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Completing fuzzy if-then rule bases by means of smoothing splines

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Abstract

A fuzzy if-then rule base may be viewed as a partial function between universes of fuzzy sets. We show how this partial function determines a total one by means of the method of smooting splines. To this end, we identify the fuzzy sets with elements of a finite-dimensional real parameter space in an approximate way, using Perfilieva's fuzzy transforms.

Key words: Fuzzy if-then rule bases, smoothing splines, fuzzy transforms

1 Introduction

Fuzzy if-then rule bases typically appear in the following context. Some controlling device is supposed to calculate from some input value, which ranges over a set $\Omega \subseteq \mathbb{R}^m$, an output value, which ranges over another set $\Psi \subseteq \mathbb{R}^n$. The dependency of the output value from the input value, however, is known only roughly. Namely, all available information is contained in finitely many rules of the form $(\mathbf{A}, \mathbf{B})$, telling us that with the fuzzy set $\mathbf{A}$ over Ω, the fuzzy set $\mathbf{B}$ over Ψ is associated.

Clearly then, an actual fuzzified input value need not appear as a left entry $\mathbf{A}$ of a pair $(\mathbf{A}, \mathbf{B})$ contained in the rule-base; hence a method of interpolation has to be defined. In most cases, a Mamdani-type inference is used. This method is usually satisfactory because the input value typically passes through three steps: first, it is fuzzified; then, the fuzzy inference is performed; and finally, the result is defuzzified. All in all, we get a function between sharp values, and this function is for most applications smooth enough.

The question addressed here is what can be done in the case that the fuzzifier and the defuzzifier are not desirable. Namely, we assume (i) that we want to process real fuzzy input values, which are not the result of an artificial fuzzification process, and (ii) that these values are to be transformed into fuzzy values, which are to be communicated together with all information they contain, that is, not just as a single point. In such a situation, Mamdani-type inference is certainly useless; what is needed is an inference method developed for genuine fuzzy data.

In the past decade, several proposals were made how to tackle this problem. For the Kóczy-Hirota interpolation method, see [KoHi] as well as e.g. [TJKVM], [Tik]. Further approaches include the one of Jenei, see [Jen]. However, many of the methods treat only fuzzy sets of a rather specific shape or with specific interrelations; moreover, the higher-dimensional cases often require special techniques. Following the method proposed in this paper, the fuzzy sets may be chosen practically arbitrarily and independently; and the method is formulated without reference to the dimension of the underlying space.

We proceed as follows. First of all, the fuzzy sets need to be parametrized by a finite set of reals. To this end, we use the fuzzy transforms which were introduced by I. Perfilieva [Per1, Per2, PeVa, StVa] (Section 2). Our task is then reduced to determine a function between the parameter spaces, which are finite products of the real unit interval. What we propose is to use the method of smoothing splines; a convenient abstract formulation of this formalism is due to Atteia [Att] (Section 3). That is, we define a function between the two parameter spaces which on the one hand is as smooth as possible and which on the other hand reproduces the given if-then rule base as well as possible.

A short discussion of advantages and disadvantages of our approach and moreover a note on the connections to the paper [Vet] is found at the end (Section 4).

2 Fuzzy transforms

In this section, we describe a way how to represent fuzzy sets by a finite number of reals. We rely on the work of Perfilieva; for proofs and further details, we refer to [Per1, StVa].

By a fuzzy set over a closed convex subset Ω of a real euclidean space $\mathbb{R}^n$, we mean a lower semicontinuous function from Ω to the real unit interval $[0, 1]$. So we view fuzzy sets in the "traditional" way rather than levelwise; and their shape remains throughout the paper unrestricted. We actually do not even have to assume lower semicontinuity; we could take any property ensuring that the integrals defined below exist.

We will now specify a class of fuzzy sets each member of which is described unambigously by just a finite set of real parameters. An arbitrary fuzzy set is approximated by a member of this class up to a certain precision, and the attainable precision depends on the number of parameters.

Definition 2.1 Let $k \geq 2$, and let Ω be a closed convex subset of $\mathbb{R}^n$. A *partition of the unity over* Ω is a k-tuple $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ of fuzzy sets over Ω such that the following holds:

(E1) for every i, $\mathbf{E}^i$ is continuous;

(E2) for every i, $\mathbf{E}^i$ attains at some point the value 1;

(E3) $\sum_i \mathbf{E}^i(x) = 1$ for all $x \in \Omega$.

In this case, we call

$$\mathcal{F}(\mathcal{E}) = \{\sum_i A^i \mathbf{E}^i\colon\ 0 \ \leq\ A^1, ..., A^k \ \leq\ 1\}$$

the *fuzzy set universe generated by* $\mathcal{E}$.

Lemma 2.2 *Let $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ be a partition of the unity over $\Omega \subseteq \mathbb{R}^n$. Then $\mathcal{F}(\mathcal{E})$, the fuzzy set universe generated by $\mathcal{E}$, consists of fuzzy sets. Moreover, each $\sum_i A^i \mathbf{E}^i \in \mathcal{F}(\mathcal{E})$ is uniquely determined by the k-tuple $(A^1, \ldots, A^k)$.*

Proof. The first assertion is clear from the fact that the $\mathbf{E}^i$ add up to the identity. Moreover, let $\sum_i A^i \mathbf{E}^i = \sum_i B^i \mathbf{E}^i$ for $A^1, ..., A^k, B^1, ..., B^k \in [0, 1]$. For any index i, there is an $x \in \Omega$ such that $\mathbf{E}^i(x) = 1$, whence $\mathbf{E}^j(x) = 0$ for $j \neq i$; it follows $A^i = B^i$. □

So by Lemma 2.2, the fuzzy set universe generated by some k-element partition of the unity is in a one-to-one correspondence with the set of k-tuples of coefficients w.r.t. that partition, that is, with the set $[0, 1]^k$.

Definition 2.3 Let $k \geq 2$, and let $\mathcal{E}$ be a k-element partition of the unity. Then we call

$$\mathcal{P}(\mathcal{E}) = [0, 1]^k$$

the *fuzzy set parameter space belonging to* $\mathcal{E}$, and we define

$$\iota_{\mathcal{E}}\colon\ \mathcal{F}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}),\ \ A^1 E^1 + \ldots + A^k E^k \mapsto (A^1, \ldots, A^k).$$

We next have to establish that the elements of a fuzzy set universe, which is generated by some partition of the unity, approximate fuzzy sets to any given precision, provided the cardinality of the partition is large enough. This was shown in the works [Per1, StVa]; we will provide here a short overview.

Definition 2.4 Let $\Omega = [a, b]$ be a closed real interval, let $k \geq 2$, and let $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ be a partition of the unity over Ω. Let $h = \frac{b-a}{k}$ and $c_0 = a,\ c_1 = a + h, \ldots, c_k = b$. Then $\mathcal{E}$ is called a *system of basic functions over* Ω if the following holds for $i = 1, \ldots, k$:

(E4) $\mathbf{E}^i(c_i) = 1$ and the support of $\mathbf{E}^i$ is within $[c_i - h, c_i + h] \cap \Omega$;

(E5) $\mathbf{E}^i$ is on $[c_i - h, c_i] \cap \Omega$ strictly increasing and on $[c_i, c_i + h] \cap \Omega$ strictly decreasing;

(E6) $\mathbf{E}^i(c_i - h') = \mathbf{E}^i(c_i + h')$ if $0 \leq h' \leq h$ and $c_i - h', c_i + h' \in \Omega$;

(E7) $\mathbf{E}^{i+1}(x) = \mathbf{E}^i(x - h)$ if $1 \leq i < k$ and $x - h, x \in \Omega$.

Furthermore, let $n \geq 1$, let $\Omega_j = [a_j, b_j]$ for every $j = 1, \ldots, n$ be a closed real interval, and let $\Omega = \Omega_1 \times \ldots \times \Omega_n$; we call such a set a *cube*. For every $j = 1, \ldots, n$, let $k_j \geq 2$, and let $(\mathbf{E}_j^1, \ldots, \mathbf{E}_j^{k_j})$ be a system of basic functions. Let

$$\mathbf{E}^{i_1, \ldots, i_n}(x) = \mathbf{E}_1^{i_1}(x) \cdot \ldots \cdot \mathbf{E}_n^{i_n}(x),\ \ x \in \Omega,$$

where $1 \leq i_1 \leq k_1, \ldots, 1 \leq i_n \leq k_n$; the $k_1 \cdot \ldots \cdot k_n$-tuple of all these functions (e.g. in the lexicographical order) is again called a *system of basic functions over* Ω.

We next see that we may associate in a canonical manner to every fuzzy set $\mathbf{A}$ an element of the fuzzy set approximation space based on a system of basic functions.

Definition 2.5 Let $n \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be a cube, and let $\mathcal{E} = (\mathbf{E}^1, \dots, \mathbf{E}^k)$ be a system of basic functions over Ω. Let $\mathbf{A}$ be a fuzzy set over Ω. Then we call the k-tuple of real numbers

$$\mathrm{F}_{\mathcal{E}}[\mathbf{A}] = (A^1, \dots, A^k)$$

the *F-transform* of $\mathbf{A}$ w.r.t. $\mathcal{E}$, where for $i = 1, \dots, k$

$$A^i = \frac{\int_\Omega \mathbf{A}(x)\mathbf{E}^i(x)dx}{\int_\Omega \mathbf{E}^i(x)dx}. \tag{1}$$

Moreover, call

$$\mathrm{IF}_{\mathcal{E}}[\mathbf{A}](x) = \sum_{i=1}^{k} A^i \mathbf{E}^i(x) \tag{2}$$

the *inverse F-transform* associated to $\mathbf{A}$ w.r.t. $\mathcal{E}$.

The point is now that a continuous fuzzy set $\mathbf{A}$ differs from its inverse F-transform $\mathrm{IF}_{\mathcal{E}}[\mathbf{A}]$ w.r.t. the supremum norm by an arbitrary small positive real. Recall furthermore that an arbitrary fuzzy set can in turn be approximated arbitrarily close by a continuous fuzzy set for instance w.r.t. the L^2-norm.

Theorem 2.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a cube, and let $\mathbf{A}$ be a continuous fuzzy set over Ω. Then for any $\varepsilon > 0$ there exists a system of basic functions $\mathcal{E}$ such that*

$$|\mathbf{A}(x) - \mathrm{IF}_{\mathcal{E}}[\mathbf{A}](x)| \leq \varepsilon \text{ for all } x \in \Omega.$$

Proof. See [Per2, Per3] or [StVa]. □

3 Smoothing splines for fuzzy if-then rule bases

Assume that we are given a fuzzy if-then rule base $(\mathbf{A}_1, \mathbf{B}_1), \dots, (\mathbf{A}_l, \mathbf{B}_l)$, where $\mathbf{A}_1, ..., \mathbf{A}_l$ are fuzzy sets over $\Omega \subseteq \mathbb{R}^m$ and $\mathbf{B}_1, ..., \mathbf{B}_l$ are fuzzy sets over $\Psi \subseteq \mathbb{R}^n$. Ω and Ψ are supposed to be bounded; thus we may assume that Ω and Ψ are actually cubes. Furthermore, by Theorem 2.6, it is no seriously restricting requirement to choose systems of basic functions $\mathcal{E}_A$ and $\mathcal{E}_B$ over Ω and Ψ, respectively, and to assume that our fuzzy sets are in the associated fuzzy set universes. The numbers r and s of elements of $\mathcal{E}_A$ and $\mathcal{E}_B$, respectively, may be chosen high enough to ensure the desired precision.

In view of Lemma 2.2 and Definition 2.3, we may then identify $\mathbf{A}_i$ with $\iota_{\mathcal{E}_A}(\mathbf{A}_i) \in [0,1]^r$ and similarly $\mathbf{B}_i$ with $\iota_{\mathcal{E}_B}(\mathbf{B}_i) \in [0,1]^s$, $i = 1, \dots, l$. As a result, the function which we have to find has the paramater space $\mathcal{P}(\mathcal{E}_A) = [0,1]^r$ as its domain and $\mathcal{P}(\mathcal{E}_B) = [0,1]^s$ as its range.

This function $f\colon\ \mathcal{P}(\mathcal{E}_A) \to \mathcal{P}(\mathcal{E}_B)$ should fulfil the following conditions:

- f should be monotonous, that is, f should preserve the usual partial order of fuzzy sets. So if $\bar{\mathbf{A}}, \bar{\mathbf{A}}' \in \mathcal{P}(\mathcal{E}_A)$, then $\bar{\mathbf{A}} \leq \bar{\mathbf{A}}'$ should imply $f(\bar{\mathbf{A}}) \leq f(\bar{\mathbf{A}}')$. Here, $\leq$ denotes the componentwise ordering of $[0,1]^r$ and $[0,1]^s$, respectively.
- The function f should be "as smooth as possible". We interprete this condition in the usual way: For some q, we assume that the q-th (generalized) derivative $D^q f$ exists and is L^2-integrable; and the L^2-norm of $D^q f$ is to be minimized.
- For $i = 1, ..., l$, $f(\mathbf{A}_i)$ should be as close as possible to $\mathbf{B}_i$. This means that the sum of the distances between $f(\mathbf{A}_i)$ and $\mathbf{B}_i$ is to be minimized as well.

These two requirements may be arbitrarily weighted relatively to each other; the controlling constant is ρ, appearing below. We note that the third requirement may in principle be strengthened; we could require $f(\mathbf{A}_i) = \mathbf{B}_i$ for all i. This would require a slightly more involved procedure, for which we refer to [Vet].

Our framework is the following. First of all, F will denote the Hilbert space containing those functions which map one fuzzy set parameter space to the other one and which are of the appropriate differentiability class. Now, in our case, these functions need to have the monotonicity property; consequently, we will restrict to a certain positive cone in F, denoted by K. Second, the differential operator δ will map F into another Hilbert space D, such that for $f \in F$ the norm of $\delta(f)$ in D will measure the degree of smoothness of f. Third, there will be another linear function ω from F to a Hilbert space V which maps $f \in F$ to the values of f at those points whose image is known from the rule base.

In what follows, we will distinguish the scalar products and the 0's of the three Hilbert spaces by indices referring to the respective space.

The following theorem provides a variant of the well-known smoothing spline formalism; for a more detailed version of its proof, see e.g. [Att, Chapter III].

Theorem 3.1 *Let F, D, and V be Hilbert spaces; let $K \subseteq F$ be convex and closed; let $\delta\colon\ F \to D$ and $\omega\colon\ F \to V$ be linear operators such that δ has a closed image and ω is surjective; let $v_0 \in V$; and let $\rho > 0$. Assume furthermore that $\omega(\mathrm{Ker}\ \delta)$ is closed in V and that $\mathrm{Ker}\ \delta \cap \mathrm{Ker}\ \omega = \{0_F\}$. Then there is unique $f \in K$ which minimizes*

$$||\delta(f)||_D + \rho||\omega(f) - v_0||_V, \ \text{ where } f \in K. \tag{3}$$

Proof. Let $D' = \delta(F)$; then D' is a closed subspace of D. Let $H = D' \times V$ the Hilbert space with the scalar product

$$((d_1, v_1), (d_2, v_2))_H \ = \ (d_1, d_2)_D + \rho(v_1, v_2)_V,$$

for $d_1, d_2 \in D'$ and $v_1, v_2 \in V$, and let

$$\ell\colon\ F \to H, \quad f \mapsto (\delta(f), \omega(f)).$$

Then by [Att, III, Lemma 2.1], ℓ is a isomorphism between F and the closed subspace $\ell(F)$. Consequently, the restriction $\ell|_K$ of ℓ to K is a homeomorphism between K and $\ell(K)$ preserving convex combinations in both directions. In particular, $\ell(K)$ is a closed convex subset of H. Let $(d, v) \in \ell(K)$ be the unique projection of $(0, v_0)$ onto $\ell(K)$. Then $f = \ell^{-1}((d, v))$ minimizes (3). □

We now apply Theorem 3.1 to functions mapping between two fuzzy set parameter spaces. For some integer $q \geq 1$ and an open set $O \subseteq \mathbb{R}^r$, we denote by $W^{q,2}(O, \mathbb{R}^s)$ the Sobolev space of functions from O to $\mathbb{R}^s$, endowed with the norm $||\cdot||_{q,2}$ [AdFo]. Note that if O fulfils the strong local Lipschitz condition and if $q \geq \frac{r+3}{2}$, then this space consists of continuous functions continuously extendible to $\bar{O}$ [AdFo].

Elements $a_1, \ldots, a_l \in \mathbb{R}^r$ are called q-unisolvent if every polynomial of degree $\leq q-1$ vanishing at $a_1, \ldots, a_l$ is 0.

Theorem 3.2 *Let $\mathcal{E}_A$ be an r-element system of basic functions over a cube $\Omega \subseteq \mathbb{R}^m$, and let $\mathcal{E}_B$ be an s-element system of basic functions over a cube $\Psi \subseteq \mathbb{R}^n$. Let $(\mathbf{A}_1, \mathbf{B}_1), \ldots, (\mathbf{A}_l, \mathbf{B}_l) \in \mathcal{F}(\mathcal{E}_A) \times \mathcal{F}(\mathcal{E}_B)$.*

Let $\bar{\mathbf{A}}_i = \iota_{\mathcal{E}_A}(\mathbf{A}_i)$ and $\bar{\mathbf{B}}_i = \iota_{\mathcal{E}_B}(\mathbf{B}_i)$ for $i = 1, \ldots, l$. Let $\mathcal{P}_A = \mathcal{P}(\mathcal{E}_A) = [0,1]^r$ and $\mathcal{P}_B = \mathcal{P}(\mathcal{E}_B) = [0,1]^s$.

Let $q \geq \frac{r+3}{2}$ and

$$\begin{aligned} F &= W^{q,2}(\overset{\circ}{\mathcal{P}}_A, \mathbb{R}^s) \\ &= \{f \in L^2(\overset{\circ}{\mathcal{P}}_A, \mathbb{R}^s)\colon\ D^t f \in L^2(\overset{\circ}{\mathcal{P}}_A, S((\mathbb{R}^r)^t, \mathbb{R}^s) \text{ for all } t \leq q\}, \\ K &= \{f \in F\colon\ f \text{ is monotonous and } f(\overset{\circ}{\mathcal{P}}_A) \subseteq \mathcal{P}_B\}, \end{aligned}$$

where $S((\mathbb{R}^r)^t, \mathbb{R}^s)$ *is the space of symmetric t-linear forms from* $\mathbb{R}^r$ *to* $\mathbb{R}^s$*. Furthermore, let*

$$\begin{aligned}
&D = L^2(\overset{\circ}{\mathcal{P}}_A, S((\mathbb{R}^r)^q, \mathbb{R}^s)), \\
&\delta\colon\ F \to D, \quad f \mapsto D^q f; \\
&V = (\mathbb{R}^s)^l, \\
&\omega\colon\ F \to V, \quad f \mapsto (\bar{f}(\bar{\mathbf{A}}_1), \ldots, \bar{f}(\bar{\mathbf{A}}_l)),
\end{aligned}$$

where $\bar{f}$ *is the continuous extension of* f *to* $\mathcal{P}_A$*.*

Assume that $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_l$ *is q-unisolvent, and let* $v_0 = (\bar{\mathbf{B}}_1, \ldots, \bar{\mathbf{B}}_l)$*. Then, for any* $\rho > 0$*, there is a unique* $f \in K$ *minimizing* (3)*.*

Proof. We claim that δ has a closed image. Indeed, $f \mapsto ||\delta(f)||_D$ is a seminorm on F, which induces a norm on $F/\mathcal{P}_q$, where $\mathcal{P}_q$ is the subspace of polynomials of degree $\leq q-1$. The assertion follows from the fact that $F/\mathcal{P}_q$ is a Hilbert space.

Moreover, ω is obviously surjective. Ker $\delta = \mathcal{P}_q$; hence Ker $\delta \cap$ Ker $\omega = \{0_F\}$. It furthermore follows that ω(Ker δ) is closed. K is obviously convex and closed. So we may apply Theorem 3.1. □

4 Discussion

We described in this paper a method how to generate from a fuzzy if-then rule base a function whose domain is the whole input range of a fuzzy value. The only serious requirement is that there must be sufficiently many rules – approximately as many as one half of the number of basic functions used to represent the input fuzzy sets.

As the main feature of this method, we should exhibit the fact that it provides the completion of a rule base according to a clear, general principle. In particular, it is applicable nearly unrestrictedly. On the other side, generality certainly has its drawback. To implement our method is as difficult as to implement the method of smoothing splines in general: it is surely technically demanding.

Let us conclude the paper by having a look to the closely related work of [Vet]. In contrast to the present approach, fuzzy sets are parametrized in [Vet] as follows: finitely many level sets are chosen and the support functions of these level sets are approximated by their values at finitely many points of the unit sphere. The advantage is that these parameters are compatible with the arithmetic operations of fuzzy sets, that is, with the addition and with the multiplication with positive reals. As a consequence, when the values in a rule base become sharper and sharper, the resulting function will in some sense approach the function which you would get in the crisp case. On the other side, it is a disadvantage in [Vet] that the parameter set is not described so easily. Finally, only fuzzy-convex fuzzy sets (in the sense of [DiKl]) are taken into account.

So in contrast to [Vet], F-transforms provide a rather direct picture of fuzzy sets, which are considered as functions from the base set to the real unit interval; and the parameter set has the easiest possible form. Moreover, we take into account a larger class of fuzzy sets. However, we should not leave unmentioned that the parameters representing the fuzzy sets are unrelated to the arithmetical operations and that consequently the structure of the base sets remains unused. We may say that our method is "vertically" (truth-value) oriented, as opposed to the "horizontal" (base-set) orientation in [Vet].

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