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1 Introduction

Fuzzy logic is now one of the leading and most successful methodologies for the treatment of the vagueness phenomenon. It is a well-established sound formal system with numerous applications. In recent years, several significant books have been published where fuzzy logic is investigated deeply and so, its applications are given sound mathematical basis (see (Hájek, 1998; Novák, Perfilieva, & Močkoř, 1999; Gerla, 2001; Gottwald, 2000)). In this contribution we give a brief overview of fuzzy logic and discuss its role in modeling of vagueness phenomenon and of a class of natural language expressions. We also show the description of *sorites* paradox offered by fuzzy logic in narrow sense with evaluated syntax.

Vagueness is understood as one of facets of the *indeterminacy* phenomenon. Note that another facet is *uncertainty* that emerges due to lack of knowledge about the occurrence of some event (see (Novák et al., 1999), Chapter 1). Vagueness occurs in the process of grouping together objects which have some property ϕ (e.g. tallness). Quite often is such a grouping imprecise, which means that there exist borderline elements — elements having the property ϕ only to some extent. There are various approaches trying to model this phenomenon, both inside or outside of classical logic. One of important properties of vagueness is its *continuity.* It means that for similar objects, the extent in which they have the property ϕ should also be similar.

There are numerous applications of fuzzy logic in control, decision making, data mining etc. The software system LFLC 2000 is under development at our institute (see `http://ac030.osu.cz/irafm`), (Dvořák, Habiballa, Novák, & Pavliska, 2003). Among other things, it provides an implementation of the model of evaluating linguistic expressions and of the deduction over linguistic descriptions (sets of linguistically stated IF-THEN rules) (see Section 3.2).

2 Theories of Vagueness

Vagueness is a phenomenon which intrigued scientists for a long time. For the review of various theories of vagueness see (Keefe, 2000). However, we think that the presentation in this book is somewhat biased against many-valued logical approaches. According to Keefe, vague phenomena can be characterized by three main features: they admit borderline cases, lack clear boundaries and are susceptible to sorites-like paradoxes. *Borderline cases* are such cases where it is not clear whether the predicate applied to some object gives truth or falsity. *Lack of clear boundaries* means that there appears to be no sharp boundary between cases giving truth and cases giving falsity. Naturally, these two features are closely related. For the discussion about *sorites paradox,* see the end of this section and Section 4.

The main theories challenging the vagueness phenomenon are (again according to Keefe) the following:

(i) Epistemic account,

(ii) pragmatic account,

(iii) supervaluations,

(iv) many-valued logics.

In *epistemic* view of vagueness (see (Williamson, 1996)), borderline case predications are always true or false, and it is our ignorance which is responsible for the impossibility to decide it in some individual situation. Classical two-valued syntax and semantics are fully retained.

Similarly, *pragmatic* view retains classical syntax and semantics, too. Vagueness is this time present only in the process of pragmatic relation between language users and language itself, see (Burns, 1991).

Supervaluational approach (Fine, 1975) retains classical syntax but adopts non-classical semantics which permits to be not true neither false for some sentences. It means that we are facing so-called *truth-value gaps*. Proposition involving vague predicate is in this approach true (false) if it is true (false) on all the ways in which it could be make precise (so-called *precisifications*). It is also possible to consider theories in which borderline case predications are both true and false (it amounts to *truth-value gluts*). It can be formalized by means of some paraconsistent logical system (Hyde, 1997).

In many-valued logical systems, borderline case predications are assigned to truth values from some ordered algebraic structure, typically from so-called *residuated lattice*. For the detailed discussion of many-valued (or fuzzy) logics see Section 3.

Another important attempt to cover vagueness phenomenon and sorites-like paradoxes represents concept of *semiset* (Vopěnka & Hájek, 1972) and the, so called, *Alternative set theory* (AST), see (Vopěnka, 1979). The semiset is a subclass of some set, but it may not necessarily be a set itself. For example, the semiset Fn of *finite natural numbers* is closed under successor, addition and multiplication, but does not contain all natural numbers. Induction fails for some formulas containing semiset variables on the constant Fn. But for each individual natural number n, AST proves that $\bar{n} \in Fn$ ($\bar{n}$ is the n-th numeral). The alternative set theory is equiconsistent with Zermelo-Fraenkel set theory.

3 Fuzzy Logic — an Overview

We can distinguish fuzzy logic in narrow sense (FLn), fuzzy logic in broader sense (FLb), and higher-order fuzzy logic.

3.1 Fuzzy logic in narrow sense

Fuzzy logic in narrow sense is a special many-valued logic whose aim is to provide means which can be used for modeling of various aspects of the vagueness phenomenon.

We understand fuzzy logic in narrow sense as *truth-functional* logical system. This means that a truth value of a compound formula can be computed from truth values of its subformulas using truth functions of connectives. The algebraic foundation for fuzzy logic in narrow sense is an appropriate structure of truth values. We can quote P. Hájek saying "Fuzzy logic is the logic of comparative notion of truth". Consequently, as the basic and most general structure of truth values we consider the *residuated lattice*. It is a structure $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ where $\mathcal{L} = \langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with $\mathbf{0}, \mathbf{1}$, $\langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid and the, so called adjunction condition, $a \otimes b \leq c$ iff $a \leq b \to c$ is fulfilled.

As was mentioned in Section 1, one of the most important features of vagueness is its *continuity*. Therefore, it is natural to require the continuity of truth functions of (some) logical connectives, considering for example real unit interval $[0, 1]$ as the support of the structure of truth values. If we require that the operation $\otimes$ has to be continuous in the unit interval, we arrive at the notion of BL-algebra introduced in (Hájek, 1998). BL-algebra is a residuated lattice satisfying two additional conditions, namely $(a \to b) \vee (b \to a) = \mathbf{1}$ (prelinearity) and $a \otimes (a \to b) = a \wedge b$ (divisibility).

If we fix the support of residuated lattice to be the unit interval and require the continuity of $\otimes$, then we can distinguish three important BL-algebras: (i) *product algebra* $\mathcal{L}_P = \langle [0, 1], \vee, \wedge, \otimes_P, \to_P, 0, 1 \rangle$, where $a \otimes_P b = a \cdot b$ and

$$a \to_P b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{otherwise,} \end{cases}$$

(ii) *Gödel algebra* $\mathcal{L}_G = \langle [0, 1], \vee, \wedge, \otimes_G, \to_G, 0, 1 \rangle$, where $a \otimes_G b = a \wedge b$ and

$$a \to_G b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise,} \end{cases}$$

and (iii) *Lukasiewicz MV-algebra* $\mathcal{L}_{\text{L}} = \langle [0,1], \otimes, \oplus, \neg, 0, 1 \rangle$ where $a \otimes b = 0 \vee (a+b-1)$, $a \oplus b = 1 \wedge (a+b)$ and $\neg a = 1-a$. MV-algebras were originally introduced in (Chang, 1958) and they nontrivially generalize the concept of boolean algebra. The Łukasiewicz MV-algebra is the most prominent example of MV-algebra and it becomes a residuated lattice (i.e. the structure of the same type as $\mathcal{L}_P$ and $\mathcal{L}_G$) if we define the residuation operation by $a \rightarrow_L b = 1 \wedge (1 - a + b)$. Note that this operation is continuous, which is not the case for $\rightarrow_P$ and $\rightarrow_G$.

The formal logical system based on BL-algebras is *basic fuzzy logic* (BL-logic) developed by P. Hájek in (Hájek, 1998). It has classical syntax (extended by additional connectives) which differs from classical boolean logic by the set of axioms. Hence, all notions well known from classical logic (terms, formulas, deduction rules, provability, formal theory) are identical with corresponding classical ones.

The semantics of BL-logic is non-classical which means that formulas may be assigned a general truth value taken from L. More precisely, the *$\mathcal{L}$-structure* for the language J is

$$\mathcal{M} = \langle M, (r_P)_P, (m_c)_c \rangle$$

where $r_P : M^n \rightarrow L$ are n-ary fuzzy relations (for each predicate P). In other words, interpretations of predicate symbols P can be understood as fuzzy subsets $r_P \underset{\sim}{\subset} M^n$. $m_c \in M$ are objects from M assigned to object constants c.

The completeness theorem then says that a formula A of some theory T is provable iff it is true in the degree 1 in each model of T. The completeness theorem holds both for propositional as well as predicate basic fuzzy logic.

There are many logics with classical syntax and semantics generalized in the way hinted above, for example monoidal logic, MTL-, IMTL-, SBL-logic etc. There are three important special logics derived from BL-logic. They correspond to three algebras $\mathcal{L}_P$ (Product logic), $\mathcal{L}_G$ (Gödel logic) and $\mathcal{L}_{\text{L}}$ (Łukasiewicz logic). The completeness theorem again holds in all of these logics (for predicate logics, however, the recursive completeness may not hold).

A more radical departure from classical logic is the *fuzzy logic with evaluated syntax* developed by J. Pavelka (Pavelka, 1979) and V. Novák (see (Novák, 1990; Novák et al., 1999)). It is based on the Łukasiewicz MV-algebra $\mathcal{L}_{\text{L}}$ of truth values.

Axioms in this logic may be only partially true, thus forming a fuzzy set of formulas. Consequently, the main concept is that of *evaluated formula* a/A where A is a formula and a is its *syntactic evaluation* taken from $\mathcal{L}$. The notions of formal theory, inference rule and proof are modified and the *degree of provability* is introduced. The semantics of this logic is also non-classical constructed analogously as in the case of BL-logic. Thus, each formula is assigned two values: syntactical evaluation and a truth value (semantical evaluation) both taken from the same structure of truth values.

Completeness theorem both in propositional as well as predicate fuzzy logic with evaluated syntax says that for every consistent theory T, the provability degree of formula A in the theory T coincides with the degree of truth of A in T. This is a highly non-trivial generalization of the classical completeness theorem.

3.2 Fuzzy logic in broader sense

Fuzzy logic in broader sense is an extension of fuzzy logic in narrow sense. It provides a model of some class of natural language expressions and the means for the reasoning based on the latter. We can e.g. successfully model the meaning of the so-called *evaluating linguistic expressions*, and deduction over sets of conditional expressions including them (the so called fuzzy *IF-THEN rules*) (Novák, 2001; Dvořák, 2003). Evaluating linguistic expressions are special natural language expressions, which characterize sizes, distances, etc. In general, they characterize a position on an ordered scale. Among them, we distinguish *atomic evaluating expressions* which include any of the adjectives *small*, *medium*, or *big* (and possibly other adjectives of the same kind, such as *cold*, *hot*, etc.), or fuzzy quantity *approximately* z. The latter is a linguistic expression characterizing some quantity z from an ordered set.

Atomic evaluating expressions usually form pairs of antonyms, i.e. the pairs

$$\langle \text{nominal adjective} \rangle \text{ --- } \langle \text{antonym} \rangle.$$

Of course, there are a lot of pairs of antonyms, for example *young — old*, *ugly — nice*, *stupid — clever*, etc. When completed by the middle term, such as *medium*, *average*, etc., they form the so-called *basic linguistic trichotomy*. Let us stress that the basic linguistic trichotomy "*small*, *medium*, *big*" should be taken as canonical, which represents a lot of other corresponding trichotomies, such as "*short*, *average*, *long*", "*deep*, *medium deep*, *shallow*", etc.

Simple evaluating expressions are expressions of the form

$$\langle\text{linguistic hedge}\rangle\langle\text{atomic evaluating expression}\rangle.$$

Examples of simple evaluating expressions are *very small*, *more or less medium*, *roughly big*, *about twenty five*, *approximately* z, etc. Linguistic hedges are special adjectives modifying the meaning of adjectives before which they stand (Lakoff, 1973). In general, we speak about linguistic hedges with *narrowing effect* (*very*, *significantly*, etc.) and those with *widening effect* (*more or less*, *roughly*, etc.).

Let $\mathcal{A}$ be an evaluating expression then

$$\langle\text{noun}\rangle \text{ is } \mathcal{A} \tag{1}$$

is called *evaluating linguistic predication*. If $\mathcal{A}$ is simple then (1) is called *simple evaluating predication*. Examples of simple evaluating predications are, e.g., *temperature is very high* (here *high* is taken instead of *big*), *pressure is roughly small*, *income is roughly three million*, etc. Various linguistic expressions can be connected by the connectives *AND* and *OR*. The fuzzy IF-THEN rule is a natural language conditional clause

$$\mathcal{R} := \text{ IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

It characterizes relation between two evaluating predications. In the sequel, we will denote by $\tilde{\mathcal{S}}$ a set of evaluating linguistic expressions and fuzzy IF-THEN rules introduced above.

A linguistic expression may, in general, be understood as a name of some property. In the linguistic theory, we speak about its *intension* (instead of property named by it). Furthermore, we have to consider a *possible world* (cf. (Tichý, 1988; Novák, 1992)), which can be informally understood as "a particular state of affairs". For us, the possible world is a set of objects, which may carry the properties in concern. Therefore, we will speak in the following about linguistic context instead of possible world. The intension of the linguistic expression determines in each linguistic context its *extension*, i.e. a grouping of objects having the given property. Since there can exist infinite number of linguistic contexts, one intension may lead to a class of extensions.

We will formalize these concepts using the means of predicate FLn with evaluated syntax. The level of formal syntax is identified with the syntax of FLn and the semantic level is identified with the semantics of FLn. In the sequel, we suppose some fixed predicate language J. By F_J we denote the set of well-formed formulas and by M the set of all closed terms of J (we will further suppose that M contains at least two elements).

To define the mathematical model of the *intension* of a linguistic expression $\mathcal{A}$, we start by assigning some formula $A(x) \in F_J$ to $\mathcal{A}$. However, this is not sufficient since this does not grasp inherent vagueness of the property represented by $\mathcal{A}$. This can be accomplished in FLn using the concept of *evaluated formula*. Namely, if $A(x)$ is a formula with one free variable then the evaluated formula $a/A_x[t]$ means that some object represented by the term t has the property A in the degree at least $a \in L$.

The *extension* is characterized on the semantic level, which is identified with the semantics of FLn. Hence, the concept of *linguistic context* is understood as a special structure $\mathcal{V}$ for J

$$\mathcal{V} = \langle V, P_{\mathcal{V}}, \ldots\rangle.$$

We will usually suppose that all fuzzy relations assigned to predicate symbols of J in the possible world $\mathcal{V}$ have continuous membership function. Moreover, some further assumptions on $\mathcal{V}$ can be made, for example the unimodality of some membership functions, specific topological structure defined on the support V, etc. When dealing with evaluating linguistic expressions, V is assumed to be a *linearly ordered interval* $V = [{}^l v, {}^r v]$.

Definition 1 *Let* $\mathcal{A} \in \tilde{\mathcal{S}}$ *be a natural language expression and let it be assigned a formula* $A(x)$*.*

(i) The intension of $\mathcal{A}$ is a set of evaluated formulas (also called multiformula)

$$\mathrm{Int}(\mathcal{A}) = \mathbf{A}_{\langle x \rangle} = \left\{ {}^{a_t}\!/A_x[t] \,\middle|\, t \in M, a_t \in L \right\}. \tag{2}$$

(ii) The extension of $\mathcal{A}$ in the context $\mathcal{V}$ is the satisfaction fuzzy set

$$\mathrm{Ext}_{\mathcal{V}}(\mathcal{A}) = \left\{ {}^{\mathcal{V}(A_x[\mathbf{v}])}\!/v \,\middle|\, v \in V \right\}. \tag{3}$$

Using the analysis of the meaning of evaluating linguistic expressions, we can analyze the properties of the so-called *linguistic descriptions*, i.e. finite sets of IF-THEN rules. Also, we can describe the deduction over such linguistic descriptions, so called *perception-based fuzzy logic deduction* (Novák & Perfilieva, 2004).

3.3 Higher order fuzzy logic

Fuzzy logic presented so far covers propositional and predicate logic of first order. The most advanced possibility is the *fuzzy type theory* (FTT) (Novák, in press). Two essential structures of truth values seem to fit best the outlined requirements: the linearly ordered IMTL-algebra (a residuated lattice with prelinearity and involutive negation) and its strengthening, the Łukasiewicz MV-algebra. It is also necessary to introduce the, so called, Baaz delta operation Δ which maps all truth values to 0 except for 1.

In FTT, we work with formulas of various types. A type is a sequence of symbols composed of the elementary types o (truth values) and ϵ (objects). Each formula is a sequence of symbols of certain type and it is itself assigned also a type. A theory T of FTT is a set of formulas of type o. The strong completeness theorem can be proved saying that for every theory T and a formula A_o of type o,

$$T \vdash A_o \text{ iff } T \models A_o$$

where $T \models A_o$ means that a formula A_o is true in the degree 1 in every model of T.

4 Sorites in Fuzzy Logic

The *sorites* (heap) paradox is one of the most famous logical paradoxes. For a general survey of sorites, see (Vásconez, 2002). Sorites is related to vagueness and to its various theories and descriptions. For many authors, it serves as a benchmark for testing their approaches and theories of vagueness. The standard form of sorites is the following:

> One grain does not form a heap. Adding one grain to what is not a heap does not make a heap. Consequently, there are no heaps.

There are also other forms of it, e.g. *falakros* (bald man) paradox: If a non-bald man looses one hair, he is still not bald. Consequently, there are no bald men. A common feature of these paradoxes is the vagueness of the concept behind the paradox (baldness, heap-likeness etc.). We are, in principle, unable to distinguish when the given property started (or ceased) to exist.

The symbolic form of sorites can be described as follows: for a given predicate F and a sequence of objects $x_0, x_1, \ldots$, premises

$$F(x_0) \tag{4}$$

$$(\forall i)(F(x_i) \Rightarrow F(x_{i+1})) \tag{5}$$

seem to be true but the conclusion

$$F(x_n) \tag{6}$$

obtained by repeated application of (5) seems false. According to (Keefe, 2000), there are four basic possibilities how the sorites paradox can be solved:

1. Deny the validity of the argument.

2. Question the strict truth of the inductive premise (5).

3. Accept the validity of the argument and the truth of inductive premise (5) but question the supposed truth of premise (4) or the falsity of conclusion (6).

4. Claim that argument form is valid, accept the premises and deny the conclusion, therefore conclude that the vague predicate F is incoherent.

The solution offered here relies on option 2. We believe that it is the most intuitive one. It is provided within *fuzzy logic with evaluated syntax* already described in Section 3.1. For other possibilities see (Hájek & Novák, 2003). Recall that a/A denotes a formula A together with syntactic evaluation a. If it occurs in the specification of a fuzzy theory, it means that A is an axiom of the theory in the degree a.

Theorem 1 *Let T_{PA} be a fuzzy Peano theory, i.e. its fuzzy set of special axioms consists of Peano axioms accepted in the degree 1. Furthermore, let $0 < \epsilon \leq 1$ and $\mathbf{Fe} \notin J(T_{PA})$ be a new predicate. Then the fuzzy theory*

$$T_{Fe} = T_{PA} \cup \left\{ 1/\mathbf{Fe}(\bar{0}), 1 - \epsilon/(\forall x)(\mathbf{Fe}(x) \Rightarrow \mathbf{Fe}(S(x))), 1/(\exists x)\neg\mathbf{Fe}(x) \right\} \tag{7}$$

is a consistent conservative extension of T_{PA}.

For proof, see (Hájek & Novák, 2003). The predicate **Fe** means "is feasible". Intuitively, this can be interpreted as "to be small", "to be finite", "to be inside the horizon", etc. The $\bar{0}$ is the literal for the natural number 0 and S is the successor function symbol.

Corollary 1 *For each $n \in \mathbb{N}$*

$$T_{Fe} \vdash_{e(n)} \mathbf{Fe}(\bar{n}) \tag{8}$$

where

$$e(n) = 0 \vee (1 - n\epsilon).$$

It is also possible to consider some general function $\nu : \mathbb{N} \to [0, 1]$ such that $\nu(n) \leq \epsilon$ for all $n \in \mathbb{N}$. It can be seen that $T_{Fe} \vdash \mathbf{Fe}(\bar{0})$ and that there is a number n_0 such that $T_{Fe} \vdash_0 \mathbf{Fe}(\overline{n_0})$. It holds in any model $\mathcal{D}$ of T_{Fe} that there are natural numbers m_0 and n_0 such that $m_0 < n_0$, $Fe_{\mathcal{D}}(m_0) = 1$ and $Fe_{\mathcal{D}}(n_0) = 0$ ($Fe_{\mathcal{D}}$ is an interpretation of the predicate **Fe** in structure $\mathcal{D}$).

5 Conclusion

We presented fuzzy logic in narrow sense both with classical as well as with evaluated syntax and demonstrated that this is a well developed logical system able to deal with the vagueness phenomenon. Our treatment systematically takes into account one important feature of vagueness often neglected in literature, namely its *continuity*. In Section 4 we presented the description of sorites paradox using the idea that the main inductive premise is taken not to be fully true (convincing) but true only in some sufficiently high degree. It was showed that the sorites theory T_{Fe} is a conservative consistent extension of T_{PA} — the theory of Peano arithmetic in fuzzy logical system. Consequently, there is no paradoxical conclusion or inconsistency involved in the "sorites paradox".

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