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System of fuzzy relation equations with inf $\rightarrow$ composition: solvability and solutions

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Abstract

In this paper I investigate the System of fuzzy relation equations with inf $\rightarrow$ composition. I compare this system with the system of fuzzy relation equations with $sup - *$-composition. The purpose of this contribution is to show that there are many things which are common for both systems with $sup - *$ and $inf \rightarrow$-composition operations.

K e y w o r d s: System of fuzzy relation equations; $sup - *$-composition and inf $\rightarrow$ composition; solvability of fuzzy relation equation system.

1 INTRODUCTION

System of fuzzy relation equations is traditionally understood with respect to the $sup - *$-composition where $*$ is usually a continuous t-norm. This comes back to early Zadeh's papers [10, 11] where he introduced the compositional rule of inference and considered it's realization with help of $sup - min$-composition. This type of composition has been successively used in formalization of the generalized Modus Ponens rule of inference and then in many models of approximate reasoning based on this rule (see e.g. [5]).

The $sup - *$ is not the only possible composition between two fuzzy relations. In [9], Sanchez suggested the dual (in some sense) composition expressed by $inf \rightarrow$ operations. The properties of this composition have been investigated in [2, 1] where the necessary and sufficient condition of solvability and the characterization of a set of all solutions have been presented. However, the $inf \rightarrow$ composition is not so popular in applications as its counterpart and therefore, only basic things regarding the solvability of systems of fuzzy relation equations are known. The purpose of this contribution is to show that there are many things which are common for both systems with $sup - *$ and $inf \rightarrow$-composition operations. For example, in a specific, but a very often case where input and output fuzzy sets are fuzzy points, the necessary and sufficient condition of solvability is the same for both systems. However, in general it is not true that the one system is solvable if and only if the dual one is solvable as well. A certain equivalence between systems of fuzzy relation equations has been discovered in this contribution.

2 PRELIMINARIES

2.1 Residuated lattice, fuzzy sets and fuzzy relations

We choose a complete residuated lattice as the basic algebra of operations.

Definition 1
A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$. *with four binary operations and two constants such that*

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ *is a lattice where the ordering* $\leq$ *defined using operations* $\vee, \wedge$ *as usual, and* $\mathbf{0}, \mathbf{1}$ *are the least and the greatest elements;*
- $\langle L, *, \mathbf{1}\rangle$ *is a commutative monoid, that is,* $*$ *is a commutative and associative operation with the identity* $a * \mathbf{1} = a$*;*
- *the operation* $\rightarrow$ *is a residuation operation with respect to* $*$*, i.e.*

$$a * b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The following unary operation of negation and binary operation of biresiduation can be additionally defined:

$$\begin{aligned} \neg x &= x \rightarrow 0, \\ x \leftrightarrow y &= (x \rightarrow y) \wedge (y \rightarrow x). \end{aligned}$$

The negation function is *involutive* if $\neg\neg x = x$ (the law of double negation).

The well known examples of residuated lattices are boolean algebras, Gödel, Łukasiewicz and product algebras. In particular case $L = [0, 1]$, multiplication $*$ is known as a t-norm.

We accept here a mathematical definition of a fuzzy set. In the rest of this paper we suppose that a complete residuated lattice $\mathcal{L}$ with a support L is fixed, and $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary non-empty sets. Then a *fuzzy set* or better, a fuzzy subset A of $\mathbf{X}$, is identified with a function $A : \mathbf{X} \longrightarrow L$. This function is known as *membership function* of the fuzzy set A. The set of all fuzzy subsets of $\mathbf{X}$ is denoted by $\mathcal{F}(\mathbf{X})$. A fuzzy set $A \in \mathcal{F}(\mathbf{X})$ is called *normal* if $A(x_0) = \mathbf{1}$ for some $x_0 \in \mathbf{X}$.

The algebra of operations over fuzzy subsets of $\mathbf{X}$ is introduced as an induced residuated lattice on $L^{\mathbf{X}}$. This means that each operation from $\mathcal{L}$ induces the corresponding operation on $L^{\mathbf{X}}$ taken point-wise. Obviously, operations over fuzzy subsets fulfill the same properties as operations in the respective residuated lattice.

A (binary) *fuzzy relation* on $\mathbf{X} \times \mathbf{Y}$ is a fuzzy subset of the Cartesian product. $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$ denotes the set of all binary fuzzy relations on $\mathbf{X} \times \mathbf{Y}$. Analogously, an n-ary fuzzy relation can be introduced.

Let $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and $S \in \mathcal{F}(\mathbf{Y} \times \mathbf{Z})$, then the fuzzy relation $T = R \circ S$ on $\mathbf{X} \times \mathbf{Z}$

$$T(x, z) = \bigvee_{y \in \mathbf{Y}} (R(x, y) * S(y, z))$$

is called a $sup - *$-composition of R and S, and the fuzzy relation $Q = R \circ' S$ on $\mathbf{X} \times \mathbf{Z}$

$$Q(x, z) = \bigwedge_{y \in \mathbf{Y}} (R(x, y) \to S(y, z))$$

is called an $inf \to$-composition of R and S.

If A is a unary fuzzy relation on $\mathbf{X}$ or simply a fuzzy subset of $\mathbf{X}$ then the $sup - *$ ($inf \to$)-composition between A and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is the fuzzy subset of $\mathbf{Y}$ defined by analogy.

2.2 Systems of fuzzy relation equations

Let $n \geq 1$. A $sup - *$-system of fuzzy relation equations

$$A_i \circ R = B_i \quad \text{or} \quad \bigvee_{x \in \mathbf{X}} (A_i(x) * R(x, y)) = B_i(y), \quad 1 \leq i \leq n, \tag{1}$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$, is considered with respect to unknown fuzzy relation R.

Analogously, we will consider an $inf \to$-system of fuzzy relation equations

$$A_i \circ' R = B_i \quad \text{or} \quad \bigwedge_{x \in \mathbf{X}} (A_i(x) \to R(x, y)) = B_i(y), \quad 1 \leq i \leq n, \tag{2}$$

with the same description of parameters and also with respect to unknown fuzzy relation R.

Because solutions of (1) and (2) may not exist, in general, the problem to investigate necessary and sufficient, or only sufficient conditions for solvability arises. This problem has been widely studied in the literature with respect to the system (1), and some nice theoretical results have been obtained (e.g. [3, 7, 8, 9]). Unfortunately, the intensive investigation of the solvability of (2) has not been attempted, and only basic results cited in the Introduction are known.

Two types of fuzzy relations have been always assumed with respect to the problem of solvability of (1) and (2), namely

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)) \tag{3}$$

considered in Mamdani & Assilian [5], and

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \tag{4}$$

first considered in Sanchez [9].

3 COMPLETE SET OF SOLUTIONS

In this section we will investigate whether the relation $\check{R}$ is a solution of (2). The following statement connects this special type with the general condition of solvability.

Theorem 1
The system (2) is solvable if and only if the fuzzy relation $\check{R}$ is its solution. If the system (2) is solvable then $\check{R}$ is its smallest solution.

The proof of this theorem can be obtained from the proof of the analogous statement concerning the solvability of the system (1) (see e.g. [4]).

It is easy to see that if system (2) is solvable then the set of solutions forms a $\wedge$-semi-lattice, i.e. a fuzzy relation $R_1 \wedge R_2$ is a solution to (2) whenever R_1 and R_2 are its solutions. Moreover, this semi-lattice has the least element and in the case of finite universes $\mathbf{X}$ and $\mathbf{Y}$, it has maximal elements (see [1]).

Set of solution of the system (1) frames a Root system with stem $\hat{R}$ and set of offshoots $\hat{O}$, where $\hat{R}$ is the greatest solution and $\hat{O}$ is a set of the minimal solutions.

$$\mathcal{R}_{sup} = \bigcup_{R \in \hat{O}} [R, \hat{R}]$$

Set of solution of the system (2) frames a Crown system with stem $\check{R}$ and set of offshoots $\check{O}$, where $\check{R}$ is the smallest solution and $\check{O}$ is a set of the maximal solutions.

$$\mathcal{R}_{inf} = \bigcup_{R \in \check{O}} [\check{R}, R]$$

4 SOLVABILITY WITH RESPECT TO $\hat{R}$

We have mentioned in the Introduction that two types of fuzzy relations have been always assumed with respect to the problem of solvability of (1) and (2), namely $\hat{R}$ and $\check{R}$. If (1) is solvable then $\hat{R}$ is the greatest solution and dually, if (2) is solvable then $\check{R}$ is the least solution. However, if system (1) or (2) is solvable, it does not necessary follow that the relation $\check{R}$, respectively $\hat{R}$, is its solution. Therefore, the additional investigation should be attempted.

In [3], the necessary and sufficient condition of the solvability of (1) with respect to $\check{R}$ is given (that is shown in brackets). We will show in this section that the same condition is necessary and sufficient too for the solvability of (2) with respect to $\hat{R}$.

Theorem 2
Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal. Then the fuzzy relation $\hat{R}$ [$\check{R}$] is a solution to (2) [(1)] if and only if for all $i, j = 1, \ldots, n$ the following inequality

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow B_j(y)) \tag{5}$$

holds.

PROOF: Suppose that the conditions of the theorem are fulfilled. Then, due to the normality of fuzzy sets A_i, it is not difficult to prove that for all $i = 1, \ldots, n$,

$$A_i \circ' \hat{R} \leq B_i$$

holds. Therefore, it is sufficient to prove that the opposite inequality is equivalent to (5). This follows from the following chain of equivalences:

$$
\begin{array}{ll}
(\forall i) & \left(A_i \circ' \hat{R} \geq B_i\right) \quad \Leftrightarrow \\
(\forall i)(\forall y) & \left(\bigwedge\limits_{x\in \mathbf{X}} \left(A_i(x) \to \bigwedge\limits_{j=1}^{n} (A_j(x) \to B_j(y))\right) \geq B_i(y)\right) \quad \Leftrightarrow \\
(\forall i,j)(\forall x)(\forall y) & ((A_i(x) \to (A_j(x) \to B_j(y))) \geq B_i(y)) \quad \Leftrightarrow \\
(\forall i,j)(\forall x)(\forall y) & (A_i(x) * A_j(x) \to B_j(y) \geq B_i(y)) \quad \Leftrightarrow \\
(\forall i,j)(\forall x)(\forall y) & (B_j(y) \geq A_i(x) * A_j(x) * B_i(y)) \quad \Leftrightarrow \\
(\forall i,j)(\forall x)(\forall y) & (B_i(y) \to B_j(y) \geq A_i(x) * A_j(x)) \, .
\end{array}
$$

□

Although solvability and $\hat{R}$-solvability of system (2) are not in general equivalent, this is true under the assumption about semi-partitioning of $\mathbf{X}$.

5 A SIMPLE CRITERION

In the above, we have considered the general case which does not put any restrictions on the given fuzzy sets: A_i and B_i. However, in a practice, we often meet with special fuzzy sets which are the so called fuzzy points or classes of equivalence of some fuzzy equivalence. In this case, we expect to have simpler conditions for solvability of (2) in general as well as in particular, with respect to $\hat{R}$.

It turned out that the condition of solvability of (2) which we are going to introduce in this section, is the same as the condition of solvability of (1) under the same assumptions (see[6]). The proof is also similar and therefore, it is omitted. All the details, concerning the notions of fuzzy equivalence (or similarity) and their classes of equivalence, can be found in [6].

Theorem 3
Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal (so that there exist $x_i \in \mathbf{X}$ and $y_i \in \mathbf{Y}$ which make true the following: $A_i(x_i) = 1$, $B_i(y_i) = 1$). Further, let fuzzy equivalence E on $\mathbf{X}$ and fuzzy equivalence F on $\mathbf{Y}$ exist so that all the fuzzy sets A_i are fuzzy points with respect to x_i and E, and all the fuzzy sets B_i are fuzzy points with respect to y_i and F, i.e.

$$(\forall x)A_i(x) = E(x_i, x) \quad \text{and} \quad (\forall y)B_i(y) = F(y_i, y).$$

Then the system (2) (or (1)) is solvable if and only if

$$(\forall i)(\forall j)A_i(x_j) \leq B_i(y_j). \tag{6}$$

6 SOLVABILITY OF SYSTEMS WITH $\sup *$ AND $\inf \to$ COMPOSITIONS

In this section, we will investigate the question: if the system (1) is solvable, would be the system (2) with the same data solvable as well? We will give the positive answer in the case where the law of double negation holds in the underlying lattice.

Theorem 4
Let $\mathcal{L}$ be a complete residuated lattice and the negation function be involutive and let the system (1) with known parameters $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ be given. Then this system is solvable if and only if the system (2) with the parameters $A_i \in \mathcal{F}(\mathbf{X}), \neg B_i \in \mathcal{F}(\mathbf{Y})$ is also solvable.

PROOF: Let $1 \leq i \leq n$ and y be some fixed element from $\mathbf{Y}$. The proof of the theorem can be easily obtained from the following chain of equalities:

$$
\begin{aligned}
B_i(y) &= \bigvee_{x \in \mathbf{X}} \left(A_i(x) * \bigwedge_{j=1}^{n} (A_j(x) \to B_j(y)) \right) = \\
&= \neg \bigwedge_{x \in \mathbf{X}} \left(\neg \left(A_i(x) * \bigwedge_{j=1}^{n} (A_j(x) \to B_j(y)) \right) \right) = \\
&= \neg \bigwedge_{x \in \mathbf{X}} \left(A_i(x) \to \neg \bigwedge_{j=1}^{n} (A_j(x) \to B_j(y)) \right) = \\
&= \neg \bigwedge_{x \in \mathbf{X}} \left(A_i(x) \to \bigvee_{j=1}^{n} (A_j(x) * \neg B_j(y)) \right).
\end{aligned}
$$

$\square$

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