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## Which logic is the real fuzzy logic?

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## Abstract

This paper is a contribution to the discussion of the problem, whether there is a fuzzy logic that can be considered as the real fuzzy logic. We give reasons for taking IMTL, BL, LII and  $\text{Ev}_L$  (fuzzy logic with evaluated syntax) as those fuzzy logics that should be indeed taken as the real fuzzy logics.

*Key words:* Mathematical fuzzy logic, fuzzy type theory, fuzzy logic with evaluated syntax, algebra of truth values.

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## 1 What makes many-valued logic a fuzzy logic

In this position paper, I will discuss a question that arose during several discussions, that took place at conferences in Vienna 2004 and Linz 2005: *What, in fact, is fuzzy logic?* After the famous P. Hájek's book [9], it turned out that there are many well established formal systems of fuzzy logic (FL in the sequel) that may well claim to be “the real” fuzzy logic. I want to show that about many possibilities, there are, in my opinion, fuzzy logic systems that can be taken as the most outstanding to fulfil the tasks put on them. Of course, one may hardly expect just one “The” fuzzy logic but still, a few may be indeed picked up.

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The history of FL has been nicely summarized by P. Hájek in [10] and so, I will not repeat it. Instead, I will start with the following informal characterization.

*Fuzzy logic is a special many-valued logic addressing the vagueness phenomenon and developing tools for its modeling via truth degrees taken from an ordered scale. It is expected to preserve as many properties of classical logic as possible.*

No concrete formal system is mentioned in this characterization, but it emphasizes well the aim of FL and pre-determines the way how a suitable system of initially many-valued logic should be developed. The leading concepts are *degrees* (of truth) and *vagueness*. The category of vagueness has been enough discussed already in [21] (cf. also [24]) and so, we will only very briefly remind the following.

The *vagueness phenomenon* raises when trying to *group* together objects that have a certain property  $\varphi$ . The result is an *actualized* grouping of objects; we can write it as

$$X = \{o \mid o \text{ is an object having property } \varphi\}. \quad (1)$$

In general,  $X$  in (1) *cannot be taken as a set* since the property  $\varphi$  may be *vague*, i.e. it may be impossible to characterize the grouping  $X$  precisely and unambiguously; there can exist *borderline* elements  $o$  for which it is unclear whether they have the property  $\varphi$  (and thus, whether they belong to  $X$ ), or not.

Vagueness should be distinguished from uncertainty. This difference corresponds to the difference between actuality and potentiality. While vagueness rises from the *actualized non-sharply delineated* groupings, uncertainty is encountered when dealing with still non-actualized groupings of objects. In the latter case, we speculate about the whole  $X$ , but only part of it indeed exists. Once an actualized (i.e. already existing) grouping of objects is at our disposal, it makes sense to speak about the *truth* of the fact that some element belongs to it. Indeed, let an object  $y$  be created (at least in our mind). If we learn that it has a property  $\varphi$  in (1), we know that it falls into (the existing part of)  $X$ , i.e. we *know the truth of*  $y \in X$ . In general, however, it is *uncertain* whether  $y$  *will be* created (will exist) or not and so, in the latter case, we cannot speak about the truth of  $y \in X$ , but only about its possibility or probability.

It follows from this discussion that the truth degrees provide a reasonable means for dealing with vagueness. On the other hand, they have little use alone: Imagine, e.g., the sentence “I love you in the degree 0.954867283<sup>†</sup>.” Of course, nobody will ever say such a sentence. On the other hand, it is quite natural to say “I love you *very much*.”

We may argue that during the talk, people implicitly use degrees<sup>†</sup>. Thus, it is important to compare the degrees. The concrete values are of little relevance, important is their tendency; shapes of the corresponding fuzzy sets.

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<sup>†</sup>) This part is based on the discussion with P. Hájek.

<sup>†</sup>) This argument is due to A. Di Nola (personal communication).

The history teaches us that fuzzy logic offers a working mathematical model of the vagueness phenomenon and of situations where vagueness plays an important role. The following is required from FL:

- (a) FL should be a well established sound formal system to have its applications well justified.
- (b) FL should be an open system. It must be possible to extend it by new connectives and by generalized quantifiers. Moreover, some specific phenomena of natural language semantics should also be expressible, such as non-commutativity of conjunction and disjunction.
- (c) FL has a specific agenda, special techniques and concepts. Among them we can rank evaluating linguistic expressions, linguistic variables, fuzzy IF-THEN rules, fuzzy quantification, defuzzification, fuzzy equality, etc.
- (d) FL should enable to develop special inference schemes including sophisticated inference schemes of human reasoning (e.g., compositional rule of inference, reasoning based natural language expressions, non-monotonic reasoning, abduction, etc.).

Note that many of these requirements are already fulfilled by the available formal logical systems.

The fundamental classification of FL is *fuzzy logic in narrow sense* (FLn) and that in *broad sense*. The latter has been coined by L. A. Zadeh to denote all kinds of applications that use fuzzy sets<sup>‡</sup>). Since this is too extensive, I have proposed in [18] (and elsewhere) a middle concept of *fuzzy logic in broader sense* (FLb) as an extension of FLn whose aim is to develop a formal theory of *human way of reasoning that would include a mathematical model of the meaning of some expressions of natural language (evaluating linguistic expressions), the theory of generalized quantifiers and their use in human reasoning*. The foreseen goal is to develop a formal logic that could be applied in human-like behaving robots.

## 2 Two fundamental approaches to fuzzy logic in narrow sense

We will now focus on fuzzy logic in narrow sense as a formal mathematical theory. Let us stress that the first mathematically deep and advanced formal system was published by J. Pavelka in [26]<sup>§</sup>). Surprisingly, his work went rather far and still, it is a cause for dispute. Namely, though not stated explicitly in his work, he, in fact, established a limit generalization of logic by allowing evaluation of formulas also in syntax. After the seminal monograph of P. Hájek [9], we now distinguish two fundamental approaches to formal theory of fuzzy logic.

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<sup>‡</sup>) Personal communication as well as his talks at many international conferences.

<sup>§</sup>) This was his dissertation defended already in 1976.

## 2.1 Fuzzy logic with traditional (classical) syntax

This approach is less radical and it is promoted by many mathematicians, starting by P. Hájek and followed by F. Esteva, S. Gottwald, L. Godo, F. Montagna, D. Mundici, and others.

The crucial question in fuzzy logic is, what is the structure of truth values. It has been generally agreed that it must be a residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \quad (2)$$

where  $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$  is a bounded lattice,  $\langle L, \otimes, \mathbf{1} \rangle$  is a commutative monoid and  $\rightarrow$  is a binary *residuation* operation joined with the *product*  $\otimes$  by adjunction ( $a \otimes b \leq c$  iff  $a \leq b \rightarrow c$ ). Adding various conditions, we obtain stronger structures (for the details see [5,8,9,24] and elsewhere). Then, depending on the structure of truth values, we can distinguish *basic fuzzy logic* (BL), MTL-logic (MTL), IMTL-logic (IMTL), Łukasiewicz logic (Ł), Gödel logic (G), product logic  $\Pi$ , and others. A more significant departure is LII-logic whose structure of truth values has two products and two (different) implications. The list is by no means complete and there are various intermediate cases.

All these logics generalize syntax of classical logic only in adding a new connective of *strong conjunction* ( $\&$ ) (and, possible, some other ones) and modifying special axioms. Furthermore, they have a finite list of schemes of logical axioms and inference rules of *modus ponens* and *generalization*. The fundamental concept of provability is classical, i.e. a formula  $A$  is provable if there exists its formal proof. Let  $V \subseteq F_J$  be a set of formulas and  $R$  be a set of inference rules. We say that  $V$  is *closed with respect to*  $r \in R$  if

$$A_1, \dots, A_n \in V \quad \text{implies} \quad r(A_1, \dots, A_n) \in V \quad (3)$$

for all  $A_1, \dots, A_n \in \text{dom}(r)$ . Then we may define a *syntactic consequence operation* by

$$\mathcal{C}^{\text{syn}}(X) = \bigcap \{V \subseteq F_J \mid X \subseteq V, V \text{ is closed w.r.t. all } r \in R\}. \quad (4)$$

### Theorem 1

For every formula  $A \in F_J$ ,  $A \in \mathcal{C}^{\text{syn}}(X)$  iff  $A$  is provable from  $X$ .

The above concepts are nicely generalized in Pavelka's work outlined below.

## 2.2 Fuzzy logic with evaluated syntax

As mentioned, J. Pavelka allowed in his work evaluation of formulas also in syntax simply by assuming that axioms may not be fully convincing, i.e., not fully true. Consequently, axioms form a fuzzy set. But this means a departure of syntax from the traditional conception. There is a good reason to call the resulting logic a *logic with evaluated syntax*.

The fundamental concept of a formula is in this logic generalized to that of *evaluated formula*  $a/A$  where  $A \in F_J$  is a formula and  $a \in L$  is its syntactic evaluation. Further basic principles are the following.

- (i) The designated truth values are replaced by the *maximality principle*: if the same formula is assigned more truth values then its final truth assignment is equal to the maximum (supremum) of all of them. Besides others, this means that all truth values are equally important.
- (ii) It is possible to make syntactical derivations concerning *any truth value*.

It can be demonstrated that these principles lead to transparent generalization of classical logic both in semantics as well as in syntax.

Let us stress considering fuzzy sets of axioms is not merely a cheap generalization but it reflects the character of our knowledge when dealing with vagueness. This becomes apparent when analyzing the sorites paradox (a typical manifestation of vagueness) since this paradox is solved when considering the evaluated formula  $1 - \varepsilon / (\forall n)(\mathbb{F}\mathbb{N}(n) \Rightarrow \mathbb{F}\mathbb{N}(n + 1))$ ,  $\varepsilon > 0$  as a special axiom where  $\mathbb{F}\mathbb{N}$  is a predicate expressing “ $n$  does not form a heap”<sup>†</sup> (for the details see [11]).

First of all, we must extend  $J$  by a set of logical (truth) constants that are names of all truth values  $\{\mathbf{a} \mid a \in L\}$ . Note that this is a generalization of classical logic where we consider just the logical constants  $\perp, \top$ .

Furthermore, we must extend inference rules to manipulate with evaluated formulas. They assign a new evaluated formula to the given formulas  $a_i/A_i$ ,  $i = 1, \dots, n$  as follows:

$$r : \frac{a_1/A_1, \dots, a_n/A_n}{r^{evl}(a_1, \dots, a_n) / r^{syn}(A_1, \dots, A_n)} \quad (5)$$

where  $r^{syn} : F_J^n \longrightarrow F_J$  is a partial operation on formulas and  $r^{evl} : L^n \longrightarrow L$  is a join-preserving evaluation operation on truth values.

Note that in traditional syntax, we may also speak about evaluated formulas; however, each formula is evaluated either by 1, or by 0 (that is, not written at all). Hence, we naturally introduce the concept of *evaluated formal proof* of an evaluated formula  $a/A$  that is a sequence of evaluated formulas

$$w_A := a_0/A_0, a_1/A_1, \dots, a_n/A_n \quad (:= a/A).$$

The last evaluation  $a_n = a$  is a *value*,  $\text{Val}_T(w_A) = a_n$ , of the proof  $w_A$ .

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<sup>†</sup> Namely, the implication “if  $n$  does not form a heap then  $n + 1$  also does not form it” cannot be fully convincing.

Let  $V \underset{\sim}{\subseteq} F_J$  be a fuzzy set of formulas and  $R$  be a set of inference rules of the form (5). We say that  $V$  is closed with respect to  $r$  if

$$V(r^{syn}(A_1, \dots, A_n)) \geq r^{evl}(V(A_1), \dots, V(A_n))$$

for all formulas  $A_1, \dots, A_n \in \text{dom}(r^{syn})$ . The *fuzzy set of syntactic consequences* of  $X \underset{\sim}{\subseteq} F_J$  is given by

$$\mathcal{C}^{syn}(X)(A) = \bigwedge \{V(A) \mid V \underset{\sim}{\subseteq} F_J, X \leq V \text{ and } V \text{ is closed w.r.t. to all } r \in R\} \quad (6)$$

(cf. (6) with (4)).

## Theorem 2

Let  $X \underset{\sim}{\subseteq} F_J$  be a fuzzy set of formulas. Then

$$\mathcal{C}^{syn}(X)(A) = \bigvee \{\text{Val}(w_A) \mid w_A \text{ is proof of } A \text{ from } X\}.$$

This theorem is apparently a generalization of Theorem 1 which nicely generalizes the concept of provability. Indeed, it is clear that there exist more proofs of the same formula  $A$ . Since each such proof has a value, in concord with the principle (i), we take supremum of them. In traditional syntax, this is the same. However, since the only (non-zero) evaluation is 1, finding one proof is sufficient.

It is natural to call  $\mathcal{C}^{syn}(X)(A)$  a *provability degree* of  $A$  from the fuzzy set  $X$ . A *fuzzy theory*  $T$  is determined by a fuzzy set  $\text{LAX} \cup \text{SAX} \underset{\sim}{\subseteq} F_J$  of logical and special axioms. Then  $a = \mathcal{C}^{syn}(\text{LAX} \cup \text{SAX})(A)$  is a provability degree of  $A \in F_J$  in the fuzzy theory  $T$ . Alternatively, we say that  $A$  is a *theorem in the degree*  $a$  in  $T$  and write

$$T \vdash_a A$$

to stress that this is a generalization of the classical concept. Note that P. Hájek in [9] uses the symbol  $|A|_T$  for the provability degree and takes Theorem 2 as the definition<sup>†)</sup>.

The semantics is straightforward generalization of classical definition. A *model* of  $T$  is a truth evaluation  $\mathcal{M}$  of formulas such that  $\text{SAX}(A) \leq \mathcal{M}(A)$  holds for all  $A \in F_J$ . Then  $A$  is true in a degree  $a$  in  $T$  if

$$a = \bigwedge \{\mathcal{M}(A) \mid \mathcal{M} \models T\}$$

and write again  $T \models_a A$  to stress that this is a generalization of the classical concept.

The following theorem holds both for propositional as well as for predicate case:

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<sup>†)</sup> In fact, he identifies evaluated formulas with formulas  $\mathbf{a} \Rightarrow A$  and does not introduce them explicitly. This is possible since it is useful to introduce a special rule  $r_{LC} : \frac{a/A}{a \rightarrow a/\mathbf{a} \Rightarrow A}$  using which evaluated formulas can be represented by ordinary ones.

**Theorem 3 (completeness)**

For every fuzzy theory  $T$  and every formula  $A \in F_{J(T)}$

$$T \vdash_a A \quad \text{iff} \quad T \models_a A.$$

Note that  $a$  both in  $T \vdash_a A$  as well as in  $T \models_a A$  is uniquely defined and so, both symbols are sound. We prefer them to see that we deal with the nice generalization of classical concept. P. Hájek in [9] writes  $\|A\|_T$  instead of  $T \models_a A$  and then, Theorem 3 is written as  $|A|_T = \|A\|_T$  and called Pavelka-style completeness.

A significant limitation of FL with evaluated syntax is expressed in the following theorem (cf. [24,26]).

**Theorem 4**

*Let Theorem 3 hold for a formal logical system with evaluated syntax. Then the support  $L$  of the residuated lattice of truth values is either finite, or it is isomorphic with  $[0, 1]$  and the the interpretation  $\rightarrow$  of the implication connective is continuous.*

Since the only residuated lattice (up to isomorphism) on  $[0, 1]$  with continuous residuation is Łukasiewicz MV-algebra, fuzzy logic with evaluated syntax is bound to it. Therefore, we denote this logic by  $\text{Ev}_L$ .

Let me comment this result. From one side, this is unpleasant since we are very limited in choosing the structure of truth values. It demonstrates that we cannot go too far with the generalization. There are possibilities how to overcome this limitation by adding a special infinitary inference rule but this is not too convincing. However, do we really need more general structures of truth values? MV-algebra is a beautiful structure since it nontrivially generalizes boolean algebra but keeps most of its important properties. Why we are not satisfied with it and search other possibilities? The uniqueness, on the other hand, together with the fact that a great deal of properties of classical logic are nicely generalized seems to be strong argument for taking  $\text{Ev}_L$  as the real fuzzy logic.

It is also notable that, as P. Hájek has shown in [9],  $\text{Ev}_L$  is representable in Łukasiewicz logic (it is even its conservative extension — cf. [12]). On the other hand, the Łukasiewicz logic is clearly a special case of  $\text{Ev}_L$  (because the provability degree includes also the classical provability). Another objection is that both in Pavelka's work as well as in [24], the logical constants form a continuum. This has been simplified by Hájek in [9] to only rational constants in the Łukasiewicz representation of  $\text{Ev}_L$  and recently also directly in  $\text{Ev}_L$  in [22]. Because of the outstanding position of  $\text{Ev}_L$ , fuzzy set theory developed from this unique formal point of view is presented in [19].



### 3 What is the real fuzzy logic?

Let us now turn to the main question of this paper. As one may expect, however, our answer will not be definite. One point of view is that the real FL should have evaluated both semantics as well as syntax. Then the real fuzzy logic is only  $Ev_L$  and we are done.

Another, more important, point of view is to take as the real fuzzy logic such logic that fulfils the aims discussed in Section 1 in the best way. Then more logics can be found suitable. We may agree with the general principles of P. Cintula and L. Běhounek in [2] trying to define the class of fuzzy logics mathematically. From this point of view, it seems that the fundamental fuzzy logic is MTL. Consequently, we claim that fuzzy logic is any logic that is a schematic extension of MTL. But not all of them can really serve well to fulfil the agenda of FL and to be a good basis for the development of FLb.

The MTL itself seems to be rather weak. Much better for this purpose is IMTL that is, MTL extended by the law of double negation ( $\neg\neg A \Rightarrow A$ ). The latter is a necessary if we want to prove various non-trivial properties.

Another beautiful logic is BL. In a sense, this logic is essential logic for the development of fuzzy set theory (for a successful attempt see [13]) because it covers all continuous t-norms. Unfortunately, the double negation cannot be introduced in BL. When adding it to BL, we obtain Łukasiewicz logic. In this logic, we have almost everything we should claim (and still not to collapse into classical logic). When compared with  $Ev_L$ , however, we see that  $Ev_L$  is stronger and so, better for fulfilling the agenda of FL. It is notable that only  $Ev_L$  is capable at modeling of the concept of intension on the first-order level (this is impossible in traditional syntax) — see the first version in [18] and elsewhere (later also in [24]).

An important operation needed, e.g. for considerations in fuzzy approximation theory (a concept significantly elaborated in the frame of fuzzy logic by I. Perfilieva, cf. [27]) and elsewhere is ordinary product of reals. A logic that uses it as interpretation of strong conjunction is product logic  $\Pi$ . But this logic has some strange properties, is more special than BL but lacks properties present in  $L$  or  $Ev_L$ . A special case of BL is also Gödel logic. Some properties of it are very nice (e.g. it has classical deduction theorem). Unpleasant is its implication that has infinite number of discontinuities; the negation is the same as in  $\Pi$  logic, i.e. it does not preserve double negation.

A very nice possibility is to join everything good from all logics into one — the result is LII logic. This logic has been established by F. Esteva et al in [6] and further elaborated by P. Cintula [3,4]. Its only disadvantage is great complexity because of plethora of connectives. But still, I think that this logic is the best to fulfil the agenda of FL, to be applied in fuzzy approximation (see Perfilieva [28]), and possibly also, to be used as the general frame for the theory of FLb. Whenever possible, of course, we may bind to some of its reducts (e.g. IMTL or  $L$  (possibly  $Ev_L$ )).

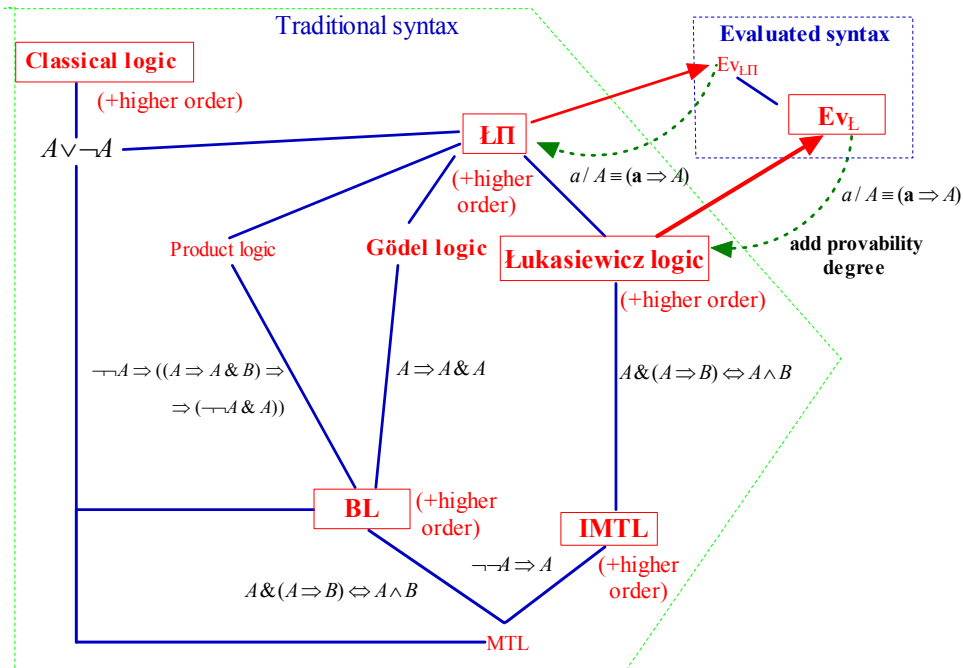


Fig. 1. Scheme of the most relevant fuzzy logics.

For the development of FLb, the most powerful logical system seems to be fuzzy type theory as a higher-order fuzzy logic (see [20]). Among others, it enables us direct generalization of various consideration from classical linguistic (e.g. Montague grammar [15]). An important possibility is also elaboration of fuzzy IF-THEN rules as conditional linguistic statements (see [23]) that heads towards developing sophisticated inference schemes of human reasoning. We should also mention a recent attempt of L. Běhounek and P. Cintula to develop a sound mathematical basis for fuzzy mathematics (see [1]). Because of space limit, we cannot discuss all the raised questions in great details.

At the end, we present a general scheme of fuzzy logics on Figure 1 that is a modification of that from [7]. In the scheme, the most distinguished fuzzy logics discussed above are emphasized. Classical logic is also included as a limit crisp case of FL. Some of the intermediate logics are omitted. In the scheme, axioms that must be added to get to another logical system are (in most cases) depicted. The scheme depicts also the shift from  $\mathbf{L}$  to evaluated syntax, and in a sense, also similar shift from  $\mathbf{LII}$  (this requires the mentioned infinitary rule). The dotted arrows back express representability in the corresponding logic without evaluated syntax. Existence of higher order fuzzy logic (fuzzy type theory) for each of the emphasized logics (including, of course, classical one) is also depicted; such logic is ready for extension to FLb.

This discussion picks up some of fuzzy logics that, according to their properties and strength, seem to fit best the aims put on FL in general. Of course, future will only prove, whether is this picture indeed relevant (and complete? ).

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