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Approximation of Fuzzy Functions by Extended Fuzzy Transforms

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Abstract

A fuzzy approximation method called fuzzy transforms for approximation of continuous function is presented in this paper. It is shown how can be fuzzy transforms naturally generalized for functions with more variables. A fuzzy function as an approximated mapping is considered. This leads to an extension of fuzzy transforms for fuzzy function as well as to an extension of generalized fuzzy transforms for fuzzy functions with more variables. It is shown how the proposed method can be used as so called learning to obtain a fuzzy rule base for fuzzy control.

Keywords: Fuzzy sets, Approximation, Fuzzy approximation, Fuzzy transforms, Normal forms, Fuzzy control

1 Introduction

Fuzzy transforms (in short F-transforms) have been already several times introduced in a number of publications. Perfilieva I. presented this technique of approximate representation of continuous functions in [4], its application to numeric methods of integrations and solution of ordinary differential equations in [1, 2]. Another application has been published in [6].

The main idea consists in the replacement of an continuous function on a real closed interval by its discrete representation (using the direct F-transform). Afterwards, the discrete representation is transformed back to the space of continuous functions (using the inverse F-transform). The result, obtained by applying both F-transforms is a good simplified approximation of an original function.

In fuzzy control we work with imprecise data and a crisp function describing some proces is described by a fuzzy relation. And any fuzzy relation can be viewed as a fuzzy function. This leads to an idea to extend the method of F-transforms for fuzzy functions to be able to apply it in fuzzy control.

2 Fuzzy Transforms

This section is devoted to Fuzzy transforms - fuzzy approximation method, first published by Perfilieva I. and Chaldeeva E. in [4]. This technique belongs to the area called numerical methods on the basis of fuzzy approximation models. An interval $[a, b]$ of real numbers will be considered as a common domain of all functions in this section.

Definition 1 *Let* $x_i = a + h(i-1)$ *be nodes on* $[a, b]$ *where* $h = (b-a)(n-1)$, $n \geq 2$ *and* $i = 1, ..., n$. *We say that functions* $A_1(x), \ldots A_n(x)$ *defined on* $[a, b]$ *are basic functions if each of them fulfills the following conditions:*

- $A_i : [a, b] \to [0, 1],\ A_i(x_i) = 1,$
- $A_i(x) = 0$ *if* $x \notin (x_{i-1}, x_{i+1})$ *where* $x_0 = a,\ x_{n+1} = b,$
- $A_i(x)$ *is continuous,*
- $A_i(x)$ *strictly increases on* $[x_{i-1}, x_i]$ *and strictly decreases on* $[x_i, x_{i+1}]$,
- $\sum_{i=1}^{n} A_i(x) = 1$, *for all* x,
- $A_i(x_i - x) = A_i(x_i + x)$, *for all* $x \in [0, h],\ i = 2, \ldots, n-1,\ n > 2,$
- $A_{i+1}(x) = A_i(x - h)$, *for all* $x,\ i = 2, \ldots, n-2,\ n > 2.$

We can say that functions $A_i(x)$ determine a fuzzy partition of real interval $[a, b]$.

The technique of fuzzy transforms is based on two transforms - the direct one and the inverse one. The direct fuzzy transform is a mapping which maps continuous functions on $[a, b]$ into the space of real vectors. The inverse F-transform maps a real vector back to the space of continuous functions. We repeat definitions given in [4].

Definition 2 *1. Let $f(x)$ be arbitrary continuous function on $[a, b]$ and $A_1, \ldots, A_n$ basic functions determining a fuzzy partition of $[a, b]$. We say that an n-tuple of real numbers $[F_1, \ldots F_n]$ is the direct F-transform of f with respect to $A_1, \ldots, A_n$ if*

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}. \tag{1}$$

2. Let $f(x)$ be a function known at nodes $x_1, \ldots x_r \in [a, b]$ and $A_1, \ldots, A_n$ basic functions determining a fuzzy partition of $[a, b]$. We say that an n-tuple of real numbers $[F_1, \ldots F_n]$ is the direct F-transform of f with respect to $A_1, \ldots, A_n$ if

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}. \tag{2}$$

Remark: When the basic functions are fixed we denote the direct F-transform of f by $F[f]$ and write $F[f] = [F_1, \ldots F_n]$.

Definition 3 *Let $F[f] = [F_1, \ldots, F_n]$ be a direct F-transform of a function $f(x)$ with respect to $A_1(x), \ldots, A_n(x)$. The function*

$$f_n^F(x) = \sum_{i=1}^{n} F_i A_i(x) \tag{3}$$

will be called an inverse F-transform.

Function f_n^F given by (3) may be considered as an approximation of function $f(x)$. It has been proved (See e.g. [4],[2]) that the sequence of such approximations given by the inverse F-transforms uniformly converges to the original function.

Moreover, it has been shown that components $F_1, ..., F_n$ given by (2), minimize the following piecewise integral least square criterion

$$\Psi(c_1, ..., c_n) = \int_a^b \left(\sum_{i=1}^{n} (f(x) - c_i)^2 A_i(x) \right) dx \tag{4}$$

and therefore determine the best approximation of $f(x)$ in the following class of approximating functions:

$$\sum_{i=1}^{n} c_i A_i(x), \tag{5}$$

where $c_1, ..., c_n$ are arbitrary real coefficients.

It is worth to mention, that a natural generalization of F-transforms for functions with two and more variables has been done in [7]. This generalization preserves all properties and is based on the following idea.

For simplicity, let us consider a function $f(x, y)$ with two variables which is continuous on $[a, b] \times [c, d]$. At first, we choose basic functions $A_1, ..., A_n$ determining a fuzzy partition of $[a, b]$ and analogously $B_1, ..., B_m$ determining a fuzzy partition of $[c, d]$.Then, formulas for the direct F-transform and the inverse F-transform are as follows:

$$F_{ij} = \frac{\int_c^d \int_a^b f(x, y)A_i(x)B_j(y)dxdy}{\int_c^d \int_a^b A_i(x)B_j(y)dxdy}, \tag{6}$$

$$f_{n,m}^F(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} F_{ij} A_i(x)B_j(y), \tag{7}$$

respectively.

3 Extension for Fuzzy Functions

In fuzzy control, we work with an imprecise data in fuzzy control and a crisp control function $f : X \to Y$ is described by a binary fuzzy relation $\Phi : X \times Y \to [0,1]$ where X, Y are closed real intervals.

Any binary fuzzy relation can be viewed as a mapping (fuzzy function)

$$\Phi : X \to [0,1]^Y, \tag{8}$$

which assigns a fuzzy set $\Phi(x)$ on Y to each node on $x \in X$.

It is a natural idea to extend the technique of F-transforms for approximate representation of control functions to be able to use it in fuzzy control.

At first, let us put some requirements on fuzzy function Φ. We restrict the usage of extended F-transforms only for fuzzy functions with "some" continuity and convexity.

Definition 4 *Let $\Phi : X \times Y \to [0,1]$ be a fuzzy function where X, Y are closed real intervals. Moreover, let $\Phi(x)$ be a convex fuzzy set for all $x \in X$. Then Φ will be called α-continuous if for all $\alpha \in [0,1]$: $\varphi_\alpha^+(x)$ and $\varphi_\alpha^-(x)$ are continuous on X where*

$$\varphi_\alpha^+(x) = \bigvee_{y \in Y} \{y \mid y \in \Phi(x)_\alpha\} \tag{9}$$

$$\varphi_\alpha^-(x) = \bigwedge_{y \in Y} \{y \mid y \in \Phi(x)_\alpha\}. \tag{10}$$

Moreover, if $\Phi(x)$ is a fuzzy set with one-element kernel for all $x \in X$ then $\varphi_1^+(x) = \varphi_1^-(x)$ and we write only $\varphi(x)$.

Applying Zadeh's extensional principal we obtain the following formula for components of extended fuzzy transforms.

Definition 5 *Let $\Phi : X \to [0,1]^Y$ be an α-continuous fuzzy function and $A_1(x), \ldots, A_n(x)$ basic functions forming fuzzy partition of X. We say that n-tuple of fuzzy sets $[\mathcal{F}_1(y), \ldots \mathcal{F}_n(y)]$ on Y is an extended direct F-transform of Φ with respect to $A_1(x), \ldots, A_n(x)$ if*

$$\mathcal{F}_i(y) = \bigvee_{\substack{\xi \in Y^X \\ \frac{\int_X \xi(x) A_i(x) dx}{\int_X A_i(x) dx} = y}} \left(\bigwedge_{x \in X} \Phi(x)(\xi(x)) \right). \tag{11}$$

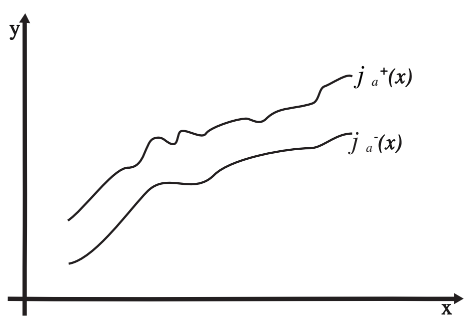

Figure 1: α-continuity of a fuzzy function

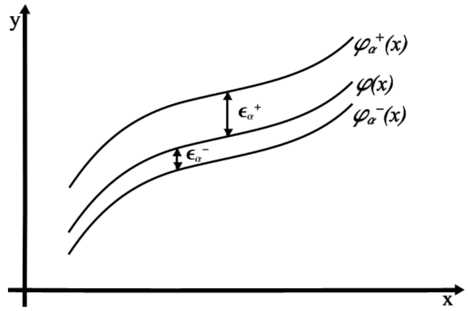

Figure 2: α-continuity of a fuzzy function with one-element kernel and additional smoothness

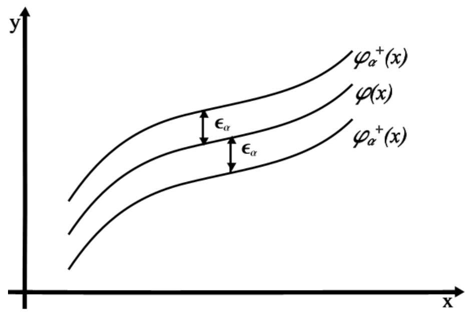

Figure 3: α-continuity of a fuzzy function with one-element kernel and additional symmetry

Remark: Due to the fact that in Def 5 we require α-continuity of Φ, it is not necessary to consider all functions $\xi \in Y^X$. It is sufficient to consider functions ξ which are either equal to $\varphi_\alpha^+(x)$ or $\varphi_\alpha^-(x)$ for some $\alpha \in [0, 1]$.

This simplifies the formula (11), and we obtain

$$\mathcal{F}_i(y^+) = \alpha \quad \text{where} \quad y^+ = \frac{\int_X \varphi_\alpha^+(x) A_i(x) dx}{\int_X A_i(x) dx}, \tag{12}$$

$$\mathcal{F}_i(y^-) = \alpha \quad \text{where} \quad y^- = \frac{\int_X \varphi_\alpha^-(x) A_i(x) dx}{\int_X A_i(x) dx}. \tag{13}$$

Below, we suggest two special simplified cases where we require not only α-continuity, but also some smoothness we additionally require and again one-element kernel of $\Phi(x)$. In the first case, for all $\alpha \in (0, 1)$: $\varphi_\alpha^+(x) = \varphi(x) + \varepsilon_\alpha^+$ and $\varphi_\alpha^-(x) = \varphi(x) + \varepsilon_\alpha^-$ where $\varepsilon_\alpha^+, \varepsilon_\alpha^-$ are real numbers.

From which the following implies (See Figure 2)

$$y^+ = \frac{\int_X \varphi_\alpha^+(x) A_i(x) dx}{\int_X A_i(x) dx} = \frac{\int_X \varphi(x) A_i(x) dx}{\int_X A_i(x) dx} + \frac{\int_X \varepsilon_\alpha^+ A_i(x) dx}{\int_X A_i(x) dx} = F_i + \varepsilon_\alpha^+, \tag{14}$$

$$\text{where} \quad F_i = \frac{\int_X \varphi(x) A_i(x) dx}{\int_X A_i(x) dx}.$$

Similarly, $y^- = F_i - \varepsilon_\alpha^-$.

The second case assumes the same conditions as the first one and moreover, the symmetry should hold truth which means $\varepsilon_\alpha^+ = \varepsilon_\alpha^-$ (see Figure 3).

Finally, we have to define an extended inverse F-transform to obtain an approximate representation of the original fuzzy function Φ.

Definition 6 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be an extended direct F-transform of fuzzy function Φ w.r.t given basic functions $A_1, \ldots, A_n$. Then the fuzzy function*

$$\Phi_n^F(x)(y) = \sum_{i=1}^{n} \mathcal{F}_i(y) A_i(x) \tag{15}$$

is called an extended inverse F-transform.

4 Fuzzy Functions with More Variables

In fuzzy control, we usually meet the situation where control function depends on more than one variable. That is why we have to generalize extended F-transforms for fuzzy functions with more variables.

For the simplicity we introduce the case of fuzzy functions with two variables. Let us be given an α-continuous fuzzy function $\Phi : U \times V \to Y$ where U, V, Y are closed real intervals.

The extension is constructed in the same way as in the case of fuzzy functions with one variable. Having in mind the generalization of F-transform for functions with more variables presented in [7], we construct basic functions $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ determining fuzzy partitions of U, V, respectively. Then, we obtain the following formula

$$\mathcal{F}_{ij}(y) = \bigvee_{\substack{\xi \in Y^{U \times V} \\ \frac{\int_V \int_U \xi(u,v) A_i(u) B_j(v) du dv}{\int_V \int_U A_i(u) B_j(v) du dv} = y}} \left(\bigwedge_{(u,v) \in U \times V} \Phi(u,v)(\xi(u,v)) \right), \tag{16}$$

for the extended direct F-transform of $\Phi(u, v)$.

For the extended inverse F-transform of $\Phi(u, v)$ we use the following formula

$$\Phi_{n,m}^F(u,v)(y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathcal{F}_{ij}(y) A_i(u) B_j(v). \tag{17}$$

5 Data-Based Model

As we have mentioned above and it is published e.g. in [1] it is possible to construct approximate representation of the original function given by the inverse F-transform even if we do not have the full knowledge of the original function. Suppose that f is known at some nodes $x_1, \ldots, x_r$.

This can be viewed as a data-based model with data $(x_k, f(x_k))$ where $k = 1, \ldots, r$. Then we compute the direct F-transform as follows

$$F_i = \frac{\sum_{k=1}^{r} f(x_k) A_i(x_k)}{\sum_{k=1}^{r} A_i(x_k)}. \tag{18}$$

Similarly, we can construct the extended F-transforms based on given data. Let us be given the following sequence of ordered pairs $(x_k, \Phi(x_k))$ where $k = 1, \ldots, r$ and $\Phi(x_k)$ is a fuzzy set on Y. Moreover, let $A_1(x), \ldots, A_n(x)$ be basic functions determining a fuzzy partition. In that case we construct the extended direct F-transform as follows

$$\mathcal{F}_i(y) = \frac{\sum_{k=1}^{r} \Phi(x_k)(y) A_i(x_k)}{\sum_{k=1}^{r} A_i(x_k)}. \tag{19}$$

Of course, the extended inverse F-transform is defined in the same way as formula (15).

Analogously, we construct data-based model for approximation of fuzzy function with more variables. For the case $n = 2$ and data $(u_k, v_l, \Phi(u_k, v_l))$, $k = 1, \ldots, r$ and $l = 1, \ldots s$, we continue as follows

$$\mathcal{F}_{ij}(y) = \frac{\sum_{k=1}^{r} \sum_{l=1}^{s} \Phi(u_k, v_l)(y) A_i(u_k) B_j(v_l)}{\sum_{k=1}^{r} \sum_{l=1}^{s} A_i(u_k) B_j(v_l)}. \tag{20}$$

Let us stress, that data where u_k, v_l are real numbers and $\Phi(u_k, v_l)$ are fuzzy sets are quite natural. These data may be obtained by questioning some expert. For example in fuzzy control of a dynamic robot we know distance E between the robot and the wall at each moment and we know its derivation dE. We ask some expert about control action. Answer could be either a linguistic expression or a fuzzy number.

Because of this, we obtain the following $n \times m$ fuzzy rules

$$\text{IF } E \text{ is } \mathcal{A}_i \text{ AND } dE \text{ is } \mathcal{B}_j \text{ THEN } y \text{ is } \mathcal{F}_{ij} \tag{21}$$

comprising a fuzzy rule base for fuzzy control of a dynamic robot. This implies that this method can be used as a learning method.

The model of these IF-THEN rules can be constructed according to formula (17). Due to the choice of basic functions and properties of fuzzy functions, ordinary sums in formula (17) can be replaced by Łukasiewicz sums. Moreover, because basic functions can be represented with fuzzy similarity relation with finite number of fixed nodes, this formula can be taken as an additive normal form with orthogonal condition. It lies between well known conjunctive and disjunctive normal forms (see [3], [5]).

6 Conclusions

We have recalled a numerical method on the basis of fuzzy approximating model called fuzzy transforms. The construction of basic functions (special fuzzy sets determining a fuzzy partition) has been repeated as well. The method of fuzzy transforms has been introduced as a universal approximation method which is possible to use for a (discrete) data-based model.

This method has been successfully extended for fuzzy functions what can be useful in fuzzy control. Furthermore, a generalized F-transforms for functions with more variables have been similarly extended as well. Moreover, a discrete case of extended fuzzy transforms as a data-based model has been introduced. It has been also shown how extended F-transforms can be used as a learning method.

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