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Abstract

A general theory of fuzzy decision systems (GFDS) over some decision space $(\mathbf{U}, C)$ is introduced, where $\mathbf{U}$ is a set of variants and C is a set of criterions, which is based on formal notions of fuzzy implications between fuzzy sets and some formal knowledge base. A pre-order relation is defined on the set of GFDS over some decision space and some properties of this relation are presented.

Let a system $\mathcal{U}$ of variants be given. A *decision function* over $\mathcal{U}$ is a function $g : \mathcal{U} \longrightarrow [0,1]$ (i.e., a fuzzy set in $\mathcal{U}$). This decision function represents, in fact, a final result of a complex procedure of a solution (i.e., element of $\mathcal{U}$) searching, which is frequently based on various systems of goals, elementary or complex evaluation of elements of $\mathcal{U}$ and another systems of criterions. A lot of decission support systems exist, resulting in a global function g The interesting reader is referred to the fundamental paper of Bellman and Zadeh [1] or to Rommelfanger [5], Young-Jou Lai and Ching-Lai Hwang [4] or Zimmermann [6], where survays of most of existing methods and models are presented. The applicability of these systems is not, however, defined precisely and it could happen that two different methods applied on the same decission situation will result in quite different global evaluation functions. On the other hand, to compare results and applicability of different decission support methods we need a common theoretical background (or frame) for these methods, which is not a very natural requirement up to date. Hence, to compare various (fuzzy) decision suppost methods we need to derive at first a general theory of these DSS. In this paper a variant of such general theory will be described enabling us to handle these complex systems. Using this general theory we can then compare various DSS and, moreover, DSS are then normal mathematical structures and it is possible to investigate their properties by using mathematical methods.

Throughout the paper $\mathcal{F}(U)$ denotes the fuzzy power set on an universe U. Any fuzzy set $A \in \mathcal{F}(U)$ will be identified with its membership function $A : U \longrightarrow [0,1]$.

1 Fuzzy inclusion relations

In order to characterize an optimal alternative, various fuzzy sets are to be compared by means of fuzzy inclusion relation $s \underset{\sim}{\subset} \mathcal{F}(U) \times \mathcal{F}(U)$, which is an analogy of an inclusion relation used for crisp sets. This relation is not unique, but it has to satisfy some conditions which represent some fuzzy analogy of a classical inclusion relation. Moreover, the value $s(A,B)$ of this fuzzy inclusion relation between two fuzzy sets A, B has to be in some relationship with some deduction rule. This intuitive condition follows from an interpretation of a meaning of this fuzzy inclusion relation. Intuitively, a fuzzy inclusion is closely connected with a following deduction rule

$$\frac{A, B \subseteq A}{B \subseteq A \cap B}.$$

Hence, if A, B are two fuzzy sets describing some property in an universe U, then "truth values" $\|A\|$,$\|B\|$ of A, B and a value $s(A,B)$ should be connected by the following condition:

$$\|A\| \wedge s(B,A) \le s(B, B \cap A).$$

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To introduce such a fuzzy inclusion relation we present the following definition. Recall that for two fuzzy sets $A, B \in \mathcal{F}(U)$ a relation $\subseteq$ is defined as follows:

$$A \subseteq B \Longleftrightarrow (\forall x \in U) A(x) \leq B(x).$$

Definition 1.1 *Let U be an universe. A fuzzy inclusion relation in U is a fuzzy set $s \underset{\sim}{\subset} \mathcal{F}(U) \times \mathcal{F}(U)$ which satisfies the following conditions:*

1. $(\forall A, B \in \mathcal{F}(U)) A \subseteq B \Longrightarrow s(A, B) = 1$,
2. $(\forall x \in U)(\forall B \in \mathcal{F}(U)) s(\{x\}, B) = B(x)$,
3. $(\forall A, B \in \mathcal{F}(U)) A \cap B = \emptyset \Longrightarrow s(A, B) = 0$.

We present firstly several examples of fuzzy inclusions.

Lemma 1.2 *Let $A, B \in \mathcal{F}(U)$. We set*

$$s_1(A, B) = \frac{\bigvee_{x \in U}(A(x) \wedge B(x))}{\bigvee_{x \in U} A(x)}$$

$$s_2(A, B) = \bigvee_{\substack{\alpha \in [0,1) \\ A_\beta \subseteq B_\beta, \forall \beta \in [0,\alpha]}} \alpha$$

$$s_3(A, B) = 1 - \frac{1}{|A|} \sum_{\substack{x \in U \\ A(x) > B(x)}} (A(x) - B(x))$$

where $|A| = \sum_{x \in supp(A)} A(x)$ and $A_\alpha = \{x \in U : A(x) > \alpha\}$. Then s_i, $i = 1, 2, 3$, are fuzzy inclusions.

Proof. To show that the above functions s_i, $i = 1, 2, 3$, satisfy axioms from Definition 1.1 is only a straightforward computation and it will be omitted. □

As we have mentioned above, a relation s has to satisfy some inequality describing interpretation of the above mentioned deduction rule. This relationship depends, in general, on a truth value $\|A\|$ of a fuzzy set A. In the following proposition we prove this property for fuzzy inclusion relations presented in Lemma 1.2.

Proposition 1.3 *Let $A, B \in \mathcal{F}(U)$.*

1. *Let for $X \in \mathcal{F}(U)$ a truth value $\|X\|$ of X be defined such that $\|X\| = \bigvee_{x \in U} X(x)$. Then*

$$\|A\| \wedge s_i(B, A) \leq s_i(B, A \cap B); \quad i = 1, 2.$$

2. *Let for $X \in \mathcal{F}(U)$ a truth value $\|X\|$ of X be defined such that $\|X\| = \frac{1}{|U|} \sum_{x \in U} X(x)$, where $|U|$ is a measure of a set U. Then*

$$\|A\| \wedge s_3(B, A) \leq s_3(B, A \cap B).$$

In these cases we say that the truth value function $\|..\| \underset{\sim}{\subset} \mathcal{F}(U)$ is connected with a fuzzy inclusion relation s.

Proof. For example, let us consider a statement (i) for $i = 2$. Then for any fuzzy sets A, B we have

$$\|A\| \wedge s_2(B, A) = \bigvee_{x \in U} A(x) \wedge (\bigvee_{\substack{\alpha \in [0,1) \\ B_\beta \subseteq A_\beta, \forall \beta \leq \alpha}} \alpha) =$$

$$= \bigvee_{x \in U} A(x) \wedge (\bigvee_{\substack{\alpha \in [0,1) \\ B_\beta \subseteq (A \cap B)_\beta, \forall \beta \leq \alpha}} \alpha) \leq (\bigvee_{\substack{\alpha \in [0,1) \\ B_\beta \subseteq (A \cap B)_\beta, \forall \beta \leq \alpha}} \alpha) = s_2(B, A \cap B).$$

The rest of the proof can be done similarly. □

It should be observed that although fuzzy inclusion relations s_i from Lemma 1.2 satisfy all required axioms, they behave very differently. In fact, let us consider, for example, classical intervals $[a,1]$ and $[b,1]$ in $U=[0,1]$. Then we have

$$s_1([a,1],[b,1]) = 1, \quad \forall a,b \in (0,1)$$
$$s_2([a,1],[b,1]) = \begin{cases} 1, & \text{if} \quad a \geq b \\ 0, & \text{if} \quad a < b \end{cases}$$
$$s_3([a,1],[b,1]) = \begin{cases} 1, & \text{if} \quad a \geq b \\ \frac{1-b}{1-a}, & \text{if} \quad a < b \end{cases}$$

It seems reasonable that for such special fuzzy sets, s_3 is the best fuzzy inclusion relation.

From several fuzzy inclusions another ones can constructed according to the following lemma.

Lemma 1.4 *Let s_1, s_2 be two fuzzy inclusions in U. Let us define the following function, where $A, B \in \mathcal{F}(U)$.*

1. $s_\wedge(A,B) = s_1(A,B) \wedge s_2(A,B)$,
2. $s_\vee(A,B) = s_1(A,B) \vee s_2(A,B)$,
3. $s_\gamma(A,B) = \gamma \cdot s_1(A,B) + (1-\gamma)\cdot s_2(A,B)$, *where* $\gamma \in [0,1]$.

Then these functions are fuzzy inclusions.

The proof of this lemma is straightforward and it will be omitted.

Lemma 1.5 *Let fuzzy inclusions s_1, s_2 be connected with a same truth value function $\|..\|$. Then fuzzy inclusions $s_\wedge, s_\vee$ and s_γ from Lemma 1.4 are connected with the same truth value.*

Proof. For all fuzzy sets A, B we have

$$\|A\| \wedge s_\wedge(B,A) \leq \|A\| \wedge s_i(B,A) \leq s_i(B, A\cap B); \quad i = 1,2$$

and it follows that $\|A\| \wedge s_\wedge(B,A) \leq s_\wedge(B, A\cap B)$. Further, we have

$$\|A\| \wedge s_\vee(B,A) = \|A\| \wedge (s_1(B,A) \vee s_2(B,A)) =$$
$$\big(\|A\| \wedge s_1(B,A)\big) \vee \big(\|A\| \wedge s_2(B,A)\big) \leq$$
$$\leq s_1(B, A\cap B) \vee s_2(B, A\cap B) = s_\vee(B, A\cap B).$$

Finally, by using an inequality $a \wedge (b * c) \leq (a\wedge b) * (a \wedge c)$, where $*$ represents operations $+,.$ in interval $[0,1]$, we obtain

$$\|A\| \wedge s_\gamma(B,A) = \|A\| \wedge \big(\gamma \cdot s_1(B,A) + (1-\gamma)\cdot s_2(B,A)\big) \leq$$
$$\leq \big(\|A\| \wedge \gamma \cdot s_1(B,A)\big) + \big(\|A\| \wedge (1-\gamma)\cdot s_2(B,A)\big) \leq$$
$$\leq (\|A\| \wedge \gamma)\cdot \big(\|A\| \wedge s_1(B,A)\big) + \big(\|A\| \wedge (1-\gamma)\big)\cdot \big(\|A\| \wedge s_2(B,A)\big) \leq$$
$$\leq \gamma \cdot s_1(B, A\cap B) + (1-\gamma) s_2(B, A\cap B) = s_\gamma(B, A\cap B).$$

□

2 General fuzzy decision systems

We begin with a definition of a general fuzzy decision systems (GFDS). Let us suppose that a discrete set $\mathbf{U}$ of variants is given and let C be an abstract set of criterions we want to apply to. Then a pair $(\mathbf{U}, C)$ is called a *decision space*. For our purposes this space will be constant and we will be working only *inside* this space.

The following definition then introduces a general notion of a general fuzzy decision system as follows.

Definition 2.1 *Let $(\mathbf{U}, C)$ be a decision space. Then a general fuzzy decision system $\mathbf{R}$ over $(\mathbf{U}, C)$ is a set*

$$\mathbf{R} = (\{G_g : g \in K\}, \{w_c : c \in C\}, \{\mathcal{V}_g : g \in K\}, s, f),$$

where

1. *K is a finite set,*
2. *$G_g \subsetneqq [0,1]$ for any $g \in K$,*
3. *For any $c \in C, w_c : \mathbf{U} \longrightarrow \mathcal{F}([0,1])$,*
4. *$\mathcal{V}_g \subseteq \{(F_{c_1}, \dots, F_{c_n}, T) : T, F_{c_i} \subsetneqq [0,1], \{c_1, \dots, c_n\} \subseteq C\}$,*
5. *s is a fuzzy inclusion relation,*
6. *$f : [0,1]^C \longrightarrow [0,1]$ is an aggregation function which has to satisfy the axiom*

$$(\forall \vec{u} = (u_c)_{c \in C}) \min_{c \in C} u_c \le f(\vec{u}) \le \max_{c \in C} u_c$$

An intutitive interpretation of this abstract definition could be done as follows:

1. A finite set K is a set of *goals* we want to deal with. We have to emphasise that sometimes it is useful to make a difference between goals and criterions. In fact, goals could be more aggregated entities.
2. For any goal $g \in K$ a level of a satisfaction of this goal could be prescribe which depends on an importance of the goal g. A fuzzy set G_g then represents a required level of a satisfaction of a goal $g \in K$. A more this fuzzy set G_g is similar to the crisp value 1 in the interval $[0,1]$, (i.e. a crisp set $\{1\} \subset [0,1]$) a more higher satisfaction of this goal g is required.
3. A function $w_c : \mathbf{U} \longrightarrow \mathcal{F}([0,1])$ then describes a level ofa satisfaction of a criterion $c \in C$ by a variant from $\mathbf{U}$. This level of satisfaction is expressed , in general, as a fuzzy set. A more this fuzzy set $w_c(u)$ is similar to the crisp set $\{1\}$ in the interval $[0,1]$, the more a variant $u \in \mathbf{U}$ satisfies a criterion $c \in C$.
4. Any element $(F_{c_1}, \dots, F_{c_k}, T)$, where $F_{c_i} \subsetneqq [0,1]$, $T \subsetneqq [0,1]$ and $c_1, \dots, c_k \in C$, represents, in fact, some expert knowledge about the influences of level of satisfaction of criterions c onto level of satisfaction of a goal $g \in K$. In this notation, an expert (for example) states that if a level of a satisfaction of a criterion c_i can be expressed as a fuzzy set $F_{c_i} \subsetneqq [0,1]$ (recall that more this fuzzy set is similar to a crisp set $\{1\}$ in [0,1] the more a criterion c_i is satisfied), then a level of a satisfaction of a goal g can be expressed by a fuzzy set $T \subsetneqq [0,1]$.
5. An aggregation function f then enables us to calculate a total utility function of any variant $u \in \mathbf{U}$.

Intuitively the most complicated notion in this definition is the set $\mathcal{V}_g$ for $g \in K$.This set represents frequently some expert knowledge which can be represented in a form

$$(F_{c_1}, \dots, F_{c_n}, T) \in \mathcal{V}_g \iff$$

$$\textbf{IF} \Big[\big(\text{level of satisfaction of a criterion } c_1 \text{ is } F_{c_1} \subsetneqq [0,1]\big) \textbf{ AND} \dots$$

$$\dots \textbf{ AND } \big(\text{level of satisfaction of a criterion } c_n \text{ is } F_{c_n} \subsetneqq [0,1]\big)\Big]$$

$$\textbf{THEN}\big(\text{level of satisfaction of a goal } g \text{ is } T \subsetneqq [0,1]\big)$$

or, in symbols

$$(F_{c_1},\dots,F_{c_n},T)\in\mathcal{V}_g \Longleftrightarrow$$
$$\Longleftrightarrow \big(\|c_1\| = F_{c_1}\wedge\dots\wedge\|c_n\| = F_{c_n} \Longrightarrow \|g\| = T\big).$$

By $\mathcal{R}(\mathbf{U},C)$ we denote the set of all general fuzzy decision systems $\mathbf{R}$ over decision space $(\mathbf{U},C)$. Moreover, for any $\mathbf{R}\in\mathcal{R}(\mathbf{U},\mathbf{C})$ we can define a local utility function $h_{\mathbf{R},g}:\mathbf{U}\longrightarrow[0,1]$ of a goal $g\in K$. This function then represents a local utility of a variant $u\in\mathbf{U}$ with respect to the local goal $g\in K$. Such a function can be defined in several ways, we present here only basic examples of such function.

Variant I.

$$h_{\mathbf{R},g}(u) = \bigvee_{\substack{\mathbf{V}\in\mathcal{V}_g\\ \mathbf{V}=(F_{\mathbf{V},c_1},\dots,F_{\mathbf{V},c_n},T_{\mathbf{V}})}} \Big[f\Big(s\big(w_{c_1}(u),F_{\mathbf{V},c_1}\big),\dots,s\big(w_{c_n}(u),F_{\mathbf{V},c_n}\big)\Big)\cdot s(T_{\mathbf{V}},G_g)\Big]$$

Hence, to apply this local utility function for a goal $g\in G$ we take an element $\mathbf{V}=(F_{\mathbf{V},c_1},\dots,F_{\mathbf{V},c_n},T_{\mathbf{V}})$ from a knowledge base $\mathcal{V}_g$. We firstly calculate how levels of satisfaction of criterions c_i (which appear in this element $\mathbf{V}$)(i.e. fuzzy sets $w_{c_i}(u)\subsetsim[0,1]$) correspond to assumptions of expert knowledge in this element $\mathbf{V}$, i.e. how these fuzzy sets $w_{c_i}(u)$ are in fuzzy inclusion relation with the assumptions $F_{\mathbf{V},c_i}$. The level of satisfaction of these assumptions is hence defined as a value of a fuzzy inclusion relation $\alpha_i(u)=s(w_{c_i}(u),F_{\mathbf{V},c_i})\in[0,1]$. Then an aggregated value $\alpha=f(\alpha_1(u),\dots,\alpha_n(u))$ represents a level in which an element of expert knowledge $\mathbf{V}$ corresponds to the real situation, which is represented by levels of satisfaction of corresponding criterions in a variant u. A more higher is this aggregated value α a more appropriate is a concrete knowledge $\mathbf{V}$ for a description of a given situation.

Secondly, we have to modify this level α by a level in which the expert conclusion (i.e. a fuzzy set $T_{\mathbf{V}}$) corresponds to the required level G_g of a satisfaction of a goal g. This is a resonable requirement since the expert knowledge $\mathbf{V}$ the result of which $T_{\mathbf{V}}$) is far from a requirement level of satisfaction G_g of a goal g is not worthwhile.

Variant II.

$$h_{\mathbf{R},g}(u) = \frac{\sum_{\mathbf{V}\in\mathcal{V}_g} f\Big(s\big(w_{c_1}(u),F_{\mathbf{V},c_1}\big),\dots,s\big(w_{c_n}(u),F_{\mathbf{V},c_n}\big)\Big)\cdot s(T_{\mathbf{V}},G_g)}{\sum_{\mathbf{V}\in\mathcal{V}_g} f\Big(s\big(w_{c_1}(u),F_{\mathbf{V},c_1}\big),\dots,s\big(w_{c_n}(u),F_{\mathbf{V},c_n}\big)\Big)}$$

Variant III.

$$h_{\mathbf{R},g}(u) = \bigvee_{\substack{\mathbf{V}\in\mathcal{V}_g\\ \mathbf{V}=(F_{\mathbf{V},c_1},\dots,F_{\mathbf{V},c_n},T_{\mathbf{V}})}} \Big[f\Big(s\big(w_{c_1}(u),F_{\mathbf{V},c_1}\big),\dots,s\big(w_{c_n}(u),F_{\mathbf{V},c_n}\big)\Big)\wedge s(T_{\mathbf{V}},G_g)\Big]$$

The intepretation of these local utility functions can be done analogously. Let us consider several intuitive examples of well known decision situations. We will try to translate these situations using our notations. In these examples we will always assume that a decision space $(\mathbf{U},C)$ is given and it is the same for all examples.

Example 1.

We want to select an element $u\in\mathbf{U}$ which satisfies any criterion $c\in C$ with at least crisp minimal level $a_c\in[0,1]$ of satisfaction and, moreover, for any $u\in\mathbf{U}$ we can estimate at least minimal level of satisfaction of u in a criterion $c\in C$. Although this example is rather trivial, we translate this classical situation into our notation.

1. $K := C$,
2. $(\forall c \in C) G_c = [a_c, 1] \subseteq [0,1]$
3. $(\forall c \in C, \forall u \in \mathbf{U}) w_c(u) = [v_c(u), 1] \subseteq [0,1]$
4. $\mathcal{V}_c = \{(\|c\| = [a,1] \Longrightarrow \|c\| = [a,1]) : a \in [0,1]\}$
5. Since s is restricted on pairs $([a,1],[b,1])$ only, we will take $s = s_3$ from Lemma 1.2
6. f is not necessary at all.

Hence, since this decision situation is rather simple, it requires formaly only a trivial knowledge base $\mathcal{V}_g$ and any criterion is simultaneously a goal. This knowledge base says that if a criterion $c \in C$ is satisfied in a level of satisfaction $[a,1]$, then in the same level of satisfaction is satisfied the goal c. In this case it is also simple to calculate the local utility function $h_{\mathcal{R},c}$ from Variant I. In fact, we have

$$s([a,1],[b,1]) = \begin{cases} 1, & \text{if} \quad a \geq b \\ \frac{1-b}{1-a}, & \text{if} \quad a < b \end{cases}$$

and

$$h_{\mathbf{R},c}(u) = \begin{cases} 1, & \text{if} \quad v_c(u) \geq a_c \\ \frac{1-a_c}{1-v_c(u)}, & \text{if} \quad v_c(u) < a_c \end{cases}$$

A calculation of a local utility function from Variant II. is more complicated. For example, if $b = v_c(u) > a_c$, then we have

$$h_{\mathbf{R},c}(u) = \frac{\sum_{a\in[0,1]} s([b,1],[a,1]) \cdot s([a,1],[a_c,1])}{\sum_{a\in[0,1]} s([b,1],[a,1])} =$$

$$= \frac{\int_0^{a_c} \frac{1-a_c}{1-a} da + \int_{a_c}^{b} da + \int_b^1 \frac{1-a}{1-b} da}{\int_0^{a_c} da + \int_{a_c}^{b} da + \int_b^1 \frac{1-a}{1-b}} = 1 - \frac{2a_c - 2(a_c-1)\ln(1-a_c)}{1+b}$$

It is clear that $\lim_{\substack{a_c \to 1 \\ b \to 1}} h_{\mathbf{R},c}(u) = 1$.

Finally, for a function $h_{\mathbf{R},c}$ from a Variant III. we have

$$h_{\mathbf{R},c}(u) = \bigvee_{a\in[0,1]} \Big(s([v_c(u),1],[a,1]) \wedge s([a,1],a_c,1]) \Big).$$

If $v_c(u) \geq a_c$ that it is clear that $h_{\mathbf{R},c}(u) = 1$. But for a case $v_c(u) < a_c$ we obtain

$$h_{\mathbf{R},c}(u) =$$

$$= \bigvee_{a\in[0,v_c(u)]} \frac{1-a_c}{1-a} \vee \bigvee_{a\in[v_c(u),a_c]} \left(\frac{1-a}{1-v_c(u)} \wedge \frac{1-a_c}{1-a} \right) \vee \bigvee_{a\in[a_c,1]} \frac{1-a}{1-v_c(u)} >$$

$$> \frac{1-a_c}{1-v_c(u)}$$

as it can be proved by simple computation. Hence, this value in Variant III. is greater than value in Variant III.

Example 2.

We want to choose an element $u \in \mathbf{U}$ which is optimal from a point of view of some another goal $g \notin C$ (i.e. $K = \{g\}$) and there is an expert knowledge describing relationships between levels of satisfaction of criterions $c \in C$ and level of satisfaction of a goal g. Moreover, asssume that for any $u \in \mathbf{U}$ we can estimate at least minimal level of satisfaction $v_c(u)$ of u in a criterion $c \in C$. Finally, we require that an optimal choise u has to satisfy this goal with a level of satisfaction at least equel to some crisp number $\alpha \in [0,1]$.

Analogously as we did in the previous case we can do some principal identification of this situation $\mathbf{R} \in \mathcal{R}(\mathbf{U}, C)$, where

1. $K := \{g\}$,
2. $G_g = [\alpha, 1] \subseteq [0, 1]$,
3. $(\forall c \in C, \forall u \in \mathbf{U}) w_c(u) = [v_c(u), 1] \subseteq [0, 1]$
4. We will take $s = s_3$ again from Lemma 1.2
5. $f = \min$ for example.

For identifying this decision situation it is most important to define an expert knowledge base $\mathcal{V} = \mathcal{V}_g$. Let us suppose that a fuzzy linquistic variable $\mathcal{L} = ([0,1], \mathbf{T}, M)$ is given, where $[0,1]$ is the universum of $\mathcal{L}$, $\mathbf{T}$ is the set of terms and M is an interpretation of $\mathcal{L}$, i.e., $M : \longrightarrow \mathcal{F}([0,1])$. This variable then describes values of satisfaction of criterions $c \in C$ and a goal g simultaneously (in general, different variables can be used for different criterions and goals). The expert knowledge can be then described in a following way:

$$\begin{aligned}
\mathbf{V}_1 &= \big(\|c_1\| = t_{1,1} \wedge \|c_2\| = t_{1,2} \wedge \quad \dots \quad \wedge \|c_n\| = t_{1,n} \Longrightarrow \|g\| = t_1\big) \\
&\qquad\qquad\qquad\qquad \dots \\
\mathbf{V}_m &= \big(\|c_1\| = t_{m,1} \wedge \|c_2\| = t_{m,2} \wedge \quad \dots \quad \wedge \|c_n\| = t_{m,n} \Longrightarrow \|g\| = t_m\big)
\end{aligned}$$

where $t_{i,j}, t_i \in \mathbf{T}$ are terms describing value of satisfaction. Hence, in this case we have

$$\mathcal{V}_g = \big\{\big(M(t_{i,1}), \dots, M(t_{i,n}\big), M(t_i)) : i = 1, \dots, m\big\}.$$

In this case a local utility function (Variant I.) is defined as follows:

$$h_{\mathbf{R},g}(u) = \bigvee_{i=1}^{m} \Big(s\big([v_{c_1}(u), 1], M(t_{i,1})\big) \wedge \dots \wedge \big([v_{c_n}(u), 1], M(t_{i,n})\big)\Big) \cdot s\big(M(t_i), [\alpha, 1]\big).$$

3 Relations between GFDS

As we have mentioned in the introduction, any GFDS is a classical mathematical system and, hence, we can define various relations between these systems. At this part we will be dealing with some kind of pre-ordering in the set $\mathcal{R}(\mathbf{U}, C)$ and we will investigate some principal properties of this pre-ordering. A definition of this pre-order relation is done as follows:

Definition 3.1 *Let $R_1, R_2 \in \mathcal{R}(\mathbf{U}, C)$ be two general fuzzy decision systems with the same goals, i.e. $K = K_1 = K_2$ and $G_{1,g} = G_{2,g}$ for all $g \in K$. Then we say that R_1 is* **coarser than** *R_2 (in symbol, $R_1 \le R_2$), if for any goal $g \in K$ and any variants $u, v \in \mathbf{U}$ one of the following conditions holds:*

$$\begin{aligned}
h_{R_2,g}(u) - h_{R_2,g}(v) \ge h_{R_1,g}(u) - h_{R_1,g}(v) \ge 0 \\
h_{R_2,g}(v) - h_{R_2,g}(u) \ge h_{R_1,g}(v) - h_{R_1,g}(u) \ge 0.
\end{aligned}$$

It is clear that a quality of GFDS depends in some sense on its ability to separate elements from $\mathbf{U}$. Hence a GFDS which can better separate these elements will be called *finer*. It is clear that $(\mathcal{R}(\mathbf{U}, C), \le)$ is a pre-order set. In the following lemma we investigate conditions under which two GFDS from this pre-ordered set are mutually comparable.

Lemma 3.2 *Let $R_1, R_2 \in \mathcal{R}(\mathbf{U}, C)$ be with the same goals. Then $R_1 \le R_2$ and $R_2 \le R_1$ if and only if the following condition holds:*

$$(\forall g \in K)(\exists \alpha_g \in [-1, 1])(\forall u \in \mathbf{U}) h_{R_1,g}(u) = h_{R_2,g}(u) + \alpha_g.$$

Proof. In fact, if $R_1 \leq R_2 \leq R_1$ then for any $u, v \in \mathbf{U}$ we have

$$h_{R_2,g}(u) - h_{R_2,g}(v) = h_{R_1,g}(u) - h_{R_1,g}(v).$$

Hence, $\alpha_g = h_{R_1,g}(u) - h_{R_2,g}(u) = h_{R_1,g}(v) - h_{R_2,g}(v)$ satisfies a required condition. The converse implication is also trivial. □

As it can be expected a trivial decision situation are also contained in a set $\mathcal{R}(\mathbf{U}, C)$. In the following lemma we show that for any GFDS from $\mathcal{R}(\mathbf{U}, C)$ there exists the worst possible R_0 such that $R_0 \leq R'$ for any $R' \in \mathcal{R}(\mathbf{U}, C), R' \leq R$.

Lemma 3.3 *Let $R \in \mathcal{R}(\mathbf{U}, C)$. Then there exists $R_0 \in \mathcal{R}(\mathbf{U}, C)$ such that $R_0 \leq R'$ for any $R' \in \mathcal{R}(\mathbf{U}, C)$ such that $R' \leq R$.*

Proof. Let $\{G_g : g \in K\}$ be goals of R. Then $R_0 = \big(\{G_g : g \in K\}, \{w_c : c \in C\}, \{\mathcal{V}_g : g \in K\}, s, f\big)$ be defined such that

1. $w_c(u) = \{1\}$ for any $u \in \mathbf{U}$,
2. $\mathcal{V}_g = \big\{\big(\{1\}, \ldots, \{1\}, G_g\big)\big\}$ for any $g \in K$.

Then we have

$$h_{R,g}(u) = f\big(s(\{1\}, \{1\}), \ldots, s(\{1\}, \{1\})\big) \cdot s(G_g, G_g) = 1, \quad \forall u \in \mathbf{U}.$$

Hence, for any $R' \leq R$ we have $R' \geq R_0$. □

The more important are GFDS which are in some sense maximal. In the next proposition we show some properties of these maximal systems. At first we will need some notation. Let $R \in \mathcal{R}(\mathbf{U}, C)$, where

$$R = \big(\{G_g : g \in K\}, \{w_c : c \in C\}, \{\mathcal{V}_g : g \in K\}, s, \min\big).$$

Since $\mathcal{V}_g$ is a finite set, we put $\mathcal{V}_g = \{V_1, \ldots, V_{m_g}\}$, where any V_i is a following system of expert knowledge

$$V_i = \big((\wedge_{c \in A_i} \|c\| = F_{i,c}) \Longrightarrow \|g\| = T_i\big),$$

where $A_i \subseteq C$ is an appropriate set and $F_{i,c}, T_i$ are fuzzy sets in $[0, 1]$. Moreover, in what follows we will use a local utility function $h_{R,c}$ from Variant I.

Proposition 3.4 *Let $R \in \mathcal{R}(\mathbf{U}, C)$.*

1. *Let $0, 1 \in h_{R,g}(\mathbf{U})$ for some $g \in K$. Then R is a maximal element in $(\mathcal{R}(\mathbf{U}, C), \leq)$.*
2. *Let R be a maximal element in $(\mathcal{R}(\mathbf{U}, C), \leq)$ and let there exists $g \in K$ such that for any $V_i \in \mathcal{V}_g, c \in C$ and $u \in \mathbf{U}$ we have $s(w_c(u), F_{i,c}) < 1$. Then $0, 1 \in h_{R,g}(\mathbf{U})$.*

Proof. 1. Let $\mathbf{U} = \{u_1, \ldots, u_n\}$ be an enumeration of elements from $\mathbf{U}$ such that $0 = h_{R,g}(u_1) \leq \ldots \leq h_{R,g}(u_n) = 1$. Let $\Delta_i = h_{R,g}(u_{i+1}) - h_{R,g}(u_i)$ for $i = 1, \ldots, n-1$. Then $\sum_i \Delta_i = 1$. If there exists $R' \in \mathcal{R}(\mathbf{U}, C)$ such that $R' > R$ and $R \not\geq R'$ then for any $i = 1, \ldots, n-1$ we have $\Delta'_i = h_{R',g}(u_{i+1}) - h_{R',g}(u_i) \geq \Delta_i$ and there exists i_0 such that $\Delta'_{i_0} > \Delta_{i_0}$. Hence, $1 \geq \sum_i \Delta' i > \sum_i \Delta_i = 1$, a contradiction. It follows that R is maximal.

2.Let us suppose that $1 \notin h_{R,g}(\mathbf{U})$. Let elements of $\mathbf{U}$ be enumerated such that $0 \leq h_{R,g}(u_1) \leq \ldots \leq h_{R,g}(u_n) < 1$, $\mathbf{U} = \{u_1, \ldots, u_n\}$. Then there exist maps $\varphi : \mathbf{U} \longrightarrow C$ and $\psi : \mathbf{U} \longrightarrow \{1, \ldots, m\}$ such that

1. For any $u \in \mathbf{U}$, $\varphi(u) \in A_{\psi(u)}$ holds,
2. For any $u \in \mathbf{U}$ we have $h_{R,g}(u) = s(w_{\varphi(u)}(u), F_{\psi(u),\varphi(u)}) \cdots (T_{\psi(u)}, G_g)$.

It is clear that such maps really exist. According to the assumption we have $s(w_{\varphi(u_n)}(u_n), F_{\psi(u_n),\varphi(u_n)}) < 1$. Then we define a new system $\{w'_c : c \in C\}$ of functions $\mathbf{U} \longrightarrow \mathcal{F}([0,1])$ such that

$$w'_c(u) = \begin{cases} w_c(u), & \text{if} \quad c \neq \varphi(u_n) \\ w_{\varphi(u_n)}(u), & \text{if} \quad c = \varphi(u_n), u \neq u_n \\ F_{\psi(u_n),\varphi(u_n)}, & \text{if} \quad c = \varphi(u_n), u = u_n. \end{cases}$$

Now, let $R' = \big(\{G_g : g \in K\}, \{w'_c : c \in C\}, \{\mathcal{V}_g : g \in K\}, s, \min\big) \in \mathcal{R}(\mathbf{U}, C)$. We show that $R' > R, R \not\geq R'$.

In fact, since for any $c \in A_{\psi(u_n)}$ we have

$$s(w'_c(u_n), F_{\psi(u_n),c}) \geq s(w_c(u_n), F_{\psi(u_n),c}),$$

then we have

$$\begin{aligned} h_{R',g}(u_n) &= \bigvee_{i=1}^{m_g} \Big(\big(\bigwedge_{c \in A_i} s(w'_c(u_n), F_{i,c}) \big) \cdot s(T_i, G_g) \Big) \geq \\ &\geq \Big(\bigwedge_{c \in A_{\psi(u_n)}} s\big(w'_c(u_n), F_{\psi(u_n),c}\big) \Big) \cdot s(T_{\psi(u_n)}, G_g) = \\ &= s\big(w'_{\varphi(u_n)}(u_n), F_{\psi(u_n),\varphi(u_n)}\big) \cdot s(T_{\psi(u_n)}, G_g) > \\ &> s\big(w_{\varphi(u_n)}(u_n), F_{\psi(u_n),\varphi(u_n)}\big) \cdot s(T_{\psi(u_n)}, G_g) = h_{R,g}(u_n). \end{aligned}$$

Moreover, for any $u \neq u_n$ we have $h_{R',g}(u) = h_{R,g}(u)$. Then for any $u, v \in \mathbf{U}$ we have $|h_{R',g}(u) - h_{R',g}(v)| \geq |h_{R,g}(u) - h_{R,g}(v)|$ and for $u = u_n, v = u_{n-1}$ a strict inequality there holds. Hence, $R' > R \not\geq R$, a contradiction.

Let us suppose that $0 \notin h_{R,g}(\mathbf{U})$ for some $g \in K$. Then for any $c \in C$ we can define a function w'_c such that

$$w'_c(u) = \begin{cases} w_c(u), & \text{if} \quad u \neq u_1 \\ \emptyset, & \text{if} \quad u = u_1 \end{cases}$$

Then for $R' = (\{G_g : g \in K\}, \{w'_c : c \in C\}, \{\mathcal{V}_g : g \in K\}, s, \min)$ we have

$$h_{R',g}(u_1) = 0 < h_{R,g}(u_1), \quad h_{R',g}(u) = h_{R,g}(u), \quad u \in \mathbf{U}, u \neq u_1.$$

Hence, for any $u, v \in \mathbf{U}$ we have $|h_{R',g}(u) - h_{R',g}(v)| \geq |h_{R,g}(u) - h_{R,g}(v)|$ and for $u = u_1, v = u_2$ a strict inequality there holds. Hence, $R' > R \not\geq R$, a contradiction. □

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