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Numerical Solution of Partial Differential Equations with Help of Fuzzy Transform

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Abstract

The paper is devoted to a fuzzy approach to numerical solutions of partial differential equations. Three main types of partial differential equations have been considered to demonstrate the algorithms with help of the fuzzy transform. We have introduced an example of a reasonable application of the fuzzy transform in this area. The justification of our approach including the convergence theorem has been presented as well.

1 Introduction

Differential equations are used for modeling various physical phenomena. Unfortunately, many problems are dynamical and too complicated and developing an accurate differential equation model for such problems requires complex and time consuming algorithms hardly implementable in practice. Thus, a usage of fuzzy mathematics seems to be appropriate.

A lot of work has been done in the field of fuzzy differential equations see e.g. [2], [3] or [12]. On the other hand, a fuzzy approach to numerical solution of differential equations has not been investigated so deeply yet. Some results have been published e.g. in [7].

Fuzzy transform have been introduced as an approximation method but and it has been successfully applied into other mathematical problems as well. It has been shown that if the original function is replaced by an approximation model, then a certain simplification of complex computations could be achieved.

In [4] I. Perfilieva introduced an application to the Cauchy problem. The main idea consists in applying the fuzzy transform to both sides of the first ordered ordinary differential equation. This turns the given differential equation to an algebraic one which is easily solvable by existing numerical methods. Then the obtained numerical solution is transformed returned back to the space of continuous functions. The application of fuzzy transform to ordinary differential equations is published in [4, 5] as well as other numerical methods using this technique.

2 Preliminaries

We recall fundamental facts about the technique of fuzzy transform (F-transform). The sequential application of the F-transform and its inversion to an original function provides its simplified approximate representation given by a linear combination of basis functions see [4], [5], [9] or [11] for details.

Definition 1 Let $x_i = a + h(i-1)$ be nodes on $[a, b]$ where $h = (b-a)/(n-1),\ n \geq 2$ and $i = 1, \dots, n$. We say that functions $A_1(x), \dots, A_n(x)$ defined on $[a, b]$ are *basis functions* if each of them fulfills the following conditions:

- $A_i : [a, b] \to [0, 1],\ A_i(x_i) = 1,$
- $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$ where $x_0 = a,\ x_{n+1} = b,$
- $A_i(x)$ is continuous,
- $A_i(x)$ strictly increases on $[x_{i-1}, x_i]$ and strictly decreases on $[x_i, x_{i+1}]$,
- $\sum_{i=1}^{n} A_i(x) = 1$, for all $x \in [a, b]$,
- $A_i(x_i - x) = A_i(x_i + x)$, for all $x \in [0, h],\ i = 2, \dots, n-1,\ n > 2,$
- $A_{i+1}(x) = A_i(x - h)$, for all $x \in [a + h, b],\ i = 2, \dots, n-2,\ n > 2.$

We say that basis functions determine a uniform fuzzy partition of interval $[a, b]$.

Definition 2 Let $f(x)$ be a continuous function on $[a, b]$ and $A_1(x), \dots, A_n(x)$ be basis functions determining a uniform fuzzy partition of $[a, b]$. The n-tuple of real numbers $[F_1, \dots, F_n]$ such that

$$F_i = \frac{\int_a^b f(x) A_i(x) dx}{\int_a^b A_i(x) dx}, \quad i = 1, \dots, n, \tag{1}$$

will be called the F-transform of f w.r.t. the given basis functions.

Reals F_i are called *components* of the F-transform and they can be viewed as an aggregated representation of the function f. Moreover, they will be used in a construction of a simplified continuous approximate representation of f.

Definition 3 Let $A_1, \ldots, A_n$ be basis functions and let $F_n[f] = [F_1, \ldots, F_n]$ be the F-transform of f w.r.t. $A_1, \ldots, A_n$. The function

$$f_n^F(x) = \sum_{i=1}^{n} A_i(x) F_i \tag{2}$$

will be called the inverse F-transform.

The basis properties of F-transform have been already discussed in [4], [5], and many applications e.g. in [4], [9], [11], [10], [6] have been suggested. That is why we avoid repeating it and we briefly recall a definition of F-transform for functions with more variables presented in [11]. Here, because of a lack of space, we present this definition just for functions of two variables.

Definition 4 Let $f(x, y)$ be an arbitrary continuous function on $\mathcal{D} = [a, b] \times [c, d]$ and let basis functions $A_1(x), ..., A_n(x)$ on $[a, b]$ and $B_1(y), ..., B_m(y)$ on $[c, d]$ form uniform fuzzy partitions, not necessarily the same. We say that a matrix $[F_{ij}]_{n\times m}$ of real numbers is the *F-transform* of $f(x, y)$ w.r.t. the given basis functions if

$$F_{ij} = \frac{\int_c^d \int_a^b f(x, y) A_i(x) B_j(y) dx dy}{\int_c^d \int_a^b A_i(x) B_j(y) dx dy}. \tag{3}$$

Analogously to the 1-dimensional case we define the inverse F-transform.

Definition 5 Let $[F_{ij}]_{n\times m}$ be the F-transform of a function f. Then the function

$$f_{n,m}^F(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} F_{ij} A_i(x) B_j(y) \tag{4}$$

will be called the *inverse F-transform.*

3 Partial Differential Equations

In this section we will see how F-transform can be used in a numerical solution of partial differential equations. We consider three main types of partial differential equations with physical backgrounds - heat equation, wave equation and Poisson's equation. Without any loss of generality we consider the 2-dimensional cases of these equations.

In general, each mentioned partial differential equation on a domain $\mathcal{D} = X \times Y$ can be written in the following form:

$$\mathbf{L}\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = q(x, y) \tag{5}$$

where L is a linear form. Moreover, in order to have a unique solution, some specific initial and boundary conditions are given.

Remark: The first two types of partial differential equations are sometimes called evolutionary equations where the second variable is supposed to be a time variable. That is why in the next two subsections we will use a domain $\mathcal{D} = X \times T$ and all the symbols y will be replaced by t.

The method of F-transform consists in applying the F-transform (w.r.t. to some given basis functions) to both sides of equation (5). This leads to an algebraic equation which is going to be solved. The solution of this algebraic equation is a discrete representation of an analytical solution of equation (5) and by the inverse F-transform, this will be returned back to the space of continuous functions $C(\mathcal{D})$.

Before we start to analyze concrete types of equations, let us fix some basis functions $A_i(x), B_j(y)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, with equidistant step h_x on X and h_y on Y where $x \in X$, $y \in Y$. All the F-transforms considered in the next subsections are meant w.r.t. these basis functions.

3.1 Heat Equation

Let the domain $\mathcal{D}$ be a Cartesian product of two real intervals $X = [0,1]$ and $T = [0,R]$. Let $u(x,t)$ be a continuous solution of the following parabolic equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t), \quad \alpha \in \mathbb{R}^+ \tag{6}$$

with the following initial and boundary conditions

$$u(x,0) = f(x), \tag{7}$$

$$u(0,t) = T_1(t), \quad u(1,t) = T_2(t). \tag{8}$$

Equation (6) is turned to the following algebraic equation

$$U_{ij}^t - \alpha U_{ij}^{xx} = Q_{ij} \tag{9}$$

where Q_{ij}, U_{ij}^t and U_{ij}^{xx} are the F-transform components of the functions q, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, respectively w.r.t. the given basis functions. Partial derivatives on the left hand side of equation (6) are supposed to be replaced by their finite differences (see [8]), before we apply the F-transform, which means

$$\frac{\partial u}{\partial t} \quad \text{by} \quad \frac{(u(x,t+h_t) - u(x,t))}{h_t} \quad \text{and similarly}$$

$$\frac{\partial^2 u}{\partial x^2} \quad \text{by} \quad \frac{(u(x+h_x,t) - 2u(x,t) + u(x-h_x,t))}{h_x^2}.$$

Then we can estimate U_{ij}^t as follows:

$$\begin{aligned}
U_{ij}^t &= \frac{\int_c^d \int_a^b \frac{\partial u(x,t)}{\partial t} A_i(x) B_j(t) dx dt}{\int_c^d \int_a^b A_i(x) B_j(t) dx dt} \\
&\approx \frac{\int_c^d \int_a^b \frac{(u(x,t+h_t) - u(x,t))}{h_t} A_i(x) B_j(t) dx dt}{\int_c^d \int_a^b A_i(x) B_j(t) dx dt} \\
&= \frac{1}{h_t} \frac{\int_c^d \int_a^b u(x,t+h_t) A_i(x) B_j(t) dx dt}{\int_c^d \int_a^b A_i(x) B_j(t) dx dt} \\
&+ \frac{1}{h_t} \frac{\int_c^d \int_a^b u(x,t) A_i(x) B_j(t) dx dt}{\int_c^d \int_a^b A_i(x) B_j(t) dx dt} \\
&= \frac{1}{h_t}(U_{i(j+1)} - U_{ij}).
\end{aligned} \tag{10}$$

Similarly we obtain

$$U_{ij}^{xx} \approx \frac{1}{h_x^2}(U_{(i-1)j} - 2U_{ij} + U_{(i+1)j}). \tag{11}$$

By (10) and (11), we obtain the following recursive equation

$$U_{i(j+1)} = rU_{(i-1)j} + (1-2r)U_{ij} + rU_{(i+1)j} + h_t Q_{ij} \tag{12}$$

where $r = (\alpha h_t)/h_x^2$ and $i = 2, \dots, n-1$, $j = 1, \dots, m-1$.

Remark: In order to have a stability of the method, we must keep the following inequality: $0 < r \leq 1/2$ (see [8]).

We will use the initial condition (7) to set up the starting values

$$U_{i1} = f((i-1)h_x), \quad i = 1, \dots, n.$$

Boundary conditions (8) will be also used for U_{ij} as follows:

$$U_{1j} = T_1((j-1)h_t), \ U_{nj} = T_2((j-1)h_t), \ j = 1, \dots, m.$$

Now, the computational process described recursively by equation (12), can be started. It produces a matrix $[U_{ij}]n \times m$, which gives us a discrete representation of the solution $u(x,t)$ of equation (6) and which is going to be turned back to the space $C(\mathcal{D})$ by the inverse F-transform.

3.2 Wave equation

Let the domain $\mathcal{D}$ be given by a Cartesian product of two real intervals $X = [0,1]$ and $T = [0, R]$. Let $u(x,t)$ be a continuous solution of the following hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^2 u}{\partial t^2} = q(x,t) \quad \alpha \in \mathbb{R}^+ \tag{13}$$

with the following initial and boundary conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \tag{14}$$

$$u(0,t) = T_1(t), \quad u(1,t) = T_2(t). \tag{15}$$

Similarly to the heat equation we turn this equation (13) into the following algebraic equation

$$U_{ij}^{xx} - \alpha U_{ij}^{tt} = Q_{ij} \tag{16}$$

where Q_{ij}, U_{ij}^{tt} and U_{ij}^{xx} are the F-transform components of the functions q, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$, respectively w.r.t. the given basis functions.

Similarly to (11) we obtain

$$U_{ij}^{tt} \approx \frac{1}{h_t^2}(U_{i(j-1)} - 2U_{ij} + U_{i(j+1)}). \tag{17}$$

This leads to a recursive equation expressed as follows:

$$\begin{aligned} U_{i(j+1)} = {} & r^2 U_{(i-1)j} + 2(1-r^2)U_{ij} + \\ & + r^2 U_{(i+1)j} - U_{i(j-1)} - h_t^2 Q_{ij} \end{aligned} \tag{18}$$

where $r = (\alpha h_t)/h_x$ and $i = 1, \dots n$, $j = 1, \dots, m$.

Remark: To keep the stability of the algorithm, we must put $0 < r \leq 1$ (see [8]).

Analogously to the heat equation, the initial and boundary conditions must be taken into account. From the boundary conditions we obtain $U_{1j} = T_1((j-1)h_t)$ and $U_{nj} = T_2((j-1)h_t)$.

From the first of initial conditions (14) we get $U_{i1} = f((i-1)h_x)$ and the last unknown expressions which occur in algorithm (18) are U_{i0} for $i = 2, \dots, n-1$.

Here, we use the following finite difference

$$\frac{\partial u}{\partial t} \approx \frac{(u(x, t+h_t) - u(x, t-h_t))}{2h_t}.$$

If we apply to the previous approximation of partial derivative to the second initial condition (14) we get

$$g((i-1)h_x) = \frac{U_{i2} - U_{i0}}{2h_t}$$

which can be rewritten

$$U_{i0} = -2h_t g((i-1)h_x) + U_{i2}.$$

3.3 Poisson's equation

In the case of Poisson's equation (or elliptic equation in general), the finite difference scheme leads to a large set of linear algebraic equations with respect to a complete set of unknowns. There is no step-by-step algorithm analogous to parabolic or hyperbolic equations which computes the unknown parameters (see [1]).

Let the domain $\mathcal{D}$ be given by a Cartesian product of two real intervals $X = [0, L]$ and $Y = [0, R]$. Let $u(x,y)$ be a continuous solution of the following hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -q(x,y), \quad \alpha \in \mathbb{R}^+ \tag{19}$$

with the following Dirichlet boundary condition

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\mathcal{D} \tag{20}$$

where $\partial\mathcal{D}$ is a boundary of the domain $\mathcal{D}$.

Because both derivatives on the left hand side of equation (19) are of the same order we usually set up the same step on both axes i.e. $h = h_x = h_y$.

Analogously to both previous cases we approximate the second order derivatives on the left hand side by finite differences. Then we apply the F-transform to both sides of equation (19) whereby we obtain the following system of algebraic equations:

$$U_{ij}^{xx} + U_{ij}^{yy} = -Q_{ij} \tag{21}$$

where Q_{ij}, U_{ij}^{yy} and U_{ij}^{xx} are the F-transform components of the functions q, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$, respectively w.r.t. the given basis functions.

Equations (21) leads to

$$\frac{1}{h^2}(4U_{ij} - U_{(i-1)j} - U_{(i+1)j} - U_{i(j-1)} - U_{i(j+1)}) = Q_{ij} \tag{22}$$

for $i = 2, \ldots, n-1$ and $j = 2, \ldots, m-1$.

From boundary condition (20) we get $U_{ij} = G_{ij} \equiv g((i-1)h, (j-1)h)$ at the nodes with indices $i \in \{1, n\}$ and $j \in \{1, m\}$. The rest of unknown elements U_{ij} can be found from the following system of linear equations in matrix form:

$$\mathbf{K}_h \mathbf{U}_h = \mathbf{Q}_h \tag{23}$$

where

$$\begin{gathered} \mathbf{U}_h = [U_{11}, \ldots, U_{2(m-1)}; U_{32}, \ldots, U_{3(m-1)}; \ldots \\ \ldots; U_{(n-1)2}, \ldots, U_{(n-1)(m-1)}]^T \end{gathered}$$

where

$$\begin{gathered} \mathbf{Q}_h = [\mathcal{Q}_{22}, \ldots, \mathcal{Q}_{2(m-1)}; \mathcal{Q}_{32}, \ldots, \mathcal{Q}_{3(m-1)}; \ldots \\ \ldots; \mathcal{Q}_{(n-1)2}, \ldots, \mathcal{Q}_{(n-1)(m-1)}]^T \end{gathered}$$

and

$$\begin{aligned}
\mathcal{Q}_{ij} &= h^2 Q_{ij}, \\
\mathcal{Q}_{22} &= h^2 Q_{22} + G_{12} + G_{21}, \\
\mathcal{Q}_{2(m-1)} &= h^2 Q_{2(m-1)} + G_{2m} + G_{1(m-1)}, \\
\mathcal{Q}_{(n-1)2} &= h^2 Q_{(n-1)2} + G_{n2} + G_{(n-1)1}, \\
\mathcal{Q}_{(n-1)(m-1)} &= h^2 Q_{(n-1)(m-1)} + G_{n(m-1)} + G_{(n-1)m}, \\
\mathcal{Q}_{i2} &= h^2 Q_{i2} + G_{i1}, \\
\mathcal{Q}_{i(m-1)} &= h^2 Q_{i(m-1)} + G_{im}, \\
\mathcal{Q}_{2j} &= h^2 Q_{2j} + G_{1j}, \\
\mathcal{Q}_{(n-1)j} &= h^2 Q_{(n-1)j} + G_{nj},
\end{aligned}$$

for $i = 3, \ldots n-2$, $j = 3, \ldots, m-2$.

Matrix $\mathbf{K}_h$ has the following three-diagonal block structure

$$\mathbf{K}_h = \begin{bmatrix} H & -E & 0 & \ldots & 0 \\ -E & H & -E & \ldots & 0 \\ 0 & -E & H & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -E & H & -E \\ 0 & \ldots & 0 & -E & H \end{bmatrix}$$

where matrices H are square matrices of type $(m-2)\times(m-2)$ given by

$$H = \begin{bmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \dots & 0 \\ 0 & -1 & 4 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 4 & -1 \\ 0 & \dots & 0 & -1 & 4 \end{bmatrix}$$

and matrices E are unit matrices of type $(m-2)\times(m-2)$ given by

$$E = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

All the details and can be found in [8].

4 Properties

In this section, we will present and recall the main properties which relate to applications in partial differential equations. The first property which we would like to recall is the linearity of the F-transform mentioned and proved in [4].

In physics, integral means some energy, e.g. heat. For demonstration let us consider the heat equation. The solution $u(x,t)$ means a temperature at point x in time t. The convergence property is required but it is not enough because e.g. in case of convergence of a numerical solution to the analytical one from below the computed heat flow would be too far from the real heat flow, although the temperatures would be quite close to each other.

At second, the convergence property is required. Let us present some results in this field including a convergence theorem. The latter will be proved only for the heat equation because in the other two cases the technique is analogous.

Now, let us introduce the convergence theorem which justifies algorithm (12) for an approximate solution of equation (6). Besides the convergence, this theorem confirms that the mentioned algorithm has the second order of accuracy in h_x and the first order accuracy in h_x.

Theorem 1 *Let $u(x,t)$ be a solution of equation (6) which is four times continuously differentiable function w.r.t. x and twice w.r.t. t. Let us denote $u_{ij} = u(x_i, t_j)$. Let $U_{ij},\ i = 1,\dots,n,\ j = 1,\dots,m$, be the approximate solution given by algorithm (12) w.r.t. some given basis functions where $r = (\alpha h_t)/h_x^2$. Let $0 < r \le 1/2$. Then the norm of error of our approximate solution can be estimated as follows*

$$\| u_{ij} - U_{ij} \| \equiv \max_{\substack{i=1,\dots,n \\ j=1,\dots,m}} | u_{ij} - U_{ij} | = O(h_t + h_x^2). \tag{24}$$

Proof. By properties of finite differences the equality

$$\frac{U_{i(j+1)} - U_{ij}}{h_t} - \alpha \frac{U_{(i-1)j} + 2U_{ij} + U_{(i+1)j}}{h_x^2} = 0 \tag{25}$$

holds up to the value $O(h_t + h_x^2)$. Values Q_{ij} given by the F-transform approximate values $q_{ij} = q(x_i, t_j)$ with an error of order $O(h_t^2 + h_x^2)$.

Then values u_{ij} fulfil equation (12) with an accuracy $h_t O(h_t + h_x^2)$ i.e.

$$u_{i(j+1)} = r u_{(i-1)j} + (1-2r) u_{ij} + r u_{(i+1)j} + h_t q_{ij} + h_t O(h_t + h_x^2).$$

Now, let us define the error of approximate solution at node (x_i, t_j) by $e_{ij} = \mid u_{ij} - U_{ij} \mid$ and continue as follows

$$e_{i(j+1)} = re_{(i-1)j} + (1-2r)e_{ij} + re_{(i+1)j} + h_t O(h_t + h_x^2). \tag{26}$$

From initial (7) and boundary (8) conditions we get $e_{i1} = 0,\ i = 1, \dots, n$ and $e_{1j} = e_{nj} = 0,\ j = 1, \dots, m$.

If we consider the column norm $\| e_j \| = \max_{i=1,\dots,n} \mid e_{ij} \mid$ we easily obtain that $\| e_1 \| = 0$ which is going to be used at the end of this proof.

Because we have chosen $0 < r \le 1/2$, all the coefficients on the right hand side of equation (26) are non-negative and their sum equals to 1.

$$\begin{aligned} \mid e_{i(j+1)} \mid = r \mid e_{(i-1)j} \mid + (1-2r) \mid e_{ij} \mid + r \mid e_{(i+1)j} \mid \\ + h_t O(h_t + h_x^2) \le \| e_j \| + h_t O(h_t + h_x^2) \end{aligned}$$

and therefore

$$\| e_{i(j+1)} \| \le \| e_j \| + h_t O(h_t + h_x^2).$$

This implies

$$\begin{aligned} \| e_j \| \le \| e_{j-1} \| + h_t O(h_t + h_x^2) \le \\ \le \| e_{j-2} \| + 2h_t O(h_t + h_x^2) \le \cdots \le \\ \le \| e_1 \| + (j-1) h_t O(h_t + h_x^2) \end{aligned}$$

and with the help of the proved equality $\| e_1 \| = 0$ and the fact that $(j-1)h_t$ is a finite number less than R, we can write

$$\| e_j \| = O(h_t + h_x^2)$$

which proves the theorem.

□

Now, let us prove the convergence of the inverse F-transform of the numerical solution U_{ij} to the precise solution of (6). Since $u(x,t)$ is continuous on $\mathcal{D}$ then $\forall \varepsilon > 0\ \exists \delta > 0 :$

$$\| (x_1, t_1) - (x_2, t_2) \| < \delta \Rightarrow \mid u(x_1, t_1) - u(x_2, t_2) \mid < \varepsilon.$$

For some fixed ε we choose $n_\varepsilon, m_\varepsilon$ such that $h_x < \frac{\delta}{\sqrt{2}}$ and $h_t < \frac{\delta}{\sqrt{2}}$.

Moreover, we choose arbitrary $(z_1, z_2) \in [x_i, x_{i+1}] \times [t_j, t_{j+1}]$ where $2 \le i \le n_\varepsilon - 2$ and $2 \le j \le m_\varepsilon - 2$. Because $h_x < \frac{\delta}{\sqrt{2}}$ and $h_t < \frac{\delta}{\sqrt{2}}$ we obtain

$$\| (x_1, t_1) - (x_2, t_2) \| < \sqrt{\frac{\delta^2}{2} + \frac{\delta^2}{2}} = \delta$$

which implies $\mid u(z_1, z_2) - u(x_i, t_j) \mid < \varepsilon$.

Let us estimate the following absolute difference

$$\begin{aligned} \mid u(z_1, z_2) - U_{ij} \mid \le \mid u(z_1, z_2) - u(x_i, t_j) \mid + \\ \mid u(x_i, t_j) - U_{ij} \mid = \varepsilon + O(h_t + h_x^2). \end{aligned}$$

In the same way we obtain

$$\begin{aligned} \mid u(z_1, z_2) - U_{(i+1)j} \mid \le \varepsilon + O(h_t + h_x^2), \\ \mid u(z_1, z_2) - U_{i(j+1)} \mid \le \varepsilon + O(h_t + h_x^2), \\ \mid u(z_1, z_2) - U_{(i+1)(j+1)} \mid \le \varepsilon + O(h_t + h_x^2). \end{aligned}$$

Finally, we can evaluate the following difference

$$
\begin{aligned}
&\mid u(z_1, z_2) - u^F_{n,m}(z_1, z_2) \mid = \\
&= \mid u(z_1, z_2) - \sum_{i=1}^{n}\sum_{j=1}^{m} A_i(z_1)B_j(z_2)U_{ij} \mid \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{m} A_i(z_1)B_j(z_2) \mid u(z_1, z_2) - U_{ij} \mid \\
&< \sum_{i=1}^{n}\sum_{j=1}^{m} A_i(z_1)B_j(z_2)(\varepsilon + O(h_t + h_x^2)) \\
&= \varepsilon + O(h_t + h_x^2).
\end{aligned} \tag{27}
$$

Therefore, the inverse F-transform uniformly converges to the precise solution.

Let us discuss the advantage of a numerical solution based on the F-transform in comparison with an ordinary numerical solution. Let $f(x, y)$ be an arbitrary continuous function on the given domain $\mathcal{D}$ and let $A_i(x), B_j(y)$ $i = 1, \ldots, n$, $j = 1, \ldots, m$ be basis functions on $\mathcal{D}$. Let us consider the following class of approximating functions:

$$
\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} A_i(x)B_j(y). \tag{28}
$$

Then the following weighted least square criterion

$$
\Psi(c_{ij}) = \int_c^d \int_a^b \sum_{i=1}^{n}\sum_{j=1}^{m} (f(x, y) - c_{ij})^2 A_i(x)B_j(y) dxdy \tag{29}
$$

is minimized by the components F_{ij} given by (3).

Criterion (29) characterizes an integral proximity of the original approximated function and arbitrary function from class (28). The uniform convergence in $C(\mathcal{D})$ in combination with a minimization of the previous integral criterion provide a powerful tool which we are going to explain.

When the right hand side $q(x, y)$ of equation (6) is damaged by a noise (inaccuracies in measurements), the ordinary finite difference method is not stable.

For more results and details we suggest the reader to see [6]. For a concrete demonstration with graphical outputs we have prepared section 5.

5 Demonstration

We are going to find the numerical solutions of the following heat equation

$$
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 5(-4(x - 1/2)^2 + 1)e^{-t/2}, \quad x \in [0, 1], t \in [0, 1], \tag{30}
$$

with the following boundary conditions

$$
u(0, t) = 0, \quad u(1, t) = 0, \quad t \in [0, 1] \tag{31}
$$

and the following initial condition

$$
u(x, 0) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1 - x), & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{32}
$$

This equation describes a distribution of temperature $u(x, t)$ of a metal rod at point x and time t while both ends of this rod are kept in contact with melting ice. The right hand side $5(-4(x - 1/2)^2 + 1)e^{-t/2}$ gives us the information about heat sources. The solution of equation (30) is displayed on Figure 1.

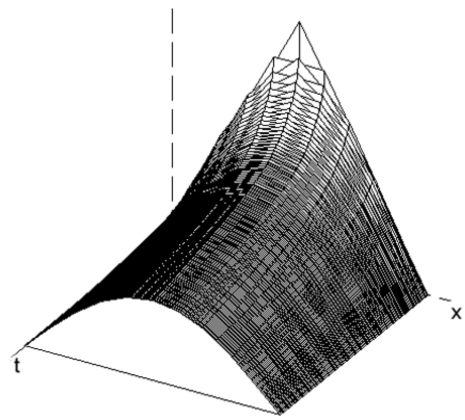

Figure 1: Solution of equation (30)

The difference between the finite difference method and the solution based on the F-transform is almost not visible and the finite difference method is even faster because it uses just values of the heat source, whereas our approach requires computations to obtain the F-transform values of the heat source.

On the other side, let us consider a situation (in practice very usual) when the right hand side of equation (30) is obtained by some measurements with inaccuracies. These inaccuracies can be modeled by adding a random noise.

For the demonstration, we have added the random noise with Gaussian distribution with 0 mean and standard deviation equal to 2, to the right hand side of equation (30). Thus modified equation has been again numerically solved by both methods: the finite difference method and by the F-transform technique. In both cases we have used 11 nodes on axis x and 200 nodes on time axis.

The difference is significant. In the case of the F-transform the solution is practically the same as the previous without any noise (see Figures 1 and 2). On the other hand, the solution based on ordinary technique is visibly influenced by the noise (see Figure 3).

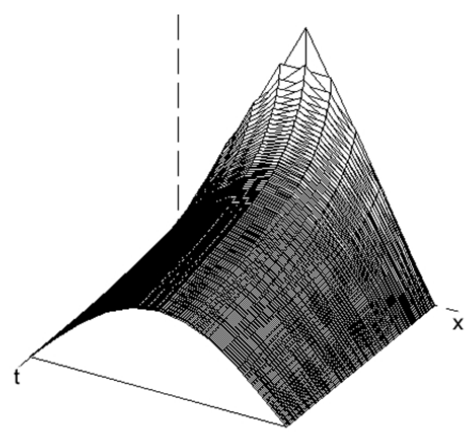

Figure 2: Numerical solution of modified equation (30). Finite difference method with help of F-transform.

This advocates in favour of the F-transform technique.

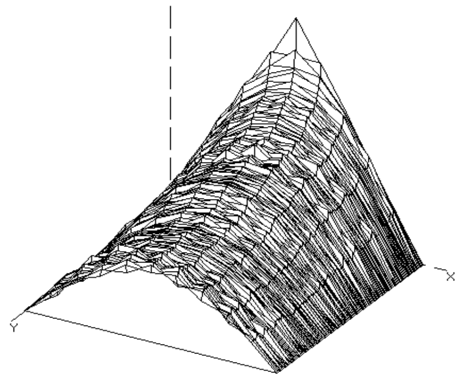

Figure 3: Numerical solution of modified equation (30). Finite difference method.

6 Conclusion

In this paper devoted to a fuzzy approach to numerical solutions of partial differential equations we recalled existing fuzzy approximating model called the F-transform published e.g. in [4], [5] and generalized in [11]. We followed the main idea from [4] of a usage of the F-transform which consisted in replacing continuous functions in differential equations by their discrete representations. After such discretization we obtained systems of algebraic equations which are solvable by existing numerical methods. Finally, the numerical solutions were transformed back into the continuous ones.

Three main types of partial differential equations have been considered to demonstrate the algorithms with help of the F-transform. Although we have used existing numerical techniques there is a significant difference between our results and result obtained by the classical approach. In section 5 we have introduced an example of a reasonable application of the F-transform in this area. The justification of our approach including the convergence theorem has been presented as well.

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