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Logical Structure of Fuzzy IF-THEN Rules

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Abstract

This paper provides a logical basis for manipulation with *fuzzy IF-THEN rules*. Our theory is wide enough and it encompasses not only finding a conclusion by means of the *compositional rule of inference* due to Lotfi A. Zadeh but also other kinds of approximate reasoning methods, e.g. perception-based deduction, provided that there exists a possibility to characterize them within a formal logical system.

In contrast with other approaches employing variants of *multiple-valued first order logic*, the approach presented here employs *fuzzy type theory* of V. Novák which has sufficient expressive power to present the essential concepts and results in a compact, elegant and justifiable form.

Within the effectively formalized representation developed here, based on a complete logical system, it is possible to reconstruct numerous well-known properties of CRI-related fuzzy inference methods, albeit not from the *analytic* point of view as usually presented, but as formal derivations of the logical system employed.

The authors are confident that eventually all relevant knowledge about fuzzy inference methods based on fuzzy IF-THEN rule bases will be represented, formalized and backed up by proof within the well-founded logical representation presented here. An immediate positive consequence of this approach is that suddenly all elements of a fuzzy inference method based on fuzzy IF-THEN rules are 'first class citizens´ of the representation: There are clear, logically founded definitions for fuzzy IF-THEN rule bases to be *consistent*, *complete*, or *independent*.

Key words: Fuzzy logic, type theory, fuzzy relation equations, fuzzy type theory, fuzzy IF-THEN rules, compositional rule of inference.

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1 Introduction

This paper has been inspired by the works of H. Thiele [34, 35], who proposed to use the concepts of logic such as model, consistency, extension, consequence and others for the analysis of fuzzy IF-THEN rules. Our main goal in this paper is to formalize the meaning of fuzzy IF-THEN rules. There is an extensive literature devoted to this task (see, for example, [2, 9, 12, 15, 26, 28, 29] and elsewhere). Fuzzy IF-THEN rules are usually taken as specific characterization of dependencies among objects. The basic tool is then either pure fuzzy set theory or the predicate fuzzy logic. For example, P. Hájek in his book [9] developed a theory of fuzzy IF-THEN rules within many-sorted first order BL-fuzzy logic extended by a special binary fuzzy equivalence predicate. Similarly, I. Perfilieva shows in [28, 29] that sets of fuzzy IF-THEN rules can be modeled using special formulas of fuzzy predicate logic being generalization of the classical boolean conjunctive and disjunctive normal forms. The goal in these and similar works is to provide a working theory suitable for technical applications, the main problem of which is to find a sufficiently precise approximation of some function. It should be stressed that none of the cited authors considered fuzzy IF-THEN rules as linguistic expressions.

For the advanced applications, for example in robotics or artificial intelligence, however, it is a challenge to take fuzzy IF-THEN rules as genuine expressions of natural language and to capture their meaning from this point of view. A good reason for this effort is the goal to develop human-like behaving robots. First order predicate logic, however, is insufficient for this task because it does not allow to formalize the concepts of intension and extension. Though fuzzy IF-THEN rules have fairly simple linguistic structure and so, first order logic can sometimes overcome this problem, further extension to more complicated linguistic phenomena would hardly be possible.

This is the main reason why we decided to employ the *fuzzy type theory* of V. Novák [16] which has sufficient expressive power to present the essential concepts and results in a compact, elegant and justifiable form. Our wider goal is to develop a concise and sufficiently general theory of the fuzzy IF-THEN rules that would encompass not only the above mentioned technical means for approximation of functions but also formalization of the linguistic meaning of these rules when taken as special expressions of natural language. Our theory also fits well the idea of developing a *precisiated natural language* (PNL) presented by L. A. Zadeh in [37], i.e. to find a reasonable mathematical model of the meaning of a certain subset of natural language expressions that could be used in various applications and that would at the same time comply with the human's way of understanding them.

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the Collaborative Research Center "Computational Intelligence" (531).

When dealing with the meaning of fuzzy IF-THEN rules, one cannot avoid the problem of solvability of fuzzy relation equations. This theory has been originated by Sanchez in [33] and further elaborated by many authors, for example Di Nola et al. [4], Gottwald [7], Gottwald et al. [8], Klawonn [11], Novák et al. [27], Perfilieva and Lehmke [30], Perfilieva and Tonis [31] and others. The authors in [30] show that the problem of solvability of fuzzy relation equations has a deeper meaning, namely that it corresponds to consistency of the description of the given problem. These results became motivation for part of our analysis contained in this paper. We will show that this problem is connected with inherent meaning of the text (we speak about implicit intension of the linguistic description). Having sound formal logical theory at disposal, we may study its relation to the meaning (intension) of the rules themselves. Moreover, we may introduce clear, logically founded definitions for fuzzy IF-THEN rule bases to be *consistent*, *complete*, or *independent* and to consider also *dependent* IF-THEN rules that can be derived from rule bases (and so, be proved to be redundant).

Our theory is wide enough and it encompasses not only finding a conclusion by means of the *compositional rule of inference* due to L. A. Zadeh [36] but also other kinds of approximate reasoning methods, e.g. the perception-based logical deduction (see [21, 25]), provided that it is possible to characterize them within a formal logical system. Within the effectively formalized representation developed here (based on a complete logical system), it is possible to reconstruct numerous well-known properties of CRI-related fuzzy inference methods, albeit not from the *analytic* point of view as usually presented, but as formal derivations of the logical system employed.

The paper is organized as follows: First we characterize the fuzzy IF-THEN rules as special linguistic expressions starting with their components. In Section 4 we briefly remind some main concepts of the fuzzy type theory (FTT). However, for full understanding to this paper, we suppose the reader to be acquainted with the papers [16, 17]. Section 5 is devoted to formalization of the meaning of a single fuzzy IF-THEN rule and the following section to formalization of the meaning of systems of them.

It is important to stress that our paper in a certain sense summarizes and systematizes results scattered over the literature on fuzzy IF-THEN rules and also the related problem of fuzzy relation equations. Most theorems are proved syntactically, some of them are syntactical formulation of the known properties that were derived on the basis of more narrow assumptions and often without logical considerations. Our results often contribute to deeper understanding of the given phenomenon in a wider context.

2 Preliminaries

By a fuzzy set we mean a function $A : U \longrightarrow L$ where U is a universe and L is a support of a given structure of truth values. The latter set coincides with the set of truth values for the fuzzy type theory (see Section 4). To express that A is a fuzzy set in U, we will sometimes write $A \underset{\sim}{\subset} U$. A (binary) fuzzy relation is a fuzzy set $R \underset{\sim}{\subset} U \times V$, i.e. a function $R : U \times V \longrightarrow L$.

The composition of a fuzzy set $A \underset{\sim}{\subset} U$ and a fuzzy relation $R \underset{\sim}{\subset} U \times V$ is a fuzzy set $B \underset{\sim}{\subset} V$ given by

$$B(v) = \bigvee_{u \in U} (A(u) \otimes R(u, v)) \tag{1}$$

where $\otimes$ is a suitable product operation (a t-norm) — see Section 4. We usually write (1) as

$$B = A \circ R. \tag{2}$$

A *fuzzy relation equation* is the equation (2) for the unknown fuzzy relation R. If such R exists then we say that the *fuzzy relation equation (2) is solvable*.

Let a set of couples of fuzzy sets $\langle A_i, B_i \rangle$, $i = 1, \ldots, m$ be given. This set leads to a system of fuzzy relation equations

$$B_i = A_i \circ R, \qquad i = 1, \ldots, m \tag{3}$$

for the unknown fuzzy relation R. If such R exists then we say that the system (3) *is solvable*.

3 Fuzzy IF-THEN rules as special linguistic expressions

3.1 Adjectival predications

In this paper, we will consider a specific part of natural language, namely adjectival clauses and adjectival predications. Adjectival clause consists of an adjective and, possibly, a modifier. For example, *red, very big, full of spare parts, rather sad, totally empty, roughly medium*, etc.

A significant constituent of adjectival clauses are the, so called, *evaluating linguistic expressions* (e.g., very small, extremely long, roughly medium, etc.) that are special expressions used for characterization of a position on some bounded ordered scale (see [19, 22, 26]). Note that we may also consider adjectival phrases[†)] that are phrases with an adjective as their head (e.g. full of spare parts). Adjectival phrases and clauses are, in essence, names of properties of objects.

Adjectival clauses and phrases may occur as premodifiers to a noun (*a very tall man, a container full of spare parts*), or as *adjectival predications*, i.e. expressions affiliating adjectives to a noun, for example *the temperature is very high, the container is full of spare parts, my friend is extremely intelligent*, etc. We will take both these expressions as synonymous and, in the sequel, deal with adjectival predications only. We will not go into further details of their structure since they are unnecessary for the general theory of fuzzy IF-THEN rules discussed further. Throughout this paper, we will denote the considered expressions of natural language by script letters $\mathcal{A}, \mathcal{B}, \mathcal{R}, \ldots$

Definition: Let $\mathcal{A}$ be an adjectival clause.

(i) The linguistic expression

$$\langle\text{noun}\rangle \ \mathsf{is}\ \mathcal{A} \tag{4}$$

is an *adjectival predication*.

(ii) Let $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ be adjectival predications. Then the following are *compound adjectival predications*:

(a) $\mathcal{R}^A := \bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$,
(b) $\mathcal{R}^O := \bar{\mathcal{A}}$ or $\bar{\mathcal{B}}$,
(c) $\mathcal{R}^I :=$ IF $\bar{\mathcal{A}}$ THEN $\bar{\mathcal{B}}$.

In the case of $\mathcal{R}^I$, the part before THEN is called *antecedent* and after it *consequent* (or succedent). Note that in more complicated cases, the antecedent may be formed from $\mathcal{R}^A$ as well as $\mathcal{R}^O$, for example "IF temperature is high or pressure is very low THEN ...", or "IF road is wide and very well preserved or bridge is new or strong THEN ...", etc.

The adjectival predications $\mathcal{R}^A$ and $\mathcal{R}^I$ linguistically characterize (or express existence of) some relation between arbitrary objects x and y named by the respective nouns, that have properties named by $\mathcal{A}$ and $\mathcal{B}$, respectively. The difference between $\mathcal{R}^A$ and $\mathcal{R}^I$ consists in the extent of their operation. Namely, $\mathcal{R}^I$ characterizes *arbitrary* relation: it assures that *if* objects have the property named by $\mathcal{A}$ *then* they are in relation with objects having the property named by $\mathcal{B}$; if not then the relation is possible but not specified.

On the other hand, $\mathcal{R}^A$ characterizes only *positive* relation — just objects having the property named by $\mathcal{A}$ are in relation with objects having the property

[†)] Informally, a phrase is a group of words in a sentence that functions somewhat like as single word.

named by $\mathcal{B}$. Thus, $\mathcal{R}^A$ can be understood as a special (more important) case of $\mathcal{R}^I$.

By $\mathcal{S}$, we denote the set of linguistic expressions consisting of adjectival clauses and adjectival predications. Note that in the applications of fuzzy logic, only that part of $\mathcal{S}$ consisting of *evaluating* linguistic expressions and predications is till now employed (mostly without the attempt to penetrate into their linguistic meaning).

3.2 Fuzzy IF-THEN rules

The '⟨noun⟩' is often unimportant in the applications. Therefore, we will replace it by some variable X whose values are not the objects themselves but only their *features*, such as *height, volume, tension, size, abstract degrees of beauty, temperature*, etc. This leads us to the concept of a fuzzy IF-THEN rule which is a kind of "abstracted" compound adjectival predication.

Definition: *By* fuzzy IF-THEN rule *we understand either of the compound adjectival predications $\mathcal{R}^I$ and $\mathcal{R}^A$ taken in the form*

$$\mathcal{R}^A := X \textsf{ is } \mathcal{A} \textsf{ AND } Y \textsf{ is } \mathcal{B}, \tag{5}$$

$$\mathcal{R}^I := \textsf{ IF } X \textsf{ is } \mathcal{A} \textsf{ THEN } Y \textsf{ is } \mathcal{B} \tag{6}$$

where X, Y represent features of objects.

The symbols $\mathcal{R}^A$ and $\mathcal{R}^I$ will be used as metavariables for the corresponding forms of fuzzy IF-THEN rules (5) and (6), respectively. If the concrete form of a fuzzy IF-THEN rule can be arbitrary then we simply write $\mathcal{R}$.

In addition, we may also introduce a simplified form of the compound adjectival predication $\mathcal{R}^O$ by

$$\mathcal{R}^O := X \textsf{ is } \mathcal{A} \textsf{ OR } Y \textsf{ is } \mathcal{B}. \tag{7}$$

This predication, however, characterizes a very weak form of a relation between X and Y and so, we will not take it as a fuzzy IF-THEN rule.

3.3 Topic and focus of fuzzy IF-THEN rules

One of the most important linguistic phenomena is division of the linguistic meaning of sentences into topic and focus. Informally, the *topic* is the theme, i.e. what is spoken about while *focus* is that part which conveys a new information. Every sentence is assumed to have a focus since otherwise it could not convey relevant information. However, there can be sentences without topic.

The division into topic and focus for one sentence is not unique. It depends on several aspects, which we will not discuss here. Let us only remark that this fact is one of the sources of extreme power of natural language to characterize real world phenomena. The interested reader is referred to [10, 32].

Since the fuzzy IF-THEN rules are also special natural language sentences, they have topic and focus. Their structure, of course, is very simple and so, we can define them unambiguously as follows.

Definition: Let $\mathcal{R}$ be a fuzzy IF-THEN rule (5) or (6). Then its *topic* is formed by the adjectival predication 'X is $\mathcal{A}$' and its *focus* by 'Y is $\mathcal{B}$'.

4 Fuzzy type theory

The main tool for the logical analysis of fuzzy IF-THEN rules in this paper is the fuzzy type theory (FTT). In this section, we will very briefly overview some of the main points of the fuzzy type theory. The detailed explanation of (FTT) can be found in [16–18]. The classical type theory is in details described in [1].

4.1

Let ϵ, o be distinct objects. The set of types is the smallest set *Types* satisfying:

(i) $\epsilon, o \in$ *Types*,
(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

The type ϵ represents elements, o truth values.

A set of formulas of type $\alpha \in$ *Types*, denoted by $Form_\alpha$, is a smallest set satisfying:

(i) Variables $x_\alpha \in Form_\alpha$ and constants $c_\alpha \in Form_\alpha$,
(ii) if $B \in Form_{\beta\alpha}$ and $A \in Form_\alpha$ then $(BA) \in Form_\beta$,
(iii) if $A \in Form_\beta$ then $\lambda x_\alpha\, A \in Form_{\beta\alpha}$,

If $A \in Form_\alpha$ is a formula of the type $\alpha \in$ *Types* then we usually write A_α. On the other hand, to make the notation more transparent, we will often write in advance that, e.g. $x \in Form_\alpha$ and then write simply x instead of x_α. Note that in FTT, all elements of the syntax are formulas (in the literature, formulas are quite often called λ-terms, instead) including variables or connectives.

The formulas of type o (truth value) can be joined by the following connectives: $\vee$ (disjunction), $\wedge$ (conjunction), $\&$ (strong conjunction), ∇ (strong disjunction), $\Rightarrow$ (implication). There are also general ($\forall$) and existential ($\exists$) quantifiers defined. For the details about their definition and semantics — see [16].

4.2 Notation

To simplify the notation as much as possible, which means especially to minimize the number of brackets, we will use the following conventions in the sequel.

Priority of logical connectives:

(1) $\neg, \boldsymbol{\Delta}$.
(2) $\&, \nabla$.
(3) $\wedge, \vee$.
(4) $\Rightarrow$.
(5) $\equiv$.

Furthermore, we will also use the dot convention as follows: the formula $A \cdot B$ is equivalent to $A(B)$. For example, the formula $\lambda x \cdot Ax \wedge \cdot Bx \Rightarrow C$ is equivalent to $\lambda x\, (Ax \wedge (Bx \Rightarrow C))$.

4.3 The structure of truth values

The structure of truth values is supposed to form a complete IMTL$_\Delta$-algebra (see [5]), which is a complete residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \Delta, \mathbf{0}, \mathbf{1} \rangle \tag{8}$$

fulfilling the prelinearity condition

$$(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}, \qquad a, b \in L,$$

and, moreover, its negation function $\neg a = a \rightarrow \mathbf{0}$ is involutive, i.e. $\neg\neg a = a$ holds for all $a \in L$. The (Baaz) Δ is a special unary operation of $\mathbf{0} - \mathbf{1}$ projection. For its general definition, see, e.g. [9]. In case that $\mathcal{L}$ is linearly ordered, Δ is defined by

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{9}$$

It is known that MTL-algebras are algebras (the negation needs not be involutive) of left-continuous t-norms. An example of left-continuous t-norm with involutive negation is *nilpotent minimum*. In this paper we often suppose a stronger structure of $\mathcal{L}$, and namely, that it is the Łukasiewicz MV-algebra. In this case, $L = [0, 1]$ and $\otimes$ is the operation of Łukasiewicz conjunction and $\rightarrow$ that of Łukasiewicz implication.

4.4 Syntax

The syntax of FTT consists of definitions of fundamental formulas, axioms and inference rules. A more detailed presentation would be too extensive and so, we will repeat only some of the main points.

The FTT has 17 axioms that may be divided into the following subsets: fundamental equality axioms, truth structure axioms, quantifier axioms and axioms of descriptions.

Fundamental equality axioms are the following:

(FT$_I$1) $\mathbf{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$
(FT$_I$2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$
(FT$_I$2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$
(FT$_I$3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$ where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).
(FT$_I$4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

Further axioms characterize structure of truth values. These axioms (altogether 11) assure that the predicate IMTL (or Łukasiewicz) fuzzy logic with the $\mathbf{\Delta}$ connective is included in FTT, i.e. all theorems of the former are provable also in FTT. Specific and important axiom for provability properties among them is the axiom

(FT$_I$6) $(A_o \equiv \top) \equiv A_o$

The quantifier axiom is

(FT$_I$16) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$ where x_α is not free in A_o.

A special constant used in FTT is the description operator $\iota_{\alpha(o\alpha)}$. For its interpretation see below. The description operator is tied with the fuzzy equality $\mathbf{E}_{(o\alpha)\alpha}$ by means of the following axioms of descriptions:

(FT$_I$17) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon.$

There are two inference rules in FTT, namely:

(R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then, infer B' where B' comes from B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

(N) *Let $A_o \in Form_o$. Then infer $\mathbf{\Delta} A_o$ from A_o.*

A theory T is a set of formulas of type o (determined by a subset of special axioms, as usual). Provability is defined as usual. The inference rules of modus ponens and generalization are derived rules of FTT.

4.5 Semantics

Let J be a language of FTT. A *frame* for J is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{10}$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ is a structure of truth values (IMTL$_\Delta$ or Łukasiewicz$_\Delta$ algebra).
(ii) $=_\alpha$ is a fuzzy equality on M_α and $=_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

Recall that if $\beta\alpha$ is a type then the corresponding set $M_{\beta\alpha}$ contains (not necessarily all) functions $f : M_\alpha \longrightarrow M_\beta$.

We put $M_o = L$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.

Let p be an assignment of elements from $\mathcal{M}$ to variables from the language J. An interpretation $\mathcal{I}^\mathcal{M}$ is a function that assigns every formula A_α, $\alpha \in$ *Types* and every assignment p a corresponding element, that is, a function (or element) of the type α. With respect to the above remark, a general model is a frame $\mathcal{M}$ such that

$$\mathcal{I}_p^\mathcal{M}(A_\alpha) \in M_\alpha \tag{11}$$

holds true. This means that each set M_α from the frame $\mathcal{M}$ has enough elements so that the interpretation of each formula $A_\alpha \in Form$ is always defined in $\mathcal{M}$.

A special constant of the language J is the above mentioned description operator $\iota_{\alpha(o\alpha)}$. Its interpretation a certain operation assigning to each normal fuzzy set in the universe M_α an element from its kernel (cf. [17]). Such an operation is in fuzzy set theory called *defuzzification*.

We also define a special operator

$$\imath z_\alpha A_o := \iota_{\alpha(o\alpha)}(\lambda z_\alpha A_o)$$

that picks up an element of type α such that the formula A_o is true in the degree 1 for it (i.e., it belongs to the kernel of the fuzzy set $\lambda z_\alpha A_o$).

In this paper we will also use speak about contexts (possible worlds) that will refer to specific types. In case of evaluating expressions, these types have the form αo (cf. Subsection 5.6). In general, these types may be even more complicated. Because of larger variety of possibilities, we will write use the general symbol W for the set of all contexts.

4.6

We will often use the equality theorem of FTT in the following form (cf. [16], Theorem 11).

Lemma
Let $A_{\beta\alpha}, B_{\beta\alpha}$ be formulas of type $\beta\alpha$. Then

$$\vdash (\forall x_\alpha)(A_{\beta\alpha}x_\alpha \equiv B_{\beta\alpha}x_\alpha) \equiv (\lambda y_\alpha\, A_{\beta\alpha}y_\alpha \equiv \lambda y_\alpha\, B_{\beta\alpha}y_\alpha).$$

PROOF: This follows from the lambda conversion axiom $\vdash C_{\beta\alpha}x_\alpha \equiv \lambda y_\alpha\, C_{\beta\alpha}y_\alpha \cdot x_\alpha$. □

4.7 Special properties of FTT

In the paper, we will use a special formula which determines a nonzero truth value:

$$\Upsilon_{oo} := \lambda z_o(\neg(\boldsymbol{\Delta}(\neg z_o))).$$

It can be easily proved that for any interpretation $\mathcal{I}^{\mathcal{M}}$ and any assignment p to variables,

$$\mathcal{I}_p^{\mathcal{M}}(\Upsilon z_o) = \mathbf{1} \quad \text{iff} \quad p(z_0) = a > \mathbf{0}.$$

Further properties of FTT are summarized in the following lemma.

Lemma

(a) Let $T \vdash (\exists x_\alpha)\boldsymbol{\Delta}B$. Then $T \cup \{B_{x_\alpha}[\mathbf{u}_\alpha]\}$ is a conservative extension of T (rule C) where $\mathbf{u}_\alpha \notin J(T)$.
(b) Let $T \vdash (\exists x_\alpha)\boldsymbol{\Delta}B_{o\alpha}x_\alpha$. Then $T \vdash B_{o\alpha} \cdot \iota_{\alpha(o\alpha)}B_{o\alpha}$.
(c) Let $T \vdash (\exists x_\alpha)A_o$. Then $T \vdash (\exists x_\alpha)A_o^n$ for all $n \geq 1$.
(d) $\vdash (\exists x)\boldsymbol{\Delta}A_o \Rightarrow \boldsymbol{\Delta}(\exists x)A_o$ and $\vdash (\exists x)\boldsymbol{\Delta}A_o \Rightarrow (\exists x)A_o$.
(e) $\vdash (\exists x)(\exists y)\boldsymbol{\Delta}A \equiv (\exists x)\boldsymbol{\Delta}(\exists y)\boldsymbol{\Delta}A$.
(f) $\vdash \boldsymbol{\Delta}(x_o \,\&\, y_o) \equiv \boldsymbol{\Delta}x_o \,\&\, \boldsymbol{\Delta}y_o$.
(g) Let $T \vdash \Upsilon z_o \,\&\, (z_o \Rightarrow y_o)$. Then $T \vdash \Upsilon y_o$.
(h) For all $\alpha \in$ Types,

$$\vdash (x_\alpha \equiv z_\alpha) \equiv (\exists y_\alpha)((x_\alpha \equiv y_\alpha) \,\&\, (y_\alpha \equiv z_\alpha)).$$

(i) $(\forall x_\alpha)(A_{o\alpha}x_\alpha \equiv B_{o\alpha}x_\alpha) \Rightarrow \cdot(\forall x_\alpha)A_{o\alpha}x_\alpha \equiv (\forall x_\alpha)B_{o\alpha}x_\alpha$.

(j) $(\forall x_\alpha)(A_{o\alpha}x_\alpha \equiv B_{o\alpha}x_\alpha) \Rightarrow \cdot(\exists x_\alpha)A_{o\alpha}x_\alpha \equiv (\exists x_\alpha)B_{o\alpha}x_\alpha$.

In the sequel, we will often refer to axioms and various other proved facts from the fuzzy type theory. Since it is not possible to list all of them here, we will simply write "by properties of FTT" and refer the reader to [16].

4.8 Extensionality

We say that a formula $A_{o\alpha}$ is *strongly extensional* in a theory T, if

$$T \vdash x_\alpha \equiv y_\alpha \Rightarrow A_{o\alpha}x_\alpha \equiv A_{o\alpha}y_\alpha. \tag{12}$$

Lemma
A formula $A_{o\alpha}$ is strongly extensional in T iff

$$T \vdash A_{o\alpha}y_\alpha \equiv (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}x_\alpha). \tag{13}$$

PROOF: By properties of FTT, we have $\vdash A_{o\alpha}y_\alpha \equiv \cdot y_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}y_\alpha$, i.e. $\vdash A_{o\alpha}y_\alpha \Rightarrow \cdot y_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}y_\alpha$ and, consequently,

$$\vdash A_{o\alpha}y_\alpha \Rightarrow (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}x_\alpha). \tag{14}$$

Let $A_{o\alpha}$ be strongly extensional. Then

$$T \vdash (x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}x_\alpha) \Rightarrow A_{o\alpha}y_\alpha.$$

Using rule of generalization and properties of quantifiers, we obtain

$$T \vdash (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}x_\alpha) \Rightarrow A_{o\alpha}y_\alpha$$

which together with (14) implies (13).

Conversely, let (13) hold. Then $T \vdash (\exists x_\alpha)(x_\alpha \equiv y_\alpha \,\&\, A_{o\alpha}x_\alpha) \Rightarrow A_{o\alpha}y_\alpha$, and by properties of quantifiers, substitution and properties of FTT we obtain $T \vdash (x_\alpha \equiv y_\alpha) \Rightarrow (A_{o\alpha}x_\alpha \Rightarrow A_{o\alpha}y_\alpha)$. Similarly, we proceed with the variables x_α and y_α exchanged and so, obtain (12) by properties of FTT. □

5 Formalization of the meaning of a single fuzzy IF-THEN rule

5.1

Our main goal in this paper is to formalize the meaning of fuzzy IF-THEN rules. As mentioned in the Introduction, there already exist formal logical theories of them (see, for example, [9, 28, 29]). However, none of them covers their linguistic meaning and, in fact, it describes only extensions (see below). More focus on the linguistic side of the fuzzy IF-THEN rules can be found in the book [26]. The theory of their meaning is there also formed within first order fuzzy logic, but in the generalized form that adds evaluation to formulas also on the syntactic level (we speak about fuzzy logic with *evaluated syntax*). It is demonstrated that this logic can already capture the distinction between intension and extension. The formalism is clear but more complicated. Because further extension of these theories to more complicated linguistic phenomena is hardly possible we decided to use fuzzy type theory in this paper.

First, we will fix some formal language J of FTT. For our explanation, it is unnecessary to define explicitly all its special symbols. Therefore, when writing $A \in J$ for some symbol A we silently assume, that such a symbol has already been introduced in the language J.

Furthermore, we will suppose to work in some formal theory T of FTT within the language J. This means that certain specific formulas are provable in T. However, we will not specify explicitly axioms of T since we do not need them. Hence, T is arbitrary formal theory in which the formulas in concern are provable. If the reader feels uncomfortable with this, he/she may assume that if a new formula is said to be provable but without explicit reference to its proof then it can be just taken as an axiom of T.

To simplify notation, we will use special variables (formulas) for specific types only. Namely, we will introduce special types $\omega, \omega', \dots$ for *contexts* (*possible worlds*) and use variables $w, w', \dots$ for formulas of this type, i.e. $w \in Form_\omega$, $w' \in Form_{\omega'}$, ... More precise explanation is provided in Subsections 5.6 and 5.7 below. Furthermore, variables x, y are always formulas of some types $\alpha, \beta, \dots$ that represent objects of various kinds.

5.2 Intension of adjectival clauses

When speaking about meaning of an adjectival clause, we must distinguish between its *intension* and *extension*. We will follow the generally accepted way how these concepts are mathematically modeled (cf., e.g. [6] and elsewhere), i.e. intension is a function assigning to each possible world and to each object a truth value. In other words, each possible world is assigned a class of objects. Extension is then the class of objects in a concrete possible world. This simple idea fits the requirement that *intension does not depend* on a concrete possible world while *extension does*. For the reasons that become clear later, we will prefer the term *context* to *possible world*.

Let $\mathcal{A}$ be an adjectival clause. We will interpret it by a formula $A \in Form_{(o\alpha)\omega}$ which represents a function assigning to each context (possible world) a fuzzy set. Clearly, the latter is extension of $\mathcal{A}$ in the given context.

Definition: Let $\mathcal{A}$ be a linguistic expression and $w \in Form_\omega$ be a context. Then *intension* of $\mathcal{A}$ is a formula

$$\mathrm{Int}(\mathcal{A}) := \lambda w\, \lambda x \cdot A_{(o\alpha)\omega} wx, \tag{15}$$

and *extension* of $\mathcal{A}$ in the context w is a formula

$$\mathrm{Ext}_w(\mathcal{A}) := \mathrm{Int}(\mathcal{A})w. \tag{16}$$

Note that we immediately get

$$\vdash \mathrm{Ext}_w(\mathcal{A}) \equiv \lambda x \cdot A_{(o\alpha)\omega} wx \tag{17}$$

using λ-conversion.

Interpretation of the intension (15) is a function from a set of all contexts W into the set of fuzzy sets in the universe M_α. Interpretation of the extension (16) is a fuzzy subset of M_α. More precisely is the interpretation described in Subsection 5.17.

5.3 Intension of compound adjectival clauses

Let $\mathcal{A}, \mathcal{B}$ be adjectival clauses (not predications!) such that

$$\mathrm{Int}(\mathcal{A}) = \lambda w\, \lambda x \cdot A_{(o\alpha)\omega} wx, \tag{18}$$

$$\mathrm{Int}(\mathcal{B}) = \lambda w'\, \lambda x \cdot B_{(o\alpha)\omega'} w'x. \tag{19}$$

where $x \in Form_\alpha$. We may form compound clauses using the connectives ‘and’ and ‘or’. For example, we may consider expressions such as “small and stout”, “red or green”, “beautiful and clever”, “very hot and heavy”, etc. Then

$$\mathrm{Int}(\mathcal{A} \text{ and } \mathcal{B}) = \lambda w\, \lambda w'\, \lambda x \cdot A_{(o\alpha)\omega} wx \wedge B_{(o\alpha)\omega'} w'x \tag{20}$$

$$\mathrm{Int}(\mathcal{A} \text{ or } \mathcal{B}) = \lambda w\, \lambda w'\, \lambda x \cdot A_{(o\alpha)\omega} wx \vee B_{(o\alpha)\omega'} w'x \tag{21}$$

where $\wedge$ can be also replaced by $\&$, and $\vee$ by ∇ [†)] .

[†)] The difference between the two conjunctions $\wedge$ and $\&$ (disjunctions $\vee$, ∇) comes out from their mathematical interpretation and their role in the formal theory. The conjunction $\&$ (sometimes called strong conjunction) must be used when combining formulas in modus ponens while $\wedge$ has rather auxiliary role. From the linguistic point of view, $\wedge$ is a *phrasal* conjunction ($\vee$ is a phrasal disjunction), i.e. it should primarily be used for joining phrases. The strong conjunction $\&$ (disjunction ∇) is *sentential*, i.e. it should be used for joining sentences. However, precise theory giving rules, how and where these connectives should be used is not yet elaborated.

Note that we deal here with different contexts w, w' of the corresponding types ω and ω'. They may be interpreted, for example, as "height" and "weight" of a "man", respectively where "man" is represented by object $x \in Form_\alpha$.

5.4 Intension of adjectival predications

In this paper we usually do not consider concrete objects represented by nouns but replace them by variables $X, Y, \dots$ whose values are *features of objects* discussed in Subsection 3.2. This simplification enables us to simplify also the model of meaning of the adjectival predications. For example, a *small* object, say a tree, a man, a barrel, etc. means that some feature measured on it (for example, *size, volume*, etc.) is close to a certain minimal value (e.g. 0). Similarly, *green* means that some feature (i.e. wavelength of the reflected light) falls in a fuzzy set of wavelengths that characterize the property of being "green" (something around 560nm). Quite analogous reasoning can be made for other adjectives and adjectival clauses, such as *full, very large, not too intelligent, extremely tasty*, etc. Note that concrete objects are involved not directly but vicariously by *their features*!

The linguistic predication "X is $\mathcal{A}$" should be read as "each value of X has the property named by $\mathcal{A}$". Hence, we conclude that its intension is

$$\mathrm{Int}(X \text{ is } \mathcal{A}) = \mathrm{Int}(\mathcal{A}) = \lambda w\, \lambda x \cdot A_{(o\alpha)\omega}\, wx. \tag{22}$$

Of course, we may think of one specific element, say x. This gives an intension

$$\mathrm{Int}(X \text{ is } \mathcal{A}) = \lambda w \cdot A_{(o\alpha)\omega}\, wx.$$

However, if not stated otherwise, we will always consider the case (22). Let us stress that the variable X stands either for a feature of an object, or for the object itself.

5.5 Explicit definition of intension of compound adjectival predications

Compound adjectival predications, in general, characterize relations between elements of different kinds having the respective properties. Let the linguistic predications 'X is $\mathcal{A}$' and 'Y is $\mathcal{B}$' have the intensions (18) and (19), respectively (this assumption is justified by (22)) where, however, $x \in Form_\alpha$ and $y \in Form_\beta$ for some, not necessarily different, types α and β, i.e. the elements in the predications may possibly be of different kinds. Then, we put:

$$\mathrm{Int}(\mathcal{R}^A) = \mathrm{Int}(X \text{ is } \mathcal{A} \textsf{ and } Y \text{ is } \mathcal{B}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot A_{(o\alpha)\omega} wx \mathbin{\&} B_{(o\beta)\omega'} w'y, \tag{23}$$

$$\mathrm{Int}(\mathcal{R}^O) = \mathrm{Int}(X \text{ is } \mathcal{A} \textsf{ or } Y \text{ is } \mathcal{B}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot A_{(o\alpha)\omega} wx \vee B_{(o\beta)\omega'} w'y, \tag{24}$$

$$\mathrm{Int}(\mathcal{R}^I) = \mathrm{Int}(\textsf{ IF } X \textsf{ is } \mathcal{A} \textsf{ THEN } Y \textsf{ is } \mathcal{B}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot A_{(o\alpha)\omega} wx \Rightarrow B_{(o\beta)\omega'} w'y. \tag{25}$$

It is possible to replace $\&$ by $\wedge$, and $\vee$ by ∇.

In general, we will write *intension of a fuzzy IF-THEN rule* $\mathcal{R}$ (i.e. of the compound predication (5) or (6)) as

$$\mathrm{Int}(\mathcal{R}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot A_{(o\alpha)\omega} wx \,\square\, B_{(o\beta)\omega'} w'y \tag{26}$$

where $\square$ is either of the connectives $\&$, $\wedge$ or $\Rightarrow$, and $x \in Form_\alpha$ and $y \in Form_\beta$.

5.6 Formal definition of contexts

Let us now discuss the meaning of the elements $w \in W$. In the literature on intensional logic (cf., for instance, [6, 14]), w is taken as a *possible world.* This term has various interpretations such as "a state of the world at the given time moment and place", "the particular context in which the linguistic expression is used", or "a maximal consistent set of known (possible) facts". All of these interpretations are very wide and rough characterizations which can be arbitrarily specified to more and more details. However, mathematically speaking, w is some parameter which distinguishes concrete manifestations of the property in concern, disregarding its real meaning. In this paper, we will take w in a narrower (and more realistic) way: it is just a certain parameter characterizing a *context* (situation) in which the given linguistic expression is used.

As a special case, when dealing with evaluating expressions only then we may even specify context as follows: The type ω representing contexts is $\omega = \alpha o$ where α is arbitrary type. Each context is in this case a function from the set of truth values to a set of objects of type α. The reason for this definition is basically technical since it enables us to transfer properties to objects of *various kinds* without necessity to define them explicitly again and to specify once and again properties (such as ordering) that are defined already for truth values (for the details see [22]). In the sequel, we will not provide a detailed interpretation of contexts. In the formal theory, they will be identified with certain chosen types, whatever kind they are.

5.7 Local and general contexts

Let us now turn to the definitions (23)–(25). One may be surprised by the fact that they represent functions which assign some truth value to each *couple* of contexts w, w' and to each couple of elements x, y. This seems to contradict the general definition of intension in (15) where only one context (possible world) is considered. However, our definition means that each element x in the context w is in relation with some element y in the context w' (in various degrees), i.e. these intensions represent a certain *relation*. Hence, couples of contexts in (23)–(25) can be naturally explained. Namely, we can distinguish local and general contexts.

Local contexts are the parameters w. *General contexts* are couples or more generally, tuples of contexts $\langle w, w', \ldots\rangle$. Hence, we may read intensions (23)–(25) as *functions from general contexts into the set of fuzzy sets*, in general on $M_\alpha \times M_\beta$ (i.e., fuzzy relations). Consequently, we returned to the generally accepted understanding to the concept of intension.

Let us remark that this enables us to consider general contexts even as certain complex structures over the set of contexts W. However, we will not develop this idea in this paper.

5.8 Remark

The theory of linguistic evaluating expressions developed in [22] where context were identified with types αo enables us to express in elegant form the meaning of fuzzy IF-THEN rules. For example, let us consider a rule

$$\mathsf{IF}\ X\ \mathsf{is}\ \text{roughly small}\ \mathsf{THEN}\ Y\ \mathsf{is}\ \text{very small}$$

where X, Y are variables for elements of different kinds (recall that these may be some features such as temperature, speed, depth, etc.).

Let $w \in \mathit{Form}_{\alpha o}$, $w' \in \mathit{Form}_{\beta o}$ be formulas representing contexts and $x \in \mathit{Form}_\alpha$, $y \in \mathit{Form}_\beta$ represent elements. Then intension of the above rule can be written as

$$\lambda w\, \lambda w'\, \lambda x\, \lambda y\, \cdot \mathit{Sm}_\nu(\sigma_w x) \Rightarrow \mathit{Sm}_{\nu'}(\sigma_{w'} y) \tag{27}$$

where ν interprets the hedge *roughly* and ν' the hedge *very* and $\sigma_w x \equiv \imath t_o \cdot x \equiv w t_o$ (for the detailed explanation of these formulas — see [22]). It is noticeable on (27) that we have obtained a clear, unified and explicit interpretation of a very large class of fuzzy IF-THEN rules which consist of adjectives of the same kind (i.e., small, big, etc.) but refer to objects of different kinds; (27) can be intension of a conditional expression, for example

IF temperature is roughly small THEN pressure is very small.

This is in perfect accordance with the way how people understand such expressions.

5.9 Special types

We will consider *one fixed type* α for all objects (or their features — cf. the discussion above) whenever possible in the sequel. Furthermore, to simplify the notation, we will introduce the following symbols for special types:

- $\varphi := (o\alpha)\omega$ — a type for intension, i.e. a function assigning a fuzzy set in a set of elements of type α to each context.
- $\rho := ((o\alpha)\alpha)\omega\omega$ — a type for intension of a relation, i.e. a function assigning a fuzzy relation to each couple of contexts that is, to each general context in the sense of Subsection 5.7).

5.10 Normality of intensions

Recall that we suppose a certain theory T to be given. We say that the intension $\mathrm{Int}(\mathcal{A})$ of a linguistic expression $\mathcal{A}$ is *normal* if the following is provable:

$$T \vdash (\forall w)(\exists x)\mathbf{\Delta} A_{(o\alpha)\omega} wx. \tag{28}$$

This means that in each context w we can find an element x that surely has the property named by the linguistic expression $\mathcal{A}$. Note that if $\mathrm{Int}(\mathcal{A})$ is normal then by Lemma 4.7(d), also

$$T \vdash (\forall w)(\exists x) A_{(o\alpha)\omega} wx.$$

The normality assumption seems to be quite natural. Namely, it says that *in each context* there exists an element *typical* for the given property; it is a *prototype* of the latter. Indeed, we always can find a typical "red colour", "small value", "deep sea", "clever man", etc.

To characterize normality of the compound intension $\mathrm{Int}(\mathcal{R})$ (recall that it characterizes a relation), note that there are symmetric and non-symmetric relations. We say that a formula R_ρ represents a *symmetric* relation if the following is provable:

$$T \vdash (\forall w)(\forall w')(\forall x)(\forall y)(R_\rho ww'xy \equiv Rww'yx).$$

The intension $\mathrm{Int}(\mathcal{R}) \equiv \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot R_\rho ww'xy$ is symmetric if the formula R_ρ represents a symmetric relation. Obviously, (23) and (24) are symmetric while (25) is not. Then we define normality of intension $\mathrm{Int}(\mathcal{R})$ as follows:

(i) If $\mathrm{Int}(\mathcal{R})$ is *symmetric* then it is *normal* if

$$T \vdash (\forall w)(\forall w')(\exists x)(\exists y)\mathbf{\Delta} R_\rho ww'xy.$$

(ii) If $\mathrm{Int}(\mathcal{R})$ is *asymmetric* then it is *normal* if

$$T \vdash (\forall w)(\forall w')(\forall x)(\exists y)\mathbf{\Delta} R_\rho ww'xy.$$

On the other hand, if the structure of R_ρ is determined by a suitable logical connective, then we may prove the following lemma on the basis of (28).

Lemma
Let intensions $\mathrm{Int}(\mathcal{A})$, $\mathrm{Int}(\mathcal{B})$ *be normal.*

(a) If $\mathrm{Int}(\mathcal{R})$ *has the form (25) then*

$$T \vdash (\forall w)(\forall w')(\forall x)(\exists y)\mathbf{\Delta}(A_{(o\alpha)\omega}wx \Rightarrow B_{(o\beta)\omega}w'y).$$

(b) If $\mathrm{Int}(\mathcal{R})$ *has the form (23) then*

$$T \vdash (\forall w)(\forall w')(\exists x)(\exists y)\mathbf{\Delta}(A_{(o\alpha)\omega}wx \,\Box\, B_{(o\beta)\omega}w'y)$$

where $\Box$ *is either of the connectives* **&** *or* $\wedge$.

PROOF: (a) Let $\mathbf{v} \notin \mathit{Form}_T$, be a new constant of type α. It follows from Lemma 4.7(a) that $T' = T \cup \{B_{(o\beta)\omega}w'\mathbf{v}\}$ is a conservative extension of T. Then $T' \vdash A_{(o\alpha)\omega}wx \Rightarrow B_{(o\beta)\omega}w'\mathbf{v}$ and so, $T' \vdash \mathbf{\Delta}(A_{(o\alpha)\omega}wx \Rightarrow B_{(o\beta)\omega}w'\mathbf{v})$ and finally, $T \vdash (\forall x)(\exists y)\mathbf{\Delta}(A_{(o\alpha)\omega}wx \Rightarrow B_{(o\beta)\omega}w'y)$ by properties of FTT and conservativeness of T', which implies (a) using generalization.

(b) Similarly as above, let $\mathbf{u}, \mathbf{b} \notin \mathit{Form}_T$ be new constants of type α. Then $T' = T \cup \{A_{(o\alpha)\omega}w\mathbf{u}, B_{(o\beta)\omega}w'\mathbf{v}\}$ is a conservative extension of T. Using Lemma 4.7(f) and properties of FTT, we prove $T' \vdash \mathbf{\Delta}(A_{(o\alpha)\omega}w\mathbf{u} \,\Box\, B_{(o\beta)\omega}w'\mathbf{v})$. Then we obtain (b) by similar arguments as in the case of (a). □

In this paper we will suppose that all intensions including intensions of fuzzy IF-THEN rules are normal.

5.11 Explicit form of intension as a function of its parts

The generally accepted assumption required already by G. Frege is that intension of an expression should be composed of intensions of simpler expressions. We will show that this is the case also of our definitions.

Let us consider the following formula:

$$S_{(\rho\varphi)\varphi} := \lambda z_\varphi\, \lambda z'_\varphi \cdot \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot (z_\varphi w)x \,\Box\, (z'_\varphi w')y \tag{29}$$

where $\square$ is any of the connectives $\wedge$, $\&$, $\Rightarrow$.

Lemma
Let T be a theory in which Int($\mathcal{A}$), Int($\mathcal{B}$), Int($\mathcal{R}$) *are defined in (18), (19) and (26), respectively. Then*

$$T \vdash \mathrm{Int}(\mathcal{R}) \equiv S_{(\rho\varphi)\varphi}\, \mathrm{Int}(\mathcal{A}) \cdot \mathrm{Int}(\mathcal{B}). \tag{30}$$

PROOF: We start with the provable formula $\vdash \mathrm{Int}(\mathcal{R}) \equiv \mathrm{Int}(\mathcal{R})$. Then, using three times λ-conversion and necessary renaming of the variables, we obtain

$$\vdash \mathrm{Int}(\mathcal{R}) \equiv \lambda w\, \lambda w'\, \lambda x\, \lambda y\, \cdot (\lambda \bar{w}\, \lambda \bar{x}\, A_{(o\alpha)\omega} \bar{w}\bar{x} \cdot wx \,\square\, \lambda \bar{w}\, \lambda \bar{x}\, A_{(o\alpha)\omega} \bar{w}\bar{y} \cdot w'y)$$

which is the right hand side of (30) after rewriting. □

It follows from this lemma that we can express intension of a compound linguistic predication (and thus, of a fuzzy IF-THEN rule) as a composition of intensions of its parts. This complies with the G. Frege's requirement.

Let us remark that we obtain traditional interpretation of fuzzy IF-THEN rules as certain fuzzy relations, as has been proposed by many authors (see the discussion in Introduction) immediately when either we fix one specific context, or omit variables for context from (29) at all. We demonstrate this explicitly in Subsection 5.17.

5.12 Implicit definition of intension of a fuzzy IF-THEN rule

In Sections 5.5 and 5.11, we explicitly constructed intension of a compound evaluating predication as a composition of intensions of its parts. In Subsection 3.1, we noted that the predications $\mathcal{R}^A$ and $\mathcal{R}^I$ characterize certain relation between properties of elements. Namely, they state that elements of X that have the property characterized by the intension Int($\mathcal{A}$) are in relation with those elements of Y that have the property characterized by the intension Int($\mathcal{B}$). Therefore, when considering the former, we should be able to obtain the latter. Consequently, the relation in concern should provide equality of these intensions.

This reasoning leads us to the requirement that there should exist a formula $U_{\varphi\varphi} \in Form_{\varphi\varphi}$ in the language $J(T)$ such that the following is provable in the theory T:

$$T \vdash \mathrm{Int}(\mathcal{B}) \equiv U_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A}). \tag{31}$$

Let r_ρ be a variable of the type ρ. We put

$$T \vdash \bar{U}_{(\varphi\varphi)\rho} \equiv \lambda r_\rho\, \lambda z_\varphi\, \lambda w'\, \lambda y\, (\exists x)(z_\varphi wx \,\&\, r_\rho ww'xy). \tag{32}$$

Now, let $R_\rho \in Form_\rho$ be a formula and put

$$U_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho} R_\rho.$$

By lambda conversion, the searched formula $U_{\varphi\varphi}$ is

$$U_{\varphi\varphi} := \lambda z_\varphi \cdot \lambda w' \lambda y \, (\exists x)(z_\varphi wx \,\&\, R_\rho ww'xy). \tag{33}$$

The formula $U_{\varphi\varphi}$ in (33) is just a syntactical form of the, so called, *compositional rule of inference* (cf., e.g., [36]) widely used, in fuzzy control and elsewhere.

If (31) is provable for $U_{\varphi\varphi}$ from (33) then we say that

$$\mathrm{Int}(\mathcal{R}) := \lambda w \, \lambda w' \, \lambda x \, \lambda y \cdot R_\rho ww'xy \tag{34}$$

is an *implicit definition* of an intension of a fuzzy IF-THEN rule containing the predications 'X is $\mathcal{A}$' and 'Y is $\mathcal{B}$'. Note that in this case, we cannot distinguish between $\mathcal{R}^A$ and $\mathcal{R}^I$. In the sequel, when speaking about intension of a fuzzy IF-THEN rule without its more precise specification, we will generally refer to (34).

5.13 Extensionality of $U_{\varphi\varphi}$

The formula $U_{\varphi\varphi}$ in (33) is a function, i.e.

$$T \vdash u_\varphi \equiv U_{\varphi\varphi} z_\varphi \,\&\, u'_\varphi \equiv U_{\varphi\varphi} z_\varphi \Rightarrow \cdot u_\varphi \equiv u'_\varphi.$$

can be proved using transitivity of $\equiv$. It is weakly extensional due to axioms of FTT. However, we can prove that it is even strongly extensional.

Lemma
The function $U_{\varphi\varphi}$ is strongly extensional, i.e.

$$T \vdash z_\varphi \equiv z'_\varphi \Rightarrow \cdot U_{\varphi\varphi} z_\varphi \equiv U_{\varphi\varphi} z'_\varphi$$

PROOF:

(L.1) $T \vdash R_\rho ww'xy \equiv R_\rho ww'xy$ (reflexivity, properties of FTT)
(L.2) $T \vdash (z_\varphi \equiv z'_\varphi) \equiv (\forall w)(\forall x)(z_\varphi wx \equiv z'_\varphi wx)$ (equivalence theorem)
(L.3) $T \vdash (\forall w)(\forall x)(z_\varphi wx \equiv z'_\varphi wx) \Rightarrow \cdot z_\varphi wx \equiv z'_\varphi wx$ (substitution)
(L.4) $T \vdash (z_\varphi \equiv z'_\varphi) \Rightarrow (z_\varphi wx \,\&\, R_\rho ww'xy \equiv z'_\varphi wx \,\&\, R_\rho ww'xy)$
(L.1, L.2, L.3, properties of FTT)
(L.5) $T \vdash (z_\varphi \equiv z'_\varphi) \Rightarrow \cdot(\exists x)(z_\varphi wx \,\&\, R_\rho ww'xy) \equiv (\exists x)(z'_\varphi wx \,\&\, R_\rho ww'xy)$
(L.4, generalization, Lemma 4.7(j), properties of FTT)

□

This property is quite important since it demonstrates that the compositional rule of inference which is obtained when interpreting $U_{\varphi\varphi}$ is some model has good properties. On the other hand, as will be seen in below, these properties may be too strong so that if we try to construct implicit intension of a linguistic description (that is, a system of fuzzy IF-THEN rules as a whole) then such a function may not exist in the given formal system.

5.14

Lemma
Let $\mathrm{Int}(\mathcal{R})$ *be the formula (34) and* $U_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho}\,\mathrm{Int}(\mathcal{R})$*. Then*

(a) $T \vdash U_{\varphi\varphi} \equiv \lambda z_\varphi \cdot \lambda w'\, \lambda y \cdot (\forall w)(\exists x)(z_\varphi wx \,\&\, R_\rho ww'xy)$.
(b) (31) is provable in T *iff*

$$T \vdash B_\varphi w'y \equiv (\exists x)(A_\varphi wx \,\&\, R_\rho ww'xy). \tag{35}$$

(c) If (31) is provable in T *then*

$$T \vdash B_\varphi w'y \equiv (\forall w)(\exists x)(A_\varphi wx \,\&\, R_\rho ww'xy). \tag{36}$$

PROOF: (a) is easily obtained using λ-conversion.

(b) After rewriting (31) using (a), we obtain

$$T \vdash \lambda w'\, \lambda y\, B_\varphi w'y \equiv \lambda w'\, \lambda y\, (\exists x)((\lambda \bar{w}\, \lambda \bar{x}\, A_\varphi \bar{w}\bar{x})wx \,\&\, R_\rho ww'xy).$$

Then (35) is obtained by λ-conversion and the equality theorem.

If (35) holds then we use the rule of generalization for w' and y and Lemma 4.7(i). Finally, we apply the equality theorem.

(c) follows from (b) by generalization. □

5.15

Let us denote

$$\hat{U}_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho} \cdot \mathrm{Int}(\mathcal{R}^I),$$
$$\breve{U}_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho} \cdot \mathrm{Int}(\mathcal{R}^A).$$

Lemma
(a) $T \vdash \hat{U}_{\varphi\varphi} \equiv \lambda z_\varphi\, \lambda w'\, \lambda y \cdot (\exists x)(z_\varphi wx \,\&\, A_\varphi wx \Rightarrow B_\varphi w'y)$.

(b) $T \vdash \breve{U}_{\varphi\varphi} \equiv \lambda z_\varphi\, \lambda w'\, \lambda y \,\cdot (\exists x)(z_\varphi wx \,\&\, A_\varphi wx \,\&\, B_\varphi w' y).$

PROOF: This follows from Lemma 5.14(a) by λ-conversion. □

5.16

The following theorem is a syntactic formulation of the basic theorem for solution of one fuzzy relation equation that was first proved by E. Sanchez in [33]. Syntactically in BL-fuzzy logic, it has been proved in [27]. Our theorem below is still generalization of these results.

Theorem
Let $A_\varphi, B_\varphi \in Form_\varphi$. Then the following is equivalent:

(a) (31) is provable for the formula $U_{\varphi\varphi}$ determined in (33) by some formula R_ρ.
(b) (31) is provable for the formula $U_{\varphi\varphi} := \hat{U}_{\varphi\varphi}$.

PROOF: (a)$\Rightarrow$ (b): Using Lemma 5.14, the properties of equivalence (i.e. equality for formulas of type o) and properties of FTT we obtain

$$T \vdash (A_\varphi wx \,\&\, R_\rho ww'xy) \Rightarrow B_\varphi w'y.$$

Then, using properties of FTT, we get

$$T \vdash R_\rho ww'xy \Rightarrow \cdot A_\varphi wx \Rightarrow B_\varphi w'y,$$

which leads to

$$T \vdash (\exists x)(A_\varphi wx \,\&\, R_\rho ww'xy) \Rightarrow (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w'y).$$

From this and (35) we obtain

$$T \vdash B_\varphi w'y \Rightarrow (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w'y).$$

The converse implication follows from the provable formula $\vdash x_o \,\&(x_o \Rightarrow y_o) \Rightarrow y_o$. Then (b) follows from the properties of FTT and the equality theorem.

The implication (b)$\Rightarrow$ (a) is immediate using Lemma 5.15(a). □

5.17 Interpretation of formulas

In this subsection, we will consider some fixed frame $\mathcal{M}$ for the language $J(T)$ defined in (10). As a special case, we will denote the set of contexts by $M_\omega = W$. This may consist of arbitrary elements of certain given type. Recall that in case of evaluating expressions, the elements $w^0 \in W$ are specific functions $w^0 : L \longrightarrow M_\alpha$.

Furthermore, we will consider a theory T in which the formula (31) is its only axiom. Finally, we denote by $\mathcal{I}_p^{\mathcal{M}}$ an interpretation for some assignment p to variables of $J(T)$. Namely, we will take $p(w) = w^0 \in W$.

(i) Interpretation of context variables are elements

$$\mathcal{I}_p^{\mathcal{M}}(w) = w^0, \qquad \mathcal{I}_p^{\mathcal{M}}(w') = w'^0$$

(actually, these are given by the initial assignment p).

(ii) Interpretation of intension of a linguistic expression $\mathcal{A}$ is

$$\mathcal{I}_p^{\mathcal{M}}(\mathrm{Int}(\mathcal{A})) = \mathcal{I}_p^{\mathcal{M}}(\lambda w\, \lambda x \cdot A_\varphi wx) = A : W \longrightarrow L^{M_\alpha}. \tag{37}$$

Clearly, $A \in M_\varphi$ is a function which assigns to each context $w^0 \in W$ a fuzzy set in M_α.

It follows from the definition (17) that interpretation of extension of $\mathcal{A}$ in the context $w^0 \in W$ is just the fuzzy set

$$\mathcal{I}_p^{\mathcal{M}}(\mathrm{Ext}(\mathcal{A})w) = A(w^0) \underset{\sim}{\subset} M_\alpha.$$

(iii) Interpretation of a fuzzy IF-THEN rule (34) is

$$\mathcal{I}_p^{\mathcal{M}}(\lambda w'\, \lambda w \cdot \lambda x\, \lambda y\, R_\rho ww'xy) = R : W \times W \longrightarrow L^{M_\alpha \times M_\alpha},$$

i.e. it is a function which assigns to each couple of contexts $w^0, w'^0 \in W$ a fuzzy relation $R(w^0, w'^0) \underset{\sim}{\subset} M_\alpha \times M_\alpha$, that is, $R \in M_\rho$.

(iv) As a special case of (iii), interpretation of a fuzzy IF-THEN rule of the form (26) is a function

$$\mathcal{I}_p^{\mathcal{M}}(\lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot A_{(o\alpha)\omega} wx \,\square\, B_{(o\alpha)\omega'} w'y) = R \tag{38}$$

where

$$R(w^0, w'^0) = A(w^0) \bullet B(w'^0), \qquad w^0, w'^0 \in W, \tag{39}$$

$A(w^0) \underset{\sim}{\subset} M_\alpha$, $B(w'^0) \underset{\sim}{\subset} M_\alpha$, and $\bullet$ is either of the operations $\otimes$, $\wedge$ or $\rightarrow$ extended to the fuzzy sets $A(w^0), B(w'^0)$ pointwise [†)].

[†)] For example, $A(w^0) \rightarrow B(w'^0) = A(w^0)(u) \rightarrow B(w'^0)(v)$ for all $u, v \in M_\alpha$.

Clearly, the fuzzy relation $A(w^0) \bullet B(w'^0) \underset{\sim}{\subseteq} M_\alpha \times M_\alpha$ is *extension* of the fuzzy rule $\mathcal{R}$ in the contexts w^0, w'^0 and it is just the interpretation offered by the other cited authors.

(v) Interpretation of the formula $U_{\varphi\varphi}$ is a function

$$\mathcal{I}_p^{\mathcal{M}}(U_{\varphi\varphi}) = \mathcal{I}_p^{\mathcal{M}}(\lambda z_\varphi \cdot \lambda w' \, \lambda y \, (\exists x)(z_\varphi w x \,\&\, R_\rho w w' x y)) = F : M_\varphi \longrightarrow \left(L^M\right)^W$$

that assigns to each intension $\mathcal{I}_p^{\mathcal{M}}(z_\varphi) = \bar{z} \in M_\varphi$ the intension given by

$$F(\bar{z})(w'^0, y) = \bigvee_{u \in M_\alpha} (\bar{z}(w^0, u) \otimes R(w^0, w'^0, u, y)) \tag{40}$$

for all $w^0, w'^0 \in W$, $y \in M_\alpha$.

(vi) Let the interpretation of $\mathrm{Int}(\mathcal{A})$ be the function A in (37) and let the context $w^0 \in W$ and the fuzzy set $A(w^0)$ be fixed. Then the interpretation of the formula $U_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A})$ in $\mathcal{M}$ is a function $G : W \longrightarrow L^{M_\alpha}$ that to each context $w'^0 \in W$ assigns a fuzzy set given by the membership function

$$G(w'^0, y) = \bigvee_{u \in M_\alpha} (A(w^0, u) \otimes R(w^0, w'^0, u, y)). \tag{41}$$

In other words, each context w'^0 and each element $y \in M_\alpha$ are assigned the truth value given by (41). The fuzzy relation R occurring in (40) and (41) says that in each couple of contexts $w^0, w'^0 \in W$ the elements u, y are in relation R (in some degree taken from L).

5.18

Let T be a theory from 5.17 and $\mathcal{M}$ be a frame for $J(T)$. Furthermore, let the interpretation $\mathcal{I}_p^{\mathcal{M}}(\mathrm{Int}(\mathcal{B})) = B : W \longrightarrow L^{M_\alpha}$.

Lemma
If $\mathcal{M}$ is a model of T then for each couple of contexts $w^0, w'^0 \in W$ the following equality holds true:

$$B(w'^0, y) = \bigvee_{u \in M_\alpha} (A(w^0, u) \otimes R(w^0, w'^0, u, y)). \tag{42}$$

for all $y \in M_\alpha$.

PROOF: Immediately from the definition of interpretation of the formula (31) due to 5.17 and the fact that (31) is true in the degree 1. □

Note that (42) is a composition of fuzzy relations defined in (1). However, here it follows from interpretation of the formula of FTT which defines implicit construction of intension of a fuzzy IF-THEN rule.

5.19

The following theorem verifies by logical means the idea of I. Perfilieva and S. Lehmke [30] about relation between solvability of fuzzy relation equations and consistency of the theory induced by them (recall that fuzzy relations stand behind interpretation of fuzzy IF-THEN rules). It is extended to systems of fuzzy IF-THEN rules in Subsection 6.7.

Theorem
Let the fuzzy sets A, B, R and the theory T be as in Subsections 5.17 and 5.18. Further, let $\mathcal{M}$ be a frame for the language $J(T)$ (cf. Subsection 4.8). Then the following is equivalent.

(a) For each couple of contexts $w^0, w'^0 \in W$ the fuzzy relation equation

$$B(w'^0) = A(w^0) \circ R(w^0, w'^0) \tag{43}$$

is solvable for the unknown fuzzy relation $R(w^0, w'^0)$.

(b) A frame $\mathcal{M}$ is a model of T.

(c) The theory T is consistent.

PROOF:

(a)$\Rightarrow$ (b) If (43) is solvable then (31) is true in the degree 1 and so, $\mathcal{M}$ is a model of T.

(b) $\Rightarrow$ (c) due to completeness of FTT.

(c) $\Rightarrow$ (a) If T is consistent then it has a model, i.e. (31) is true in the degree 1 in it. Since $\mathcal{M}$ is a frame in which all constructions have been correctly done, we conclude that $\mathcal{M}$ is also a model of T. But then (43) is solvable in it. □

5.20

On the basis of 5.5, 5.16 we see that the explicit form (25) of intension of a fuzzy IF-THEN rule provides necessary and sufficient condition for the existence of its implicit form. But even more holds true. Recall that all the intensions are supposed to be normal.

Theorem
Let T, A, B be as above.

(a) $T \vdash \lambda w'\, \lambda y\, B_\varphi w' y \equiv \cdot \lambda w'\, \lambda y\, (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w' y)$,

(b) $T \vdash \lambda w'\, \lambda y\, B_\varphi w' y \equiv \cdot \lambda w'\, \lambda y\, (\exists x)(A_\varphi wx \,\&\, A_\varphi wx \,\&\, B_\varphi w' y)$.

PROOF: (a)

(L.1) $T \vdash B_\varphi w'y \equiv B_\varphi w'y$ (reflexivity, properties of FTT)
(L.2) $T \vdash B_\varphi w'y \equiv (\exists x)A_\varphi wx \wedge B_\varphi w'y$ (normality of intension of $\mathcal{A}$, properties of FTT)
(L.3) $T \vdash B_\varphi w'y \equiv (\exists x)(A_\varphi wx \wedge B_\varphi w'y)$ (L.2., properties of FTT)
(L.4) $T \vdash B_\varphi w'y \equiv (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w'y)$ (L.3., properties of FTT)
(L.5) $T \vdash (\forall y)(B_\varphi w'y \equiv (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w'y))$ (L.4., generalization)
(L.6) $T \vdash \lambda y\, B_\varphi w'y \equiv \lambda y\, (\exists x)(A_\varphi wx \,\&\, \cdot A_\varphi wx \Rightarrow B_\varphi w'y)$ (L.5., Lemma 4.6, axioms)

The rest follows from L.6 using rule of generalization and Lemma 4.6.

(b)

(L.1) $T \vdash (\exists x)(A_\varphi wx)^2$ (normality assumption, Lemma 4.7(c))
(L.2) $T \vdash B_\varphi w'y \equiv B_\varphi w'y$ (reflexivity, properties of FTT)
(L.3) $T \vdash B_\varphi w'y \equiv (\exists x)(A_\varphi wx)^2 \,\&\, B_\varphi w'y$ (L.1, L.2, properties of FTT)
(L.4) $T \vdash B_\varphi w'y \equiv (\exists x)(A_\varphi wx) \,\&\, A_\varphi wx) \,\&\, B_\varphi w'y)$ (L.3, properties of FTT)

The rest is the same as in the case (a). □

5.21

A consequence of Subsection 5.20 is the following theorem.

Theorem
(a) $T \vdash \mathrm{Int}(\mathcal{B}) \equiv \hat{U}_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A})$,
(b) $T \vdash \mathrm{Int}(\mathcal{B}) \equiv \breve{U}_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A})$.

PROOF: (a) Using Lemma 5.15(a) and λ-conversion we show that

$$T \vdash \hat{U}_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A}) \equiv \lambda w'\, \lambda y\ \cdot (\exists x)(A_\varphi wx \,\&\, A_\varphi wx \Rightarrow B_\varphi w'y)$$

Then use Theorem 5.20(a) and definition of $\mathrm{Int}(\mathcal{B})$.

The proof of (b) is analogous. □

It follows from this theorem, that the explicit definition of intension of a fuzzy IF-THEN rule taken either as $\mathcal{R}^I$ or $\mathcal{R}^A$ is also its implicit definition.

5.22

Let $\mathrm{Int}(\mathcal{R}) \equiv \lambda w\, \lambda w'\, \lambda x\, \lambda y \cdot R_\rho ww'xy$ be a normal intension. The following theorem is a simple consequence of Lemma 4.7(a), (b).

Theorem
Let $\mathbf{u}^0 \in Form_\alpha$ *be a new constant and put*

$$T' = T \cup \{(\forall w)(\forall w')(\exists y)\mathbf{\Delta} R_\rho ww'\mathbf{u}^0 y\}.$$

Then

$$T' \vdash (\forall w)(\forall w') \cdot R_\rho ww'\mathbf{u}^0(\imath y \cdot R_\rho ww'\mathbf{u}^0 y) \tag{44}$$

and T' *is a conservative extension of* T.

PROOF:

(L.1) $T \vdash (\exists x)(\exists y)\mathbf{\Delta} R_\rho ww'xy$ (normality of $\mathrm{Int}(\mathcal{R})$, substitution)
(L.2) $T \vdash (\exists x)\mathbf{\Delta}(\exists y)\mathbf{\Delta} R_\rho ww'xy$ (L.1, Lemma 4.7(e))
(L.3) $T' \vdash (\exists y)\mathbf{\Delta} R_\rho ww'\mathbf{u}^0 y$ (L.2, Lemma 4.7(a))
(L.4) $T' \vdash R_\rho ww'\mathbf{u}^0(\imath y \cdot R_\rho ww'\mathbf{u}^0 y)$ (L.3., Lemma 4.7(b))

Formula (44) is obtained using rule of generalization. □

According to this theorem, if an element $\mathbf{u}^0$ from the kernel of the fuzzy set represented by a formula $\lambda x\, (\exists y)\mathbf{\Delta} R_\rho ww'xy$ is given for each couple of contexts w, w' then it is in relation R with the element $\imath y \cdot R_\rho ww'\mathbf{u}^0 y$. Recall that the operator $\imath y$ is interpreted as a defuzzification. Consequently, this theorem verifies what we expect — that the defuzzification gives an element which is in relation characterized by a fuzzy IF-THEN rule and so, it can be taken as a result of reasoning based on the latter.

6 Systems of fuzzy IF-THEN rules

In this section, we will focus on systems of fuzzy IF-THEN rules; we will call them *linguistic descriptions*. The ground, of course, will be the interpretation of one rule developed above. As one can expect, the situation is more complicated because there raise more possibilities how to grasp this problem. One possibility is to take a linguistic description as a text consisting of separate rules. The general topic is the same but semantically, the linguistic description leads to a set of intensions rather than to one specific intension only. We will speak about *explicit intension* in this case. The second possibility is to seek a certain hidden intension of the whole linguistic description. This will be called *implicit intension* and we will show that this may not always exist. There is also a possibility to join rules forming the linguistic description by a connective ('and' or 'or') and take it as one (long) sentence that, of course, has its own intension. However, this intension may be different from the implicit intension. Conditions for existence of an implicit intension are studied. This question turns out to be related with the old problem of solving fuzzy relation equations studied in the fuzzy set literature (see the citations).

6.1 Linguistic description and its interpretation

Let us consider a system of fuzzy IF-THEN rules

$$\begin{array}{l} \mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}_1\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}_1, \\ \mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}_2\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}_2, \\ \dots\dots\dots\dots\dots\dots \\ \mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}_m\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}_m. \end{array} \tag{45}$$

If all $\mathcal{A}, \mathcal{B}$'s are evaluating linguistic expressions then this system can be understood as a description of some strategy (e.g. control or decision-making), behaviour of a system, etc. Therefore, we will call the system (45) a *linguistic description* and denote it by *LD*. We can also understand it as a *special text*.

The problem now is how such a linguistic description should be analysed, what does it provide? In correspondence with the explanation above, (45) can be seen from two sides: *first*, each rule is a compound evaluating expression which has its own intension and thus, the linguistic description is a set of intensions providing us with some knowledge. We will speak about *explicit knowledge* coming out of the linguistic description, or simply, explicit knowledge from the linguistic description.

Second, the linguistic description *as a whole* represents a certain kind of a relation raising from the intensions of *all* the evaluating expressions occurring in (45). We will speak about *implicit knowledge* from the linguistic description.

6.2 Explicit knowledge from the linguistic description

In this case, the linguistic description is simply a list of intensions

$$LD = \{\mathrm{Int}(\mathcal{R}_1), \dots, \mathrm{Int}(\mathcal{R}_m)\}, \tag{46}$$

each of them defined in (26) and formed within a language $J(T)$ of some theory T. We will write LD^A if all rules $\mathcal{R}$ have the form (23) and LD^I if all rules $\mathcal{R}$ have the form (25).

The knowledge contained in the linguistic description comes out from the intensions of all the rules. More specifically, due to the normality assumption for $\mathrm{Int}(\mathcal{R})$, we require

$$T \vdash (\forall w)(\forall w')(\exists x)(\exists y) \cdot \boldsymbol{\Delta}(A_{\varphi,j}wx \,\square\, B_{\varphi,j}w'y), \tag{47}$$
$$T \vdash (\forall w)(\forall w')(\forall x)(\exists y) \cdot \boldsymbol{\Delta}(A_{\varphi,j}wx \Rightarrow B_{\varphi,j}w'y) \tag{48}$$

for all $j = 1, \dots, m$ where $\square$ is either $\&$ or $\wedge$. Thus, the explicit knowledge means that the linguistic description *provides a list of formulas* (47) or (48) (depending on the form of fuzzy IF-THEN rules occurring in LD) which *must be provable* in T. Such a theory T will be called a *theory determined by the linguistic description LD*. Note that they may be taken as axioms of T but in general, this is unnecessary. Therefore, we will not specify the theory T precisely and focus only on some properties which should be fulfilled by it.

Furthermore, the text is a set of sentences and so, we may distinguish its topic and focus (cf. Subsection 3.3). Namely, the *topic of a linguistic description* is a set of intensions

$$\mathit{Topic}^{LD} = \{\mathrm{Int}(\mathcal{A}_j) \mid j = 1, \dots, m\}. \tag{49}$$

This means that the topic is determined by the set of linguistic predications, each of them forming the first part the respective fuzzy IF-THEN rule. Similarly, its focus is

$$\mathit{Focus}^{LD} = \{\mathrm{Int}(\mathcal{B}_j) \mid j = 1, \dots, m\}. \tag{50}$$

Note that the common characteristics both for the topic as well as for the focus of a linguistic description, which ties together all the fuzzy IF-THEN rules present in it, are the variables X and Y, respectively.

6.3 Linguistic description consisting of implications

First, we will interpret the fuzzy IF-THEN rules in the linguistic description as logical implications. Let us put

$$\mathit{Eval}_{o(\varphi\alpha\omega)} := \lambda w\, \lambda x_\alpha\, \lambda z_\varphi\, (\exists z_o)(\Upsilon z_o \,\&\, \cdot z_o \Rightarrow z_\varphi w x_\alpha). \tag{51}$$

Let $\mathrm{Int}(\mathcal{A})$ be intension of some evaluating expression $\mathcal{A}$. We say that an element x in the context w is *evaluated* by the expression $\mathcal{A}$ if the following formula is provable:

$$T \vdash \mathit{Eval}_{o(\varphi\alpha\omega)} \cdot wx\, \mathrm{Int}(\mathcal{A}). \tag{52}$$

Recall that Υz_o means that z_o is a non-zero truth value. Then (52) means that *x in the context w is evaluated by $\mathcal{A}$*. In other words, the truth value of the statement "x in the context w has the property named by $\mathcal{A}$" is non-zero.

For tangibility, we will introduce a constant $\mathbf{w}^0$ to represent some definite context and a constant $\mathbf{u}^0 \in Form_\alpha$ to represent some definite element. With respect to T, we may suppose that these are new constant symbols included in the language $J(T)$.

Furthermore, let us also introduce a new constant $\mathbf{b}_i^0 \in Form_o$ and put $T \vdash \mathbf{b}_i^0 \equiv A_{\varphi,i}\mathbf{w}^0\mathbf{u}^0$. Then the explicit knowledge leads to the following theorem.

Theorem
Let T be a theory determined by the linguistic description LD^I. Furthermore, let $T \vdash \Upsilon\mathbf{b}_i^0$ and $T \vdash \hat{\mathbf{y}} \equiv \imath y \cdot \mathbf{b}_i^0 \Rightarrow B_{\varphi,i}w'y$. Then

$$T \vdash (\forall w') \cdot Eval_{o(\varphi\alpha\omega)}\, w'\hat{\mathbf{y}} \cdot B_{\varphi,i}.$$

Moreover, $T \vdash (\forall w')\Upsilon(B_{\varphi,i}w'\hat{\mathbf{y}})$.

PROOF:

(L.1) $T \vdash (\forall w)(\forall x)(\exists y)\mathbf{\Delta}(A_{j,\varphi}\, wx \Rightarrow B_{j,\varphi}\, w'y)$ (normality assumption)
(L.2) $T \vdash (\exists y)\mathbf{\Delta}(A_{j,\varphi}\, \mathbf{w}^0\mathbf{u}^0 \Rightarrow B_{j,\varphi}\, w'y)$ (L.1, substitution)
(L.3) $T \vdash A_{j,\varphi}\, \mathbf{w}^0\mathbf{u}^0 \Rightarrow B_{\varphi,j}\, w'y(\imath y \cdot A_{\varphi,j}\, \mathbf{w}^0\mathbf{u}^0 \Rightarrow B_{\varphi,j}\, w'y)$
(L.2, Lemma 4.7(b), substitution)
(L.4) $T \vdash \Upsilon\mathbf{b}_i^0 \,\&(\mathbf{b}_i^0 \Rightarrow B_{\varphi,i}w'\hat{\mathbf{y}})$ (L.3, assumption, properties of FTT)
(L.5) $T \vdash \mathrm{Int}(\mathcal{B}_i) \equiv \lambda w'\, \lambda y\, B_{\varphi,i}w'y$ (definition of intension)
(L.6) $T \vdash (\exists z_o) \cdot \Upsilon z_o \,\&(z_o \Rightarrow B_{\varphi,i}w'\hat{\mathbf{y}})$ (L.4, substitution, modus ponens)
(L.7) $T \vdash B_{\varphi,i}w'\hat{\mathbf{y}} \equiv (\lambda\bar{w}'\, \lambda y\, B_{\varphi,i}\bar{w}'y)w'\hat{\mathbf{y}}$ (L.5, λ-conversion)
(L.8) $T \vdash (\exists y_o) \cdot \Upsilon z_o \,\&(z_o \Rightarrow \mathrm{Int}(\mathcal{B}_i)w'\hat{\mathbf{y}})$
(L.6, L.7, properties of FTT, definition of $\mathrm{Int}(\mathcal{B}_i)$)
(L.9) $T \vdash (\forall w') \cdot Eval_{o(\varphi\alpha\omega)}\, w'\hat{\mathbf{y}} \cdot B_{\varphi,i}$ (L.8, definition of *Eval*, generalization)

The rest follows from Lemma 4.7(g). □

This theorem has the following interpretation. Let us consider a linguistic description LD^I (i.e., the fuzzy IF-THEN rules are implications). If we find a formula $\mathrm{Int}(\mathcal{A}_i) \in Topic^{LD}$ and an element $\mathbf{u}^0$ in the context $\mathbf{w}^0$ such that $A_{\varphi,i}\mathbf{w}^0\mathbf{u}^0$ has a non-zero truth degree in all contexts then we conclude that the element $\hat{\mathbf{y}}$, *typical for the formula* $\mathbf{b}_i^0 \Rightarrow B_{\varphi,i}w'y$, is evaluated by the linguistic expression $\mathcal{B}_i$ in every context w', where $\mathrm{Int}(\mathcal{B}_i) \in Focus^{LD}$.

In other words, by this theorem, each fuzzy IF-THEN rule from LD^I provides a typical element $\hat{\mathbf{y}}_j$. When learning that some element $\mathbf{u}^0$ is evaluated by a concrete antecedent $\mathrm{Int}(\mathcal{A}_i)$ from the topic of LD^I, we derive the corresponding

$\hat{\mathbf{y}}_i$. This element is then the result of our reasoning based on the given linguistic description.

6.5 Implicit knowledge from the linguistic description

When seeing a linguistic description as a text, we may take it as a certain complicated property having its own intension. Then the following question arises: does such intension exist? Under which conditions it can be formed and what are its properties?

Let a linguistic description LD be given and consider the the formula $\bar{U}$ in (33). Furthermore, let $T \vdash U_{\varphi\varphi} \equiv \bar{U}_{(\varphi\varphi)\rho} R_\rho$. We say that the *formula R_ρ represents an implicit knowledge contained in the linguistic description LD* if

$$T \vdash \mathrm{Int}(\mathcal{B}_i) \equiv U_{\varphi\varphi} \cdot \mathrm{Int}(\mathcal{A}_i) \tag{53}$$

holds for all the linguistic expressions forming the fuzzy IF-THEN rules $\mathcal{R}_i$, $i = 1, \dots, m$ in the linguistic description LD. Hence, the implicit knowledge can be, in accordance with Subsection 5.12, considered as a kind of *implicit intension* of the whole linguistic description LD.

Note that, analogously to Lemma 5.14, (53) is equivalent to

$$T \vdash B_{i,\varphi} w' y \equiv (\exists x)(A_{i,\varphi} wx \mathbin{\&} R_\rho w w' x y), \qquad i = 1, \dots, m. \tag{54}$$

Analogously to Subsection 5.15, we denote

$$\hat{U}_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho} \cdot \bigwedge_{j=1}^{m} A_{j,\varphi} wx \Rightarrow B_{j,\varphi} w' y,$$
$$\breve{U}_{\varphi\varphi} := \bar{U}_{(\varphi\varphi)\rho} \cdot \bigvee_{j=1}^{m} A_{j,\varphi} wx \mathbin{\&} B_{j,\varphi} w' y.$$

Obviously,

(a) $T \vdash \hat{U}_{\varphi\varphi} \equiv \lambda z_\varphi\, \lambda w'\, \lambda y \cdot (\exists x)(z_\varphi wx \mathbin{\&} \bigwedge_{j=1}^{m} A_{j,\varphi} wx \Rightarrow B_{j,\varphi} w' y)$.
(b) $T \vdash \breve{U}_{\varphi\varphi} \equiv \lambda z_\varphi\, \lambda w'\, \lambda y \cdot (\exists x)(z_\varphi wx \mathbin{\&} \bigvee_{j=1}^{m} A_{j,\varphi} wx \mathbin{\&} B_{j,\varphi} w' y)$.

6.6

The following theorem is a syntactic and very general form of the main result in fuzzy relation equations theory, originally proved by Sanchez in [33]. Other authors who significantly contributed to it are, for example Di Nola et al. [4], Gottwald [7], Gottwald et al. [8], Klawonn [11], Novák et al. [27], Perfilieva and Lehmke [30], Perfilieva and Tonis [31] and others.

Theorem
Let $A_{i,\varphi}, B_{i,\varphi} \in Form_\varphi$*,* $i = 1, \dots, m$*. Then the following is equivalent:*

(a) (53) holds for the formula $U_{\varphi\varphi}$ *determined in (33) by some formula* R_ρ*,* $i = 1, \dots, m$*.*
(b) (53) holds for the formula $U_{\varphi\varphi} := \hat{U}_{\varphi\varphi}$*,* $i = 1, \dots, m$*.*

PROOF: (a)⇒ (b): Using (54), the properties of equivalence (i.e. equality for formulas of type o) and properties of FTT we obtain

$$T \vdash (A_{i,\varphi}wx \,\&\, R_\rho ww'xy) \Rightarrow B_{i,\varphi}w'y$$

for all $i = 1, \dots, m$. Then, using properties of FTT, we get

$$T \vdash R_\rho ww'xy \Rightarrow \bigwedge_{i=1}^{m} A_{i,\varphi}wx \Rightarrow B_{i,\varphi}w'y,$$

which, by generalization and distribution of quantifiers, leads to

$$T \vdash (\exists x)(A_\varphi wx \,\&\, R_\rho ww'xy) \Rightarrow (\exists x)(A_\varphi wx \,\&\, \bigwedge_{i=1}^{m} A_{i,\varphi}wx \Rightarrow B_{i,\varphi}w'y).$$

From this and (35) we obtain

$$T \vdash B_\varphi w'y \Rightarrow (\exists x)(A_\varphi wx \,\&\, \bigwedge_{i=1}^{m} A_{i,\varphi}wx \Rightarrow B_{i,\varphi}w'y).$$

The converse implication follows from the provable formula $\vdash x_o \,\&(x_o \Rightarrow y_o) \Rightarrow y_o$. Then (b) follows from the properties of FTT and the equality theorem.

The implication (b)⇒ (a) is immediate. □

From our point of view, the above theorem demonstrates that an implicit intension of a linguistic description may not exist. Its existence depends on the way, how formulas $A_{i,\varphi}, B_{i,\varphi}$ comply with the requirement that $U_{\varphi\varphi}$ must be a strongly extensional function — cf. Subsection 5.13.

6.7

Analogously to Subsection 5.19, we may formulate the following theorem. We suppose that T is a theory in which the formulas (53), $i = 1, \dots, m$ are its only axioms. Let $\mathcal{M}$ be a frame for the language $J(T)$ and $\mathcal{I}_p^{\mathcal{M}}$ be interpretation for some assignment p to variables. Let A_i, B_i, R, $i = 1, \dots, m$ be functions constructed similarly as in Subsections 5.17 and 5.18.

Theorem
The following is equivalent.

(a) *A frame $\mathcal{M}$ is a model of T.*
(b) *For each couple of contexts $w^0, w'^0 \in W$ the system of fuzzy relation equations*

$$B_i(w'^0) = A_i(w^0) \circ R(w^0, w'^0), \qquad i = 1, \ldots, m \tag{55}$$

is solvable for the unknown fuzzy relation $R(w^0, w'^0)$.
(c) *The theory T is consistent.*

6.8

Theorem
Let $T \vdash (\forall w)(\exists x)A_{i,\varphi}wx$ for all $i = 1, \ldots, m$. Then the following is equivalent.

(a) *(53) holds for the formula $U_{\varphi\varphi} := \breve{U}_{\varphi\varphi}$, $i = 1, \ldots, m$.*
(b) *For all $i, j = 1, \ldots, m$,*

$$T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx) \Rightarrow B_{i,\varphi} \equiv B_{j,\varphi}.$$

PROOF: First, we verify that

$$T \vdash B_{i,\varphi}w'y \Rightarrow (\exists x)(A_{i,\varphi}wx \,\&\, \bigvee_{j=1}^{m} A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \tag{56}$$

for all $i = 1, \ldots, m$.

(L.1) $T \vdash B_{i,\varphi}w'y \Rightarrow B_{i,\varphi}w'y$ (properties of FTT)
(L.2) $T \vdash B_{i,\varphi}w'y \Rightarrow (\exists x)A_{i,\varphi}wx \,\&\, B_{i,\varphi}w'y$
(L.1, assumption, properties of FTT)
(L.3) $T \vdash B_{i,\varphi}w'y \Rightarrow (\exists x)(A_{i,\varphi}wx)^2 \,\&\, B_{i,\varphi}w'y$ (L.2, Lemma 4.7(c))
(L.4) $T \vdash B_{i,\varphi}w'y \Rightarrow (\exists x)(A_{i,\varphi}wx)^2 \,\&\, B_{i,\varphi}w'y \vee (\exists x) \bigvee_{\substack{j=1 \\ j\neq i}}^{m} A_{i,\varphi}wx \,\&$
$A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y)$ (L.3, properties of FTT)
(L.5) $T \vdash B_{i,\varphi}w'y \Rightarrow (\exists x)(A_{i,\varphi}wx \,\&\, \bigvee_{j=1}^{m} A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y)$
(L.4,properties of FTT)

Hence, it is sufficient to prove the theorem only for the opposite implication in (56).

(a) $\Rightarrow$ (b):

(L.1) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, \bigvee_{j=1}^{m} A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i = 1, \ldots, m$
(assumption)

(L.2) $T \vdash (\exists x)(\bigvee_{j=1}^{m} A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i = 1,\dots,m$
(L.1, properties of FTT)

(L.3) $T \vdash (\bigvee_{j=1}^{m}(\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y)) \Rightarrow B_{i,\varphi}w'y, \quad i = 1,\dots,m$
(L.2, properties of FTT)

(L.4) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i,j = 1,\dots,m$
(L.3, properties of FTT)

(L.5) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx) \Rightarrow \cdot B_{j,\varphi}w'y \Rightarrow B_{i,\varphi}w'y, \qquad i,j = 1,\dots,m$
(L.4, properties of FTT)

(b) ⇒ (a):

(L.1) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx) \Rightarrow \cdot B_{j,\varphi}w'y \Rightarrow B_{i,\varphi}w'y, \qquad i,j = 1,\dots,m$
(assumption)

(L.2) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i,j = 1,\dots,m$
(L.1, properties of FTT)

(L.3) $T \vdash (\bigvee_{j=1}^{m}(\exists x)(A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y)) \Rightarrow B_{i,\varphi}w'y, \quad i = 1,\dots,m$
(L.2, properties of FTT since L.2 holds for all i, j)

(L.4) $T \vdash (\exists x)(\bigvee_{j=1}^{m} A_{i,\varphi}wx \,\&\, A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i = 1,\dots,m$
(L.3, properties of FTT)

(L.5) $T \vdash (\exists x)(A_{i,\varphi}wx \,\&\, \bigvee_{j=1}^{m} A_{j,\varphi}wx \,\&\, B_{j,\varphi}w'y) \Rightarrow B_{i,\varphi}w'y, \qquad i = 1,\dots,m$
(L4, properties of FTT)

The theorem then follows from the definition of $\equiv$. □

6.9

The following result is a very general syntactic formulation of the results found by I. Perfilieva and presented in [24, 30].

Theorem
Let (53) be provable. Then

$$T \vdash A_{i,\varphi} \equiv A_{j,\varphi} \Rightarrow \cdot B_{i,\varphi} \equiv B_{j,\varphi}$$

for all $i,j = 1,\dots,m$.

PROOF: Immediately from Lemma 5.13 and (54). □

It follows from this theorem that if a system of fuzzy IF-THEN rules has an implicit intension then nearness (in the sense of the fuzzy equality $\equiv$) of intensions $\mathrm{Int}(\mathcal{A}_i)$ and $\mathrm{Int}(\mathcal{A}_j)$ necessarily implies also nearness of the corresponding intensions $\mathrm{Int}(\mathcal{B}_i)$ and $\mathrm{Int}(\mathcal{B}_j)$.

6.10 Outline of further development

It follows from the above analysis that fuzzy IF-THEN rules can be seen as linguistic (i.e. surface) representations of special formulas of fuzzy intensional logic. When taken in this way, they lead to formal theories with special properties. This opens a wide field for pure logical study. Various specific concepts may thus be introduced, for example as follows.

Let T be a consistent theory based on LD in the sense of Subsection 6.7. Let $\mathcal{R}^0 \notin LD$ be a new fuzzy IF-THEN rule. Then $\mathcal{R}^0$ extends LD if

$$T \cup \{\lambda w' \, \lambda y \, B_\varphi w' y \equiv \lambda w' \, \lambda y \, U_{\varphi\varphi} A_\varphi\}$$

is consistent where A_φ, B_φ interpret adjectival predications in the antecedent and succedent of $\mathcal{R}$, respectively. A theory T is maximal if there is no fuzzy IF-THEN rule $\mathcal{R}$ which extends it. We may now study conditions under which this is possible.

We may also study ways of deriving conclusions from linguistic descriptions. A specific principle is perception-based logical deduction described in [21, 25].

Interesting problem raises when realizing that we may consider also compound linguistic expressions of the form

$$\hat{\mathcal{R}} := \mathcal{R}_1^I \text{ and } \cdots \text{ and } \mathcal{R}_m^I, \tag{57}$$

$$\breve{\mathcal{R}} := \mathcal{R}_1^A \text{ or } \cdots \text{ or } \mathcal{R}_m^A. \tag{58}$$

We may find explicit intensions of these expressions in the way analogous to that described in Subsections 5.3 and 5.5:

$$\mathrm{Int}(\hat{\mathcal{R}}) = \lambda w \, \lambda w' \cdot \lambda x \, \lambda y \cdot \bigwedge_{j=1}^{m} A_{j,\varphi} w x \Rightarrow B_{j,\varphi} w' y, \tag{59}$$

$$\mathrm{Int}(\breve{\mathcal{R}}) = \lambda w \, \lambda w' \cdot \lambda x \, \lambda y \cdot \bigvee_{j=1}^{m} A_{j,\varphi} w x \,\&\, B_{j,\varphi} w' y. \tag{60}$$

However, intensions (59) or (60) need not be equivalent to the implicit intensions considered in Subsection 6.5 and further. Conditions for existence of the latter, relation between explicit and implicit intensions, their deduction potential, and other questions should be studied.

Is there a difference between (45), and (57) or (58) (alternatively, in terms of intensions, between (46) and (59) or (60))? The answer is yes: (45) is a text consisting of separate sentences while (57) or (58) are specific compound expressions whose parts (i.e. fuzzy IF-THEN rules) are set to be related to each other using a specific connective, namely 'and' or 'or'. Thus, the latter is a stronger proposition than the former. We see from Subsections 6.5–6.9 that

(57) and (58) are linguistic characterizations of strongly extensional functions. Unfortunately, this is a rather big limitation and we will demonstrate elsewhere that the use of evaluating linguistic expressions is in most realistic situations excluded (cf. [20]).

Note that interpretation of (59) and (60) are functions that assign to each couple of contexts $w^0, w'^0 \in W$ one of the respective fuzzy relations

$$\bigwedge_{j=1}^{m} (A_j(w^0, u) \rightarrow B_j(w'^0, v), \tag{61}$$

or

$$\bigvee_{j=1}^{m} (A_j(w^0, u) \otimes B_j(w'^0, v), \tag{62}$$

where $A_j(w^0) \underset{\sim}{\subseteq} M_\alpha$, $B_j(w'^0) \underset{\sim}{\subseteq} M_\alpha$, $j = 1, \ldots, m$ and $A_j(w^0, u)$, $B_j(w'^0, v)$ are the corresponding membership degrees for $u, v \in M_\alpha$.

6.11 Disregarding natural language

Let us briefly discuss the relation between the interpretation of fuzzy IF-THEN rules as presented in the literature on fuzzy set theory (cf. [2, 9, 15, 28–30]) and the theory developed in this paper.

The essential difference lays in taking fuzzy IF-THEN rules as a specific way for description of a function and ignoring their character as natural language expressions. Namely, the whole linguistic description is assigned one of two possible formulas of first-order fuzzy logic:

$$\mathrm{DNF}(x, y) = \bigvee_{j=1}^{m} (A_j(x) \mathbin{\&} B_j(y)), \tag{63}$$

or

$$\mathrm{CNF}(x, y) = \bigwedge_{j=1}^{m} (A_j(x) \Rightarrow B_j(y)) \tag{64}$$

where DNF stands for *disjunctive*, and CNF for *conjunctive normal form* because due to I. Perfilieva [26, 28–30], these forms are obtained as generalization of the corresponding classical concepts. There are no special requirements on the formulas $A_j(x), B_j(y)$ from the point of view of linguistics. It should be noted that in practice, most frequent is (63). Then the formula

$$(\exists x)(A'(x) \mathbin{\&} \mathrm{DNF}(x, y)),$$

corresponds to our implicit knowledge from linguistic description (cf. Subsection 6.5) and its interpretation is called Mamdani (or Mamdani-Assilian) rule of inference (the connective **&** is often interpreted by minimum) — cf. [3, 13].

In the light of Subsection 5.17, (63) and (64) correspond to (60) and (59) when a couple of contexts $w^0, w'^0 \in W$ is fixed with the interpretation being one of the respective fuzzy relations (62) and (61). It is significant, that another interpretation of (63) and (64) is obtained when *changing the model* while in case of (60) and (59), another interpretation is obtained in the *same model* when changing the contexts w^0, w'^0. Still more can be obtained when changing also the model. Moreover, FTT includes predicate fuzzy logic †) and so, everything proved in the latter can be proved also in FTT. Consequently, our theory of fuzzy IF-THEN rules is more general and comprehensive.

7 Conclusion

In this paper, we made a step to the development of a concise formal theory of fuzzy IF-THEN rules. This theory should encompass the formal theories based on fuzzy predicated logics as well as theories focusing only on semantical models. The outcome of our approach is high generality which enables us to cover not only technical side of the rules, but also to model their meaning when taken as special expressions of natural language. We are confident that eventually all relevant knowledge about fuzzy inference methods based on fuzzy IF-THEN rule bases will be represented, formalized and backed up by proof within the well-founded logical representation presented here. Thus, we get closer to the ability to equip robots with intelligence so that they could understand, at least, part of our language and process linguistic instructions in a way analogous to the way how people normally do it.

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†) The systems of FTT can be developed also for stronger structures of truth values than IMTL_Δ, e.g. Łukasiewicz$_\Delta$-, BL_Δ- or ŁΠ-algebras.

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