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Algebraic Analysis of Fuzzy Systems [1]

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Abstract

In this paper, we have developed an algebraic theory, suitable for the analysis of fuzzy systems. We have used the notions of semiring and semimodule, introduced the notion of semilinear space and gave numerous examples of them and defined also the notions of linear dependence and independence. Then, we have shown that the composition operation, which plays an essential role in the analysis of fuzzy systems because of its role in the compositional rule of inference, can be interpreted as a homomorphism between special semimodules. Consequently, this operation is, in a certain sense, a linear operation. This property formally explains why fuzzy systems are attractive for the applications.

Key words: fuzzy logic, MV-algebra, semimodule, semilinear space, semiring, fuzzy systems

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1 Introduction

In performing complex tasks or in modeling complex systems or complex decision situations, we very often recur to rules that have the form of a conditional statements

$$\mathcal{R} = \ \textsf{IF } x \textsf{ is } A \textsf{ THEN } x \textsf{ is } B. \tag{1}$$

The part before THEN is called *antecedent* (a conditional part) and the part after it is called *consequent* (conclusion or action part). The antecedent usually describes some input characteristics (independent parameters) while consequent describes the output (dependent) ones. Using rules, we can formalize rule based systems that can be taken as a special case of knowledge based systems. In practice, however, we very often face a problem that the phenomena we want to describe are known only imprecisely, or are even hereditarily vague. Then, another formalism is necessary to cope with such a situation. Namely, fuzzy logic is a reasonable tool which proved its usefulness in many real world applications. In this case, A and B in the rules of the form (1) are some vague expressions (e.g. fuzzy numbers) and therefore, (1) are called *fuzzy IF-THEN rules*.

A crucial question is how fuzzy IF-THEN rules can be mathematically modeled. In fuzzy logic theory, two essential approaches can be recognized. The *first* one, originally proposed by L. A. Zadeh [24, 25] and elaborated further by many authors (cf., e.g. [6, 12, 13]), is *algebraic*. In this case, the antecedent of a fuzzy rule is represented by a fuzzy set $A \underset{\sim}{\subset} X$ and the consequent by a fuzzy set $B \underset{\sim}{\subset} Y$ where X, Y are some sets. Each rule $\mathcal{R}$ expresses a relation between elements from X and Y and so, it is modeled by a fuzzy relation $R \underset{\sim}{\subset} X \times Y$ that is usually constructed using the fuzzy sets A and B. If more rules $\mathcal{R}_i$, $i \in I$, are considered then the corresponding fuzzy relations R_i are joined (usually using the union operation) into one fuzzy relation R which represents a *model* of the given system and which is often called a *fuzzy system*.

The *second* model of fuzzy rules (1) is based on *formal logic*, possibly extended even by linguistic aspect. This approach has been elaborated in some special formal fuzzy logic systems, for example, by G. Gerla in [10] or in BL-fuzzy logic by P. Hájek [14] without linguistics and with linguistic considerations in fuzzy logic with evaluated syntax by V. Novák et al. [18], or newly in fuzzy type theory by V. Novák and S. Lehmke [17].

In this paper, however, we will consider only the algebraic model of the fuzzy IF-THEN rules. To apply the fuzzy system R, let us consider a situation that a new knowledge is given by a fuzzy set $A' \underset{\sim}{\subset} X$. Then a conclusion based on A' and R is a new fuzzy set $B' \underset{\sim}{\subset} Y$ computed using the equality

$$B'(y) = (A' \circ R)(y) = \bigvee_{x \in X} (A'(x) \odot R(x, y)), \qquad y \in Y \tag{2}$$

where $\odot$ is a suitable operation in the chosen algebra of membership degrees (usually a t-norm, or more generally, a product operation in a residuated lattice). Equation (2) has been called by L. A. Zadeh (see [24, 25]) a *compositional rule of inference* because of its close relation to the modus ponens inference rule in classical logic.

From the algebraic point of view, compositional rule of inference is a composition of a fuzzy set A' and a fuzzy relation R. Note that for finite universes X, Y, equation (2) reduces to a matrix product analogous to the classical matrix product where the summation is replaced by the supremum, and the product by the operation $\odot$.

Our aim in this paper is to clarify the algebraic motivation supporting the choice of (2). Although fuzzy logic provides a basis for the approximate description of different dependencies, including, of course, nonlinear ones, we will see how the above formula can be understood as a model of a fuzzy function. Furthermore, a very important concept in classical system theory is that of linearity since linear models demonstrate the best (most transparent) behavior. Unfortunately, fuzzy systems are from this point of view non-linear. Therefore, a surprising and important conclusion of this paper is that fuzzy systems behave in a way similar to that of classically linear systems. We are convinced that this fact well justifies the use of fuzzy systems and helps in explanation why they behave so well in applications.

We will use the machinery of semimodules theory in the MV-algebraic setting. In this context, linearity means a natural morphism between algebraic structures of same type. In fact, we will show that using the semimodule structure living in the MV-algebraic setting, some dependencies *prima facie* nonlinear, i.e., nonlinear in the sense of classical algebra, become linear in the MV-semimodule framework. Let us remark that an analogous phenomenon can be found also in Maslov's idempotent analysis, see [15].

The paper is organized as follows. In Section 2, we overview the basic concepts of MV-algebras, fuzzy sets and fuzzy systems. Section 3 is devoted to the concept of semirings, semimodules and semilinear spaces. We will show that these concepts naturally generalize the concepts of module and linear space where the latter is crucial for characterization of classical linearity. These concepts are further used in Section 4 for elaboration of the concept of MV-semimodule homomorphism. Finally, the latter is applied in Section 5 to show that behavior of fuzzy systems can be described as a homomorphism between semimodules and so that it is semilinear (close to the classical linearity).

2 Preliminaries

2.1 MV-algebras

A crucial algebraic structure which stands in the basis of the development of fuzzy logic as a formal system is that of *residuated lattice* (see, e.g. [10, 14, 18]). It is an algebraic structure of type $(2, 2, 2, 2, 0, 0)$

$$\mathcal{L} = \langle L, \vee, \wedge, \odot, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{3}$$

where $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$. The operation $\odot$ is a multiplication operation such that $\langle L, \odot, \mathbf{1} \rangle$ is a commutative monoid and $\rightarrow$ is a residuation operation fulfilling the adjunction property

$$x \odot y \leq z \quad \text{iff} \quad x \leq y \rightarrow z, \qquad x, y, z \in L. \tag{4}$$

An MV-algebra (see [2, 3]) is an algebraic structure of type $(2, 2, 1, 0, 0)$

$$\mathcal{L} = \langle L, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1} \rangle \tag{5}$$

satisfying the following axioms:

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(ii) $x \oplus y = y \oplus x$,
(iii) $x \oplus 0 = x$,
(iv) $\neg\neg x = x$,
(v) $x \oplus \mathbf{1} = \mathbf{1}$,
(vi) $\neg \mathbf{0} = \mathbf{1}$,
(vii) $x \odot y = \neg(\neg x \oplus \neg y)$,
(viii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

for all $x, y \in L$.

Let $\mathcal{L}$ be an MV-algebra and put

$$\begin{aligned} x \vee y &= (x \odot \neg y) \oplus y, \\ x \wedge y &= (x \oplus \neg y) \odot y, \qquad x, y \in L. \end{aligned}$$

Then, as proved in [3], $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a bounded distributive lattice and we will refer to it as the *reduct lattice* of the MV-algebra $\mathcal{L}$. As usual, we will define

$$x \leq y \quad \text{iff} \quad x \wedge y = x.$$

We say that an MV-algebra is *complete*, if the reduct lattice $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is complete.

Proposition 1
Let $\mathcal{L}$ be an MV-algebra (5) and put

$$x \rightarrow y = \neg x \oplus y \tag{6}$$

for all $x, y \in L$. Then

$$\mathcal{L} = \langle L, \vee, \wedge, \odot, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

is a residuated lattice.

MV-algebras originated from an algebraic analysis of Łukasiewicz many-valued logic done by C. C. Chang in [2]. They are non-idempotent generalizations of Boolean algebras. Indeed, Boolean algebras are just the MV-algebras obeying the additional equation $x \oplus x = x$.

Let $B(L) = \{x \in L \mid x \oplus x = x\}$ be the set of all idempotent elements of L. Then, $\mathcal{B}_L = \langle B(L), \oplus, \odot, \mathbf{0}, \mathbf{1}\rangle$ is a subalgebra of $\mathcal{L}$, which is also a Boolean algebra. Moreover, it is the greatest Boolean subalgebra of $\mathcal{L}$.

MV-algebras have been studied with respect to their relationship to other parts of mathematics, as well as to various structures. D. Mundici in [16] proved that there exists an equivalence functor Γ between the category of MV-algebras and the category of lattice-ordered abelian groups (abelian ℓ -groups) with strong unit. For every abelian ℓ-group with strong unit $\langle G, +, 0, u\rangle$ the functor Γ equips the unit interval $[0, u]$ with the following operations:

(i) $\mathbf{0} = 0$,
(ii) $\mathbf{1} = u$,
(iii) $x \oplus y = u \wedge (x + y)$,
(iv) $x \odot y = 0 \vee (x + y - u)$,
(v) $\neg x = u - x$.

It is easy to see that the resulting structure $\mathcal{L} = \langle [0, u], \oplus, \odot, \neg, 0, u\rangle$ is an MV-algebra.

Let $L = [0, 1]$ (interval of real numbers) and define the MV-operations on $[0, 1]$ as follows:

(i) $\mathbf{0} = 0$,
(ii) $\mathbf{1} = 1$,
(iii) $x \oplus y = \min\{1, x + y\}$,
(iv) $x \odot y = \max\{0, x + y - 1\}$
(v) $\neg x = 1 - x$.

Then the resulting structure $\mathcal{L}_{Ł} = \langle [0, 1], \oplus, \odot, \neg, 0, 1\rangle$ is MV-algebra that is called *Łukasiewicz MV-algebra*. It can be proved that the variety $\mathcal{MV}$ of MV-algebras coincides with the variety $HSP([0, 1])$, generated by the MV-algebra $\Gamma(\mathbb{R}, +, 0, 1)$, where $(\mathbb{R}, +, 0)$ is the totally ordered group of the real numbers (i.e. by Łukasiewicz MV-algebra). More details about MV-algebras can be found in the book [3].

2.2 Fuzzy sets and fuzzy systems

Let X be an arbitrary set called a *universe* and $\mathcal{L}$ be a residuated lattice (3). A *fuzzy set* A is identified with the function

$$A : X \longrightarrow L. \tag{7}$$

The function (7) is also called the *membership function* of the fuzzy set A (i.e. fuzzy set is identified with its membership function). We will occasionally use the symbol $A \underset{\sim}{\subset} X$ to express that A is a fuzzy set in the universe X. The set of all fuzzy sets is the set L^X of all functions (7). If $L = [0, 1]$ or it is known from the context then we will simply write $\mathcal{F}(X)$.

Operations on fuzzy sets are defined pointwise using the operations from the residuated lattice $\mathcal{L}$. This means that for arbitrary fuzzy sets A, B we put $(A \cup B)(x) = A(x) \vee B(x)$ and similarly for the other operations. As a special case, we may take $\mathcal{L}$ to be an MV-algebra and define pointwise the operations $\oplus$, $\odot$, $\neg$ on fuzzy sets. Then

$$\langle L^X, \oplus, \odot, \neg, \mathbf{0}_X, \mathbf{1}_X \rangle \tag{8}$$

is an MV-algebra of all fuzzy sets on X. The zero $\mathbf{0}_X$ and unit $\mathbf{1}_X$ elements of this algebra are functions identically equal to $\mathbf{0}$ and $\mathbf{1}$, respectively.

We may define also ordering of fuzzy sets by

$$A \leq B \quad \text{iff} \quad A(x) \leq B(x)$$

for all $A, B \underset{\sim}{\subset} X$ and $x \in X$.

Let $r \in L$, $A \in L^X$ be a fuzzy set and let $\circ$ denote either $\odot$ or $\oplus$. The external operation rA is defined by

$$(rA)(x) = r \circ A(x)$$

for all $x \in X$.

If $X \times Y$ is a cartesian product of universes then the fuzzy set $R \underset{\sim}{\subset} X \times Y$ is a *fuzzy relation*. We will write $R(x, y)$ instead of $R(\langle x, y \rangle)$. A special fuzzy relation is the cartesian $\odot$-product of fuzzy sets $A \underset{\sim}{\subset} X$ and $B \underset{\sim}{\subset} Y$ defined by

$$(A \times_{\odot} B)(x, y) = A(x) \odot B(y).$$

The *domain* and *codomain* (range) of a fuzzy relation R are fuzzy sets given by the membership functions

$$\begin{aligned} \operatorname{dom}(R)(x) &= \bigvee_{y \in Y} R(x, y), \\ \operatorname{rng}(R)(y) &= \bigvee_{x \in X} R(x, y). \end{aligned}$$

Let $R \underset{\sim}{\subset} X \times Y$, $S \underset{\sim}{\subset} Y \times Z$ be two fuzzy relations. Then their *composition* is a fuzzy relation $R \circ S \underset{\sim}{\subset} X \times Z$ defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} (R(x, y) \odot S(y, z)). \tag{9}$$

Let $A \underset{\sim}{\subset} X$ be a fuzzy set and $R \underset{\sim}{\subset} X \times Y$ a fuzzy relation. A relevant role in the theory of fuzzy systems is played by an *image* of A under the fuzzy relation R, that is, a fuzzy set defined by

$$B(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)). \tag{10}$$

Note (10) can be also seen as a special case of a composition (9) of a fuzzy set and a fuzzy relation. Clearly, (10) coincides with (2) and it is a principal formula characterizing behavior of fuzzy systems constructed using fuzzy IF-THEN rules (cf. Examples 3 and 4 in Section 5).

Another view on (10) is to understand it as a function g assigning every element $A \in \mathcal{F}(X)$ an element $g(A) = B \in \mathcal{F}(Y)$ so that B is defined by the equation (10) (see Perfilieva [19]). Let us stress that this definition can be understood as a generalization of the classical concept of a function. Indeed, let g be a function from a finite set $X = \{x_1, \ldots, x_n\}$ to a finite set $Y = \{y_1, \ldots, y_m\}$ and let $G = \{(x, g(x)) \mid x \in X\} \subset X \times Y$ be the graph of g. Obviously, $(x, y) \in G$ iff $y = g(x)$. Hence the graph of g is a (boolean) binary relation defined by

$$I(x, y) = 1 \quad \text{iff} \quad y = g(x).$$

Thus, by identifying every $x_i \in X$ and every $g(x_i) \in Y$ with their characteristic functions χ_{x_i} and $\chi_{g(x_i)}$, respectively, we obtain

$$\chi_{g(x_i)}(y) = \bigvee_{x \in X} (\chi_{x_i}(x) \circ I(x, y))$$

which is a special case of (10).

3 Semirings, semimodules and semilinear spaces

In this section we will further develop special algebraic means useful for analysis of fuzzy systems. Namely we will concentrate on the concepts of a semimodule and a semilinear space.

3.1 Semirings and semimodules

Definition 1
A semiring $\mathcal{R} = \langle R, +, \cdot, \mathbf{0}_R, \mathbf{1}_R \rangle$ is an algebraic structure (cf. [4, 11]) such that:

(i) $\langle R, +, \mathbf{0}_R \rangle$ *is a commutative monoid.*
(ii) $\langle R, \cdot, \mathbf{1}_R \rangle$ *is a monoid.*
(iii) $r \cdot (s + t) = r \cdot s + r \cdot t$ *holds for all* $r, s, t \in R$.
(iv) $\mathbf{0}_R \cdot r = r \cdot \mathbf{0}_R = \mathbf{0}_R$ *holds for all* $r \in R$.

A semiring is called *commutative* if $\langle R, \cdot, \mathbf{1}_R \rangle$ is a commutative monoid.

Proposition 2 ([4])
Let $\mathcal{L}$ be an MV-algebra. Then the reducts $\mathcal{L}^{\vee} = \langle L, \vee, \odot, \mathbf{0}, \mathbf{1} \rangle$ and $\mathcal{L}^{\wedge} = \langle L, \wedge, \oplus, \mathbf{1}, \mathbf{0} \rangle$ are commutative semirings.

PROOF: This follows from the definition. □

The following definition of a semimodule is taken from J. S. Golan [11].

Definition 2
Let $\mathcal{R} = \langle R, +, \cdot, \mathbf{0}_R, \mathbf{1}_R \rangle$ be a semiring. A left $\mathcal{R}$-semimodule is a commutative monoid $\mathcal{A} = \langle A, +_A, \mathbf{0}_A \rangle$ for which an external multiplication $R \times A \longrightarrow A$ denoted by ra is defined with the following properties: all $r, r' \in R$ and $a, a' \in A$ satisfiy the equalities

(i) $(r \cdot r')a = r(r'a)$,
(ii) $r(a +_A a') = ra +_A ra'$,
(iii) $(r + r')a = ra +_A r'a$,
(iv) $\mathbf{1}_R\, a = a$,
(v) $\mathbf{0}_R\, a = r\, \mathbf{0}_A = \mathbf{0}_A$.

The definition of *right* $\mathcal{R}$-semimodule is analogous, where the external multiplication is defined as a function $A \times R \longrightarrow A$. An $\mathcal{R}$-*bisemimodule* is both right as well as left $\mathcal{R}$-semimodule, i.e. it satisfies the equality $(ra)r' = r(ar')$. A nonempty subset N of a left $\mathcal{R}$-semimodule M is called a *subsemimodule* if N is closed under addition and external multiplication.

Example 1
(a) *Let $\mathcal{L} = \langle L, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1} \rangle$ and $\mathcal{B} = \langle B, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1} \rangle$ be MV-algebras, $\mathcal{L}^{\vee} = \langle L, \vee, \odot, \mathbf{0}, \mathbf{1} \rangle$ the semiring reduct of $\mathcal{L}$ and $\langle B, \vee, \mathbf{0} \rangle$ the sup monoid reduct of $\mathcal{B}$. Moreover, let $h : L \longrightarrow B$ be an MV-homomorphism and define the external multiplication by*

$$pb = h(p) \odot b$$

for all $p \in L$ and $b \in B$. Then $\langle B, \vee, \mathbf{0} \rangle$ is an $\mathcal{L}^{\vee}$-semimodule.

(b) *Let $\mathcal{L}$ be an MV-algebra and $\mathcal{L}^{\vee}$ its semiring reduct. Put $\mathbf{A} = L^n$ to be the set of all n-dimensional vectors for some $n \geq 1$ and define*

$$(a_1, \ldots, a_n) +_A (b_1, \ldots, b_n) = (a_1 \vee b_1, \ldots a_n \vee b_n),$$
$$p(a_1, \ldots, a_n) = (p \odot a_1, \ldots, p \odot a_n)$$

where $a_i, b_i \in L, i = 1, \ldots, n$ and $p \in L$. Then $\mathcal{A} = \langle \mathbf{A}, +_A, \mathbf{0}_A \rangle$ is an $\mathcal{L}^{\vee}$-semimodule where the zero (neutral) element is the vector $\mathbf{0}_A = (0, \ldots, 0)$.

(c) *Let $X \neq \emptyset$, $\mathcal{L}$ be an MV-algebra and $\mathcal{L}^{\vee} = \langle L, \vee, \odot, \mathbf{0}, \mathbf{1}\rangle$ its semiring reduct. Put $A = L^X = \{f \mid f : X \longrightarrow L\}$ and for all $f, g \in A$ define*

$$f(x) +_A g(x) = f(x) \vee g(x),$$
$$p\, f(x) = p \odot f(x), \qquad x \in X, p \in L.$$

The zero element $\mathbf{0}_A$ is the function $\mathbf{0}_A : x \mapsto \mathbf{0}$. Then $\mathcal{A}_X = \langle A, +_A, \mathbf{0}_A\rangle$ is an $\mathcal{L}^{\vee}$-semimodule.

(d) *Let $\mathcal{L} = \langle L, \vee, \wedge, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1}\rangle$ be an MV-algebra, $\mathcal{L}^{\wedge} = \langle L, \wedge, \oplus, \mathbf{1}, \mathbf{0}\rangle$ its semiring reduct. Put $\mathbf{A} = L^n$ to be a set of all n-dimensional vectors for some $n \geq 1$ and*

$$(a_1, \ldots, a_n) +_A (b_1, \ldots, b_n) = (a_1 \wedge b_1, \ldots a_n \wedge b_n),$$
$$p(a_1, \ldots, a_n) = (p \oplus a_1, \ldots, p \oplus a_n),$$

where $a_i, b_i \in L, i = 1, \ldots, n$ and $p \in L$. Then $\mathcal{A} = \langle \mathbf{A}, +_A, \mathbf{0}_A\rangle$ is an $\mathcal{L}^{\wedge}$-semimodule where the zero element is the vector $\mathbf{0}_A = (1, \ldots, 1)$.

(e) *Let $X \neq \emptyset$, $\mathcal{L}$ be an MV-algebra and $\mathcal{L}^{\wedge} = \langle L, \wedge, \oplus, \mathbf{1}, \mathbf{0}\rangle$ its semiring reduct. Put $A = L^X = \{f \mid f : X \longrightarrow L\}$ and for all $f, g \in A$ define*

$$f(x) +_A g(x) = f(x) \wedge g(x),$$
$$p\, f(x) = p \oplus f(x), \qquad x \in X, p \in L.$$

The zero element $\mathbf{0}_A$ is the function $\mathbf{0}_A : x \mapsto \mathbf{1}$. Then $\mathcal{A}_X = \langle A, +_A, \mathbf{0}_A\rangle$ is an $\mathcal{L}^{\wedge}$-semimodule.

(f) *Let $\mathcal{L} = \langle L, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1}\rangle$ be an MV-algebra and $\mathcal{L}^{\wedge} = \langle L, \wedge, \oplus, \mathbf{1}, \mathbf{0}\rangle$ its reduct semiring. Furthermore, let X be a non-empty set and put $A = L^X = \{f \mid f : X \longrightarrow L\}$. Let us now define*

$$f(x) +_A g(x) = f(x) \vee g(x),$$
$$p\, f(x) = \neg p \odot f(x), \qquad x \in X, p \in L.$$

where $f, g \in A$.

The zero element $\mathbf{0}_A$ is the function $\mathbf{0}_A : x \mapsto \mathbf{0}$. Then the monoid $\mathcal{A}_X = \langle A, +_A, \mathbf{0}_A\rangle$ is an $\mathcal{L}^{\wedge}$-semimodule.

We will prove the last claim by verification of the properties (i)–(v) of Definition 2. Let $p, q \in L$ and $f, g \in L^X$. Then

(i) $(pq)f = \neg(p \oplus q) \odot f = \neg p \odot (\neg q \odot f) = p(\neg q \odot f) = p(qf)$,
(ii) $p(f +_A g) = \neg p \odot (f \vee g) = (\neg p \odot f) \vee (\neg p \odot g) = pf +_A pg$,
(iii) $(p \wedge q)f = \neg(p \wedge q) \odot f = (\neg p \vee \neg q) \odot f = (\neg p \odot f) \vee (\neg q \odot f) = pf +_A qf$,
(iv) $\mathbf{1}\, f = \neg\mathbf{0} \odot f = f$,
(v) $\mathbf{0}f = \neg\mathbf{1} \odot f = \mathbf{0}_A = \neg p \odot \mathbf{0}_A = p\, \mathbf{0}_A$.

Examples (c), (e) and (f) demonstrate that it is possible to introduce a semimodule structure on the set L^X of all fuzzy sets on X. Therefore, any of the resulting semimodules will be called an *MV-semimodule associated with $\mathcal{L}$*. More precisely, the semimodules (c) and (f) will be called MV-sup-semimodules and denoted by $\mathcal{A}^{\vee}_X = \langle L^X, \vee, \mathbf{0}_X\rangle$. Similarly, the semimodule (e) will be called MV-inf-semimodule and denoted by $\mathcal{A}_X^\wedge = \langle L^X, \wedge, \mathbf{1}_X \rangle$. Note that due to the commutativity of the operations on MV-algebra, all these semimodules are at the same time bisemimodules.

3.2 Semilinear spaces

A central role in mathematics and its applications is played by the concept of a linear space. This is a special case of a module which has been introduced to characterize the linearity property in an abstract way. Since the concept of semimodule is weakening of the concept of module, we may introduce *semilinear space* as a semimodule over an MV-algebra $\mathcal{L} = \langle L, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1} \rangle$.

From the point of view applications, these spaces may be suitable for solving semilinear equations and systems of semilinear equations with fuzzy coefficients. Moreover, as shown in this paper, they play also important role in the demonstration that the fuzzy systems can be taken as semilinear and so, they serve as a powerful tool useful in applications. The concepts semilinear space, linear dependence and basis, as well as some results of this section are taken from [20].

To define a linear space, it is necessary to have at disposal a commutative group accomplished with the inverse operation with respect to the main operation of addition. In the case of a semiring (and not a group), we may consider the operation of residuum $\rightarrow$ (introduced by (6)) in order have the semi-inverse operation with respect to $\odot$. Therefore we may state the following definition.

Definition 3
Let semiring $\mathcal{R}$ be a reduct of an MV-algebra $\mathcal{L} = \langle L, \oplus, \odot, \rightarrow, \neg, \mathbf{0}, \mathbf{1} \rangle$ extended by the residuation operation $\rightarrow$ (6). Then a semimodule over $\mathcal{L}$ is called a semilinear space.

Elements of a semilinear space will be called *vectors* and elements of an MV-algebra *scalars*. The former will be denoted by bold letters to distinguish them from scalars. Examples of semilinear space are in Example 1.

3.3 Linear dependence and independence

The notions of linear dependence and independence can be extended also to semilinear spaces. However, some properties and criteria become easier if we confine to semilinear spaces with linearly ordered semirings. Therefore, we will formulate all basic results for both cases in parallel.

Let $\mathcal{L}$ be an MV-algebra and $\mathcal{A} = \langle A, +_A, \mathbf{0}_A \rangle$ be some of its semilinear spaces. By a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in A$ we mean the expression

$$\alpha_1 \cdot \mathbf{a}_1 + \cdots + \alpha_n \cdot \mathbf{a}_n$$

where $\alpha_1, \ldots, \alpha_n \in R$ are scalars (called also coefficients). Each linear combination uniquely determines a certain vector from A.

Definition 4
A single vector $\mathbf{a}$ is linearly independent. Vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $n \geq 2$, are linearly independent if none of them can be represented by a linear combination of the others. Otherwise, we say that vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly dependent. An infinite set of vectors is linearly independent if any finite subset of it is linearly independent.

Example 2

1. *Let $\mathcal{L}$ be an MV-algebra and $\mathcal{A}$ the $\mathcal{L}^{\vee}$-semilinear space of n-dimensional vectors $A = L^n$. Then the following vectors are linearly independent:*

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0) \\ &\ldots\ldots\ldots\ldots\ldots \\ \mathbf{e}_n &= (0, 0, 0, \ldots, 1) \end{aligned} \tag{11}$$

2. *Let $\mathcal{L}$ be an MV-algebra and A the $\mathcal{L}^{\wedge}$-semilinear space of n-dimensional vectors $A = L^n$. Then the following vectors are linearly independent:*

$$\begin{aligned} \mathbf{f}_1 &= (0, 1, 1, \ldots, 1) \\ \mathbf{f}_2 &= (1, 0, 1, \ldots, 1) \\ &\ldots\ldots\ldots\ldots\ldots \\ \mathbf{f}_n &= (1, 1, 1, \ldots, 0) \end{aligned}$$

In the rest of this section, we will confine ourselves to the semilinear space L^n over $\mathcal{L}^{\vee}$ where $\mathcal{L}$ is an MV-algebra.

The following theorem characterizes coefficients of a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in L^n$ representing a vector $\mathbf{b}$ (provided that the latter is expressible by at least one of such combinations).

Theorem 1
Let $A = L^n$ be the $\mathcal{L}^{\vee}$-semilinear space of n-dimensional vectors where $\mathcal{L}$ is an MV-algebra. Let the vector $\mathbf{b} \in L^n$ be represented by a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in L^n$. Then $\mathbf{b}$ can be represented by the linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_m$ with coefficients

$$\hat{x}_i = \bigwedge_{j=1}^{n} (a_{ij} \rightarrow b_j), \quad i = 1, \ldots, m. \tag{12}$$

PROOF: Let us form the $(m \times n)$ matrix $\mathbf{A}$ from components of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{13}$$

By the assumption, the vector $\mathbf{b} \in L^n$ can be represented by

$$\mathbf{b} = x_1 \odot \mathbf{a}_1 \vee \cdots \vee x_m \odot \mathbf{a}_m$$

with coefficients constituting the vector $\mathbf{x} = (x_1, \dots, x_m)$. Then we can rewrite the last equality in the matrix form

$$\mathbf{b} = \mathbf{x} \circ \mathbf{A}$$

where $\circ$ stands for the sup $-\odot$ composition.

From the theory of fuzzy relation equations and their solvability (see e.g. [5, 23]) it follows that the vector $\hat{\mathbf{x}}$ with components

$$\hat{x}_i = \bigwedge_{j=1}^{n} (a_{ij} \to b_j),$$

$i = 1, \dots, m$, is the greatest solution of the matrix equality. □

It is worth noticing that if a vector $\mathbf{b} \in L^n$ can be represented by a linear combination of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ then the representation is not necessarily unique.

Corollary 1
Let the conditions of Theorem 1 be fulfilled and $\mathbf{b} = (b_1, \dots, b_n) \in L^n$ *be a vector represented by a linear combination of* $\mathbf{a}_1, \dots, \mathbf{a}_m$. *Then for each* $j = 1, \dots, n$

$$b_j \le a_{1j} \vee \cdots \vee a_{mj}. \tag{14}$$

PROOF: By the assumption, the vector $\mathbf{b}$ can be represented by a linear combination of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$:

$$b_j = x_1 \odot a_{1j} \vee \cdots \vee x_m \odot a_{mj}.$$

where $j = 1, \dots n$. On the basis of the inequality $x \odot a \le a$ we easily come to the conclusion that

$$b_j \le \bigvee_{i=1}^{m} a_{ij}, \quad j = 1, \dots n.$$

□

Corollary 2
Let $A = L^n$ be the $\mathcal{L}^\vee$-semilinear space of n-dimensional vectors where $\mathcal{L}$ is an MV-algebra. Then the zero vector $\mathbf{0} = (0, \dots, 0) \in L^n$ is representable by the linear combination of arbitrary vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ with the respective coefficients

$$\hat{x}_i = \bigwedge_{j=1}^{n} \neg a_{ij}, \quad i = 1, \dots, m. \tag{15}$$

PROOF: The statement easily follows from Theorem 1, formula (12) and is substantiated by the fact that the zero vector is always representable by the linear combination with zero coefficients and the fact that $\neg a = a \rightarrow 0$. □

Remark 1
We refer again to the theory of systems of fuzzy relation equations (cf. [5, 23]) and can state even more: coefficients (15) are the largest among all those which give zero linear combination of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$.

By the criterion suggested below, it is possible to investigate whether the given system of vectors is linearly independent.

Theorem 2
Let $A = L^n$ be the $\mathcal{L}^\vee$-semilinear space of n-dimensional vectors where $\mathcal{L}$ is an MV-algebra. Vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$, $m \geq 2$, are linearly independent if and only if

$$(\forall l \in \{1, \dots, m\})(\exists i \in \{1, \dots, n\}) \left(a_{li} \not\leq \bigvee_{j=1, j \neq l}^{m} a_{ji} \odot \left(\bigwedge_{k=1}^{n} a_{jk} \rightarrow a_{lk} \right) \right). \tag{16}$$

PROOF: Suppose that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ are linearly dependent. Then at least one of them is representable by a linear combination of the others. Therefore, there exists a vector of coefficients $\mathbf{x} = (x_1, \dots, x_m)$ such that

$$(\exists l \in \{1, \dots, m\})(\forall i \in \{1, \dots, n\}) \left(a_{li} = \bigvee_{j=1, j \neq l}^{m} x_j \odot a_{ji} \right). \tag{17}$$

Let us fix the value l for which (17) holds. Analogously to the proof of Theorem 1, the vector $\hat{\mathbf{x}}$ with components

$$\hat{x}_j = \bigwedge_{i=1}^{n} (a_{ji} \rightarrow a_{li}), \quad j = 1, \dots, m, j \neq l,$$

fulfils the equality (17), i.e.

$$(\exists l \in \{1, \dots, m\})(\forall i \in \{1, \dots, n\}) \left(a_{li} = \bigvee_{j=1, j \neq l}^{m} \hat{x}_j \odot a_{ji} \right). \tag{18}$$

It is easy to verify that the following inequality holds in general:

$$\bigvee_{j=1,j\neq l}^{m} \hat{x}_j \odot a_{ji} = \bigvee_{j=1,j\neq l}^{m} \left(\left(\bigwedge_{k=1}^{n} (a_{jk} \to a_{lk}) \right) \odot a_{ji} \right) \leq a_{li}.$$

Therefore, (18) is equivalent to the inequality

$$a_{li} \leq \bigvee_{j=1,j\neq l}^{m} \left(\left(\bigwedge_{k=1}^{n} (a_{jk} \to a_{lk}) \right) \odot a_{ji} \right). \tag{19}$$

Summarizing the above reasoning, we come to the conclusion that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ are linearly dependent if and only if (19) holds true. This implies that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ are linearly independent if and only if the negation of (19) holds true. It remains to realize that the negation of (19) is precisely the condition (16). □

Corollary 3
Let $A = [0,1]^n$ be the $\mathcal{L}^\vee$-semilinear space of n-dimensional vectors and $\mathcal{L}$ be an MV-algebra on $[0,1]$. Vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ are linearly independent if and only if

$$(\forall l \in \{1, \dots, m\})(\exists i \in \{1, \dots, n\}) \left(a_{li} > \bigvee_{j=1,j\neq l}^{m} a_{ji} \odot \left(\bigwedge_{k=1}^{n} a_{jk} \to a_{lk} \right) \right). \tag{20}$$

Remark 2

(1) Theorems 1, 2 and their Corollaries 1 and 3 are valid also in the case of a $\mathcal{L}^\vee$-semilinear space of L-valued functions $A = L^X$ where $\mathcal{L}$ is an MV-algebra.

(2) Let us remind that in the case of a linear space, we distinguish linearly dependent and linearly independent vectors by analyzing coefficients of their linear combinations leading to zero vectors. As we will see below, this characterization is unhelpful in the case of semilinear spaces where we care about the expressibility property. To exemplify this claim, let us take the reduct $\mathcal{L}^\vee$ of Łukasiewicz algebra on $[0,1]$ and for $a \in (0,1)$ consider the following set of linearly independent vectors from $[0,1]^n$:

$$\begin{aligned} \mathbf{a}_1 &= (a, 0, 0, \dots, 0) \\ \mathbf{a}_2 &= (0, a, 0, \dots, 0) \\ &\dots\dots\dots\dots \\ \mathbf{a}_n &= (0, 0, 0, \dots, a). \end{aligned} \tag{21}$$

It is easy to see that the linear combination

$$\neg a \odot \mathbf{a}_1 \vee \dots \vee \neg a \odot \mathbf{a}_n = \mathbf{0}$$

with non-zero coefficients $\neg a$ gives the zero vector.

On the other hand, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_1 + \mathbf{a}_2$ are linearly dependent, and again, their linear combination with all coefficients equal to $\neg a$ gives the zero vector. Therefore, independently on the fact whether the vectors are linearly dependent or not (in the sense of our definition), their linear combination with non-zero coefficients *may be* equal to the zero vector. *Note that this may happen if at least one of the coefficients given by (15) is non-zero.*

3.4 Basis in a semilinear space

In this subsection, we will show that the concept of basis known from the theory of linear spaces can be introduced in semilinear ones. However, we must be careful since not all properties are transferred in a straightforward way.

Let us fix an MV-algebra $\mathcal{L}$ and consider the $\mathcal{L}^{\vee}$-semilinear space of n-dimensional vectors $A = L^n$.

Definition 5
A linear independent set of generators of a semi-linear space A is called a basis of A.

An example of a basis in L^n is given by vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ in Example 2 (cf. (11)). It immediately follows from the last definition that an arbitrary element from a semi-linear space L^n can be represented by a linear combination of elements of its basis.

We investigate conditions assuming or guaranteeing that vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in L^n$ form a basis. Let us denote

$$\mathbf{a}_i = (a_{i1}, \dots, a_{in}), \qquad i = 1, \dots, m.$$

Suppose that a vector $\mathbf{b} = (b_1, \dots, b_n) \in L^n$ is represented by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_m$. Then there exist coefficients $x_1, \dots, x_m \in L$ such that

$$\mathbf{b} = x_1 \odot \mathbf{a}_1 \vee \cdots \vee x_m \odot \mathbf{a}_m; \tag{22}$$

coordinatewise, (22) is equivalent to

$$b_j = x_1 \odot a_{1j} \vee \cdots \vee x_m \odot a_{mj}, \qquad j = 1, \dots n.$$

If we take $\mathbf{b} = (1, \dots, 1)$ then from the last equality, we easily obtain the following *necessary condition the the basic vectors*:

$$a_{1j} \vee \cdots \vee a_{mj} = 1, \quad j = 1, \dots n. \tag{23}$$

It turns out that when L is linearly ordered, a criterion that vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ constitute a basis of L^n has a very simple form.

Theorem 3
Let a support set L of an MV-algebra be the interval $[0, 1]$. Then the unique basis in the semilinear space of n-dimensional vectors $[0, 1]^n$ over the semiring $\mathcal{L}^{\vee}$ is given by (11).

PROOF: Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in L^n$ be basic vectors in $[0,1]^n$ and $i, 1 \leq i \leq n$, be a fixed natural number. The vector $\mathbf{e}_i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$ can be represented by a linear combination of basic vectors. Moreover, the components

$$\hat{x}_k = \bigwedge_{j=1}^{n} (a_{kj} \rightarrow e_{ij}), \quad k = 1, \ldots, m, \tag{24}$$

constitute the vector $\hat{\mathbf{x}}$ which is a solution of the equation

$$\mathbf{e}_i = \mathbf{x} \circ \mathbf{A} \tag{25}$$

where $\mathbf{x}$ is an unknown vector and the matrix A is given by (13).

We will prove that for the chosen i there exists $l, 1 \leq l \leq m$, such that $\mathbf{e}_i = \mathbf{a}_l$. Substituting the coefficients given by (24) into (25), we obtain for the i-th component:

$$\begin{aligned} 1 = \hat{x}_1 \odot a_{1i} \vee \cdots \vee \hat{x}_m \odot a_{mi} = \\ &= (\neg a_{11} \wedge \cdots \wedge \underbrace{1}_{i} \wedge \cdots \wedge \neg a_{1n}) \odot a_{1i} \vee \cdots \\ &\qquad \cdots \vee (\neg a_{m1} \wedge \cdots \wedge \underbrace{1}_{i} \wedge \cdots \wedge \neg a_{mn}) \odot a_{mi}. \end{aligned}$$

Therefore, there exists l, $1 \leq l \leq m$, such that

$$1 = (\neg a_{l1} \wedge \cdots \wedge \underbrace{1}_{i} \wedge \cdots \wedge \neg a_{ln}) \odot a_{li}$$

which implies

$$a_{l1} = 0, \ldots a_{li} = 1, \ldots, a_{ln} = 0.$$

Thus,

$$\mathbf{e}_i = \mathbf{a}_l$$

which proves the theorem. □

Let us investigate the problem whether a system of linearly independent vectors can be extended to a basis. The following simple example shows that this is not always the case: Let us take the reduct $\mathcal{L}^{\vee}$ of Łukasiewicz algebra on $[0,1]$ and for $a \in (0,1)$ consider the set of linearly independent vectors (21) from L^n. By Theorem 3, the only basis for this space is given by (11). However, we cannot add any of the vectors from (11) to the set (21) without destroying their linear independency. On the basis of this example, we may prove the following lemma.

Lemma 1
Let $A = L^n$ be the $\mathcal{L}^{\vee}$-semilinear space of n-dimensional vectors where $\mathcal{L}$ is an MV-algebra. Then the system of linearly independent vectors (21) cannot be extended to a basis of L^n.

PROOF: Suppose that contrary to the conclusion of the lemma, there are vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$, $k \geq 1$, which together with vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ from (21) form a basis of L^n. Then either the vector $\mathbf{e}_1 = (1, 0, \ldots, 0)$ is among the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$, or it can be represented by a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_k$. In the first case, the vector $\mathbf{a}_1$ is represented by $\mathbf{a}_1 = a \odot \mathbf{e}_1$. In the second case, $\mathbf{e}_1$ can be represented by the linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_k$ which does not include the vector $\mathbf{a}_1$ so that $\mathbf{a}_1$ can be represented by the same linear combination multiplied by a. In both cases, the obtained extension $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_k$ cannot be a basis. This contradiction proves the lemma. □

4 MV-semimodule homomorphisms

In this section, we will study the concept of semimodule homomorphism that plays a crucial role when characterizing semilinearity, analogously as is the role of the concept of module homomorphism when characterizing classical linearity. Besides the general definition of semimodule homomorphism, we will define also the concepts of sup- and inf-semicontinuous homomorphisms that occur when dealing with fuzzy systems. We will also show that homomorphisms give rise to a new semimodule.

Definition 6
Let $\mathcal{R} = \langle R, +, \cdot, \mathbf{0}_R, \mathbf{1}_R \rangle$ be a semiring and $\langle M, +_M, \mathbf{0}_M \rangle$ and $\langle N, +_N, \mathbf{0}_N \rangle$ $\mathcal{R}$-semimodules. A mapping $H : M \longrightarrow N$ is an $\mathcal{R}$-homomorphism if the followings holds:

(i) $H(m +_M m') = H(m) +_N H(m')$, *for all* $m, m' \in M$.
(ii) $H(rm) = rH(m)$, *for all* $m \in M$ *and* $r \in R$.

The *kernel* of H is the set

$$\mathrm{Ker}(H) = \{m \in M \mid H(m) = \mathbf{0}_N\}.$$

Note that $\mathrm{Ker}(H) \neq \emptyset$. Indeed, choose $r = \mathbf{0}_R$. Then by (ii) of Definition 6 and (v) of Definition 2, we obtain $H(\mathbf{0}_M) = \mathbf{0}_N$.

The set of all $\mathcal{R}$-homomorphisms $M \longrightarrow N$ is denoted by $\mathrm{Hom}(M, N)$. Note that $\mathrm{Hom}(M, N) \neq \emptyset$ because the function $H_0 : m \mapsto \mathbf{0}_N$, $m \in M$, is an element of $\mathrm{Hom}(M, N)$.

Let us now define the addition $+_H$ on $\mathrm{Hom}(M, N)$ by

$$(H +_H K)(m) = H(m) +_N K(m) \tag{26}$$

for every $H, K \in \mathrm{Hom}(M, N)$ and $m \in M$. Furthermore, let us define the external multiplication by

$$(r \circ_H H)(m) = rH(m) \tag{27}$$

for every $H \in \mathrm{Hom}(M, N)$, $m \in M$ and $r \in R$.

Lemma 2
If $\mathcal{R}$ is a commutative semiring and $r \in R$ then $H +_H K$ and $r \circ_H H$ are elements of $\mathrm{Hom}(M,N)$.

PROOF: It is immediate that

$$(H +_H K)(m+m') = (H +_H K)(m) + (H +_H K)(m').$$

Moreover,

$$(H +_H K)(rm) = H(rm) + K(rm) = rH(m) + rK(m) = r((H +_H K)(m)).$$

Thus $H +_H K \in \mathrm{Hom}(M,N)$.

Consider now $r \circ_H H$. Then

$$(r \circ_H H)(m+m') = rH(m+m') = rH(m) + rH(m') = (r \circ_H H)(m) + (r \circ_H H)(m').$$

Moreover,

$$\begin{aligned}(r \circ_H H)(sm) = rH(sm) = r(sH(m)) = (s\cdot r)H(m) = s(rH(m)) = \\ = s((r \circ_H H)(m))\end{aligned}$$

and so, $r \circ_H H \in \mathrm{Hom}(M,N)$. □

Theorem 4
Let $\mathcal{R}$ be a commutative semiring and the addition "$+_H$" and the external multiplication "$\circ_H$" be defined by (26) and (27), respectively. Then

$$\langle \mathrm{Hom}(M,N), +_H, H_0\rangle$$

is an $\mathcal{R}$-semimodule.

PROOF: Clearly, $\langle \mathrm{Hom}(M,N), +_H, H_0\rangle$ is a commutative monoid. Let us now verify the properties (i)–(v) of Definition 2.

Let $r,s \in R$ and $H,K \in \mathrm{Hom}(M,N)$. Then for every $m \in M$, we have the following:

(i)

$$\begin{aligned}((r\cdot s) \circ_H H)(m) = (s\cdot r)H(m) = H(s(rm)) = sH(rm) \\ = (s \circ_H H)(rm) = r \circ_H (s \circ_H H)(m).\end{aligned}$$

Hence, $((r\cdot s) \circ_H H) = r \circ_H (s \circ_H H)$.

(ii)

$$
\begin{aligned}
(r \circ_H (H +_H K))(m) &= r(H +_H K)(m) = r(H(m) + K(m)) \\
&= rH(m) + rK(m) = (r \circ_H H)(m) + (r \circ_H K)(m).
\end{aligned}
$$

Hence, $r \circ_H (H +_H K) = (r \circ_H H) +_H (r \circ_H K)$.

(iii)

$$
\begin{aligned}
((r+s) \circ_H H)(m) &= (r+s)H(m) = rH(m) + sH(m) \\
&= (r \circ_H H)(m) + (s \circ_H H)(m)
\end{aligned}
$$

Hence, $(r+s) \circ_H H) = r \circ_H H +_H s \circ_H H$.

(iv) $(\mathbf{1}_R \circ_H H)(m) = H(\mathbf{1}_R m) = H(m)$. Hence, $\mathbf{1}_R \circ_H H = H$.

(v) $(r \circ_H H_0)(m) = rH_0(m) = r\,\mathbf{0}_N = \mathbf{0}_N$. Hence, $r \circ_H H_0 = H_0$. Finally, $(\mathbf{0}_R \circ_H H)(m) = \mathbf{0}_R\, H(m) = \mathbf{0}_N$. Hence, $\mathbf{0}_R \circ_H H = H_0$.

□

Corollary 4
Let $\mathcal{L}$ be an MV-algebra and $\langle L^X, +, \mathbf{0}\rangle$, $\langle L^Y, +, \mathbf{0}\rangle$ be MV-semimodules associated with $\mathcal{L}$. Then $\langle \mathrm{Hom}(L^X, L^Y), +_H, H_0\rangle$ is a semimodule.

PROOF: This follows from Lemma 2, Theorem 4 and Examples 1 (c), (e), (f). □

We will now introduce a special kind of semimodule homomorphism that preserves either supremum or infimum.

Definition 7
Let $H \in \mathrm{Hom}(L^X, L^Y)$. If

$$H(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} H(f_i)$$

holds for every family $\{f_i \mid i \in I\}$ of elements in L^X such that $\bigvee_{i \in I} f_i$ exists then H is called sup-semicontinuous. *If*

$$H(\bigwedge_{i \in I} f_i) = \bigwedge_{i \in I} H(f_i)$$

holds for every family $\{f_i \mid i \in I\}$ such that $\bigwedge_{i \in I} f_i$ exists then H is called inf-semicontinuous.

If a homomorphism is both sup- as well as inf-semicontinuous then we will call it *continuous.*

The set of all the sup-semicontinuous homomorphism from L^X to L^Y is denoted by $\mathrm{Hom}_{sc}(L^X, L^Y)$ and inf-semicontinuous ones by $\mathrm{Hom}_{ic}(L^X, L^Y)$. It holds that both $\mathrm{Hom}_{sc}(L^X, L^Y)$ as well as $\mathrm{Hom}_{ic}(L^X, L^Y)$ are nonempty because they both contain the homomorphism H_0 (note that in the second case, the H_0 denotes a homomorphism assigning to each fuzzy set $f \in L^X$ the function $\mathbf{0}_{L^Y}$ identically equal to $\mathbf{1}$).

Proposition 3
Let $\mathcal{L}$ be an MV-algebra, $\mathcal{A}_X^{\vee} = \langle L^X, \vee, \mathbf{0}_X \rangle$ and $\mathcal{A}_Y^{\vee} = \langle L^Y, \vee, \mathbf{0}_Y \rangle$ the associated MV-sup-semimodules. Then $\langle \mathrm{Hom}_{sc}(L^X, L^Y), \vee_H, H_0 \rangle$ is a subsemimodule of

$$\langle \mathrm{Hom}(L^X, L^Y), \vee_H, H_0 \rangle.$$

PROOF: Let $H, K \in \mathrm{Hom}_{sc}(L^X, L^Y)$ and $\{f_i \mid i \in I\}$ be a set of elements in L^X such that $\bigvee_{i\in I} f_i$ exists and $r \in L$. Then, using sup-continuity, we can prove:

$$(H \vee_H K)(\bigvee_{i\in I} f_i) = H(\bigvee_{i\in I} f_i) \vee K(\bigvee_{i\in I} f_i) = \bigvee_{i\in I} H(f_i) \vee \bigvee_{i\in I} K(f_i) = \\ = \bigvee_{i\in I}(H(f_i) \vee K(f_i)) = \bigvee_{i\in I}(H \vee_H K)(f_i).$$

This implies that $H \vee_H K \in \mathrm{Hom}_{sc}(L^X, L^Y)$.

Let $r \circ_H H \in \mathrm{Hom}_{sc}(L^X, L^Y)$. Then

$$(r \circ_H H)(\bigvee_{i\in I} f_i) = q \odot H(\bigvee_{i\in I} f_i) = q \odot \bigvee_{i\in I} H(f_i) = \\ = \bigvee_{i\in I}(q \odot H(f_i)) = \bigvee_{i\in I}(r \circ_H H)(f_i)$$

where $q = p$ or $q = \neg p$ dependently on the semimodule of Example 1(c) or (f). This implies that $r \circ_H H \in \mathrm{Hom}_{sc}(L^X, L^Y)$. $\square$

Proposition 4
Let $\mathcal{L}$ be an MV-algebra, $\mathcal{A}_X^{\wedge} = \langle L^X, \wedge, \mathbf{1}_X \rangle$ and $\mathcal{A}_Y^{\wedge} = \langle L^Y, \wedge, \mathbf{1}_Y \rangle$ the associated MV-inf-semimodules. Then $\langle \mathrm{Hom}_{ic}(L^X, L^Y), \wedge_H, H_0 \rangle$ is a subsemimodule of

$$\langle \mathrm{Hom}(L^X, L^Y), \wedge_H, H_0 \rangle.$$

PROOF: We will proceed analogously to the proof of Proposition 3. Let $H, K \in \mathrm{Hom}_{ic}(L^X, L^Y)$ and $\{f_i \mid i \in I\}$ be a set of elements in L^X such that $\bigwedge_{i\in I} f_i$ exists and $r \in L$. Then, using inf-continuity, we can prove:

$$(H \wedge_H K)(\bigwedge_{i\in I} f_i) = H(\bigwedge_{i\in I} f_i) \wedge K(\bigwedge_{i\in I} f_i) = \bigwedge_{i\in I} H(f_i) \wedge \bigwedge_{i\in I} K(f_i) = \\ = \bigwedge_{i\in I}(H(f_i) \wedge K(f_i)) = \bigwedge_{i\in I}(H \wedge_H K)(f_i).$$

This implies that $H \wedge_H K \in \mathrm{Hom}_{sc}(L^X, L^Y)$.

Let $r \circ_H H \in \mathrm{Hom}_{ic}(L^X, L^Y)$. Then

$$(r \circ_H H)(\bigwedge_{i\in I} f_i) = r \oplus H(\bigwedge_{i\in I} f_i) = r \oplus \bigwedge_{i\in I} H(f_i) =$$

$$= \bigwedge_{i\in I}(r \oplus H(f_i)) = \bigwedge_{i\in I}(r \circ_H H)(f_i).$$

This implies that $r \circ_H H \in \mathrm{Hom}_{ic}(L^X, L^Y)$. □

5 Compositional rule of inference as an MV-semimodule homomorphism

5.1 General theory

In this section we will show that the composition of fuzzy relations can be seen as a semimodule homomorphism. Since this operation is the core of semantics of the compositional rule of inference that is the main tool for manipulation with fuzzy systems, we may conclude from our results that the latter is a semilinear operator whose behavior, as can be seen from our results in previous sections, is close to linear. Recall that by a fuzzy system, we understand a special fuzzy relation. We have proved that fuzzy systems form a semimodule that is isomorphic to the semimodule of semicontinuous semilinear operators. Consequently, fuzzy systems can be treated as (almost) linear that is a good news for the applications (and their justification). This opens a space for application, at least some of important results from the functional analysis in the theory of fuzzy systems.

Let $\mathcal{L} = \langle L, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1}\rangle$ be a complete MV-algebra and X, Y two nonempty sets. Note that in the case, the MV-algebras (8) with the respective supports L^X and L^Y are complete as well.

Let us now consider the MV-sup-semimodules $\mathcal{A}_X^\vee = \langle L^X, \vee, \mathbf{0}_X\rangle$ and $\mathcal{A}_Y^\vee = \langle L^Y, \vee, \mathbf{0}_Y\rangle$. We will now show how the theory presented above allows us to interpret the composition operation as a homomorphism between $\mathcal{A}_X^\vee$ and $\mathcal{A}_Y^\vee$.

Let $A \underset{\sim}{\subset} X$ and $R \underset{\sim}{\subset} X \times Y$ (i.e. $A \in L^X$ and $R \in L^{X\times Y}$). Set

$$H_R(A)(y) = \bigvee_{x\in X} (A(x) \odot R(x, y)) = B(y), \qquad y \in Y, \tag{28}$$

i.e. $B \in L^Y$. It is easy to verify that $H_R \in \mathrm{Hom}(L^X, L^Y)$.

Lemma 3
Let $H \in \mathrm{Hom}(L^X, L^Y)$. Then $r \odot H(A) = H(r \odot A)$ holds for every $r \in L$ and $A \in L^X$.

PROOF: By (ii) of Definition 6, $r \odot H(A) = qH(A) = H(qA) = H(r \odot A)$ where $q = r$

or $q = \neg r$ dependently on the semimodule of Example 1(c) or (f). $\square$

Theorem 5
The mapping H_R given in (28) is a homomorphism from the semimodule $\mathcal{A}_X^\vee = \langle L^X, \vee, \mathbf{0}_X\rangle$ to the semimodule $\mathcal{A}_Y^\vee = \langle L^Y, \vee, \mathbf{0}_Y\rangle$.

PROOF: Let $A, B \in L^X$ and $r \in L$. We will verify the properties (i) and (ii) of Definition 6. For all $y \in Y$ we prove:

(i)

$$\begin{aligned} H_R(A \vee B)(y) = \bigvee_{x\in X} ((A \vee B)(x) \odot R(x, y)) = \\ = \bigvee_{x\in X} ((A(x) \odot R(x, y)) \vee (B(x) \odot R(x, y))) = \\ = \bigvee_{x\in X} (A(x) \odot R(x, y)) \vee \bigvee_{x\in X} (B(x) \odot R(x, y)) = H(A)(y) \vee H_R(B)(y). \end{aligned}$$

Thus, $H_R(A \vee B) = H_R(A) \vee H_R(B)$.

(ii)

$$\begin{aligned} H_R(rA)(y) = H_R(q \odot A)(y) = \bigvee_{x\in X} ((q \odot A(x)) \odot R(x, y)) = \\ = q \odot \bigvee_{x\in X} (A(x) \odot R(x, y)) = q \odot H_R(A)(y) = rH_R(A)(y). \end{aligned}$$

where $q = r$ or $q = \neg r$ dependently on the semimodule of Example 1(c) or (f). Thus $H_R(rA) = rH_R(A)$.

$\square$

We have shown that a fuzzy composition bears a homomorphism between semimodules so that indeed, the compositional rule of inference can be taken as a (semi)linear mapping.

Now we will study the kernel $\mathrm{Ker}(H_R)$. Recall that $H_R(\mathbf{0}_X) = \mathbf{0}_Y$. Thus $\mathbf{0}_X \in \mathrm{Ker}(H_R)$, so that $\mathrm{Ker}(H_R) \neq \emptyset$. The following proposition tells us explicitly what is the content of this kernel.

Proposition 5
Put $B(x) = \bigwedge_{y\in Y} \neg R(x, y)$ for all $x \in X$. Then

$$\mathrm{Ker}(H_R) = \{A \mid A \in L^X, \mathbf{0}_X \leq A \leq B\}.$$

PROOF: Observe that the fuzzy relation equation $\xi \circ R = \mathbf{0}_X$ (with respect to the unknown ξ) is solvable. Therefore, B defined in the assumption is the greatest solution.

Let $A \leq B$. Then we get $H_R(A) \leq H_R(B) = \mathbf{0}_X$, which implies $H_R(A) = \mathbf{0}_X$. $\square$

An interesting question is the following: given a homomorphism $F : L^X \longrightarrow L^Y$, what conditions on F assure us that there exists a fuzzy relation $R(x, y)$ such that $F = H_R$. If such a relation exists, we say that F is *representable by a fuzzy relation* R or, shortly, that F is *representable*. We will answer this question in the following theorem.

Let $a \in X$. By f_a we denote the singleton fuzzy set on a, i.e. it is an element of L^X defined by

$$f_a(x) = \begin{cases} \mathbf{1} & \text{if } x = a, \\ \mathbf{0} & \text{if } x \neq a, \end{cases}$$

for all $x \in X$. Let $F \in \mathrm{Hom}(L^X, L^Y)$. We put $R_F(x, y) = F(f_x)(y)$ for all $(x, y) \in X \times Y$.

Theorem 6
If $F \in \mathrm{Hom}(L^X, L^Y)$ is sup-semicontinuous then F is representable and $F = H_{R_F}$.

PROOF: The function $(H_{R_F})(A)$ is defined by

$$(H_{R_F})(A)(y) = \bigvee_{x \in X} (A(x) \odot F(A_x)(y)), \qquad y \in Y.$$

Fix $x_0 \in X$. Since F is a homomorphism, by Lemma 3, $A(x_0) \odot F(f_{x_0}) = F(A(x_0) \odot f_{x_0})$. From this, we conclude that

$$\bigvee_{x \in X} (A(x) \odot F(f_x)) = \bigvee_{x \in X} (F(A(x) \odot f_x)) = F\left(\bigvee_{x \in X} (A(x) \odot f_x)\right) = F(A)$$

by the semicontinuity assumption. Consequently, $H_{R_F}(A)(y) = F(A)(y)$ holds for all $y \in Y$ and so, F is representable. $\square$

Lemma 4
The function $\mathcal{H} : R \mapsto H_R$ where $R \in L^{X \times Y}$ and $H_R \in \mathrm{Hom}(L^X, L^Y)$ is an injective homomorphism from the $\mathcal{R}$-semimodule $\mathcal{A}^{\vee}_{X \times Y} = \langle L^{X \times Y}, \vee, \mathbf{0} \rangle$ to the $\mathcal{R}$-semimodule $\langle \mathrm{Hom}(L^X, L^Y), \vee_H, H_0 \rangle$.

PROOF: First, we will verify the properties (i) and (ii) of Definition 6. Let $R, S \in L^{X \times Y}$ and $A \in L^X$.

(i) Since $\mathcal{H}(R \vee S) = H_{R \vee S}$, we get for $y \in Y$

$$\begin{aligned} \mathcal{H}(R \vee S)(A)(y) &= \bigvee_{x \in X} (A(x) \odot (R(x, y) \vee S(x, y))) = \\ &= \bigvee_{x \in X} ((A(x) \odot R(x, y)) \vee (A(x) \odot S(x, y)) = \\ &= \bigvee_{x \in X} (A(x) \odot R(x, y)) \vee \bigvee_{x \in X} (A(x) \odot S(x, y)) = H_R(A)(y) \vee H_S(A)(y). \end{aligned}$$

(ii) Let $r \in L$. Since $\mathcal{H}(rR) = H_{rR}$, we get for $y \in Y$

$$\mathcal{H}(rR)(A)(y) = \bigvee_{x \in X} (A(x) \odot (q \odot R(x,y))) =$$
$$q \odot \bigvee_{x \in X} (A(x) \odot R(x,y)) = q \odot H_R(A)(y) = r\, H_R(A)(y)$$

where $q = r$ or $q = \neg r$ dependently on the semimodule of Example 1(c) or (f).

Now will will show that $\mathcal{H}$ is an injective map. Assume $R \neq S$. Hence, there is a couple $(x_0, y_0) \in X \times Y$ such that $R(x_0, y_0) \neq S(x_0, y_0)$. Then we have

$$H_R(f_{x_0})(y_0) = R(x_0, y_0) \neq S(x_0, y_0) = H_S(f_{x_0})(y_0).$$

Thus, $H_S(f_{x_0}) \neq H_R(f_{x_0})$, which implies $H_R \neq H_S$. □

Theorem 7
The semimodules $\mathcal{A}^{\vee}_{X \times Y} = \langle L^{X \times Y}, \vee, \mathbf{0}_{X \times Y} \rangle$ and $\langle \mathrm{Hom}_{sc}(L^X, L^Y), \vee_H, H_0 \rangle$ are isomorphic.

PROOF: In Lemma 4, we proved that $\langle \mathcal{H}(L^{X \times Y}), \vee, \mathbf{0}_{X \times Y} \rangle$ is a subsemimodule of $\langle \mathrm{Hom}(L^X, L^Y), \vee, H_0 \rangle$ (up to isomorphism). Moreover, by Theorem 6, it is the subsemimodule of all semicontinuous homomorphisms from L^X to L^Y and each semicontinuous homomorphism is representable by some fuzzy relation. The theorem then follows from Lemma 4. □

Note that by Theorem 6, $\mathcal{H}(L^{X \times Y})$ contains the subsemimodule of all semicontinuous homomorphisms from L^X to L^Y. Then we obtain the following theorem.

Theorem 8
Let X, Y be finite sets, $R \in L^{X \times Y}$ and $H_R \in \mathrm{Hom}(L^X, L^Y)$. Then the function $\mathcal{H} : R \mapsto H_R$ is an isomorphism from the $\mathcal{R}$-semimodule $\mathcal{A}^{\vee}_{X \times Y} = \langle L^{X \times Y}, \vee, \mathbf{0}_{X \times Y} \rangle$ to the $\mathcal{R}$-semimodule $\langle \mathrm{Hom}(L^X, L^Y), \vee, H_0 \rangle$.

PROOF: This follows from Theorem 6 and Lemma 4. □

5.2 Examples

Example 3
Let $X, Y \subset \mathbb{R}$ be compact sets, $h : X \longrightarrow Y$ be a continuous function and $\varepsilon > 0$ be a real number. Using the methods described in [18], Chapter 5, we may find finite sets $\{A_j \mid A_j \underset{\sim}{\subset} X, j \in J\}$ and $\{B_j \mid B_j \underset{\sim}{\subset} X, j \in J\}$ such that each fuzzy relation

$$R_j = A_j \times_{\odot} B_j,$$

$j \in J$, represents a fuzzy rule and the following properties are fulfilled:

(a) $\bigcup_{j\in J} \mathrm{Supp}(A_j) = X$, $\bigcup_{j\in J} \mathrm{Supp}(B_j) = \mathrm{rng}\, h$.
(b) If $R_j(x,y) > 0$ *then* $|y - h(x)| < \varepsilon$ *for all* $x \in X$ *and* $y \in Y$.

Since $R_j \in L^{X\times Y}$ *is an element of the semimodule* $\mathcal{A}^{\vee}_{X\times Y}$ *the fuzzy relation* $R = \bigcup_{j\in J} R_j$ *is its element as well. Then by Theorems 7 and 5, R determines a (semicontinuous) semilinear operator* H_R.

Let us consider a fuzzy function $F : L^X \longrightarrow L^Y$ *such that* $F(A) = H_R(A)$ *for all* $A \underset{\sim}{\subset} X$. *Since* H_R *is a semilinear operator, the fuzzy function F is semilinear as well.*

Take $x_0 \in X$ *and let* $A' \underset{\sim}{\subset} X$ *be a fuzzy set such that* $A'(x_0) = 1$ *(i.e. A' is normal) and there is* $j \in J$ *such that* $\mathrm{Supp}(A') \subseteq \mathrm{Supp}(A_j)$ *(we can take, for example, the singleton* $A' = f_{x_0}$*). Then the construction of R described in [18] assures that* $F(A')(h(x)) > 0$ *and so,* $|y - h(x)| < \varepsilon$ *for every* $y \in \mathrm{Supp}(F(A'))$. *Hence, each* $y \in \mathrm{Supp}(F(A'))$ *is a reasonable approximation of $h(x)$. We may conclude that a generally nonlinear function h is approximated by some semilinear fuzzy function with arbitrary precision* $\varepsilon > 0$. *Note that for* $A' = f_{x_0}$*, we have, in fact, obtained the classical approximation of the function h by means of the technique of fuzzy approximation theory (cf. also Perfilieva [21]).*

Example 4
Let us consider the inverted pendulum control problem. This system can be approximately modeled by a non-linear differential equation

$$\ddot{y} - 10 \sin y = 0$$

where y depends on time t. The system can be controlled using a non-linear function $u(t) = h(e(t), \dot{e}(t))$ *where* $e(t) = -y(t)$ *is a pendulum deviation from the vertical position. We will replace it by means of a fuzzy PD-controller.*

In our terms this means that we must construct a fuzzy system $R \underset{\sim}{\subset} X \times Y \times Z$ *where X is the range of deviation $e(t)$, Y is the range of its velocity $\dot{e}(t)$ and Z is the range of control action $u(t)$ (this is a momentum sent to the inverted pendulum). For the control, we have used the software system LFLC (see [7]). The universes are set as follows:* $X = [-1.5, 1.5]$, $Y = [-10, 10]$, *and* $Z = [-90, 90]$. *The fuzzy system (i.e. fuzzy PD-controller) consists of the following fuzzy IF-THEN rules:*

number	$e(t)$	$\dot{e}(t)$	$u(t)$
1	*Ze*	*Ze*	*Ze*
2	*RoZe*	*- Sm*	*+VeSm*
3	*RoZe*	*+ Sm*	*-VeSm*
4	*-NoZe*	*-NoZe*	*+Me*
5	*+NoZe*	*+NoZe*	*-Me*
6	*-NoSm*	*+NoZe*	*+Ro17*
7	*+NoSm*	*-NoZe*	*-Ro17*

Each rule is a fuzzy relation of the form $R_j = A_j \times_\odot B_j \times_\odot C_j$ *where* $\odot$ *is in this case either the operation* $\wedge$ *or the MV-product, and* A_j, B_j, C_j, $j = 1, \ldots, 7$ *are fuzzy sets occurring in the corresponding rule. Their shapes are depicted on Figure 1. They are, in fact, meanings of the corresponding linguistic expressions* zero (Ze), roughly zero (RoZe), not zero (NoZe), small (Sm), very small (VeSm), not small (NoSm), roughly 20% (Ro20) *(the latter is a fuzzy number).*

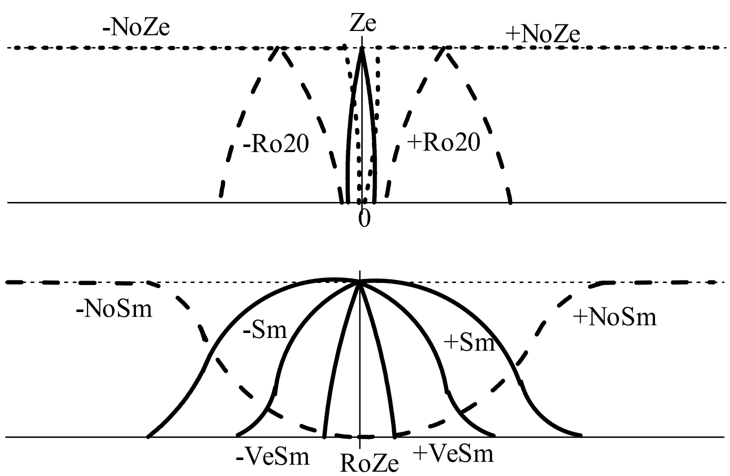

Fig. 1. Shapes of fuzzy sets for the fuzzy control of inverted pendulum. They are symmetrically placed around 0.

The fuzzy control is given by a semilinear fuzzy function F *defined by*

$$F(f_{e(t)}, f_{\dot{e}(t)}) = H_R(f_{e(t)}, f_{\dot{e}(t)})$$

where $f_{e(t)} \in X, f_{\dot{e}(t)} \in Y$ *are singletons and for each* t, $F(f_{e(t)}, f_{\dot{e}(t)}) \underset{\sim}{\subsetneq} Z$ *is a fuzzy set that approximates value of the control function* $u(t) = h(e(t), \dot{e}(t))$, *i.e. each element* $z \in Z$ *belonging to it in non-zero membership degree approximates* $u(t)$. *Concretely, we take* $z = z_0$ *where* z_0 *is obtained by a defuzzification of* $F(f_{e(t)}, f_{\dot{e}(t)})$ *(usually Center Of Gravity).*

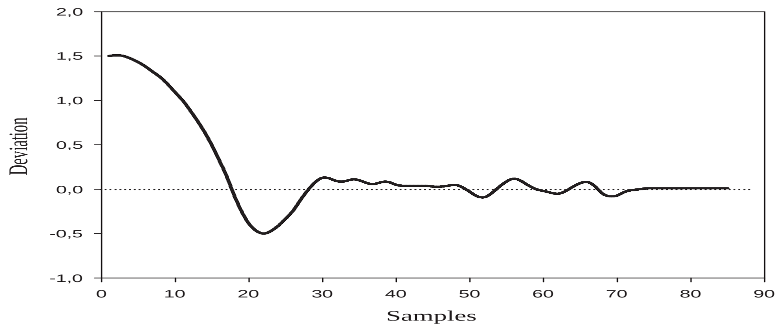

Fig. 2. Fuzzy control of inverted pendulum.

Figure 2 demonstrates that this simple fuzzy system indeed controls the inverted pendulum. Of course, the control can be much improved when adding more rules. Our goal, however, was to demonstrate the method and not the control itself. Again, we have approximated nonlinear function by a semilinear fuzzy function.

6 Conclusion

In this contribution, we have developed an algebraic theory suitable for the analysis of fuzzy systems. Namely, we have used the notions of semiring and semimodule and introduced the notion of semilinear space. We gave numerous examples of these notions, defined the notions of linear dependence and independence and suggested conditions (necessary or sufficient) when vectors form a basis of a semilinear space. Then, we have considered the composition operation of fuzzy relations which plays an essential role in the analysis of fuzzy systems because using it, the compositional rule of inference is defined. We have shown that it can be interpreted as a homomorphism between special semimodules. Because the classical concepts of module and linear space are abstract characterization of the linearity property, it follows from our results that the compositional rule of inference can be taken as a linear operation as well. This fact makes it even more attractive for the applications because linearity is a desirable property that makes models simpler and more transparent.

The basic algebraic structure considered in this paper is MV-algebra. However, it is clear from all the constructions that we can take even a more general algebra such as BL- or IMTL-algebra.

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