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Fuzzy Type Theory As Higher Order Fuzzy Logic

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Abstract

In the paper, logical axioms, inference rules, semantics, and some specific properties including the completeness theorems of four kinds of fuzzy type theory are presented. This theory is a higher order fuzzy logic that can be used for precise formalization, for example, of the theory of computing with words, fuzzy IF-THEN rules, approximate reasoning, and others.

I. Introduction

Successful development of the mathematical theory of fuzzy logic started in 1979 by the seminal paper of J. Pavelka [14]. He developed a propositional fuzzy logic that has been extended to first order by V. Novák in [9] (see also [13]). It is specific for this logic (called *fuzzy logic with evaluated syntax* and denoted by $\mathrm{Ev}_{Ł}$) that it makes possible to consider axioms that are not fully convincing. Consequently, both semantics as well as syntax of $\mathrm{Ev}_{Ł}$ are evaluated. The concept of provability is generalized and the completeness theorem stating that the provability degree of a formula is equal to its truth degree is proved.

After the seminal book of P. Hájek [7], mathematical fuzzy logic achieved a rapid development. Many formal systems covering both propositional as well as predicate logic and differing in the structure of truth values appeared.

To be able to capture semantics of natural language, with the goal to develop a formal theory of human reasoning (the, so called, *fuzzy logic in broader sense*), fuzzy logic penetrated also into higher order, and a formal theory of *fuzzy type theory* (FTT) has been developed (see [10], [11]). This theory follows the development of classical type theory, as elaborated by A. Church [4] and L. Henkin [8], and later continued by P. Andrews in [1].

Because of a large variety of possibilities, a discussion about, what is fuzzy logic, is now in progress. Based on the expressive power and various experiences, several distinguished kinds of fuzzy logic took privilege over the other ones. Such logics are Łukasiewicz logic, Basic fuzzy logic (BL), and $\mathrm{L}\Pi$ fuzzy logic. All these logics have propositional as well as predicate versions.

It turns out that also fuzzy type theory can be developed in parallel ways. The original theory in [10] has been developed on the basis of IMTL_{Δ}-algebra that is, a residuated lattice with prelinearity and double negation extended, moreover, by a special unary operation of Baaz delta. In this paper we will present fuzzy type theories based on four different structures of truth values, namely IMTL_{Δ}, Łukasiewicz$_{\Delta}$, BL_{Δ} and $\mathrm{L}\Pi$ algebras. All these theories enjoy the generalized completeness property (i.e. completeness w.r.t. generalized models). It should be stressed that the fundamental connective in all of them is a fuzzy equality. Because of essential importance of this connective, the resulting theory is elegant and philosophically interesting.

II. Truth values

All above discussed fuzzy logics are based on extensions of residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle. \tag{1}$$

In this lattice, we define negation by $\neg a = a \rightarrow \mathbf{0}$. Another important operation is *biresiduation* defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

The residuated lattice (1) is an IMTL-algebra if it is prelinear, i.e.

$$(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}, \qquad a, b \in L, \tag{2}$$

and the double negation $\neg\neg a = a$ holds true for all $a \in L$. It is a BL-algebra if it is prelinear (2) and divisible, i.e.

$$a \otimes (a \rightarrow b) = a \wedge b, \qquad a, b \in L. \tag{3}$$

A Π-algebra is a BL-algebra fulfilling

$$\neg\neg a \leq (a \rightarrow a \otimes b) \rightarrow b \otimes \neg\neg b, \qquad a, b \in L.$$

An MV-algebra is a BL-algebra fulfilling the law of double negation.

Finally, ŁΠ-algebra is the algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \odot, \overset{Ł}{\to}, \overset{\Pi}{\to}, \mathbf{0}, \mathbf{1}\rangle \tag{4}$$

where

(i) $\langle L, \vee, \wedge, \otimes, \overset{Ł}{\to}, \mathbf{0}, \mathbf{1}\rangle$ is an MV-algebra,
(ii) $\langle L, \vee, \wedge, \odot, \overset{\Pi}{\to}, \mathbf{0}, \mathbf{1}\rangle$ is a Π-algebra,
(iii) $a \odot (b \ominus c) = (a \odot b) \ominus (a \odot c)$
holds for all $a, b, c \in L$ where

$$a \ominus b = a \otimes \overset{Ł}{\neg} b$$
$$\overset{Ł}{\neg} a = a \overset{Ł}{\to} \mathbf{0}.$$

We will further define $\overset{\Pi}{\neg} a = a \overset{\Pi}{\to} \mathbf{0}$ and $\Delta a = \overset{\Pi}{\neg} \overset{Ł}{\neg} a$. The algebra of truth values is *complete* if its lattice reduct is complete.

A very important operation for the development of fuzzy type theory is the *Baaz delta operation*, which is a unary operation on L fulfilling the following properties:

$$\Delta a \vee \neg\Delta a = \mathbf{1}, \qquad \Delta(a \vee b) \leq \Delta a \vee \Delta b,$$
$$\Delta a \leq a, \qquad \Delta a \leq \Delta\Delta a,$$
$$\Delta(a \to b) \leq \Delta a \to \Delta b, \qquad \mathbf{\Delta 1} = \mathbf{1}.$$

For the details concerning properties of the above structures of truth values, see [5], [6], [7], [13].

A. Examples

Typical (standard) examples of the above introduced algebras have the support $L = [0, 1]$ so that the residuated lattice is complete and has the general form

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle. \tag{5}$$

Example of IMTL-algebra is (5) where

$$a \otimes b = \begin{cases} a \wedge b, & \text{if } a + b > 1, \\ 0 & \text{otherwise} \end{cases} \qquad \textit{(nilpotent minimum)}$$

$$a \to b = \begin{cases} 1, & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise} \end{cases} \qquad \textit{(residuum)}$$

$$\neg a = 1 - a \qquad \textit{(negation)}$$

$$a \leftrightarrow b = (\neg a \wedge \neg b) \vee (a \wedge b). \qquad \textit{(biresiduation)}$$

The Baaz delta operation is

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad \textit{(Baaz delta)}$$

Example of BL-algebra is any residuated lattice (5) where $\otimes$ is a continuous t-norm. For the case when $\otimes$ is product or minimum, we get

$$\neg a = a \to \mathbf{0} = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{0}, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \neg\neg a = \begin{cases} \mathbf{0} & \text{if } a = \mathbf{0}, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Łukasiewicz MV-algebra is a residuated lattice (5) where

$$a \otimes b = 0 \vee (a + b - 1), \qquad \textit{(Łukasiewicz conjunction)}$$

$$a \to b = 1 \wedge (1 - a + b). \qquad \textit{(Łukasiewicz implication)}$$

Furthermore, $\neg a = 1 - a$, and $a \leftrightarrow b = 1 - |a - b|$.

Example of ŁΠ-algebra is

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \cdot, \overset{Ł}{\to}, \overset{\Pi}{\to}, \mathbf{0}, \mathbf{1}\rangle$$

where

$$a \overset{Ł}{\to} b = 1 \wedge (1 - a + b), \qquad a \otimes b = 0 \vee (a + b - 1),$$
$$a \overset{\Pi}{\to} b = \begin{cases} 1 & a \leq b, \\ \frac{b}{a} & \text{otherwise}, \end{cases} \qquad \overset{Ł}{\neg} a = 1 - a,$$
$$\overset{\Pi}{\neg} a = \begin{cases} 1 & a = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad a \ominus b = 0 \vee (a - b).$$

All these examples are algebras considered in fuzzy logic theory. It has been proved that IMTL, BL and Łukasiewicz propositional fuzzy logics are standard complete, i.e. their formulas are provable iff they are true in all the above standard algebras.

B. Fuzzy equality and extensionality of functions

Important fuzzy relation is that of a fuzzy equality

$$\doteq : M \times M \longrightarrow L$$

which must fulfil the following properties:

(i) reflexivity $[m \doteq m] = \mathbf{1}$,
(ii) symmetry $[m \doteq m'] = [m' \doteq m]$,
(iii) $\otimes$-transitivity $[m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m'']$

for all $m, m', m'' \in M$ where $[m \doteq m']$ denotes a *truth value* of $m \doteq m'$.

Example of a fuzzy equality on $M = [u, v] \subset \mathbb{R}$ with respect to IMTL$_\Delta$-algebra is

$$[m =_\epsilon n] = \begin{cases} 1, & \text{if } m = n, \\ \frac{1}{v-u}\,((v - m) \wedge (v - n)) \\ \qquad\qquad \vee((m - u) \wedge (n - u)), \end{cases}$$

$m, n \in [u, v]$.

Let $F : M_\alpha \longrightarrow M_\beta$ be a function and $=_\alpha, =_\beta$ be fuzzy equalities in the respective domains M_α and M_β. Then F is extensional w.r.t $=_\alpha$ and $=_\beta$ if there is a natural number $q \geq 1$ such that

$$[m =_\alpha m']^q \leq [F(m) =_\beta F(m')], \qquad m, m' \in M_\alpha$$

where the power is taken with respect to $\otimes$. We say that F is *strongly extensional* if $q = 1$. It is *weakly extensional* if

$$[m =_\alpha m'] = \mathbf{1} \quad \text{implies that} \quad [F(m) =_\beta F(m')] = \mathbf{1}.$$

This equivalent to the condition

$$\Delta[m =_\alpha m'] \leq [F(m) =_\beta F(m')].$$

It is easy to prove that each fuzzy equality $=_\alpha$ (as a binary function) is strongly extensional w.r.t. itself and $\leftrightarrow$. The $\leftrightarrow$ is a fuzzy equality on L strongly extensional w.r.t. itself; $\wedge$ is strongly, and Δ is weakly extensional w.r.t. $\leftrightarrow$.

III. Syntax and semantics of IMTL-FTT

A. Syntax

a) Basic syntactical elements: Let ϵ, o be distinct objects. Then *Types* is the smallest set of symbols fulfilling the conditions

(i) $\epsilon, o \in$ *Types*,
(ii) If $\alpha, \beta \in$ *Types* then $\alpha\beta \in$ *Types*.

Each formula in the fuzzy type theory is assigned a type $\alpha \in$ *Types* written as its subscript. Then we introduce primitive symbols, that are variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$, $\iota_{\epsilon(o\epsilon)}, \iota_{o(oo)}$ (*description operators*), the symbol λ and various kinds of brackets. Among special constant we put $\mathbf{E}_{(o\alpha)\alpha}$ (*equality*), $\mathbf{C}_{(oo)o}$ (*conjunction*) and $\mathbf{D}_{oo}$ (*Baaz delta*).

b) Formulas: the set of formulas is the smallest set *Form* fulfilling for all $\alpha, \beta \in$ *Types* the conditions

(i) $x_\alpha \in$ *Form*,
(ii) $c_\alpha \in$ *Form*,
(iii) if $B_{\beta\alpha} \in$ *Form* and $A_\alpha \in$ *Form* then $(B_{\beta\alpha} A_\alpha) \in$ *Form*,
(iv) if $A_\beta \in$ *Form* then $\lambda x_\alpha\, A_\beta \in$ *Form*.

Let us remark that formulas are also called *lambda-terms*, especially in fuzzy type theories developed for the applications in computer science (cf. [2] and elsewhere).

To simplify the notation, we will often write $A \in Form_\alpha$ to stress that A is a formula of type α (without writing explicitly its type as a subscript).

The following are basic definitions of symbols of FTT:

(a) *Fuzzy equivalence/fuzzy equality*

$$\equiv \;:=\; \lambda x_\alpha\, (\lambda y_\alpha\, \mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha.$$

(b) *Conjunction* $\wedge \;:=\; \lambda x_o\, (\lambda y_o\, \mathbf{C}_{(oo)o}\, y_o) x_o$.
(c) *Baaz delta* $\boldsymbol{\Delta} \;:=\; \lambda x_o\, \mathbf{D}_{oo}\, x_o$.
(d) *Representation of truth* $\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o)$
and *falsity* $\bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top)$.
(e) *Negation* $\neg := \lambda x_o\, (\bot \equiv x_o)$.
(f) *Implication* $\Rightarrow \;:=\; \lambda x_o\, (\lambda y_o\, ((x_o \wedge y_o) \equiv x_o))$.
(g) *Special connectives*

$$\vee := \lambda x_o(\lambda y_o(((x_o \Rightarrow y_o) \Rightarrow y_o) \wedge ((y_o \Rightarrow x_o) \Rightarrow x_o))), \qquad \textit{(disjunction)}$$

$$\& := \lambda x_o(\lambda y_o(\neg(x_o \Rightarrow \neg y_o))). \qquad (\textit{strong conjunction})$$

(h) *General quantifier* $(\forall x_\alpha) A_o := (\lambda x_\alpha\, A_o \equiv \lambda x_\alpha\, \top)$.

B. Semantics

Semantics of FTT is a generalization of the semantics of first-order fuzzy logic. Let D be a set of objects and L be a set of truth values. A *basic frame* is a system of sets $(M_\alpha)_{\alpha \in Types}$ where $M_\epsilon = D$ is a set of objects, $M_o = L$ is a set of truth values and if $\gamma = \beta\alpha$ then $M_\gamma \subseteq M_\beta^{M_\alpha}$. A *frame* is a system

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}\,, \mathcal{L} \rangle$$

where $\mathcal{L}$ the algebra of truth values; namely either of IMTL$_\Delta$, Łukasiewicz$_\Delta$, BL$_\Delta$ or ŁΠ-algebra depending on the concrete fuzzy type theory (see further explanation). Furthermore, $=_\alpha$ is a fuzzy equality on M_α and for $\alpha \neq o, \epsilon$, each function $F \in M_\alpha$ is weakly extensional.

A *general model* is a frame such that every formula A_α, $\alpha \in$ *Types*, has interpretation in it (i.e. there is an element in the corresponding set M_α of the frame that interprets A_α).

Because of lack of space, we will omit precise definition of interpretation of formulas. The reader may find it in [10]. Let us remark only that the formula of the form $\lambda x_\alpha\, A_\beta$ is in $\mathcal{M}$ interpreted as a function assigning to every $m \in M_\alpha$ an element from M_β that is obtained as an interpretation of A_β in which all occurrences of x_α are replaced by the corresponding m. For example, interpretation of $\lambda x_\alpha\, A_o$ is a fuzzy set on M_α determined by the property represented by the formula A_o.

IV. Logical axioms and inference rules of IMTL-FTT

In this section, we will overview logical axioms of IMTL-FTT that have been in details presented in [10]. Logical axioms of the other three FTT will be obtained as a modification of them and presented in the next section.

The logical axioms of IMTL-FTT are divided into several groups.

Fundamental equality axioms

(FT$_I$1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$
(FT$_I$2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$
(FT$_I$2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$
(FT$_I$3) $(\lambda x_\alpha B_\beta) A_\alpha \equiv C_\beta$
where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).
(FT$_I$4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

Truth structure axioms

(FT$_I$5) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$

(FT$_I$6) $(A_o \equiv \top) \equiv A_o$
(FT$_I$7) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$
(FT$_I$8) $(A_o \Rightarrow (B_o \Rightarrow C_o)) \equiv (B_o \Rightarrow (A_o \Rightarrow C_o))$
(FT$_I$9) $((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o)$
(FT$_I$10) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$
(FT$_I$11) $A_o \wedge B_o \equiv B_o \wedge A_o$
(FT$_I$12) $A_o \wedge B_o \Rightarrow A_o$
(FT$_I$13) $(A_o \wedge B_o) \wedge C_o \equiv A_o \wedge (B_o \wedge C_o)$
(FT$_I$14) $(g_{oo}(\mathbf{\Delta} x_o) \wedge g_{oo}(\neg \mathbf{\Delta} x_o)) \equiv (\forall y_o)\, g_{oo}(\mathbf{\Delta} y_o)$
(FT$_I$15) $\mathbf{\Delta}(A_o \wedge B_o) \equiv \mathbf{\Delta} A_o \wedge \mathbf{\Delta} B_o$

Quantifier axiom

(FT$_I$16) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$ where x_α is not free in A_o.

Axioms of descriptions

(FT$_I$17) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon$

c) Inference rules and provability.: There are two inference rules in all the discussed fuzzy type theories.

(R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then, infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

(N) *Let $A_o \in Form_o$. Then infer $\mathbf{\Delta} A_o$ from A_o.*

V. Formal theories in IMTL-FTT

A *theory* T over FTT is set of formulas of type o (truth value). The provability in the theory T is defined as usual; $T \vdash A_o$ means that A_o is provable in T.

The following theorems summarize some of the important properties of IMTL-FTT. For the precise proofs of them see [10].

Theorem 1
(a) If $\vdash A_o$ then $\vdash (\forall x_\alpha) A_o$ (Rule of generalization).
(b) $\vdash (\forall x_\alpha) B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$ (Substitution axioms).
(c) If $T \vdash A_o$ and $T \vdash A_o \Rightarrow B_o$ then $T \vdash B_o$ (Rule of modus ponens).

It can be proved that IMTL-FTT includes formal system of the first-order IMTL-logic.

Theorem 2 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

Theorem 3 (Equality theorem)

$$\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta) \Rightarrow \mathbf{\Delta}((f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta))$$

Theorem 4 (Description operator theorem)

$$\vdash \iota_{\gamma(o\gamma)}(\mathbf{E}_{(o\gamma)\gamma}\, y_\gamma) \equiv y_\gamma$$

holds for every type $\gamma \in Types$.

Note that interpretation of $p_{o\alpha}$ is a fuzzy set. It can be demonstrated that interpretation of $\iota_{\alpha(o\alpha)} p_{o\alpha}$ is a *typical element (prototype)* of $p_{o\alpha}$, namely, an arbitrary but fixed element from its kernel (if this fuzzy set is subnormal then there is no such element).

Let T be a theory of IMTL-FTT.

(i) T is *contradictory* if

$$T \vdash \bot.$$

Otherwise it is *consistent*.

(ii) T is *maximal consistent* if each its extension T', $T' \supset T$ is inconsistent.

(iii) T is *complete* if

$$T \vdash A_o \Rightarrow B_o \quad \text{or} \quad T \vdash B_o \Rightarrow A_o.$$

holds for every two formulas A_o, B_o.

(iv) T is *extensionally complete* if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ implies that there is a closed formula C_α such that $T \not\vdash A_{\beta\alpha} C_\alpha \equiv B_{\beta\alpha} C_\alpha$.

Theorem 5

Every consistent theory T can be extended to a maximal consistent theory $\overline{T}$ which is complete. To every consistent theory T there is an extensionally complete consistent theory $\overline{T}$ which is an extension of T.

Theorem 6 (Completeness)

(a) A theory T of fuzzy type theory is consistent iff it has a general model $\mathcal{M}$.

(b) For every theory T and a formula A_o,

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

VI. Other fuzzy type theories

In this section, we will present systems of logical axioms of the other three fuzzy type theories, namely Ł-FTT (Łukasiewicz fuzzy type theory), BL-FTT and ŁΠ-FTT. Their syntax and semantics is essentially the same. They differ in some of the definitions and by logical axioms that are obtained as modification of the logical axioms of IMTL-FTT.

A. Łukasiewicz-fuzzy type theory

This is another important kind of fuzzy type theory that differs from IMTL-FTT by the definition of disjunction:

$$\vee := \lambda x_o(\lambda y_o(x_o \Rightarrow y_o) \Rightarrow y_o).$$

Logical axioms of Ł-FTT are (FT$_I$1)–(FT$_I$17) and, further, the axiom

(FT$_Ł$18) $(A_o \vee B_o) \equiv (B_o \vee A_o)$.

There is also simpler alternative which uses Rose-Rosser implication axioms for characterization of the structure of truth values. Then, axioms (FT$_I$7)–(FT$_I$10) should be replaced by the following axioms.

Implication axioms

(FT$_Ł$7) $A_o \Rightarrow (B_o \Rightarrow A_o)$

(FT$_Ł$8) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$

(FT$_Ł$9) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$

(FT$_Ł$10) $(A_o \vee B_o) \equiv (B_o \vee A_o)$

Completeness of Ł-FTT formulated in Theorem 6 is provable.

B. BL-fuzzy type theory

Recall that BL stands for *basic fuzzy logic* developed by P. Hájek in [7]. Since the law of double negation does not hold in BL-algebra, we must introduce a new special constant $\mathbf{S}_{o(oo)}$ interpreted by the operation of supremum. Moreover, we must introduce the following definition of the disjunction connective:

$$\vee := \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o}\, y_o) x_o.$$

Obviously this is a formula of type $(oo)o$. In the same spirit as above, we will write $x_o \vee y_o$ instead of $(\vee x_o) y_o$.

Axioms of BL-FTT are (FT$_I$1)–(FT$_I$17) and, moreover,

(FT$_{BL}$18) $(A_o \wedge B_o) \Rightarrow A \,\&\, (A_o \Rightarrow B_o)$

(FT$_{BL}$19) $\vdash B_{o,x_\alpha}[A_\alpha] \Rightarrow (\exists x_\alpha) B_o$.

(FT$_{BL}$20) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha) A_o \Rightarrow B_o)$

(FT$_{BL}$21) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha) A_o \vee B_o)$

On the basis of Theorem 5, it is possible to prove the following lemma:

Lemma 1

Let T be an extensionally complete theory of BL-FTT. Then it is Henkin: if $T \not\vdash (\forall x_\alpha) A_o$ holds for a closed formula $(\forall x_\alpha) A_o$ then there is a closed formula C_α such that $T \not\vdash A_{o,x_\alpha}[C_\alpha]$.

This lemma enables to prove completeness of BL-FTT in the form of Theorem 6. However, the item (a) holds w.r.t. *safe general models*, that is, models in which all the necessary suprema and infima exist (cf. [7]). The reason is that completion of the structure of truth values is not generally possible.

C. ŁΠ-fuzzy type theory

In this case, we must consider two fuzzy equalities, namely $\overset{Ł}{\equiv}$ and $\overset{\Pi}{\equiv}$. Further, we will consider a new special constant $\mathbf{P}_{(oo)o}$ interpreted by the special product $\odot$ (in the case that the support of truth values is $L = [0, 1]$ then $\odot$ is the ordinary product of reals).

Let $\tau, \nu \in \{Ł, \Pi\}$. The following are definitions of special symbols:

$$\begin{aligned}
\blacklozenge &:= \lambda x_o\, (\lambda y_o\, \mathbf{P}_{oo} y_o) x_o \\
\overset{\tau}{\Rightarrow} &:= \lambda x_o\, \lambda y_o\, \cdot x_o \wedge y_o \overset{\tau}{\equiv} x_o, \\
\overset{\tau}{\neg} &:= \lambda x_o\, (x_o \overset{\tau}{\Rightarrow} \bot) \\
\& &:= \lambda x_o\, \lambda y_o\, \cdot \overset{Ł}{\neg}(x_o \overset{Ł}{\Rightarrow} \overset{Ł}{\neg}\, y_o) \\
\mathbf{\Delta} &:= \lambda x_o\, \cdot \overset{\Pi}{\neg} \overset{Ł}{\neg}\, x_o \\
\vee &:= \lambda x_o\, \lambda y_o\, \cdot (x_o \overset{Ł}{\Rightarrow} y_o) \overset{Ł}{\Rightarrow} y_o, \\
\overset{G}{\Rightarrow} &:= \lambda x_o\, \lambda y_o\, \cdot \mathbf{\Delta}(x_o \overset{Ł}{\Rightarrow} y_o) \vee y_o, \\
(\forall x_\alpha) A_o &:= (\lambda x_\alpha\, A_o \overset{Ł}{\equiv} \lambda x_\alpha\, \top),
\end{aligned}$$

D. Logical axioms

The following are logical axioms of ŁΠ-FTT where again $\tau, \nu \in \{Ł, \Pi\}$.

Fundamental equality axioms

(FT$_{Ł\Pi}$1) $\mathbf{\Delta}(x_\alpha \overset{\tau}{\equiv} y_\alpha) \overset{Ł}{\Rightarrow} (f_{\beta\alpha}\, x_\alpha \overset{\nu}{\equiv} f_{\beta\alpha}\, y_\alpha)$,
where $\tau \neq \nu$,

(FT2$_{Ł\Pi 1}$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \overset{\tau}{\equiv} g_{\beta\alpha}\, x_\alpha) \overset{Ł}{\Rightarrow} (f_{\beta\alpha} \overset{\tau}{\equiv} g_{\beta\alpha})$

(FT2$_{Ł\Pi 2}$) $(f_{\beta\alpha} \overset{\tau}{\equiv} g_{\beta\alpha}) \overset{Ł}{\Rightarrow} (f_{\beta\alpha}\, x_\alpha \overset{\tau}{\equiv} g_{\beta\alpha}\, x_\alpha)$

(FT$_{Ł\Pi}$3) $(\lambda x_\alpha\, B_\beta) A_\alpha \overset{\tau}{\equiv} C_\beta$
where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

(FT$_{Ł\Pi}$4) $(x_\epsilon \overset{\tau}{\equiv} y_\epsilon) \overset{\tau}{\Rightarrow} ((y_\epsilon \overset{\tau}{\equiv} z_\epsilon) \overset{\tau}{\Rightarrow} (x_\epsilon \overset{\tau}{\equiv} z_\epsilon))$

Truth structure axioms

(FT$_{Ł\Pi}$5) $(x_o \overset{\tau}{\equiv} y_o) \overset{Ł}{\equiv} ((x_o \overset{\tau}{\Rightarrow} y_o) \wedge (y_o \overset{\tau}{\Rightarrow} x_o))$

(FT$_{Ł\Pi}$6) $(A_o \overset{\tau}{\equiv} \top) \overset{Ł}{\equiv} A_o$

(FT$_{Ł\Pi}$7) $(A_o \overset{\tau}{\Rightarrow} B_o) \overset{\tau}{\Rightarrow} ((B_o \overset{\tau}{\Rightarrow} C_o) \overset{\tau}{\Rightarrow} (A_o \overset{\tau}{\Rightarrow} C_o))$

(FT$_{Ł\Pi}$8) $(A_o \overset{\tau}{\Rightarrow} (B_o \overset{\tau}{\Rightarrow} C_o)) \overset{Ł}{\equiv} (B_o \overset{\tau}{\Rightarrow} (A_o \overset{\tau}{\Rightarrow} C_o))$

(FT$_{Ł\Pi}$9) $((A_o \overset{\tau}{\Rightarrow} B_o) \overset{\tau}{\Rightarrow} C_o) \overset{\tau}{\Rightarrow} (((B_o \overset{\tau}{\Rightarrow} A_o) \overset{\tau}{\Rightarrow} C_o) \overset{\tau}{\Rightarrow} C_o)$

(FT$_{Ł\Pi}$10) $(\overset{Ł}{\neg}\, B_o \overset{Ł}{\Rightarrow} \overset{Ł}{\neg}\, A_o) \overset{Ł}{\equiv} (A_o \overset{Ł}{\Rightarrow} B_o)$

(FT$_{Ł\Pi}$11) $A_o \wedge B_o \overset{Ł}{\equiv} B_o \wedge A_o$

(FT$_{Ł\Pi}$12) $A_o \wedge B_o \overset{Ł}{\Rightarrow} A_o$

(FT$_{Ł\Pi}$13) $(A_o \wedge B_o) \wedge C_o \overset{Ł}{\equiv} A_o \wedge (B_o \wedge C_o)$

(FT$_{Ł\Pi}$14) $\overset{\Pi}{\neg} \overset{\Pi}{\neg}\, A_o \overset{\Pi}{\Rightarrow} ((A_o \overset{\Pi}{\Rightarrow} A_o \blacklozenge B_o) \overset{\Pi}{\Rightarrow} (\overset{\Pi}{\neg} \overset{\Pi}{\neg}\, B_o \blacklozenge B_o))$

(FT$_{Ł\Pi}$15) $A_o \blacklozenge (A_o \overset{\Pi}{\Rightarrow} B_o) \overset{Ł}{\equiv} A_o \wedge B_o$

Quantifier axiom

(FT$_{Ł\Pi}$16) $(\forall x_\alpha)(A_o \overset{Ł}{\Rightarrow} B_o) \overset{Ł}{\Rightarrow} (A_o \overset{Ł}{\Rightarrow} (\forall x_\alpha) B_o)$
where x_α is not free in A_o

Axioms of descriptions

(FT$_{Ł\Pi}$17) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \overset{Ł}{\equiv} y_\alpha, \qquad \alpha = o, \epsilon$

The inference rule (R) from Section IV is formulated only for $\tau = Ł$. Similarly as in the case of BL-fuzzy type theory, completeness Theorem 6 holds w.r.t. *safe general models*.

VII. Conclusion

In this paper, we have formed four kinds of fuzzy type theories as fuzzy logics of higher order. These logics are necessary when we want to model natural human reasoning which is specific by the use of natural language. Because of the complexity of the latter, first-order (fuzzy) logic is not sufficient. However, fuzzy type theory is extremely powerful formal system which can be used in arbitrary mathematical model.

The reason why we confined just to IMTL-FTT, Ł-FTT, BL-FTT and ŁΠ-FTT lays in their potential for various kinds of applications. The IMTL-FTT should be taken as fundamental fuzzy type theory so that the other ones are obtained as specific extensions of it. The most powerful is ŁΠ-FTT which, on the other hand, is the most complicated. Therefore, if the product connective $\blacklozenge$ is unnecessary, it is better to confine to Ł-FTT only. If our main task is the fuzzy approximation theory then BL-FTT is convenient. Its drawback, however, is the lack of the law of double negation that is very important when modeling natural human reasoning. Note that ŁΠ-FTT contains all the other three fuzzy type theories. Let us also remark that a slightly different formal system of ŁΠ-FTT that is not based on λ-notation has been developed in [3] with the goal to establish formal background for fuzzy mathematics.

It is probable that we can develop more kinds of FTT than the above four ones. This should be done, however, only when there is essential substantiation for them.

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