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On the Semantics of Perception-Based Fuzzy Logic Deduction*)

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Abstract

In this paper, we return to the problem of derivation of a conclusion on the basis of fuzzy IF-THEN rules. The, so called, Mamdani method is well elaborated and widely applied. In this paper, we present an alternative to it. The fuzzy IF-THEN rules are here interpreted as genuine linguistic sentences consisting of the, so called, evaluating linguistic expressions. Sets of fuzzy IF-THEN rules are called linguistic descriptions. Linguistic expressions derived on the basis of an observation in a concrete context are called perceptions. Together with the linguistic description, they can be used in logical deduction, which we will call a *perception-based logical deduction*.

We focus on semantics only and confine ourselves to one specific model. If the perception-based deduction is repeated and the result interpreted in appropriate model, we obtain a piece-wise continuous and monotonous function. Though the method has already proved to work well in a lot of applications, the non-smoothness of the output may sometimes lead to problems. We propose in this paper a method how the resulting function can be made smooth so that the output preserves its good properties. The idea consists in post-processing the output using a special fuzzy approximation method called F-transform.

Keywords: Perception-based deduction, evaluating linguistic expression, intension, extension, fuzzy intensional logic, F-transform.

1 Introduction

Though there are already hundreds of papers focusing on elaboration of fuzzy IF-THEN rules and on the theory of approximate reasoning (among many of them, let us only mention[7,10,21]), we return in this paper again to this problem. It is usual that fuzzy IF-THEN rules are interpreted as special fuzzy relations (special formulas) which imprecisely characterize some dependence. The fuzzy relations are derived from fuzzy sets, which for better understandability, are labelled by some expressions of natural language but without pretension that these fuzzy sets are indeed an apt explication of the meaning of the latter. The reason is that in this case, the goal of approximate reasoning is to approximate some function being characterized by these fuzzy relations. Thence, the proper meaning of the labelling linguistic expressions is unimportant. The extensively used Mamdani approximate reasoning method (and its variants) is thus mainly manipulation with fuzzy relations and no formal deduction proceeds. One of important arguments in favour of using it in fuzzy control is continuity and smoothness of its output.

However, we are convinced that the potential of fuzzy set theory is large enough to model the genuine meaning of, at least part of natural language expressions and that fuzzy IF-THEN rules can be taken and interpreted as special conditional sentences. We confine to the case when they consist of the, so

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called, *evaluating linguistic expressions* (expressions like "big", "roughly medium", etc.). The relation between expressions forming a fuzzy IF-THEN rule is understood as a *linguistically characterized logical implication*. When finding an appropriate formal logical system, we may translate fuzzy IF-THEN rules into formulas of it and, hence, we are open to formal manipulation common in logic.

We will deal with *sets of fuzzy IF-THEN rules*, which we call *linguistic descriptions*. Each linguistic description can be translated into a set of formulas of some chosen logical system. The translation, however, must fit the genuine linguistic meaning of the occurring linguistic expressions as best as possible.

There are two formal systems where such a translation has been described, namely fuzzy logic with evaluated syntax[3,4,11,17] and a fuzzy intensional logic[12]. In both of them, a linguistic description is used for establishing a formal logical theory within which logical deduction is possible.

To speak about logical deduction, we suppose to be given an observation which for us is some element (value) encountered in a concrete context (in the paper, we speak about possible world). Then, we can characterize the observation by an evaluating linguistic expression, which becomes our *perception* of the given value. Consequently, the perception can be translated into a formula of the above mentioned formal system and together with the linguistic description, a logical deduction can proceed. The derived conclusion is a formula which represents certain evaluating expression characterizing values in various contexts. Finally, choosing a specific context, we can find a value which is a result of the deduction. We will call this procedure a *perception-based deduction*.

The perception-based deduction often leads at each step to firing of one fuzzy IF-THEN rule only. This is in accordance with our intuition and we argue that when a linguistic description of some situation is given then our method gives results which are close to conclusion derived by people when facing the same situation. This fact, together with clear understandability of linguistic descriptions to people (because they use simple expressions of natural language) make perception-based deduction attractive for applications, especially when expert-related solutions are necessary (among them we can rank also fuzzy control).

Let us consider a specific model. When repeating the perception-based deduction for all elements of the model and interpret the results of formal derivation back in it, we obtain a function, which, unlike Mamdani method is in general only piece-wise continuous and monotonous. Applications of perception-based deduction in fuzzy control demonstrated that this is not a principal obstacle (see, e.g.[15]). However, since the output is not enough smooth, the behaviour of the active element need not be desirable. Therefore, we proposed an improved perception-based deduction whose output is continuous and smooth but at the same time it preserves its main advantages mentioned above. The main idea is to post-process the output using the so called F-transform. The latter is a fuzzy approximation technique, which can be applied in various tasks, e.g. smooth filtering of data, solving differential equations, etc. (see[19,20]).

In this paper, we prefer fuzzy intensional logic as the basic formal logical system since its means enable us to formulate the theory of meaning of natural language in most elegant way. However, this paper is not logical and so, the reader will not find here precise definitions of formal language, formula, interpretation, or description of all formal steps of the perception-based deduction. The interested reader is referred to[12,13]. Instead, we confine to one specific model and focus only on the semantics. The reason is that we are interested more in the results which perception-based deduction can bring us and our goal is to improve them.

In Section 2, we first briefly remind the concept of evaluating linguistic expressions, outline mathematical model of their semantics and show how to model the meaning of the above considered fuzzy IF-THEN rules. In Section 3 we show, how perceptions can be modelled and introduce the perception-based logical deduction. In Section 4, we briefly describe the method of F-transform and then show how the perception-based deduction and F-transform can be joined together. In Section 5, we compare the behaviour of the original and smooth perception-based deduction on some examples and demonstrate their power and advantages.

A fuzzy set A in the universe V, in symbols $A \underset{\sim}{\subset} V$, is identified with a function $A : V \longrightarrow [0,1]$ (the function A is also called the *membership function* of the fuzzy set A). By $\mathcal{F}(V)$ we denote the set of all fuzzy sets on V.

Elements from $[0,1]$ are interpreted as truth values. We suppose the that they form Łukasiewicz algebra (understood as a residuated lattice)

$$\mathcal{L}_{\text{L}} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle$$

where $\vee$ is the operation of *maximum*, $\wedge$ that of *minimum*, $a \otimes b = 0 \vee (a+b-1)$ is *Łukasiewcz conjunction* and $a \rightarrow b = 1 \wedge (1 - a + b)$ is *Łukasiewicz implication* ($a, b \in [0,1]$). As a special case, we introduce the operation of *negation* by $\neg a = a \rightarrow 0 = 1 - a$.

2 The Theory of Evaluating Linguistic Expressions

The perception-based logical deduction assumes that the linguistic description is formulated using the so called *evaluating linguistic expressions*, for example "very large, extremely deep, roughly one thousand, more or less hot", etc. In this paper, we confine only to the simplest case of these expressions. More about them and their theory can be found in[11,17,18].

The class of simple evaluating linguistic expressions. The simple evaluating expressions have the form

$$\langle \text{linguistic hedge} \rangle \langle \text{atomic evaluating expression} \rangle$$

where *atomic evaluating expressions* are words of type "small", "medium", or "big". Let us stress that these words should be taken as *canonical* and can be replaced by any other cases such as "thin", "thick", "old", "new", etc. Specific are also fuzzy quantities, namely "approximately x_0". Atomic evaluating expressions usually form *pairs of antonyms* and when completed by a middle term, such as "medium", "average", etc., they form the so called *fundamental linguistic trichotomy.*

Linguistic hedges (introduced by L. A. Zadeh[23]) are special adverbs which modify the meaning of adjectives before which they stand (cf. also[1,10]). We distinguish hedges with *narrowing effect* (very, significantly, etc.) and *widening effect* (more or less, roughly, etc.).

It is important that missing linguistic hedge is understood as presence of an *empty linguistic hedge.* Hence, all simple evaluating expressions can be treated equally. In the sequel, we will use script letters $\mathcal{A}, \mathcal{B}, \ldots$ to denote evaluating expressions.

Evaluating linguistic predications are expressions of the form "$\langle$noun$\rangle$ is $\mathcal{A}$" where $\mathcal{A}$ is an evaluating linguistic expression. Since in our considerations, we are usually not interested in specific nouns, we replace "$\langle$noun$\rangle$" by some variable X and assume that its values are real numbers.

The fuzzy IF-THEN rule is a conditional linguistic clause of natural language characterizing relation between two evaluating predications, which has the form

$$\mathcal{R} := \ \ \mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}. \tag{1}$$

The part before THEN is called *antecedent* and the part after it is called *succedent* (for simplicity, we confine to one antecedent variable only in this paper; generalisation to more variables is straightforward).

Informal characterisation of the semantics of evaluating linguistic expressions. We outline here some ideas of the construction of mathematical model of the meaning of evaluating expressions.

In any model of the semantics of linguistic expressions, we must make a distinction between their intension and extensions in various possible worlds. A *possible world* is a state of our world at a given time moment and place. A possible world can also be understood as a *particular context* in which the linguistic expression is used. Let us stress that the term "possible world" intension and extension (see further) have been introduced by Carnap[2]. In this paper, we will prefer to use the term *linguistic context* (or simply *context*) instead of possible world.

Linguistic expressions can be generally taken as names of properties. Then an *intension* of a linguistic expression is an abstract construction which conveys a property denoted by the expression. Consequently, we can take linguistic expressions to be names of intensions. Intension is invariant with respect to various contexts (possible worlds) and so, each linguistic expression is a name of just one intension[*)].

An *extension* of a linguistic expression is a class of objects determined by its intension in a given context (possible world). Thus, it depends on a particular context of use and it changes whenever the context (time, place) is changed.

For example, the expression "deep" is the name of an intension being a certain property of depth, which in a concrete context may mean 1 cm when a beetle needs to cross a puddle, 3 m in a small lake, but 3 km or more in the ocean.

Let us now formalise the previous reasoning. We will begin with more general definitions but soon make them specific only for the case of evaluating expressions.

We consider a sufficiently large set V, the elements of which will be taken to form extensions of all thinkable linguistic expressions. In case of evaluating expressions, we may put $V = \mathbb{R}$. Furthermore, let W be a set of elements which will represent contexts. The original idea of R. Carnap[2] was to define *intension* as a *function* from the set of contexts (possible worlds) to the set of extensions. Following it, we will define intension of an evaluating expression as a function

$$A : W \longrightarrow \mathcal{F}(V), \tag{2}$$

i.e. the *extension* in the given context $w \in W$ is a *fuzzy set*

$$A(w) \underset{\sim}{\subset} V.$$

In other words, extension is a functional value of the intension A in the given context w. Let us remark that R. Carnap has been led to his simple definition of intension by the generally accepted requirement that extension of an expression should be fully definable from its intension.

We will often write A_w instead of $A(w)$. Summing up the previous considerations, we conclude that each linguistic expression is assigned some intension of the form (2).

For example, "small" is a name of an intension $\mathrm{Sm} : W \longrightarrow \mathcal{F}(\mathbb{R})$. In each context $w \in W$, the extension of "small" is a certain fuzzy set of real numbers.

It is difficult to specify what does a concrete context mean. Hence, in mathematical model of linguistic meaning the set W is taken as a set of some abstract parameters representing various contexts without attempt to specify them more closely. Fortunately, the evaluating linguistic expressions seem to be simple enough to make possible an explicit definition of the set of contexts. Namely, we will put

$$W = \{\langle v_L, v_S, v_R\rangle \mid v_L, v_S, v_R \in [0, \infty) \quad \text{and} \quad v_L < v_S < v_R\}. \tag{3}$$

[*)] Of course, homonyms behave as if having more intensions but these should be taken as various expressions having equal surface form.

Elements $u \in [0, \infty]^{\dagger)}$ may belong to various contexts. If $w \in W$ is a context, $w = \langle v_L, v_S, v_R \rangle$, then $u \in w$ means that $u \in [v_L, v_R]$ and we say that u belongs to the context w.

The justification for taking contexts as elements of the set (3) is based on ideas of P. Vopěnka[22]. Each context $w \in W$ is delineated by two distinguished points v_L, v_R, which represent a *left limit* and a *right limit*, respectively. All values which fall in the context w lay between them. Since (in the given context!), nothing can be smaller than v_L, or bigger than v_R, these points are the "most typical" small value and the "most typical" big value, respectively. The properties of being "small" and "big" are the, so called, primary recordable properties, which are vague. We can point a small value (for example, v_L) but there does not exist the last small value. The only thing we know is that small values run somewhere towards a certain point which is the horizon of our seeing of small values. Everything which is beyond this point is surely not small. Note that this reasoning contains the sorites paradox*).

Quite similarly, starting from v_R and going in the opposite direction, we find a horizon of big values such that everything beyond it is surely not big. In our model, we will identify both horizons with one point v_S which lays somewhere between v_L and v_R. This point represents a *central limit*. However, we cannot be sure about precise position of v_S. Therefore, we specify our uncertainty about this using *degrees of truth* expressing that "we find ourselves still before the horizon.

Consequently, extensions of the evaluating expressions characterizing small values lay between v_L, v_S and those characterizing big values lay between v_S, v_R (with the direction from v_R to v_S).

The expressions characterising medium values are determined by the point v_S which is the "most typical" medium and their extensions lay around it.

Mathematization of the semantics of evaluating linguistic expressions. To model mathematically the above reasoning, we start with fuzzy logic model of the sorites paradox (for the detailed analysis of it see[6,17]). The simplest semantic interpretation of the sorites paradox leads to a couple of linear functions $L, R : W \times \mathbb{R} \longrightarrow [0, 1]$ defined in each context $w \in W$ by

$$L_w(x) = \left(\frac{v_S - x}{v_S - v_L} \right)^*,$$
$$R_w(x) = \left(\frac{x - v_S}{v_R - v_S} \right)^*$$

where the star means cut of the values to the interval $[0, 1]$. The functions L_w, R_w describe the idea of running towards horizon since $L_w(v_L) = 1$ and $L_w(u) = 0$ for $u \geq v_S$. If the truth value $L_w(u) > 0$ for $u \in [v_L, v_S]$ then u lays still before the horizon v_S and so, it may fall into extension of some evaluating expression characterizing small values. If $L_w(u) = 0$ then u lays beyond the horizon. Analogous interpretation for big values has R_w.

There are also values which are *neither small nor big*. These are characterized by the function

$$M_w(x) = \neg L_w(x) \wedge \neg R_w(x) = \left(\frac{x - v_L}{v_S - v_L} \right)^* \wedge \left(\frac{v_R - x}{v_R - v_S} \right)^*.$$

The functions L, R, M are fundamental in further construction of our model of the meaning of evaluating expressions. For each possible world, extensions of the expressions characterizing the respective

†)In this paper, we will consider only positive numbers to fall into the meaning of extensions of evaluating expressions. Generalisation to the whole $\mathbb{R}$ is possible and requires fine model which reverses ordering when speaking about "negative small", "negative very big", etc.

*)One grain does not form a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.

small, big, or *medium* values lay before the corresponding horizon. To mathematise this, we proceed as follows.

Let us consider a class $\mathbf{Hf}$ of functions which we will call *abstract hedges* (horizon deformations). Elements of $\mathbf{Hf}$ are continuous functions $\nu_{a,b,c} : [0,1] \longrightarrow [0,1]$ determined by some parameters $a < b < c$ so that $\nu_{a,b,c}(y) = 0$ for $y \leq a$, $\nu_{a,b,c}(y) = 1$ for $c \leq y$ and it is increasing otherwise. We explicitly put

$$\nu_{a,b,c}(y) = \begin{cases} 1, & c \leq y, \\ 1 - \frac{(c-y)^2}{(c-b)(c-a)}, & b \leq y < c, \\ \frac{(y-a)^2}{(b-a)(c-a)}, & a \leq y < b, \\ 0, & y < a. \end{cases} \tag{4}$$

Note that there are infinitely many possible functions $\nu_{a,b,c}$. Our goal was to consider the simplest nonlinear one since according to known psychological investigations, the shapes of membership functions should be nonlinear. However, we cannot reject the function $\nu_{a,b,c}$ to be, possibly even linear.

Now we define the following classes of functions which will serve as possible intensions of evaluating expressions. We will distinguish three type of functions according to their future role as intensions:

(i) S-intensions:

$$\mathbf{Sm} = \{\mathrm{Sm}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Sm}_{\nu,w}(x) = \nu(L_w(x)), \nu \in \mathbf{Hf}\}.$$

(ii) M-intensions:

$$\mathbf{Me} = \{\mathrm{Me}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Me}_{\nu,w}(x) = \nu(M_w(x)), \nu \in \mathbf{Hf}\}.$$

(iii) B-intensions:

$$\mathbf{Bi} = \{\mathrm{Bi}_\nu : W \longrightarrow \mathcal{F}(\mathbb{R}) \mid \mathrm{Bi}_{\nu,w}(x) = \nu(R_w(x)), \nu \in \mathbf{Hf}\}.$$

Let $\mathcal{A}$ be an evaluating expression. Then its intension $\mathrm{Int}(\mathcal{A})$ is a function from one of the above classes, i.e.

$$\mathrm{Int}(\mathcal{A}) \in \mathbf{Sm} \cup \mathbf{Me} \cup \mathbf{Bi}\,. \tag{5}$$

Let $w \in W$ be a context. Then the extension of $\mathcal{A}$ in the context w is

$$\mathrm{Ext}_w(\mathcal{A}) = \mathrm{Int}(\mathcal{A})(w) \subsetapprox [v_L, v_R]. \tag{6}$$

The construction of extensions of evaluating expressions is depicted on Figure 1 where

$$\begin{aligned} c_{Sm} &= L_w^{-1}(c), & a_{Sm} &= L_w^{-1}(a), \\ c_{Me}^1 &= (\neg L_w)^{-1}(c), & a_{Me}^1 &= (\neg L_w)^{-1}(a), \\ c_{Me}^2 &= (\neg R_w)^{-1}(c), & a_{Me}^2 &= (\neg R_w)^{-1}(a), \\ c_{Bi} &= R_w^{-1}(c), & a_{Bi} &= R_w^{-1}(a). \end{aligned}$$

In the sequel, we will often use a general variable Ev which represents intension of some evaluating linguistic expression, i.e.

$$\mathrm{Ev} \in \mathbf{Sm} \cup \mathbf{Me} \cup \mathbf{Bi}\,. \tag{7}$$

Similarly as above, will usually write Ev_w for the extension of Ev in a context w, instead of $\mathrm{Ev}(w)$. By abuse of language, we will also quite often blur the distinction between evaluating expression and its intension, i.e. we may say "the evaluating expression Ev" having on mind either the expression together with its intension, or only its intension. We hope that this will not cause misunderstanding.

Let us remark that the meaning of fuzzy quantities is modelled similarly as that of the expressions of type "medium". For simplicity, we have omitted details in this paper.

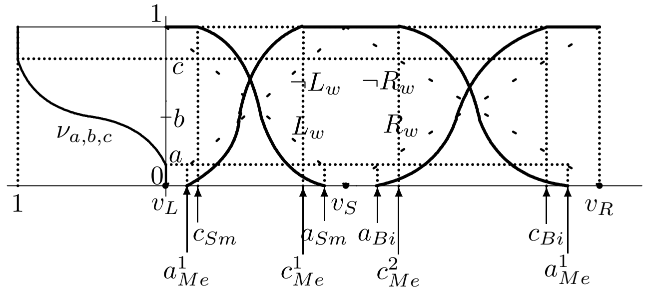

Figure 1:

Specificity ordering of evaluating expressions. In the perception-based logical deduction described further, an important role is played by natural ordering of evaluating expressions, which we call the *specificity ordering*. This is determined by the fact that we can distinguish hedges with *narrowing* and *widening* effect. The latter make the meaning of the atomic expression before which they stand more precise while the former make it opposite.

In applications described in Section 5, we work with several concrete hedges, namely *"extremely (Ex), significantly (Si), very (Ve), empty hedge, more or less (ML), roughly (Ro), quite roughly (QR), very roughly (VR)"*. Among them, the hedges Ex, Si, and Ve have narrowing effect and ML, Ro, QR and VR have widening effect. In[18], we have defined empirical values of the parameters a, b, c of these hedges. We have chosen these hedges because they are very common in ordinary speech. However, our theory is general enough to include many other concrete examples of hedges.

The distinction between hedges with narrowing and widening effect induces an ordering $\preceq$ of specificity between them:

$$\text{Ex} \preceq \text{Si} \preceq \text{Ve} \preceq \langle\text{empty hedge}\rangle \preceq \text{ML} \preceq \text{Ro} \preceq \text{QR} \preceq \text{VR}. \tag{8}$$

Definition (8) means that all values in some context, that are *extremely* small (or big), are also *significantly* small (or big), etc.

We can generalise the definition of specificity ordering to all hedges. Of course, there exist also hedges which have neither narrowing, nor widening effect (for example *rather*), and so, the ordering $\preceq$ is in general, only partial.

On the basis of that, we can extend the specificity ordering to all evaluating expressions, namely: the expressions of the form "⟨linguistic hedge⟩ small" are the *most specific*, and "⟨linguistic hedge⟩ big" are the *least specific*. If two expressions differ only in hedges then they are ordered by the specificity ordering of hedges introduced above. Consequently, with respect to definition (6), extensions of the evaluating expressions containing hedges from (8) make in each possible world a nested sequence of fuzzy sets.

In the sequel, we will suppose that the specificity ordering between evaluating expressions in concern is fixed.

3 Linguistic Description and Logical Deduction

Linguistic description and its meaning. A linguistic description is a set of fuzzy IF-THEN rules

$$\mathcal{R} := \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m\} \tag{9}$$

where each $\mathcal{R}_i$ has the form (1).

Let us now consider one fuzzy IF-THEN rule $\mathcal{R}$ and let

$$\begin{aligned}\mathrm{Int}(X \text{ is } \mathcal{A}) &= \mathrm{Ev}^A,\\ \mathrm{Int}(Y \text{ is } \mathcal{B}) &= \mathrm{Ev}^S\end{aligned}$$

where $\mathrm{Ev}^A, \mathrm{Ev}^S$ are intensions (7) of some evaluating expressions. Then intension of $\mathcal{R}$ is a function $\mathrm{Int}(\mathcal{R}) : W \times W \longrightarrow \mathcal{F}(\mathbb{R} \times \mathbb{R})$ given by

$$\mathrm{Int}(\mathcal{R})(w, w') = \mathrm{Ev}^A_w \to \mathrm{Ev}^S_{w'}, \qquad w, w' \in W, \tag{10}$$

where the formula on the right-hand side of (10) represents a function (fuzzy relation) composed of the functions Ev^A_w, $\mathrm{Ev}^S_{w'}$ and the Łukasiewicz implication $\to$.

The formula (10) gives at the same time rules for computation of extension of $\mathcal{R}$ in the contexts w, w'. Indeed, let extensions of the evaluating predications in the antecedent and succedent in the respective contexts w, w' be fuzzy sets

$$\mathrm{Ext}_w(X \text{ is } \mathcal{A}) = \mathrm{Ev}^A_w, \tag{11}$$

$$\mathrm{Ext}_{w'}(Y \text{ is } \mathcal{B}) = \mathrm{Ev}^S_{w'}\,. \tag{12}$$

Then the extension of $\mathcal{R}$ in a couple of contexts $w, w' \in W$ is a fuzzy relation defined by

$$\mathrm{Ext}_{\langle w,w'\rangle}(\mathcal{R})(v, v') = \mathrm{Ev}^A_w(v) \to \mathrm{Ev}^S_{w'}(v'), \qquad v \in w, v' \in w'. \tag{13}$$

Defuzzification. For dealing with evaluating expressions, a special defuzzification operation DEE (Defuzzification of Evaluating Expressions) is necessary. This operation is a realisation of the, so called, description operator described in[14]. In general, it leads to a very simple conclusion that the result of defuzzification can be any element from the kernel of the given fuzzy set. For applications, however, we need a systematic result. Therefore, we will define the defuzzification as follows.

Let Ev be intension of some evaluating expression and $w \in W$ be a context. Let c be the parameter of the corresponding linguistic hedge $\nu_{a,b,c}$ and $\sigma \in (0, 1]$. Then we put

$$\mathrm{DEE}_\sigma(\mathrm{Ev}_w) = \begin{cases} v_L + \sigma(1-c)(v_S - v_L), & \text{if } \mathrm{Ev} \in \mathbf{Sm},\\ v_S + \frac{\sigma(1-c)}{2}(v_L + v_R - 2v_S), & \text{if } \mathrm{Ev} \in \mathbf{Me},\\ v_R + \sigma(1-c)(v_S - v_R), & \text{if } \mathrm{Ev} \in \mathbf{Bi}\,. \end{cases} \tag{14}$$

The parameter σ is a global characteristics of the defuzzification and it should be set close to 1. A schematic picture demonstrating the behaviour of the DEE defuzzification method is given in Fig. 2. It is easy to see that this defuzzification is a generalisation of three methods, namely *Last of Maxima* for "small", *First of Maxima* for "big" *Center of Gravity* for "medium" values.

Let us remark that, similarly as all the other defuzzification methods, the general theoretical frame of the DEE_σ method is fuzzy logic, but the concrete formula has been derived on the basis of practical experience. Our conclusion that its result should be arbitrary element from the kernel of the fuzzy set is in accordance with the general result in the fuzzy approximation theory (cf.[17], Chapter 5) where the concrete defuzzification has no influence on the precision of approximation. Practical testing of the DEE_σ method demonstrates that it gives results which are in accordance with the human way of reasoning with evaluating expresions.

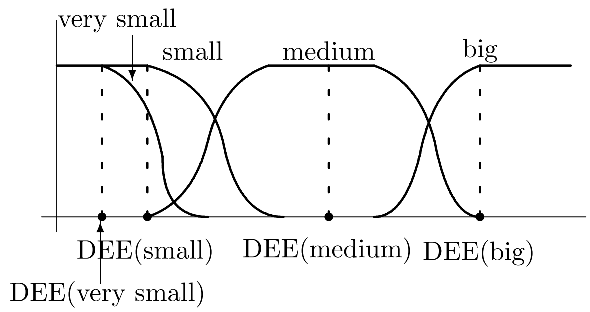

Figure 2: Scheme of the Defuzzification of Evaluating Expressions DEE_σ method for $\sigma = 1$.

Perception-based logical deduction. The way how people make inferences on the basis of linguistic description can be explained on an example. Let us consider a linguistic description, which consists of two rules:

$$\mathcal{R}_1 := \ \textsf{IF}\ X\ \textsf{is}\ \textit{small}\ \textsf{THEN}\ Y\ \textsf{is}\ \textit{big}$$
$$\mathcal{R}_2 := \ \textsf{IF}\ X\ \textsf{is}\ \textit{big}\ \textsf{THEN}\ Y\ \textsf{is}\ \textit{small}.$$

Furthermore, let the contexts for the respective variables X, Y be $w = w' = \langle 0, 0.5, 1\rangle$. Then small values are some values around 0.3 (and smaller) and big ones some values around 0.7 (and bigger). We know from the linguistic description that small input values correspond to big output ones and vice-versa. Therefore, given an input, e.g. $X = 0.3$, we expect the result $Y \approx 0.7$ due to the rule $\mathcal{R}_1$. The reason is that with respect to the above linguistic description, our perception of 0.3 (in the given context) is "small", and thus, in this case the output value of Y should be "big". Similarly, for $X = 0.75$ we expect the result $Y \approx 0.25$ due to the rule $\mathcal{R}_2$.

This intuitive procedure can be characterized using precise formal means of fuzzy logic. This has been done in the frame of fuzzy intensional logic in[12], and in the frame of fuzzy logic with evaluated syntax in[11,17]. We refer the reader to these papers since as mentioned in Introduction, in this paper we focus on semantics only and so, we have not developed here appropriate formal means.

Less formally, we recall that the meaning of linguistic description is represented by a set of intensions

$$\mathrm{Int}(\mathcal{R}_1), \dots, \mathrm{Int}(\mathcal{R}_m), \tag{15}$$

where each $\mathrm{Int}(\mathcal{R}_i)$ is defined in (10). The logical deduction may proceed, if we learn an observation leading to an intension, which is equal to some of the antecedents Ev_i^A occurring in the given linguistic description. This is done as follows.

The observation is some value $u \in w$ in a context $w = \langle v_L, v_S, v_R\rangle$. We must first transform it into a suitable *perception* using a function

$$\mathrm{Suit} : \mathbb{R} \times W \longrightarrow \mathbf{Sm} \cup \mathbf{Me} \cup \mathbf{Bi}\,. \tag{16}$$

The result of $\mathrm{Suit}(u, w)$ is (intension of) such an evaluating expression Ev, that the observation $u \in w$ is the *most specific and typical* for its extension Ev_w. To be *typical* means that the membership degree

$\mathrm{Ev}_w(u)$ is non-zero and greater than some reasonable threshold a^0 (we usually put $a^0 = 0.9$ or even $a^0 = 1$). To be *most specific* means that the evaluating expression Ev is the most specific (sharpest) one in the sense of the natural ordering $\preceq$ mentioned in Section 2 (cf. (8)). This definition of Suit can be justified by the empirical finding that in the given context, each value can be classified by some evaluating expression. Since the expressions are more, or less specific, the most specific one gives the most precise information. If there is no evaluating expression being most specific and typical then Suit gives nothing.

Let us now fix an observation $u_0 \in w$. If the linguistic description (15) is given then we expect Suit to give an (intension of) evaluating expression

$$\mathrm{Suit}(w, u_0) = \mathrm{Ev}_i^A \tag{17}$$

(provided that it exists) where $1 \leq i \leq m$. This, together with (15) is used in the deduction. We say that the corresponding rule $\mathcal{R}_i$ *fired*. Note that if the most specific and typical extension $\mathrm{Ev}_{i,w}^A$ is found then Suit extends it to the whole intension Ev_i^A.

Let $u_0 \in w$ be an observation such that the rule

$$\mathcal{R}_i = \ \mathsf{IF}\ X\ \mathsf{is}\ \mathcal{A}_i\ \mathsf{THEN}\ Y\ \mathsf{is}\ \mathcal{B}_i$$

fired where $\mathrm{Int}(\mathcal{A}_i) = \mathrm{Ev}_i^A$ and $\mathrm{Int}(\mathcal{B}_i) = \mathrm{Ev}_i^S$. Then $\mathrm{Ev}_{i,w}^A(u_0) \in [0, 1]$ is a non-zero truth value. Let us denote it by b (i.e. $b = \mathrm{Ev}_{i,w}^A(u_0)$). Then for each context $w' \in W$, it can be demonstrated (cf.[12]) that the perception-based deduction gives a fuzzy set

$$(\mathrm{Ev}_i^S)'(w') = b \rightarrow \mathrm{Ev}_i^S(w'). \tag{18}$$

Each element $v \in w'$ such that $(\mathrm{Ev}_i^S)'(w', v) = 1$ surely *belongs* to the extension $\mathrm{Ev}_{w',i}^S$ and can be taken as the result of perception-based deduction. The appropriate element is obtained by the DEE_σ defuzzification.

If we repeat the just described procedure for all elements $u \in w$ then we conclude that in the case of one fuzzy IF-THEN rule, $\mathcal{R}_i$ it determines in each couple of contexts w, w' a function $f_\mathcal{R} : w \longrightarrow w'$ given by

$$f_{\mathcal{R}_i}(u) = \mathrm{DEE}_\sigma(\mathrm{Ev}_{i,w}^A(u) \rightarrow \mathrm{Ev}_{i,w'}^S)), \qquad u \in w. \tag{19}$$

If a linguistic description consists of more fuzzy IF-THEN rules then the perception-based logical deduction provides a function $f_\mathcal{R}$ which is piece-wise continuous and its pieces consist of parts of the functions (19). The points of discontinuity depend on the perceptions determined by the Suit function (17).

4 Fuzzy Transform and Smooth Perception-Based Logical Deduction

4.1 Fuzzy transform

The fuzzy transform (F-transform) is a technique developed by I. Perfilieva[19,20] which can be ranked among fuzzy approximation techniques. It works with a continuous function f defined on an interval of real numbers $w = [v_L, v_R] \subset \mathbb{R}$. There are several purposes, in which F-transform can be used. The purpose important in this paper is to use it for approximation of f with sufficient precision and to filter its possible noise.

Let us choose some *points* $p_1, \ldots, p_N \in w$ in which the function f is computed. Furthermore, let the interval w be divided into a set of equidistant *nodes* $x_k = v_L + h(k-1)$, $k = 1, \ldots, n$ where $N > n$ and $h = \frac{v_R - v_L}{n-1}$ is the fixed length. Obviously, $x_1 = v_L$ and $x_n = v_R$. The F-transform has two phases.

Direct F-transform. We define n basic functions $A_1, \ldots, A_n$, which cover w and divide it into n vague areas. The basic functions must fulfil the following conditions ($k = 1, \ldots, n$):

1. $A_k : w \longrightarrow [0, 1]$, $A_k(x_k) = 1$,
2. $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where we formally put $x_0 = x_1 = v_L$, $x_{n+1} = x_n = v_R$,
3. $A_k(x)$ is continuous,
4. $A_k(x)$ monotonously increases on $[x_{k-1}, x_k]$ and monotonously decreases on $[x_k, x_{k+1}]$,
5. $\sum_{k=1}^{n} A_k(x) = 1$, for all $x \in w$.

Using the basic functions, we transform the given function f into n-tuple of real numbers $[F_1, \ldots, F_n]$ defined by

$$F_k = \frac{\sum_{j=1}^{N} f(p_j) A_k(p_j)}{\sum_{j=1}^{N} A_k(p_j)}, \qquad k = 1, \ldots, n. \tag{20}$$

Inverse F-transform. The result of the direct F-transform is a vector of numbers $[F_1, \ldots, F_n]$. This set contains information about the original function f and can be used to obtain a function

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k \cdot A_k(x). \tag{21}$$

The function (21) is called an inverse F-transform. It can be proved that if n increases then $f_{F,n}(p_j)$ converges to $f(p_j)$, $j = 1, \ldots, N$. It is clear that the function $f_{F,n}$ is continuous.

The F-transform has (besides others) the following properties important for its use in this paper:

(a) It has nice filtering properties.

(b) It is easy to compute.

(c) Let the function f be given. Then the F-transform is stable with respect to the choice of the points $p_1, \ldots, p_N$, provided that the number of nodes is fixed. This means that when choosing other points p_k (and possibly changing their number N), the resulting function $f_{F,n}$ does not significantly change. Note that this is not true for many classical numerical methods.

The detailed formal description of the F-transform including theorems characterizing its behaviour can be found in[19,20].

4.2 Smoothing perception-based logical deduction

Recall that logical deduction gives a piece-wise continuous function $f_{\mathcal{R}}$ consisting of pieces defined in (19). We can make it continuous by joining perception-based deduction with the F-transform. Of course, we could use also other (classical) numerical technique, for example splines or the least square method. In our case, F-transform surpasses these techniques for the following reasons: we need a sufficiently simple technique which keeps the main properties of the given function and its main role is to make it smooth. For example, the least squares would be very difficult to use because the course of $f_{\mathcal{R}}$ can vary significantly and so, it is not clear in advance, which kind of the polynomial should be used; and of course, it is far from being simple. Important for us is the stability mentioned at item c) above. Significant is also the fact that F-transform is a fuzzy technique and thus, it is integral with the perception-based deduction in concern.

Let a linguistic description $\mathcal{R}$. Then the function $f_{\mathcal{R}}$ takes the role of the function to be filtered using the F-transform. We choose some *smoothing number*, which is the number of nodes n.

Clearly, there are infinitely many possibilities for choosing the basic functions considered in the direct F-transform. In our case, we want to follow the theory of linguistic evaluating expressions, among which we rank also fuzzy numbers which are expressions of the form

$$\langle\text{linguistic hedge}\rangle[\textit{approximately}]x_0.$$

Extension of the fuzzy number is a fuzzy set Fn_{ν,x_0} where $\nu_{a,b,c}$ (see (4)) is a linguistic hedge and x_0 is the central point around which the fuzzy number is defined. Without going into details, which are not important for our purpose, we only remark that it is analogous to the membership function of the extension $\text{Me}_{\nu,w}$. We explicitly set

$$\text{Fn}_{\nu,x_0}(x) = \begin{cases} 1, & x \in [c^L_{x_0}, c^R_{x_0}], \quad c^L_{x_0} = x_0 - (1-c)h, \\ & \qquad\qquad\qquad\quad c^R_{x_0} = x_0 + (1-c)h, \\ 1 - \frac{(c^L_{x_0} - x)^2}{K_1 h^2}, & x \in [b^L_{x_0}, c^L_{x_0}), \quad b^L_{x_0} = x_0 - (1-b)h, \\ 1 - \frac{(x - c^R_{x_0})^2}{K_1 h^2}, & x \in (c^R_{x_0}, b^R_{x_0}], \quad b^R_{x_0} = x_0 + (1-b)h, \\ \frac{(x - a^L_{x_0})^2}{K_2 h^2}, & x \in (a^L_{x_0}, b^L_{x_0}), \quad a^L_{x_0} = x_0 - (1-a)h, \\ \frac{(a^R_{x_0} - x)^2}{K_2 h^2} & x \in (b^R_{x_0}, a^R_{x_0}), \quad a^R_{x_0} = x_0 + (1-a)h, \\ 0 & x \le a^L_{x_0}, \quad x \ge a^R_{x_0}, \end{cases} \tag{22}$$

Note that $\text{Fn}_{\nu,x_0}(x_0) = 1$ and $\text{Fn}_{\nu,x_0}(x_0 \pm h) = 0$. Thus, one fuzzy number is spread over three neighbouring nodes $x_0 - h, x_0, x_0 + h$. Furthermore,

$$\sum_{k=1}^{n} \text{Fn}_{\nu,x_{0k}}(x) = 1$$

holds for each $x \in w$. Consequently, each $x \in w$ is covered by exactly two neighbouring fuzzy numbers $\text{Fn}_{\nu,x_{0k}}, \text{Fn}_{\nu,x_{0k+1}}$. This means that $x_{0k} \le x \le x_{0k+1}$ and $\text{Fn}_{\nu,x_{0k}}(x) + \text{Fn}_{\nu,x_{0k+1}}(x) = 1$. This property is used below for smooth logical deduction.

Let an observation $u_0 \in w$ be given. Since only one data item is to be filtered, namely $(u_0, f_{\mathcal{R}}(u_0))$, only two numbers F_k, F_{k+1} in (20) are needed for each two nodes x_{0k}, x_{0k+1} such that $x_{0k} \le u_0 \le x_{0k+1}$. To do it we have to choose a step $r > 0$, compute a sequence of values $u_1 = x_{0k-1}, u_2 = u_1 + r, u_3 = u_1 + 2r, \ldots, u_M = x_{0k+2}$ laying between the nodes x_{0k-1} and x_{0k+2} and generate a sequence of auxiliary data

$$\begin{aligned} &(u_1, f_{\mathcal{R}}(u_1)), \\ &\ldots\ldots\ldots\ldots \\ &(u_M, f_{\mathcal{R}}(u_M)) \end{aligned} \tag{23}$$

where each $f_{\mathcal{R}}(u_j)$, $j = 1, \ldots, M$ is a result of the logical deduction (19).

Now, the smooth logical deduction consists of two steps. First we compute two numbers F_k, F_{k+1} according to (20)

$$F_k = \frac{\sum_{j=1}^{M} f_{\mathcal{R}}(u_j) \cdot \mathrm{Fn}_{\nu,x_{0k}}(u_j)}{\sum_{j=1}^{M} \mathrm{Fn}_{\nu,x_{0k}}(u_j)}. \tag{24}$$

Second, compute the resulting smoothed output using the formula

$$f_{\mathcal{R},n}(u_0) = F_k \cdot \mathrm{Fn}_{\nu,x_{0k}}(u_0) + F_{k+1} \cdot \mathrm{Fn}_{\nu,x_{0k+1}}(u_0). \tag{25}$$

The principle of smooth logical deduction is depicted on Figure 3.

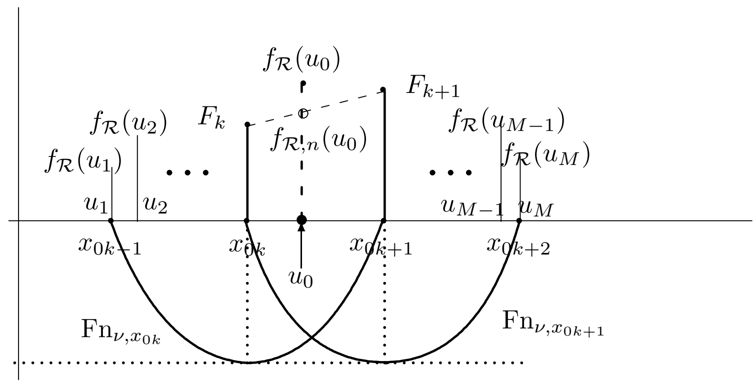

Figure 3: Scheme of the smooth perception-based logical deduction.

From the point of view of the perception-based logical deduction, the above described smoothing behaves as if a new special defuzzification operation. This will be demonstrated in the next section.

5 Demonstration of perception-based logical deduction

In this section, we will demonstrate the power of perception-based logical deduction and compare its original behaviour with the smooth one.

Demonstration 1. First, let us consider a simple linguistic description

Rule	X ⇒ Y
1	VeSm ⇒ RoBi
2	Sm ⇒ Bi
3	Me ⇒ Me
4	Bi ⇒VeSm

We learn from this description that for *very small* input values, the output should be *roughly big*, for *small* it should be *big* (i.e. bigger than for very small), *medium* for *medium* input values and *very small*

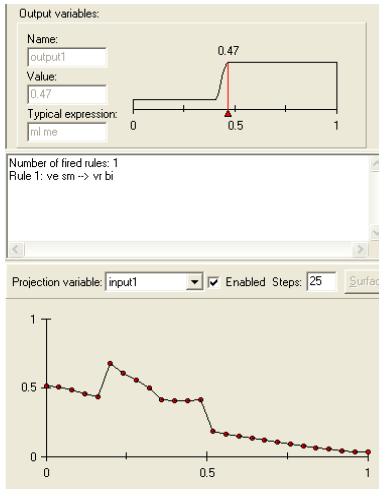

Figure: 4 Result of original (non-smooth) perception-based deduction. In the upper part, the output fuzzy set (extension in the context w') with the result of the DEE defuzzification and the corresponding fired rule for the input $X = 0.1$ are depicted.

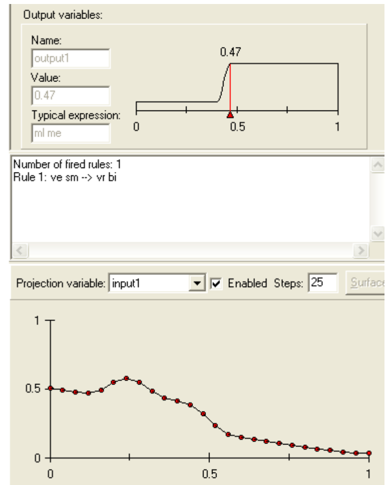

Figure: 5 Result of smooth perception-based deduction. Upper part the same as in Figure 4.

for *big* input values. This behaviour is clearly seen from Figure 4 for the case of the linguistic context, which is the same for both variables, namely $w = w' = \langle 0, 0.4, 1\rangle$. Then indeed, Rule 1 is fired if the input values are *very small*, i.e. around 0.1–0.2. Then the output is *roughly big* (around 0.5–0.6). If they are greater, i.e. *small*, then Rule 2 is fired and the output is *big* (around 0.7). If the input values are medium (around 0.4–0.5), the output is also medium, and if they are *big* (around 0.7 and greater) then the output is *very small* (around 0.2 and smaller). Let us stress that this behavior is independent on the chosen context, i.e. when changing it, the general behaviour will be the same which means that the output values will be different but again corresponding to perceptions of *big, roughly big, medium*, and *very small* in the new context.

Note also, that the output values decrease. This is caused by the evaluation since, e.g. if Rule 1 fires then the truth of the perception *very small* for the input value is computed. The greater is the input value, the less is true that it is indeed "very small" and thus, the less it is true that the output is *roughly big*. Of course, analogous but opposite behavior would is obtained in the case of Rule 4.

It should be stressed that different values (i.e. *small, very small, medium, big*) are distinguished and the output is appropriate. However, all values that are *very small* are at the same time also *small* (but not vice-versa). Hence, if Rule 1 is deleted then the general behaviour is not changed but the output provided by Rule 1 is now taken over by Rule 2. This is well seen from Figure 6.

The result of smooth perception-based deduction on the basis of the same linguistic description is depicted on Figure 5. It is important that the essential shape of the output function has not changed i.e., the above expected behaviour is preserved. On the other hand, it is continuous.

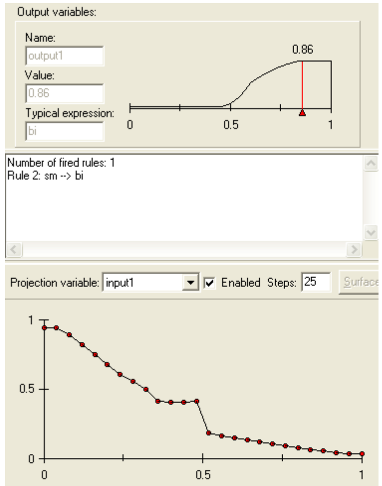

Figure: 6 Result of original (non-smooth) perception-based deduction after Rule 1 is deleted (input $X = 0.1$).

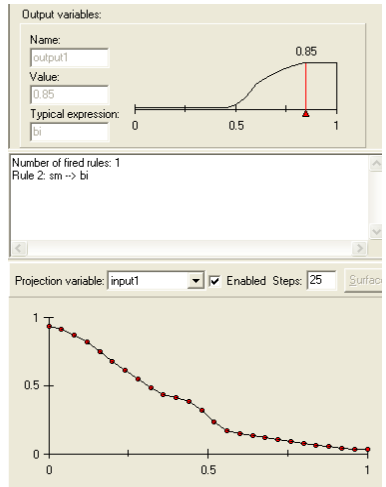

Figure: 7 Result of smooth perception-based deduction after Rule 1 is deleted (input $X = 0.1$).

Let us delete Rule 1. Then its role is taken by Rule 2, i.e. the output is *big* but not *roughly big*; this means that the output values are bigger than in the previous case. The difference is depicted on Figure 6. The smooth deduction on the same description is on Figure 6. Again, the essential behaviour is the same.

Demonstration 2. The following problem has been discussed in literature on fuzzy logic applications. The task is to avoid some obstacle given instructions that, *if the obstacle is very near then we should turn to the left, if it is near then turn to the right, otherwise do nothing.* The perception-based deduction is able to cope with this problem by means of the linguistic description

Rule	X ⇒ Y
1	ExSm ⇒-ExBi
2	VeSm ⇒ -Bi
2	Sm ⇒ +Bi
3	Me ⇒ zero
4	Bi ⇒ zero

where X is the distance of obstacle and Y is turn of the steering wheel.

The result is on Figure 8. One can see that the obstacle is avoided to the left if the perception of the distance is *extremely* or *very small*, otherwise it is avoided to the right. Figure 9 demonstrates the same in the case of smooth perception-based deduction. Position of steering wheel is in both cases changed rapidly (as expected depending of the realised perception), but in smooth deduction, the change is not abrupt.

The behaviour of the perception-based deduction *does not significantly change* also when Rule 1 is deleted, as can be seen from Figures 10 and 11. This is due to the fact that *extremely small* values

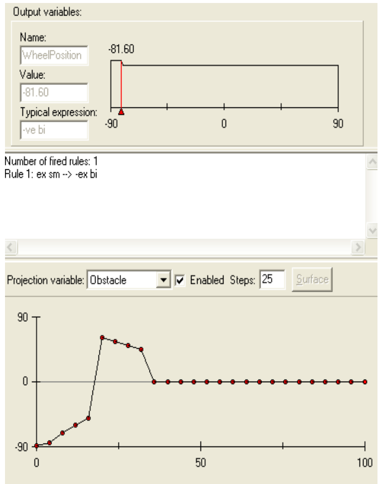

Figure: 8 Avoiding of an obstacle (non-smooth perception-based deduction, input $X = 6$).

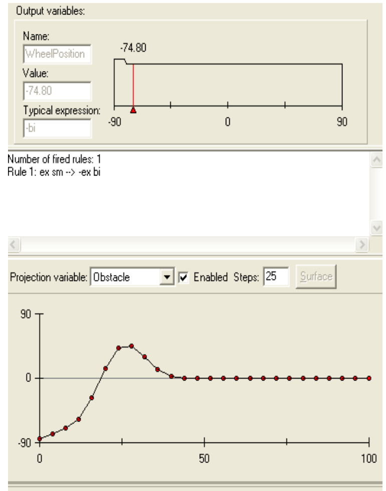

Figure: 9 Avoiding of an obstacle (smooth perception-based deduction, input $X = 6$).

are also *very small*. Let us stress that this behaviour is very important (cf. also Demonstration 1). It clearly shows that the perception-based logical deduction obeys the information contained in the linguistic description, both in the original (non-smooth) as well as smooth case.

Discussion. We have demonstrated behaviour of the perception-based logical deduction on two different linguistic descriptions. The reader could see that this method is able to mimic the way of human reasoning when dealing with conditional statements consisting of evaluating expressions. The disadvantage of piece-wise, smooth and continuous output can be overcome by using the F-transform so that the obtained output is finally continuous and smooth.

Let us note that the demonstrated behaviour in avoiding the obstacle is not possible when applying the usual Mamdani method with COG defuzzification (which, of course, is continuous). If the shapes of fuzzy sets from Fig. 1 are used then the result of Mamdani method necessarily leads to striking the obstacle. To avoid it, we must use symmetrical and little overlapping membership functions, as depicted on Fig. 12. However, these membership functions cannot be extensions of the corresponding evaluating expressions for, at least, two reasons: **first**, these functions break the property that each *extremely small* value is at the same time also *very small*, and thus also *small*. **Second**, it follows from these shapes that there are values, which are very small with the high membership degree and then even *smaller* values which are very small with *smaller membership degree*. Similar counterintuitive conclusion follows for *small*, and also for comparison of the expressions *extremely small*, *very small* and *small*. We conclude that symmetric fuzzy sets cannot be used as extensions of evaluating linguistic expressions from the fundamental linguistic trichotomy. Therefore, Mamdani method is not suitable for modelling of human reasoning based on the use of evaluating expressions and of linguistic descriptions containing them.

We conclude that there are two essential approaches to elaboration of fuzzy IF-THEN rules. The first is Mamdani method which is suitable for approximation of some function and has a lot of applications (e.g. in fuzzy control it proved to be very effective). The second one is the perception-based logical

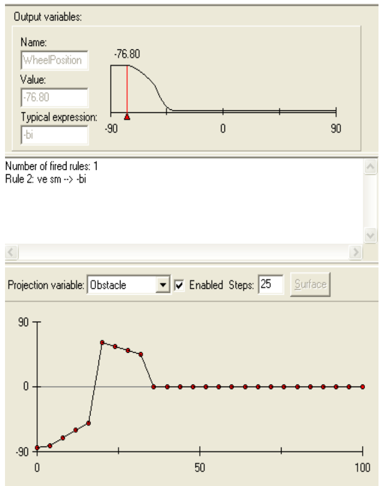

Figure: 10 Avoiding of an obstacle when Rule 1 is deleted (input $X = 6$).

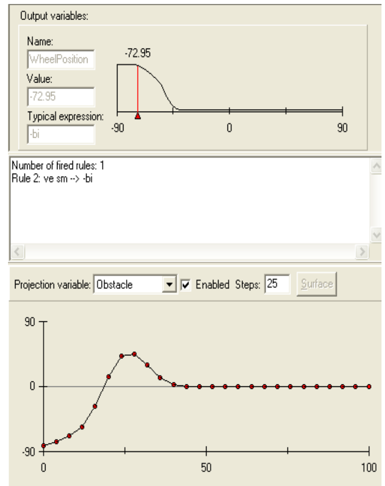

Figure: 11 (Smooth) avoiding of an obstacle when Rule 1 is deleted (input $X = 6$).

deduction. Let us remark that it has great application potential in various decision problems (such as the discussed avoiding the obstacle) but it has been used for fuzzy control, too (see[15]). The possibility for smoothing the output makes it even more attractive.

6 Conclusion

In this paper, we described a method for derivation of a conclusion on the basis of information provided using genuine linguistically characterized fuzzy IF-THEN rules. We call it perception-based logical deduction. Its advantage is the possibility to use linguistic expressions, which are interpreted in accordance with the human way of understanding to them. Thus, the perception-based deduction mimics human way of reasoning.

The disadvantage of this method is that it provides, in general, only piecewise-continuous function. This disadvantage can be overcome when joining this method with special technique of fuzzy approximation called F-transform. The result is a function, which keeps the mentioned advantages and, moreover, it is continuous and smooth.

The perception-based logical deduction has been implemented in the LFLC 2000 software package developed in the University of Ostrava and many times successfully applied to control and decision-making problems (see[5]).

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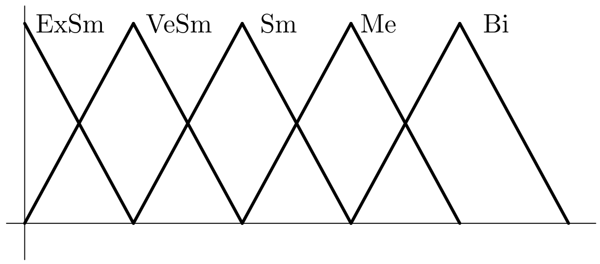

Figure 12: Shapes of extensions of evaluating expressions for the obstacle avoiding problem using Mamdani method.

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