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Genetic algorithms in fuzzy approximation

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Abstract

Three universal approximation formulas given by so called disjunctive, conjunctive and additive normal forms are recalled. Properties and relations between all three formulas are studied. The quality of approximation is shown and the role of genetic algorithms in the field of fuzzy approximation is explained.

Keywords: Fuzzy approximation, Additive normal form, Conjunctive normal form, Disjunctive normal form, Genetic algorithm.

1 Introduction

Originally, the notion of normal form has been introduced for Boolean functions. There, normal forms had a simplifying as well as unifying character. Moreover, they are equal to the original function.

The generalization considered below is based on the shape of the classical normal forms for Boolean functions, where the operations on $\{0,1\}$ are replaced by generalized ones on $[0,1]$. And moreover, fuzzy relations are used instead of Boolean functions, see e.g. [9, 5, 10]. Having such a generalization, we can formulate problems whose solutions clarify the relationship between normal forms and the initial fuzzy relation.

In this work, we will follow the approach introduced in [9] and generalized in [2]. There, normal forms in disjunctive and conjunctive form are considered. Moreover, we will generalize the additive normal form introduced in [11] and elaborate its properties. We will establish a family of additive normal forms which will include the original motivation case published in [11]. Moreover, the conditional equivalence of this additive normal forms will be partially clarified.

Additionally, we are interested in formulas minimizing some fixed criterion. It leads to the formulation of an optimization problem, which we will solve using genetic algorithms.

2 Preliminaries

2.1 t-norms

In this subsection, we recall basic facts from the theory of t-norms. For more details we refer to [7].

Definition 1 *Let $* : [0,1]^2 \to [0,1]$ be a commutative, associative, non-decreasing mapping such that $x * 1 = x$ for all $x \in [0,1]$. Then $*$ is called triangular norm.*

Typical t-norms are the *minimum*, the *product* and the *Łukasiewicz* t-norm $\otimes$:

$$x \otimes y = \max(x + y - 1, 0).$$

It follows from the definition of a t-norm that it is a monoidal operation on $[0,1]$. Furthermore, $\langle [0,1], \wedge, \vee \rangle$ is a complete lattice. Therefore, we can define the residuation operation (also residuum) in the following form.

Definition 2 *Let $*$ be a left-continuous t-norm. The residuation operation $\to_*$: $[0,1]^2 \to [0,1]$ is defined by*

$$x \to_* y = \bigvee \{z \in [0,1] \mid x * z \leq y\}. \tag{1}$$

Moreover, we will use the following operations

$$x \leftrightarrow_* y = (x \to_* y) \wedge (y \to_* x) \tag{2}$$

called biresiduation (also biresiduum) and

$$x \oplus y = \min(1, x + y) \tag{3}$$

called Łukasiewicz t-conorm. It is worth to mention the following equality

$$(1 - x) \oplus y = x \to_\otimes y. \tag{4}$$

2.2 Extensionality of fuzzy relations

Below, we will introduce generalized definition of the extensionality property, where we assume that the description of an interrelation between elements $\mathbf{x}$ and $\mathbf{y}$ expressed by $R(\mathbf{x}, \mathbf{y})$ may be arbitrary.

Let M be some nonempty set of objects.

Definition 3 *A fuzzy relation $P : M^n \to [0,1]$ is extensional w.r.t. binary fuzzy relation R on M^n and t-norm $*$ if for each $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n) \in M$,*

$$R(\mathbf{u}, \mathbf{v}) * P(\mathbf{u}) \leq P(\mathbf{v}). \tag{5}$$

Usually, the extensionality is defined w.r.t. a similarity relation. The given definition of the generalized extensionality has been already motivated and discussed in [3, 4].

Remark 1 *For more details on this topic we refer to [6, 4, 5, 10].*

3 Normal forms

First of all, we recall discrete disjunctive and conjunctive normal forms introduced in [9, 10] for an n-ary fuzzy relation. For more details we refer to [5, 11].

In the sequel, we will assume that $*$ is a left-continuous t-norm, F is an n-ary fuzzy relation on M, R is a binary fuzzy relation on M^n, and $\mathbf{C}_k = \{\mathbf{c}_i \in M^n \,|\, i = 1, \ldots, k\}$, $k \in \mathbb{N}$.

The discrete disjunctive and conjunctive normal forms are given as follows:

$$F_{\mathrm{DNF},*}(\mathbf{x}) = \bigvee_{\mathbf{c} \in \mathbf{C}_k} (R(\mathbf{c}, \mathbf{x}) * F(\mathbf{c})), \tag{6}$$

$$F_{\mathrm{CNF},*}(\mathbf{x}) = \bigwedge_{\mathbf{c} \in \mathbf{C}_k} (R(\mathbf{x}, \mathbf{c}) \to_* F(\mathbf{c})), \tag{7}$$

respectively.

The following definition of an additive normal form is taken from [11] and extended. There, the approximating formula consists of two main parts: (1) description of neighborhoods of some fixed nodes in the form of similarity relations (reflexive, symmetric, and $*$-transitive) joined by the Łukasiewicz t-norm, (2) the concrete values of a given function in these nodes. Moreover, parts (1) and (2) are joined by the product t-norm, and finally, they are tagged together by the addition (Łukasiewicz t-conorm). Below, an arbitrary t-norm is considered instead of the product.

Definition 4 *Let F be an n-ary fuzzy relation on M and R be a binary fuzzy relation on M^n. The additive normal form of F w.r.t. a t-norm $*$ is given by the following formula*

$$F_{\mathrm{ANF},*}(\mathbf{x}) = \bigoplus_{\mathbf{c} \in \mathbf{C}_k} (R(\mathbf{c}, \mathbf{x}) * F(\mathbf{c})). \tag{8}$$

It has been already shown [2] that the discrete disjunctive and conjunctive normal forms give lower and upper approximation of an extensional fuzzy relation, respectively i.e.

$$F_{\mathrm{DNF},*}(\mathbf{x}) \leq F(\mathbf{x}) \leq F_{\mathrm{CNF},*}(\mathbf{x}). \tag{9}$$

We do not obtain similar result for the discrete additive normal form. But we can at least compare $F_{\mathrm{ANF},*}(\mathbf{x})$ with $F_{\mathrm{DNF},*}(\mathbf{x})$ and $F_{\mathrm{CNF},\otimes}(\mathbf{x})$ whenever R fulfils an additional property called orthogonality.

Definition 5 *Let R be a binary fuzzy relation on M^n. We say that R fulfils the orthogonality property if*

$$\bigoplus_{\mathbf{c} \in \mathbf{C}_k, \mathbf{c} \neq \mathbf{d}} R(\mathbf{c}, \mathbf{x}) = 1 - R(\mathbf{d}, \mathbf{x}), \tag{10}$$

is valid for each $\mathbf{x} \in M^n$ and $\mathbf{d} \in \mathbf{C}_k$.

The relationship between normal forms is shown in the following proposition.

Proposition 1 *Let R be a binary fuzzy relation on M^n and F be an n-ary fuzzy relation on M. If R is symmetric and fulfils orthogonality condition (10) then*

$$F_{\mathrm{DNF},*}(\mathbf{x}) \leq F_{\mathrm{ANF},*}(\mathbf{x}), \tag{11}$$

$$F_{\mathrm{ANF},*}(\mathbf{x}) \leq F_{\mathrm{CNF},\otimes}(\mathbf{x}), \tag{12}$$

for all $\mathbf{x} \in M^n$.

PROOF: The technique of the proof is analogous to that of [11]. □

Let us illustrate the relationships between normal forms on the following example.

Example 1 *Let us consider the following one-dimensional case where the approximated fuzzy relation*

$$F(\mathbf{x}) = 0.4 \sin(4\mathbf{x}) + 0.4$$

is defined on $M = [0, 1]$ and let $k = 10$. Binary fuzzy relation R is given as

$$R(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \leftrightarrow_{\otimes} \mathbf{y})^9,$$

while the nodes $\mathbf{c}_i$ are defined as $\mathbf{c}_i = (i - 1)/k$ for $i = 1, \ldots, k$. Finally, let $*$ be the product t-norm $\odot$.

Then, we obtain a relationship between the conjunctive, the disjunctive and the additive normal forms which is illustrated on Figure 1. From Figure 2, it is clear that the additive normal form is absolutely the best approximation formula from the set of normal forms for the fuzzy relation F with respect to R and the given number and distribution of nodes $\mathbf{c}_i$ over M and its error is even not visible.

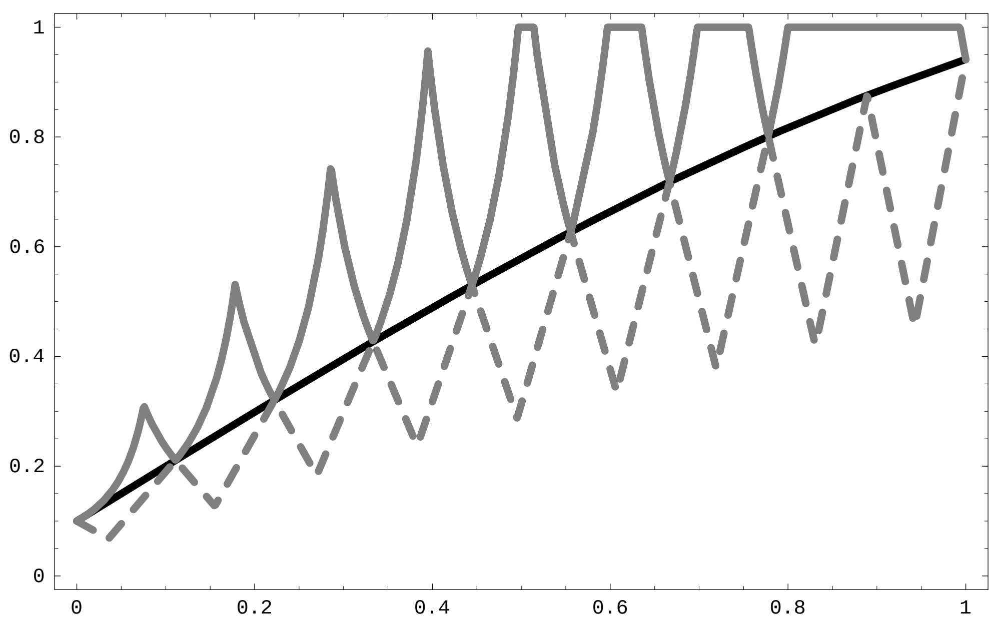

Figure 1: Relationship between normal forms from Example 1. The black line represents $F_{\mathrm{ANF},\odot}$, the dashed gray line is for $F_{\mathrm{DNF},\odot}$ and the solid gray line belongs to $F_{\mathrm{CNF},\odot}$.

Normal forms give an approximate representation of a given fuzzy relation. This fact is further expressed on the basis of the biresidual operation. Since the biresiduum belongs to the class of similarity relations (also called equivalences), therefore in the sequel, we will use the term equivalence instead of biresiduum. The next result is taken from [5] and it shows that an arbitrary extensional formula is equivalent to its normal form. This equivalence is bounded from below by a function independent of the original fuzzy relation.

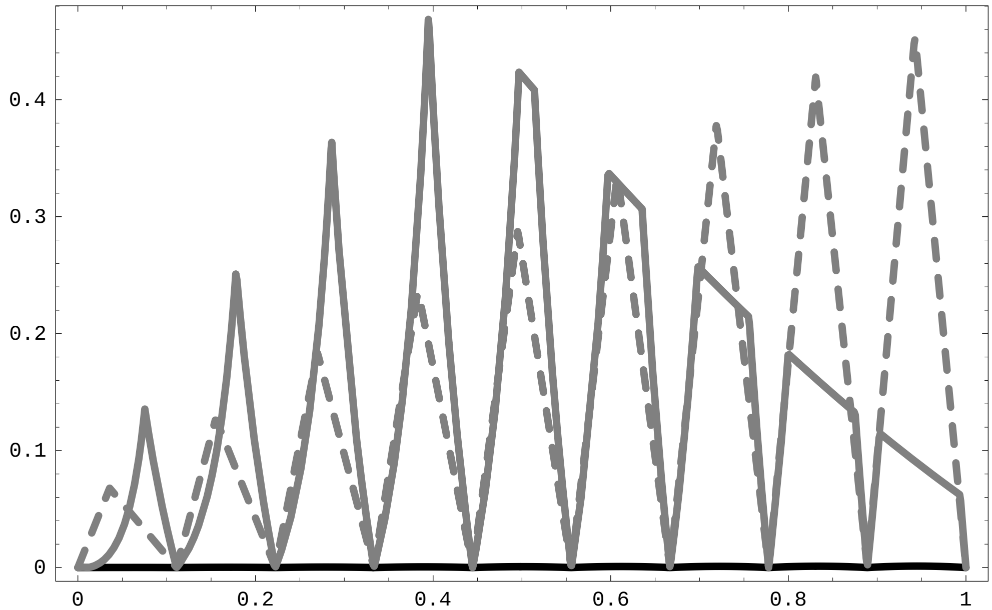

Figure 2: Error of approximation from Example 1. The black line represents $F_{\mathrm{ANF},\odot}$, the dashed gray line is for $F_{\mathrm{DNF},\odot}$ and the solid gray line belongs to $F_{\mathrm{CNF},\odot}$.

Theorem 1 *Let R be a binary fuzzy relation on M^n and F be an n-ary fuzzy relation on M. If F is extensional w.r.t. R and a left-continuous t-norm $*$ then*

$$F_{\mathrm{DNF},*}(\mathbf{x}) \leftrightarrow_* F(\mathbf{x}) \geq C_*(\mathbf{x}), \tag{13}$$
$$F_{\mathrm{CNF},*}(\mathbf{x}) \leftrightarrow_* F(\mathbf{x}) \geq C_*(\mathbf{x}), \tag{14}$$

for all $\mathbf{x} \in M^n$, where

$$C_*(\mathbf{x}) = \bigvee_{\mathbf{c}\in\mathbf{C}_k} (R(\mathbf{x},\mathbf{c}) * R(\mathbf{c},\mathbf{x})). \tag{15}$$

The similar result can be obtained for a special class of $F_{\mathrm{ANF},*}$'s. This class is specified by properties of $*$ and it is known that the class of t-norms is partially ordered set w.r.t. the pointwise order (see [7]). To formulate and prove Theorem 2, the notion of weaker and stronger t-norm will be used.

We say that a t-norm $*_1$ is weaker than $*_2$ if $a *_1 b \leq a *_2 b$ for all $a, b \in [0,1]$. Then, we write $*_1 \leq *_2$. Analogously, we say that $*_1$ is stronger than $*_2$ if $a *_1 b \geq a *_2 b$ for all $a, b \in [0,1]$, and we write $*_1 \geq *_2$ (see [7]).

Theorem 2 *Let R be a symmetric binary fuzzy relation on M^n and R fulfils orthogonality condition (10). Let F be extensional w.r.t. R and $\otimes$ and moreover, let F be extensional w.r.t. R and a left-continuous t-norm $*$.*

- *If $*$ is weaker than $\otimes$, then*

$$F_{\mathrm{ANF},*}(\mathbf{x}) \leftrightarrow_* F(\mathbf{x}) \geq C_*(\mathbf{x}), \tag{16}$$

  for $\mathbf{x} \in M^n$.

- *If $*$ is stronger than $\otimes$ then*

$$F_{\mathrm{ANF},*}(\mathbf{x}) \leftrightarrow_\otimes F(\mathbf{x}) \geq C_\otimes(\mathbf{x}), \tag{17}$$

  for $\mathbf{x} \in M^n$.

PROOF: Using Theorem 1 we get

$$F(\mathbf{x}) \to_* F_{\mathrm{DNF},*}(\mathbf{x}) \geq C_*(\mathbf{x}),$$

$$F_{\mathrm{CNF},\otimes}(\mathbf{x}) \to_\otimes F(\mathbf{x}) \geq C_\otimes(\mathbf{x})$$

and Proposition 1 implies

$$F(\mathbf{x}) \to_* F_{ANF,*}(\mathbf{x}) \geq C_*(\mathbf{x}), \tag{18}$$

$$F_{\mathrm{ANF},*}(\mathbf{x}) \to_\otimes F(\mathbf{x}) \geq C_\otimes(\mathbf{x}). \tag{19}$$

If $*$ is weaker than $\otimes$ then

$$F_{\mathrm{ANF},*}(\mathbf{x}) \to_* F(\mathbf{x}) \geq F_{\mathrm{ANF},*}(\mathbf{x}) \to_\otimes F(\mathbf{x})$$

and

$$C_\otimes(\mathbf{x}) \geq C_*(\mathbf{x})$$

and therefore

$$F_{\mathrm{ANF},*}(\mathbf{x}) \to_* F(\mathbf{x}) \geq C_*(\mathbf{x})$$

which together with (18) proves (16).

If $*$ is stronger than $\otimes$ then

$$F(\mathbf{x}) \to_\otimes F_{\mathrm{ANF},*}(\mathbf{x}) \geq F(\mathbf{x}) \to_* F_{\mathrm{ANF},*}(\mathbf{x})$$

and

$$C_*(\mathbf{x}) \geq C_\otimes(\mathbf{x})$$

and therefore

$$F(\mathbf{x}) \to_\otimes F_{\mathrm{ANF},*}(\mathbf{x}) \geq C_\otimes(\mathbf{x})$$

which together with (19) proves (17). □

4 Genetic algorithm in construction of normal forms

In this section, we focus on the problem of the proper distribution of the nodes used in the construction of normal forms by minimizing an error function. In [4], there has been shown that for the same error function the distribution of the nodes used in the construction of F_{CNF} and F_{DNF} is different. Therefore, each of three normal forms must be constructed separately and independently of the other ones.

Let $R(\mathbf{x},\mathbf{y}) = R_1(x_1,y_1) * \cdots * R_n(x_n,y_n)$ and fuzzy relations $R_i(x,y)$ be given by

$$R_i(x,y) = (x \to_* y)^{k_i(x,y)} * (y \to_* x)^{l_i(x,y)}$$

and k_i, l_i are determined on the basis of Theorem 4 from [3]. Then for introduced normal forms the error functions, which are to be minimized, are given as follows:

$$\mathbf{e}_{\mathrm{D(C)NF},*} = \sup_{\mathbf{x}\in M^n} g(F_{\mathrm{D(C)NF},*}(\mathbf{x}) \leftrightarrow_* F(\mathbf{x})), \tag{20}$$

$$\mathbf{e}_{\mathrm{ANF},*} = \sum_{\mathbf{x}\in M^n} (g(F_{\mathrm{ANF},*}(\mathbf{x}) \leftrightarrow_* F(\mathbf{x}))^2, \tag{21}$$

where g is an additive generator of the t-norm $*$. It is worth mentioning that while optimizing the distribution of the nodes for $F_{\mathrm{ANF},*}$ the orthogonality property (10) should be kept.

We have chosen genetic algorithms as an optimization tool for this particular problem because of their advantageous behavior in the case of complex problems. At this stage of investigation the simple genetic algorithm (SGA) is considered and classical binary coding is used (see [1, 8]).

Now, we are able to describe formally the algorithm searching for a normal form (NF) approximating the given fuzzy relation with desired accuracy ε.

Algorithm:
Inputs: ε, MaxNumber

```
begin
    NumberOfNodes = 0;
    BestNodes = [ ];
    e_{NF,*} = 1;
    while (e_{NF,*} > ε) and
    (NumberOfNodes < MaxNumber) do
    begin
        NumberOfNodes = ...
           NumberOfNodes + 1;
        [BestNodes, e_{NF,*}] = ...
           SGA(NumberOfNodes);
    end
end
```

Output: BestNodes

Remark 2 *The standard genetic algorithm called in the algorithm above has the following attributes:*

- *the initial population is generated randomly,*
- *an array of nodes is coded the same as multidimensional point (see [8]) representing one individual,*
- *the best representative of a population is the one with the minimal evaluation value,*
- *the set of nodes forming parameter* $\mathbf{C}$ *of the normal forms is given as a set of all* n *- dimensional permutations from* $\{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$, $\mathbf{c}_i \in M$. *It implies that we construct normal forms in* k^n *nodes,*
- *the stopping condition in SGA is analyzed from* p *previous generations (including the actual one) on the basis of the following characteristic*

$$\triangle(p) = \sum_{i=j-p}^{j-1} |\mathbf{e}^{i}_{\mathrm{NF},*} - \mathbf{e}^{i+1}_{\mathrm{NF},*}|, \tag{22}$$

  where j *is the index of the actual generation and* $\mathbf{e}^{i}_{\mathrm{NF},*}$ *is the error of approximation of the best individuum in the* i*'th population. Then, SGA runs either until* $\triangle(p) < d$ *(parameter* d *is set by a user) or until the number of populations reaches its maximum value.*

Example 2 *Let us consider* $n = 1$ *and* $F(\mathbf{x}) = 0.25\sin(20\mathbf{x}) + 0.5$, $\mathbf{x} \in [0, 1]$ *and the Łukasiewicz t-norm* $* = \otimes$. *We have applied the genetic algorithm in the form described above to minimize error functions (20) and (21), respectively.*

The additive normal form for F *constructed using the genetic algorithm is depicted on Fig. 3 by the dashed line. Fuzzy relations* $R(\mathbf{c}_i, \mathbf{x})$ *with nodes* $\mathbf{c}_i$ *found by the genetic algorithm are shown on Fig. 4. Finally, the error of approximation is displayed on Fig. 5. Concerning the disjunctive and conjunctive normal forms constructed using the genetic algorithm (described above) we refer to [4].*

5 Conclusion

We have presented the specific approach to approximation of extensional fuzzy relations given by the so-called normal forms. Besides already introduced disjunctive and conjunctive normal forms we have extended this class of approximating formulas by the generalized additive normal form.

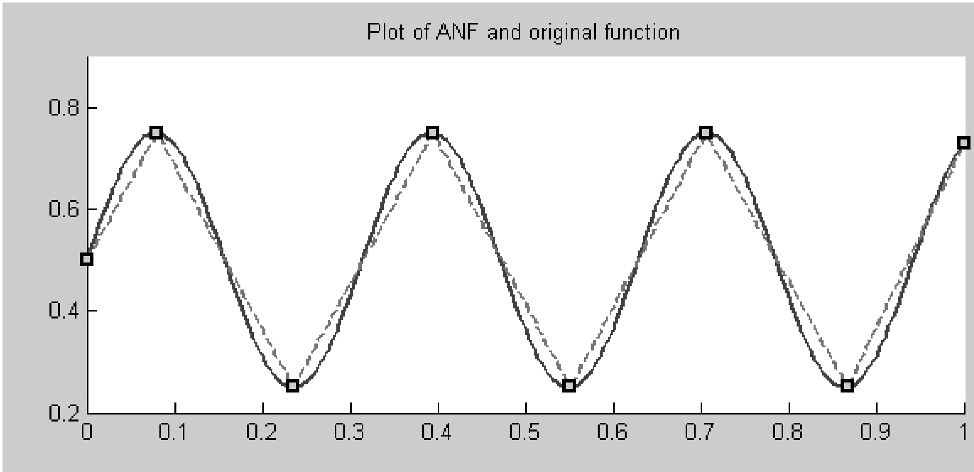

Figure 3: F (solid line) and its $F_{\mathrm{ANF},\otimes}$ (dashed line). The dots represent $F_{\mathrm{ANF},\otimes}(\mathbf{c}_i)$.

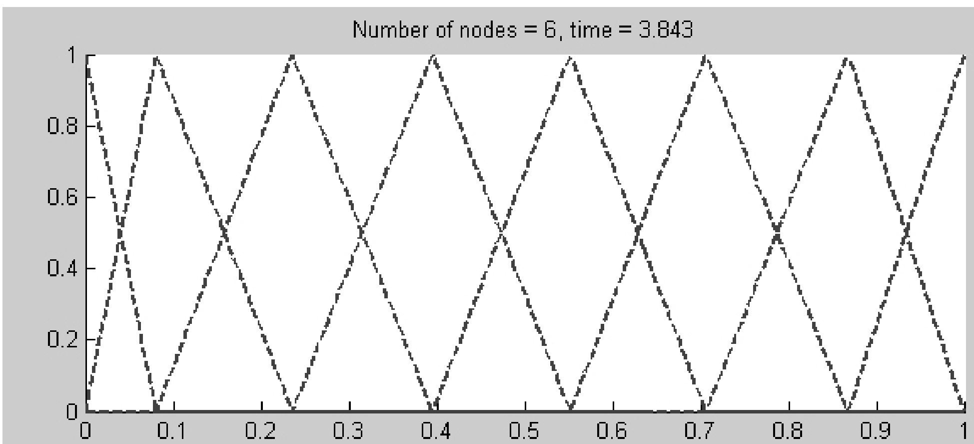

Figure 4: Fuzzy relations $R(\mathbf{c}_i, \mathbf{x})$ with nodes $\mathbf{c}_i$ found by the genetic algorithm.

Properties of such approximate representation of a fuzzy relation by normal forms have been presented as well, namely the conditional equivalence property and the fact that disjunctive (conjunctive) normal forms are lower (upper) bounds of the original fuzzy relation. Conditions, under which introduced additive normal forms lie between these bounds, have been formulated as well as conditions under which the additive normal forms are conditionally equivalent.

Finally, optimized automatic construction of any of normal forms with help of the simple genetic algorithm has been justified. The greatest advantage of this approximation method together with the above given algorithm lies in automatization of building an approximation formula for given fuzzy relation without complicated coding of: 1) fuzzy sets covering the domain, and 2) fuzzy relations joining them, into the genetic information. It follows from the fact that we work with pre-set description of neighborhoods for each node $\mathbf{c}_i \in M^n$ which eliminates a time-consuming process of tricky coding and decoding of fuzzy sets describing the domain of the approximated fuzzy relation F.

Acknowledgment

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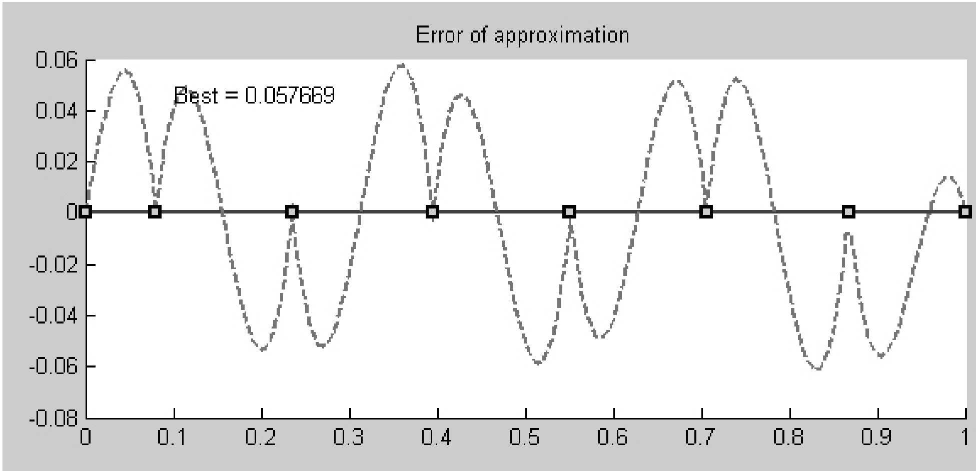

Figure 5: Error of $F_{\mathrm{ANF},\otimes}$. The dots represent the error in $\mathbf{c}_i$.

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