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Logical Foundations of Rule-Based Systems

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Abstract

This paper is focused on the evolution of the theory of fuzzy IF-THEN rules and its contribution to the establishment of fuzzy logic. I advocate that Hájek's fuzzy logic is a right methodology for development of special logical theories. A theory of fuzzy IF-THEN rules (as a special theory in this sense) is proposed. This theory is for all practitioners who want to create a system of fuzzy IF-THEN rules free of conflicts (logically consistent) and rich enough to be able to make non-trivial conclusions or answer inquires.

Key words: Fuzzy logic, deduction theory, fuzzy IF-THEN rules, system of fuzzy relation equations

1 Introduction

The theory of fuzzy sets introduced and elaborated by Lotfi A. Zadeh attracted researches from many areas. Due to the brightly formulated ideas and postulates, it became the basis of many modern branches of computer science and mathematics. However, the mostly known and used notion is not the notion of fuzzy set, but the notion of fuzzy logic which is used whenever one operates with fuzzy sets or simply uses lattice operations over the interval $[0, 1]$. This makes pure logicians upset, but on the other hand, this fact realizes expectations of the great logician Alfred Tarski who foresaw a remarkable impact of a logic especially on applications. By this, he meant the methodological aspect of logic:

⋆ This paper is dedicated to Lotfi A. Zadeh whose ideas inspired logicians to break limits.

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treatment of notions, establishment of a correct system of definitions, deduction, analysis of consistency and completeness, etc.

The aim of this contribution as well as of the whole special issue is to stop for a moment and look back at what has been achieved on a wide field of fuzzy logic. Of course, this particular contribution will concern only one aspect of this direction, leaving other aspects to other contributors. I will concentrate on the evolution of a theory of fuzzy IF-THEN rules and on its contribution to the establishment of fuzzy logic. Moreover, I will consider also the question "What is fuzzy logic?" in a light of the above mentioned evolution.

At the very beginning the theory of fuzzy IF-THEN rules was the closest one to a formal logical theory. In fact, it had all attributes of a formal calculus: axioms (IF-THEN rules themselves) and deduction rules (Compositional Rule of Inference, Generalized Modus Ponens and others) which however, have been introduced semantically. The hope to elaborate a formal theory of fuzzy IF-THEN rules has been explicitly expressed by Lotfi A. Zadeh when he characterized the agenda of fuzzy logic in narrow sense. However, many attempts to build a rigorous formal logic theory of IF-THEN rules failed or were not completed. In my opinion, the main source of difficulties stemmed from assigning truth values to each rule independently from a truth value which can be calculated on the basis of the principle of *truth functionality*. Therefore, a logical analysis of axioms (rules) which are not absolutely true was impossible. Some exceptions are worth to be remarked - the V. Novák's formal logic theory of IF-THEN rules based on the fuzzy logic with evaluated syntax [8] and Gottwald's theory of fuzzy relation equations [3].

In [8], each rule is represented by a special fuzzy set of closed formulas whose membership degrees are then propagated to their provability degrees so that a consistent use of semantics in the syntactic theory is then enabled. The resulting theory is focused on a logical analysis of a meaning of natural language expressions comprising also fuzzy IF-THEN rules.

In [3], a system of fuzzy IF-THEN rules is characterized by formulas of a first order language and modeled by a fuzzy relation. However, a deduction theory has not been elaborated.

To conclude, there was no unique logical frame for elaboration on its base special logical theories including the theory of fuzzy IF-THEN rules. The situation has changed in 1998 with issuing the book of Peter Hájek "Metamathematics of Fuzzy Logic" [4]. This fundamental book put fuzzy logic on the platform of a classical deduction theory and showed that fuzzy logic is a specific many-valued logic. In Hájek's book vague (fuzzy) objects are put inside the deduction theory which is constructed traditionally using the rules of Modus Ponens and Generalization. Only two truth constants are required: $\top$ and $\bot$. Totally true statements containing (fuzzy) parameters, estimated on an extended truth scale, are picked up as theorems. To be able to process fuzzy objects, some new logical connectives (other than those used in classical logic) are involved. This book has been taken with a great enthusiasm by a mathematical community and a lot of research work has been done, and is doing on the basis of the new mathematical structures discovered in it. Many new fuzzy logics and algebraic structures appeared and continue to appear. A certain kind of a competition (who will find a weaker fuzzy logic) has been captured by Peter Hájek himself in his paper about flea logics. Therefore, a reasonable question arises: is this the right direction that fuzzy logic should follow? Is it enough to establish a system of axioms and prove the completeness property with respect to specially constructed structures of truth values? The most important question raised to the class of thus developed fuzzy logics concerns the methodology that any logic is expected to provide (in general). What would I gain or loose if my reasoning would proceed using BL logic and not using MTL logic? (To be frank, the Hájek's book provided methodology of construction new fuzzy logics (calculi).)

Luckily, there are other attempts to take advantage of the new methodology proposed by Peter Hájek in his "Metamathematics of Fuzzy Logic". By this I mean theoretical aspects of a fuzzy control elaborated by Hájek himself to demonstrate the new methodology in work, a fuzzy type theory [7], a fuzzy mathematics [1], a theory of fuzzy approximation [9], etc.

Summarizing,

> *Fuzzy logic is a special many-valued logic based on the classical deduction theory and focused on dealing with vaguely delineated propositions. Fuzzy logic provides a methodology of analyzing knowledge based systems characterized by fuzzy IF-THEN rules.*

I am advocating in favour of Hájek's fuzzy logic regarding it as a suitable methodology for analyzing related to it subjects. However, the place for a theory of fuzzy IF-THEN rules (as a special logical theory) is still open and in the rest of this paper, I want to suggest my contribution to this specific problem. I have been motivated by the following two facts:

(i) there is a permanent interest in this topic, but not having a good theory, people act blindly or, better say, intuitively;
(ii) having rich experience in the theory of fuzzy relation equations, I have realized that it may be a right theoretical basis for the theory of fuzzy IF-THEN rules.

One chapter in Hájek's book is devoted to the logical analysis of fuzzy IF-THEN rule systems. In this chapter, the status quo has been fixed and some new aspects connecting such systems with fuzzy functions have been realized. However, there was no dynamics in the proposed presentation in the sense of changing or completing such systems. Therefore, his theory of fuzzy IF-THEN rules has never been applied in practice.

Below, I would like to suggest another approach to the construction of the logical theory of fuzzy IF-THEN rules which is based on Hájek's BL-logic (and by this, verifies the methodological message of his book). This theory is for all practitioners who want to create a system of fuzzy IF-THEN rules free of conflicts (logically consistent) and rich enough to be able to make non-trivial conclusions or answer inquires. This theory uses the knowledge acquired from investigation of the problem of solvability of a system of fuzzy relation equations. Moreover, this theory demonstrates that the deduction rule known as Generalized Modus Ponens works correctly only if a system of fuzzy IF-THEN rules is modeled by a solvable system of fuzzy relation equations.

The paper is organized as follows: Section 2 gives a brief introduction to fuzzy IF-THEN rules and their models, Section 3 introduces language and formulas of a logical theory of IF-THEN rules and Section 4 characterizes consistency and completeness of the logical theory.

2 Fuzzy Relation as a Model of Fuzzy IF–THEN Rules

By a system of fuzzy IF–THEN rules we mean the following set of formal expressions:

$$\begin{aligned} \mathcal{R}_1 &: \ \mathsf{IF}\ X\ \mathsf{is}\ A_1\ \mathsf{THEN}\ Y\ \mathsf{is}\ B_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{R}_n &: \ \mathsf{IF}\ X\ \mathsf{is}\ A_n\ \mathsf{THEN}\ Y\ \mathsf{is}\ B_n \end{aligned} \tag{1}$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $\mathcal{F}(\mathbf{X})$, $\mathcal{F}(\mathbf{Y})$ are universes of fuzzy subsets on $\mathbf{X}$, respectively $\mathbf{Y}$ (see below for the precise definition).

Let us agree to model this system of fuzzy IF–THEN rules in a class of fuzzy relations on $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$. To be able to express the relationship between system (1) and its model, we need to choose an appropriate algebra of operations over fuzzy subsets. For this purpose, let us choose a BL-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{2}$$

with four binary operations and two constants (see [4] for details) extended by the binary operation $\leftrightarrow$ of equivalence:

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The universes of fuzzy subsets on $\mathbf{X}$ and $\mathbf{Y}$ will be defined as universes of functions (membership functions) as follows:

$$\mathcal{F}(\mathbf{X}) = [0,1]^{\mathbf{X}} \quad \text{and} \quad \mathcal{F}(\mathbf{Y}) = [0,1]^{\mathbf{Y}}.$$

Fuzzy relations will be identified with fuzzy sets on cartesian products, for example binary fuzzy relations are elements from $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$, etc.

Saying that a fuzzy relation $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is a model of fuzzy IF–THEN rules (and therefore, of a dependence, partially given by them), we specify how this model can be used in computation.

Definition 1
We say that a fuzzy set $B \in \mathcal{F}(\mathbf{Y})$ *is an output of the (fuzzy) model* $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ *given input* $A \in \mathcal{F}(\mathbf{X})$ *if*

$$B(y) = \bigvee_{x \in \mathbf{X}} (A(x) * R(x, y)) \tag{3}$$

(in short, $B = A \circ R$ *and in words,* B *is the result of* $\sup -*$ *composition between a fuzzy set* A *and a fuzzy relation* R*).*

Recall that equation (3) realizes the Compositional Rule of Inference (CRI) first introduced by L.A. Zadeh in [14].

Till now, we did not put any restriction on a fuzzy relation which models a set of fuzzy IF-THEN rules. We are going to do it below, considering different relations between the original data contained in fuzzy IF-THEN rules and their model.

Definition 2
We say that a model $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ *is a correct model of fuzzy IF-THEN rules (1) which are based on the data* (A_i, B_i), $i = 1, \ldots, n$, *(cf. [5]) if for all* $i = 1, \ldots, n$

$$A_i \circ R = B_i. \tag{4}$$

It is easy to see that a model is correct if and only if it gives a solution to the system of fuzzy relation equations expressed by (4) where the fuzzy sets A_i and B_i are given by (1). At this point, we may refer to different criteria of solvability (see Introduction) which tell us when we may expect to have a correct model. We recall the following one to which we will refer later.

Proposition 1
If system (4) is solvable with respect to an unknown fuzzy relation R *then the relation*

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \tag{5}$$

is the greatest solution to (4) (see [3,13]).

3 Logical Theory of IF–THEN Rules. Language and Formulas

We are going to construct a special predicate theory of fuzzy IF-THEN rules. This is supposed to be a formal deduction theory to be able to infer logically supported conclusions.

For the basic predicate calculus we chose the Hájek's BL∀ [4] which means that basic logical connectives will be interpreted by operations of a some BL-algebra and quantifiers by the operations of supremum and infimum (see the details below). Some unexplained notation is taken from [4] as well.

For simplicity, we consider the theory of fuzzy IF-THEN rules with one-place antecedent only but it is not difficult to generalize the theory to the more complex case. For the same reason, the language will contain unary and binary special predicates only.

Suppose that our language $\mathcal{J}_n$, $n \geq 1$, consist of six subsets:

(i) At most countable set of predicate symbols $\mathcal{P}$ each together with its *arity*; we will distinguish special unary predicate symbols $A_1, \ldots, A_n \in \mathcal{P}$ and $B_1, \ldots, B_n \in \mathcal{P}$, and a special binary predicate symbol $R \in \mathcal{P}$.
(ii) A set $\mathcal{X} = \{x, y, \ldots\}$ of object variables.
(iii) A set $\mathcal{A} = \{a, b, \ldots\}$ of object constants.
(iv) A set $\mathcal{C} = \{\wedge, \vee, \&, \rightarrow, \neg, \equiv, \circ\}$ of connectives.
(v) The set $\mathcal{Q} = \{\forall, \exists\}$ of quantifiers.
(vi) The set $\mathcal{TC} = \{\bot, \top\}$ of truth constants.

The notions of *term* and *formula* are defined as in the classical predicate logic with the following additional abbreviation: if $\varphi(x)$ is a formula where x is a free variable then the following construction is a formula too:

$$(\varphi \circ R)(y) = (\exists x)(\varphi(x) \& R(x, y)).$$

Let $\mathcal{J}_n$ be a predicate language as above and let $\mathbf{L}$ be a BL-algebra

$$\langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

extended by the operations $\neg, \leftrightarrow$. An $\mathbf{L}$-*structure for* $\mathcal{J}_n$ is a tuple $\mathbf{M} = \langle M, \{r_P \mid P \in \mathcal{P}\}, \{m_a \mid a \in \mathcal{A}\} \rangle$ where $M \neq \emptyset$ and for each k-ary predicate $P \in \mathcal{P}$ (including special ones), $r_P : M^k \longrightarrow \mathbf{L}$ is an k-ary $\mathbf{L}$-fuzzy relation on M, for each object constant $a \in \mathcal{A}$, m_a is an element of M. An $\mathbf{M}$-*evaluation* of object variables and values of terms and formulas are defined as in [4].

The structure $\mathbf{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, i.e. $\|\varphi\|^L_{M,v}$ is defined for all φ, v.

Remark 1
For simplicity, we did not use many-sorted predicates in our language though it would be better to assume that special predicates A_i and B_i contain variables of different sorts. We refer to [4] for the extension to a many-sorted predicate calculus.

4 Logical Theory of IF–THEN Rules. Consistency and Completeness

The *special theory* $\mathcal{R}_n$ of n IF-THEN rules (over a given predicate language $\mathcal{J}_n$) consists of:

- all axioms of the Hájek's BL∀,
- special axioms:
  SA1 $R(x,y) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))$
  SA2$_i$ $B_i(y) \to (A_i \circ R)(y)$
  where $i = 1, \ldots, n$ (so that we have n axioms of type **SA2**),
- deduction rules:
  MP (modus ponens): from φ, and $\varphi \to \psi$ infer ψ,
  Gen (generalization): from φ infer $(\forall x)\varphi$,
  CRI (compositional rule of inference): from φ, and R infer $\varphi \circ R$.

Let **L** be a linearly ordered BL-algebra and **M** an **L**-safe structure for $\mathcal{J}_n$ where we additionally assume that fuzzy relations interpreting A_i, B_i, R are denoted by the same symbols. We say that **M** is an **L**-*model* for the theory $\mathcal{R}_n$ if all axioms of $\mathcal{R}_n$ are **1**-true in **M**. Two models of the same theory $\mathcal{R}_n$ are *data-connected* if they differ in interpretation of the binary predicate R. The set of data-connected models of $\mathcal{R}_n$ will be denoted by $\mathcal{M}_{\mathcal{R}_n}$. We may prove the following:

Proposition 2
M *is an* **L**-model *for the theory* $\mathcal{R}_n$ *if and only if the binary fuzzy relation* $R(x,y)$ *solves the system of fuzzy relation equations*

$$B_i(y) = \bigvee_{x \in M} (A_i(x) * X(x,y)), \qquad i = 1, \ldots, n \tag{6}$$

(in short, $B_i = A_i \circ R$*) with respect to an unknown* X*.*

We say that a theory is *consistent* if it has a model. By Proposition 2, the theory $\mathcal{R}_n$ is consistent because there always exist unary fuzzy relations (fuzzy sets) $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ and binary fuzzy relation R such that the system (6) is solvable (see e.g. [10]).

Remark 2
By using different symbols $A_1, \ldots, A_n$ *for antecedents, we insure the theory* $\mathcal{R}_n$ *against inconsistency. If there were two rules with the same antecedents, but different consequents, we cannot find a model which meets all requirements, and therefore, such a theory cannot be consistent.*

We present some examples of provable formulas. Let A be an arbitrary unary predicate, then for each $i = 1, \ldots, n$, $\mathcal{R}_n$ proves the following:

$$(\forall y)(B_i(y) \equiv (A_i \circ \bigwedge_{i=1}^{n}(A_i \to B_i))(y)), \tag{7}$$

$$R(x,y) \to (A(x) \to (A \circ R)(y)), \tag{8}$$

$$(\forall x)(A(x) \equiv A_i(x)) \to (\forall y)((A \circ R)(y) \equiv B_i). \tag{9}$$

By (7), in each **L**-*model* for $\mathcal{R}_n$ the fuzzy relation $\bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))$ solves the system (6). By (8), we are able to make a conclusion $(A \circ R)$ from the assumption (input) given by A. Precisely, the implication $A(x) \to (A \circ R)(y)$ can be consistently added to the rule base R comprised of fuzzy IF-THEN rules $A_i(x) \to B_i(y)$. Formula (9) establishes the property which can be recognized as the *Generalized Modus Ponens* [14]. According to it, a conclusion $(A \circ R)$ made from the assumption A is close to one of B_i if A is close to one of A_i. However, we prefer to call this property the *Principle of Relative Distinguishability* and reformulate it as follows:

> *We distinguish a conclusion $(A \circ R)$ among different B_i at the degree not greater than the degree of distinguishability A among A_i.*

There are other general and specific properties which characterize the system of fuzzy IF-THEN rules and which may be proved formally in $\mathcal{R}_n$. We will leave the details for a special paper, focusing here on a *completeness* of $\mathcal{R}_n$. In our treatment, completeness will be tightly joined with a possibility of extension. Let us explain what we have in mind.

Logical theory of fuzzy IF-THEN rules aims to make inferences with antecedents different from those formalized by A_i (cf. (8)). For this purpose, the deduction rule **CRI** has been proposed. However, if we cannot relate a given antecedent, say A, to any of A_i, the conclusion, obtained by the deduction, is so general that it does not express any specific property or constraint. In this case, some additional information (in the form of a new fuzzy IF-THEN rule) is required. Logically, this means that we would like to extend our special theory by adding new special axioms. This must be done carefully, keeping the consistency of the original theory. In case when an eligible extension does not keep the consistency of the original consistent theory, we say that it is complete. Formally, we will define the notions of extended theory and complete theory as follows.

Definition 3
Let the language $\mathcal{J}_n$ be extended to $\mathcal{J}_{n+1}$ by adding two special unary predicate symbols A_{n+1} and B_{n+1}. We say that the theory $\mathcal{R}_{n+1}$ extends $\mathcal{R}_n$ if it has two more axioms:
$\mathbf{SA1}_{n+1}$ $\quad R(x, y) \to (A_{n+1}(x) \to B_{n+1}(y))$,
$\mathbf{SA2}_{n+1}$ $\quad B_{n+1}(y) \to (A_{n+1} \circ R)(y)$.

Proposition 3
Let **M** *be an* **L**-model *for the theory $\mathcal{R}_n$ where fuzzy relations, interpreting A_i, B_i, R, are denoted by the same symbols. Then* **M** *can be* expanded *to an* **L**-model *for the theory $\mathcal{R}_{n+1}$ in the language $\mathcal{J}_{n+1}$.*

This proposition easily follows when expanding **M** by an arbitrary interpretation of A_{n+1} and by the interpretation $(A_{n+1} \circ R)$ of B_{n+1}. This expansion of **M** will be further called *trivial*.

Corollary 1
The theory $\mathcal{R}_{n+1}$ extends $\mathcal{R}_n$ conservatively.

Definition 4
A theory $\mathcal{R}_k$ is complete with respect to its model **M** *if its extension to $\mathcal{J}_{k+1}$ admits only trivial expansion of* **M**.

In case $\mathbf{M}_1$ and $\mathbf{M}_2$ are data-connected models of $\mathcal{R}_n$, we can construct non-trivial expansions of both models. Denote by R_1 and R_2 different fuzzy relations interpreting the binary predicate R in $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. Then expand $\mathbf{M}_1$ by an arbitrary interpretation of A_{n+1} and the interpretation of B_{n+1} by $(A_{n+1} \circ R_2)$ and similarly for $\mathbf{M}_2$. This type of expansion will be called *trivial expansion within* $\mathcal{M}_{\mathcal{R}_n}$.

Definition 5
A theory $\mathcal{R}_k$ *is* complete with respect to all data-connected models *if its extension to* $\mathcal{J}_{k+1}$ *admits only trivial expansions within* $\mathcal{M}_{\mathcal{R}_n}$.

To illustrate, why the notion of completeness is important for the development of the theory of fuzzy IF-THEN rules, let us consider the example which L. Zadeh uses in his lectures.

> "Usually Robert returns from work at about 6 pm. What is the probability that Robert is home at about 6.15 pm?"

This data may be rewritten in the following IF-THEN form:

> IF the time is "*about 6 pm*" THEN *Robert returns* (home) from his work.
> The time is "*about 6.15 pm*".

What can be inferred from this (incomplete!) rule base about Robert's position at about 6.15 pm?

Let us agree to use the truth estimation instead of the probabilistic one. According to our *Principle of Relative Distinguishability* we can infer that the degree of truth of the conclusion "at *about 6.15 pm* Robert is at home" is not greater than the degree of distinguishability between the propositions "the time is *about 6 pm*" and "the time is *about 6.15 pm*". This conclusion is the only one which can be formally inferred from our special theory. In order to be more precise, we shall *complete* the rule base. Then the specification of the Robert position at the time *about 6.15 pm*" will be related to the additionally obtained information about him at some later time instance.

In general, the idea of completion of the rule base system in order to make justified conclusions is intuitively pursued by any researcher who aims at creation of a meaningful data-base system, cf. [2]. The latter reference considers one practical example containing an assertion extracted from the real economic report. In order to analyze the assertion and be able to answer a question related to it, Diaz and Takagi in [2] created a knowledge base formed by fuzzy IF-THEN rules. This rule base contains explanation of all used special terms and their behavior with respect to the term in the question. By doing this, they intuitively followed the procedure which in logical terminology means *completion of a set of special axioms*. Moreover, the completion process had not been taken absolutely, but relatively, to be able to answer the inquiry. This means that the *Principle of Relative Distinguishability* has been applied in [2] (again intuitively). On the basis of this, the authors were able to infer (with help of CRI) the non-trivial answer to their inquiry. The methodology which the authors of [2] followed is now a subject of the special logical theory presented here. Let me express the hope that this special theory contributes to the goal that fuzzy logic should consider with respect to applications.

5 Conclusion

Our main claim is that a fuzzy logic should provide a methodology for analysis of knowledge based systems formed by fuzzy IF-THEN rules. An axiomatic approach to the construction of a logical theory of fuzzy IF-THEN rules based on Hájek's BL-logic has been presented. The proposed theory aims at creation of a system of fuzzy IF-THEN rules free of conflicts (logically consistent) and rich enough to be able to make non-trivial conclusions or answer inquires. This theory uses the knowledge acquired from investigation of the problem of solvability of a system of fuzzy relation equations. Moreover, this theory demonstrates that the deduction rule known as Generalized Modus Ponens works correctly only if a system of fuzzy IF-THEN rules is modeled by a solvable system of fuzzy relation equations. Some examples are included.

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