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1 Introduction

This paper presents a model theory respective theory of omissions of types, where the basic mathematical theory is fuzzy logic in narrow sense with evaluated syntax. For this work we assume fuzzy logic with evaluated syntax which is based on Łukasiewicz MV-algebra of truth values (see [3]), but this theory can be constructed for other fuzzy logic with evaluated syntax for example fuzzy logic with evaluated syntax extended by product where the corresponding structure of truth values is Łukasiewicz MV-algebra extended by product. For further discussion of this logic (see [4],[5]) where was shown that this logic is complete and the model theory was studied (substructure, elementary substructure, chain of models).

In this paper we introduce the basic definitions and results of this theory where we suppose special set of evaluated formulas which will be formulated later.

2 Preliminaries

Syntax. We will work with first-order fuzzy logic with functional symbols. The set of truth values is supposed to form the Łukasiewicz MV-algebra

$$\mathcal{L} = \langle L, \otimes, \oplus, \neg, 0, 1 \rangle$$

where $L = [0,1]$, $\otimes$ is the Łukasiewicz conjunction defined by $a \otimes b = 0 \vee (a + b - 1)$, $\oplus$ is called the Łukasiewicz disjunction definded by $a \oplus b = 1 \wedge (a + b)$ and $\neg$ is the negation operation defined by $\neg a = 1 - a$ for all $a, b \in [0,1]$. Other lattice operations can be introduced by

$$a \vee b = (a \otimes \neg b) \oplus b, \quad a \wedge b = (a \oplus \neg b) \otimes b.$$

Moreover, put $a \to b = \neg a \oplus b = 1 \wedge (1 - a + b)$ which means that $\to$ is the Łukasiewicz implication.

The corresponding language of fuzzy logic for my work is denoted by J. The connectives are $\boldsymbol{\vee}$ (disjunction) interpreted by $\vee$, $\boldsymbol{\nabla}$ (Łukasiewicz disjunction) interpreted by $\oplus$, $\wedge$ (conjuction) interpreted by $\wedge$, $\&$ (Łukasiewicz conjuction) interpreted by $\otimes$, $\Rightarrow$ (Łukasiewicz implication) interpreted by $\rightarrow$ and $\neg$ (negation) interpreted by $\neg$. The set of all the well-formed formulas for the language J is denoted by F_J and the set of all the closed terms by M_J. The couple $({}^a/A)$ where $a \in L$ and $A \in F_J$ is called an *evaluated formula*. Furthermore, the language J is supposed to contain *logical constants* $\mathbf{a}$ being names of all the truth values $a \in L$. We write $\top, \bot$ instead of the logical constant $\mathbf{1}, \mathbf{0}$, respectively.

A fuzzy theory T is a fuzzy set of formulas $T \underset{\sim}{\subsetneq} F_J$ given by the triple

$$T = \langle \mathrm{LAx}, \mathrm{SAx}, \mathrm{R} \rangle$$

where $\mathrm{LAx} \underset{\sim}{\subsetneq} F_J$ is a fuzzy set of logical axioms, $\mathrm{SAx} \underset{\sim}{\subsetneq} F_J$ is a fuzzy set of special axioms, and R is a set of sound inference rules which contains the rules of modus ponens (r_{MP}), generalization (r_G) and logical constant introduction (r_{LC}). The example of fuzzy theory can be found in [3] (on the page 141).

Given a fuzzy theory T and a formula A. If w_A is its proof with the value $\mathrm{Val}(w_A)$ then $T \vdash_a A$ means that A is provable in the fuzzy theory T in the degree

$$a = \bigvee \{ \mathrm{Val}(w) | \, w \text{ is a proof of } A \}.$$

If there is a proof w_A such that $\mathrm{Val}(w_A) = a$ then we say that A is effectively provable in T in the degree a.

Definition 1. A fuzzy theory T is *contradictory* if there is a formula A and proofs w_A and $w_{\neg A}$ of A and $\neg A$, respectively, such that

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) > 0.$$

It is *consistent* in the opposite case.

The proof of the following lemma and other details and precise definitions can be found in [3].

Lemma 1. *Let T be a consistent fuzzy theory and $T \vdash_a A$. Then $T \vdash_b \neg A$ where $b \leq \neg a$.*

Semantics. The semantics is defined in by generalization of the classical semantics of predicate logic. The *structure* for the language J is

$$\mathcal{D} = \langle D, P_D, \ldots, f_D, \ldots, u, \ldots \rangle$$

where D is a set, $P_D \underset{\sim}{\subsetneq} D^n$ are n-ary fuzzy relations assigned to each n-ary predicate symbol $P, \ldots$, f_D are ordinary n-ary functions on D assigned to each n-ary functional symbol f, and $u, \ldots \in D$ are designated elements which are assigned to each object constant $\mathbf{u} \in J$. Interpretation of terms and formulas — see [3].

We say that the structure $\mathcal{D}$ is a *model* of the fuzzy theory T and write $\mathcal{D} \models T$, if $\mathrm{SAx}(A) \leq \mathcal{D}(A)$ holds for all formulas $A \in F_J$.

3 Symbols and basic definitions

We introduce definitions and some conventions which will be necessary in the rest part of this paper. In this section we consider some structure $\mathcal{V}$ for the language J introduced above. By symbol $\Sigma(x_1, \ldots, x_n)$ we denote a set of evaluated formulas (a fuzzy set of formulas) where each formula $A(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$ has free variables among $x_1, \ldots, x_n$. It is necessary to say that Σ is *infinite*. In the following we will sometimes write Σ instead of $\Sigma(x_1, \ldots, x_n)$. If a formula A belongs to Σ in the degree $a > 0$ which is *fixed* for every formula, then we will write $A \in_a \Sigma$. Alternatively, we can also write $\Sigma(A) = a$. By A' we denote an instance of the formula $A \in_a \Sigma$ for some fixed $v_1, \ldots, v_n \in V$ where V is a support of the structure $\mathcal{V}$. We can write

$$A' = A({}^{v_1}/x_1, \ldots, {}^{v_n}/x_n). \tag{1}$$

For the next lemmas we put

$$\hat{a} = \bigwedge\{a \mid A \in_a \Sigma\ ,\ \forall a > 0\} \tag{2}$$

and

$$\check{a} = \bigvee\{a \mid A \in_a \Sigma\ ,\ \forall a > 0\}. \tag{3}$$

Satisfiable of formula, interpretation and omission of Σ.

Definition 2. (A is satisfiable in T) Let $A \in F_J$ and T be a fuzzy theory. We say that A is *satisfiable* in T if there exists some $v_1, \ldots, v_n \in V$ and some $\mathcal{V} \models T$ such that

$$\mathcal{V}(A({}^{v_1}/x_1, \ldots, {}^{v_n}/x_n)) = a_{\mathcal{V}} > 0 \tag{4}$$

where $a_{\mathcal{V}} > 0$ is denoted *the degree of satisfiability.*

It is *insatisfiable* in T in the opposite case. In the next definition we will define the maximal degree of satisfiability of formula $A \in F_J$ in the fuzzy theory T.

Definition 3. Let $A \in F_J$ and T be a fuzzy theory . Put

$$a^{\max} = \bigvee\{a_{\mathcal{V}} \mid \mathcal{V} \models T \text{ and } \exists v_1, \ldots, v_n \in V, \mathcal{V}(A({}^{v_1}/x_1, \ldots, {}^{v_n}/x_n)) = a_{\mathcal{V}}\}. \tag{5}$$

Then $a^{\max}$ is denoted *the maximal degree of satisfiability* of A in T.

The minimal degree of satisfiability have no meaning to suppose because $a^{\min}$ (defined as in (5) where besides $\bigvee$ will be $\bigwedge$) can be equal to zero.

Definition 4. (interpretation) We say that Σ is *interpreted* in the structure $\mathcal{V}$ in the degree $c > 0$ if there are $v_1, \ldots v_n \in V$ such that for each evaluated formula $A \in_a \Sigma$

$$\mathcal{V}(A') \geq c \vee a \tag{6}$$

holds where A' is the instance (1) of the formula A.

Definition 5. (omission Σ) Let Σ be a fuzzy set introduced above and $\mathcal{V}$ be a structure. We say that Σ is *a-omitting* if to each n-tuple $v_1, \dots v_n \in V$ of elements there is a formula $A \in_a \Sigma$ such that

$$\mathcal{V}(A') < a \tag{7}$$

holds.

Consistency and inconsistency of Σ with T. Now we introduce the definitions of consistency, inconsistency and isolation of Σ with respect to some fuzzy theory T.

Definition 6. (consistency of formula) Let $A \in_a \Sigma$. We say that A is *consistent* if there exist some structure $\mathcal{V}$ for the language J and $v_1, \dots, v_n \in V$ such that $\mathcal{V}(A(^{v_1}/x_1, \dots, ^{v_n}/x_n)) \geq a$ holds.

Remark 1. (consistency of Σ) We say that Σ is consistent if there exist some structure $\mathcal{V}$ for J and $v_1, \dots, v_n \in V$ such that for every formula $A \in_a \Sigma$, $\mathcal{V}(A(^{v_1}/x_1, \dots, ^{v_n}/x_n)) \geq a$ holds.

Definition 7. (consistency of formula with T) Let $A \in_a \Sigma$ and T be a fuzzy theory. We say that $A(x_1, \dots, x_n)$ is *consistent* with T if there are a model $\mathcal{V} \models T$ and $v_1, \dots, v_n \in V$ such that $\mathcal{V}(A(^{v_1}/x_1, \dots, ^{v_n}/x_n)) \geq a$ holds. Let $A(x_1, \dots, x_n) \in F_J$. We say that A is *consistent* with T if there are a model $\mathcal{V} \models T$ and $v_1, \dots, v_n \in V$ such that $\mathcal{V}(A(^{v_1}/x_1, \dots, ^{v_n}/x_n)) > 0$ holds.

Remark 2. (consistency of Σ with T) We say that Σ is *consistent* with T if there are a model $\mathcal{V} \models T$ and $v_1, \dots, v_n \in V$ such that for every formula $A \in_a \Sigma$, $\mathcal{V}(A(^{v_1}/x_1, \dots, ^{v_n}/x_n)) \geq a$ holds.

Definition 8. (inconsistency of formula with T) Let $A \in_a \Sigma$. We say that A is *inconsistent* with T if it is omitted in *every* model of T, i.e. for every $\mathcal{V} \models T$ and for all n-tuples of elements $v_1, \dots, v_n \in V$, $\mathcal{V}(A') < a$ holds.

Remark 3. (inconsistency of Σ with T) We say that Σ is *inconsistent* with T if every formula from $\mathrm{Supp}(\Sigma)$ is omitted in *every* model of T.

Type and type over T.

Definition 9. (type) Let J be a language and $\Gamma(x_1, \dots, x_n)$ set of evaluated formulas from J. We say that $\Gamma(x_1, \dots, x_n)$ is n-type if it is *maximal consistent* fuzzy set of formulas with T with free variables among $x_1, \dots, x_n$.

Definition 10. (type over T) Let T be a fuzzy theory with the language J and $\Gamma(x_1, \dots, x_n)$ set of evaluated formulas from J. If Γ is *maximal consistent* fuzzy set of formulas with free variables among $x_1, \dots, x_n$. Then Γ is n-type over fuzzy theory T

Complete formula and isolation of Σ in T. Now we will study the definition of complete formula and the definition of isolation of Σ in T which will be generated by some evaluated consistent formula ${}^a/A(x_1, \ldots, x_n)$ with T.

Definition 11. (complete formula in T) Let ${}^a/A(x_1, \ldots, x_n)$ be an evaluated formula and T be a fuzzy theory. We say that A is *complete* in T if there exists the degree of satisfiability $a_{\mathcal{V}} > 0$ of A in T such that $a \leq a_{\mathcal{V}}$ and

$$\Sigma(x_1, \ldots, x_n) = \{{}^b/B(x_1, \ldots, x_n) \mid T \models_b A(x_1, \ldots, x_n) \Rightarrow B(x_1, \ldots, x_n)\} \tag{8}$$

is n-type over T.

It means that $\Sigma(x_1, \ldots, x_n)$ in (8) is a maximal and every $B \in_b \Sigma$ is consistent with T.

Definition 12. (isolation of Σ in T) Let T be a fuzzy theory and Σ be a fuzzy set defined above. Then we say that Σ is *isolated* in a degree $d > 0$ in T if there exists a formula ${}^a/A(x_1, \ldots, x_n)$ which is consistent with T and

$$T \models_c A(x_1, \ldots, x_n) \Rightarrow B(x_1, \ldots, x_n) \tag{9}$$

holds for all $B \in_b \Sigma$ where $d \leq c$ and $b \leq a \otimes c$.

¿From the definition of consistency of Σ and consistency of formula with T follows that T has some model then T must be consistent. From completeness theorem we get $T \vdash_c A(x_1, \ldots, x_n) \Rightarrow B(x_1, \ldots, x_n)$ which mens that $A(x_1, \ldots, x_n) \Rightarrow B(x_1, \ldots, x_n)$ is consistent with T.

If Σ is *not isolated* in a degree $d > 0$ in T then there exists some evaluated formula ${}^a/A(x_1, \ldots, x_n)$ which is consistent with T and for all $B \in_b \Sigma$,

$$T \models_{c'} \neg(A(x_1, \ldots, x_n) \Rightarrow B(x_1, \ldots, x_n)),$$

respectively,

$$T \models_{c'} (A(x_1, \ldots, x_n) \,\&\, \neg B(x_1, \ldots, x_n)$$

holds where $c' \leq \neg c \leq \neg d$ (it must hold for consistency of fuzzy theory T). In the same way we can show that

$$A(x_1, \ldots, x_n) \,\&\, \neg B(x_1, \ldots, x_n)$$

is consistent with T.

4 Results

¿From the special symbols defined above and the definition of interpretation follows the next lemmas.

Lemma 2. *Let $\mathcal{V}$ be a structure and Σ be a fuzzy set defined above. Let $\mathcal{V} \models \Sigma$ and $c \leq \hat{a}$. Then Σ is interpreted in $\mathcal{V}$ in the degree $c > 0$.*

Proof. ¿From the assumption of the lemma we suppose that there exists some model of Σ which means that for every formula $A \in_a \Sigma$ and every $v_1, \ldots, v_n \in V$, $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$ holds. From this $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq \hat{a} \geq c$ holds for every instance (1). Using this we get that

$$\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a \vee c$$

which means that Σ is interpreted in the model $\mathcal{V} \models \Sigma$. □

Before the next lemmas we remind the following property. If there are $v_1, \ldots, v_n \in V$ and structure $\mathcal{V}$ for J such that for every formula $A \in_a \Sigma$, $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$ holds, then $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$.

Lemma 3. *Let $\mathcal{V}$ be a structure and Σ be a fuzzy set defined above. Let Σ be interpreted in $\mathcal{V}$ in the degree $c > 0$ and $c \geq \check{a}$. Then there are $v_1, \ldots, v_n \in V$ such that $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$.*

Proof. Let $\mathcal{V}$ be a structure in which is Σ interpreted in some degree $c > 0$. We want to show that there exists some $v_1, \ldots, v_n \in V$ and for every formula $A \in_a \Sigma$ the inequality $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$ holds. Because Σ is interpreted in the $\mathcal{V}$ in the degree $c > 0$ then we know that there exists some $v_1, \ldots, v_n \in V$ and such that for every formula $A \in_a \Sigma$, $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq c \vee a$ holds. From this we obtain

$$\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq c \geq \check{a} \geq a.$$

for some $v_1, \ldots, v_n \in V$. Using this we get that $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$. □

Lemma 4. *Let T be a fuzzy theory, $\mathcal{V}$ be some model of T. Let Σ be an consistent with T. Then there are $v_1, \ldots, v_n \in V$ such that $\mathcal{V}$ is a model of $\Sigma(v_1, \ldots, v_n)$.*

Proof. Let T be a fuzzy theory, $\mathcal{V}$ be some model of T and Σ be an consistent with T. Then from the definition of consistency of Σ with T there exists some $v_1, \ldots, v_n \in V$ such that for every formula $A \in_a \Sigma$, $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$ holds. Consequently, we get $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$. □

Lemma 5. *Let $^a/A(x_1, \ldots, x_n)$ be a complete formula in fuzzy theory T then A is consistent with T.*

Proof. It follows from the definition of complete formula in T and from the definition of the degree of satisfiability of formula $^a/A(x_1, \ldots, x_n)$. □

In the lemma below we can see the next property of Σ if it is isolated in the fuzzy theory T.

Lemma 6. *If Σ be isolated in the fuzzy theory T then Σ is consistent with T.*

Proof. Put

$$A' = A(^{v_1}/x_1, \ldots, ^{v_n}/x_n) \text{ and } B' = (^{v_1}/x_1, \ldots, ^{v_n}/x_n).$$

Let there exists Σ which is isolated in T. Then, using the definition of isolation of Σ there exist some formula ${}^{a}/A(x_1, \ldots, x_n)$ which is consistent with T, $\mathcal{V} \models T$ and $v_1, \ldots, v_n \in V$ such that $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$. From the definition of isolation of Σ in T we know that for every model $\mathcal{V}' \models T$ and every $v_1, \ldots, v_n \in V'$,

$$\mathcal{V}'(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \Rightarrow \mathcal{V}'(B(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq c.$$

Because $\mathcal{V} \models T$ there exist some $v_1, \ldots, v_n \in V$ such that

$$\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \Rightarrow \mathcal{V}(B(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq c$$

and $\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$ hold. From these two formulas for some elements $v_1, \ldots, v_n \in V$ we get

$$\mathcal{V}(B') \geq \mathcal{V}(A') \otimes (\mathcal{V}(A') \Rightarrow \mathcal{V}(B')) \geq a \otimes c \geq b.$$

Consequently, every $B \in_b \Sigma$ is consistent with T then Σ is consistent with T. □

From the lemmas above you can see the relation between a model of fuzzy theory T and a model of $\Sigma(v_1, \ldots, v_n)$ if Σ is isolated in T.

If Σ is isolated in T then Σ is consistent with T which means that there exists some model of T ($\mathcal{V} \models T$) such that $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$ (using lemma (4.)).

Corollary 1. *If Σ is isolated in fuzzy theory T then there are $v_1, \ldots, v_n \in V$ such that $\mathcal{V} \models \Sigma(v_1, \ldots, v_n)$ where $\mathcal{V} \models T$.*

Lemma 7. *Let T be a fuzzy theory and Σ be inconsistent with T then Σ is not isolated in T.*

Proof. Let Σ be isolated in T. Then there exists some model of T ($\mathcal{V} \models T$) in which is Σ consistent with T (using lemma (4.)). It means that there exists some $v_1, \ldots, v_n \in V$ such that for every formula $A \in_a \Sigma$

$$\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) \geq a$$

holds. We showed that there exists some model $\mathcal{V} \models T$ and $v_1, \ldots, v_n \in V$ such that

$$\mathcal{V}(A(^{v_1}/x_1, \ldots, ^{v_n}/x_n)) < a$$

does not hold for every $v_1, \ldots, v_n \in V$ and every $\mathcal{V} \models T$. It is contradictory with inconsistency of Σ. □

5 Conclusion

This paper studies the theory of omissions of types in fuzzy logic with evaluated syntax. We showed some properties for special set of formulas. The next idea extending this work is to construct theorem about omissions of types as in classical logic.Further question is how to describe this theory for other fuzzy logics (for example for BL-logic which is studied in [2]) where we do not suppose set of evaluated formulas.

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