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Fuzzy Transform as an Additive Normal Form

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Abstract

Fuzzy transform from [15] has been originally presented as an approximation method for continuous functions. In fact, one can say that only the name indicate the connection to fuzzy methods, otherwise, it belongs to the classical approximation techniques.

In this paper, we introduce a class of approximating formulas. Since, they arose as a generalization of the classical normal forms, consequently, we denote them by the same term. Moreover, we will show that fuzzy transform from [15] is a part of the larger group of formulas which are aimed at collecting a partial knowledge about some real system or process. In this way, we vindicate its appertain to the class of fuzzy methods.

Keywords: Disjunctive, conjunctive and additive normal forms; Fuzzy relation; Approximation Fuzzy transform

1 Introduction

The notion of normal forms is well known from the classical logic and has been generalized into fuzzy logic in many different ways. As an example, let us remind the results of P. Cintula and B. Gerla [3], where the introduced normal forms represent formulas of propositional Gödel logic. The representation of functions represented by formulas of propositional Łukasiewicz and Goguen logics can be found in the work [1]. From the other valuable results in this field we can recommend [11, 7, 2].

In this paper, we consider a direct generalization of the normal forms for the boolean functions associated with the formulas of classical propositional logic

$$\mathrm{DNF}_f(\mathbf{x}) = \bigvee_{f(\mathbf{x})=1} x_1^{\sigma_1} \wedge \ldots \wedge x_n^{\sigma_n}, \tag{1}$$

$$\mathrm{CNF}_f(\mathbf{x}) = \bigvee_{f(\mathbf{x})=0} x_1^{\sigma_1} \vee \ldots \vee x_n^{\sigma_n}, \tag{2}$$

into the fuzzy case, such that we exchange $\{0,1\}$-valued operations by $[0,1]$-valued ones. This generalization, as we will see later, can be extended and further used for an approximate representation of extensional fuzzy relations (interpretation of fuzzy predicate formulas). Approximate representation in logical framework means that a formula is equivalent to its normal form (disjunctive, conjunctive or additive) under the special condition. The schematic representation to which we refer as *conditional equivalence* or *logical approximation*, looks as follows:

$$\text{Condition} \leq (\text{Extensional Formula} \leftrightarrow \text{Normal Form}). \tag{3}$$

In fact, normal forms aggregate available local information about a fuzzy relation. This local information consists of two parts combined by conjunction: the first part characterizes a local domain by its membership function, and the second part describes a value of the fuzzy relation provided that its arguments lie inside the respective local domain. Thus, the normal forms can be viewed as collections of fuzzy **IF-THEN** rules and consequently they relate to the problematic of the approximate inference. The fuzzy approach is recently widely used in practical applications where robustness of a system is demanded.

We will introduce two types of normal forms, namely infinite and discrete (or finite) normal forms (see [6, 5]). We will see that the first one could be viewed as an precise representation of the initial fuzzy relation while the second one serves us as an "universal" approximation formula. On the algebraic level we do not have the notion of a limit at disposal. This is the main argument for having normal forms of the infinite type which serves there as an limit element of a sequence of the finite normal forms where the number of nodes specifying them increase. The condition which implies the equivalence (estimation of an approximation error) of normal forms and initial formula will be presented as well.

Finally, we will show the concrete method for construction of additive normal forms using F-transform. F-transform appeared to be elegant and very powerful tool for approximation of continuous functions. Here, the F-transform is introduced as a part of much general class of formulas, namely normal forms, and it is used for approximation of extensional fuzzy relations. The detailed presentation of F-transform can be found in [12, 15].

2 Preliminaries

A fuzzy relation is nothing else than a fuzzy subset of a Cartesian product of non-empty sets. Its values are interpreted as degrees to which are particular individuals in relation.

Definition 1 Let M be a non-empty set of objects. A function $R : M^n \to L$ is called n-ary *L-fuzzy relation* on M.

Very natural is to put $L = [0, 1]$. In this case we will use notions fuzzy set and fuzzy relation instead $[0, 1]$-fuzzy set and $[0, 1]$-fuzzy relation, respectively.

2.1 Elements form the analysis of t-norms

Original motivation for introducing the class of generalized multiplications known as *triangular norms* (t-norms) was not logical. The main idea was to generalize the concept of the triangular inequality. Since t-norms preserve the fundamental properties of the crisp conjunction, consequently they become to be interesting for fuzzy logic as a natural generalization.

Definition 2 A function $* : [0, 1]^2 \to [0, 1]$ is called *triangular norm* (t-norm) if it is commutative, associative, non-decreasing mapping fulfilling boundary condition, i.e. if for all $x, y, z \in [0, 1]$:

$$\begin{array}{cl} x * y = y * x & \text{(commutativity)}, \\ x * (y * z) = (x * y) * z & \text{(associativity)}, \\ x \le y \Longrightarrow x * z \le y * z & \text{(monotonicity)}, \\ x * 1 = x & \text{(boundary condition)}. \end{array}$$

Example 1 *Below, we show the most known examples of continuous t-norms which serve as natural interpretations of a generalized conjunction:*

(1) *Minimum t-norm* $x * y = x \wedge y$,

(2) *Product t-norm* $x * y = x \cdot y$,

(3) *Lukasiewicz t-norm* $x * y = \max(0, x + y - 1)$.

A concept associated with the t-norm is called t-conorm which corresponds due to its behavior to a generalization of the classical connective 'or'.

Definition 3 The t-conorm is a binary operation $\nabla : [0, 1] \times [0, 1] \to [0, 1]$ which has the properties of commutativity, associativity and monotonicity from Definition 2 and fulfills the following boundary condition for all $x \in [0, 1]$:

$$0 \nabla x = x.$$

A t-conorm dual to the given t-norm $*$ is given by

$$a \nabla b = 1 - (1 - a) * (1 - b).$$

Example 2 *The most important t-conorms dual to the t-norms from Example 1 are:*

(1) *Maximum t-conorm (dual to minimum)* $x \nabla y = x \vee y$,

(2) *Product t-conorm (dual to product)* $x \nabla y = x + y - x \cdot y$,

(3) *Lukasiewicz t-conorm (dual to Lukasiewicz t-norm)* $x \nabla y = \min(1, x + y)$.

Let us stress that *maximum* is the least t-conorm i.e. $x \vee y \le x \nabla y$ for all $x, y \in [0, 1]$ and for any t-conorm ∇ (see [11]).

It follows from the definition of the t-norm that it is a monoidal operation on $[0, 1]$. Furthermore, $< [0, 1], \wedge, \vee >$ is a complete lattice. Therefore, we can introduce the residuation operation in the following form.

Definition 4 Let $*$ be a t-norm. The residuation operation $\rightarrow_*$: $[0,1]^2 \rightarrow [0,1]$ is defined by

$$x \rightarrow_* y = \bigvee\{z \,|\, x * z \leq y\}. \tag{4}$$

Let us remind that the only necessary condition for an existence of the unique residuation operation is that the respective t-norm is left-continuous (see [9]).

Moreover, we will use the following derived operations

$$x^n = \underbrace{x * \ldots * x}_{n-times},$$
$$x \leftrightarrow_* y = (x \rightarrow_* y) \wedge (y \rightarrow_* x).$$

In the sequel, we denote Łukasiewicz operations t-norm, t-conorm and residuation as $\otimes$, $\oplus$ and $\rightarrow_{\text{Ł}}$, respectively. It is worth to mention the following relation between Łukasiewicz t-conorm and residuation:

$$(1 - x) \oplus y = x \rightarrow_{\text{Ł}} y \tag{5}$$

Lemma 1 *Let $*$ be a t-norm and $\rightarrow_*$ its residuation. Then the following properties hold for all $x, y, z \in [0,1]$:*

$$x * y \leq x, \tag{6}$$
$$x * y \leq z \Longleftrightarrow y \leq x \rightarrow_* z, \tag{7}$$
$$x \leq y \Longrightarrow x \rightarrow_* y = 1, \tag{8}$$
$$x \leq y \Longrightarrow y \rightarrow_* z \leq x \rightarrow_* z, \tag{9}$$
$$x \leq y \Longrightarrow z \rightarrow_* x \leq z \rightarrow_* y, \tag{10}$$

Interesting t-norms are those having additive generators.

Definition 5 Let $g : [0,1] \rightarrow [0,\infty]$ be a continuous strictly decreasing function such that $g(1) = 0$ and $*$ is a t-norm. Then g is an additive generator of $*$ if

$$x * y = g^{(-1)}(g(x) + g(y)) \tag{11}$$

holds for all $x, y \in [0,1]$. Moreover, the function $g^{(-1)} : [0,\infty] \rightarrow [0,1]$ such that

$$g^{(-1)}(y) = \begin{cases} g^{-1}(y) & \text{if } y \in [0, g(0)] \\ 0 & \text{if } y \in (g(0), \infty] \end{cases}$$

is called the pseudoinverse of g.

Additive continuous generators for t-norms are determined uniquely up to a positive multiplicative constant.

Example 3 *The following are examples of additive generators for continuous t-norms:*

(1) $g_{\text{Ł}}(x) = 1 - x$ *generates Łukasiewicz t-norm,*

(2) $g_P(x) = -\ln x$ *generates product t-norm.*

For the t-norm $*$ generated by a continuous additive generator g, the corresponding residuation is given by

$$x \rightarrow_* y = g^{(-1)}(\max(0, g(y) - g(x))), \tag{12}$$

and the corresponding biresiduation operation by

$$x \leftrightarrow_* y = g^{(-1)}(|g(x) - g(y)|). \tag{13}$$

Definition 6 A t-norm $*$ is called Archimedean if for every $x, y \in (0,1)$ there exists $n \in \mathbb{N}$ such that $x^n < y$.

The following theorem (see [9]) characterizes the class of generated t-norms.

Theorem 1 *A t-norm* $* : [0,1]^2 \to [0,1]$ *is a continuous Archimedean t-norm if and only if it has an additive generator.*

Let us remind the result in [9], where the authors proved that arbitrary continuous t-norm could be approximated with arbitrary precision by a t-norm constructed as an combination of those having additive generators.

Definition 7 Let $*$ be a t-norm generated by the additive generator g. Then

- $*$ is called nilpotent if $g(0) < +\infty$,
- $*$ is called strict if $g(0) = +\infty$.

Theorem 2 demonstrates that continuous Archimedean t-norms can be divided in two disjoint classes, namely nilpotent and strict (see [9] or Theorem 2.10 in [11]).

Theorem 2 *Let* $*$ *be a continuous Archimedean t-norm. Then,* $*$ *is nilpotent if and only if* $*$ *is not strict.*

Remark 1 *Let* $*$ *be a t-norm with with an additive generator then*

$$\lim_{n\to\infty} (x \to_* y)^n = \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote

$$(x \to_* y)^\infty = \lim_{n\to\infty} (x \to_* y)^n \tag{14}$$

and additionally

$$(x \to_* y)^0 = 1. \tag{15}$$

2.2 Extensional fuzzy relations

Extensionality is well known notion from the classical set theory. A generalized version of this notion has been introduced by F. Klawonn and R. Kruse in [8]. There, the extensional fuzzy relations are defined w.r.t. a similarity relation on their domain.

Below, we present a more general case of extensionality. The reason comes from the fact that extensional fuzzy relations defined w.r.t. similarity have properties relating to Lipschitz continuity. Let us recall the paper [10] where it has been proved that in a t-norm based algebra, the extensionality of a fuzzy relation w.r.t. a similarity is equivalent to Lipschitz continuity w.r.t. the pseudo-metric induced by the similarity.

The following generalized notion describes better character of a given relation.

Let M be some nonempty set of objects and L be some scale of truth values such that it include $\mathbf{0}$ as minimal element and $\mathbf{1}$ as maximal one.

Definition 8 Let E be a binary fuzzy relation on M and $f(x_1, \dots, x_n)$ be an L-valued function and let $c_1, \dots, c_n \in M$. We say that f is extensional w.r.t E and $*, \circledast$ if for all $c_1, \dots, c_n, x_1, \dots, x_n \in M$ the following holds truth

$$E(x_1, c_1) * \cdots * E(x_n, c_n) \circledast f(x_1, \dots, x_n) \le f(c_1, \dots, c_n), \tag{16}$$

where $*, \circledast$ are arbitrary left-continuous t-norms. If $* = \circledast$ then we speak about extensionality of f w.r.t E and $*$.

Later on, we will work also with L-fuzzy relations extensional w.r.t. binary L-fuzzy relations which are supposed to fulfill some of the following properties.

Definition 9 Let $*$ be a t-norm and consider a binary L-fuzzy relation R on a domain M. Then

1. R is called *reflexive* if
$$R(x,x) = \mathbf{1}, \text{for all } x \in M,$$
2. R is called *symmetric* if
$$R(x,y) = R(y,x), \text{for all } x, y \in M,$$
3. R is called $*$-*transitive* if
$$R(x,y) * R(y,z) \leq R(x,z), \text{for all } x, y, z \in M.$$

The extensionality property is closely related to a Lipschitz continuity. Later, we will use the result from [10].

Theorem 3 *Let $*$ be a t-norm generated by a continuous additive generator g. Moreover, let $f(\mathbf{x})$ be an n-ary fuzzy relation on M and $g \circ f \circ T^{-1} \circ g^{-1}$ has bounded partial derivatives on $[0, g(0)]$. Then $f(\mathbf{x})$ is extensional w.r.t. similarity S and $*$, where*

$$S(x,y) = (T(x) \leftrightarrow T(y))^{\tilde{k}}. \tag{17}$$

The parameter in the relation S is computed as follows

$$k = \max_{i=1,\ldots,n} k_i, \text{ where}$$
$$k_i = \max_{\mathbf{x} \in [0,g(0)]^n} \left| \frac{\partial (g \circ f \circ T^{-1} \circ g^{-1})}{\partial x_i} (\mathbf{x}) \right|$$

for all $i = 1, \ldots, n$. Moreover, $g(S(x,y))$ is Lipschitz continuous on M. An operation $\tilde{x}$ rounds up x to the nearest integer.

A necessary condition for f to be extensional is the continuity of f. A sufficient condition is the boundedness of the first partial derivatives of $g \circ f \circ T^{-1} \circ g^{-1}$ on $(0, g(0))$.

3 Representation of fuzzy relations by infinite normal forms

In this section, we are going to introduce special formulas of the infinite type. We show that these formulas can be viewed as universal representation formulas having on mind the extensionality property of an original fuzzy relation. But from the other side, we want to stress that they are not suitable for representation of the given fuzzy relation because it brings no simplification. Normal forms of the infinite type mirror here limit elements of the normal forms of the discrete type.

Definition 10 Let $f(x_1, \ldots, x_n)$ be an n-ary fuzzy relation, E be a binary fuzzy relation on M and $*, \circledast$ be left-continuous t-norms.

The following formulas are disjunctive normal form of f, conjunctive normal form of f and additive normal form of f

$$\overline{f_{DNF}}(x_1, \ldots, x_n) = \bigvee_{c_1, \ldots, c_n \in M} (E(c_1, x_1) * \cdots * E(c_n, x_n) \circledast f(c_1, \ldots, c_n)) \tag{18}$$

$$\overline{f_{CNF}}(x_1, \ldots, x_n) = \bigwedge_{c_1, \ldots, c_n \in M} (E(x_1, c_1) * \cdots * E(x_n, c_n) \rightarrow_{\circledast} f(c_1, \ldots, c_n)), \tag{19}$$

$$\overline{f_{ANF}}(x_1, \ldots, x_n) = \bigoplus_{c_1, \ldots, c_n \in M} (E(c_1, x_1) * \cdots * E(c_n, x_n) \circledast f(c_1, \ldots, c_n)), \tag{20}$$

respectively.

Later, we will need a special property of the binary fuzzy relation being a part of an additive normal form. It can be viewed as a generalization of the classical orthogonality.

Definition 11 Let E be a binary fuzzy relation on M and $*$ be a t-norm. We say that E fulfils the orthogonality property of the infinite type if

$$\bigoplus_{\substack{c_1,\ldots,c_n\in M\\ c_1,\ldots,c_n\neq d_1,\ldots,d_n}} (E(c_1,x_1)*\cdots*E(c_n,x_n)) = 1-(E(d_1,x_1)*\cdots*E(d_n,x_n)). \tag{21}$$

is valid for each $x_1,\ldots,x_n\in M$.

The following theorem relates to the properties of disjunctive and conjunctive normal forms of the infinite type with respect to an original fuzzy relation. In fact, it shows that based on the extensionality property the original formula is equal to its normal form.

Theorem 4 *Let $f(x_1,\ldots,x_n)$ be an n-ary fuzzy relation and $E(x,y)$ be a reflexive binary fuzzy relation on M. If f is extensional w.r.t E and left-continuous t-norms $*,\circledast$ then*

$$f(x_1,\ldots,x_n) = \overline{f_{DNF}}(x_1,\ldots,x_n), \tag{22}$$

$$f(x_1,\ldots,x_n) = \overline{f_{CNF}}(x_1,\ldots,x_n). \tag{23}$$

for all $x_1,\ldots,x_n\in M$.

Proof. 1 *It is sufficient to consider the case n=2.*
Since $f(x_1,x_2)\le f(x_1,x_2)$ and E is reflexive, we obtain

$$f(x_1,x_2)\le E(x_1,x_1)*E(x_2,x_2)\circledast f(x_1,x_2),$$

$$E(x_1,x_1)*E(x_2,x_2)\circledast f(x_1,x_2)\le \bigvee_{c_1,c_2\in M}(E(c_1,x_1)*E(c_2,x_2)\circledast f(c_1,c_2)),$$

$$f(x_1,x_2)\le \overline{f_{DNF}}(x_1,x_2).$$

On the other hand, from extensionality of f we have

$$E(c_1,x_1)*E(c_2,x_2)\circledast f(c_1,c_2)\le f(x_1,x_2),$$

$$\bigvee_{c_1,c_2\in M}(E(c_1,x_1)*E(c_2,x_2)\circledast f(c_1,c_2))\le f(x_1,x_2).$$

And that is why $f(x_1,x_2)=\overline{f_{DNF}}(x_1,x_2)$.
Similarly, the following proves the second formula.

$$E(x_1,x_1)*E(x_2,x_2)\rightarrow_{\circledast} f(x_1,x_2)\le E(x_1,x_1)*E(x_2,x_2)\rightarrow_{\circledast} f(x_1,x_2),$$

$$\bigwedge_{c_1,c_2\in M}(E(x_1,c_1)*E(x_2,c_2)\rightarrow_{\circledast} f(c_1,c_2))\le E(x_1,x_1)*E(x_2,x_2)\rightarrow_{\circledast} f(x_1,x_2),$$

due to (1) and reflexivity of E we obtain

$$E(x_1,x_1)*E(x_2,x_2)\circledast\bigwedge_{c_1,c_2\in M}(E(x_1,c_1)*E(x_2,c_2)\rightarrow_{\circledast} f(c_1,c_2))\le f(x_1,x_2),$$

$$\overline{f_{CNF}}(x_1,x_2)\le f(x_1,x_2).$$

On the other hand, from extensionality we have

$$E(x_1,c_1)*E(x_2,c_2)\circledast f(x_1,x_2)\le f(c_1,c_2),$$

and because of (1)

$$f(x_1,x_2)\le E(x_1,c_1)*E(x_2,c_2)\rightarrow_{\circledast} f(c_1,c_2),$$

$$f(x_1,x_2)\le \bigwedge_{c_1,c_2\in M}(E(x_1,c_1)*E(x_2,c_2)\rightarrow_{\circledast} f(c_1,c_2)),$$

which implies that $f(x_1,x_2)=\overline{f_{CNF}}(x_1,x_2)$.

Now, we are going to prove an analogous result for additive normal forms. We find out that extensionality of f w.r.t. E is deficient requirement in this case and we have to demand some kind of orthogonality from the reflexive binary fuzzy relation E.

Theorem 5 *Let $f(x_1, \ldots, x_n)$ be an n-ary fuzzy relation and $E(x, y)$ be a reflexive binary fuzzy relation on M. Moreover, let E fulfills the orthogonality condition of the infinite type. Then*

$$\overline{f_{ANF}}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n). \tag{24}$$

for all $x_1, \ldots, x_n \in M$.

Proof. 2 *The case $n = 2$. Since*

$$E(x_1, x_1) * E(x_2, x_2) = 1$$

we obtain the following equality

$$\bigoplus_{\substack{c_1, c_2 \in M \\ c_1, c_2 \neq x_1, x_2}} (E(c_1, x_1) * E(c_2, x_2)) = 0. \tag{25}$$

From (25), reflexivity of E and boundary condition of a any t-norm we have

$$\overline{f_{ANF}}(x_1, x_2) = \bigoplus_{c_1, c_2 \in M} ((E(c_1, x_1) * E(c_2, x_2)) \circledast f(c_1, c_2)) =$$

$$\bigoplus_{\substack{c_1, c_2 \in M \\ c_1, c_2 \neq x_1, x_2}} ((E(c_1, x_1) * E(c_2, x_2)) \circledast f(c_1, c_2)) \oplus$$

$$\oplus (E(x_1, x_1) * E(x_2, x_2) \circledast f(x_1, x_2)) = f(x_1, x_2).$$

4 Discrete normal forms and their approximation abilities

Normal forms from this section are introduced especially with the aim to have approximations of a fuzzy relation with arbitrary precision. The information about the error of approximation is contained in the below proved condition of conditional equivalence.

Definition 12 Let $f(x_1, \ldots, x_n)$ be an n-ary fuzzy relation, $E(x, y)$ be a binary fuzzy relation on M and $*, \circledast$ be left-continuous t-norms.

The following formulas are the discrete disjunctive normal form of f, the discrete conjunctive normal form of f and the discrete additive normal form of f

$$f_{DNF}(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_n = 1}^{k} (E(c_{i_1}, x_1) * \cdots * E(c_{i_n}, x_n) \circledast f(c_{i_1}, \ldots, c_{i_n})) \tag{26}$$

$$f_{CNF}(x_1, \ldots, x_n) = \bigwedge_{i_1, \ldots, i_n = 1}^{k} (E(x_1, c_{i_1}) * \cdots * E(x_n, c_{i_n}) \rightarrow_{\circledast} f(c_{i_1}, \ldots, c_{i_n})), \tag{27}$$

$$f_{ANF}(x_1, \ldots, x_n) = \bigoplus_{i_1, \ldots, i_n = 1}^{k} (E(c_{i_1}, x_1) * \cdots * E(c_{i_n}, x_n) \circledast f(c_{i_1}, \ldots, c_{i_n})), \tag{28}$$

respectively.

Similarly to the infinite case, we introduce a generalized orthogonality of the finite type.

Definition 13 Let E be a binary fuzzy relation on M and $*$ be a t-norm. We say that E fulfills the orthogonality property of the finite type if

$$\bigoplus_{\substack{i_1, \ldots, i_n = 1 \\ i_1, \ldots, i_n \neq j_1, \ldots, j_n}}^{k} (E(c_{i_1}, x_1) * \cdots * E(c_{i_n}, x_n)) = 1 - (E(c_{j_1}, x_1) * \cdots * E(c_{j_n}, x_n)). \tag{29}$$

is valid for each $x_1, \ldots, x_n \in M$.

As we will see, the discrete disjunctive and/or conjunctive normalforms give lower and/or upper approximation of an extensionalfuzzy relation, respectively. Moreover, the reflexivity requirement on E is not necessary in this case.

Proposition 1 *Let $f(x_1,\dots,x_n)$ be an n-ary fuzzy relation, $E(x,y)$ be a binary fuzzy relation on M and $*,\circledast$ be left-continuous t-norms.*

If f is extensional w.r.t E and $*,\circledast$ then

$$f(x_1,\dots,x_n) \geq f_{DNF}(x_1,\dots,x_n), \tag{30}$$
$$f(x_1,\dots,x_n) \leq f_{CNF}(x_1,\dots,x_n). \tag{31}$$

for all $x_1,\dots,x_n \in M$.

Proof. 3 *It is sufficient to prove the case $n=2$. From extensionality of f we have*

$$E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}) \leq f(x_1,x_2),$$

$$\bigvee_{i_1,i_2=1}^{k} E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}) \leq f(x_1,x_2),$$

and that is why $f(x_1,x_2) \geq f_{DNF}(x_1,x_2)$.

Similarly,

$$E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(x_1,x_2) \leq f(c_{i_1},c_{i_2}),$$

and (1) implies

$$f(x_1,x_2) \leq E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \rightarrow_{\circledast} f(c_{i_1},c_{i_2}),$$

$$f(x_1,x_2) \leq \bigwedge_{i_1,i_2=1}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \rightarrow_{\circledast} f(c_{i_1},c_{i_2})),$$

i.e. $f(x_1,x_2) \leq f_{CNF}(x_1,x_2)$.

Considering the symmetry of the binary fuzzy relation E, we are able to prove the following relationship between disjunctive and conjunctive normal forms with respect to additive normal form.

Proposition 2 *Let $f(x_1,\dots,x_n)$ be an extensional n-ary fuzzy relation w.r.t. binary reflexive and symmetric fuzzy relation E and left-continuous t-norms $*,\circledast$ and moreover w.r.t. E and $*,\otimes$, where $\otimes$ is Lukasiewicz t-norm. Furthermore, the orthogonality condition (29) holds truth. Then*

$$f_{DNF}(x_1,\dots,x_n) \leq f_{ANF}(x_1,\dots,x_n), \tag{32}$$
$$f_{ANF}(x_1,\dots,x_n) \leq f_{CNF}(x_1,\dots,x_n), \tag{33}$$

for all $x_1,\dots,x_n \in M$, where f_{ANF}, f_{DNF} are built with help of $*,\circledast$, while f_{CNF} is constructed with help of $*,\otimes$.

Proof. 4 *Since the fact that supremum is the least t-conorm, we obtain that $f_{DNF}(x_1,\dots,x_n) \leq f_{ANF}(x_1,\dots,x_n)$.*

It is sufficient to consider the case $n=2$.

$$E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2}) \leq E(x_1,c_{i_1}) * E(x_2,c_{i_2}),$$

$$\bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_1})) \leq \bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2})),$$

and easily from the orthogonality assumption we have

$$\bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2})) \leq 1 - E(x_1,c_{j_1}) * E(x_2,c_{j_2}),$$

$$\bigoplus_{i_1,i_2=1}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2})) \leq$$
$$(1 - E(x_1,c_{j_1}) * E(x_2,c_{j_2})) \oplus (E(x_1,c_{j_1}) * E(x_2,c_{j_2}) \circledast f(c_{j_1},c_{j_2})),$$
$$\bigoplus_{i_1,i_2=1}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2})) \leq (1 - E(x_1,c_{j_1}) * E(x_2,c_{j_2})) \oplus f(c_{j_1},c_{j_2}).$$

From the property (5) we obtain

$$\bigoplus_{i_1,i_2=1}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2})) \leq E(x_1,c_{j_1}) * E(x_2,c_{j_2}) \rightarrow_{L} f(c_{j_1},c_{j_2}),$$

and finally

$$\bigoplus_{i_1,i_2=1}^{k} (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \circledast f(c_{i_1},c_{i_2})) \leq \bigwedge_{j_1,j_2=1}^{k} (E(x_1,c_{j_1}) * E(x_2,c_{j_2}) \rightarrow_{L} f(c_{j_1},c_{j_2})),$$

which implies

$$f_{ANF}(x_1,\ldots,x_n) \leq f_{CNF}(x_1,\ldots,x_n).$$

Let us illustrate the relationships between normal forms on the following example.

Example 4 *Let us consider the following one-dimensional case where the approximated fuzzy relation*

$$f(x) = \sin(x) + 0.1$$

is defined on $M = [0,1]$. A binary fuzzy relation E is given as

$$E(x,y) = (x \leftrightarrow_{L} y)^9,$$

while the nodes c_i are defined as $c_i = (i-1)/k$ for $i = 1,\ldots,10$. Finally, let $\circledast$ be product t-norm.

Then, we obtain a relationship between conjunctive, disjunctive and additive normal forms which is illustrated on Figure 1. From Figure 1(c), it is clear that additive normal form is absolutely the best approximation formula from the set of normal forms for the function f with respect to E and such a number and distribution of the nodes c_i over M. This fact immediately follows from Proposition 1 and Proposition 2.

It has been mentioned that a conditional equivalence of the form (3) gives a lower boundary for the value of an equivalence (biresiduum) between the normal forms and the original fuzzy relation. Considering a t-norm with the additive generator g, we can rewrite (3) into the following form

$$|\, g(\text{Extensional Formula}) - g(\text{Normal Form})| \leq \text{Error} = g(\text{Condition}),$$

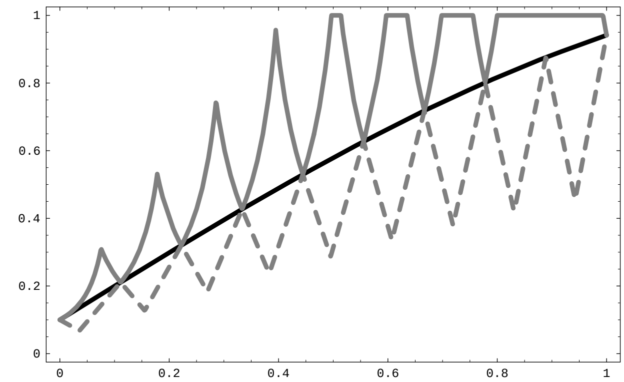
which allows us to speak about an approximation on the pseudo-metric space generated by g. Let us remind that $g(S(x,y))$ defines a pseudo-metric on M if S is a similarity relation on M.

Theorem 6 *Let all the assumptions of Proposition 1 be valid.*
If f is extensional w.r.t E and t-norms $*, \circledast$ then

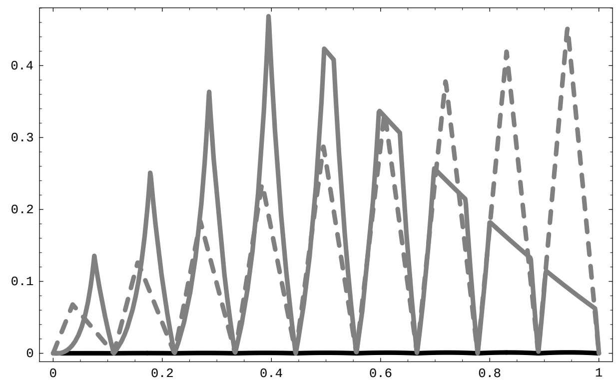
$$f_{DNF}(x_1,\ldots,x_n) \leftrightarrow_{\circledast} f(x_1,\ldots,x_n) \geq C(x_1,\ldots,x_n), \quad (34)$$
$$f_{CNF}(x_1,\ldots,x_n) \leftrightarrow_{\circledast} f(x_1,\ldots,x_n) \geq C(x_1,\ldots,x_n) \quad (35)$$

for all $x_1,\ldots,x_n \in M$, where

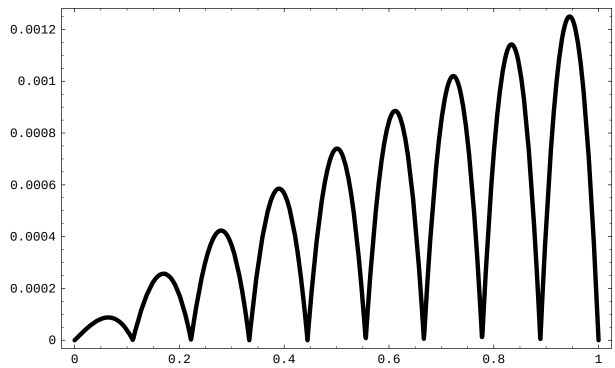
$$C(x_1,\ldots,x_n) =$$
$$\bigvee_{i_1,\ldots,i_n}^{k} (E(x_1,c_{i_1}) * \cdots * E(x_n,c_{i_n})) \circledast (E(c_{i_1},x_1) * \cdots * E(c_{i_n},x_n)). \quad (36)$$



(a) A relationship between normal forms



(b) An error of approximation



(c) An error of approximation by additive normal form

Figure 1: An illustration of the approximation abilities of all three given normal forms for the fuzzy relation from Example 4. The black line represents $f_{ANF}(x)$, the dashed gray line is for $f_{DNF}(x)$ and the smooth gray line belongs to $f_{CNF}(x)$.

Proof. 5 *Only the case* $n = 2$ *will be considered. From extensionality*

$$(E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast f(x_1, x_2) \leq f(c_{i_1}, c_{i_2})$$

and monotonicity of t-norms we obtain

$$(E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \circledast (E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast f(x_1, x_2) \leq (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \circledast f(c_{i_1}, c_{i_2}),$$

$$(E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \circledast (E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast f(x_1, x_2) \leq f_{DNF}(x_1, x_2).$$

Now, we apply (1)

$$\bigvee_{i_1, i_2 = 1}^{k} (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \circledast (E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \leq f(x_1, x_2) \rightarrow_{\circledast} f_{DNF}(x_1, x_2),$$

and since

$$f_{DNF}(x_1, x_2) \leq f(x_1, x_2),$$

we obtain

$$1 \leq f_{DNF}(x_1, x_2) \rightarrow_{\circledast} f(x_1, x_2)$$

what proves (34).

Using $(\varphi_1 \rightarrow_{\circledast} \psi_1) \circledast (\varphi_2 \rightarrow_{\circledast} \psi_2) \leq (\varphi_1 \circledast \varphi_2) \rightarrow_{\circledast} (\psi_1 \circledast \psi_2)$ *we obtain*

$$(E(x_1, c_{i_1}) * E(x_2, c_{i_2}) \rightarrow_{\circledast} f(c_{i_1}, c_{i_2})) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2) \rightarrow_{\circledast} E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \leq$$

$$((E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2))) \rightarrow_{\circledast} (f(c_{i_1}, c_{i_2}) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2))).$$

Since the property (1)

$$(E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \rightarrow_{\circledast} (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) = 1$$

and applying (1) twice gives us

$$(E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \leq$$

$$((E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \rightarrow_{\circledast} f(c_{i_1}, c_{i_2})) \rightarrow_{\circledast} (E(c_{i_1}, x_1) * E(c_{i_2}, x_2) \circledast f(c_{i_1}, c_{i_2})).$$

Because of the property (1) of the residuation operation

$$(E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \leq$$

$$\bigwedge_{i_1, i_2 = 1}^{k} ((E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \rightarrow_{\circledast} f(c_{i_1}, c_{i_2})) \rightarrow_{\circledast} (E(c_{i_1}, x_1) * E(c_{i_2}, x_2) \circledast f(c_{i_1}, c_{i_2})).$$

Finally, from extensionality

$$(E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \circledast f(c_{i_1}, c_{i_2}) \leq f(x_1, x_2)$$

and with help of (1) we obtain

$$\bigvee_{i_1, i_2 = 1}^{k} (E(x_1, c_{i_1}) * E(x_2, c_{i_2})) \circledast (E(c_{i_1}, x_1) * E(c_{i_2}, x_2)) \leq f_{CNF}(x_1, x_2) \rightarrow_{\circledast} f(x_1, x_2).$$

And similarly to the proof of (34) since

$$f(x_1, x_2) \leq f_{CNF}(x_1, x_2)$$

we obviously obtain the proof of (35).

Conditional equivalence for additive normal form can be proved under the much strict requirements. Additionally, this equivalence is only of the Łukasiewicz sense.

Theorem 7 *Let all the assumptions of Proposition 1 be valid. Moreover, let f is extensional w.r.t E and $*, \circledast$, f is extensional w.r.t. E and t-norms $*, \otimes$, where $\otimes$ is Łukasiewicz t-norm, and let the orthogonality condition of the finite type holds truth. Then*

$$f_{ANF}(x_1,\dots,x_n) \leftrightarrow_L f(x_1,\dots,x_n) \geq C(x_1,\dots,x_n), \tag{37}$$

for $x_1,\dots,x_n \in M$, where C is given by (36).

Proof. 6 *The case $n=2$ is sufficient. Easily, due to the fact that maximum is the least t-conorm*

$$f_{ANF}(x_1,x_2) \geq \bigvee_{i_1,i_2=1}^{k} E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}),$$

from extensionality w.r.t. $E, *, \otimes$: $E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \otimes f(x_1,x_2) \leq f(c_{i_1},c_{i_2})$ we obtain

$$f_{ANF}(x_1,x_2) \geq \bigvee_{i_1,i_2=1}^{k} (E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast (E(x_1,c_{i_1}) * E(x_2,c_{i_2}) \otimes f(x_1,x_2))),$$

$$f_{ANF}(x_1,x_2) \geq \bigvee_{i_1,i_2=1}^{k} ((E(c_{i_1},x_1) * E(c_{i_2},x_2)) \circledast (E(x_1,c_{i_1}) * E(x_2,c_{i_2}))) \otimes f(x_1,x_2),$$

and from (1)

$$f(x_1,x_2) \rightarrow_L f_{ANF}(x_1,x_2) \geq C(x_1,x_2).$$

The other side can be proven as follows

$$E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}) \leq E(c_{i_1},x_1) * E(c_{i_2},x_2),$$

$$\bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}) \leq \bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} E(c_{i_1},x_1) * E(c_{i_2},x_2),$$

$$\bigoplus_{\substack{i_1,i_2=1\\ i_1,i_2\neq j_1,j_2}}^{k} E(c_{i_1},x_1) * E(c_{i_2},x_2) \circledast f(c_{i_1},c_{i_2}) \leq (1 - E(c_{j_1},x_1) * E(c_{j_2},x_2)),$$

$$f_{ANF}(x_1,x_2) \leq (1 - E(c_{j_1},x_1) * E(c_{j_2},x_2)) \oplus (E(c_{j_1},x_1) * E(c_{j_2},x_2) \circledast f(c_{j_1},c_{j_2})),$$

what is due to (5)

$$f_{ANF}(x_1,x_2) \leq (E(c_{j_1},x_1) * E(c_{j_2},x_2)) \rightarrow_L (E(c_{j_1},x_1) * E(c_{j_2},x_2) \circledast f(c_{j_1},c_{j_2})),$$

and because of (1)

$$(E(c_{j_1},x_1) * E(c_{j_2},x_2)) \otimes f_{ANF}(x_1,x_2) \leq E(c_{j_1},x_1) * E(c_{j_2},x_2) \circledast f(c_{j_1},c_{j_2}).$$

Now, we apply extensionality w.r.t. $E, *, \circledast$

$$(E(c_{j_1},x_1) * E(c_{j_2},x_2)) \otimes f_{ANF}(x_1,x_2) \leq f(x_1,x_2),$$

and again since 1

$$(E(c_{j_1},x_1) * E(c_{j_2},x_2)) \leq f_{ANF}(x_1,x_2) \rightarrow_L f(x_1,x_2),$$

$$(E(c_{j_1},x_1) * E(c_{j_2},x_2)) \circledast (E(x_1,c_{j_1}) * E(x_2,c_{j_2})) \leq f_{ANF}(x_1,x_2) \rightarrow_L f(x_1,x_2),$$

which follows

$$C(x_1,x_2) \leq f_{ANF}(x_1,x_2) \rightarrow_L f(x_1,x_2),$$

If we work with symmetrical E and Łukasiewicz t-norm then we obtain a concrete case of the introduced conditional equivalence of the additive normal form as a straight corollary of the Proposition 2. The fact that the additive normal form lies between the disjunctive and the conjunctive normal forms directly implies the following result.

Corollary 1 *Let f be an extensional fuzzy relation w.r.t. binary fuzzy relation E and the left-continuous t-norms $*, \otimes$, where $\otimes$ is Łukasiewicz t-norm. Moreover, let E fulfills the orthogonality condition (29). If E is reflexive and symmetric then*

$$f_{ANF}(x_1, \ldots, x_n) \leftrightarrow_{\circledast} f(x_1, \ldots, x_n) \geq C(x_1, \ldots, x_n) \tag{38}$$

for all $x_1, \ldots, x_n \in M$, where C is a modification of (36), where $\circledast$ is replaced by $\otimes$.

5 F-transform as an example of additive normal forms

In this section we deal with a fuzzy approximation method called fuzzy transform (F-transform) [12, 13, 15]. The main idea of F-transform consists in the replacement of an original continuous function by its simplified discrete representation in complex computations. Results of such computations are later transformed back to the space of continuous functions and reflect in additive normal form.

5.1 Original concept of F-transform

This subsection consist of the original definitions of F-transform (direct and inverse) mainly taken from [12, 13] for the 1-dimensional case and from [15] for its generalization to the higher dimension. In the sequel, an interval $[a, b]$ of real numbers will be denoted by symbol M.

Definition 14 Let $c_i = a + h \cdot (i-1)$ be nodes on M where $h = (b-a)(k-1),\ k \geq 2$ and $i = 1, \ldots, k$. We say that functions $A_1(x), \ldots A_k(x)$ defined on M are *basic functions* if each of them fulfills the following conditions:

- $A_i : M \to [0, 1],\ A_i(c_i) = 1$,
- $A_i(x) = 0$ if $x \notin (c_{i-1}, c_{i+1})$ where $c_{-1} = a,\ c_{k+1} = b$,
- $A_i(x)$ is continuous,
- $A_i(x)$ strictly increases on $[c_{i-1}, c_i]$ and strictly decreases on $[c_i, c_{i+1}]$,
- $\sum_{i=1}^{k} A_i(x) = 1$, for all $x \in M$

These basic functions forming together a fuzzy partition of M play a crucial role in further definitions.

Definition 15 Let $f(x)$ be any continuous function on M and $A_1, \ldots, A_k(x)$ are basic functions forming fuzzy partition. We say that the k-tuple of real numbers $[F_1, \ldots, F_k]$ is the *F-transform* of f with respect to $A_1(x), \ldots, A_k(x)$ if

$$F_i = \frac{\int_a^b f(x) A_i(x)\, dx}{\int_a^b A_i(x)\, dx} \tag{39}$$

The *components* F_i of the F-transform serve us as a discrete representation of values of f above the non-zero domains of A_i. In fact, we are averaging all the values above such intervals $[c_{i-1}, c_{i+1}]$ and these fuzzy sets A_i are used as weights in this averaging.

To obtain an approximation of function f we must somehow transform the discrete representation back to the space of continuous functions. For this purpose an inverse F-transform is used.

Definition 16 Let $[F_1, \ldots, F_k]$ be the F-transform of a function $f(x)$ with respect to $A_1(x), \ldots, A_k(x)$. The function

$$f_k^F(x) = \sum_{i=1}^{k} F_i A_i(x) \tag{40}$$

will be called the *inverse F-transform.*

The concept of F-transform can be straightly generalized for functions with more variables. For instance, let us consider a continuous function with two variables $f(x_1, x_2)$ defined on a domain $M = [a, b]^2$. Then the formula defining the F-transform is modified into the following one

$$F_{ij} = \frac{\int_a^b \int_a^b f(x_1, x_2) A_i(x_1) A_j(x_2) dx_1 dx_2}{\int_a^b \int_a^b A_i(x_1) A_j(x_2) dx_1 dx_2}. \tag{41}$$

And analogously to the one-dimensional case, the inverse F-transform is given as follows

$$f_n^F(x_1, x_2) = \sum_{i,j=1}^{k} A_i(x_1) A_j(x_2) F_{ij}. \tag{42}$$

5.2 F-transform for fuzzy relations

Here, we debunk F-transform as a special case of additive normal form and so, we can bring to bear all the results from the theory of the normal forms introduced above on this special additive normal forms.

Let $M = [a, b] \subseteq \mathbb{R}$ and f be a fuzzy set $f : M \to [0, 1]$ and let E_k be defined as follows

$$E_k(x, y) = (T(x) \leftrightarrow_{\text{L}} T(y))^k, \tag{43}$$

where $k \in \mathbb{N}$ and $T : M \to [0, 1]$ is given by

$$T(x) = \frac{x - a}{b - a}. \tag{44}$$

It is clear that E_k is similarity relation for each k. In general, T can be arbitrary continuous strictly increasing function such that $T(a) = 0$ and $T(b) = 1$.

As the next step, we will introduce a generalized F-transform for a class of fuzzy sets with continuous membership functions. Taking into account the phenomenon of vagueness then the continuity of fuzzy sets is a natural requirement.

Definition 17 Let f be a fuzzy set on M with the continuous membership function and E_k be a binary fuzzy relation defined by (43). Then a fuzzy set F_k given by

$$F_k(x) = \frac{\int_a^b E_k(x, y) \odot f(y) dy}{\int_a^b E_k(x, y) dy}, \tag{45}$$

is called the F-transform of $f(x)$ w.r.t. E_k.

The following lemma relates to the extensionality of the F-transform.

Lemma 2 *Let E_k and F_k be as above. Then, F_k is extensional w.r.t. E_k and product t-norm $\odot$.*

Proof. 7 *From the transitivity and symmetry of E_k, we obtain the following inequalities*

$$E_k(x, y) \odot E_k(x, z) \leq E_k(y, z) \quad \textit{and}$$
$$E_k(x, y) \odot E_k(y, z) \leq E_k(x, z),$$

the monotonicity of $\odot$ *implies that*

$$E_k(x,y) \odot E_k(x,z) \odot f(z) \leq E_k(y,z) \odot f(z),$$

and then

$$\int_a^b E_k(x,y) \odot E_k(x,z) \odot f(z)\,dz \leq \int_a^b E_k(y,z) \odot f(z)\,dz,$$

$$\int_a^b E_k(x,y) \odot E_k(y,z)\,dz \leq \int_a^b E_k(x,z)\,dz,$$

because we integrate over z*, we have*

$$E_k(x,y) \odot \int_a^b E_k(x,z) \odot f(z)\,dz \leq \int_a^b E_k(y,z) \odot f(z)\,dz,$$

$$E_k(x,y) \odot \int_a^b E_k(y,z)\,dz \leq \int_a^b E_k(x,z)\,dz,$$

the property (1) of $\odot$ *follows*

$$E_k(x,y) \leq \int_a^b E_k(x,z) \odot f(z)\,dz \rightarrow_\odot \int_a^b E_k(y,z) \odot f(z)\,dz,$$

$$E_k(x,y) \leq \int_a^b \frac{1}{E_k(x,z)}\,dz \rightarrow_\odot \int_a^b \frac{1}{E_k(y,z)}\,dz,$$

and thus

$$E_k(x,y) \leq \left(\int_a^b E_k(x,z) \odot f(z)\,dz \rightarrow_\odot \int_a^b E_k(y,z) \odot f(z)\,dz \right) \odot$$
$$\odot \left(\int_a^b \frac{1}{E_k(x,z)}\,dz \rightarrow_\odot \int_a^b \frac{1}{E_k(y,z)}\,dz \right),$$

since $\odot$ *satisfies* $(a_1 \rightarrow_\odot a_2) \odot (b_1 \rightarrow_\odot b_2) \leq (a_1 \odot b_1 \rightarrow_\odot a_2 \odot b_2)$ *then*

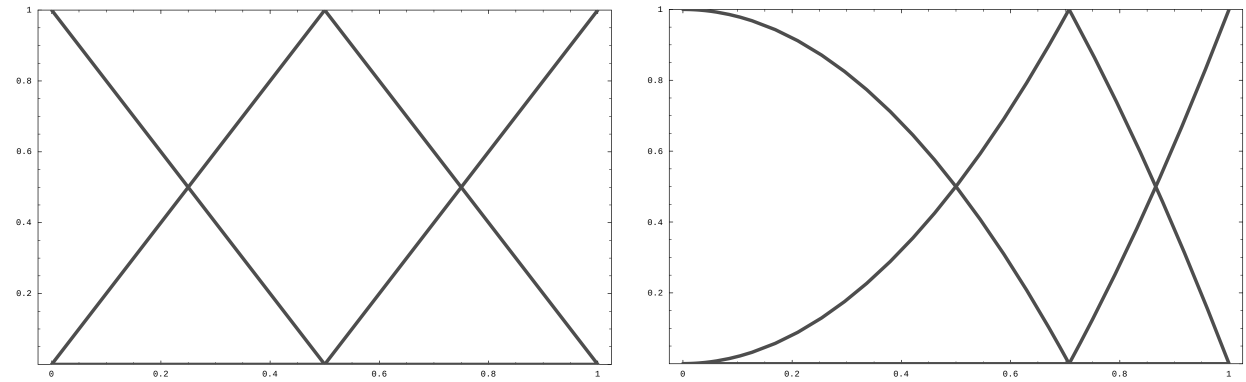
$$E_k(x,y) \leq \frac{\int_a^b E_k(x,z) \odot f(z)\,dz}{\int_a^b E_k(x,z)\,dz} \rightarrow_\odot \frac{\int_a^b E_k(y,z) \odot f(z)\,dz}{\int_a^b E_k(y,z)\,dz},$$

or equivalently

$$E_k(x,y) \leq F_k(x) \rightarrow_\odot F_k(y).$$

Now, let us repeat the formula defining the orthogonality property for E_k

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{k+1} E_k(c_i,x) = 1 - E_k(c_j,x). \tag{46}$$



(a) $M = [0, 1]$ and $T(x)$ is given by (44) (b) $M = [0, 1]$ and $T(x) = x^2$

Figure 2: An example of basic functions given as $E_k(c_i, x)$, where c_i for $i = 1, \ldots, k + 1$ are distributed over $[a, b]$ such that E_k fulfills the orthogonality property.

This leads to $(k + 1)$ equidistant nodes $\hat{c}_i = (i - 1)/k$, $i = 1, \ldots, (k + 1)$ on $[0, 1]$, which define nodes $c_i \in M$ as $c_i = T^{-1}(\hat{c}_i)$.

It is worth of mentioning that fuzzy relation $E_k(c_i, x)$, where nodes c_i are chosen to hold the orthogonality condition (46), determine so called basic functions from Definition 14 of the triangular shape (see figure 2). Moreover, let us stress that values $F_k(c_i)$ exactly correspond to the components of the F-transform F_i from Definition 15.

Lemma 3 *Let E_k be a binary fuzzy relation on $M = [a, b]$ given by (43) and $c_1 = a$. Then, E_k fulfills the orthogonality property of the finite type (46) if and only if $c_i = T^{-1}((i - 1)/k)$, $i = 1, \ldots, (k + 1)$.*

Proof. 8 *Let us denote $\hat{x} = T(x)$ and $\hat{y} = T(y)$ and evaluate relation $E_k(x,y)$. Keeping in mind that $x^k = x \otimes x^{(k-1)}$ we get that*

$$E_k(x, y) = (\hat{x} \leftrightarrow_L \hat{y})^k = 0 \wedge (1 - k|\hat{x} - \hat{y}|).$$

Now, we assume that there exist $\hat{c}_i \in [0, 1]$ for $i = 1, \ldots, k+1$ such that they the following orthogonality condition

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{k+1} E_k(c_i, x) = 1 - E_k(c_j, x),$$

is valid for arbitrary j and $x \in M$.

For the left hand side we write

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{k+1} E_k(c_i, x) = 1 \wedge \sum_{\substack{i=1 \\ i \neq j}}^{k+1} (0 \vee (1 - k|\hat{c}_i - \hat{x}|)),$$

while for the right hand side the following holds

$$1 - E_k(c_j, x) = 1 - (0 \wedge (1 - k|\hat{c}_j - \hat{x}|))$$

and both sides are equal.

This equality must hold for each $x \in M$ and thus also for $x = c_j$, where $j \in \{1, \ldots, k + 1\}$. Let us fix some j and put $x = c_j$ i.e. $\hat{x} = \hat{c}_j$. Then, the right hand side obviously equals to 0. Further, the left hand side equals to 0 if and only if all the summands from the left hand side equal to 0 i.e.

$$0 \wedge (1 - k|\hat{c}_i - \hat{c}_j|) = 0 \quad \textit{for all } i = 1, \ldots, k + 1,\ i \neq j,$$

which implies $|\hat{c}_i - \hat{c}_j| \geq 1/k$. *Since* j *has been chosen arbitrarily we obtain*

$$|\hat{c}_i - \hat{c}_j| \geq 1/k \quad \text{for all } i, j = 1, \ldots, k+1,\ i \neq j. \tag{47}$$

If we fix $\hat{c}_1 = 0$ *then only a distribution of the nodes* $\hat{c}_i$ *given by* $\hat{c}_i = (i-1)/k$, *for* $i = 1, \ldots, k+1$, *fulfills the condition (47), which proves the claim of this lemma.*

Let us use the following denotations

$$E_\infty(x, y) = \lim_{k \to \infty} E_k(x, y) = \begin{cases} 1 & x = y, \\ 0 & \text{otherwise.} \end{cases}$$
$$F_\infty(x) = \lim_{k \to \infty} F_k(x) = f(x),$$

If we specify an additive normal form of the infinite type such that it gives the formula of the F-transform then we can formulate the following result.

Proposition 3 *The infinite variant of the orthogonality (21) is fulfilled only by* E_∞ *and additionally*

$$\overline{f_{FT}}(x) = \overline{f_{ANF}}(x) = \bigoplus_{c \in M} (E_\infty(c, x) \odot F_\infty(c)) = f(x). \tag{48}$$

Proof. 9 *It is easy to see that* $\overline{f_{FT}}(x) = f(x)$.

Let us prove the first claim by contradiction. Assume that the orthogonality (21) is fulfilled by E_k

$$\bigoplus_{\substack{c \in M \\ c \neq d}} E_k(c, x) = 1 - E_k(d, x)$$

for some finite $k < \infty$. *Let us choose* $d = x$. *Then the right hand side of the infinite orthogonality is as follows*

$$1 - E_k(d, x) = 1 - E_k(x, x) = 0.$$

That means that

$$\bigoplus_{\substack{c \in M \\ c \neq d}} E_k(c, x) = 0.$$

It is possible if and only if $E_k(c, x) = 0$ *for all* $c \neq x$. *But then* $k = \infty$, *which contradicts with the assumption. Thus, (21) holds only for* E_∞.

For practical applications we need to have a finite discrete case of an approximating formula.

Definition 18 Let $f : M \to [0, 1]$ be a fuzzy set, $T : M \to [0, 1]$ be a transformation function and E_k be given by formula (43). Furthermore, let $F_k(x)$ be the F-transform of f w.r.t. E_k. Then the additive normal form of $F_k(x)$ w.r.t. E_k and the product t-norm $\odot$

$$f_{FT}^k(x) = \bigoplus_{i=1}^{(k+1)} (E_k(c_i, x) \odot F_k(c_i)) \tag{49}$$

will be called the discrete F-transform of f w.r.t. E_k.

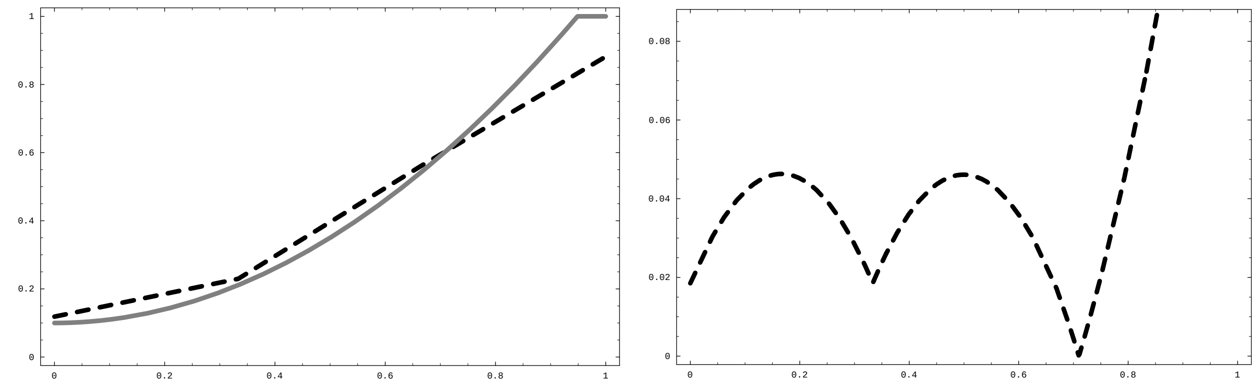
In the following proposition, we show the conditional equivalence for F-transform.

Proposition 4 *Let* f, F_k *and* E_k *be as above and* E_k *satisfies the orthogonality requirement for* $c_1, \ldots, c_{k+1}$ *then*

$$C(x) \leq F_k(x) \leftrightarrow_L f_{FT}^k(x), \tag{50}$$

for all $x \in M$. *The condition* $C(x)$ *is given by*

$$C(x) = \bigvee_{i=1}^{k+1} E_k^2(c_i, x).$$



(a) Original function and its approximation for $k = 3$

(b) An error of approximation given by $e(x) = |f^2_{FT}(x) - f(x)|$

Figure 3: An example of f^k_{FT} given by (49) for $f(x) = 1 \wedge (x^2 + 0.1)$ on $M = [0, 1]$.

Proof. 10 *From Lemma 2, we know that F_k is extensional w.r.t. E_k and $\odot$. Using the fact that*

$$E_k(x, y) \otimes p \leq E_k(x, y) \odot p,$$

holds for arbitrary p, we obtain that F_k is also extensional w.r.t. E_k and $\otimes$. Finally, applying Theorem 7, we have

$$\bigvee_{i=1}^{k+1} E^2_k(c_i, x) \leq F_k(x) \leftrightarrow_{\otimes} f^k_{FT}(x). \tag{51}$$

All the results from this section are established for 1-dimensional case with the aim of having better transparency. Nevertheless, a generalization is straightforward and leads to the following formula

$$f^k_{FT}(x_1, \ldots, x_n) = \bigoplus_{i_1, \ldots, i_n = 1}^{(k+1)} (E_k(c_{i_1}, x_1) \odot \cdots \odot E_k(c_{i_n}, x_n) \odot F_k(c_{i_1}, \ldots, c_{i_n})),$$

where F_k is given by

$$F_k(x_1, \ldots, x_n) = \frac{\int_a^b \cdots \int_a^b E_k(x_1, y_1) \odot \cdots \odot E_k(x_n, y_n) \odot f(y_1, \ldots, y_n)\, dy_1 \ldots, dy_n}{\int_a^b \int_a^b E_k(x_1, y_1) \odot \cdots \odot E_k(x_n, y_n)\, dy_1 \ldots, dy_n}.$$

Let us illustrate properties of F-transform f_{FT} on the following example.

Example 5 *Let $f(x) = 1 \wedge (x^2 + 0.1)$ be a fuzzy relation on $M = [0, 1]$ and let $\circledast$ be product t-norm. Considering $k = 3$, we obtain that*

$$|F_k(x) - f^3_{FT}(x)| \leq 1 - \bigvee_{i=1}^{4} E^2_3(c_i, x).$$

The final approximation is depicted on Figure 3.

6 Conclusion

In this paper, we established the basis for further investigations of approximating abilities of normal forms. Such approach to an approximation of extensional fuzzy relations brings the new view on this problematic and simplifies a further exploration of its properties.

From the results of Section 4, it follows that the symmetry plays a significant role in this field of research. This fact has been widely used in the last section concerning F-transform as an special case of additive normal forms.

Moreover, it has been shown that we can estimate an error of the approximation by F-transform for the case of a function with bounded partial derivatives. The estimation is given as a limitation of the condition C from the inequality of the conditional equivalence.

The main goal of this work lies in introducing of F-transform as an eminent part of a larger group of formulas intended for approximations of extensional fuzzy relations.

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