University of Ostrava

Institute for Research and Applications of Fuzzy Modeling

International Conference

# The Logic of Soft Computing IV

&

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Edited by
Vilém Novák, Martin Štěpnička

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# Editorial

A lot of conferences on soft computing and related topics are organized each year all over the world. However, only few of them became the tradition that can be ranked among most productive. As a rule, these conferences are smaller in number of participants and so, less formal relations lead to discussions that often lead to conceptually new ideas and solutions. I am convinced that the conferences with the general title "The Logic of Soft Computing" that are usually organized in parallel with workshops of the ERCIM working group on soft computing are of this kind. Till now, they took place in Italy (Capri, Gargnano, Sienna) and in Vienna in Austria. I am very proud that the fourth time we have privilege to organize this conference in Ostrava in the Czech Republic. This choice is not accidental. In 1996, the Institute for Research and Applications of Fuzzy Modeling of the University of Ostrava has been established as one of few working places focused entirely on the topic of fuzzy modeling and soft computing. Starting with 7 workers in the beginning, the results of this small institute now reached more than 260 publications including 4 books. Besides that, our results include also special software for fuzzy modeling (Linguistic Fuzzy Logic Controller; LFLC 2000) using which we have realized a number of models and applications of various kinds.

I sincerely hope that this conference will be also very productive and successful and that all participants will leave Ostrava with the feeling that they spent nice time which brought them some new friendships and especially, new hints and ideas for their own work. I want to thank to rector of the University of Ostrava, Prof. Vladimír Baar, for taking the auspices over the conference, to the municipal government of Ostrava for its generous support and to all who spent their time to organize this conference.

Ostrava, October 4, 2005

Vilém Novák

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# *Conference Program*

## Tuesday, October 4, 2005

**18:30**
**Welcome dinner**

## Wednesday, October 5, 2005

**9:00 – 9:30**
**Opening**



**10:40 – 11:00**
**Coffee break**



**12:45 – 14:00**
**Lunch**

15:00 – 15:35
Best Locations for River Water Quality Monitoring Sensors through Fuzzy Interpolation


15:35 – 16:10
Fuzzy Petri Nets in Modelling Business Processes


**16:10 – 16:30**
**Coffee break**

16:30 – 17:05
On Symmetric MV–polynomials


17:05 – 17:40
Product Logic and Probabilistic Ulam Games


17:40 – 18:15
Game Semantics for Product Logic - Part II


**18:30**
**Dinner**

## Thursday, October 5, 2005

9:00 – 10:00
**Invited Speaker**
Hajek's Basic Logic and Multichannel Games with Lies


10:00 – 10:35
Complexity of T–norm Based Fuzzy Logics with Rational Truth Constants


10:35 – 11:10
The Enigma of Quantifying Vague Information


**11:10 – 11:30**
**Coffee break**

## 12:40 – 14:00
## Lunch



## 16:45 – 17:05
## Coffe break



## 19:30
## Banquet

## Friday, October 7, 2005

**16:30 – 16:50**
**Coffee break**

16:50 – 17:25
Logical Theory of Fuzzy IF-THEN Rules
*Perfilieva, Irina* ........................................................................................................ *94*

17:25 – 18:00
Skolem Functions in Fuzzy Logics
*Baaz, Matthias*........................................................................................................... *95*

**18:00 – 18:30**
**Closing**

**18:30**
**Dinner**

# Axiomatizations of Fuzzy Set Theory: A Critical Survey

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## 1 Introduction

It was now exactly 40 years ago, in 1965, that the first three papers appeared in print which discussed sets with a graded membership relation: two of them by the now famous US-American Lotfi Zadeh, and one by a rather unknown German mathematician Dieter Klaua.

The basic ideas of both authors differed slightly—but unessentially from a mathematical point if view: Zadeh chose the real unit interval for the membership degrees, Klaua considered arbitrary finite subsets of equidistant points from $[0, 1]$ because he from the very beginning embedded his considerations into the language of the (finitely valued) Łukasiewicz logics.

And for both approaches potential applications had been constitutive. Zadeh clearly explained them: they arose out of his system theoretic investigations. For Klaua[1] the stimulation came out of discussions with Karl Menger.

## 2 Classifying the Approaches

The approaches which have been presented toward the main problems of the foundations for fuzzy set theory are rather inhomogeneous. So it is not suitable to discuss them in chronological order. Instead we classify the approaches into some types:

- axiomatic approaches toward fuzzy set theory;
- model oriented approaches toward some (kind of) standard model of the universe of fuzzy sets;
- category theoretic approaches.

In this paper the first type of approaches shall be surveyed and explained with their basic ideas and core results.

[1]Personal communication to this author.

## 3 Axiomatic Approaches

The paradigmatic classical system for these axiomatizations of fuzzy set theory is almost always the system ZF. Only occasionally the system NBG is the reference system. The approaches differ mainly w.r.t. the formalization of the generalized membership predicate: either as ternary or even quaternary predicate treated via classical logic, or as a binary predicate inside an elementary theory over some suitable system of many-valued logic.

There is a wealth of such proposals summed up in the following list (which is—hopefully—essentially complete). Some of these approaches shall be considered a bit more in detail.

- Membership degrees as fuzzy sets as proposed by E.W. Chapin Jr. [NDJFL **15** (1974), **16** (1975)].
- Semi-lattices as degree structures as proposed by A.J. Weidner [Ph.D. Thesis, Univ. Notre Dame 1974; FSS **6** (1981)].
- An adaptation of the ZF-axiomatization proposed by H. Toth [J. Fuzzy Math. **1** (1993)].
- An adaptation of the Bernays axiomatization by M. Demirci/D. Çoker [FSS **60** (1993)].
- An NBG-like axiomatization sketched by V. Novák [FSS **3** (1980)].
- An embedding into NBG proposed by D.E. Tamir, Z.-Q. Cao, A. Kandel and J.L. Mott [Information Sci. **52** (1990)].
- An axiomatization of so-called fuzzy objects by N. Prati [Stochastica **12** (1988)].
- An approach toward fuzzy sets as multisets as proposed by J. Lake [J. London Math. Soc.(2) **12** (1976)] and realized by W.D. Blizard [FSS **33** (1989)].
- An axiomatization of fuzzy sets in ŁΠ-logic offered by Běehounek and Cintula [FSS **154** (2005)].

## 4 A General Remark

Summing up, there is a wealth of different proposals for axiomatizations of the theory of fuzzy sets. However, up to now there is no common agreement whether one of these proposals offers the right ideas, or whether all of them fall short of what is needed for an axiomatization of the area of fuzzy sets.

# Remarks on residuated lattices

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## 1 Introduction

Many-valued logics based on residuated bounded lattices were intensively studied, [10–12, 17, 21, 24, 25], even if the underlying lattice is not a chain (a first attempt in this direction is described in [10, Section 15.2], compare [2, 7] and also the *paraconsistent logic* in [5]).

Throughout this paper, let $\mathbf{L} = (L, \leq, *, \rightarrow, 0, 1)$ be an integral commutative residuated $\ell$-monoid with bottom element 0 and top element 1, and we shall briefly call it a *residuated bounded lattice*.

The following types of residuated bounded lattices have been considered so far:

- $\mathbf{L}$ is an infinite bounded chain (quite often represented by $[0, 1]$): then $*$ is a left-continuous t-norm (with the additional requirement of divisibility, i.e., $x * (x \rightarrow y) = x \wedge y$, $*$ is even a continuous t-norm). Note that the complete characterization of all left-continuous t-norms on $[0, 1]$ is still an open problem (for continuous t-norms, their structure as an ordinal sum of continuous Archimedean t-norms is well-known, see e.g. [16,20,23,15]). For an overview of construction methods for left-continuous t-norms see [13,19].

- **L** is a finite chain: then any t-norm $*$ on $L$ induces a residuated bounded lattice (note that there are exactly $2^{\mathrm{card}(L)-2}$ divisible t-norms on $L$ [18]).
- **L** is an abstract bounded lattice, $* = \wedge$ and the adjoint operation $\rightarrow$ exists.

However, a deeper study of bounded lattices admitting the structure of a residuated bounded lattice is still missing. The aim of this contribution is a closer look at this problem.

# 2 Triangular norms on bounded lattices

Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T\colon L^2 \to L$ which turns $L$ into an ordered commutative semigroup with neutral element 1 will be called a *triangular norm* or, briefly, a *t-norm* on $L$ [6]. In fact, the triangular norms on $L$ considered here are commutative semigroups satisfying Conditions A and B in [9], for concrete examples see [9, Examples 1.1–1.4].

For each bounded lattice $(L, \leq, 0, 1)$, the following functions are t-norms:

- the strongest t-norm $T^L_{\mathbf{M}}$ defined by $T^L_{\mathbf{M}}(x, y) = x \wedge y$,
- the weakest t-norm (drastic t-norm) $T^L_{\mathbf{D}}$ defined by $T^L_{\mathbf{D}}(x, y) = 0$ whenever $x \neq 1$ and $y \neq 1$, and $T^L_{\mathbf{D}}(x, y) = x \wedge y$ otherwise.

Observe that $T^L_{\mathbf{M}} \neq T^L_{\mathbf{D}}$ whenever $\mathrm{card}(L) > 2$.

**Proposition 2.1** *Let $(L_i, \leq_i, 0_i, 1_i)_{i\in I}$ be a family of bounded lattices. Then:*

(i) *If $0_i = 0$ and $1_i = 1$ for all $i \in I$ and if $L_i \cap L_j = \{0, 1\}$ for $i \neq j$, then the horizontal sum $(L, \leq, 0, 1) = H(\langle L_i, \leq_i, 0, 1\rangle, i \in I)$, given by $L = \cup_{i\in I} L_i$ and $x \leq y$ if and only if $x, y \in L_i$ and $x \leq_i y$, is a bounded lattice.*

(ii) *if $I$ is a bounded chain and if for $i < j$, $L_i \cap L_j \subseteq \{1_i\} \cap \{0_j\}$, and if $L_i \cap L_j \neq \emptyset$ implies $L_k = \{1_i\}$ for each $k \in I$, $i < k < j$, then the vertical sum $(L, \leq, 0, 1) = V(\langle L_i, \leq_i, 0_i, 1_i\rangle, i \in I)$, given by $L = \cup_{i\in I} L_i$ and $x \leq y$ if and only if $x, y \in L_i$ and $x \leq_i y$, or $x \in L_i$, $y \in L_j$ and $i < j$, $0 = 0_{i_*}$ and $1 = 1_{i^*}$, where $i_*$ and $i^*$ are the bottom and the top element of $I$, respectively, is a bounded lattice.*

In the case that on each bounded lattice $(L_i, \leq_i, 0_i, 1_i)$, $i \in I$, we have a t-norm $T_i$ acting on $L_i$, we have the following results, see also [22].

**Proposition 2.2** *Let $(L, \leq, 0, 1) = H(\langle L_i, \leq_i, 0, 1\rangle, i \in I)$ be a horizontal sum of bounded lattices and let, for each $i \in I$, $T_i : {L_i}^2 \to L_i$ be a t-norm on $L_i$. Then the function $T : L^2 \to L$ given by*

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in L_i^2, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{1}$$

*is a t-norm.*

Note that, in the second case of (1), we always get $x \wedge y = 0$ and that the above construction method is the only way how to construct t-norms on horizontal sums of bounded lattices, i.e., each t-norm $T$ on such a lattice can be represented in the form (1). For example, the drastic t-norm $T_{\mathbf{D}}^L$ on a horizontal sum $L$ has just the form (1) with $T_i = T_{\mathbf{D}}^{L_i}$.

**Proposition 2.3** *Let $(L, \leq, 0, 1) = V(\langle L_i, \leq_i, 0_i, 1_i\rangle, i \in I)$ be a vertical sum of bounded lattices and let, for each $i \in I$, $T_i : {L_i}^2 \to L_i$ be a t-norm on $L_i$. Then the function $T : L^2 \to L$ given by (1) is a t-norm (and it is called the ordinal sum of the t-norms $T_i$).*

Note that the ordinal sum t-norm $T$ given by (1) is the strongest t-norm on $L$ such that its restriction to ${L_i}^2$ coincide with $T_i$, and thus there are, in general, t-norms acting on a vertical sum $L$ of bounded lattices which cannot be represented as an ordinal sum of t-norms. For example, the drastic t-norm $T_{\mathbf{D}}^L$ on a vertical sum $L$ of bounded lattices is an ordinal sum of drastic t-norms $T_{\mathbf{D}}^{L_i}$ only if $L = L_i$ for some $i \in I$ (i.e., $L$ is a trivial vertical sum).

# 3 Bounded lattices admitting residuals

For finite lattices, we have the following characterization.

**Proposition 3.1** *Let $(L, \leq, 0, 1)$ be a finite lattice. Then $(L, \leq, \wedge, \to, 0, 1)$ is a residuated bounded lattice if and only if for all $u, v, y, z \in L$ with $u \wedge y \leq z$ and $v \wedge y \leq z$ we have $(u \vee v) \wedge y \leq z$.*

As an easy corollary we see that each finite distributive lattice is a bounded residuated lattice (with $* = T_{\mathbf{M}}^L$). We pose an open problem here: are there non-distributive finite lattices admitting residuals?

Although $T_{\mathbf{M}}^L$ is a t-norm on $L$ for each bounded lattice $\mathbf{L}$, for $* = T_{\mathbf{M}}^L$ there is not always an adjoint operator $\to$ such that $(L, \leq, *, \to, 0, 1)$ is a residuated bounded lattice. Especially for lattices which are horizontal sums the residuation property is quite restrictive.

**Proposition 3.2** *Let $(L, \leq, 0, 1) = H(\langle L_i, \leq_i, 0, 1\rangle, i \in I)$ be a horizontal sum of bounded lattices where each $L_i$ is a proper subset of $L$. If $*$ is an operation on $L$ such that the lattice $\mathbf{L} = (L, \leq, *, 0, 1)$ admits residuals then $\mathbf{L}$ is the diamond lattice whose Hasse diagram is given in Figure 1 (left), and $* = T_{\mathbf{M}}^L$.*

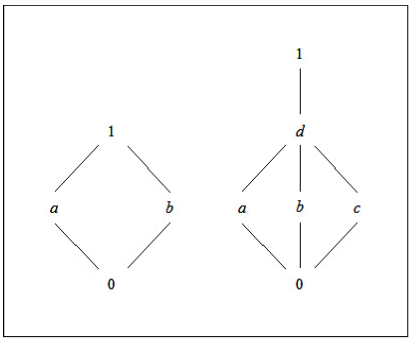

Figure 1: The diamond lattice (left) and the lattice from Example 3.4

In all other cases, i.e., for all horizontal sums which are proper supersets of the diamond lattice, there is no t-norm $*$ admitting residuals.

On the other hand, for vertical sums of lattices and the corresponding ordinal sums of t-norms we have the following positive result.

**Proposition 3.3** *Let* $(L, \leq, 0, 1) = V(\langle L_i, \leq_i, \rangle, 0_i, 1_i\rangle, i \in I)$ *be a vertical sum of bounded lattices and let* $* : L^2 \to L$ *be an ordinal sum of t-norms on* $L$*, i.e.,* $* = (\langle L_i, *_i\rangle, i \in I)$*. Then* $(L, \leq, *, \to, 0, 1)$ *is a bounded residuated lattice if and only if* $(L_i, \leq_i, *_i, \to_i, 0_i, 1_i)$ *is a bounded residuated lattice for each* $i \in I$*.*

Observe that the ordinal sum is not only a construction method of bounded residuated lattices, but in specific cases we have also a representation of bounded residuated lattices as ordinal sums of special bounded residuated lattices. For example, each divisible BL-algebra [11] is an ordinal sum of divisible Archimedean BL-algebras and of trivial singleton BL-algebras (compare a similar result for BL-chains [1, 4]).

Note that in the proposition above the fact that $*$ is an ordinal sum of t-norms is crucial.

**Example 3.4** Consider the lattice $(L, \leq, 0, 1)$ whose Hasse diagram is given in Figure 1 (right), and let $* = T^L_{\mathbf{D}}$. Then $(L, \leq, *, \to, 0, 1)$ is a bounded residuated lattice, although $\mathbf{L}$ is a vertical sum of a horizontal sum not admitting residuals and of a trivial 2-element lattice. Observe also that $T^L_{\mathbf{M}}$ does not possess an adjoint operation $\to$ in this case.

# 4 Conclusion

We have discussed the structural problems of bounded residuated lattices which serve as the truth-values range for several approaches to many-valued logics. We have shown that the horizontal sum construction of lattices is not compatible (up to the trivial case leading to the diamond) with the structure of residuated lattices. On the other hand, the ordinal sum of residuated lattices (i.e., ordinal sum of t-norms acting on a vertical sum of lattices) always yields a residuated lattice. There are still many open problems in this field, as indicated in the last example. Observe that $\mathbf{L}$ discussed there possesses the interval $[0, a]$ on which no residuals exist, although the entire lattice $L$ admits residuals, thus excluding the "genuine" necessary condition for the existence of residuals — to require this property from each subinterval $[x, y]$ of $L$.

## Acknowledgment

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# On the Failure of Strong Standard Completeness in ΠMTL

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It is well-known that Hájek's basic fuzzy logic (BL), Łukasiewicz logic, and product logic are not strongly standard complete, i.e. there is a theory $T$ and a formula $\varphi$ such that in each standard algebra $\mathbf{L}$ we have $e(\varphi) = 1$ for any $\mathbf{L}$-model $e$ of $T$ but $T \not\vdash \varphi$. On the other hand, the monoidal t-norm based logic (MTL), which arises from BL by omitting the divisibility axiom, enjoys the strong standard completeness theorem. This is valid also for involutive monoidal t-norm based logic (IMTL) which is an extension of MTL by the law of involution. Thus it is natural to ask whether ΠMTL (i.e. the extension of MTL obtained by adding pseudocomplementation and cancellation) satisfies this theorem as well. In this talk we are going to show that ΠMTL is not strongly standard complete like the product logic.

Let $p, q, r$ be propositional variables and

$$T' = \{\neg\neg r, p \to q, \neg p \to q\} \cup \{(p^n \to r) \to q \mid n \in \mathbb{N}\}.$$

Further, let $M$ be the set of all formulas constructed only from $p, r$ and

$$T'' = \{\varphi \& (\varphi \to \psi) \equiv \varphi \wedge \psi \mid \varphi, \psi \in M\}.$$

Finally, let $T = T' \cup T''$. We claim that $e(q) = 1$ for each $\mathbf{L}$-model $e$ of $T$ in any standard ΠMTL-chain $\mathbf{L}$. Clearly, if $e(p) = 1$ or $e(p) = 0$ then $p \to q$ or $\neg p \to q$ does the job. Thus suppose that $e(p) \neq 0, 1$. Further, $e(r) \neq 0$ because of the formula $\neg\neg r$. Finally, if $e(p^n) \leq e(r)$ for some $n \in \mathbb{N}$ then $(p^n \to r) \to q$

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ensures that $e(q) = 1$. Hence it is sufficient to prove that there is always $n \in \mathbb{N}$ such that $e(p^n) \leq e(r)$. In other words, this means that $e(p)$ and $e(r)$ belong to the same Archimedean class provided that $e(p) > e(r)$. The rest of the proof is based on the following lemma.

**Lemma 1** *Let* $\mathbf{L} = ([0,1], *, \Rightarrow, \leq, 0, 1)$ *be a standard* $\Pi$MTL*-chain,* $x, y \in [0,1]$ *such that* $x^n > y$ *for all* $n \in \mathbb{N}$*. Then there is* $m \in \mathbb{N}$ *such that* $x^m \Rightarrow y = \max\,[y]_{F(x)}$*, where* $F(x)$ *is a filter generated by* $x$ *and* $[y]_{F(x)}$ *is the equivalence class w.r.t.* $F(x)$ *containing* $y$*.*

Thus suppose that $e(p^n) > e(r)$ for all $n$. Let $a = e(p)$ and $b = e(r)$. By the latter lemma we know that there is $m \in \mathbb{N}$ such that $a^m \Rightarrow b = \max\,[b]_{F(a)}$. The maxima of the equivalence classes has the following property.

**Lemma 2** *Let* $\mathbf{L} = (L, *, \Rightarrow, \leq, 0, 1)$ *be an MTL-chain,* $y \in L$*, and* $F$ *be a filter in* $\mathbf{L}$*. If* $y = \max\,[y]_F$ *then* $x \Rightarrow y = y$ *for all* $x \in F$*.*

If we let $\varphi = p$ and $\psi = p^m \to r$, we get $e(\varphi \to \psi) = e(\psi)$ by the latter lemma since $e(p^m \to r) = a^m \Rightarrow b$. It follows that

$$e(\psi \& (\varphi \to \psi)) = e(p) * e(p^m \to r) < e(p^m \to r) = e(\psi) = e(\varphi \wedge \psi)\,,$$

where the strict inequality follows from cancellativity of $*$. But it is a contradiction with the fact that $e$ is a $\mathbf{L}$-model of $T$ since $e(\varphi\&(\varphi \to \psi) \equiv \varphi \wedge \psi) < 1$. Thus $e(q) = 1$ for each model $e$ of $T$ in all standard $\Pi$MTL-chains.

Finally, we will show that $T \not\vdash q$. For this purpose it is sufficient to prove that for any finite subtheory $T_{fin} \subseteq T$ there is an $\mathbf{L}$-model $e$ of $T_{fin}$ such that $e(q) < 1$ for some $\Pi$MTL-algebra $\mathbf{L}$. Let $\mathbf{L}$ be the standard product algebra $[0,1]_\Pi$ and $m \in \mathbb{N}$ be the maximal natural number such that $(p^m \to r) \to q \in T_{fin}$. We evaluate the propositional variables as follows:

$$e(p) = e(q) = \frac{1}{2}\,, \qquad e(r) = \frac{1}{2^{m+1}}\,.$$

Then we have $e(\neg\neg r) = e(p \to q) = e(\neg p \to q) = 1$ and for all $n \leq m$

$$e(p^n \to r) = \frac{1}{2^n} \Rightarrow \frac{1}{2^{m+1}} = \frac{1}{2^{m+1-n}} \leq \frac{1}{2} = e(q)\,.$$

Thus $e((p^n \to r) \to q) = 1$ for all possible $n$ which may appear in $T_{fin}$. From the above-mentioned facts it follows that $e$ is a $[0,1]_\Pi$-model of $T_{fin}$ and $e(q) < 1$. Hence the following result follows.

**Theorem 3** $\Pi$MTL *does not enjoy the strong standard completeness theorem.*

Even since we have found the counterexample for $T_{fin}$ in $[0,1]_\Pi$, we know that any logic between $\Pi$MTL and the product logic cannot be strongly standard complete.

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# Two notions of fuzzy lattice completion

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In the framework of Henkin-style higher-order fuzzy logic we define two notions of fuzzy lattice completion. Our attention is restricted to dense linear crisp orderings, which are important for the theory of fuzzy real numbers. We investigate the properties of both notions and compare them with some results from the literature.

**The framework.** In [2], the Henkin-style higher-order fuzzy logic ŁΠ has been defined, and proposed as a foundational theory for formal fuzzy mathematics. Recall that it is an axiomatic theory over the multi-sorted first-order logic ŁΠ, with sorts for fuzzy classes of any finite order, tuples, comprehension terms, crisp identity $=$, and typed membership predicate $\in$, axiomatized by the class comprehension schemes and the extensionality axioms for classes of all orders. It can easily be generalized to other fuzzy logics: the present notions can in fact be defined and the results proved already in the Henkin-style 3rd-order logic $\mathrm{BL}_\Delta$, which will be our framework in the present paper. We shall use standard definitions of [2] and usual abbreviations of classical mathematics.

**Cones, suprema, and infima.** We fix an arbitrary binary fuzzy relation $\leq$. Although the following notions are most meaningful for fuzzy (quasi)orderings, we need not impose any restrictions on $\leq$.

By standard definitions one can define the *upper cone* $A^{\uparrow} =_{\mathrm{df}} \{x \mid (\forall a \in A)(a \leq x)\}$ (and dually the lower cone $A^{\downarrow}$). Several properties of cones known from classical mathematics can be proved, e.g. that the transitivity of $\leq$ implies that $A^{\uparrow}$ is upper in $\leq$, the antitony of $^{\uparrow}$ w.r.t. $\subseteq$, and the closure and stability properties of $A^{\uparrow\downarrow}$. (All properties and definitions can of course be dualized.)

The usual definition of *suprema* as least upper bounds can then be formulated as $\operatorname{Sup} A =_{\mathrm{df}} A^{\uparrow} \cap A^{\uparrow\downarrow}$, and dually for $\operatorname{Inf} A$ (where $\cap$ denotes the *strong* intersection $A \cap B =_{\mathrm{df}} \{x \mid x \in A \,\&\, x \in B\}$). Notice that they are fuzzy classes, since the property of being a bound is graded. Nevertheless, if $\leq$ is antisymmetric w.r.t. a relation $E$, then the suprema and infima are $E$-unique; if furthermore $\operatorname{Ker}(E)$ is the identity relation, the unique element of $\operatorname{Ker}(\operatorname{Sup} A)$ is *the* supremum of $A$, denoted by $\sup A$.

Some properties known from classical mathematics hold for fuzzy suprema, e.g., the monotony w.r.t. $\subseteq$ (antitony for infima). The property of *lattice completeness* is defined as the existence of a supremum for any class; the existence of all infima then already follows, since suprema and infima are interdefinable by $\operatorname{Sup} A = \operatorname{Inf} A^{\uparrow}$ and $\operatorname{Inf} A = \operatorname{Sup} A^{\downarrow}$.

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**Fuzzy lattice completions of dense linear crisp orders.** We restrict our attention to *linear crisp* domains, since we aim at a formal theory of fuzzy numbers constructed in the usual way, i.e. over some system of crisp numbers. For simplicity, we only consider *dense* orderings here (the treatment of non-dense orderings would need some special adjustments). The theory of fuzzy lattice completions of dense linear crisp domains is directly applicable in the construction of fuzzy reals over crisp rationals or reals (cf. [1]).

We distinguish two methods of fuzzy lattice completion for linear crisp domains, which generally differ in fuzzy logic (unlike in classical logic): the *Dedekind* completion by (fuzzy) Dedekind cuts, and the *MacNeille* completion by (fuzzy) stable sets. Both methods directly generalize the classical Dedekind–MacNeille completion by admitting fuzzy sets into the construction. Both methods yield complete lattices and preserve the existing suprema and infima. However, the resulting lattices cannot generally be characterized as the least complete lattice extending the original order. (The latter is, of course, the crisp Dedekind–MacNeille completion, since we start from a crisp order; the former are just the least completions containing all fuzzy cuts or all fuzzy stable sets.) We define the constructions for lower sets (both can of course be dualized for upper sets).

**Fuzzy MacNeille completion.** We call $A$ *(lower) stable* iff $A^{\uparrow\downarrow} = A$. The *fuzzy MacNeille completion* $\mathcal{M}(X)$ of $X$ is the (crisp) class of all stable subclasses of $X$, ordered by inclusion. It is a complete lattice into which $X$ is embedded by assigning $\{x\}^{\downarrow}$ to $x \in X$; the embedding preserves all suprema and infima that already existed in $X$. The suprema and infima in $\mathcal{M}(X)$ are unique w.r.t. bi-inclusion; due to the extensionality axiom, there is a unique $\sup \mathcal{A} \in \mathrm{Ker}(\mathrm{Sup}\,\mathcal{A})$ for any $\mathcal{A} \subseteq \mathcal{M}(X)$ (dtto for infima). Furthermore, $\inf \mathcal{A} = \bigcap \mathcal{A}$ and $\sup \mathcal{A} = (\bigcup \mathcal{A})^{\uparrow\downarrow}$, as in classical mathematics.

**Fuzzy Dedekind completion.** We call $A$ a *(lower) Dedekind cut* iff it satisfies the following two axioms:

$$\Delta(\forall x, y)[x \leq y \rightarrow (y \in A \rightarrow x \in A)], \quad \Delta(\forall x)[(\forall y < x)(y \in A) \rightarrow x \in A].$$

Thus fuzzy cuts are lower, right-closed subsets of $X$ (i.e., their membership functions are non-increasing and left-continuous). The conditions reflect the intuitive motivation that the membership $x \in A$ expresses (the truth value of) the fact that $x$ *minorizes* the fuzzy lattice-element represented by $A$. (The $\Delta$'s express the fact that any imperfection would strictly violate the motivation.)

*Fuzzy Dedekind completion* $\mathcal{D}(X)$ of $X$ is the set of all Dedekind cuts on $X$, ordered by inclusion. The properties of MacNeille completions listed above hold for Dedekind completions as well. The soundness of the axioms of Dedekind cuts w.r.t. the intuitive motivation can be proved: it holds for Dedekind cuts that $x \in A \leftrightarrow \{x\}^{\downarrow} \subseteq A$.

**Comparing the two notions.** In classical mathematics, $\mathcal{M}(X) = \mathcal{D}(X)$. This is also true in Łukasiewicz logic (where negation is involutive). Generally, in any logic containing $\mathrm{BL}_\Delta$ we can prove $\mathcal{M}(X) \subseteq \mathcal{D}(X)$, i.e., any stable class is a Dedekind cut. The converse inclusion, however, is not valid over $\mathrm{BL}_\Delta$. If negation is strict, all cones (and therefore, all stable sets) are crisp, since $y \in$

$A^{\uparrow} \equiv (\forall a > y)\neg(a \in A)$ for dense linear crisp $\leq$. Thus in $\mathrm{SBL}_\Delta$ (or stronger), fuzzy MacNeille completions coincide with crisp MacNeille completions; fuzzy Dedekind completions of non-empty sets, on the other hand, always contain fuzzy cuts (in non-crisp models).

In Łukasiewicz logic, fuzzy completions of dense crisp linear orders show rather special properties: not only $\mathcal{M}(X) = \mathcal{D}(X)$, but also $\subseteq$ is a weak (i.e., with strong disjunction) linear order on $\mathcal{M}(X)$ $(= \mathcal{D}(X))$. Again, this property cannot be proved generally (in logics where co-norm disjunction is present): in particular, it fails for the Gödel co-norm $\vee$ (max-linearity is equivalent to the excluded middle).

**Comparison with results from the literature.** Höhle's paper [5] and a chapter in Bělohlávek's book [3] study the minimal lattice completion of *fuzzy* orderings (by the construction that we call here the MacNeille completion). In our present setting we are, on the other hand, concerned with lattice completions of *crisp* orders by fuzzy sets (the latter is also studied towards the end of [5]).

Both [3] and [5] arrive at essentially the same results as the present paper wherever our areas of interest intersect (fuzzy MacNeille completions of crisp domains in [5], fuzzy suprema and infima in both), even though their definitions slightly differ from ours (in [5], the setting is further complicated by considering fuzzy domains of $\leq$). The most important difference is in the definitions of antisymmetry, where both works use $\wedge$ instead of & (in [3], $\wedge$ for & is also used in the definition of suprema and infima); such definitions are narrower, and thus our results are more general. Reasons can be given why & rather than $\wedge$ should (from the point of view of formal fuzzy logic) be used in the definitions; the results of [3] and [5] are then well-motivated only in Gödel logic.

Incidentally, both notions defined in the present abstract satisfy Dubois and Prade's requirement of [4] that the cuts of fuzzy notions be the corresponding crisp notions. This is a rather general feature of classical definitions transplanted to formal fuzzy logic (which is a methodology of [2], foreshadowed already in [5]).

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# Weakly implicative predicate fuzzy logics

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There are two classes of propositional logics related to the area of mathematical fuzzy logics proposed in [3] (see also joint paper by the author and Libor Běhounek [1] where philosophical, methodological, and pragmatical reasons for introducing these two classes appear.)

After we recall same basic definitions we turn our attention to the first-order variants of these two classes of logics. The results presented here are mainly from the author's thesis [2] and prepared paper [4]. Because of the lack of space we present the basic definitions and theorems only and we completely disregard the important concept of Baaz delta.

## 1 Weakly implicative (fuzzy) logics

The class of *weakly implicative logics* extends the well-known class of Rasiowa's implicative logics (see [7]) by omitting the rule $\varphi \vdash \psi \to \varphi$. A logic (represented by the deductive closure $\vdash$) is weakly implicative iff it contains a (definable) connective $\to$ that satisfies the following conditions:

$$\begin{array}{rcll} & \vdash & \varphi \to \varphi & \\ \varphi, \varphi \to \psi & \vdash & \psi & \\ \varphi \to \psi, \psi \to \chi & \vdash & \varphi \to \chi & \\ \varphi \to \psi, \psi \to \varphi & \vdash & c(\ldots, \varphi, \ldots) \to c(\ldots, \psi, \ldots) & \text{for all connectives } c \end{array}$$

Weakly implicative logics can be characterized as those which are complete w.r.t. a class of *ordered matrices* (in which the set $D$ of designated values is upper), if the ordering of the elements of the matrix is defined as $x \leq y \ \equiv_{\mathrm{df}} \ x \to y \in D$.

By (weakly implicative) *fuzzy* logics we call such weakly implicative logics that are complete w.r.t. a class *linearly* ordered matrices. It turns out that this class approximates well the bunch of logics studied in so-called "fuzzy logic in narrow sense". For logics with finitary rules only, the class of weakly implicative fuzzy logics can be equivalently characterized by the following conditions:

- Each **L**-matrix is a subdirect product of linear ones. *(Subdirect representation property)*
- Each theory in **L** can be extended to one whose Lindenbaum-Tarski matrix is linear. *(Linear extension property)*
- $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ entails $T \vdash \chi$. *(Prelinearity property)*

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Let us recall that BCI is an implicational fragment of intuitionistic linear logic, it is not implicative logic, and it has the following presentation:

| | |
|---|---|
| $\mathcal{B}$ | $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| $\mathcal{C}$ | $\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$ |
| $\mathcal{I}$ | $\vdash \varphi \to \varphi$ |
| (MP) | $\varphi, \varphi \to \psi \vdash \psi.$ |

Let **L** be a finitary logic expanding BCI. Then the following are equivalent:

- **L** has LDT (Local Deduction Theorem: for each theory $T$ and formulae $\varphi, \psi$: $T, \varphi \vdash \psi$ iff there is $n$ such that $T \vdash \varphi^n \to \psi$.)
- **L** is pure (**L** has an axiomatic system where Modus Ponens is the only deduction rule).

# 2 Predicate weakly implicative fuzzy logics

Now we move to the first-order logics. Our approach is inspired by the classical first-order logic and by its modifications, the main sources are Hájek's treatment of basic predicate fuzzy logic (for details see [6]) and Rasiowa's approach to first-order implicative logics (see [7]).

We assume that the reader is familiar with the syntax and semantics of some fuzzy predicate logic (see [6]). Let us just mention that by $\models_{\mathbf{L}\forall^-}$ ($\models_{\mathbf{L}\forall}$) we understand a consequence relation given by all safe **B**-models for all (linearly) ordered **L**-matrices **B**. Let us fix a predicate language $\mathbf{\Gamma}$. For weakly implicative logic **L** we define logic $\mathbf{L}\forall^-$ as:

| | |
|---|---|
| (P) | the axioms and rules resulting from the axioms and rules of **L** by the substitution of the propositional variables by the formulas $\mathbf{\Gamma}$. |
| ($\forall$1) | $(\forall x)\varphi(x) \to \varphi(t)$, where $t$ is substitutable for $x$ in $\varphi$. |
| ($\exists$1) | $\varphi(t) \to (\exists x)\varphi(x)$, where $t$ is substitutable for $x$ in $\varphi$. |
| ($R\forall$2) | $(\forall x)(\chi \to \varphi) \vdash (\chi \to (\forall x)\varphi)$, where $x$ is not free in $\chi$. |
| ($R\exists$2) | $(\forall x)(\varphi \to \chi) \vdash ((\exists x)\varphi \to \chi)$, where $x$ is not free in $\chi$. |
| ($Gen$) | $\varphi \vdash (\forall x)\varphi$. |

A logic $\mathbf{L}\forall^-$ is *nice* if rules ($R\forall$2) and ($R\forall$2) can be replaced by their usual axiomatic forms $((\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi))$. It can be shown that logic $\mathbf{L}\forall^-$ is nice if **L** has the structural rule of exchange and (definable) residual conjunction.

For each weakly implicative logic **L** we can easily get:

$$T \vdash_{\mathbf{L}\forall^-} \varphi \quad \text{IFF} \quad T \models_{\mathbf{L}\forall^-} \varphi.$$

For weakly implicative *fuzzy* logics, we further define stronger variant of first-order calculi—the logic $\mathbf{L}\forall$—by adding the axiom of *constant domains*:

$$(\forall x)(\varphi \vee \psi) \to (\varphi \vee (\forall x)\psi) \quad x \text{ not free in } \varphi.$$

Of course, this definition assumes that there is a connective $\vee$ in the language (new results by Petr Hájek show how to avoid this obstacle). We say that $\mathbf{L}\forall$ is complete if for each predicate language $\mathbf{\Gamma}$:

$$T \vdash_{\mathbf{L}\forall} \varphi \quad \text{IFF} \quad T \models_{\mathbf{L}\forall} \varphi.$$

We are not able to prove completeness of $\mathbf{L}\forall$ for all fuzzy logics $\mathbf{L}$ (so far). However we present few rather strong sufficient (and necessary) conditions. For finitary fuzzy logic $\mathbf{L}$ the following conditions are equivalent:

- $\mathbf{L}\forall$ is complete.
- $T \vdash_{\mathbf{L}\forall} \varphi$ IFF $T \models_{\mathbf{L}\forall} \varphi$, for all *at most countable* predicate languages $\mathbf{\Gamma}$.
- $T, \varphi \to \psi \vdash_{\mathbf{L}\forall} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}\forall} \chi$ entails $T \vdash_{\mathbf{L}\forall} \chi$.

Observe that the last condition (prelinearity property) surely holds for $\mathbf{L}$ (because $\mathbf{L}$ is propositional fuzzy logic) but we cannot prove (in general) that this property is preserved by transition to the first order. We are able to it in some cases only; finitary predicate fuzzy logic $\mathbf{L}\forall$ is complete if:

- $\mathbf{L}\forall$ has LDT (LDT is defined as in the propositional case).
- $\mathbf{L}\forall^-$ is nice and $\mathbf{L}$ has LDT.
- $\mathbf{L}\forall^-$ is nice and $\mathbf{L}$ is pure logic extending BCI.
- $\mathbf{L}\forall$ is an axiomatic extension of some complete fuzzy logic.
- $\mathbf{L}\forall$ is the intersection of an arbitrary class of complete fuzzy logics.

As corollary we get completeness for the following known fuzzy logics (for their definitions see survey [5]): MTL, SMTL, IMTL, ΠMTL, WNM, NM, BL, SBL, Łukasiewicz, product, Gödel, and the "hoop" variants of all these logics. By a small modification of the presented results we could get completeness also for all above-mentioned logic with Baaz delta.

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# Fuzzy Description Logics and the Semantic Web

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**Extended Abstract:** In the last decade a substantial amount of work has been carried out in the context of Description Logics (DLs). DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. Nowadays, DLs have gained even more popularity due to their application in the context of the Semantic Web. Ontologies play a key role in the Semantic Web and major effort has been put by the Semantic Web community into this issue. Informally, an ontology consists of a hierarchical description of important concepts in a particular domain, along with the description of the properties (of the instances) of each concept. DLs play a particular role in this context as they are essentially the theoretical counterpart of the Web Ontology Language OWL DL, the state of the art language to specify ontologies. Web content is then annotated by relying on the concepts defined in a specific domain ontology.

However, OWL DL becomes less suitable in all those domains in which the concepts to be represented have not a precise definition (which on the Web is likely the rule rather than an exception).

We present the current state of the art of fuzzy description logics and present open issues to be addressed to make them appealing for the Semantic Web.

**Key Words :** *fuzzy logic, description logics, ontology representation*

# BEST LOCATIONS FOR RIVER WATER QUALITY MONITORING SENSORS THROUGH FUZZY INTERPOLATION

ANGELO MARCELLO ANILE, SALVATORE SPINELLA, AND MARCO OSTOICH

Abstract. This work concerns the interpolation of environmental data using B-splines fuzzy in order to monitor water quality in a river. Sparse fuzzy interpolated model is then queried in order to retrieve information useful for planning precautionary measures. Moreover the information retrieved can be used to improve the distribution of the monitoring sensors on the basin area to optimize the coverage.

Geographical data concerning environment pollution consist of a large set of temporal measurements (representing, e.g. hourly measurements for one year) at a few scattered spacial sites. In this case the temporal data at a given site must be summarized in some form in order to employ it as input to build a spatial model. Summarizing the temporal data (data reduction) will necessarily introduce some form of uncertainty which must be taken into account. Fuzzy numbers can represent this uncertainty in a conservative way without any statistical "a priori" hypothesis. This method has been employed for ocean floor geographical data by [Patrikalakis et al 1995] (in the interval case) and [Anile 2000] (for fuzzy numbers) and to environmental pollution data by [Anile and Spinella 2004].

Fuzzy interpolation is carried out with B-splines to get a deterministic model for environmental pollution data. Then the model is interrogated by fuzzy queries to find the sites exceeding a quality threshold. The results suggest the areas of the basin which should be subjected to a further rigorous examinations.

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*Key words and phrases.* Uncertain, Fuzzy Number, Fuzzy Interpolation, Fuzzy Queries, B-spline, Sparse Data, Environment Pollution.

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# Fuzzy Petri nets in modelling business processes

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**Abstrakt.** To describe the „behaviour“ of business processes are used final automats which have a lot of restrictions. This can be simply solved by Petri nets which are more suitable because of their precision and exact specification. In case of extensive real business processes, where connections between individual activities it's possible to describe only vaguely would be more suitable to use classical Petri nets with applied fuzzy logic. This article is about description a application of fuzzy Petri nets for modelling these business processes.

**Key words:** business processes, fuzzy modelling, Petri nets.

## 1 Business processes modelling

During business processes modelling is important to devotedly describe associations between activities and roles represented by abilities of participants involved in the process as an activity we understand on atomic (no more divisible) step in the process execution. Role is set of skills which mutually supplement each other. Roles are assigned to individual activities to let them full fill in scope of process execution.

Generally we have three basic approaches for process modelling which are based on elemental types of used abstraction [5]:

**1. Functional approach** which is aimed at functions, their structures, inputs and outputs.

**2. Approach of behaviour specifications** aimed at operating aspect of process execution by setting up the events and conditions according to which these individual activities can be executed.

**3. Structural approach** is aimed at static aspect of process. The goal is to affect entities and sources appearing in process within their attributes, activities (services) and mutual relations.

To describe the behaviour of business processes are used the **final automats**, which have a lot of restrictions e.g. in number of statuses in modelling complicated processes. In order to that are often used Petri nets which were created for extension purposes of modelling possibilities of final automats.

As an advantage of business process modelling by Petri nets we see their formal description which supplements the graphic illustration. Thereby is permit precise and exact specification of the process and so is possible to remove definiteness, uncertainty and contradiction. Except clear graphic expression Petri nets have also very well defined mathematic basics which can be used in various software tools for specification and analysis of business processes solved by IT.

But anyway classical Petri nets can have certain problems in modelling of real and complicated processes. From this reason were created extensions aimed at procuration of increase of modelling power. It is about possibilities for:

- hierarchization,
- Petri nets with additional time,
- Coloured Petri nets.

## 2 Fuzzy modelling

As another approach of description of real business processes is application of **fuzzy modelling** [4]. If we would like to describe complicated reality then we can decide between relevance of information, which is less exact, and accuracy of information which will be less relevant. If you would increase the exact of processes description we get at the point when accuracy and relevance become mutually contradictable characteristics. For instance process of car production is possible to describe by few sentences where we globally describe individual parts of car and assembly sequence. We found out this way how to assemble a car but we won't know anything about the relations between the individual components, machines and people. If we would like to know more details we have to add data about machines' permeability, performance of people, order of tables etc. But the amount of information is increasing in this case. And they are more exact that mean we will know more but just about a small part of processes in company. If we would like to describe these all processes in company into such details it would end up with huge amount of detail information which nobody would be able to read. A if so, than to understand to such amount of information he would need to use natural language so I would refer to vague characteristic. In other case he would get lost in such exact details because human mind is limited. We can see that accuracy is just illusion, for it is essentially attainable. All these facts are in the background of considerations of fuzzy logic founders [6]. Fuzzy logic basically comes from theory of fuzzy sets and is concerned on vagueness described by mathematics.

In this context there is fuzzy set defined as a set which except of full or no membership permits also partial membership. That means that the item belongs into a set with some particular degree of membership. Function, which links to every single item from universum a degree of membership, is called membership function. Fuzzy theory tries to cover the reality in its vagueness and uncertainty. During nearly 40 years existence is worthy of many solutions of technical problems which was impossible to solve by other tools in practise. To every single item is possible to add the Degree of membership which expresses the measure of membership to particular item into fuzzy set. For instance: when you try to manage complaint of supplier you can set up the measure of membership of the same type of bug into fuzzy sets. You can decide which parts are "good", which parts is possible to "process yet" and which parts is necessary to "scrap". For classical deciding is in this case possible to set up limits of what is still admissible and what is not any more too hard. We can add number from interval $<0,1>$, which express measure of our conviction. Fuzzy theory notices vaguely specified requirements in question and adequately calculate for that the degree of membership. Fuzzy logic let us use vagueness directly and knows also how to represent it easily.

## 3 Fuzzy Petri nets

Integration of fuzzy logic into classical Petri nets is possible to implement as following. Let's use definition of fuzzy logic Petri net.

FLPN = (P,T,F,$M_0$,D,h,a, $\theta$, l) where

P = {$p_1$, …, $p_n$} is final set of places,

T = {$t_1$, …, $t_m$} is final set of transitions,

F $\subseteq$ (P x T) $\cup$ (T x P) is flow relation, where is

$$\forall t \in T \ \exists p, q \in P : (p,t) \vee (t,q) \in F ,$$

$M_0$: $P \rightarrow \{0,1\}$ is initial marking,

D is final set of statements – $P \cap D = T \cap D = \emptyset, |P| = |D|$,

h: $P \rightarrow D$ is associated function representing bijection from place to statement,

a: $P \rightarrow [0,1]$ is associated function representing a value in place from set of real numbers from 0 to 1,

$\theta$, l: $T \rightarrow [0,1]$ is associated function representing transition value from set of 0 to 1.

For $\forall$ x$\in (P \cup T)$

$^{\bullet}x = \{y \mid yFx\}$ , input set (preset) element x

$x^{\bullet} = \{y \mid xFy\}$, output set (postset) element x

For $\forall p \in P$, valid for following:

$M'(p) = M(p) + 1,$ if $p \in t^{\bullet} - {}^{\bullet}t$ ;

$M'(p) = M(p) - 1,$ if $p \in {}^{\bullet}t - t^{\bullet}$ ;

$M'(p) = M(p),$ otherwise,

$\alpha(p) = \lambda_t \alpha(p')$ if $\alpha_1 \geq \theta_t \wedge p \in t^{\bullet} \wedge p' \in {}^{\bullet}t$ .

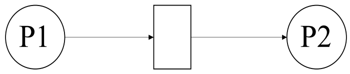
Pro $t \in T^{AND}$ is $\alpha(p) = \lambda_t \min\limits_{\forall p' \in {}^{\bullet}t} \alpha(p')$ if

$$\min_{\forall p' \in {}^{\bullet}t} \{\alpha(p')\} \geq \theta_t \wedge p \in t^{\bullet}$$

a pro $t \in T^{OR}$ is $\alpha(p) = \lambda_t \max\limits_{\forall p' \in {}^{\bullet}t} \alpha(p')$ if

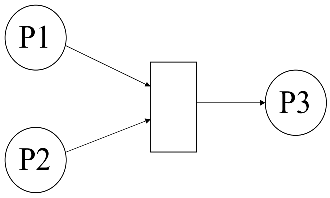
$$\max_{\forall p' \in {}^{\bullet}t} \{\alpha(p')\} \geq \theta_t \wedge p \in t^{\bullet}.$$

Now let's express IF-THEN rules and their transformation into fuzzy logic by Petri nets.
Rule IF p1 THEN p2 let's express



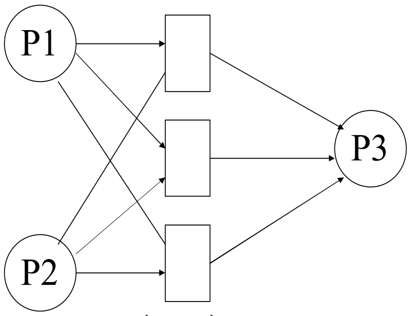

And in fuzzy logic $\alpha_2 = \lambda_t \alpha_1$ if $\alpha_1 \geq \theta_t$.

Rule IF p1 AND p2 THEN p3 is expressed:



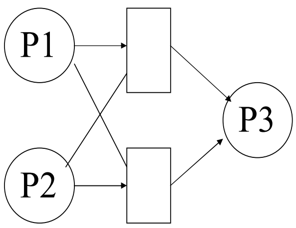

And in fuzzy logic $\alpha_3 = \lambda_t \min\limits_{\alpha_i \geq \theta_{t\,AND}} \{\alpha_1 \alpha_2\}$ pro i=1 $\wedge$ 2.

Rule IF p1 OR p2 THEN p3 we express by inhibitive edges




And in fuzzy logic $\alpha_3 = \lambda_{t\,OR} \max\limits_{\alpha_i \geq \theta_{t\,OR}} \{\alpha_1 \alpha_2\}$ pro i=1 $\vee$ 2.

Rule IF p1 XOR p2 THEN p3 express by inhibitive edges




And in fuzzy logic $\alpha_3 = \lambda_{t\,XOR} \alpha_1$ if $\alpha_1 \geq \theta_{t\,XOR} \wedge \alpha_2 = 0$,

$\alpha_3 = \lambda_{t\,XOR} \alpha_2$ if $\alpha_2 \geq \theta_{t\,XOR} \wedge \alpha_1 = 0$.

By the application of fuzzy logic into Petri nets spring up strong tool for modelling of real business processes especially for:

- Easy comprehensibility and elaborate mathematic devise,

- Quite easy and simple proposal,
- Modulability of solution – it is possible to add and delete individual modules without necessity of recreating the whole system,
- Robustness of suggestion that means system is not necessary to modify in case of change of solution parameters of task in frame of particular surroundings.

## 4 Integration of system for modelling of business processes with information system QI

For securing good quality of company management is advantageous to integrate information system with process system. This integration let us to do change in incorporation of information system. His functions are machine-controlled and run by process system that means that information system purvey an order of functions to users and at the same time hand over reports to process system which evaluate them and according to the results process the movement in process map. This way is implemented the run of functions of information system by process system.

University of Ostrava in Ostrava in cooperation with company DCC a.s. generate currently a tool for process management and its implementation into information system QI [2]. The goal is to create inside of the information QI a tool which will work on the basics of fuzzy Petri nets.

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# On Symmetric MV-polynomials

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## 1 Abstract

Let $\mathcal{L}$ be the poset, under $\subseteq$, of subalgebras of the MV-algebra $[0,1]$. $\mathcal{L}$ then has a unique maximal element, $[0,1]$.

$\mathcal{L}$ also contains atoms, that is subalgebras $A \subseteq [0,1]$ such that $A' \subseteq A$, then $A' = \{0,1\}$ or $A' = A$. The algebra $\{0, \frac{1}{2}, 1\}$ is such an atom.

Since, for a maximal ideal $M$ of an MV-algebra $A$, $\frac{A}{M} \in \mathcal{L}$, we have a method to refine the structure of the maximal ideal space $Max(A)$. Heuristically, the "smaller" the quotient $\frac{A}{M}$, the "larger" the maximal ideal $M$. In effect this provides a pre-order on the set of maximal ideals.

In this work we shall study these ideas for the set of maximal ideals of "finite type", that is maximal ideals $M$ with $\frac{A}{M}$ finite.

We shall first look at "super maximal" ideals $M$, that is, those maximal $M$ such that $\frac{A}{M}$ is a "small" as possible, namely $\frac{A}{M} = \{0,1\}$ Next we shall look at some classes of "big" maximal ideals $M$, that is, those maximal ideals $M$ which if not "super maximal" are sucht hat $\frac{A}{M}$ is an atom of $\mathcal{L}$.

Given a subalgebra $S \subseteq [0,1]$ it is easy to find MV-algebras $A$ with a maximal ideal $M$ such that $\frac{A}{M} \cong S$. One need only form a product $A' \times S$.

What we will show in this work is that given any MV-algebra $A$ there are subalgebras that contain maximal ideals of finite type, in fact of a prescribed finite type.

Here our study will use a class of MV-polynomials we call "symmetric" which will permit us to construct the appropriate subalgebras.

The first part of this work concerns supermaximal ideals (considering the Boolean algebra as an MV-algebra).

Given an MV-algebra, its set of idempotents, $B(A)$, is a subalgebra which is a Boolean algebra. We shall examine extensions of $B(A)$ in $A$, that is subalgebras $A'$ of $A$ such that $B(A) \subseteq A' \subseteq A$, that have supermaximal ideals.

The second part of this work will take up the case of certain extensions of $B(A)$ which may have big maximal ideals, and we shall study some properties of these algebras.

Both of these parts will be presented as a special case of subalgebras determined by certain symmetric MV-polynomials.

Finally, as an application of a class of symmetric polynomials here described, we show that some projective $MV$-subalgebras of the one-generated free $MV$-algebra can be obtained via symmetric polynomials. Actually, such projective $MV$-algebras are algebras of compositions of a given symmetric polynomial and the McNaughton functions of one variable.

# Product logic and probabilistic Ulam games

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Connections between games and many-valued logic have been shown first by Mundici [M] for the case of the Ulam-Rènyi game and Łukasiewicz logic. In fact, Mundici showed that it is possible to code the information contained in a sequence $\sigma$ of questions-answers (called *record* in [CM]) by means of a function $f_\sigma$ from $\Omega$ into $[0,1]$ called the *truth-value function* corresponding to $\sigma$. More precisely, if up to $n$ lies are allowed, then for every $x \in \Omega$, $f_\sigma(x) = \frac{n+1-h}{n+1}$, where $h$ is the numbers of questions answers which falsify $x$ (repetitions count, of course). Truth-value functions have a logic, which is precisely the $n$-valued Łukasiewicz logic if $n$ is the upperbound to the number of lies allowed by the game. Moreover if one considers the logic of all truth-value functions corresponding to all Ulam-Rènyi games with an arbitrary number $n$ of lies, then the underlying logic is just the infinite-valued Lukasiewicz logic.

One may ask if similar games can be found for other many-valued logics. A positive answer for the case of Hàjek's Basic Logic $BL$ was given by Cicalese and Mundici in [CM]. There the authors propose a multichannel variant of the Ulam-Rènyi game and prove that truth-value functions in such variant constitute a complete game sematics for $BL$. Since a semantics for Gödel logic can be obtained as a particular case, what still remains open is a game semantics for product logic.

Pelc's game $G(p, \Omega)$ is a probabilistic variant of Ulam game in which the secret number is in $\Omega$ and Responder gives the correct answer with probability $p \geq \frac{1}{2}$. Such a game seems to be a good candidate for a semantics for product logic, in that the most natural interpretation of the conjunction of truth-value functions is given by their pointwise product. In this game, the truth-value function $f_\sigma$ associated to a sequence $\sigma$ of questions-answers, is the function from $\Omega$ into $[0,1]$ such that for each $x \in \Omega, f_\sigma(x)$ expresses the conditional probability of (the answers to the questions contained in) $\sigma$ given that the secret is $x$. Note that, using the Bayes formula, $f_\sigma$ allows us to compute the inverse conditional probability, that is, the probability that the secret is $x$ given the sequence $\sigma$. Thus a truth function gives us a complete information about what we know on the ground of the sequence $\sigma$.

Truth functions have an algebraic structure: if $\tau$ is the justaposition of two records $\sigma$ and $\rho$, then $f_\tau$ is the pointwise product of $f_\sigma$ and $f_\rho$. Moreover, we can define a partial order between truth-value functions: $f_\sigma \leq f_\rho$ if for all $x \in \Omega$, $f_\sigma(x) \leq f_\rho(x)$. Thus truth-value functions constitute a partially ordered monoid, the identity being the constantly 1 function $f_\emptyset$.

**Lemma 1** *Let $p \in [\frac{1}{2}, 1]$. A function $f$ from $\Omega$ into $[0,1]$ is a truth-value function iff there are a natural numbers $n$, $k_x : x \in \Omega$ such that for every $x$, $k_x \leq n$ and $f(x) = p^{k_x} \cdot (1-p)^{n-k_x}$.*

Since the conjunction (justaposition) of two records is interpreted as the product of their truth-value functions, one might expect that the logic of such truth-value functions could be product logic. However, this is not the case. More precisely, we have the following situation:

**Lemma 2** *Let $N = Card(\Omega)$, and $\mathcal{M}$ be the partially ordered monoid of all truth-value functions corresponding to $G(\Omega, p)$. Then:*

*(a) $\mathcal{M}$ can be equipped with the structure of a prelinear residuated lattice iff either $p \in \{1, \frac{1}{2}\}$ or $N = 1$.*

*(b) $\mathcal{M}$ has a minimum iff $p = 1$.*

*(c) If $N = 1$, then $\mathcal{M}$ is divisible iff $1 - p$ is a power of $p$.*

*(d) If $N > 1$, then there are values of $p$ such that $\mathcal{M}$ is not residueted. However, its partial order is an inverse well quasi order, but possibly not a lattice order.*

Thus, $\mathcal{M}$ is a product algebra only of $p = 1$ (in fact, in this case it is a Boolean algebra), and it can be a cancellative hoop only in some uninteresting cases.

In this paper we investigate a variant of probabilistic Ulam game in which with an additional cost, depending on the size of $S$, Questioner can require that the answer is absolutely reliable (i.e., correct with probability 1) if the unknown number belongs to some set $S$ and reliable with probability $p$ otherwise.

# 1 A variant of Pelc's game

We now consider a variant of Pelc's game. For $\frac{1}{2} \leq p < 1$, the game $G^*(\Omega, p)$ is defined as follows: 0nce again, Questioner has to guess a secret number contained in a known finite search space $\Omega$. Each time Questioner chooses a subset $S$ of $\Omega$ and, with an additional cost $c(S)$, which we will assume to be proportional to the size of $S$, he can obtain a true answer with probability 1 if the unknown number is in $S$ and a true answer with probability $p$ otherwise. Unlike the case of Pelc's game $G(\Omega, p)$, Questioner has to guess the secret number with probability 1. Even though the correct answer is required with probability 1, the total cost (number of questions needed plus the sum of all additional costs) when Questioner uses a given guessing strategy $\Sigma$ is a random variable $X$. The aim of the game consists in finding strategies $\Sigma$ such that the expected value $E(X)$ when Questioner play startegy $\Sigma$ is minimum.

Formally speaking, every question *is your number in* $X$? is accompanied by a reliability set $S \subseteq \Omega$, which means that if the secret number is in $S$, then the answer must be truthful with probability 1. Also in this case, to every record $\sigma$ (i.e., to everey sequence $\sigma$ of triples $(Q_i, S_i, A_i) : i = 1, ..., n$, where $Q_i, S_i$ and $A_i$ denote the $i^{st}$ question, the $i^{st}$ reliability set and the $i^{st}$ answer respectively, we associate a truth-value function $f_\sigma$ from $\Omega$ into $[0,1]$ such that for all $x \in \Omega$, $f_g(x)$ represents the probability of (the answers to the questions in) $\sigma$ given that the secret number is $x$.

Also in the case of $G^*(\Omega, p)$, truth-value functions play a very important role because for $x \in \Omega$ the probability that $x$ is the unknown secret given a sequence $\sigma$ can be computed from $f_\sigma$ using the Bayes formula. Moreover, it is readily seen that also in this case the truth-value function $f_\tau$ corresponding to the justaposition $\tau$ of two records $\sigma$ and $\rho$ is the pointwise product of $f_\sigma$ and $f_\rho$.

# 2 The algebra of truth-value functions of the game $G^*(\Omega, p)$.

Also in this case, truth-value functions of the game $G^*(\Omega, p)$ can be characterized:

**Lemma 3** *A function $f$ from $\Omega$ into $[0,1]$ is a truth-value functions of the game $G^*(\Omega, p)$ iff for all $x \in \Omega$, either $f(x) = 0$ or there are natural numbers $n, k$ such that $f(x) = p^n(1-p)^k$. Thus if $\mathcal{H}$ denotes the submonoid of $([0,1], \cdot, 1, \leq)$ generated by $p$, $1-p$ and $0$, then the set $\mathbf{S}$ of truth-value functions is just $\mathcal{H}^\Omega$.*

Once again, it is readily seen that the truth-value function $f_\tau$ of the justaposition $\tau$ of two records $\sigma$ and $\rho$ is the pointwise product of $f_\sigma$ and $f_\rho$. Moreover truth-value functions can be partially ordered pointwise, exactly as in the case of games $G(\Omega, p)$.

We recall the following lemma, proved by Hórčic [H]:

**Lemma 4** *Any finitely generated totally ordered monoid is residuated.*

Thus $\mathcal{H}$ is a residuated lattice. Since the restriction of product to its non-zero elements is cancellative $\mathcal{H}$ is a $\Pi MTL$-algebra. Finally, since $\mathbf{S} = \mathcal{H}^\Omega$, and both product and order in it are pointwise, we can give it the structure of a $\Pi MTL$-algebra as well, just taking pointwise joins, meets and residuals. Of course the residual of two truth-value functions $f_\sigma$ and $f_\rho$ is the greatest truth-value functions We will denote it by $\mathcal{S}$.

**Theorem 1** *$\mathcal{S}$ is a product algebra iff $1-p$ is a power of $p$. In this case, $\mathcal{S}$ generates the whole variety of product algebras. If $1-p$ is not a power of $p$, then $\mathcal{S}$ is a $\Pi MTL$-algebra which is not a product algebra and which does not generate the whole variety of $\Pi MTL$-algebras.*

We now consider two kinds of algebraic interpretation of formulas of many-valued logics. The first one, which we call interpretation of type 1, is as follows:

- We fix an arbitrary $n > 0$, and we take $p$ to be a solution in $[\frac{1}{2}, 1]$ of the equation $p^n + p - 1 = 0$. Note that such a solution exists and that it is $\frac{1}{2}$ iff $n = 1$.
- We fix an arbitrary finite non-empty set $\Omega$ and we consider the game $G^*(\Omega, p)$.
- We interprete atoms as arbitrary truth-value functions of $G^*(\Omega, p)$, and falsum as the identically 0 function on $\Omega$.
- We interprete conjunction & as product, $\vee$ as join, $\wedge$ as meet and implication $\to$ as residual in $\mathcal{S}$.

Interpretations of type 2 are defined as interpretations of type 1, with the difference that $p$ is a number in $[\frac{1}{2}, 1]$ such that $1 - p$ is not a power of $p$.

**Theorem 2** *The logic $L_1$ of all interpretations of type 1 (i.e., the set of formulas which take value 1 under any interpretation of type 1) is product logic, and the logic $L_2$ of all interpretations of type 2 is a power of p, and it is a logic between $\Pi MTL$ and product logic. In particular, the divisibility axiom is not valid $L_2$, but the formula*

$$((A \to B) \to B)^2 \leq A \vee B \vee \neg B,$$

*although not provable in $\Pi MTL$, is a tautology of $L_2$.*

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# Game semantics for product logic
# Part II

C. Marini [*], G. Simi [*]

**Abstract** As the title suggests, this paper is the sequel of Montagna's work about game semantics for product logic. In the first part we investigate some natural strategies for $G^*(\Omega, p)$. In order to compare these strategies, we define the *cost of a question* $Q$ with reliability set $S$ as $1 + a(Card(S))$, with $a > 0$, and the *cost of a sequence* $\sigma$ of questions-reliability sets-answer as the sum of the costs of the single questions-reliability set-answers occurring in it. The *cost of a strategy* is the mean value of the costs of the sequence $\sigma$ produced when Questioner discovers the secret playing the strategy. We investigate the following cases:

1. $p = \frac{1}{2}$ (but $p$ fixed) and $N \to +\infty$;
2. $\frac{1}{2} < p \leq \frac{2}{3}$ (but $p$ fixed) and $N \to +\infty$;
3. $\frac{2}{3} < p < 1$ and $N \to +\infty$;
4. $N$ sufficiently large but fixed and $p \to 1$.

In the second part we shall introduce another variant of $G_P(N, p)$, called $G^*(N)$, which for every $N$ constitutes a complete algebraic semantics for product logic.

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# Hajek's basic logic and multichannel games with lies

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**ABSTRACT**

The only difference between traditional Renyi-Ulam games and multichannel games is that in the latter games answers can be sent on any one among m channels $c_1, ..., c_m$. Each channel $c_i$ is equipped with a parameter $e_i = 0, 1, 2, ...$, in such a way that, if more than $e_i$ erroneous answers happen to be sent on $c_i$, still they are all counted as $1 + e_i$. After asking a question, the Questioner chooses a channel $c_j$ and asks the Responder to send his reply on $c_j$. Further, for any $x$ in the search space $S$, the information about $x$ sent on channel $c_i$ supersedes all past and future information about $x$ sent on $c_j$, for $j > i$. As in the traditional game, all that the Questioner knows about the secret number is given by the multiset $M$ of received answers: $M$ is not a set in general, because two equal answers to the same repeated question carry more information than a single answer. The family of such multisets naturally determines a partially ordered monoid $M$: the monoidal operation amounts to taking the disjoint union of two multisets, and $M' < M''$ means that $M'$ contains more information than $M''$. It turns out that the order structure can be expressed using the monoidal operation together with a sort of negation operation, which is not involutive if $m > 1$. Using heavy machinery from BL-algebras, the algebras of Hajek's basic logic, we prove that the problem of deciding if two states of knowledge $M'$ and $M''$ are equivalent in any possible multichannel game amounts to deciding equality of BL-terms.

# Complexity of t-norm based fuzzy logics with rational truth constants – abstract

Petr Hájek

If a continuous t-norm on $[0,1]$ maps pairs of rationals into rationals (call it *r-admissible*) then the corresponding fuzzy propositional calculus can be extended by rational truth constants and "bookkeeping" axioms for them.

In the sequel, given an r-admissible t-norm $*$, we denote by $R\mathcal{L}(*)$ the extension of $\mathcal{L}(*)$ by adding the truth constants $\bar{r}$ for each rational $r \in [0,1]$ to the language, declaring that $\bar{r}$ denotes just $r$. Given an axiom system for $\mathcal{L}(*)$, ($*$ r-admissible) the corresponding axiom system for $R\mathcal{L}(*)$ results by adding the "bookkeeping" axioms for all rational $r, s \in [0,1]$ :

$$(\bar{r}\&\bar{s}) \equiv \overline{r * s}, \quad (\bar{r} \to \bar{s}) \equiv \overline{r \Rightarrow s},$$

For any r-admissible $*$, $TAUT(R\mathcal{L}(*))$ denotes the set of all tautologies of $R\mathcal{L}(*)$, similarly for $SAT(R\mathcal{L}(*))$ and satisfiable formulas; furthermore, $SCONS(R\mathcal{L}(*))$ is the set of all pairs $(\varphi, \psi)$ such that $\psi$ is a semantic consequence of $\varphi$ in the sense of $R\mathcal{L}(*)$.

**Theorem 1** For Łukasiewicz t-norm Ł, $TAUT(R\mathcal{L}(\text{Ł}))$ and $CSONS(R\mathcal{L}(\text{Ł}))$ are coNP-complete and $SAT(R\mathcal{L}(\text{Ł}))$ is NP-complete.

For Gödel t-norm $G$, $TAUT(R\mathcal{L}(G))$ and $SCONS(R\mathcal{L}(G))$ is co-NP complete and $SAT(R\mathcal{L}(G))$ is NP-complete.

For the product t-norm $\Pi$, the sets $TAUT(R\mathcal{L}(\Pi))$, $TAUT(R\mathcal{L}(\Pi))$ as well as $TAUT(R\mathcal{L}(\Pi))$ are in PSPACE.

For other continuous t-norms (having more than one component) we define a natural notion of being *strongly admissible.* We restrict ourselves to r-admissible t-norms with finitely many components, all having rational endpoints.

**Theorem 2** If $*$ is a strongly r-admissible t-norm with finitely many components and having no $\Pi$-component then $TAUT(R\mathcal{L}(*))$ is co-NP-complete, $SAT(R\mathcal{L}(*))$ is NP-complete (and $SCONS(R\mathcal{L}(*))$ is co-NP-complete.

If $*$ is any strongly r-admissible t-norm with finitely many components then $SAT(R\mathcal{L}(*))$, $TAUT(R\mathcal{L}(*))$ and $SCONS(R\mathcal{L}(*))$ are in $PSPACE$.

One can give an example of an r-admissible continuous t-norm with infinitely many components, all being Ł, which is as undecidable as you want.

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# The enigma of quantifying vague information

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Champions of fuzzy logic and soft computing like to emphasize that vagueness is not invariably pernicious, but — to the contrary — often adequate and welcome in conveying information at the 'right' level of detail. E.g., it is taken for granted that a fuzzy set is often more adequate than a crisp set as a formal counterpart of, say, the set of 'tall people', or 'fast cars', or 'relevant websites' etc. Similarly, fuzzy relations often seem to capture more directly than crisp (standard) relations the type and amount of information that is contained in sentences of the form 'X likes Y', 'A is relevant to B' etc.

But note that concrete syntactical descriptions of a given fuzzy set (understood here as a function with the real unit interval $[0, 1]$ as codomain) are usually *more complex* than a comparable description of a crisp set referring to the same domain of objects. To give a concrete example, think of a *direction* given as a reply to the request: 'Point to the direction of the city centre'. With respect to a fixed reference axis one may 'classically' use, e.g., an integer number between 0 and 360 to formally represent the indicated direction in terms of degrees. Fuzzy logicians, of course, will rush to point out that a more adequate representation of the presumably *vague information* involved, is achieved by a fuzzy (singleton) set of such numbers. However, it should be clear that representing a fuzzy set of, say, integers between 0 and 360 (with a total sum 1 of respective degrees of membership) requires, in general, considerably more complex syntactic objects than the representation of a single integer between 0 and 360. In other words: the *reduction* (or even elimination) of redundant information that is often intended in passing from crisp sets to fuzzy sets seems to result in an *increased* descriptive complexity; which — from a purely quantitative point of view — amounts to even more redundancy. We refer to this seemingly trivial dilemma as 'the enigma of quantifying vague information'.

Of course, there are many ways to formally deal with the challenge to quantify vague and inexact information adequately. However, many approaches that spring immediately to one's mind do not fit in smoothly with classical descriptive complexity and (Shannon style) information theory. We will use the basic machinery of classical Kolmogorov complexity to explore some relevant options in this context. However, in addressing the outlined *foundational challenge*, it seems very important to us not to rush prematurely into concrete mathematical models. We rather suggest to place the phenomenon into the wider context of the contemporary debate on theories of vagueness in analytic philosophy and philosophical logic. In particular, we explain the connection between the outlined 'enigma' and so-called higher order vagueness in light of recent literature on that topic. Moreover we hint at the possibility to employ probabilistic models of computation for quantifying vague (formal) description, in a way that is somewhat analogous to the use of bets on dispersive elementary experiments in dialogue game models of reasoning Łukasiwiecz, Gödel, and Product logic. Following ideas that originated with Robin Giles in the 1970s, the latter models have been recently studied in the context of analytic proof theory for $t$-norm based fuzzy logics by A. Ciabbatoni, G. Metcalfe, and C. Fermüller.

Maciej Farulewski

# On Nonassociative Lambek Calculus

The Nonassociative Lambek Calculus (NL) (Lambek 1961) is a substructural logic used primarily as a type reduction system for categorial grammars. This calculus was presented as a basic logic of types by Moortgat (1997). Other logic of types can be treaded as axiomatic extensions of NL (also enriched with additional operations). NL is strictly substructural logic in the sense, that its Gentzen style form admits no structural rules.

The class of languages generated by categorial grammars based on NL equals the class of context-free languages (Buszkowski 1986, Kandulski 1988), and the same holds for the Associative Lambek Calculus (L). The decision problem for NL is PTIME (de Groote 2002), while it is NP-complete for the associative calculus (Pentus 2003).

The classical version of NL disallows sequents with empty premises and uses product (multiplicative conjunction) and two residuation $\backslash, /$ (implications) operators. We consider extension of NL with additive conjunction and discjunction $\wedge, \vee$. Languages generated by the associative calculus extended with $\wedge$, $\vee$ surpass the class of context free languages (Kanazawa 1992). In (Farulewski, 2005) the Finite Model Property (FMP) of L and NL with $\wedge$ was proved by a refinement of methods used by Buszkowski (2002) for product-free systems of that kind. Here we prove FMP for NL with $\wedge, \vee$ using a modification of intutionistic phase space models from (Okada and Terui 1999). We also consider languages generated by categorial grammars based on NL with additives.

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# Non-clausal Resolution in Fuzzy Predicate Logic with Evaluated Syntax

background and implementation

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**Abstract**

The presentation deals with the refutational resolution theorem proving system for the Fuzzy Predicate Logic of First-Order (FPL) based on the general (non-clausal) resolution rule. It is based on the Fuzzy Predicate Logic with Evaluated Syntax. There is also presented an unification algorithm handling existentiality without the need of skolemization. Its idea follows from the general resolution with existentiality for the first-order logic. When the prover is constructed it provides the deductive system, where existing resolution strategies and its implementations may be used with some limitations arising from specific properties of the FPL.

Additionaly it presents recent advances in implementation of the above mentioned ideas through an experimental application - Fuzzy Predicate Logic GEneralized Resolution Deductive System. The application provides standard breadth-first search algorithm and also originally developed technique DCF (Detection of Consequent Formulas) that is able to significantly reduce the number of produced resolvents. The DCF method make the inference process practically usable since the standard breadth-first search leads to the "combinatorial explosion" during proof search.

The fuzzy predicate logic with evaluated syntax is a flexible and fully complete formalism, which will be used for below presented extension. For the purposes of fuzzy extension the Modus ponens rule was considered as an inspiration. We will suppose that set of truth values is Łukasiewicz algebra. Therefore we will assume standard notions of conjunction, disjunction etc. to be bound with Łukasiewicz operators.

**General resolution for fuzzy predicate logic** ($GR_{FPL}$)

$$r_{GR} : \frac{a/F[G_1,, ..., G_k],\ b/F'[G'_1, ..., G'_n]}{a \otimes b/F\sigma[G/\bot]\nabla F'\sigma[G/\top]} \qquad (1)$$

where $\sigma$ is the union of the most general unifiers (mgu) of the atom pairs $(G_1, G_i)$ and $(G_1, G'_j)$, $G_1, ..., G_k, G'_1, ..., G'_n, G = G_1\sigma$. F is called positive and F' is called negative premise, G represents an occurrence of a subformula (mgu applies to all atoms occurring in $F, F'$). The expression below the line represents the resolvent of premises on G.

Example: **Proof of child's happiness by** $r_{GR}$

Consider the following knowledge (significantly simplified in contrast to the reality) about child's happiness. We suppose that a child is happy in the degree 0.8 if it has mother and father. Further we suppose that a child is happy in the degree 0.5 if it has a lot of toys (we suppose parents are a bit more important for children). We will present several proofs and then we mark the best provability degree from the following axioms. It was used the automated theorem prover of the author for classical logic. Xa. steps represent application of simplification rules for $\perp$ and $\top$.

Common proof members (axioms):

| | |
|---|---|
| 1. $^{0.8}/\forall X[\exists Y[child(X,Y) \,\&\, female(Y)]$ | |
| $\&\, \exists Y[child(X,Y) \,\&\, male(Y)] \Rightarrow happy(X)]$ | (happy with parents - 0.8) |
| 2. $^{0.5}/\forall X[toys(X) \Rightarrow happy(X)]$ | (happy with toys - 0.5) |
| 3. $^{1}/child(johana, hashim)$ | (clear crisp fact) |
| 4. $^{1}/child(johana, lucie)$ | (clear crisp fact) |
| 5.$^{1}/male(hashim)$ | (clear crisp fact) |
| 6.$^{1}/female(lucie)$ | (clear crisp fact) |
| 7.$^{0.9}/toys(johana)$ | (johana has a lot of toys - 0.9) |
| 8.$^{1}/\neg happy(johana)$ | (negated goal - is johana happy?) |

Proof 1:

| | |
|---|---|
| 9. $^{0.9 \otimes 0.5}/\perp \nabla [\top \Rightarrow happy(johana)]$ | |
| 9a. $^{0.4}/happy(johana)$ | ($r_{GR}$ on 7.,2., $Sbt(X) = johana$) |
| 10. $^{1 \otimes 0.4}/\perp \nabla \neg\top$ | |
| 10a. $^{0.4}/\perp$ | ($r_{GR}$ on 9.,8.) |

(happy(johana) is provable in 0.4)

Proof 2:

| | |
|---|---|
| 9. $^{0.8 \otimes 1}/[\exists Y[child(johana,Y) \,\&\, female(Y)]$ | |
| $\&\, \exists Y[child(johana,Y) \,\&\, male(Y)] \Rightarrow \perp]\nabla\neg\top$ | |
| 9a. $^{0.8}/\neg[\exists Y[child(johana,Y) \,\&\, female(Y)]$ | |
| $\&\, \exists Y[child(johana,Y) \,\&\, male(Y)]]$ | ($r_{GR}$ on 1.,8., $Sbt(X) = johana$) |
| 10. $^{0.8 \otimes 1}/\neg[[child(johana, lucie) \,\&\, \top]$ | |
| $\&\, \exists Y[child(johana,Y) \,\&\, male(Y)]]\nabla\perp$ | |
| 10a. $^{0.8}/\neg[child(johana, lucie)$ | |
| $\&\, \exists Y[child(johana,Y) \,\&\, male(Y)]]$ | ($r_{GR}$ on 6.,9., $Sbt(Y) = lucie$) |
| 11. $^{0.8 \otimes 1}/\neg[child(johana, lucie)$ | |
| $\&[child(johana, hashim) \,\&\, \top]]\nabla\perp$ | |
| 11a. $^{0.8}/\neg[child(johana, lucie)$ | |
| $\&\, child(johana, hashim)]$ | ($r_{GR}$ on 5.,10., $Sbt(Y) = hashim$) |
| 12. $^{0.8 \otimes 1}/\neg[\top \,\&\, child(johana, hashim)]\nabla\perp$ | |
| 12a. $^{0.8}/\neg[child(johana, hashim)]$ | ($r_{GR}$ on 4.,11.) |
| 13. $^{0.8 \otimes 1}/\neg\top\nabla\perp$ | |
| 13a. $^{0.8}/\perp$ | ($r_{GR}$ on 3.,12.) |

(happy(johana) is provable in 0.8)

We have stated two different proofs and it is clear that several other proofs could be constructed. Let us note that these proofs either consist of redundant steps or they are variants of Proof 1 and Proof 2, where only the order of resolutions is different. So we can conclude that it is effectively provable that Johana is a happy child in the degree 0.8.

The *Non-clausal Refutational Resolution Theorem Prover* forms a powerful inference system for automated theorem proving in fuzzy logic, which is significantly less discovered area in contrast with classical logic. The main contribution lies in the application into fuzzy logic, which gives a formalization of the refutational proving with the resolution principle and therefore it is essential for practically successful theorem proving in such areas like logic programming in fuzzy logic. Theoretical solution of the prover needed also some new notions to be defined especially the notion of the *refutational proof* and consequent notion of the *refutation degree*. The next interesting area for the presented formalism is the field of semantic web and especially *description logic*, in which the author proposed also the usage of the resolution principle. The recent idea of fuzzy description logic is naturally suitable for further extensions with the presented inference rules and also reflects real situations as it could be observed from the last example. The last but not least further application relates to the previous author's works in the implementation of the non-clausal resolution principle. This implementation called GERDS (GEneralised Resolution Deductive System) will be extended for usage in fuzzy logic and description logic.

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# Generalized Quantifier Theory:
## an(other) area where logic meets linguistics and computer science

Dag Westerståhl

## Abstract

GQ theory is an unusually clean and tidy logical framework. Interesting mathematical facts are known about it, but it also has connections with linguistics and with computer science. Below I present some of the interaction with linguistics, not in the form of an overview but with a few chosen examples:

**1.** GQs were introduced in logic by Mostowski and Lindström in the 1950s and 60s, but Frege formulated essentially the same notion in the 1890s, and in fact Aristotle's account of the four quantifiers in the *square of opposition* readily extends to other quantifiers. There is a subtle but important – and often misunderstood – distinction between the 'modern' square and Aristotle's, and clearing that up highlights at least two facets of natural language semantics: the idea of *existential import* and the distinction between meaning and presupposition or implicature on the one hand, and the notion of *negation* on the other.

**2.** A type $\langle 1,1\rangle$ (generalized) quantifier $Q$ associates with each universe $M$ a binary relation $Q_M$ between subsets of $M$, and similarly for other types. Many natural languages contain a wide range of simple or complex *determiner* expressions that can be seen to denote type $\langle 1,1\rangle$ quantifiers: `no`, `every`, `at least six`, `all but three`, `no ...except John`, `infinitely many`, `most`, `few`, `more than two thirds`, `Mary's`, `several students'`, etc. These have two characteristic properties, *conservativity* and *extension*, and a simple but fundamental fact is that the operation of *relativization* is an isomorphism from the class of type $\langle 1\rangle$ quantifiers to the class of CONSERV and EXT type $\langle 1,1\rangle$ quantifiers (where, for type $\langle 1\rangle$ $Q$, $(Q^{\mathrm{rel}})_M(A,B) \Leftrightarrow Q_A(A\cap B)$).

**3.** One area of fruitful interaction concerns *monotonicity.* The notion of a GQ being increasing or decreasing in a given argument is standard, but NL quantifiers also exhibit more subtle monotonicity properties, such as *smoothness*: the conjunction of

(1) $Q_M(A,B)\ \&\ A' \subseteq A\ \&\ A\cap B = A'\cap B \ \Rightarrow\ Q_M(A',B)$

(2) $Q_M(A,B)\ \&\ A \subseteq A' \subseteq M\ \&\ A - B = A' - B \ \Rightarrow\ Q_M(A',B)$

There is the linguistic task of investigating the distribution of such properties, and the logical task of their mathematical characterization. In fact several familiar monotonicity properties, and also other properties of type $\langle 1, 1\rangle$ quantifiers such as *symmetry*, can be seen as combinations of basic properties like (1) and (2). Most type $\langle 1, 1\rangle$ NL quantifiers which are MON↑ (increasing in the right argument) are in fact smooth (which is stronger), and it has been conjectured that *all* of them are. But the logical representation provides a few counterexamples, not otherwise easily discoverable, e.g., *at least three of the five or more* and *at least two of most students'* are MON↑ but not smooth. Monotonicity is furthermore instrumental in systematizing various other linguistic phenomena, such as the distribution of so-called *polarity items*.

**4.** Logical GQs are usually supposed to satisfy ISOM (isomorphism closure), but many determiner denotations don't. They are nevertheless built from higher-order ISOM operations; a typical case is the *possessive* determiners – `John's`, `several professors'`, `at least two of most students'` – which can be given with an operation *Poss* (the denotation of the genitive `'s`) taking two type $\langle 1, 1\rangle$ quantifiers, a set, and a binary relation as arguments; e.g., [[`at least two of most students'`]] $= Poss(most, student, at\ least\ two, R)$. *Poss* has interesting properties; e.g. the monotonicity behavior of $Poss(Q_1, C, Q_2, R)$, which is predictable from that of $Q_1$ and $Q_2$, is worth studying.

**5.** ISOM is one aspect of *logicality*. GQ theory applied to natural languages offers a useful testing ground for an analysis of the notion of a *logical constant*, where the logicality part may be separated from the *constancy* part.

**6.** A much debated issue among linguists is how NLs express existence, in particular which determiners are adequate in *existential-there* sentences; cf.

(3) There are many/at least two/no/an even number of children in the garden.

(4) *There are every/most/all but three/the seven children in the garden.

Barwise and Cooper gave an explanation of terms of *weak* vs. *strong* Dets, and Keenan one in terms of symmetry. Again a closer a look at the logical representation of GQs reveals some examples that don't fit these explanations, but, although not (so far) discussed by linguists, turn out to occur in natural languages.

**7.** Logical GQ theory has been quite successful in establishing facts about the expressive power of *logics with generalized quantifiers*, mostly with the methods introduced by Ehrenfeucht and Fraïssé. May one draw any conclusions for natural languages? The ideas of (compositional) *translation* and of *relative expressive power* can be formulated for NLs too. These notions are always relative to a *synonymy* relation: in logic it is logical equivalence (truth in the same models), bur for NLs other notions are relevant as well. One may still show that logical equivalence has a special place among these, and go on to specify circumstances under which definability and undefinability results for logical languages transfer to natural languages.

# Fuzzy logic as a logic of information with uncertain content

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Although a great variety of logical calculi dealing with statements of fuzzy nature has been defined in recent years and although many of these systems are well-developed and perfectly understood, we can hardly claim that there is similar clarity about the nature of the formalised statements themselves. Fuzzy logic is based on the idea that a proposition can take as a truth value not only 0 and 1, but also any real number in between. So the question is immediate what a specific truth value actually expresses; an answer to this question, however, is not part of the concept. It is certainly not in all cases adequate to ask about the meaning of a single truth value; the set $[0,1]$ is a set of grades, so what should count is the order relation between pairs of truth values. However, there is probably only one fuzzy logic which is based on $[0,1]$ as a bounded dense linear order – the Gödel logic. As soon as we wish to define more interesting connectives than just the infimum and the supremum, we will make use of structure on $[0,1]$ which originates from other concepts than the order. As a disjunction, for instance, we may take the truncated addition; as a conjunction, we may take the product. However, addition and multiplication are derived from our intuition about length or area, and thus somewhat unrelated to the idea of a continuous set of grades.

Now that calculi like Łukasiewicz or product logic are quite popular not only among mathematicians, but apparently also among practitioners, we may ask about a reasonable interpretation of fuzzy statements in retrospect. Few efforts in this directions have been made. Let us mention, as an example, that finitely valued Łukasiewicz logic can be interpreted by means of Ulam games [Mun], a variant of which even covers Basic Fuzzy Logic (BL) [CiMu]. Another approach is due to J. Paris [Par] and comes probably closest to what we have in mind here; accordingly, the truth value of a proposition is the proportion of the 'arguments' which are associated to it and which are 'in favour' of it; the product or the Gödel logic finds a rather natural interpretation in this way.

In our talk, we shall propose a framework of a still different kind. The aim is to provide alternative semantics for propositional Łukasiewicz and product logic. The idea is the following. Rather than dealing with propositions which are subject to an assignment of unsharp truth values, we start with a Boolean algebra, to be understood as a system of "usual", that is, sharp propositions. A fuzzy proposition is then a subset of this algebra (fulfilling certain conditions). This is a simple way to express uncertainty; a set containing more than one element is meant to express that there is no clarity which of the elements expresses the actual situation.

More specifically, let $(\mathcal{B}; \wedge, \vee, \neg, 0, 1)$ be the Boolean algebra freely generated by some countable set $\{\varphi_1, \varphi_2, \ldots\}$. So $\mathcal{B}$ is the Lindenbaum algebra of the classical propositional calculus, $\varphi_1, \ldots$ being its variables, and every $\psi \in \mathcal{B}$ expresses logical dependencies between them. Furthermore, let $G$ be a group of automorphisms of $\mathcal{B}$.

Let a fuzzy proposition be a non-empty and proper subset of $\mathcal{B}$ which (i) is an order-filter and (ii) invariant under every $g \in G$. On the set $\mathcal{P}$ of all fuzzy propositions, define

$$\begin{aligned} \alpha \odot \beta &= \{\varphi \wedge \psi\colon\ \varphi \in \alpha,\ \psi \in \beta,\ \varphi \wedge \psi > 0\}, \\ \alpha \Rightarrow \beta &= \{\xi\colon\ \xi = 1 \ \text{ or, for all } \varphi \in \alpha,\ \varphi \wedge \xi \in \beta\}, \\ \mathbf{0} &= \mathcal{B} \backslash \{0\}, \\ \mathbf{1} &= \{1\}, \end{aligned}$$

where $\alpha, \beta \in \mathcal{P}$.

Now, let $L(\mathcal{B}, G)$ be a propositional logic with connectives $\odot, \Rightarrow$ and constant 0. Define a proposition to be true if it attains $\mathbf{1}$ under all evaluations based on $\mathcal{P}$ and the operations above.

We do not know what $L(\mathcal{B}, G)$ is like if there are no assumptions on $G$. But consider the case that $G$ consist of all automorphisms $g$: $\mathcal{B} \to \mathcal{B}$ with finite support. Here, I say that $g$ has finite support if $\mathcal{B} = \mathcal{B}_g \times \mathcal{B}'_g$ such that $\mathcal{B}_g$ is finite and invariant under $g$ and $g$ is constant on $\mathcal{B}'_g$. Then the true propositions of $L(\mathcal{B}, G)$ are the tautologies of Łukasiewicz logic.

A more special version of the same formalism goes as follows. Again, let $\mathcal{B}$

be the Boolean algebra freely generated by countably many elements. Now, we endow $\mathcal{B}$ with a measure $\mu$: $\mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}$ which is strictly positive, $\sigma$-finite, and such that $\mu(\alpha) \leq \mu(\beta)$ implies that $\alpha' \leq \beta$ and $\mu(\alpha') = \mu(\alpha)$ for some $\alpha'$. A fuzzy proposition takes the form $\{\varphi\colon\ \mu(\neg\varphi) \leq m\}$, where $m \in \mathbb{R}^+$ is from the range of $\mu$.

Proceeding in a similar way as above, we arrive at Łukasiewicz logic if the measure is totally finite, and else, after discarding the constant 0, at the falsity-free version of product logic.

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# On weakly cancellative fuzzy logics

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Hájek defined in [6] the logic BL as a common fragment of the three main fuzzy logics: Łukasiewicz logic, Product logic and Gödel logic, semantically defined from a continuous t-norm (the Łukasiewicz t-norm, the product reals and the minimum, respectively). In particular, Product logic was proved to be the axiomatic extension of BL obtained by adding:

$\varphi \wedge \neg\varphi \to 0$ ($\Pi 1$),
and
$\neg\neg\chi \to ((\varphi * \chi \to \psi * \chi) \to (\varphi \to \psi))$ ($\Pi 2$),

where first one is the law of pseudocomplementation and the second one expresses the law of cancellativity.

Actually, Hájek conjectured that BL was complete with respect to the semantics given by continuous t-norms and their residua. This was proved by Cignoli, Esteva, Godo and Torrens in [4]. Also in [6] an algebraic semantics was given for BL-logic based on the variety of BL-algebras (bounded integral commutative prelinear divisible residuated lattices).

Nevertheless, the necessary and sufficient condition for a t-norm to have a residuated implication is not the continuity, but the left-continuity. For that reason, Esteva and Godo in [5] defined a weaker logic than BL, which they called MTL (for Monoidal T-norm based Logic) aiming to capture the logic of all left-continuous t-norms and their residua. Jenei and Montagna proved in [7] that MTL was indeed complete with respect to the semantics given by the class of all left-continuous t-norms and their residua, i.e. standard complete.

Esteva and Godo gave also and algebraic semantics for MTL based on MTL-algebras (bounded integral commutative prelinear residuated lattices). This class is a variety that contains the class of BL-algebras as a proper subvariety and it is possible to prove that in fact it is an equivalent algebraic semantics for MTL logic in the sense of Blok and Pigozzi [2]. Therefore, MTL is an algebraizable logic, i.e. it belongs to the class of logics which is better studied by Abstract Algebraic Logic and for which this discipline gives a lot of important results. In particular, the study of its axiomatic extensions is equivalent to the study of varieties of MTL-algebras, and there is a correspondence between logical and algebraic properties. The structure of BL-algebras is well-known and some important parts of their lattice of subvarieties have been completely described, but in the framework of MTL, i.e. when the property of divisibility is not assumed, few algebraic studies have been done till now.

The talk is devoted to the investigation of some varieties of MTL-algebras, or equivalently to some schematic extensions of MTL. We focus our attention on the so called weakly cancellative MTL-algebras (WCMTL-algebras for short) and on their logic, WCMTL. WCMTL-algebras are MTL-algebras in which the monoid operation is either cancellative or has 0 as a result. The interest of this variety and of its corresponding logic is motivated as follows:

- Both MV-algebras and Product algebras are weakly cancellative, hence WCMTL-algebras are obtained from the join of the varieties of MV-algebras and of Product algebras by removing divisibility. Moreover, it will turn out that ΠMTL-algebras are just WCMTL-algebras without zero divisors, and that MV-algebras are just the involutive WCMTL-algebras.
- While the structure of involutive MTL-algebras seems to be very hard to describe (every MTL-algebra generates an involutive one by disconnected rotation [8], so involutive MTL-algebras can contain the zero-free reduct of any MTL-algebra), the structure of WCMTL-algebras, although not easy, seems to be more accessible. Moreover some tech-

niques introduced by Horčík for the study of ΠMTL-algebras can be successfully applied to WCMTL-algebras.

- WCMTL-chains are either indecomposable as ordinal sums or are the ordinal sum of a two-element chain and a cancellative (hence indecomposable) residuated lattice. So they constitute an interesting example of indecomposable (or almost indecomposable) MTL-algebras. This also suggest the investigation of the variety $\Omega(\mathbb{WCMTL})$ generated by all ordinal sums of zero-free subreducts of WCMTL-algebras. Interestingly, the divisible $\Omega$(WCMTL)-algebras are precisely the BL-algebras.

We prove that, as in the case of BL-algebras (see [1]), all MTL-chains have a maximum decomposition as ordinal sum of indecomposable totally ordered semihoops. Then, we introduce weak cancellation to obtain a class of those indecomposable semihoops. Moreover, some interesting properties of weak cancellation are proved, obtaining a new axiomatization for the cancellative fuzzy logics (Product logic and ΠMTL) and defining a new hierarchy of fuzzy logics. We study some properties of those logics, in particular we concentrate in the task of deciding which of them enjoy standard completeness. We finish with some concluding remarks and open problems.

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# A logic for reasoning about fuzzy events

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## Abstract

A fuzzy logical treatment of probability has been widely studied in these last years. In particular, starting form a basic idea exposed by Hájek et al. in [5] and later refined in [4], simple (i.e. unconditional) and conditional probability of *crisp* events can be studied by using various kind of modal-fuzzy logic (see [1, 2, 3, 4, 6]). The very basic idea allowing a treatment of simple probability inside a fuzzy-logical setting consists in enlarging the language of Łukasiewicz logic by means of a unary (fuzzy) modality $P$ for *probably*, and defining a set of axioms $(FP)$ reflecting those of a probability measure. In such a logic (usually denoted by $FP(Ł)$) there are two kinds of formulas: classical Boolean formulas $\varphi, \psi, \ldots$ (which are definable in Ł) and modal formulas: for each Boolean formula $\varphi$, $P\varphi$ is a modal formula and, moreover, such a class of modal formulas is taken closed under the connectives of Łukasiewicz logic. In this setting the probability of an event $\varphi$ is interpreted as the truth value of the modal formula $P\varphi$ saying "$\varphi$ is probable".

In [4] Hájek also proposed a logic over Łukasiewicz predicate calculus Ł∀ allowing a treatment of (simple) probability of *fuzzy* events (a notion early defined by Zadeh and more recently considered and developed in the context of MV-algenbras by e.g. Mundici, Riečan et al., Navara and others). This can be done by assuming the logic of events be Łukasiewicz logic and not just Boolean logic. To model probability Hájek introduces in Ł∀ a generalized fuzzy quantifier standing for *most*. In our work however we want to remain at a propositional level, using the same approach as in the above $FP(Ł)$ logic, but considering fuzzy events instead of Boolean events. We use $FP(Ł, Ł)$ to denote such a logic. This notation, even if it differs form the original Hájek's notation, it allows us to point out both, the logic of events (the first argument) and the logic which is used in order to reason about modal-formulas $P\varphi$, with $\varphi$ being a Łukasiewicz formula (the second argument).

In his monograph Hájek proposed two different (Kripke-style) probabilistic semantics for this logic: a *weak* one and a *strong* one. These two kind of models are defined as follows:

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(a) A weak-probabilistic model for $FP(\text{Ł}, \text{Ł})$ is a system $\mathcal{M}_w = \langle W, e, I\rangle$ where: $W$ is a non-empty set of possible words, $e : W \times V \to [0, 1]$ (being $V$ the set of propositional variables on which fuzzy-events are built up over) is such that, for each $w \in W$, $e(w, \cdot) : V \to [0, 1]$ is an Ł-evaluation. Finally $I : Form(\text{Ł}) \to [0, 1]$ satisfies the following:

   (i) If $\text{Ł} \vdash \varphi \equiv \psi$, then $I(\varphi) = I(\psi)$,

   (ii) $I(\overline{1}) = 1$,

   (iii) $I(\neg\varphi) = 1 - I(\varphi)$,

   (iv) $I(\varphi \oplus \psi) = I(\varphi) + I(\psi) - I(\varphi \& \psi)$.

(b) A strong-probabilistic model for $FP(\text{Ł}, \text{Ł})$ is a system $\mathcal{M}_s = \langle W, e, \pi\rangle$ where: $W$ and $e$ are defined as in the case of a weak-probabilistic model, $\pi : W \to [0, 1]$ satisfies the following:

$$\sum_{w \in W} \pi(w) = 1.$$

Given a modal formula $P\varphi$ of $FP(\text{Ł}, \text{Ł})$, and a strong-probabilistic Kripke model $\mathcal{M}_s$ for $FP(\text{Ł}, \text{Ł})$, the truth value of $P\varphi$ in $\mathcal{M}_s$ is defined as $\|P\varphi\|_{\mathcal{M}_s} = \sum_{w \in W} e(w, \varphi) \cdot \pi(w)$.

Hajék shows his logic to be Pavelka-style complete w.r.t. weak-probabilistic models. So far we have not been able yet to prove (usual) completeness results for $FP(\text{Ł}, \text{Ł})$, either w.r.t. the classes of strong or weak probabilistic models. The main difficulty which arises (for any of the two classes of models) can be summarized as follows: Let $\Gamma \cup \{\Phi\}$ be a finite set of modal formulas of $FP(\text{Ł}, \text{Ł})$ such that $\Gamma \vdash_{FP(\text{Ł},\text{Ł})} \Phi$. The usual strategy which allows to prove completeness is based on translating proofs in $FP(\text{Ł}, \text{Ł})$ into proofs within a suitable theory in Łukasiewicz logic. This is done using the the following trick:

(a) Define a *translation* $^\star$ of modal formulas into plain Łukasiewicz logic formulas. This can be done by introducing, for each atomic modal formula $P\varphi$ a new propositional variable $p_\varphi$ and then requiring that the translation $^\star$ will commute with the Łukasiewicz connectives (for instance $(P\varphi \to P\psi)^\star = (P\varphi)^\star \to (P\psi)^\star = p_\varphi \to p_\psi$ and so forth).

(b) Translate all modal formulas in $\Gamma \cup \{\Phi\}$, denote them $\Gamma^*$ and $\Phi^*$ respectively, and second translate all the (instances of) axioms for probability $(FP)$ and add all formulas $p_\varphi$ for each Ł-tautology $\varphi$, leading to a countable set of Ł-formulas denoted by $(FP)^\star$.

Now it is not difficult to prove that $\Gamma \vdash_{FP(\text{Ł},\text{Ł})} \Phi$ iff $(FP)^\star \cup \Gamma^\star \vdash_{\text{Ł}} \Phi^\star$. Now, each Ł-evaluation which is model of $(FP)^\star$ defines a weak-probabilistic model. Unfortunately Łukasiewicz logic does not have strong standard completeness for arbitrary theories (only for finite theories), that is, it is possible to find a countable Ł-theory $T$ and an Ł-formula $\varphi$ such that $T \not\vdash_{\text{Ł}} \varphi$, but $T \models_{\text{Ł}} \varphi$ (see [4] for more details). This shows that the usual technique cannot be used in order to get the desired result.

In this work we are going to show a preliminary result in this direction. In particular we have proved that, for each $n \in \mathbb{N}$, the logic $FP(Ł_n, Ł)$ is complete w.r.t. the class of its strong-probabilistic models. The logic $FP(Ł_n, Ł)$ allows us to treat the probability of only those events which can be described by using the finite valued Łukasiewicz logic $Ł_n$.

Strong-probabilistic models $\mathcal{M}_s = \langle W, e, \pi \rangle$ for $FP(Ł_n, Ł)$ can be easily defined just by stipulating that the evaluation $e$ is an $Ł_n$-evaluation. In particular $e$ will give back as output rational values: $e : W \times V \to \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$.

Clearly $FP(Ł_n, Ł)$ is a weaker logic than $FP(Ł, Ł)$, but, on the other hand, restricting ourselves to fuzzy events as formulas of $Ł_n$ logic has the advantage that we can suitably reduce the "translated set" $(FP)^\star$ to a finite one $(FP)^\circ$ such that $(FP)^\star \cup \Gamma^\star \vdash_Ł \Phi^\star$ iff $(FP)^\circ \cup \Gamma^\star \vdash_Ł \Phi^\star$. Now, since Łukasiewicz logic is complete w.r.t. finite theories, we can conclude:

$$\Gamma \vdash_{FP(Ł_n,Ł)} \Phi \text{ iff } (FP)^\circ \cup \Gamma^\star \vdash_Ł \Phi^\star \text{ iff } (FP)^\circ \cup \Gamma^\star \models_Ł \Phi^\star.$$

Now the strong probabilistic completeness of $FP(Ł_n, Ł)$ follows by using a result proved by Paris in [7] which shows that each $Ł_n$-model of $(FP)^\circ$, which easily induces a weak-probabilistic model for $FP(Ł_n, Ł)$, induces in fact a strong-probabilistic Kripke model for $FP(Ł_n, Ł)$.

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# Allowed operations in the many-valued R-S memory circuit

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**Abstract**

The finitely many-valued R-S memory circuit presented in [8] is constructed by two logical gates implementing the standard fuzzy Sheffer operation i.e. $a \overline{\wedge}_S b = 1 - \min(a, b)$. There is a question why only this operation has been chosen when an abundance of fuzzy Sheffer operations as well as other fuzzy operations is available. In this paper we find the set of all fuzzy operations with which our R-S circuit works as it is supposed to. We prove that no other operations are allowed in order to have an R-S circuit with the correct behavior.

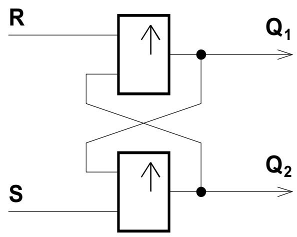


Figure 1: Scheme of an R-S memory circuit.

## 1 Introduction

A *memory circuit* is a logical circuit which is able to memorize a logical value. Two states of the circuit are defined:

- In the *open state* the value of the output is given by the input.
- In the *closed state* the value of the output is kept.

In the two-valued logic the problem of memory circuits is already well solved; for further details see e.g. [1, 2, 5]. In the many-valued logic there exist several approaches e.g. in [3, 4, 6, 7, 8, 9]. In our paper we focus on the many-valued R-S memory circuit described in [8].

The two-valued R-S memory circuit, as described e.g. in [1, 2, 5], is a logical circuit consisting of two gates implementing Sheffer (negation of conjuction) resp. Pierce operation (negation of disjunction) connected as shown in Figure 1. A generalization of this circuit to the many-valued logic is based on a generalization of Sheffer resp. Pierce operation. It can be defined as a fuzzy negation of a fuzzy conjuction resp. disjunction. Since we have an abundance of fuzzy negations, conjuctions and disjunctions, we have an abundance of fuzzy Sheffer and Pierce operations as well. There is a question which of them may be suitable for the many-valued R-S memory circuit. In this text we are going to answer this question and show which operations are possible and why.

## 2 Many-valued R-S memory circuit

Let us have a set of logical values as the real interval $[0,1]$. Let $\succcurlyeq$ be a total order of the interval $[0,1]$ and let $\bigstar$ be the greatest element defined by this order.

**Definition 2.1** *A* duality *is a unary operation* $\overline{\phantom{a}}\colon [0,1] \to [0,1]$ *that satisfies for every* $a \in [0,1]$*:*

$$\begin{array}{rcl} \overline{\overline{a}} & = & a \\ a \succcurlyeq b & \Rightarrow & \overline{b} \succcurlyeq \overline{a} \end{array}$$

Now we are going to define the behavior of the many-valued R-S memory circuit. The following properties are taken as a generalization of the two-valued R-S memory circuit.

**Definition 2.2** *A* many-valued R-S circuit*, an* R-S circuit *for short, is a logical circuit consisting of two gates implementing a fuzzy binary operation, two input ports $R$, $S$, and two output ports $Q_1$, $Q_2$, connected as shown in Figure 1.*

*We expect the following properties of an R-S circuit:*

1. Open state: *If $R = \overline{S}$ (and thus $\overline{R} = S$) then the dual values are passed to the output regardless of the previous values of the output; i.e. $Q_1 = \overline{R} = S$ and $Q_2 = \overline{S} = R$ (and thus $Q_1 = \overline{Q_2}$).*
2. Closed state: *If $R = S = \bigstar$ and $Q_1 = \overline{Q_2}$ then the values of the output are kept.*
3. *If $R \succcurlyeq \overline{S}$, $S \succcurlyeq \overline{R}$, and $Q_1 = \overline{Q_2}$ then the values of the output are kept.*

Note that the third property corresponds to the continuous transition from the open to the closed state; the remembered value must not be lost during this procedure.

Let us call the undefined binary operation in the R-S circuit an *R-S operation.* Its definition follows:

**Definition 2.3** *The R-S operation is a binary operation $\uparrow\colon [0,1] \times [0,1] \to [0,1]$ that satisfies for every $a, b, x \in [0,1]$:*

1. *If* $a = \overline{b}$ *then* $\uparrow(\uparrow(x, a), b) = \overline{b}$ *and* $\uparrow(\uparrow(x, b), a) = \overline{a}$.

   *This axiom correspods to the open state of the R-S circuit (Definition 2.2, Part 1). The variable* $x$ *stands for the previous value of the output,* $a$ *and* $b$ *for the values of the input. The axiom says that after one passing of the signal through the eight-shaped inner loop of the circuit the output will equal the input regardless of the previous value of the output.*

2. $\uparrow(a, \bigstar) = \overline{a}$

   *This axiom correspods to the closed state of the R-S circuit (Definition 2.2, Part 2) saying that if the input equals* $\bigstar$ *then the output value is kept.*

3. $a \succcurlyeq b \Rightarrow \uparrow(a, b) = \overline{b}$

   *This axiom correspods to the state of the R-S circuit described in Definition 2.2, Part 3 saying that if the input increases then the output value is kept.*

It can be proven that Property 3 is a stronger variant of both Property 1 and Property 2. Thus the only condition which operation $\uparrow$ must comply is:

$$a \succcurlyeq b \Rightarrow \uparrow(a, b) = \overline{b}$$

At this moment, we have an axiom which defines the R-S operation. If we specify the duality (Definition 2.1) and the total order, the operation $\uparrow$ is fully defined. Nevertheless both operations $\overline{\phantom{a}}$ and $\uparrow$ must be continuous in order to be physically implementable. This requirement of continuity restricts the total order $\succcurlyeq$ only to $\geq$ with the greatest element 1 resp. $\leq$ with the greatest element 0; the operation of duality becomes a fuzzy negation and the R-S operation becomes a fuzzy negation of the standard fuzzy conjunction (the minimum) resp. a fuzzy negation of the standard fuzzy disjunction (the maximum).

# 3 Conclusion

We have proven that the only suitable operations in the many-valued R-S memory circuit are the fuzzy negation of minimum $\uparrow(a, b) = \overline{a \wedge b}$ and the fuzzy negation of maximum $\uparrow(a, b) = \overline{a \vee b}$. The only possibility of minimum and maximum shows logical if we consider that they are the only continuous binary operations which equal either its first or second argument. This property is crucial since it is expected that the ouput of the circuit either equals the input or it is kept; this property also assures that the output value is not lost when changing from the open to the closed state.

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# Sets of uniqueness of left-continuous t-norms *

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The aim of this talk is twofold: First we give a geometrical characterization of commutative associative operations using the so-called rotation-invariance property [5] and the notion of quantic nuclei of Rosenthal [7]. Then we shall demonstrate how this geometrical understanding of associativity may be used to obtain new results in the following topic:

Many authors have focused on the identification of small subsets of the unit square which uniquely determine a continuous Archimedean t-norm. We briefly summarize these results. Then other subsets of the unit square are shown to admit the property that there exists a unique t-norm (either a nilpotent one or a strict one or a left-continuous one) provided that its values are given on that subset. The employed subsets are either vertical cuts of the graph of the t-norm $T$, that is, functions of the form $T(.,x)$, which can be considered as intersections of the graph of the t-norm with vertical planes, or horizontal cuts, that is, one-place functions of the form $f_c(x)$, which can be considered as limit lines of intersections of the graph of the t-norm with horizontal planes.

We shall introduce the notion of involutive elements of left-continuous t-norms. Then we prove Theorem 1, which says, roughly speaking, that two involutive elements ensure the existence of other involutive elements. As a by-product we obtain results concerning left-continuous t-norms (Theorem 2, Theorem 3) and continuous Archimedean t-norms (Theorem 4).

Let $T$ be a left-continuous t-norm. For any $c \in [0,1]$ define the mapping $f_c : [0,1] \to [0,1]$ by

$$f_c(x) = \max\{y \in [0,1] \mid T(x,y) \le c\}.$$

Let $T$ be a left-continuous t-norm. We call an element $c \in [0,1]$ *involutive* if the restriction of the mapping $f_c$ to $[c,1]$ is an involution of $[c,1]$.

**Theorem 1** *Let $T$ be a left-continuous t-norm. Assume that $c, e \in [0,1]$ are involutive. Then $a = f_e(f_c(e))$ is involutive as well, and for $x \in [0,1]$ we have*

$$f_a(x) = f_e(f_c(f_e(x))) \tag{1}$$

*Supported by the Bolyai Research Grant.

**Theorem 2** *Let $T$ be a left-continuous t-norm with the following property: There exists $\{c_n \mid n \in \mathbb{N}\} \subset ]0,1]$ such that $\lim_{n\to\infty} c_n = 0$ and*

$$f_{c_n}(x) = \min(1, 1 + c_n - x) \tag{2}$$

*holds for all $x \in [0,1]$. Then $T$ is the Łukasiewicz t-norm.*

**Remark** 1 The set $\{c_n \mid n \in \mathbb{N}\} \subset ]0,1]$ in Theorem 2 is minimal in the following sense: Of course, dropping out any subset from a convergent sequence such that the cardinality of the remaining sequence is still infinite results in a convergent sequence with the same limit, thus such a subset can always be left out from $\{c_n \mid n \in \mathbb{N}\}$. However, as shown by Example 1, antecedents of Theorem 2 can not be relaxed such that the set $\{c_n\}$ becomes finite. Moreover, not even a convergent sequence is sufficient if its limit differs from 0.

**Example** 1 Increasing bijections from $[0,1]$ to $[0,1]$ are called *automorphisms* of $[0,1]$. With any automorphism $\varphi$ and with any t-norm $T$ one can define $T_\varphi$, which is a t-norm and is called the *$\varphi$-transformation* of $T$, as follows:

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$$

Let $f$ and $g$ be two automorphisms of $[0,1]$, as depicted respectively in Figure 1. It is easy to verify that the $f$-, and $g$-transformations of the Łukasiewicz t-norm have elements, such that the corresponding level sets satisfy (2). For a visualization see Figure 2.

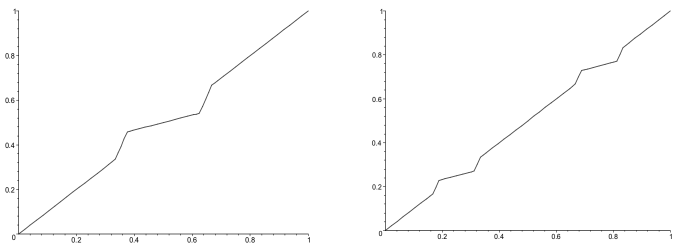


Figure 1: Two automorphisms of $[0,1]$

**Example** 2 The condition in (2) can not be relaxed by simply saying that $f_{c_n}(x)$ is involutive. A counterexample is the rotation ([?]) of the product t-norm given as follows: Let $T$ be the linear transformation of the product t-norm into $[\frac{1}{2}, 1]$, that is, $T(x,y) = \frac{(2x-1)(2y-1)+1}{2}$ and let

$$(T_{\mathbf{P}})_{\mathbf{rot}}(x,y) = \begin{cases} T(x,y) & \text{if } x,y \in ]\frac{1}{2},1] \\ 1 - \max\{t \in [\frac{1}{2},1] \mid T(x,t) \le 1-y\} & \text{if } x \in ]\frac{1}{2},1] \text{ and } y \in [0,\frac{1}{2}] \\ 1 - \max\{t \in [\frac{1}{2},1] \mid T(y,t) \le 1-x\} & \text{if } x \in [0,\frac{1}{2}] \text{ and } y \in ]\frac{1}{2},1] \\ 0 & \text{if } x,y \in [0,\frac{1}{2}] \end{cases}.$$

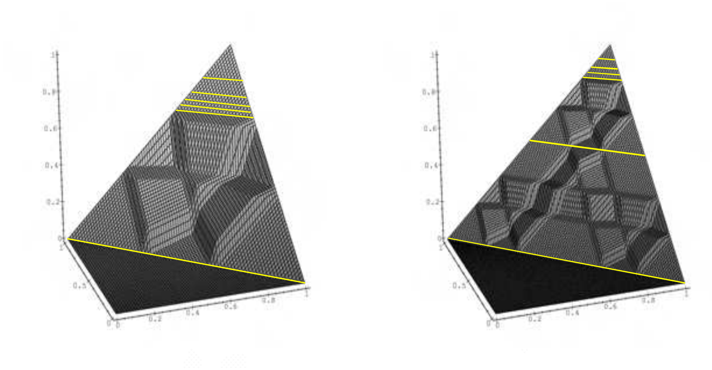

Figure 2: $T_f$ and $T_g$, see Example 1

Then each element in $[0, \frac{1}{2}[$ in involutive. The rotation of the product t-norm (depicted in Figure 3) has exactly one point of discontinuity; hence it is not isomorphic to the Łukasiewicz t-norm.

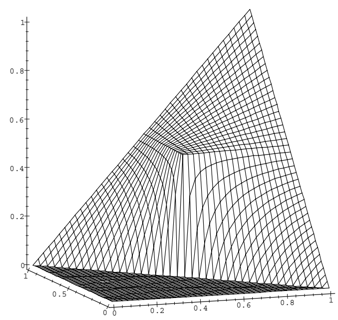

Figure 3: Rotation of the product t-norm

**Theorem 3** *Let $T$ be a left-continuous t-norm with the following property: There exists $\{c_n \mid n \in \mathbb{N}\} \subset ]0,1]$ such that $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1$ and*

$$f_{c_n}(x) = \min\left(1, \frac{c_n}{x}\right) \tag{3}$$

*holds for all $x \in [0,1]$. Then $T$ is the product t-norm.*

**Theorem 4** *Let $A = \{c_n \mid n \in \mathbb{N}\} \subset ]0,1]$ such that $\lim_{n\to\infty} c_n = 0$. Any nilpotent t-norm is determined by its $c_n$-level sets. If in addition we have $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1$ then any strict t-norm is determined by its $c_n$-level sets.*

**Remark** 2 Since the counterexamples of Examples 1, 2 are based on continuous Archimedean t-norms we obtain the minimality of the set $\{c_n \mid n \in \mathbb{N}\}$ in Theorem 4.

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# Multilattices as a Basis for Generalized Fuzzy Logic Programming

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**Extended abstract.** The possibility of weakening the structure of the underlying set of truth-values for logic programming has been extensively studied in the recent years, and there are approaches which are based either on the structure of lattice (residuated lattice [4, 13] or multi-adjoint lattice [9]), or on more restrictive structures, such as bilattices or trilattices [7], or on more general structures such as algebraic domains [11]. One can also find some attempts aiming at weakening the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" is relaxed to the "existence of *minimal* upper bounds and *maximal* lower bounds". In this direction, Benado [1] and Hansen [5] proposed definitions of a structure so-called multilattice.

Recently an alternative notion of multi-lattice was introduced [2, 8] as a theoretical tool to deal with some problems in the theory of mechanized deduction in temporal logics. This kind of structure also arises in the research area concerning fuzzy extensions of logic programming, in a natural manner. For instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [3] can be weakened only when the underlying set of truth-values is a multilattice (as far as we know, the question of providing a counter-example on a lattice remains open). Our aim in this work is to study the computational capabilities of this new structure in the framework of extended logic programming and, specifically, in relation to its fixed point semantics.

Recall that a lattice is a poset such that the set of upper (lower) bounds has a unique minimal (maximal) element, that is, a *minimum* (*maximum*). In a multilattice, this property is relaxed in the sense that minimal elements for the set of upper bounds should exist, but the uniqueness condition is dropped.

**Definition 1.** *A* complete multilattice *is a partially ordered set, $\langle M, \leq\rangle$, such that for every subset $X \subseteq M$, the set of upper (lower) bounds of $X$ has minimal (maximal) elements, which are called* multi-suprema *(*multi-infima*).*

It is remarkable that, under suitable conditions, the set of fixed points of a mapping from $M$ to $M$ does have a minimum and a maximum.

**Theorem 1.** *Let $f\colon M \longrightarrow M$ be an isotone and inflationary mapping on a multilattice, then its set of fixed points is nonempty and has a minimum element.*

Regarding computational properties of multilattices, it is interesting to impose certain conditions on the sets of upper (lower) bounds of a given set $X$. Specifically, we would like to ensure that any upper (lower) bound is greater (less) than a minimal (maximal); this condition enables to work on the set multi-suprema (multi-infima) as a set of "generators" of the bounds of $X$. This leads to *consistent* multilattices, for which the following result can be shown, stating the existence of some suprema and infima.

**Lemma 1.** *Let $M$ be a consistent multilattice without infinite antichains, then any chain in $M$ has a supremum and an infimum.*

All the hypotheses are necessary for the existence of supremum and infimum of chains; in particular, the condition on infinite antichains cannot be dropped.

We provide now a first approximation of the definition of an extended logic programming paradigm in which the underlying set of truth-values is assumed to have structure of multilattice. The proposed schema is an extension of the monotonic logic programs of [4].

The definition of logic program is given, as usual, as a set of rules and facts.

**Definition 2.** *An* extended logic program *is a set* $\mathbb{P}$ *of rules of the form* $A \leftarrow \mathcal{B}$ *such that* $A$ *is a propositional symbol of* $\Pi$*, and* $\mathcal{B}$ *is a formula of* $\mathfrak{F}$ *built from propositional symbols and elements of* $M$ *by using monotone operators.*

*An* interpretation *is an assignment of truth-values to every propositional symbol in the language.*

*A rule of an extended logic program is* satisfied *whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body.*

Every extended program $\mathbb{P}$ has the top interpretation $\triangledown$ as a model; regarding minimal models, it is possible to prove the following lemma.

**Lemma 2.**

1. *A chain of models* $\{I_k\}_{k \in K}$ *of* $\mathbb{P}$ *has an infimum in* $\mathcal{I}$ *which is a model of* $\mathbb{P}$*.*
2. *Given an extended logic program* $\mathbb{P}$*, there exist minimal models for* $\mathbb{P}$*.*

**Definition 3.** *Given an extended logic program* $\mathbb{P}$*, an interpretation* $I$ *and a propositional symbol* $A$*; we can define* $T_{\mathbb{P}}(I)(A)$ *as*

$$\operatorname{multisup}\Big(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\Big)$$

Some properties of this definition of the $T_{\mathbb{P}}$ operator are stated below, where $\sqsubseteq_S$ denotes the Smyth-ordering between subsets of a poset:

**Lemma 3.** *If* $I \sqsubseteq J$*, then* $T_{\mathbb{P}}(I)(A) \sqsubseteq_S T_{\mathbb{P}}(J)(A)$ *for all propositional symbol* $A$*.*

The definition of $T_{\mathbb{P}}$ proposed above generates some coherence problems, in that the resulting 'value' is not an element, but a subset of the multilattice. A possible solution to this problem would be to consider a *choice function* $()^*$ which, given an interpretation, for any propositional symbol $A$ selects an element in $T_{\mathbb{P}}(I)(A)$; this way, $T_{\mathbb{P}}(I)^*$ represents actually an interpretation which, by definition, is an inflationary operator. Note that, however, that for some choice functions, the resulting operator ${T_{\mathbb{P}}}^*$ might not be monotone in the set of interpretations, since it can lead to incomparable interpretations.

We are interested in computing models of our extended programs by successive iteration of ${T_{\mathbb{P}}}^*$. Therefore, we should characterize the models of $\mathbb{P}$ in terms $T_{\mathbb{P}}$. The following result, which characterizes the models of our extended programs in terms of properties of $T_{\mathbb{P}}$, can be proved:

**Lemma 4.** *The four statements below are equivalent:*

1. $I$ *is a model of* $\mathbb{P}$*.*
2. $T_{\mathbb{P}}(I)(A) = \{I(A)\}$ *for all* $A \in \Pi$*.*
3. $T_{\mathbb{P}}(I)^* = I$ *for all choice function.*
4. $I \in T_{\mathbb{P}}(I)$*, (abusing notation this means that* $I(A) \in T_{\mathbb{P}}(I)(A)$ *for all* $A \in \Pi$*).*

Note that item 4 above states that an interpretation $I$ is a model of $\mathbb{P}$ if and only if it is a fixed point of $T_{\mathbb{P}}$, viewed as a non-deterministic operator.

Regarding the iterated application of the $T_{\mathbb{P}}$ operator, the use of choice functions is essential. Let us consider a model $I$, that is, a fixed point of $T_{\mathbb{P}}$, then for all propositional variable $A$, we have that $T_{\mathbb{P}}(I)(A) = \{I(A)\}$. Lemma 3 guides us in the choice after each application of $T_{\mathbb{P}}$ as follows:

- For the base case, we have[1] $\vartriangle \sqsubseteq I$, then $T_{\mathbb{P}}(\vartriangle)(A) \sqsubseteq_S T_{\mathbb{P}}(I)(A) = \{I(A)\}$. This means that there exists an element $m_1(A) \in T_{\mathbb{P}}(\vartriangle)(A)$ such that

$$m_1(A) \leq I(A)$$

  This way we obtain an interpretation $m_1$ satisfying $m_1 \sqsubseteq I$ such that for any propositional variable $A$, $m_1(A)$ is an element of $T_{\mathbb{P}}(\vartriangle)(A)$.

[1] Here, as usual, $\vartriangle$ denotes the minimum interpretation.

– This argument applies also to any successor ordinal: given $m_k \sqsubseteq I$, there exists an element $m_{k+1}(A) \in T_{\mathbb{P}}(m_k)(A)$ such that

$$m_k(A) \leq m_{k+1}(A) \leq I(A)$$

where the first inequality holds by the definition of $T_{\mathbb{P}}$ and the second inequality follows from Lemma 3.
– For a limit ordinal $\alpha$, Lemma 1 states that for all $A$ the increasing sequence $\{m_n(A)\}$ has a supremum, which is considered, by definition, to be $m_\alpha(A)$.

As a result of the discussion above we obtain that we can choose suitable elements in the sets generated by the application of $T_{\mathbb{P}}$ in such a way that we can construct a transfinite sequence of interpretations $m_k$ satisfying

$$m_1 \sqsubseteq m_2 \sqsubseteq \ldots \sqsubseteq m_k \sqsubseteq \ldots \sqsubseteq I$$

Note that the sequence of interpretations above, can be interpreted as the Kleene sequence which allows to reach the minimal fixed point of $T_{\mathbb{P}}$ in the classical case.

Interestingly enough, if $I$ is a minimal model of $\mathbb{P}$, the previous sequence of interpretations can be proved to converge to $I$.

**Theorem 2.** *Let $I$ be a minimal model of $\mathbb{P}$, then the previous construction leads to a Kleene sequence $\{m_\lambda\}$ which converges to $I$.*

**Conclusions and future work** A fixed point semantics has been presented for multilattice-based logic programming, together with some initial and encouraging results: in particular, we have proved the existence of minimal models for any extended program and that any minimal model can be attained by some Kleene-like sequence.

However, a number of theoretical problems have to be investigated in the future: such as the constructive nature of minimal models (is it possible to construct suitable choice functions which generate convergent sequence of interpretations with limit a minimal model?). Possible answers should on a general theory of fixed points, relying on some of the ideas related to fixed points in partially ordered sets [10] or, perhaps, in fuzzy extensions of Tarski's theorem [12].

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# Fragments of First Order Gödel Logics

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## Abstract

Axiomatizability for first order Gödel logics has been completely characterized using topological properties of the underlying truth value set. In the present talk we will discuss three fragments of these logics, and characterize their axiomatizability. The fragments are

- Prenex fragment
- $\perp$-free fragment
- $\exists$-fragment

For all three fragments we will show that it is axiomatizable if and only if the underlying truth value set is either finite or uncountable.

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# Proof Theory for (Fragments of) First Order Łukasiewicz Logic

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Although the first-order Łukasiewicz logic $\forall$Ł based on the real unit interval $[0, 1]$ is famously not recursively axiomatizable, there exist both axiomatizations with infinitary rules, and interesting recursively axiomatizable fragments. In this work we begin a proof-theoretic investigation of $\forall$Ł, focussing on fragments with a natural syntactic characterization. Our starting point is the hypersequent calculus GŁ for propositional Łukasiewicz logic based on a language with connective $\rightarrow$ and constant $\bot$: [1]

Initial Sequents

$$\frac{}{A \Rightarrow A}\ (ID) \qquad \frac{}{\Rightarrow}\ (\Lambda) \qquad \frac{}{\bot \Rightarrow A}\ (\bot)$$

Structural rules

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ (EW) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (EC) \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, A \Rightarrow \Delta}\ (WL)$$

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (S) \qquad \frac{G \mid \Gamma_1 \Rightarrow \Delta_1 \quad G \mid \Gamma_2 \Rightarrow \Delta_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (M)$$

Logical Rules

$$\frac{G \mid \Gamma, B \Rightarrow A, \Delta}{G \mid \Gamma, A \rightarrow B \Rightarrow \Delta}\ (\rightarrow, l) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \quad G \mid \Gamma, A \Rightarrow B, \Delta}{G \mid \Gamma \Rightarrow A \rightarrow B, \Delta}\ (\rightarrow, r)$$

Cut

$$\frac{G \mid \Gamma_1, A \Rightarrow \Delta_1 \quad G \mid \Gamma_2 \Rightarrow A, \Delta_2}{G \mid \Gamma, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (CUT)$$

GŁ enjoys both cut-elimination and the subformula property; hence a natural first step in investigating first-order fragments of $\forall$Ł is simply to extend GŁ with the "usual" quantifier rules (i.e. the hypersequent versions of Gentzen's quantifier rules for **LK**):

$$\frac{G \mid \Gamma, A(t) \Rightarrow \Delta}{G \mid \Gamma, \forall x A(x) \Rightarrow \Delta}\ (\forall, l) \qquad \frac{G \mid \Gamma \Rightarrow A(a), \Delta}{G \mid \Gamma \Rightarrow \forall x A(x), \Delta}\ (\forall, r)$$

$a$ new

This calculus is sound and complete for the fragment of $\forall$Ł axiomatized by Hájek, but, as we show with a suitable counter-example, fails to admit cut-elimination. On the other hand, defining an infinitary rule a cut-free calculus is obtained that (using a version of Herbrand's theorem) is shown to be sound and complete for the whole of $\forall$Ł.

[1] Hypersequents are a natural generalization of sequents consisting of a multiset (intuitively, a disjunction) of sequents written $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$

# Theorem proving in fuzzy logics

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## 1 Motivation

Computer algebra allows to perform many operations which were considered difficult, e.g., factorization, integration, symbolic solution of ODEs, etc. Logical operations are not always implemented. E.g., Maple 9 has a package `logic` which was missing in several preceding versions. Except for packages for fuzzy control, there seems to be no professional software for fuzzy logical tasks. Here we summarize current situation in computer algebra support of testing tautologies in fuzzy logics.

## 2 Different approaches

Syntactical theorem proving was implemented in [9]. For a given set of axioms and deduction rules, it generates provable formulas. This extensive approach is necessarily ineffective and allows usually logical proofs of length up to 10. However, it succeeded to prove the dependence of two axioms of basic logic used in [7]. This approach is that it can only prove theorems, not recognize formulas which are not provable.

There is an easy procedure going in the opposite direction: Try random evaluations of the expression. If many trials did not found a counterexample, it is highly probable that the formula is a tautology. However, this tool does never guarantee a positive answer. It is implemented as an option in [5].

In the sequel, we deal with methods which allow to *decide* whether a formula in a fuzzy logic is a tautology or not. Testing of tautologies in the Łukasiewicz and other logics can be translated to a task of mixed integer programming (see e.g. [7, 6, 8]). This approach seems to be promising, but no practical implementation is known at the moment.

# 3 Bounds for testing of tautologies in Łukasiewicz logic

In Łukasiewicz logic, the evaluation is performed in an MV-algebra. Due to Chang's completeness theorem, a formula is a tautology of Łukasiewicz logic iff it is evaluated to 1 in the standard MV-algebra, i.e., the real unit interval $[0, 1]$ with the Łukasiewicz operations. Moreover, it is sufficient to consider finite MV-algebras (MV-chains) of the form $L_m = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\}$ for all $m \in \mathbb{N}$. Mundici [10] proved that, for a given formula $\phi$, it is sufficient to restrict attention to the cases $m \leq B(\phi)$, where $B(\phi) \in \mathbb{N}$ is a finite bound dependent on formula $\phi$. This opens a possibility to decide in a finite time whether $\phi$ is a tautology of Łukasiewicz logic. However, this has been possible only theoretically until the bound was improved by Aguzzoli and others in [1, 2, 3]. We use the following notation:

- $\phi$ is the formula to be tested,
- $M = \#\phi$ is the number of variables in formula $\phi$, including multiple occurrences,
- $n$ is the number of different variables in formula $\phi$.

**Theorem 1** *[3] Formula $\phi$ is a tautology of Lukasiewicz logic iff it is a tautology in the MV-algebra $L_{2^{M-1}}$ (with $2^{M-1} + 1$ truth values).*

The complexity of a complete test is $\left(2^{M-1} + 1\right)^n$ in the worst case (when $\phi$ is a tautology).

**Theorem 2** *[3, p. 367] Formula $\phi$ is a tautology of Łukasiewicz logic iff it is a tautology in the MV-algebras $L_m$ for all $m \leq b(M, n) = \left\lfloor \left(\frac{M}{n}\right)^n \right\rfloor$.*

Complexity arguments show that the second theorem leads to more efficient programs (although it has one additional cycle). This method has been implemented by Brůžková [5]. The complexity is $\sum_{m=1}^{b(M,n)} (m + 1)^n$ in the worst case.

# 4 Other fuzzy logics

Analogous principle is known to be applicable in Gödel logic:

**Theorem 3** *[4] Formula $\phi$ is a tautology of Gödel logic iff it is a tautology in the algebra with truth values $\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\}$ and the Gödel operations, where $m = n + 1$.*

Also this principle is implemented in [5].

Several papers deal with the possibility of testing of tautologies in the product logic. There is no non-Boolean finite-valued product logic, but the test can be reduced to a task analogous to that in Łukasiewicz logic. Again, finite complexity is not enough, it is necessary to find a small bound in order to make the program applicable. This still requires more attention.

There are also papers describing how the testing of tautologies can be performed in fuzzy logics based on ordinal sums of the above t-norms, even in the basic logic [8]. So far, their complexity seems far beyond the possibilities of current technology. However, drastic simplification might be possible and allow a reasonable implementation. This is subject to future study.

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# Convergence of sequences of sets with respect to lattice-valued possibilistic measures (extended abstract)

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Let the pair $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ denote a *complete lattice*, so that $T$ is a nonempty set and $\leq_{\mathcal{T}}$ is a partial ordering (relation) on $T$ such that, for every $A \subset T$, the supremum $\bigvee_{\mathcal{T}} A$ and the infimum $\bigwedge_{\mathcal{T}}$ w.r.to $\leq_{\mathcal{T}}$ are defined. By convention, $\bigvee_{\mathcal{T}} \emptyset = \oslash_{\mathcal{T}} (= \bigwedge_{\mathcal{T}} T)$ and $\bigwedge_{\mathcal{T}} \emptyset = \mathbf{1}_{\mathcal{T}} (= \bigvee_{\mathcal{T}} T)$ for the empty subset of $T$. The index $\mathcal{T}$ is omitted, if no misunderstanding menaces. *$\mathcal{T}$-(valued) possibilistic space* is a triple $\langle \Omega, \mathcal{A}, \Pi \rangle$, where $\Omega$ is a nonempty set, $\mathcal{A}$ is a system of subsets of $\Omega$ such that $\emptyset, \Omega \in \mathcal{A}$, and $\Pi$ is a *$\mathcal{T}$-(valued normalized) possibilistic measure on $\mathcal{A}$*, hence, $\Pi$ takes $\mathcal{A}$ into $T, \Pi(\emptyset) = \oslash_{\mathcal{T}}, \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$, and $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for each $A, B, A \cup B \in \mathcal{A}$. $\Pi$ is *complete*, if $\Pi(\bigcup \mathcal{A}_0) = \bigvee \{\Pi(A) : A \in \mathcal{A}_0\}$ for every $\mathcal{A}_0 \subset \mathcal{A}$ such that $\bigcup \mathcal{A}_0 = \bigcup_{A \in \mathcal{A}_0} A$ is in $\mathcal{A}$. Given a $\mathcal{T}$-possibilistic space over a field $\mathcal{A} \subset \mathcal{P}(\Omega)$, the $\Pi$-metric $\rho$ on $\mathcal{A}$ is defined by $\rho(A, B) = \Pi(A \div B)((= \Pi((A - B) \cup (B - A)))$ for every $A, B \in \mathcal{A}$. For each $A, B, C \in \mathcal{A}$ the relations (i) $\rho(A, A) = \oslash_{\mathcal{T}}$, (ii) $\rho(A, B) = \rho(B, A)$, and (iii) $\rho(A, C) \leq \rho(A, B) \vee \rho(B, C)$ are valid.

**Definition 1** Let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space over a $\sigma$-field $\mathcal{A}$. A sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ *converges (tends) to* $A_0 \in \mathcal{A}$ *in the* $\Pi$-*metric* $\rho(\{A_n\}_{n=1}^{\infty} \rightarrow A_0(\Pi)$, in symbols), if $\{\rho(A_n, A_0)\}_{n=1}^{\infty} \searrow \oslash_{\mathcal{T}}$ holds, i.e., if there exists a sequence $\{s_n\}_{n=1}^{\infty}$ of elements of $T$ such that, for each $n = 1, 2, \ldots, s_n \geq s_{n+1}$ and $s_n \geq \rho(A_n, A_0)$ is valid and $\bigwedge_{n=1}^{\infty} s_n = \oslash_{\mathcal{T}}$.

As a matter of fact, neither $\{A_n\}_{n=1}^{\infty} \rightarrow \bigcap_{n=1}^{\infty} A_n(\Pi)$ holds in general for any nonincreasing nested sequence $\{A_n\}_{n=1}^{\infty}$, nor $\{A_n\}_{n=1}^{\infty} \rightarrow \bigcup_{n=1}^{\infty} A_n(\Pi)$ holds in general for any nondecreasing nested sequence $\{A_n\}_{n=1}^{\infty}$. If both these relations are valid, the $\Pi$-metric $\rho$ is called *regular*. As a matter of fact, both these conditions, which can be called *regularity from above* and *regularity from below* are equivalent.

Under the notations of Definition 1, if $\{A_n\} \rightarrow A_0(\Pi)$ holds and $A_* \in \mathcal{A}$ is such that $\Pi(A_0 \div A_*)(= \rho(A_0, A_*)) = \oslash_{\mathcal{T}}$, then $\{A_n\} \rightarrow A_*(\Pi)$ holds as well.

**Definition 2** Let $\langle\Omega,\mathcal{A},\Pi\rangle$ be as in Definition 1. $\Pi$ is *continuous from above,* if for each nonincreasing sequence $\{A_n\}_{n=1}^{\infty}\subset\mathcal{A}$ the relation $\bigwedge_{n=1}^{\infty}\Pi(A_n)=\Pi(\bigcap_{n=1}^{\infty}A_n)$ holds. $\Pi$ is *continuous from below,* if for each nondecreasing sequence $\{A_n\}_{n=1}^{\infty}\subset\mathcal{A}$ the relation $\bigvee_{n=1}^{\infty}\Pi(A_n)=\Pi(\bigcup_{n=1}^{\infty}A_n)$ holds.

**Fact 1** Let $\mathcal{T}=\langle T,\leq_{\mathcal{T}}\rangle$ be a complete lattice such that $\bigwedge_{s\in S}(s\vee t)=t$ for every $t\in T$ and every $S\subset T$ such that $\bigwedge S=\oslash_{\mathcal{T}}$, let $\langle\Omega,\mathcal{A},\Pi\rangle$ be as in Definition 1. Then (i) $\Pi$ is continuous from above iff the $\Pi$-metric $\rho$ is regular, and (ii) if the $\Pi$-metric $\rho$ is regular, then $\Pi$ is continuous from below. The implication inverse to (ii) does not hold in general.

**Fact 2** Let $\mathcal{T}=\langle T,\leq_{\mathcal{T}}\rangle$ be a complete lattice satisfying this property: for every countable subsets $T_1,T_2\subset T$ such that $\bigwedge T_1=\bigwedge T_2=\oslash_{\mathcal{T}}$ and for every $t\in T$ the identities $\bigwedge\{t_1\vee t_2: t_1\in T_1,t_2\in T_2\}=\oslash_{\mathcal{T}}(=(\bigwedge T_1)\vee(\bigwedge T_2))$ and $\bigwedge\{t_1\vee t: t_1\in T_1\}=t(=(\bigwedge T_1)\vee t)$ are valid. Let $\langle\Omega,\mathcal{A},\Pi\rangle$ be as in Definition 1, let the $\mathcal{T}$-possibilistic measure $\Pi$ be continuous from above on $\mathcal{A}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets from $\mathcal{A}$ (not necessarily nested) such that $\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}A_m=\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}A_m$ holds, let $A_0$ denote this set. Then $\{A_n\}_{n=1}^{\infty}\to A_0(\Pi)$ holds.

If $\Pi_1,\Pi_2$ are $\mathcal{T}$-possibilistic measures on $\mathcal{A}$ such that $\Pi_1(A)\leq\Pi_2(A)$ holds for each $A\in\mathcal{A}$, then for each $\{A_n\}_{n=1}^{\infty}\subset\mathcal{A}$ such that $\{A_n\}_{n=1}^{\infty}\to A_0(\Pi_2)$ holds for some $A_0\in\mathcal{A}$, the relation $\{A_n\}_{n=1}^{\infty}\to A_0(\Pi_1)$ holds as well. Consequently, if $\{A_n\}_{n=1}^{\infty}$ is such that, for some $n_0, A_n=A_0$ for every $n\geq n_0$ (and only in this case), $\{A_n\}_{n=1}^{\infty}\to A_0(\Pi)$ holds for every $\mathcal{T}$-possibilistic measure $\Pi$ on $\mathcal{A}$.

**Fact 3** Let $\langle\Omega,\mathcal{A},\Pi\rangle$ be a $\mathcal{T}$-possibilistic space over a $\sigma$-field $\mathcal{A}\subset\mathcal{P}(\Omega)$, let $\{A_n\}_{n=1}^{\infty}$ be a nested nonincreasing (nondecreasing, resp.) sequence of sets which tends to $A_0=\bigcap_{n=1}^{\infty}A_n$ (to $A_0=\bigcup_{n=1}^{\infty}A_n$, resp.) w.r.to the $\Pi$-metric $\rho$. Then the sequence $\{\Omega-A_n\}_{n=1}^{\infty}$ tends to $\bigcup_{n=1}^{\infty}(\Omega-A_n)$ (to $\bigcap_{n=1}^{\infty}(\Omega-A_n)$, resp.) w.r.to the same $\Pi$-metric $\rho$.

**Fact 4** Let $\mathcal{T}=\langle T,\leq_{\mathcal{T}}\rangle$ be a complete lattice satisfying the conditions imposed on $\mathcal{T}$ in Fact 2, let $\langle\Omega,\mathcal{A},\Pi\rangle$ be a $\mathcal{T}$-possibilistic space over a $\sigma$-field $\mathcal{A}\subset\mathcal{P}(\Omega)$.

(i) Let $\{A_n\}_{n=1}^{\infty},\{B_n\}_{n=1}^{\infty}$ be nonincreasing sequences of sets from $\mathcal{A}$, let $A_0,B_0\in\mathcal{A}$ be such that $\bigcap_{n=1}^{\infty}A_n\supset A_0,\bigcap_{n=1}^{\infty}B_n\supset B_0,\{A_n\}_{n=1}^{\infty}\to A_0(\Pi)$ and $\{B_n\}_{n=1}^{\infty}\to B_0(\Pi)$ hold. Then $\{A_n\cup B_n\}_{n=1}^{\infty}\to(A_0\cup B_0)(\Pi)$ also holds.

(ii) Let $\{A_n\}_{n=1}^{\infty},\{B_n\}_{n=1}^{\infty}$ be nondecreasing sequences of sets from $\mathcal{A}$, let $A_0,B_0\in\mathcal{A}$ be such that $\bigcup_{n=1}^{\infty}A_n\subset A_0$, $\bigcup_{n=1}^{\infty}B_n\subset B_0$, $\{A_n\}_{n=1}^{\infty}\to A_0(\Pi)$ and $\{B_n\}_{n=1}^{\infty}\to B_0(\Pi)$ hold. Then $\{A_n\cup B_n\}_{n=1}^{\infty}\to(A_0\cup B_0)(\Pi)$ also holds.

**Definition 3** Let $\mathcal{T}=\langle T,\leq_{\mathcal{T}}\rangle$ and $\mathcal{S}=\langle S,\leq_{\mathcal{S}}\rangle$ be complete lattices, let $\oslash_{\mathcal{S}}^{C}=\mathbf{1}_{\mathcal{S}}$ and $\mathbf{1}_{\mathcal{S}}^{C}=\oslash_{\mathcal{S}}$, let $\langle\Omega,\mathcal{A},\Pi\rangle$ be a $\mathcal{T}$-possibilistic space over a $\sigma$-field $\mathcal{A}\subset\mathcal{P}(\Omega)$, let $\Delta:S\times S\to S$ be defined in the same way as $\rho$ in the case of complete lattice $\mathcal{T}$. let $f_n:\Omega\to S, n=0,1,2,\dots$ be mappings such that, for each $s\in S$ and each $n=1,2,\dots$, the sets $\{\omega\in\Omega:\Delta(f_n(\omega),f_0(\omega))>s\}$ and $\{\omega\in\Omega:\Delta(f_n(\omega),f_0(\omega)\geq s\}$ are in $\mathcal{A}$. The sequence $\{f_n\}_{n=1}^{\infty}$ of mappings *converges (tends) to* $f_0$ *w.r.to the* $\mathcal{T}$*-possibilistic measure* $\Pi$, if $\Pi(\{\omega \in \Omega : \Delta(f_n(\omega), f_0(\omega)) > \oslash_{\mathcal{S}}\}) \searrow \oslash_{\mathcal{T}}$ holds.

For each $A \subset \Omega$, its $\mathcal{S}$*-(valued) characteristic function (identifier)* $\chi_A : \Omega \to S$ is defined by $\chi_A(\omega) = \mathbf{1}_{\mathcal{S}}$, if $\omega \in A$, $\chi_A(\omega) = \oslash_{\mathcal{T}}$ otherwise. As could be expected, the convergence of a sequence of sets w.r.to the $\Pi$-metric $\rho$ is equivalent to the convergence of the corresponding $\mathcal{S}$-characteristic functions w.r.to $\Pi$.

**Fact 5** Let the notations and conditions introduced in Definition 3 hold. A sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ tends to $A_0 \in \mathcal{A}$ w.r.to the $\Pi$-metric $\rho$ iff the sequence $\{\chi_{A_n}\}_{n=1}^{\infty}$ of their $\mathcal{S}$-characteristic functions tends to $\chi_{A_0}$ w.r.to $\Pi$.

# Group-like Structures In Monoidal Categories — A Categorical Foundation Of Many-Valued Algebra

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## Abstract

Let $G$ be an ordinary group. In particular, $\cdot, e$ and $\iota$ denote respectivly the multiplication, the neutral element and the inversion in $G$. In 1971, A . Rosenfeld defines a fuzzy subgroup of $G$ as a map $\mu : G \to [0,1]$ satisfying the following axioms (cf. [6])

(G1) $\min\big(\mu(g_1), \mu(g_2)\big) \le \mu(g_1 \cdot g_2)$.
(G2) $\mu(e) = 1$.
(G3) $\mu(g) \le \mu(\iota(g))$.

In 1979, J.M. Anthony and H. Sherwood refined the concept of fuzzy subgroups by replacing the binary minimum by an arbitrary t-norm $T$. Thus a fuzzy subgroup of $G$ in the sense of Anthony and Sherwood is a map $\mu : G \to [0,1]$ satisfying (G2), (G3) and the following axiom (cf. [1])

(G1$'$) $T\big(\mu(g_1), \mu(g_2)\big) \le \mu(g_1 \cdot g_2)$.

From the point of view of many-valued logic (see also [2]) we can understand Rosenfeld's definition as subgroups of $G$ in the sense of *Gödel's logic*, while in the case of the t-norm $T_m$ determined by *Łukasiewicz' arithmetic conjunction*

$$T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0), \quad \alpha, \beta \in [0,1]$$

we can view Anthony and Sherwood's definition as subgroups of $G$ in the sense of *Łukasiewicz' logic.*
If we identify the map $\mu$ with the corresponding characteristic morphism of the topos $sh([0,1])$ of sheaves on $[0,1]$, then it is not difficult to see that Rosenfeld's fuzzy subgroups are nothing but subgroup objects of the simple sheaf $\widetilde{G}$ generated by $G$ (see also Theorem 3.5 and Remark 3.6 in [4]).
On this background we rise the following

Question: Does there exist a general categorical framework such that fuzzy subgroups in the sense of Łukasiewicz logic can be understood as subgroup objects?

The aim of this talk is to solve this problem. For this pupose we first introduce group-like structures in semimonoidal categories.

A *semimonoidal category* $\mathcal{C} = (\mathcal{C}_0, \otimes, a, \ell, r, \eta, \varepsilon)$ consists of a category $\mathcal{C}_0$ with a *terminal* object $1$, a bifunctor $\mathcal{C}_0 \times \mathcal{C}_0 \xrightarrow{\otimes} \mathcal{C}_0$ , natural isomorphisms $\otimes \circ (\otimes \times id_{\mathcal{C}_0}) \xrightarrow{a} \otimes \circ (id_{\mathcal{C}_0} \times \otimes)$ and natural transformations

$$1 \otimes \_ \xrightarrow{\ell} id_{\mathcal{C}_0}\,, \quad \_ \otimes 1 \xrightarrow{r} id_{\mathcal{C}_0}\,, \quad id_{\mathcal{C}_0} \xrightarrow{\eta} 1 \otimes \_\,, \quad id_{\mathcal{C}_0} \xrightarrow{\varepsilon} \_ \otimes 1$$

satisfying the following conditions:

(C1) For all $\mathcal{C}_0$-objects $W, X, Y, Z$ the pentagonal diagram

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a_{W \otimes X Y Z}"] \arrow[d, "a_{WXY} \otimes id_Z"] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a_{WXY \otimes Z}"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{WX \otimes YZ}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id_W \otimes a_{XYZ}"']
\end{tikzcd}$$

commutes.

(C2) For all $\mathcal{C}_0$-objects $X$ the subsequent diagrams:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "id_X"'] & 1 \otimes X \arrow[d, "\ell_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[dr, "id_X"'] & 1 \otimes X \arrow[d, "r_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes (1 \otimes X) \arrow[rr, "a^{-1}_{11X}"] \arrow[d, "id_1 \otimes \ell_X"'] & & (1 \otimes 1) \otimes X \arrow[d, "\ell_1 \otimes_X"] \\
1 \otimes X \arrow[r, "\ell_X"'] & X & 1 \otimes X \arrow[l, "\ell_X"]
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes 1) \otimes 1 \arrow[rr, "a_{X11}"] \arrow[d, "r_X \otimes id_1"'] & & X \otimes (1 \otimes 1) \arrow[d, "id_X \otimes r_1"] \\
1 \otimes X \arrow[r, "r_X"'] & X & 1 \otimes X \arrow[l, "r_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(1 \otimes X) \otimes 1) \arrow[rr, "a_{1X1}"] \arrow[d, "\ell_X \otimes id_1"'] & & 1 \otimes (X \otimes 1) \arrow[d, "id_1 \otimes r_X"] \\
X \otimes 1 \arrow[r, "r_X"'] & X & 1 \otimes X \arrow[l, "\ell_X"]
\end{tikzcd}$$

are commutative, and also $\eta_1 = \varepsilon_1$ .

**Proposition 1.1.** *Let $\mathcal{C}$ be a monoidal category with a terminal object. If the unit object is isomorphic to the terminal object, then $\mathcal{C}$ is also a semimonoidal category.*

**Example 1.2.** Let $\mathcal{C}$ be a monoidal category with a terminal object 1 s.t. the unit object is not necessarily isomorphic to 1. Then there exists an intrinsic monoid on 1 in the sense of $\mathcal{C}$, and the category of 1-biactions (cf. [5]) is a semimonoidal category which is not necessarily monoidal.

A quadruple $(X, m, e, \iota)$ is called a **group-like object** in a sense of the semi-monoidal category $\mathcal{C}$ iff $X$ is an object of $\mathcal{C}_0$ and

$$X \otimes X \xrightarrow{m} X, \quad 1 \xrightarrow{e} X, \quad X \xrightarrow{\iota} X$$

are $\mathcal{C}_0$-morphisms such that the following diagrams are commutative:

(I)
$$\begin{tikzcd}
X \arrow[r, "a_{XXX}"] \arrow[d, "m \otimes id_X"'] & X \otimes (X \otimes X) \arrow[r, "id_X \otimes m"] & X \otimes X \arrow[d, "m"] \\
X \otimes X \arrow[rr, "m"'] & & X
\end{tikzcd}$$
(*Associativity*)

(II)
$$\begin{tikzcd}
1 \otimes X \arrow[r, "e \otimes id_X"] \arrow[dr, "\ell_X"'] & X \otimes X \arrow[d, "m"] & X \otimes 1 \arrow[l, "id_X \otimes e"'] \arrow[dl, "r_X"] \\
& X &
\end{tikzcd}$$
(*Existence of Unity*)

(III)
$$\begin{tikzcd}
\Delta^*_X \arrow[r, tail, "\delta^*_X"] \arrow[d] & X \otimes X \arrow[r, "\iota \otimes id_X"] & X \otimes X \arrow[d, "m"] & X \otimes X \arrow[l, "id_X \otimes \iota"'] & \Delta^*_X \arrow[l, tail, "\delta^*_X"'] \arrow[d] \\
1 \arrow[rr, "e"'] & & X & & 1 \arrow[ll, "e"]
\end{tikzcd}$$
$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "id_X"'] & X \arrow[d, "\iota"] \\
& X
\end{tikzcd}$$
(*Existence of Antipode*)

where $\Delta^*_X \overset{\delta^*_X}{\rightarrowtail} X \otimes X$ denotes the tensorial modification of the *diagonal* $X \overset{\langle id_X, id_X \rangle}{\rightarrowtail} X \times X$ of $X$ i.e. the following pullback square holds:

$$\begin{tikzcd}
\Delta^*_X \arrow[r] \arrow[d, tail, "\delta^*_X"'] & X \arrow[d, tail, "\langle id_X, id_X \rangle"] \\
X \otimes X \arrow[r, "\langle \pi^*_1, \pi^*_2 \rangle"'] & X \times X
\end{tikzcd}$$

As an application of the these considerations we obtain the solution of the *previous question*:

Let $M$ be the canonical $MV$-algebra given by the real unit interval $[0,1]$. Then the category $M$-**SET** of $M$-valued sets (cf. [3]) is a monoidal category in which the terminal object is *not* isomorphic to the unit object. Further, fuzzy subgroups in the sense of Anthony and Sherwood are subgroup objects in the sense of the category of 1-biactions associated with $M$-**SET**.

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# WEAK REFLECTIONS IN CATEGORIES OF FUZZY SETS OVER MV-ALGEBRAS

Jiří Močkoř

In a fuzzy set theory there are several categories of fuzzy sets over a complete $MV$-algebra $\Omega = (L, \otimes, \rightarrow)$. The first one is a category $\mathbf{Set}(\Omega)$ with objects $(A, \delta)$ where $A$ is a set and $\delta : A \times A \rightarrow \Omega$ is a similarity relation such that

(i) $(\forall x \in A) \quad \delta(x, x) = 1$,
(ii) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,
(iii) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \le \delta(x, z)$.

A morphism $f : (A, \delta) \rightarrow (B, \gamma)$ in $\mathbf{Set}(\Omega)$ is a map $f : A \times B \rightarrow \Omega$ satisfying the following conditions .

(1) $(\forall x, z \in A)(\forall y \in B) \quad \delta(x, z) \otimes f(x, y) \le f(z, y)$,
(2) $(\forall x \in A)(\forall y, z \in B) \quad \gamma(y, z) \otimes f(x, y) \le f(x, z)$,
(3) $(\forall x \in A)(\forall y, z \in B) \quad f(x, y) \otimes f(x, z) \le \gamma(y, z)$,
(4) $(\forall x \in A) \quad 1 = \bigvee\{f(x, y) : y \in B\}$.

The other category $\mathbf{SetF}(\Omega)$ will have the same objects as the category $\mathbf{Set}(\Omega)$. A morphism $f : (A, \delta) \rightarrow (B, \gamma)$ in $\mathbf{SetF}(\Omega)$ is a map $f : A \rightarrow B$ such that $(\forall x, y \in A) \quad \gamma(f(x), f(y)) \ge \delta(x, y)$.

In any of these categories $\mathbf{K}$ we can introduce a *fuzzy subset* of an object $(A, \delta)$ as a morphism $s : (A, \delta) \rightarrow (\Omega, \leftrightarrow)$ in a corresponding category, in symbol $s \underset{\sim}{\subset}_{\mathbf{K}} (A, \delta)$. Let $\mathcal{F}_{\mathbf{K}}(A, \delta) = \{s : s \underset{\sim}{\subset}_{\mathbf{K}} (A, \delta)\}$ be the set of fuzzy subsets of $(A, \delta)$ in a category $\mathbf{K}$. On these sets $\mathcal{F}_{\mathbf{K}}(A, \delta)$ similarity relations can be defined such that we obtain the following subcategories of $\mathbf{K}$.

(1) A full subcategory $\mathcal{F}^{\leftrightarrow}_{\mathbf{SetF}(\Omega)}$ of a category $\mathbf{SetF}(\Omega)$ with objects $(\mathcal{F}_{\mathbf{SetF}(\Omega)}(A, \delta), \sigma)$, where $\sigma(s, t) = \bigwedge_{x \in A} s(x) \leftrightarrow t(x)$,
(2) A full subcategory $\mathcal{F}^{\otimes}_{\mathbf{SetF}(\Omega)}$ of a category $\mathbf{SetF}(\Omega)$ with objects $(\mathcal{F}_{\mathbf{SetF}(\Omega)}(A, \delta), \tau)$, where $\tau(s, t) = \bigvee_{x \in A} s(x) \otimes t(x)$,
(3) A full subcategory $\mathcal{F}^{\leftrightarrow}_{\mathbf{Set}(\Omega)}$ of a category $\mathbf{Set}(\Omega)$ with objects $(\mathcal{F}_{\mathbf{Set}(\Omega)}(A, \delta), \sigma)$.

**Theorem.** *Any of the above mentioned full subcategories is a weak reflective subcategory in a corresponding category.*

# LOGICS OF A CONTINUOUS T-NORM AND ITS RESIDUUM WITH TRUTH-CONSTANTS

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## 1 Abstract

In the context of fuzzy logical systems, introducing truth-constants in the language is an elegant means to be able to explicitely reason with partial degrees of truth. This goes back to Pavelka [9] who built a propositional many-valued logical system over Lukasiewicz logic by adding into the language a truth constant $\overline{r}$ for each real $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic (like Lukasiewicz logic) is not strongly complete, Pavelka proved that his logic, which we shall call it PL, is complete in a weaker sense. Namely, by defining the truth degree of a formula $\varphi$ in a theory $T$ as

$$|| \varphi ||_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of $\varphi$ in $T$ as

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\},$$

Pavelka proved that these degrees coincide. This kind of completeness, is usually known as Pavelka-style completeness, and strongly relies in the continuity of Lukasiewicz truth functions. Novák extended Pavelka approach to Lukasiewicz first order logic.

Later, Hájek [6] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results, indeed it is enough to extend the language only by a countable number of truth-constants, one per each *rational* in $[0, 1]$, and by two additional axiom schemata, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are Lukasiewicz t-norm and its residuum respectively. He denoted this new system Rational Pavelka Logic, RPL for short. Moreover he proved that RPL is strongly complete for finite theories.

Similar *rational* expansions for other popular fuzzy logics can be obviously defined, but note that Pavelka-style completeness cannot be obtained since Lukasiewicz logic is the only fuzzy logic with continuous truth-functions in the

*The results contained in this abstracts are mainly the results of a forthcoming paper with Lluís Godo and Carles Noguera and contain also results obtained in collaboration with Petr Savicki and Roberto Cignoli real unit interval [0, 1]. Among different works in this direction we may cite [6] where an extension of $G_\Delta$ (the extension of Gödel logic with Baaz's Delta operator) with a finite number of rational truth-constants, and [4] where the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel logic with an involutive negation) and to $\Pi$ (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation). In the case of the rational expansions of $\Pi$ and $\Pi_\sim$ an infinitary inference rule (from $\{\varphi \to \overline{r} : r \in \mathbb{Q} \cap [0,1]\}$ infer $\varphi \to \overline{0}$) is introduced in order to get Pavelka's style completeness.

Rational truth-constants have also considered in some stronger logics like in the logic Ł$\Pi\frac{1}{2}$ [2], a logic that combines the connectives from both Lukasiewicz and Product logics plus the truth-constant $\overline{1/2}$, and in the logic PŁ [7], a logic which combines Lukasiewicz logic connectives plus the Product logic conjunction (but not implication), as well as in some closely related logics.

More recently, in [1] and in [10] the authors considered the expansions with rational truth-constants of Gödel and weak Nilpotent minimum logics (and some of its extensions) in the first paper and of the product logic in the second paper. We use the fact that the corresponding logics are algebraizable and the fact that any algebra of the corresponding variety is subdirect product of linearly ordered ones. Standard (weak) completeness is shown for those logics as well as finite strong completeness when restricted to formulas of the kind $\overline{r} \to \varphi$, where $\overline{r}$ denotes the truth constant $r$ and $\varphi$ is a formula without truth-constants. Actually, this kind of formulas have been extensively considered in other frameworks for reasoning with partial degrees of truth. In particular, these formulas correspond to Novák's *evaluated* formulas in [8]. Evaluated formulas are expressions $a/A$ where $a$ is a truth value (from a given algebra) and $A$ is a formula of a language built using truth constants too. Our formula $\overline{r} \to \varphi$ would be expressed as $r/\varphi$ in Novák's syntax. They also appear in the framework of abstract fuzzy logics developed by Gerla [5] based on the notion of fuzzy consequence or deduction operators over fuzzy sets of formulas, where the membership degree of formulas are interpreted as lower bounds on their truth degrees.

In this talk we will present the results about the expansions of the three main fuzzy logics corresponding to Lukasiewicz, product and minimum t-norms and its residuum and their generalization to the expansion with truth-values of any logic of a continuous t-norm and its residuum. From [3] we know that any of these logics is finitely axiomatizable and from them we can study their expansion wiith truth-value constants. We will present general completeness results for logics defined by a t-norm that is a finite ordinal sum and its residuum. Finally we will present some partial results for logics of t-norms that are ordinal sum with infinite components and its rsiduum.

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**Fuzzy Approximation - Basic Concept and Overview of Recent Applications**

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In the contribution, we will present our attempt to the problematic of fuzzy approximation. This notion may be understand as a theory studying properties of special functions, relations or formulas having an approximating character. In the sequel, we will focus to a special class of $L$-valued functions (where $L$ is a support of ŁΠ algebra) aggregating local information about some other fixed function to be approximated. Moreover, we will show some applications where particular techniques of fuzzy approximation have been successfully applied.

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# Software system LFLC2000, its current state and future development

**Abstract**

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**Keywords:** Fuzzy logic, fuzzy control, data mining.

In this contribution we present the software system LFLC2000 (Linguistic Fuzzy Logic Controller). It is a complex tool for the design of linguistic descriptions (i.e. sets of fuzzy IF-THEN rules). These description then can be used in various application fields, e.g. in fuzzy control, decision making or data mining. One such working industrial application is described in [2].

We sketch the unique methodology and theoretical results upon which is LFLC2000 based (see [3, 1]). Then we present its current state and abilities. Finally we discuss possible directions of its future development.

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**Logical Theory of Fuzzy IF-THEN Rules**

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This paper is focused on the development of the theory of fuzzy IF-THEN rules and its contribution to the establishment of fuzzy logic. I advocate that Hájek's fuzzy logic is a right methodology for development of special logical theories. A theory of fuzzy IF-THEN rules (as a special theory in this sense) is proposed. This theory is for all practitioners who want to create a system of fuzzy IF-THEN rules free of conflicts (logically consistent) and rich enough to be able to make non-trivial conclusions or answer inquires.

# Skolem Functions in Fuzzy Logics

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