then for any $\varepsilon > 0$ there exists a triangulation τ_{h_ε} such that for all $T \in \overline{\Omega}$ the following holds true:

$$|f(T) - f_{\tau_{h_\varepsilon}}(T)| \leq \varepsilon.$$

PROOF: Function f is continuous on bounded closed set $\overline{\Omega}$, therefore f is uniformly continuous on $\overline{\Omega}$. This means

$$\forall \varepsilon > 0\, \exists \delta > 0 : \forall P, Q \in \overline{\Omega} : \|P - Q\|_{R^2} < \delta \Rightarrow |f(P) - f(Q)| < \varepsilon,$$

where $\|.\|$ is the Euclidean norm. Let us fix some $\varepsilon > 0$ and take the corresponding $\delta > 0$. Choose a discretization parameter h_ε of a triangulation such that $h_\varepsilon \leq \frac{\delta}{2}$. The proof will be split into several steps:

i) Triangulation
Let T be an arbitrary point of the area Ω. Then there exists a triangle $\widehat{K} \in \tau_{h_\varepsilon}$ such that $T \in \widehat{K}$. (If T belongs to an edge then we take an arbitrary triangle to which T belongs.) Let V_1, V_2, V_3 denote the vertices of the triangle $\widehat{K}$.

ii) Direct F-transform
For the chosen point $T \in \overline{\Omega}$ and vertex $V_i, i = 1, 2, 3$ we have

$$\begin{aligned}
|f(T) - F_{V_i}| &= \left| f(T) - \frac{\int_\Omega f(x,y) A_{V_i}(x,y)dxdy}{\int_\Omega A_{V_i}(x,y)dxdy} \right| = \\
&= \left| \frac{\int_\Omega (f(T) - f(x,y)) A_{V_i}(x,y)dxdy}{\int_\Omega A_{V_i}(x,y)dxdy} \right| \leq \\
&\leq \frac{\int_\Omega |(f(T) - f(x,y))| A_{V_i}(x,y)dxdy}{\int_\Omega A_{V_i}(x,y)dxdy}.
\end{aligned}$$

On the basis of the fact that a function A_{V_i} is non-zero only inside the triangles K, which share the vertex V_i we obtain

$$|f(T) - F_{V_i}| \leq \frac{\int_{K, V_i \in K} |(f(T) - f(x,y))| A_{V_i}(x,y)dxdy}{\int_{K, V_i \in K} A_{V_i}(x,y)dxdy}.$$

Let us denote $P = (x, y)$ as an arbitrary point which belongs to set $\bigcup_{K \in \tau_h, V_i \in K} K$. Then $\|T - P\| \leq 2h_\varepsilon < \delta$ (because the points T and P lie either in the same triangle or in the neighbouring triangles). Therefore

$$|f(T) - f(P)| < \varepsilon.$$

Hence

$$\begin{aligned}
&\frac{\int_{K, V_i \in K} |(f(T) - f(x,y))| A_{V_i}(x,y)dxdy}{\int_{K, V_i \in K} A_{V_i}(x,y)dxdy} \leq \\
&\leq \frac{\int_{K, V_i \in K} \varepsilon \cdot A_{V_i}(x,y)dxdy}{\int_{K, V_i \in K} A_{V_i}(x,y)dxdy} = \varepsilon.
\end{aligned}$$

iii) Inverse F-transform
For the chosen point T let us estimate

$$|f(T) - f_{\tau_{h_\varepsilon}}(T)| = \left| \sum_{V \in \mathcal{V}} (f(T) - F_V) A_V(T) \right| \leq$$

$$\leq \sum_{V \in \mathcal{V}} |f(T) - F_V| A_V(T) = \sum_{i=1}^{3} |f(T) - F_{V_i}| A_{V_i}(T) \leq$$

$$\leq \varepsilon \sum_{i=1}^{3} A_{V_i}(T) = \varepsilon.$$

□

3 ELEMENTARY EXAMPLE

Let us consider the following function

$$f(x, y) = \sin x \cos y$$

and let us apply the F-transform of this function on the area:

$$\Omega = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 4\}.$$

In our case we use the regular triangulation on the Figure 1. This triangulation is competent especially therefore that we can describe each vertex by means of two parameters. The first parameter indicates a number of a concentric circle where the research vertex lies. The second parameter indicates order of vertex on the concentric circle. In addition we can easily keep count of the numbers of neighbouring vertices.

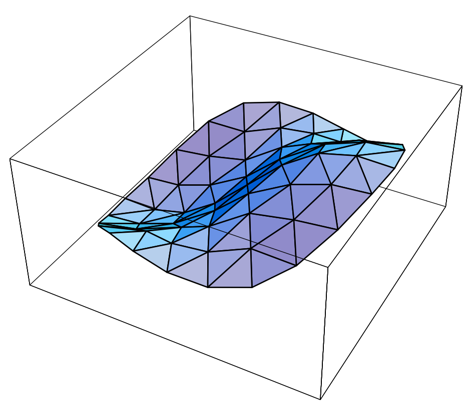

Figure 2: Approximation of function $f(x, y) = \sin x \cos y$ on the triangular area

On the basis of direct F-transform we determine the approximate values of function f in the vertices which correspond with given triangulation. We apply to the counts software

Mathematica and we depict the corresponding graph of the approximate function that obtain inverse F-transform(see in the Figure 2). We compare the result with the real graph (see in the Figure 3). The maximum difference between the accurate function and the approximate function is $0,1096$. The maximum difference in internal points of this circular is only 0,0396. This difference is small with respect to number of basic functions.

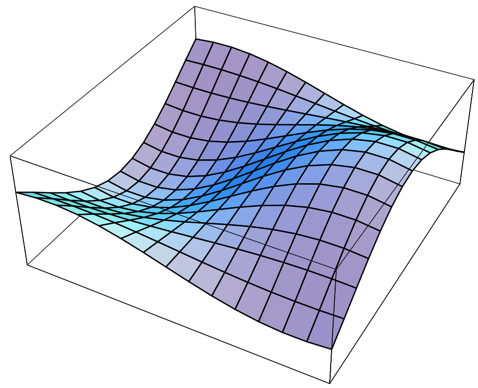

Figure 3: Graph of the function $f(x, y) = \sin x \cos y$

4 CONCLUSION

We have introduced technique of direct and inverse fuzzy transform which enables us to construct various approximating models depending on the choice of basic functions.

Above mentioned F-transform is used for elliptical and other areas too and for three and more variables. In other works we would like to use this method to compression and reconstruction pictures.

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