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How to Construct Own Continuous T-Norm

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Abstract

The unified representation of continuous t-norms and t-conorms with the help of generalized additive generators has been suggested. This representation shows similar origin of triangular operations (t-norms and t-conorms) as truncated sums of negative and positive reals, respectively. Moreover, a generalized additive generator establishes an isomorphic mapping between a standard BL-algebra $\mathcal{L}$ on $[0,1]$ and a certain algebra of negative reals with truncated operations.

I. Introduction

Triangular norms were introduced in the framework of probabilistic metric spaces. However, they are widely applied in several other fields, e.g., in fuzzy set theory, fuzzy logic, and their applications. Therefore, the problem of easy construction of a continuous t-norm using generating functions is of permanent interest (see [3]).

II. Functional Representation of Continuous t-Norms and t-Conorms

Let I be a countable and linearly ordered family of indices and

$$\mathcal{F}_I = \{f_i, \quad i \in I\},$$

be a family of continuous and strictly monotonously increasing functions such that

$$f_i : [0,1] \longrightarrow [0, c_i], c_i \leq +\infty, \quad \text{and} \quad f_i(0) = 0$$

for all $i \in I$. Each $f_i, i \in I$, determines the function

$$g_i(x) = f_i(1-x)$$

which is an additive generator of some continuous Archimedean t-norm. Therefore, we will refer to functions from $\mathcal{F}_I$ as *generating functions*.

Definition 1
For a countable and linearly ordered family of indices I, let $\mathcal{A}_I = \{a_i\}_{i\in I}$, $\mathcal{B}_I = \{b_i\}_{i\in I}$, be two families of nodes from the interval $[0,1]$ such that for all $i \in I$

$$a_i < b_i,$$

and for all $i, j \in I$

$$i < j \Rightarrow (a_i < a_j) \,\&\, (b_i < b_j) \,\&\, (b_i \leq a_j).$$

We say that the families $\mathcal{A}_I$ and $\mathcal{B}_I$ determine the partition $\mathcal{P}_{\mathcal{A},\mathcal{B},I}$ *of $[0,1]$:*

$$[0,1] = \bigcup_{i\in I}(a_i, b_i) \cup D$$

where

$$D = [0,1] \setminus \bigcup_{i\in I}(a_i, b_i).$$

On each subinterval $[a_i, b_i]$, $i \in I$, we define two linear transition functions $\varphi_i, \psi_i : [a_i, b_i] \longrightarrow [0,1]$ such that

$$\varphi_i(x) = \frac{b_i - x}{b_i - a_i}, \quad \psi_i(x) = \frac{x - a_i}{b_i - a_i}. \tag{1}$$

It is easy to see that the following equalities hold true for each $i \in I$ and for each $x \in [a_i, b_i]$:

$$\varphi_i(x) + \psi_i(x) = 1,$$
$$\varphi_i(x) = \psi_i(a_i + b_i - x).$$

The inverse functions $\varphi_i^{-1}, \psi_i^{-1} : [0,1] \longrightarrow [a_i, b_i]$ fulfil the following equality for arbitrary $x \in [0,1]$:

$$\varphi_i^{-1}(x) + \psi_i^{-1}(x) = a_i + b_i.$$

The following proposition immediately follows from the representation of an arbitrary continuous t-norm (t-conorm) as an ordinal sum of continuous Archimedean t-norms (t-conorms) (see [1]).

Proposition 1
Let I be at most countable, linearly ordered family of indices, $\mathcal{F}_I$ a family of generating functions, and $\mathcal{P}_{\mathcal{A},\mathcal{B},I}$ be a partition of $[0,1]$ determined by families $\mathcal{A}_I, \mathcal{B}_I$. Let the transition functions be defined by (1). Then

$$T(x,y) = \begin{cases} \varphi_i^{-1}(f_i^{-1}(\min(f_i(\varphi_i(x)) + f_i(\varphi_i(y)), f_i(1)))), & \\ & \textit{if } (x,y) \in (a_i,b_i)^2, \\ \min(x,y), & \textit{otherwise} \end{cases} \tag{2}$$

is a continuous t-norm and

$$S(x,y) = \begin{cases} \psi_i^{-1}(f_i^{-1}(\min(f_i(\psi_i(x)) + f_i(\psi_i(y)), f_i(1))), & \\ & \textit{if } (x,y) \in (a_i,b_i)^2, \\ \max(x,y), & \textit{otherwise} \end{cases} \tag{3}$$

is a continuous t-conorm. Moreover, each continuous t-norm (t-conorm) can be represented in the form (2) (resp. (3)).

From the properties of transition functions φ_i, ψ_i and their inverse, it is easy to prove that for the t-norm T and the t-conorm S given by (2) and (3), the following relation holds true for arbitrary $x, y \in (a_i, b_i)$:

$$S(x,y) = a_i + b_i - T(a_i + b_i - x, a_i + b_i - y).$$

III. Generalized Additive Generators of Continuous t-norms and their Residua

In this section, we will focus on the representation of continuous t-norms with help of generalized additive generators. By definition (see [1]), an additive generator $g : [0,1] \longrightarrow [0,\infty]$ of a t-norm T is a strictly decreasing function which fulfils two conditions:

(a) g is right-continuous at 0 and $g(1) = 0$,

(b)

$$T(x,y) = g^{-1}(\min(g(0), g(x) + g(y))). \tag{4}$$

It is known (cf. [1]) that it is possible to construct the representation (4) for continuous Archimedean t-norms. Moreover, if a t-norm has an additive generator then it is necessarily Archimedean. We will generalize representation (4) to the case of arbitrary continuous t-norm (see Theorem 1 below).

On the basis of Proposition 1, we may claim that a continuous t-norm T can be fully characterized by a pair $(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$ and moreover, each function f_i from $\mathcal{F}_I$ determines the additive generator

$$g_i(x) = f_i(\varphi_i(x)) \tag{5}$$

of the respective Archimedean part on interval $[a_i, b_i]$, $i \in I$. We will show that each continuous t-norm can be represented using truncated arithmetic sum of negative reals (see expression (7)). Furthermore, the operation of residuum of a continuous t-norm can be represented using truncated arithmetic subtraction of negative reals (see expression (8)).

We restrict ourselves to continuous non-Archimedean t-norms, because the representation (4) has been constructed in ([1]) for continuous Archimedean t-norms. Let a continuous non-Archimedean t-norm T be characterized by the pair $(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$ where $|I| \geq 2$, and additive generators of the respective Archimedean parts on intervals $[a_i, b_i]$ are given by (5). We consider the following set of couples of reals:

$$R_T^- = \bigcup_{a \in D} \{(a, 0)\} \cup \bigcup_{i \in I} \{b_i\} \times (0, -c_i)$$

and define the lexicographic order on R_T^-:

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1 < x_2) \vee (x_1 = x_2 \;\&\; y_1 \leq y_2)$$

as well as one-to-one mapping $g : [0,1] \longrightarrow R_T^-$:

$$g(x) = \begin{cases} (x, 0), & \text{if } x \in D, \\ (b_i, -g_i(x)), & \text{if } x \in (a_i, b_i). \end{cases} \tag{6}$$

The construction of R_T^- is fully determined by the partition $\mathcal{P}_{\mathcal{A},\mathcal{B},I}$ of $[0,1]$ and therefore, by the choice of the t-norm T. This dependence is marked by the lower index in the denotation R_T^-.

Let us introduce the operations of truncated sum and truncated subtraction on R_T^- as follows:

$(x_1, y_1) \dotplus (x_2, y_2) =$

$$\begin{cases} \min((x_1, y_1),\ (x_2, y_2)), & \text{if } x_1 \neq x_2, \\ (b_i, y_1 + y_2), & \text{if } (x_1 = x_2 = b_i)\ \&\ (y_1 + y_2 > -c_i), \\ (a_i, 0), & \text{if } (x_1 = x_2 = b_i)\ \&\ (y_1 + y_2 \leq -c_i) \\ & \text{or } (x_1 = x_2 = a_i)\ \&\ (y_1 = y_2 = 0). \end{cases}$$

and

$$(x_1, y_1) \dot{-} (x_2, y_2) = \begin{cases} (x_1, y_1 - y_2), & \text{if } x_1 = x_2\ \&\ y_1 < y_2, \\ (1, 0), & \text{if } x_1 > x_2 \\ & \text{or } x_1 = x_2\ \&\ y_1 \geq y_2, \\ (x_1, y_1), & \text{if } x_1 < x_2. \end{cases}$$

Then we can prove the following representation theorem.

Theorem 1
Let a continuous t-norm T be characterized by the pair

$$(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$$

where $|I| \geq 2$. Moreover, let I_T denote the respective residuum of the t-norm T. Then

$$T(x, y) = g^{-1}(g(x) \dotplus g(y)) \tag{7}$$

and

$$I_T(x, y) = g^{-1}(g(y) \dot{-} g(x)) \tag{8}$$

where g is the generalized additive generator (6) of the t-norm T.

We will illustrate this theorem by an example which justifies the title of this contribution, as well.

Example 1

1) *Let us determine a t-norm T by a set $\mathcal{F}_I$ of generating functions (Fig.1) and a partition $\mathcal{P}_{\mathcal{A},\mathcal{B},I}$ of $[0, 1]$ (Fig.2):*

$$\mathcal{F}_I = \{f_1, f_2\}, \mathcal{A}_I = \{0, 0.5\}, \mathcal{B}_I = \{\frac{1}{3}, 1\}$$

where

$$f_1(x) = 3x^3, f_2(x) = 2 - \log_2(4 - 2x).$$

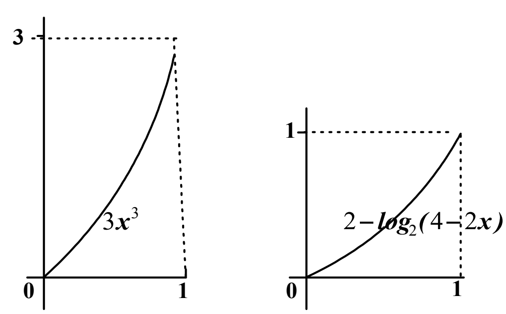

Figure 1. Generating functions

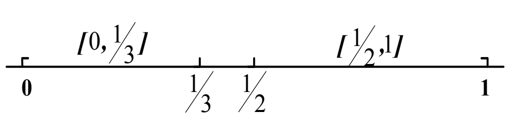

Figure 2. Partition

2) *With the help of two transition functions*

$$\varphi_1(x) = 1 - 3x, \quad \varphi_2(x) = 2 - 2x$$

we compute two components of generalized additive generator (6) of the t-norm T:

$$g_1(x) = 3\frac{\frac{1}{3}-x}{\frac{1}{3}} = 3(1-3x)^3,$$
$$g_2(x) = 2 - \log_2(4 - 2(2-2x)) = -\log_2 x$$

and their corresponding inverse functions:

$$g_1^{-1}(x) = \frac{1-\sqrt[3]{\frac{x}{3}}}{3},$$
$$g_2^{-1}(x) = 2^{-x}.$$

3) *Now the the t-norm T is fully determined by (6). To show this, let us compute $T(0.1, 0.25)$:*

$$g(0.1) = (\frac{1}{3}, -3\cdot 0.7^3), \quad g(0.25) = (\frac{1}{3}, -\frac{3}{64}),$$
$$g(0.1) \dotplus g(0.25) = (\frac{1}{3}, -3\cdot 0.7^3 - \frac{3}{64}),$$
$$g^{-1}(g(0.1) \dotplus g(0.25)) = g_1^{-1}(3\cdot 0.7^3 + \frac{3}{64}) \approx 0.096.$$

4) *Finally,*

$$T(0.1, 0.25) \approx 0.096.$$

We can prove even more if we consider a standard BL-algebra (see [2]) on $[0,1]$

$$\mathcal{L} = \langle [0,1], \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$$

where $*$ is a continuous t-norm and $\rightarrow$ is its residuum.

Theorem 2
Let $\mathcal{L}$ be a standard BL-algebra on $[0,1]$ and $*$ be a continuous t-norm characterized by the pair $(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$ where $|I| \geq 2$. Then the generalized additive generator g of $*$ defined by (6) establishes an isomorphic mapping between $\mathcal{L}$ and the following algebra on R_*^- with truncated operations:

$$\mathcal{R}_*^- = \langle R_*^-, \vee, \wedge, \dotplus, \dot{-}, (0,0), (1,0)\rangle.$$

IV. Basic Identities in $\mathcal{R}_T^-$

The following basic identities in $\mathcal{R}_T^-$ are obtained as isomorphic images of BL-algebra identities and BL-logic axioms.

Identities in $\mathcal{R}_T^-$
$x \dotplus y = y \dotplus x$
$x \dotplus (y \dotplus z) = (x \dotplus y) \dotplus z$
$(y \dot{-} x) \vee (x \dot{-} y) = \langle 1, 0\rangle$
$x \dotplus y \leq z \Leftrightarrow y \leq z \dot{-} x$
$x \dotplus (y \dot{-} x) = x \wedge y$
$x \dotplus \langle 1, 0\rangle = x$
$x \dot{-} \langle 1, 0\rangle = x$
$\langle 1, 0\rangle \dot{-} x = \langle 1, 0\rangle$
$y \dot{-} x \leq (z \dot{-} x) \dot{-} (z \dot{-} y)$
$x \dotplus y \leq x$
$z \dot{-} (x \dotplus y) = (z \dot{-} y) \dot{-} x$
$z \dot{-} (y \dot{-} x) \leq z \dot{-} (z \dot{-} (x \dot{-} y))$
$\langle 0, 0\rangle \leq x$
$x \dotplus \langle 0, 0\rangle = \langle 0, 0\rangle$
$x \dot{-} \langle 0, 0\rangle = \langle 1, 0\rangle$

V. Generalized Additive Generators of Continuous t-conorms

In this section, we will suggest the representation of continuous t-conorms with help of generalized additive generators. In Proposition 1, we have presented the representation of an arbitrary continuous t-conorm as an ordinal sum of continuous Archimedean t-conorms. Let us recall (see [1]) that the additive generator $g : [0,1] \longrightarrow [0,\infty]$ of a t-conorm S is a strictly increasing function $g : [0,1] \longrightarrow [0,\infty]$ which fulfils two conditions:

1) g is left-continuous at 1 and $g(0) = 0$,
2)
$$S(x,y) = g^{-1}(\min(g(0), g(x)+g(y))). \tag{9}$$

Similar to t-norms, it is known [1] that it is possible to construct the representation (9) for continuous Archimedean t-conorms. We will generalize this kind of representation to the case of arbitrary continuous t-conorm (see Theorem 3 below).

Let a continuous t-conorm S be characterized by the pair $(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$ where $|I| \geq 2$, and moreover, each function f_i from $\mathcal{F}_I$ determines the additive generator

$$g_i(x) = f_i(\psi_i(x)) \tag{10}$$

of the respective Archimedean part on interval $[a_i, b_i]$, $i \in I$.

We consider the following set of couples of reals:

$$R_T^+ = \bigcup_{a \in D} \{(a,0)\} \cup \bigcup_{i \in I} \{b_i\} \times (0, c_i).$$

and define the lexicographic order on R_T^+:

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow (x_1 < x_2) \vee (x_1 = x_2 \;\&\; y_1 \leq y_2)$$

as well as the one-to-one mapping $g : [0,1] \longrightarrow R_T^+$:

$$g(x) = \begin{cases} (x,0), & \text{if } x \in D, \\ (a_i, g_i(x)), & \text{if } x \in (a_i, b_i). \end{cases} \tag{11}$$

Let us introduce the operations of truncated sum on R_T^+ as follows:

$$(x_1,y_1) \dot{+} (x_2,y_2) = \begin{cases} \max((x_1,y_1),(x_2,y_2)), & \text{if } x_1 \neq x_2, \\ (a_i, y_1+y_2), & \text{if } (x_1 = x_2 = a_i) \& (y_1+y_2 < c_i), \\ (b_i, 0), & \text{if } (x_1 = x_2 = a_i) \& (y_1+y_2 \geq c_i) \\ & \text{or } (x_1 = x_2 = b_i) \& (y_1 = y_2 = 0). \end{cases}$$

Theorem 3
Let a continuous t-conorm S be characterized by the pair

$$(\mathcal{F}_I, \mathcal{P}_{\mathcal{A},\mathcal{B},I})$$

where $|I| \geq 2$. Then

$$S(x,y) = g^{-1}(g(x) \dot{+} g(y)) \tag{12}$$

where g is the generalized additive generator (11) of the t-conorm S.

VI. Conclusion

We have suggested the representation of continuous t-norms and t-conorms with help of generalized additive generators. This representation shows similar origin of triangular operations (t-norms and t-conorms) as truncated sums of negative and positive reals, respectively. Moreover, we showed that a generalized additive generator establishes an isomorphic mapping between a standard BL-algebra $\mathcal{L}$ on $[0,1]$ and the algebra on the set of couples of reals R_*^- with truncated operations.

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