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Fuzzy Transforms and Their Application on Wave Equation

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Research report No. 52

2004

Submitted/to appear:

Journal of Electrical Engineering

Supported by:

Grant IAA 1187301 of the GA AV ČR; Grant 01/2004 of the IGA FS OU

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Abstract

Fuzzy transforms are getting more and more known technique of approximate representation of functions continuous on a closed interval of real numbers. This technique based on two transforms (direct and inverse) has been successfully applied in numerical methods and in many other problems. However, it is still a lot of work to do, in particular to extend it for functions with arbitrary finite number of variables and apply it in solving of partial differential equations.

Keywords: Fuzzy sets, fuzzy transforms, approximation, fuzzy partition, partial differential equations, wave equation.

2000 Mathematics Subject Classification: 68T37

1 INTRODUCTION

Fuzzy transforms (in short F-transforms) have been already several times introduced in a number of publications and international conferences. Perfilieva I. presented this technique in [2], its application in [3] and then possible applications on ordinary differential equations in [1]. Another application has been published in [4].

However, many problems cannot be expressed and solved by ordinary differential equation and partial differential equations have to be used. This requires an extension of F-transforms for functions with more variables. The first step has been already done in [5]. The aim of this paper is to present fuzzy transforms extended for functions with $k \in \mathbb{N}$ variables and to introduce an application on concrete type of partial differential equations with physical background.

2 F-TRANSFORMS FOR FUNCTIONS WITH ONE VARIABLE

This section recalls the method published by Perfilieva I. and Chaldeeva E. in [2]. This technique is more numerical then linguistic and that is why it belongs to the area called numerical methods on the basis of fuzzy approximation models. An interval $[a, b]$ of real numbers has been used as a common domain of all functions in this publication.

Definition 1 Let $x_i = a + h(i-1)$ be nodes on $[a, b]$ where $h = (b-a)/(n-1),\ n \geq 2$ and $i = 1, \dots, n$. We say that functions $A_1(x), \dots, A_n(x)$ defined on $[a, b]$ are *basic functions* if each of them fulfills the following conditions:

- $A_i : [a, b] \to [0, 1],\ A_i(x_i) = 1$,
- $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$, where $x_0 = a,\ x_{n+1} = b$,
- $A_i(x)$ is continuous,
- $A_i(x)$ strictly increases on $[x_{i-1}, x_i]$ and strictly decreases on $[x_i, x_{i+1}]$,
- $\sum_{i=1}^{n} A_i(x) = 1$, for all $x \in [a, b]$,
- $A_i(x_i - x) = A_i(x_i + x)$, for all $x \in [0, h],\ i = 2, \dots, n-1,\ n > 2$,
- $A_{i+1}(x) = A_i(x - h)$, for all $x \in [a + h, b],\ i = 2, \dots, n-2,\ n > 2$.

In short, we can define basic functions as fuzzy sets determining a uniform fuzzy partition of real interval $[a, b]$. Moreover, each basic function $A_i(x)$ can be viewed as a fuzzy set "approximately x_i".

The technique of fuzzy transforms is based on two transforms: the direct one and the inverse one. At first, we transform an element of space of continuous functions $\mathcal{C}([a, b])$ to a vector which serves us as its discrete representation.

Definition 2 Let $f(x)$ be any continuous function on $[a,b]$ and $A_1(x),\dots,A_n(x)$ be basic functions. We say that the n-tuple of real numbers $[F_1,\dots,F_n]$ is *the direct F-transform* of f w.r.t. $A_1(x),\dots,A_n(x)$ if

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}. \tag{1}$$

When the basic functions are fixed, we denote the direct F-transform of f by $F[f]$.

Remark: If we do not have the full knowledge of function $f(x)$, but this function at some nodes we can replace the integrals in formula (1) by the sums.

The discrete representation is returned back by to the space of continuous functions to obtain a continuous approximation of function $f(x)$.

Definition 3 Let $F[f] = [F_1,\dots,F_n]$ be the direct F-transform of a function $f(x)$ with respect to $A_1(x),\dots,A_n(x)$. The function

$$f_{F,n}(x) = \sum_{i=1}^{n} F_i A_i(x) \tag{2}$$

will be called *the inverse F-transform.*

3 FUZZY TRANSFORMS FOR FUNCTIONS WITH MORE VARIABLE

We would like to mention, that the presented way of the extension for functions with more variables is not the only one existing approach, but the most natural and intuitive approach.

We will follow the idea of one common domain from [2] and [1] and we will use $[a_1,b_1]\times\cdots\times[a_k,b_k]$ as a common domain of all functions $f(x_1,...,x_k)$ with k variables considered.

Definition 4 Let $x_j^{(i_j)} = a_j + h_j(i_j-1)$ be nodes on $[a_j,b_j]$ where $h_j = (b_j-a_j)/(n_j-1)$, $n_j \geq 2$, $i_j = 1,\dots,n_j$ and $j=1,\dots,k$. We say that functions $\{A_j^{(i_j)}(x_j)\}$ determine *the system of basic functions* if the following conditions are fulfilled:

- $A_j^{(i_j)} : [a_j,b_j] \to [0,1]$, $A_j^{(i_j)}(x_j^{(i_j)}) = 1$,
- $A_j^{(i_j)}(x_j) = 0$ if $x \notin (x_j^{(i_j-1)}, x_j^{(i_j+1)})$, $x_j^0 = a_j$, $x_j^{(n_j+1)} = b_j$,
- $A_j^{(i_j)}(x_j)$ are continuous functions,
- $A_j^{(i_j)}(x_j)$ strictly increases on $[x_j^{(i_j-1)}, x_j^{(i_j)}]$ and strictly decreases on $[x_j^{(i_j)}, x_j^{(i_j+1)}]$,
- $\sum_{i_1=1}^{n_1}\cdots\sum_{i_j=k}^{n_k} A_1^{(i_1)}(x_1)\cdots A_k^{(i_k)}(x_k) = 1$, for all $x_j \in [a_j,b_j]$
- $A_j^{(i_j)}(x_j^{(i_j)} - x_j) = A_j^{(i_j)}(x_j^{(i_j)} + x_j)$, for all $x_j \in [0,h_j]$, $i_j = 2,\dots,n_j-1$, $n_j > 2$,
- $A_j^{(i_j+1)}(x_j) = A_j^{(i_j)}(x_j - h_j)$, for all $x_j \in [a_j+h_j, b_j]$, $i_j = 2,\dots,n_j-2$, $n_j > 2$.

To simplify the way how to construct basic functions, we can require the following k subconditions instead of the fifth condition

- $\sum_{i_j=1}^{n_j} A_j^{(i_j)}(x_j) = 1$, for all $x_j \in [a_j,b_j]$, where $j = 1,\dots,k$.

Remark: We do not restrict the shapes of basic functions as well as in [2], [1] or the other references. We can use well known polynomial basic functions (special case is a triangular shape) or sinusoidal shaped basic functions. Of course, it is possible to use basic functions of different shapes on different axes.

Lemma 1 *If $\{A_j^{(i_j)}(x_j)\}$ is the system of basic functions on $[a_1, b_1] \times \cdots \times [a_k, b_k]$ then*

$$\int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} A_1^{(i_1)}(x_1) \cdots A_k^{(i_k)}(x_k) dx_1 \cdots dx_k = \frac{h_1 \cdots h_k}{2^N}, \tag{3}$$

where N is the frequency of occurrence $i_j = 1$ or $i_j = n_j$ for all $j = 1, \ldots, k$.

PROOF: Because of lack of space, we introduce only the main idea. The proof is based on the following property which is a straight corollary of lemma published in [1, 2, 3, 4]:

$$\int_{a_j}^{b_j} A_j^{(i_j)}(x_j) dx_j = h_j \quad \text{for } i_j = 2, \ldots, n_j - 1,$$

and

$$\int_{a_j}^{b_j} A_j^{(1)}(x_j) dx_j = \int_{a_j}^{b_j} A_j^{(n_j)}(x_j) dx_j = \frac{h_j}{2}.$$

□

This lemma confirms that the definite integral of a product of basic functions does not depend on concrete shapes of these functions, only their positions are considered.

Definition 5 Let $f(x_1, \ldots, x_k)$ be arbitrary continuous function on $\mathcal{D} = [a_1, b_1] \times \cdots \times [a_k, b_k]$ and $\{A_j^{(i_j)}(x_j)\}$ be a system of basic functions. We say that the $n_1 \cdot n_2 \cdots n_k$-tuple $[F^{(i_1 \cdots i_k)}]$ of real numbers is *the direct F-transform* of f w.r.t. the given system of basic functions if

$$F^{(i_1 \cdots i_k)} = \frac{\int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_k) A_1^{(i_1)}(x_1) \cdots A_k^{(i_k)}(x_k) dx_1 \cdots dx_k}{\int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} A_1^{(i_1)}(x_1) \cdots A_k^{(i_k)}(x_k) dx_1 \cdots dx_k}. \tag{4}$$

And similarly to the case of functions with one variable, if the system of basic functions is fixed we denote the direct F-transform of f by $F^{(k)}[f]$.

Definition 6 Let $F^{(k)}[f]$ be the direct F-transform of a function $f(x_1, \ldots, x_k)$. Then function

$$f_F^{(n_1, \ldots, n_k)}(x_1, \ldots, x_k) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} F^{(i_1 \cdots i_k)} A_1^{(i_1)}(x_1) \cdots A_1^{(i_k)}(x_k) \tag{5}$$

will be called *the inverse F-transform.*

4 PROPERTIES OF F-TRANSFORMS

It is easy to see that if the system of basic functions is fixed, then the direct F-transform as a mapping from $\mathcal{C}(\mathcal{D})$ to $\mathbb{R}^{n_1 \cdot n_2 \cdots n_k}$ is linear, so that

$$F^k[\alpha f + \beta g] = \alpha F^k[f] + \beta F^k[g],$$

where $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathcal{C}(\mathcal{D})$. This property is an obvious corollary of the usage of definite integrals in formula (4).

The lemma below confirms that the sequence of functions $\{f_F^{(n_1, \ldots, n_k)}(x_1, \ldots, x_k)\}$ for $n_j \to \infty$ such that $j = 1, \ldots, k$, uniformly converges to $f(x_1, \ldots, x_k)$.

Lemma 2 *Let $f(x_1,\ldots,x_k)$ be arbitrary continuous function on $\mathcal{D}=[a_1,b_1]\times\cdots\times[a_k,b_k]$ and let $\{A_j^{(i_j)}(x_j)\}_{(n_j)}$ be a sequence of systems of basic functions on $\mathcal{D}$. Let $\{f_F^{(n_1,\ldots,n_k)}(x_1,\ldots,x_k)\}_{(n_j)}$ be the sequence of inverse F-transforms w.r.t. the given sequence of systems of basic functions. Then for any $\varepsilon>0$ there exist $n_j(\varepsilon)$ such that for each $n_j>n_j(\varepsilon)$ and for all $(x_1,\ldots,x_k)\in\mathcal{D}$*

$$\mid f(x_1,\ldots,x_k)-f_F^{(n_1,\ldots,n_k)}(x_1,\ldots,x_k)\mid<\varepsilon.$$

PROOF: Again, because of lack of space, we mention only the main idea. Since f is continuous, we can find for each $\varepsilon>0$ some $\delta>0$ such that for all $P,Q\in\mathcal{D}$ $\parallel P-Q\parallel<\delta$ implies $\parallel f(P)-f(Q)\parallel<\varepsilon$.

Let $n_j(\varepsilon)\geq 2$ be such that $h_j=(b_j-a_j)/(n_j(\varepsilon)-1)\leq\delta/\sqrt{k}$.

Let $(t_1,\ldots,t_k)$ be arbitrary element of $[x_1^{(i_1)},x_1^{(i_1+1)}]\times\cdots\times[x_k^{(i_k)},x_k^{(i_k+1)}]$.

Obviously,

$$\mid f(t_1,...,t_k)-F^{(i_1\ldots i_k)}\mid<\varepsilon,$$

$$\mid f(t_1,...,t_k)-F^{((i_1+1)\ldots i_k)}\mid<\varepsilon,$$

$$\cdots$$

$$\mid f(t_1,...,t_k)-F^{(i_1\ldots(i_k+1))}\mid<\varepsilon.$$

□

Besides the fact that the inverse F-transform uniformly converges to the original function, we are interested in the criterion which guarantees us that the inverse F-transform is the best approximation in some sense.

Let $A_1^{(i_1)},\ldots,A_k^{(i_k)}$ be basic functions and $f(x_1,\ldots,x_k)$ be an integrable function. Let us consider the class of approximating functions represented by the following formula

$$\sum_{i_1=1}^{n_1}\cdots\sum_{i_k=1}^{b_k}c^{(i_1\cdots i_k)}A_1^{(i_1)}(x_1)\cdots A_k^{(i_k)}(x_k), \tag{6}$$

where $c^{(i_1\cdots i_k)}\in\mathbb{R}$.

Then let us suggest the following criterion

$$\Phi\left(c^{(i_1\cdots i_k)}\right)=\int_{\mathcal{D}}\sum_{i_1=1}^{n_1}\cdots\sum_{i_k=1}^{n_k}\left(f(x_1,\ldots,x_k)-c^{(i_1\cdots i_k)}\right)^2A_1^{(i_1)}(x_1)\cdots A_1^{(i_k)}(x_k)dx_1\cdots dx_k \tag{7}$$

characterizing the integral proximity between f and a function from class (6).

This integral least-square error criterion (7) has a good sense in physics (especially in partial differential equation) where integral means some energy, heat etc.

Coefficients $F^{(i_1\cdots i_k)}$ of the direct F-transform are constructed to minimize criterion (7).

5 APPLICATION ON WAVE EQUATION

In this section we present how F-transforms can be used in solving of one concrete type of partial differential equations. For demonstration, we will consider the following equation which is called wave equation in physics. For the simplicity, we present two-dimensional case of this equation. So, let us have the following hyperbolic equation

$$\frac{\partial^2u}{\partial x^2}-\alpha\frac{\partial^2u}{\partial t^2}=q(x,t),\quad x\in[0,L] \tag{8}$$

with the following initial and boundary conditions

$$u(x,0)=f(x),\quad \frac{\partial u}{\partial t}(x,0)=g(x), \tag{9}$$

$$u(0,t)=T_1(t),\quad u(L,t)=T_2(t). \tag{10}$$

The idea of the numerical solution using F-transforms is as follows. We apply the direct F-transform to both sides of equation (8) to obtain the following algebraic equation

$$U_{xx}^{(i_1 i_2)} - \alpha U_{tt}^{(i_1 i_2)} = Q^{(i_1 i_2)} \tag{11}$$

where $U_{xx}^{(i_1 i_2)} = F^{(2)}[\frac{\partial^2 u}{\partial x^2}]$, $U_{tt}^{(i_1 i_2)} = F^{(2)}[\frac{\partial^2 u}{\partial t^2}]$ and $Q^{(i_1 i_2)} = F^{(2)}[q]$ w.r.t to any given system of basic functions Then we transform (usage of the inverse F-transform) this solution back to the space of continuous functions.

We replace partial derivatives in equation (8) by their approximations (finite differences), so we have e.g.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{(u(x+h_1,t) - 2u(x,t) + u(x-h_1,t))}{h_1^2}.$$

After some steps we obtain

$$U_{xx}^{(i_1 i_2)} \approx \frac{1}{h_1^2}(U^{((i_1-1)i_2)} - 2U^{(i_1 i_2)} + U^{((i_1+1)i_2)}),$$

and similarly for $U_{tt}^{(i_1 i_2)}$. Now, we solve given algebraic equation (11) and obtain

$$U^{(i_1(i_2+1))} = r^2 U^{((i_1-1)i_2)} + 2(1-r^2)U^{(i_1 i_2)} + r^2 U^{((i_1+1)i_2)} - U(i_1(i_2-1)) - h_2^2 Q^{(i_1 i_2)} \tag{12}$$

where $r = (\alpha h_2)/h_1$.

Finally, we use the knowledge of initial and boundary conditions. For $i_2 = 0$ we put $U^{(i_1 i_2)} = f(i_1 h_1)$ and similarly from boundary conditions (10) we obtain $U^{(0 i_2)} = T_1(i_2 h_2)$ and $U^{(n_1 i_2)} = T_2(i_2 h_2)$.

The last replacement, which is necessary to start the algorithm described by formula (12), is following $U^{(i_1(-1))} = -2h_2 \cdot g(i_1 h_1) + U^{(i_1 1)}$.

The solution of algebraic equation (11) is going to be transformed back to $\mathcal{C}(\mathcal{D})$ by the inverse F-transform.

6 CONCLUSION

We have recalled F-transforms as a method which belongs to the area called numerical methods on the basis of fuzzy approximation models. Then, we have presented the extension of this method for functions with arbitrary finite number of variables. Properties of this extended technique have been introduced as well.

Finally, we have shown our suggested approach to the numerical solution of wave equation (hyperbolic partial differential equation in general) using this technique. The usage of F-transforms is justified by the minimization of so called integral least-square criterion which is very useful, in particular, in physics and partial differential equations.

Acknowledgement

This investigation has been partially supported by grant IAA 1187301 of the GA AV ČR.

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