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Omitting Types in Fuzzy Logic with Evaluated Syntax

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Abstract

This paper is a contribution to the development of model theory of fuzzy logic in narrow sense. We consider a formal system Ev_{L} of fuzzy logic that has evaluated syntax, i.e. axioms need not be fully convincing and so, they form a fuzzy set only. Consequently, formulas are provable in some general degree. A generalization of Gödel's completeness theorem does hold in Ev_{L}. The truth values form an MV-algebra that is either finite or Łukasiewicz algebra on $[0,1]$.

The classical omitting types theorem states that given a formal theory T and a set $\Sigma(x_1,\ldots,x_n)$ of formulas with the same free variables, we can construct a model of T which omits Σ, i.e. there is always a formula from Σ not true in it. In this paper, we generalize this theorem for Ev_{L}, that is, we prove that if T is a *fuzzy theory* and $\Sigma(x_1,\ldots,x_n)$ forms a *fuzzy set* then a model omitting Σ also exists. We will prove this theorem for two essential cases of Ev_{L}: either Ev_{L} has logical (truth) constants for all truth values, or it has these constants for truth values from $[0,1]\cap\mathbb{Q}$ only.

1 Introduction

In this paper, we continue to study model theory of fuzzy logic with evaluated syntax (Ev_{L}). This logic belongs to the class of fuzzy logics in narrow sense where, however, it has a special position. Most of the known fuzzy logics, such as BL, MTL, Łukasiewicz and others have traditional syntax and many-valued semantics. The former means that the concept of a formula remains unchanged and in comparison with classical logic it is extended only by some additional connectives, such as that of strong conjunction **&**. Thus, these fuzzy logics deal with ordinary sets of axioms, i.e. each axiom is taken classically as hereditary true. Ev_{L}, on the other hand, enables to consider axioms that may be true only in some general degree (i.e. a degree smaller than 1). Formally, this means that each formula is assigned a *syntactic evaluation degree* so that we deal with a fuzzy theory determined by a *fuzzy set of axioms* instead of ordinary theory.

A typical example is axiomatic characterization of the sorites paradox[1] that was formally elaborated in [5, 11]. The solution within Ev_{L} stems from the assumption that the axiom "if n is not a heap then $n+1$ is also not a heap" is true in a degree that is close, but smaller than 1, since we cannot take it as fully convincing. We thus obtain a *sorites fuzzy theory* which is consistent and provides a reasonable model solving the sorites paradox.

Other outcome of evaluated syntax is, that it enables us to reason about specific degrees also on the level of syntax. It is notable that Ev_{L} was historically the first precisely elaborated formal system of fuzzy logic initiated in the seminal paper of J. Pavelka [12]. Surprisingly, now it is very little known though its explication power is very high.

A detailed presentation of Ev_{L} is contained in the book [11]. It is specific for Ev_{L}that the set of truth values must be the Łukasiewicz MV-algebra whose support set is either a finite set or the interval $[0,1]$ of real numbers since otherwise, the completeness theorem cannot hold. Other specific feature of Ev_{L} is presence of logical (truth) constants in its language (note that this is a direct generalization of presence of $\bot$ and $\top$ in the language of classical logic). In case that the set of truth values is $[0,1]$, we have two possibilities: either we consider logical constants for all $a\in[0,1]$ or we confine ourselves only to rational ones ($a\in[0,1]\cap\mathbb{Q}$). Unfortunately, the language in the former case is uncountable which is undesirable for many considerations.

Interesting task is to develop a model theory of Ev_{L}. The first steps have been done in [11], further continuation is presented in [9]. In this paper, we turn to important theorem of classical model theory that is the Omitting types theorem (see, e.g. [1, 2, 6]). Let us remark that the first version of this theorem in fuzzy logic has been proved in the book [3], namely for finitely-valued Łukasiewicz logic. We will further extend it, first for Ev_{L} with rational logical constants only and then, for Ev_{L} with logical constants for all the truth values from $[0,1]$. A detailed presentation of Ev_{L} with rational logical constants is contained in [10].

The paper is organized as follows: We start with preliminaries where the notation, basic concepts and properties of Ev_{L} are summarized. The main contribution of the paper is contained in Section 3 where basic notions of the omitting types theory are introduced and three theorems are proved: the Omitting types theorem for Ev_{L} with countable number of logical constants, its slight generalization to countable

[1] One grain does not form a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.

number of fuzzy sets of formulas and finally, its generalization for $\mathrm{Ev}_{\text{Ł}}$ with logical constants for all truth values.

2 Preliminaries

This section contains a brief overview of the main concepts and notation of $\mathrm{Ev}_{\text{Ł}}$. As mentioned, the set of truth values forms either a finite MV-algebra, or *Łukasiewicz MV-algebra*

$$\mathcal{L}_{\text{Ł}} = \langle [0,1], \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$$

where

$$\begin{aligned} a \otimes b &= 0 \vee (a + b - 1), \qquad a \oplus b = 1 \wedge (a + b), \\ \neg a &= 1 - a. \end{aligned}$$

We may introduce also other operations as follows:

$$\begin{aligned} a \vee b &= (a \otimes \neg b) \oplus b, \qquad & a \wedge b &= (a \oplus \neg b) \otimes b, \\ a \to b &= 1 \wedge (1 - a + b), \qquad & a \leftrightarrow b &= (a \to b) \wedge (b \to a). \end{aligned}$$

By L we denote a set of truth values being either a finite set or the interval of reals $[0,1]$. By $\mathbb{Q}$ we denote the set of all rational numbers and by $[0,1]_{\mathbb{Q}}$ the set $[0,1] \cap \mathbb{Q}$ (to have unified notation, we will usually write $L_{\mathbb{Q}}$ for the latter). The zero of L is denoted by $\mathbf{0}$ and its unit by $\mathbf{1}$.

By a fuzzy set we mean a function $D : U \longrightarrow L$ where U is some set (a universe of discourse). We will often write $D \underset{\sim}{\subset} U$ to express that D is a fuzzy set in U. The element $D(x) \in L$ for $x \in U$ is called the *membership degree* of x in D. A fuzzy singleton is a one-element set $\{a/x\}$ where $a \in L$ and $x \in U$. If $D \underset{\sim}{\subset} U$ then its *support* is the (ordinary) set

$$\mathrm{Supp}(D) = \{x \mid D(x) > \mathbf{0}\}.$$

A usual principle considered in fuzzy set theory is the *maximality principle*: the set $\{a/x \mid a \in K \subseteq L\}$ is taken as a singleton $\left\{(\bigvee_{a \in K} a)/x\right\}$. With this principle, each set of singletons is at the same time a fuzzy set.

The *language* of $\mathrm{Ev}_{\text{Ł}}$, denoted by J, consists of a set of *object variables* $x, y, \ldots$, a set of *object constants* $\mathbf{u}, \mathbf{v} \ldots$, a set of *functional symbols* $f, g \ldots$, a set of *predicate symbols* $P, Q \ldots$ where, of course, each functional as well as predicate symbol has a nonzero arity. We denote by $\mathrm{Func}(J)$ the set of all functional symbols, by $\mathrm{Pred}(J)$ the set of all predicate symbols and by $\mathrm{OC}(J)$ the set of all object constants of J.

Furthermore, J contains *implication connective* $\Rightarrow$ and the *general quantifier* $\forall$. Terms and formulas are defined in the same way as in classical logic. Other connectives than above, as well as the existential quantifier are taken as the following abbreviations of formulas:

$$\begin{aligned} \neg A &:= A \Rightarrow \bot, && \textit{(negation)} \\ A \wedge B &:= \neg((B \Rightarrow A) \Rightarrow \neg B), && \textit{(conjunction)} \\ A \vee B &:= (B \Rightarrow A) \Rightarrow A, && \textit{(disjunction)} \\ A \,\&\, B &:= \neg(A \Rightarrow \neg B), && \textit{(Łukasiewicz conjunction)} \\ A \nabla B &:= \neg(\neg A \,\&\, \neg B), && \textit{(Łukasiewicz disjunction)} \\ A \Leftrightarrow B &:= (A \Rightarrow B) \wedge (B \Rightarrow A), && \textit{(equivalence)} \\ (\exists x)A &:= \neg(\forall x)\neg A. \end{aligned}$$

Important feature of $\mathrm{Ev}_{\text{Ł}}$ is presence of *logical (truth) constants* in the language J which are names of the truth values $a \in L$. The original presentation of $\mathrm{Ev}_{\text{Ł}}$ by J. Pavelka in [12], continued also in most parts of [11] assumes logical constants for all the truth values. This assumption, in the case that

$L = [0,1]$, causes a lot of problems and so, it is desirable to get rid of all logical constants for irrational truth values. This has been done by V. Novák in [8] and by P. Hájek in [4] who proved that the generalized completeness theorem still does hold. Hence, in case that $L = [0,1]$ we will in the sequel consider two possibilities: either we take logical constants for all the truth values $a \in L$, or we consider them only for $a \in L_{\mathbb{Q}}$. If necessary, we will more precisely write J_L instead of J if the language contains logical constants *for all truth values* $a \in L$ and J_{L_Q} if the language contains logical constants only for $a \in L_{\mathbb{Q}}$. If the set of logical constants is unimportant or clear from the context, we will simply write J. By $\mathrm{LC}(J)$ we denote the set of all logical constants of the language J. The top and bottom logical constants will be written as $\top, \bot$, instead of $\mathbf{1}, \mathbf{0}$, respectively. Note that logical constants $\mathbf{a} \in \mathrm{LC}(J)$ are atomic formulas.

The set of all the well-formed formulas for the language J is denoted by F_J (we will also speak about J-formulas). A couple a/A, where $a \in L$ and A is some J-formula, is called the *evaluated formula*. If the language is J_{L_Q} then the evaluated formulas are only those couples a/A for which $a \in L_{\mathbb{Q}}$. Note that the evaluation a does not belong to the language J.

The notions of free and bound variables, the substitutable term, closed and open formula, are the same as in classical logic. As usual, $A(x_1, \ldots, x_n)$ denotes a formula whose all free variables are among $x_1, \ldots, x_n$. If $t_1, \ldots, t_n$ are terms substitutable in A for $x_1, \ldots, x_n$, respectively then $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ denotes instance of A in which all free occurrences of $x_1, \ldots, x_n$ are replaced by $t_1, \ldots, t_n$, respectively. If $x_1, \ldots, x_n$ are clear from the context then we will write simply $A[t_1, \ldots, t_n]$.

By $\Sigma(x_1, \ldots, x_n)$ we denote a fuzzy set of J-formulas such that each formula $A(x_1, \ldots, x_n)$ has all its free variables among $x_1, \ldots, x_n$. We will often write $\Sigma(\bar{x})$ instead of $\Sigma(x_1, \ldots, x_n)$. If $J = J_{L_Q}$ then all the membership degrees in Σ are supposed to be rational. If $x_1, \ldots, x_n$ are known from the context then we will write Σ instead of $\Sigma(x_1, \ldots, x_n)$. Analogously to notation for substitution of terms into one formula, $\Sigma_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ denotes a fuzzy set of formulas obtained from $\Sigma(x_1, \ldots, x_n)$ by replacing all formulas $A \in \mathrm{Supp}(\Sigma(x_1, \ldots, x_n))$ by the instances $A_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$. We will often simply write $\Sigma[t_1, \ldots, t_n]$.

Remark 1 *Fuzzy sets of formulas will be dealt with in the sequel. Quite often, however, it is useful to take them as sets of evaluated formulas. Hence, if $\Sigma \subsetsim F_J$ is such a fuzzy set then we may alternatively write $A \in \mathrm{Supp}(\Sigma)$, or $a/A \in \Sigma$ and $\Sigma(A)$ for the evaluation a.*

Logical axioms of $\mathrm{Ev_L}$ are the following schemes of evaluated formulas:

- $\mathbf{1}/A \Rightarrow (B \Rightarrow A)$, $\qquad \mathbf{1}/(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$,
- $\mathbf{1}/(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$, $\qquad \mathbf{1}/((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)$,
- $\mathbf{1}/(\mathbf{a} \Rightarrow \mathbf{b}) \Leftrightarrow (\overline{\mathbf{a} \rightarrow \mathbf{b}})$
  where $\overline{\mathbf{a} \rightarrow \mathbf{b}}$ denotes a logical constant for the truth value $a \rightarrow b$ when a and b are given (*bookkeeping axiom*).
- $\mathbf{1}/(\forall x)A \Rightarrow A_x[t]$
  for any substitutable term t,
- $\mathbf{1}/(\forall x)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x)B)$,
  provided that x is not free in A,
- $a/\mathbf{a}$ for all logical constants $\mathbf{a} \in \mathrm{LC}(J)$,
- $\mathbf{0}/A$ if A is a formula different from the formulas considered above.

Inference rules manipulate with evaluated formulas:

$$r_{MP} : \frac{a/A, b/A \Rightarrow B}{(a \otimes b)/B}, \qquad \text{(modus ponens)}$$

$$r_G : \frac{a/A}{a/(\forall x)A}, \qquad \text{(generalization)}$$

$$r_{LC} : \frac{a/A}{(a \rightarrow a)/\mathbf{a} \Rightarrow A}. \qquad \text{(logical constant introduction)}$$

A *fuzzy theory* T is a fuzzy set of formulas $T \subsetneqq_{\sim} F_J$ given by the triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, R \rangle,$$

where $\mathrm{LAx} \subsetneqq_{\sim} F_J$ is the above defined set of evaluated logical axioms, $\mathrm{SAx} \subsetneqq_{\sim} F_J$ is a fuzzy set of formulas (set of evaluated formulas) taken as special axioms, and R is a set of sound inference rules containing the rules r_{MP}, r_G, r_{LC}.

Let T be a fuzzy theory. By $J(T)$ (alternatively $J_L(T)$, $J_{L_Q}(T)$), we denote its language and by $F_{J(T)}$ the set of all the well formed formulas of $J(T)$. An *evaluated proof* w_A of a formula A in a fuzzy theory T is a finite sequence of evaluated formulas which are either axioms, or they are derived using the inference rules. The evaluation of the last formula in the proof w_A is its *value* and we denote it by $\mathrm{Val}_T(w_A)$.

A formula $A \in F_{J(T)}$ is *provable (a theorem) in the fuzzy theory T in the degree a*, in symbols $T \vdash_a A$, if

$$a = \bigvee \{\mathrm{Val}_T(w_A) \mid w_A \text{ is a proof of } A \text{ in } T\}.$$

If there exists a proof w_A such that $\mathrm{Val}_T(w_A) = a$ then we say that A is effectively provable in T in the degree a (note that this may not always be the case).

A fuzzy theory T is *contradictory* (inconsistent) if there are a formula A, the proof w_A of A with the value $\mathrm{Val}_T(w_A)$ and the proof $w_{\neg A}$ of $\neg A$ with the value $\mathrm{Val}_T(w_{\neg A})$ such that

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0.$$

It is *consistent* otherwise. Surprisingly, a contradictory fuzzy theory collapses into a degenerated theory just as in classical logic.

Theorem 1 *A fuzzy theory T is contradictory iff $T \vdash A$ holds for every formula $A \in F_{J(T)}$.*

A fuzzy theory T is *Henkin* if to every formula $A \in F_{J(T)}$ there is a constant $\mathbf{r} \in \mathrm{OC}(J(T))$ such that

$$T \vdash A_x[\mathbf{r}] \Rightarrow (\forall x)A(x). \tag{1}$$

The formula (1) is called *Henkin* and the object constant $\mathbf{r}$ is called *special*, or also *witness constant*.

A fuzzy theory is complete if it is consistent and for every closed formula A,

$$T \vdash_a A$$

implies that there is a set $H_A \subseteq \mathrm{LC}(J(T))$ such that $\bigwedge\{b \mid \mathbf{b} \in H_A\} = a$ and

$$T \vdash A \Rightarrow \mathbf{b}$$

for every $\mathbf{b} \in H_A$. If the language $J(T)$ contains logical constants for all truth values then complete theories can be characterized using the following theorem.

Theorem 2 *A consistent fuzzy theory T in the language $J_L(T)$ is complete iff the following holds for every closed formula $A \in F_{J_L(T)}$:*

$$T \vdash_a A \quad \text{iff} \quad T \vdash A \Rightarrow \mathbf{a}.$$

The following lemma (see [11]) will be used below.

Lemma 1 *Let T be a consistent fuzzy theory and $T \vdash_a A$ as well as $T \vdash_b \neg A$. Then $b \leq \neg a$.*

We will also need the following lemma.

Lemma 2 *Let T be a consistent fuzzy theory, $A \in F_{J(T)}$ and $a \in L$.*

(a) Let $T \vdash \mathbf{a}' \Rightarrow A$ hold for all $\mathbf{a}' \in \mathrm{LC}(J)$ such that $a' \leq a$ and $T \vdash A \Rightarrow \mathbf{a}''$ hold for all $\mathbf{a}'' \in \mathrm{LC}(J)$ such that $a \leq a''$. Then $T \vdash_a A$.

(b) Let $T \vdash \mathbf{a}' \Rightarrow A$ hold for all $\mathbf{a}' \in \mathrm{LC}(J)$ such that $a' \leq a$ and $T \vdash \mathbf{a}'' \Rightarrow \neg A$ hold for all $\mathbf{a}'' \in \mathrm{LC}(J)$ such that $a'' \leq \neg a$. Then $T \vdash_a A$.

Semantics of Ev_{L} is defined by generalization of the classical (Tarskian) semantics of predicate logic: a *model* (structure)[2] for the language J is

$$\mathcal{V} = \langle V, \{P_V \mid P \in \mathrm{Pred}(J)\}, \{f_V \mid f \in \mathrm{Func}(J)\}, \{u \mid \mathbf{u} \in \mathrm{OC}(J)\}\rangle,$$

where V is a set, each $P_V \underset{\sim}{\subset} V^n$ is an n-ary fuzzy relation assigned to the n-ary predicate symbol $P \in \mathrm{Pred}(J)$ (n depends on P), each f_V is an ordinary n-ary function on V assigned to the n-ary functional symbol $f \in \mathrm{Func}(J)$, and each $u \in V$ is a designated element assigned to the object constant $\mathbf{u} \in \mathrm{OC}(J)$.

The connectives $\boldsymbol{\neg}, \boldsymbol{\wedge}, \boldsymbol{\vee}, \boldsymbol{\&}, \boldsymbol{\nabla}, \boldsymbol{\Rightarrow}, \boldsymbol{\Leftrightarrow}$ are interpreted by $\neg, \wedge, \vee, \otimes, \oplus, \rightarrow, \leftrightarrow$, respectively. Terms are interpreted in the same way as in classical logic. If t is a term interpreted by an element $v \in V$ then we will write

$$\mathcal{V}(t) = v.$$

Logical constants are interpreted by the truth value they represent, i.e.

$$\mathcal{V}(\mathbf{a}) = a, \qquad a \in L$$

for all $\mathbf{a} \in \mathrm{LC}(J)$.

Let X_J be a set of all variables of the language J and $\mathcal{V}$ be a model for J. A $\mathcal{V}$-*evaluation of object variables* is a mapping $e : X_J \longrightarrow V$. Let $A(x_1, \ldots, x_n) \in F_J$ and $e(x_1) = v_1, \ldots, e(x_n) = v_n$.

The degree of satisfaction of A under the evaluation e is a truth value $a \in L$ obtained after assigning the elements $v_1, \ldots, v_n$ to the corresponding free occurrences of variables $x_1, \ldots, x_n$ in A and interpreting all the functional and predicate symbols and connectives in an appropriate way usual in logic (for the details see [4, 11]). We will write

$$\mathcal{V}(A({}^{v_1}/{x_1}, \ldots, {}^{v_n}/{x_n})) = a, \tag{2}$$

or simply $\mathcal{V}(A)(v_1, \ldots, v_n) = a$. If the variables $x_1, \ldots, x_n$ are not present or unimportant for the explanation then we will simply write $\mathcal{V}(A)$. To simplify the notation, we will often write $\bar{v} \in V^n$ for the n-tuple of elements $v_1, \ldots, v_n \in V$ and $A({}^{\bar{v}}/{\bar{x}})$ instead of $A({}^{v_1}/{x_1}, \ldots, {}^{v_n}/{x_n})$, or simply $A(\bar{v})$. Similarly when dealing with the fuzzy set $\Sigma(x_1, \ldots, x_n)$, we will write $\Sigma({}^{v_1}/{x_1}, \ldots, {}^{v_n}/{x_n})$, or $\Sigma(v_1, \ldots, v_n)$, or only $\Sigma(\bar{v})$.

Equivalent, more pedantic way how interpretation of formulas can be defined is to expand J by new object constants $\mathbf{v}$ being names for all the elements $v \in V$ and then put

$$\mathcal{V}(A_{x_1,\ldots,x_n}[\mathbf{v_1}, \ldots, \mathbf{v_n}]) = a$$

where $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are names of the elements $v_1, \ldots, v_n \in V$, respectively. In this paper, however, we will prefer (2).

Let us recall that the general quantifier is interpreted by

$$\mathcal{V}((\forall x)A(x)) = \bigwedge\{A({}^{v}/{x}) \mid e(x) = v, e \text{ is a } \mathcal{V}\text{-evaluation}\}.$$

The same definition is introduced also for a formula $A(x)$ with a free variable x.

Let T be a fuzzy theory and $\mathcal{V}$ be a model (structure) for $J(T)$. We say that $\mathcal{V}$ is a *model* of T, $\mathcal{V} \models T$, if $\mathrm{SAx}(A) \leq \mathcal{V}(A)$ holds for all formulas $A \in \mathrm{Supp}(\mathrm{SAx})$. A formula $A \in F_{J(T)}$ is *true* in T in the degree a, $T \models_a A$, if

$$a = \bigwedge\{\mathcal{V}(A) \mid \mathcal{V} \models T\}.$$

In general, let $\Sigma \underset{\sim}{\subset} F_J$ be a fuzzy set of formulas. By $\mathcal{V} \models \Sigma$ we understand a model (structure) $\mathcal{V}$ for the language J such that

$$\mathcal{V}(A) \geq \Sigma(A)$$

holds for all $A \in F_J$.

Theorem 3 (Completeness)

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_J$.

The proof can be found in [10, 11].

[2] We will follow the way common in model theory and do not distinguish between the terms "structure" and "model" for the language J.

3 Omitting types theory in $\mathbf{Ev}_{\text{Ł}}$

3.1 Basic definitions

Let J be a predicate language, $\Sigma(\bar{x})$ be a fuzzy set of J-formulas and let $\mathcal{V}$ be a model for J. We say that $\Sigma(\bar{x})$ is *realized* in $\mathcal{V}$ if there is $\bar{v} \in V^n$ such that

$$\mathcal{V}(A(\bar{v})) \geq \Sigma(A)$$

(or more precisely $\mathcal{V}(A(\bar{v}/\bar{x})) \geq \Sigma(A))$ holds for all $A \in \mathrm{Supp}(\Sigma(\bar{x}))$. Equivalently, we will also say that $\mathcal{V}$ is a $\bar{v}$-model of $\Sigma(\bar{x})$ and write $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$ (or briefly, $\mathcal{V} \models \Sigma(\bar{v})$).

We say that $\Sigma(\bar{x})$ is *omitted* by $\mathcal{V}$ if to each $\bar{v} \in V^n$ there is a formula $A \in \mathrm{Supp}(\Sigma)$ such that

$$\mathcal{V}(A(\bar{v})) < \Sigma(A). \tag{3}$$

The following is obvious.

Lemma 3 *Let $\mathcal{V}$ be a model for J and Σ be a fuzzy set of J-formulas realized in $\mathcal{V}$. Then there is $\bar{v} \in V^n$ such that $\mathcal{V}$ is a $\bar{v}$-model of Σ.*

Let T be a fuzzy theory. A fuzzy set $\Sigma \underset{\sim}{\subseteq} F_{J(T)}$ is *T-consistent* if there is $\mathcal{V} \models T$ and $\bar{v} \in V^n$ such that $\mathcal{V}$ is a $\bar{v}$-model of Σ. More precisely, we may say that Σ is T-consistent on $\bar{v} \in V^n$ and write $\mathcal{V} \models T \cup \Sigma(\bar{v})$.

The following lemma is again obvious.

Lemma 4 *Let J be a predicate language and T a fuzzy theory in J. Let $\mathcal{V} \models T$ and $\Sigma \underset{\sim}{\subseteq} F_J$ be a fuzzy set of formulas. If Σ is realized in $\mathcal{V}$ then Σ is T-consistent.*

We say that a fuzzy set $\Gamma(\bar{x}) \underset{\sim}{\subseteq} F_{J(T)}$ is a *fuzzy* n-type over fuzzy theory T if it is *maximal T-consistent* fuzzy set of formulas where maximality means that no membership degree $\Gamma(A)$, $A \in F_{J(T)}$, can be increased without harming the consistency of Γ.

3.2 Isolation and non-isolation of a fuzzy set of formulas

The concept of isolation of a set of formulas Σ in classical model theory enables to formulate necessary and sufficient condition on T to have a model which omits Σ. Note that in the literature, we can also find the term "local realization" instead of isolation (cf. [2, 3]).

In this paper, we will generalize this concept to fuzzy set of formulas. Our definition, moreover, stems from a slight generalization introduced, e.g. in [2] which considers a set of formulas instead of only one with respect to which is isolation considered. The reason is that we have to deal with fuzzy sets of formulas taken at the same time as sets of evaluated formulas and cannot assure their finiteness.

Definition 1 *Let J be a predicate language, T be a fuzzy theory in J and $\Sigma \in F_J$ be a fuzzy set of formulas. We say that Σ is* isolated *in T if there is a fuzzy set of formulas $\Phi(\bar{x}) \underset{\sim}{\subseteq} F_J$ such that*

(i) $\Phi(\bar{x})$ is T-consistent,

(ii) for every $\bar{v} \in V^n$ and a model $\mathcal{V}$ for the language J, if $\mathcal{V} \models T \cup \Phi(\bar{v})$ then $\mathcal{V} \models \Sigma(\bar{v})$.

We say that Σ is non-isolated *in T otherwise.*

(in some classical literature, e.g. [2], non-isolated set is also called "locally omitting").

Note that Σ is non isolated in T if for every fuzzy set $\Phi(\bar{x}) \underset{\sim}{\subseteq} F_{J(T)}$ and every model $\mathcal{V} \models T$ such that $\mathcal{V} \models T \cup \Phi(\bar{v})$ (i.e. Φ is T-consistent on $\bar{v} \in V^n$) there is $B \in \mathrm{Supp}(\Sigma)$ such that

$$\mathcal{V}(\neg B(\bar{v})) > \neg\Sigma(B). \tag{4}$$

From the definition of isolation of Σ in T we immediately get the following lemma.

Lemma 5 *If Σ is isolated in the fuzzy theory T then Σ is T-consistent.*

3.3 Omitting types theorem in $\mathrm{Ev_Ł}$with countable number of logical constants

The main theorem of this paper is formulated and proved in this subsection. It generalizes the Omitting types theorem of classical model theory.

Theorem 4 *Let T be a consistent fuzzy theory in a language J such that* $\mathrm{LC}(J)$ *is at most countable. Let $\Sigma(\bar{x}) \underset{\sim}{\subseteq} F_J$ be a fuzzy set of formulas non-isolated in T. Then there exists a countable model $\mathcal{V} \models T$ which omits $\Sigma(\bar{x})$.*

Proof. Without lack of generality, we will consider a fuzzy set $\Sigma(x)$ with one free variable only. Let $K = \{\mathbf{c}_0, \mathbf{c}_1, \dots\} \not\subset J$ be a countable set of new constants and put $J_K = J \cup K$. Let $A_0, A_1, \dots, A_m, \dots$ be a sequence of closed formulas of the language J_K. By induction we will construct an increasing sequence of consistent fuzzy theories $T = T_0 \subset T_1 \subset \cdots \subset T_m \subset \cdots$ in the language J_K such that for each m, the following is fulfilled:

(i) T_m is consistent and it is obtained from $T = T_0$ by extending it by a fuzzy set of special axioms which contain finite number of new variables (free or bound).

(ii) If $A_m \in F_{J(T_m)}$ and $T_m \vdash_a A_m$ for some $a \in L$ then $T_{m+1} \vdash A_m \Rightarrow \mathbf{a}'$ holds for all $\mathbf{a}' \in \mathrm{LC}(J)$ such that $a \le a'$.

(iii) If $A_m := (\forall y)C(y)$ then $T_{m+1} \vdash C[\mathbf{c}_p] \Rightarrow (\forall y)C(y)$ where $\mathbf{c}_p \in K$ and $\mathbf{c}_p \notin J(T_m)$.

(iv) There exists a formula $D \in \mathrm{Supp}(\Sigma(x))$ such that $T_{m+1} \vdash_e \neg D[\mathbf{c}_m]$ and $\neg e < \Sigma(D)$ for some $\mathbf{c}_m \in K$.

Assume that we already have the theory T_m being a consistent extension of T and let Φ' be a fuzzy set of new special axioms of T_m. We construct T_{m+1} as follows: Let $\mathbf{c}, \mathbf{c}_1, \dots, \mathbf{c}_n \in K$ be all the new constants occurring in all $B \in \mathrm{Supp}(\Phi')$ (our construction below assures a finite number of them). We replace these constants by new variables $x, y_1, \dots, y_n$, respectively and obtain new evaluated formulas ${}^{b}\!/B'$ for all ${}^{b}\!/B \in \Phi'$. Now we form a fuzzy set

$$\Phi(x) = \left\{ {}^{b}\!/(\exists y_1)\cdots(\exists y_n)B' \,\middle|\, {}^{b}\!/B \in \Phi' \right\}.$$

Because T_m is consistent, there is $\mathcal{V} \models T_m$ such that

$$\mathcal{V}((\exists y_1)\cdots(\exists y_n)B') \ge \mathcal{V}(B) \ge b$$

holds for all ${}^{b}\!/B \in \Phi'$. Consequently, $\Phi(x)$ is T_m-consistent and also T-consistent.

Since $\Sigma(x)$ is non-isolated in T, for every model $\mathcal{V} \models T$ there is a formula $D(x) \in \mathrm{Supp}(\Sigma)$ such that

$$\mathcal{V}(\neg D(v)) > \neg\Sigma(D)$$

(cf. (4)) where $v \in V$ is an element such that $\mathcal{V} \models T \cup \Phi(v)$ (i.e. $\Phi(x)$ is T-consistent on it). Let us choose a constant $\mathbf{c}_m \in K$ and define its interpretation by

$$\mathcal{V}(\mathbf{c}_m) = v \quad \text{iff} \quad \mathcal{V} \models T \cup \Phi(v)$$

where $v \in V$ is the corresponding element in every model $\mathcal{V}$ such that $\mathcal{V} \models T \cup \Phi(v)$. Put

$$e' = \bigwedge\{\mathcal{V}(\neg D[\mathbf{c}_m] \mid \mathcal{V} \models T \cup \Phi[\mathbf{c}_m]\}$$

and take $e \in L$ such that $\mathcal{V}'(\neg D(v)) \ge e > e'$ for some model $\mathcal{V}'$ considered above (if there is no such $\mathcal{V}'$ then we take $e = e'$). Note that $e > \neg\Sigma(D)$. Now we can put

$$T'_{m+1} = T_m \cup \left\{ {}^{e}\!/\neg D[\mathbf{c}_m] \right\}.$$

Since T_m is consistent (by the assumption), our construction of e ensures that T'_{m+1} is consistent as well (it has the model $\mathcal{V}'$, at least) and, moreover, item (iv) is also guaranteed.

Let $A_m \in F_{J(T_m)}$ and $T'_{m+1} \vdash_a A_m$. Then we construct a fuzzy theory

$$T''_{m+1} = T'_{m+1} \cup \{\mathbf{1}/A_m \Rightarrow \mathbf{a}'' \mid \mathbf{a}'' \in \mathrm{LC}(J), a \leq a''\}.$$

Using properties of $\mathrm{Ev_L}$ (see [11], pp. 124–142) we can prove that the theory T''_{m+1} is a consistent extension of T'_{m+1} and $T'_{m+1} \vdash \mathbf{a}' \Rightarrow A_m$ for all $a' \leq a$. Then by Lemma 2(a), $T''_{m+1} \vdash_a A_m$ and $T''_{m+1} \vdash A_m \Rightarrow \mathbf{a}''$ holds for all $\mathbf{a}'' \in \mathrm{LC}(J)$ such that $a \leq a''$. This makes item (ii). Clearly, $\bigwedge\{a'' \mid a \leq a''\} = a$.

If $A_m := (\forall y)C(y)$ is T''_{m+1}-consistent then we put

$$T_{m+1} = T''_{m+1} \cup \{\mathbf{1}/C[\mathbf{c}_p] \Rightarrow (\forall y)C(y)\}$$

where $\mathbf{c}_p$ is the first constant from K that has not yet been used. Analogously as in [11], Theorem 4.16, we can show that T_{m+1} is a consistent extension of T''_{m+1}. This assures item (iii). Let us remark that we have added to special axioms of T_m evaluated formulas with only finite number of new constants despite the possibly infinite number of the former.

Let us now put $T^+ = \bigcup_{m\in\mathbb{N}} T_m$. Then according to items (i),(ii), (iii), T^+ is consistent complete Henkin theory with the language J_K. Let $\mathcal{W}^+$ be a countable model of T^+ and let $\mathcal{V}^+$ be a submodel of $\mathcal{W}^+$ generated by the constants. We can show by induction on the complexity of a formula $A \in F_{J_K(T^+)}$ that

$$\mathcal{V}^+(A) = \mathcal{W}^+(A).$$

Thus, $\mathcal{V}^+$ is a countable model of T^+ and since T^+ is extension of T, we have

$$\mathcal{V}^+(\neg D([\mathbf{c}_m]) \geq e > \neg\Sigma(D)$$

which gives

$$\mathcal{V}^+(D[\mathbf{c}_m]) < \Sigma(D). \tag{5}$$

By restriction of $\mathcal{V}^+$ to J we obtain a model $\mathcal{V}^0 \models T$ which omits $\Sigma(x)$ because to every element $v_0 \in V^0$ which realizes the constant $\mathbf{c}_p$ we can find $B \in \mathrm{Supp}(\Sigma(x))$ which fulfills the omitting condition (3). □

The omitting types theorem can be also generalized to countably many fuzzy sets of J-formulas.

Theorem 5 (Extended Omitting Types Theorem) *Let T be a consistent fuzzy theory in a countable language J. Let for each $q < \omega$, $\Sigma_q(x_1, \ldots, x_{n_q})$ be a fuzzy set of J-formulas. If each Σ_q is non-isolated in T then there exists a countable model $\mathcal{V}$ of T which omits each Σ_q.*

3.4 Omitting types theorem in $\mathrm{Ev_L}$with all logical constants

When considering $\mathrm{Ev_L}$ with the full language J_L and $L = [0,1]$ (our considerations would hardly have sense for finite L), we face the problem that the set F_{J_L} of all formulas is uncountable. Therefore, in the proof of omitting types theorem we cannot simply consider a countable sequence of formulas to construct the complete Henkin fuzzy theory as is done in the proof of Theorem 4. A clue for solution of this problem comes from the fact that logical constants are closed formulas without variables. Therefore, we can gather formulas differing only in logical constants into sets so that we will have again only countable number of these sets.

We say that a formula $A \in F_{J_L}$ is *pure* if it is does not contain logical constants (as subformulas). It is *general* otherwise. Clearly, that there is only countable number of pure formulas.

For each pure formula A we can construct a set $Q(A)$ of all formulas obtained from A by adding logical constants to it as its new subformulas.

Definition 2 *Let A be a pure formula. Then a set $Q(A)$ is obtained by iterated application of the following items:*

(i) $A \in Q(A)$.

(ii) Let $B \in Q(A)$, C be a subformula of B and $\mathfrak{a}$ be a logical constant. Let $\bar{B}$ be a formula obtained from B when replacing C by $(\mathfrak{a} \Rightarrow C)$. Then $\bar{B} \in Q(A)$.

(iii) Let $B, C, \mathfrak{a}$ be as in item (ii) and $\bar{B}$ be a formula obtained from B when replacing C by $(C \Rightarrow \mathfrak{a})$. Then $\bar{B} \in Q(A)$.

For example, let $A(y) = (\forall x)(P(x) \Rightarrow Q(y))$ be a pure formula. Then, e.g. a formula $(\forall x)((P(x) \Rightarrow \mathfrak{a}) \Rightarrow (\mathfrak{b} \Rightarrow Q(y)) \Rightarrow \mathfrak{c})$ belongs to $Q(A(y))$.

Of course, each $Q(A)$ is an uncountable set of formulas. However, all $B \in Q(A)$ have the same variables (bound or free). This is ground for proving the omitting-types theorem in a way analogous to the proof of Theorem 4.

We will also need to consider a set $Q^{\mathbb{Q}}(A)$ which is $Q(A)$ when confining only to rational logical constants from $\mathrm{LC}(J_{L_Q})$. Clearly, $Q^{\mathbb{Q}}(A)$ is countable.

Lemma 6 *Let T be a fuzzy theory in the language J. Then to every general formula $A \in F_J$ with irrational logical constants there are general formulas $A_1, A_2 \in F_J$ obtained from A by replacing irrational logical constants by rational ones such that*

$$T \vdash A_1 \Rightarrow A \quad \textit{and} \quad T \vdash A \Rightarrow A_2.$$

Proof. Note that if $A := B \Rightarrow \mathfrak{a}$ where a is irrational then we can put $A_1 := B \Rightarrow \mathfrak{a}_1$ and $A_2 := B \Rightarrow \mathfrak{a}_2$ where $a_1 \leq a_2$ and a_1, a_2 are rational. The proposition then follows from the properties of implication (see [11], p. 123). Similarly for $A := \mathfrak{a} \Rightarrow B$. The rest follows immediately by induction on the complexity of the formula A. □

Now we are ready to prove the theorem on omitting types in $\mathrm{Ev_L}$ with all logical constants.

Theorem 6 *Let T be a consistent fuzzy theory in a language J_L. Let $\Sigma(\bar{x}) \underset{\sim}{\subset} F_{J_L}$ be a fuzzy set of formulas non-isolated in T. Then there exists a model $\mathcal{V} \models T$ which omits $\Sigma(\bar{x})$.*

Proof. We will proceed similarly as in the proof of Theorem 4 but modify some of its points as follows.

Let $A_0, A_1, \ldots, A_m, \ldots$ be a countable sequence of a pure formulas from the language $J_{L,K}$ and $Q(A_0), Q(A_1), \ldots, Q(A_m), \ldots$ be a sequence of sets due to Definition 2. We will now rearrange the first sequence as follows: for each i, if A_i has the form $A_i := (\forall x)B(x)$ then we replace A_i by a sequence of all formulas from $Q^{\mathbb{Q}}(A_i)$. For simplicity, the resulting sequence of formulas will again be denoted by $A_0, A_1, \ldots, A_m, \ldots$.

Now, we will again construct an increasing sequence of consistent fuzzy theories $T = T_0 \subset T_1 \subset \cdots \subset T_m \subset \cdots$ in the language $J_{L,K}$ such that for each m the following items are fulfilled:

(i) T_m is consistent and it is obtained from $T = T_0$ by extending it by a fuzzy set of special axioms which contain finite number of new variables (free or bound).

(ii) If $\bar{A}_m \in Q(A_m) \subset F_{J(T_m)}$ and $T_m \vdash_a \bar{A}_m$ for some $a \in L$ then $T_{m+1} \vdash_a \bar{A}_m$ and $T_{m+1} \vdash \bar{A}_m \Rightarrow \mathfrak{a}$.

(iii) If $\bar{A}_m \in Q(A_m) \subset F_{J(T_m)}$ and $\bar{A}_m := (\forall y)C(y)$ then $T_{m+1} \vdash C[\mathbf{c}_p] \Rightarrow (\forall y)C(y)$ where $\mathbf{c}_p \notin J(T_m)$.

(iv) There exists a formula $D \in \mathrm{Supp}(\Sigma(x))$ such that $T_{m+1} \vdash_e \neg D[\mathbf{c}_m]$ and $\neg e < \Sigma(D)$.

Assume that we already have the theory T_m being a consistent extension of T and let Φ' be a fuzzy set of new special axioms of T_m. We will construct T_{m+1} in three steps as in Theorem 4.

(a) item (iv) is analogous as in the proof of Theorem 4.

(b) Let $Q(A_m) \subset F_{J(T_m)}$ and $T'_{m+1} \vdash_a \overline{A_m}$ for $\overline{A_m} \in Q(A_m)$. Then we construct a fuzzy theory

$$T''_{m+1} = T'_{m+1} \cup \{\mathbf{1}/A_m \Rightarrow \mathfrak{a} \mid \overline{A_m} \in Q(A_m)\}.$$

By properties of $\mathrm{Ev_L}$, the theory T''_{m+1} is a consistent extension of T'_{m+1}, $T''_{m+1} \vdash_a A_m$ and $T''_{m+1} \vdash A_m \Rightarrow \mathfrak{a}$. This makes item (ii). Moreover, notice that we have used at most finite number of object constants because all $\overline{A_m} \in Q(A_m)$ have the same variables and object constants.

(c) Let $A_m \in Q^{\mathbb{Q}}((\forall y)C(y))$. If A_m is T''_{m+1}-consistent then we put

$$T_{m+1} = T''_{m+1} \cup \{\mathbf{1}/C[\mathbf{c}_p] \Rightarrow (\forall y)C(y)\}$$

where $\mathbf{c}_p$ is the first constant from K. Analogously as in [11], Theorem 4.16, we can show that T_{m+1} is a consistent extension of T''_{m+1}.

Now put $T^+ = \bigcup_{m\in\mathbb{N}} T_m$. First, we must check item (iii). Let $C(x) \Rightarrow (\forall y)C(y)$ contain irrational logical constants. By Lemma 6, there is a formula of the form $C_1(x) \Rightarrow (\forall y)C_1(y)$ containing rational logical constants such that

$$T^+ \vdash (C_1(x) \Rightarrow (\forall y)C_1(y)) \Rightarrow (C(x) \Rightarrow (\forall y)C(y)).$$

When replacing x by $\mathbf{c}_p$, we obtain $T^+ \vdash C[\mathbf{c}_p] \Rightarrow (\forall y)C(y)$ because the Henkin formula $C_1[\mathbf{c}_p] \Rightarrow (\forall y)C_1(y)$ has been added in step (c). Therefore, item (iii) is also assured. Consequently, according to items (i),(ii), (iii), T^+ is a consistent complete Henkin theory with the language $J_{L,K}$. The rest is the same as in the proof of Theorem 4. □

4 Conclusion

This paper is a continuation of the development of model theory of fuzzy logic with evaluated syntax that belongs among fuzzy logics in narrow sense. We focused on the theory of omitting types. The latter are special fuzzy sets of formulas $\Sigma(\bar{x})$ that can be either realized by some model of a given fuzzy theory T, or omitted, i.e. no model of T can be at the same time a model of $\Sigma(\bar{x})$. In classical logic, this enables (besides others) to construct non-standard models of the theory of natural numbers. Whether such construction is possible also in Ev_{L} and how it should be formulated is an open question for further research.

The main result of this paper is a non-trivial generalization of three omitting types theorems of classical model theory. Let us mention that we have proved in [7] analogous omitting types theorem also for BL-fuzzy logic (the fuzzy logic with traditional syntax introduced by P. Hájek in [4]).

To conclude, let us remark that our definitions are not graded, i.e. we speak about fuzzy set of formulas that is omitted without specifying a *degree* of omitting. It is questionable whether such a generalization is possible and whether it may have reasonable substantiation. This problem is left to some of the subsequent papers.

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