University of Ostrava

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Vilém Novák

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University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic

tel.: +420-597 460 234 fax: +420-597 461 478
e-mail: e-mail

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1 Motivation

Fuzzy logic has been successfully developed including its predicate version especially thanks to P. Hájek and several other authors — see [10,13,26,25]. One of the main arguments in favour of it consists in its ability to provide a working model of some manifestations of the vagueness phenomenon. The latter is encountered in all kinds of human reasoning and is a distinguished feature of the semantics of natural language.

The fact that fuzzy logic touched the vagueness phenomenon led to a large number of its successful applications. Quite often, they rely on a simple model of the meaning of some words of natural language. From linguistic point of view, however, these applications are more or less naive. Moreover, since fuzzy logic has been developed only up to first order, these applications are rather limited. If we want to apply fuzzy logic more deeply in linguistic semantics, higher order formulas have to be employed. This need comes out also in connection with the task to develop a semantic web.

In the accepted linguistic theory, an important role is played by intensional logic (cf. [1,9,29]), which is based on a simple type theory. Thus, a challenge is raised whether the latter could be generalized to fuzzy one.

Type theory has been originated by B. Russel and A. Whitehead in their *Principia matematica*. This theory has various versions, which can essentially be divided into two (overlapping) groups: a classical type theory being a logic of higher order, and a constructive type theory which has extensive applications in theoretical computer science (e.g. analysis of correctness of programs, automated proving) or constructive mathematics. The first group has been initiated especially by A. Church [6] (due to him, classical type theory is often called Church's type theory) and L. Henkin [15,16] and continued in works of P. Andrews [2–4,17] and other authors (cf. citations therein). The second group covers a lot of publications which often refer to the seminal works of P. Martin-Löf [20]. Further on this direction – see [5,11,12] and the citations therein. There are also applications of the constructive type theory in linguistics (cf. e.g. [28]).

In this paper, we will formulate *fuzzy type theory* (FTT) as a *fuzzy logic of higher order* (and thus, belonging to the first of the above discussed groups). The set of truth values is assumed to form an IMTL algebra (on $[0, 1]$, this is algebra of left continuous t-norms with involutive negation). We will follow the way of the development of the classical type theory as elaborated especially by A. Church and L. Henkin where the equivalence/equality belong among the basic connectives. In the fuzzy case this means that the equivalence is many-valued binary function and the equality is a fuzzy equality. Unfortunately, the latter leads to problems, which cannot be solved in full generality since equality as the basic connective must fulfil axioms general for all formulas. In many-valued case, however, this is impossible and in most cases it finally forces the fuzzy equality to be classical crisp one. The only way which turned out possible was to introduce in the language also a special Δ connective which puts to 0 all truth values smaller than 1. Consequently, the fuzzy type theory presented in this paper is based on a linearly ordered IMTL_Δ algebra. This decision enabled us to preserve elegance of classical type theory. Generalizations to other kinds of structures of truth values is possible but it should be done carefully keeping in mind the overall purpose of such theory.

2 Syntax and Semantics of Fuzzy Type Theory

2.1 Introduction

Essential concept in fuzzy type theory is that of *type*. Informally, types can be understood as general characteristics of formulas which precisely determine their interpretation in the semantics; namely, they are used to distinguish special sets with specific properties. A formula of the given type is then interpreted as an element belonging to the set of the given type. In computer science, type often refers to a certain data structure, such as array, record, etc. We use Greek letters to denote types.

There are elementary and complex types. Formulas of elementary types represent objects from the given sets while complex types represent functions. Thus, if α and β are types then a formula A_α of type α represents some object from a set M_α (for precise definitions see below) and similarly, B_β of type β represents some object from a set M_β. A formula $A_{\beta\alpha}$ of type $\beta\alpha$ represents a *function* $M_\alpha \longrightarrow M_\beta$ (which is an element from $M_\beta^{M_\alpha}$).

Essential elementary type is o (omicron). Formulas of type o represent truth values and thus, can be understood as propositions (statements). Thus, formulas of type $o\alpha$ represent fuzzy sets in the universe M_α and those of type $(o\alpha)\alpha$ fuzzy (binary) relations on it.

In type theory, it is a custom to express functions of n variables using only functions of one variable**. For example, a function $y_\gamma = k(x_\beta, x_\alpha)$ can be understood as a function $h_{(\gamma\beta)\alpha}$ of type $(\gamma\beta)\alpha$ since then $h_{(\gamma\beta)\alpha}(x_\alpha)$ is a function $f_{\gamma\beta}$ of type $\gamma\beta$ and thus, we can write $k(x_\beta, x_\alpha) = (h_{(\gamma\beta)\alpha}(x_\alpha))(x_\beta)$. Note that values of the variable x_α can be seen as parameters using which the functions $h_{(\gamma\beta)\alpha}(x_\alpha)$, which are functions of the variable x_β, can be distinguished. Therefore, we will confine to functions of one variable only in the sequel.

2.2 Basic syntactical elements

2.2.1 Types

Let ϵ, o be distinct objects. The set of types is the smallest set *Types* satisfying:

(i) $\epsilon, o \in$ *Types*,
(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

**This trick has been introduced by M. Schönfinkel, Über die Bausteine der mathematischen Logik. Mathematische Annalen **92**(1924), 150–182.

The type ϵ represents elements and o truth values.

Remark 1 (to notation)
In constructive type theory, it is usual to denote types as $A : \alpha$

2.2.2 *Primitive symbols*

The following symbols form the language (basic syntactical elements) from which formulas of FTT are constructed.

(i) Variables $x_\alpha, \ldots$ where $\alpha \in \mathit{Types}$.
(ii) Special constants $c_\alpha, \ldots$ where $\alpha \in \mathit{Types}$. We will consider the following concrete special constants: $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in \mathit{Types}$, $\mathbf{C}_{(oo)o}$ and $\mathbf{D}_{oo}$.
(iii) Auxiliary symbols: λ, brackets.

The given language will be denoted by J.

2.2.3 *Formulas*

The set Form_α is a set of formulas of type $\alpha \in \mathit{Types}$, which is the smallest set satisfying:

(i) $x_\alpha \in \mathit{Form}_\alpha$,
(ii) $c_\alpha \in \mathit{Form}_\alpha$,
(iii) if $B_{\beta\alpha} \in \mathit{Form}_{\beta\alpha}$ and $A_\alpha \in \mathit{Form}_\alpha$ then $(B_{\beta\alpha} A_\alpha) \in \mathit{Form}_\beta$,
(iv) if $A_\beta \in \mathit{Form}_\beta$ then $\lambda x_\alpha\, A_\beta \in \mathit{Form}_{\beta\alpha}$,

The set of all formulas is $\mathit{Form} = \bigcup_{\alpha \in \mathit{Types}} \mathit{Form}_\alpha$. If $A \in \mathit{Form}_\alpha$ is a formula of the type $\alpha \in \mathit{Types}$ then we will write A_α. If we want to specify explicitly the language of which formulas are formed then we will write Form_J.

Remark 2 (to the notation)
We will denote variables by small letters and formulas by capital letters, usually completed by their type written as a subscript. The resulting symbol should be taken as complete and unique, which means that if, e.g. we write A_α *and* A_β*, where* $\alpha \neq \beta$ *then these are treated as* two different *symbols.*

Remark 3
In some modern systems of type theory, the above introduced notation is changed into "one line notation" suitable for computer programs (cf. [5]). Thus, a formula $A_{\beta\alpha}$ is written as $A : \alpha \to \beta$. The readability of such notation, especially in case of complex formulas, is disputable. Therefore, we will stick the original one as proposed by A. Church and still used in other up-to-date logical treatises (cf. [4]). Let us also note that the term "formula" is sometimes replaced by "lambda term". Because of the purely logical character of this paper, we will stick on the former.

2.3 Semantics

To define the semantics of FTT, we have to start with characterization of the structure of truth values.

2.3.1 Truth values

We will suppose the structure of truth values to form a complete IMTL-algebra (see [8]). This is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

where $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a complete lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$. The operation $\otimes$ is multiplication such that $\langle L, \otimes, \mathbf{1} \rangle$ is a commutative monoid and $\to$ is residuation fulfilling the adjunction property

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \to c, \qquad a, b, c \in L \tag{2}$$

(the above definition means that (1) is a residuated lattice). Finally, (1) fulfils also the prelinearity condition

$$(a \to b) \vee (b \to a) = \mathbf{1}, \qquad a, b \in L, \tag{3}$$

and its negation function $\neg a = a \to \mathbf{0}$ is involutive, i.e.

$$\neg\neg a = a, \qquad a \in L. \tag{4}$$

It is known that MTL-algebras on [0, 1] are algebras of left-continuous t-norms where the negation needs not be involutive. An example of left-continuous t-norm with involutive negation is *nilpotent minimum* defined by

$$a \otimes b = \begin{cases} a \wedge b, & \text{if } a + b > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Its residuum is defined by

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise} \end{cases} \tag{6}$$

where $\neg a = 1 - a$. Thus, the algebra where $\otimes$ is the operation (5) and $\rightarrow$ the operation (6) is IMTL-algebra.

Furthermore, we define the following operations for all $a, b \in L$:

$$a \oplus b = \neg(\neg a \otimes \neg b), \qquad \textit{(strong sum)}$$

$$a^n = \underbrace{a \otimes \cdots \otimes a}_{n\text{-times}} \qquad \textit{(strong power)}$$

$$na = \underbrace{a \oplus \cdots \oplus a}_{n\text{-times}} \qquad \textit{(n-fold strong sum)}$$

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \qquad \textit{(biresiduation)}$$

Remark 4
The operation $\leftrightarrow$ has the property $a \leftrightarrow b = \mathbf{1}$ iff $a = b$. It is reflexive, symmetric and transitive in the following sense:

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq a \leftrightarrow c$$

holds true for all $a, b, c \in L$.

For nilpotent minimum, the operation $\leftrightarrow$ is defined by

$$a \leftrightarrow b = (\neg a \wedge \neg b) \vee (a \wedge b). \tag{7}$$

The IMTL-algebra of truth values is further extended by the so called *Baaz delta* (cf. [13]) defined on [0, 1] by

$$\Delta(a) = \begin{cases} 1, & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We will define IMTL$_\Delta$ algebra as the algebra (cf. [8])

$$\mathcal{L}_\Delta = \langle L, \vee, \wedge, \otimes, \rightarrow, \Delta, \mathbf{0}, \mathbf{1} \rangle \tag{9}$$

which is an IMTL-algebra fulfilling the following additional conditions:

$$\Delta a \vee \neg \Delta a = \mathbf{1}, \tag{10}$$
$$\Delta(a \vee b) \leq \Delta a \vee \Delta b, \tag{11}$$
$$\Delta a \leq a, \tag{12}$$
$$\Delta a \leq \Delta\Delta a, \tag{13}$$
$$\Delta(a \rightarrow b) \leq \Delta a \rightarrow \Delta b, \tag{14}$$
$$\Delta \mathbf{1} = \mathbf{1}. \tag{15}$$

If $\mathcal{L}_\Delta$ is linearly ordered then Δ has the property (8).

The following properties hold in IMTL$_\Delta$ algebra:

Lemma 1
Let $a, b, c \in L$. Then

(a) $a \otimes b = \neg(a \rightarrow \neg b)$,
(b) $a \rightarrow b = (a \wedge b) \leftrightarrow a$,
(c) $(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d)$,
(d) $\Delta(a \rightarrow b) \leq \Delta(a) \rightarrow \Delta(b)$.
(e) $\Delta(a \leftrightarrow b) \leq \Delta(a) \leftrightarrow \Delta(b)$.
(f) $\Delta(a \wedge b) \leq \Delta(a) \wedge \Delta(b)$.

PROOF: We will prove only (b). By definition, for all $a, b \in L$ we have

$$(a \wedge b) \leftrightarrow a = ((a \wedge b) \rightarrow a) \wedge (a \rightarrow (a \wedge b)) = a \rightarrow (a \wedge b) = a \rightarrow b$$

since by the properties of residuated lattice, $(a \wedge b) \rightarrow a) = \mathbf{1}$, $a \otimes (a \rightarrow b) \leq a \wedge b$, and $a \rightarrow (a \wedge b) \leq a \rightarrow b$ because of isotonicity of $\rightarrow$ in the second variable. □

2.3.2 Basic frame

Let D be a set of objects and L be a set of truth values. A *basic frame* based on D, L is a family of sets $(M_\alpha)_{\alpha \in Types}$ where

(i) $M_\epsilon = D$ is a set of objects,
(ii) $M_o = L$ is a set of truth values,
(iii) For each type $\gamma = \beta\alpha$, the corresponding set $M_\gamma \subseteq M_\beta^{M_\alpha}$.

We say that the basic frame is *standard* if for each γ in item (iii), $M_\gamma = M_\beta^{M_\alpha}$ holds true.

Remark 5 (to the notation)
If M_α is a set from the basic frame then we denote its elements by lower case letters also indexed by α, i.e. $m_\alpha \in M_\alpha$ means that m_α is an element from the set M_α. Note that in general, each such element is a function (except for basic types o, ε).

Furthermore, we will use the following notational convention. Let $m_{\beta\alpha} \in M_\beta^{M_\alpha}$ be an element of the type $\beta\alpha$. This means that it is a function $m_{\beta\alpha} : M_\alpha \longrightarrow M_\beta$ assigning to each element $m_\alpha \in M_\alpha$ some element $m_\beta \in M_\beta$. But given a concrete element m_α, we need to denote its functional value w.r.t. the function $m_{\beta\alpha}$. By abuse of notation, we will use the symbol $m_{\beta\alpha}(m_\alpha)$ and understand that $m_{\beta\alpha}(m_\alpha) = m_\beta$ for some (concrete) $m_\beta \in M_\beta$.

2.3.3 Fuzzy equality

The basic frame is not sufficient for the definition of the interpretation of fuzzy type theory since we work with various kinds of equality defined on its members. In general, we introduce the concept of *fuzzy equality* on a set M. This is a binary fuzzy relation $\doteq \underset{\sim}{\subset} M \times M$, i.e. a function

$$\doteq : M \times M \longrightarrow L.$$

If $m, m' \in M$ then we will usually write $m \doteq m'$ instead of $\doteq(m, m')$. To stress that $m \doteq m'$ holds in some degree $c \in L$, we will write $[m \doteq m'] = c \in L$, or simply $[m \doteq m']$.

The fuzzy equality is supposed to fulfil the following conditions:

(i) reflexivity

$$[m \doteq m] = \mathbf{1}, \qquad m \in M,$$

(ii) symmetry

$$[m \doteq m'] = [m' \doteq m], \qquad m, m' \in M,$$

(iii) $\otimes$-transitivity

$$[m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m''], \qquad m, m', m'' \in M.$$

The fuzzy equality is a generalized equivalence. It is also used as interpretation of the generalized equality predicate in fuzzy logic (hence, the name).

We say that a fuzzy equality $=$ is *1-faithful*** if

$$[m \doteq m'] = \mathbf{1} \quad \text{iff} \quad m = m'$$

holds for all $m, m' \in M$.

It follows from Remark 4 that the biresiduation $\leftrightarrow$ is a 1-faithful fuzzy equality.

Example 1
Let M be a set and the algebra of truth values be the IMTL-algebra on $[0, 1]$ with nilpotent minimum as a t-norm. Let us put

$$[m \doteq m'] = \begin{cases} \mathbf{1}, & \text{iff } m = m', \\ a, & \text{otherwise} \end{cases}$$

for all $m, m' \in M$ where $0 < a \leq \frac{1}{2}$. Then $\doteq$ is a 1-faithful fuzzy equality on M.

In correspondence with Remark 5, if M_α is a set from the basic frame then a fuzzy equality on it is denoted by $=_\alpha$.

Lemma 2
Let $=_\beta$ be a fuzzy equality. Then the function

$$=_{\beta\alpha}: M_\beta^{M_\alpha} \times M_\beta^{M_\alpha} \longrightarrow L$$

defined for every $m_{\beta\alpha}, m'_{\beta\alpha} \in M_\beta^{M_\alpha}$ by

$$[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] = \bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] \tag{16}$$

**This term has been proposed by P. Hájek.

is a fuzzy equality. If $=_\beta$ is 1-faithful then $=_{\beta\alpha}$is also 1-faithful.

PROOF: Let $m_\alpha, m'_\alpha, m''_\alpha \in M_\alpha$.

(a) *reflexivity:*

$$[m_{\beta\alpha} =_{\beta\alpha} m_{\beta\alpha}] = \bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m_{\beta\alpha}(m_\alpha)] = \bigwedge_{m_\alpha \in M_\alpha} \mathbf{1} = \mathbf{1}.$$

(b) *symmetry:*

$$[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] = \bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] = \\ = \bigwedge_{m_\alpha \in M_\alpha} [m'_{\beta\alpha}(m_\alpha) =_\beta m_{\beta\alpha}(m_\alpha)] = [m'_{\beta\alpha} =_{\beta\alpha} m_{\beta\alpha}]$$

because of the symmetry of $=_\beta$.

(c) *transitivity:*

$$[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] \otimes [m'_{\beta\alpha} =_{\beta\alpha} m''_{\beta\alpha}] = \\ = \bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] \otimes \bigwedge_{m_\alpha \in M_\alpha} [m'_{\beta\alpha}(m_\alpha) =_\beta m''_{\beta\alpha}(m_\alpha)] \leq \\ = \bigwedge_{m_\alpha \in M_\alpha} \bigwedge_{\bar{m}_\alpha \in M_\alpha} \left([m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] \otimes [m'_{\beta\alpha}(\bar{m}_\alpha) =_\beta m''_{\beta\alpha}(\bar{m}_\alpha)]\right). \tag{17}$$

Since $=_\beta$ is $\otimes$-transitive, we have

$$[m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] \otimes [m'_{\beta\alpha}(\bar{m}_\alpha) =_\beta m''_{\beta\alpha}(\bar{m}_\alpha)] \leq [m_{\beta\alpha}(m_\alpha) =_\beta m''_{\beta\alpha}(\bar{m}_\alpha)]$$

for all $m_\alpha = \bar{m}_\alpha$. Consequently, the right-hand side of (17) is not greater than

$$\bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m''_{\beta\alpha}(m_\alpha)] = [m_{\beta\alpha} =_{\beta\alpha} m''_{\beta\alpha}]$$

by the properties of infimum.

Let $=_\beta$ be 1-faithful. Then $[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] = \mathbf{1}$ iff for all $m_\alpha \in M_\alpha$ it holds that $[m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m_\alpha)] = \mathbf{1}$ iff $m_{\beta\alpha}(m_\alpha) = m'_{\beta\alpha}(m_\alpha)$ iff $m_{\beta\alpha} = m'_{\beta\alpha}$. $\square$

Example 2
Let $M_\epsilon = [u, v] \subset \mathbb{R}$ be an interval of real numbers and the algebra of truth values be the IMTL-algebra on $[0, 1]$ with nilpotent minimum as a t-norm. Let us put $d = v - u$ and

$$[m =_\epsilon n] = \frac{1}{d} \cdot ((v - m) \wedge (v - n)) \vee ((m - u) \wedge (n - u))$$

for all $m, n \in M_\epsilon$. Then $=_\epsilon$ is a 1-faithful fuzzy equality on M_ϵ and $[m =_\epsilon n] \in (0, 1)$ holds for all $m \neq n$, $m, n \in (u, v)$. If $\mu, \mu' \in M_o^{M_\epsilon}$ are fuzzy sets in M_ϵ then (16) defines a 1-faithful fuzzy equality between them.

Let μ_m be a fuzzy set in M_ϵ defined by

$$\mu_m(z) = \frac{q_m - |z - m|}{q_m}, \qquad q_m < d.$$

This is a triangular fuzzy set with the kernel m and span q_m. If μ_m and $\mu_{m'}$ are triangular fuzzy sets such that $m' - q_{m'} < m$ and $m' < m + q_m$ (i.e. they are close enough) then $[\mu_m =_{o\epsilon} \mu_{m'}] \in (0, 1]$.

2.3.4 Extensional functions

Using the fuzzy equality, we can define the important concept of extensionality of functions (cf [13,19,27]). Let $F : M_{\alpha_1} \times \cdots \times M_{\alpha_n} \longrightarrow M_\beta$ be a function. In the following definition, we use the strong power defined in Subsection 2.3.1.

We say that F is *extensional w.r.t fuzzy equalities* $=_{\alpha_1}, \ldots, =_{\alpha_n}, =_\beta$ if there are exponents $q_1, \ldots, q_n \geq 1$ such that the inequality

$$[m_{\alpha_1} =_{\alpha_1} m'_{\alpha_1}]^{q_1} \otimes \cdots \otimes [m_{\alpha_n} =_{\alpha_n} m'_{\alpha_n}]^{q_n} \leq \\ \leq [F(m_{\alpha_1}, \ldots, m_{\alpha_n}) =_\beta F(m'_{\alpha_1}, \ldots, m'_{\alpha_n})] \quad (18)$$

holds for all $m_{\alpha_i}, m'_{\alpha_i} \in M_{\alpha_i}$, $i = 1, \ldots, n$.

We will distinguish the following types of extensionality. If $q_1 = \cdots = q_n = 1$ holds in (18) then we say that F is *strongly extensional* otherwise it is simply *extensional*. We say that F is *weakly extensional* if

$$[m_{\alpha_1} =_{\alpha_1} m'_{\alpha_1}] = \ldots = [m_{\alpha_n} =_{\alpha_n} m'_{\alpha_n}] = \mathbf{1}$$

implies that

$$[F(m_{\alpha_1}, \ldots, m_{\alpha_n}) =_\beta F(m'_{\alpha_1}, \ldots, m'_{\alpha_n})] = \mathbf{1}.$$

It is easy to see that a strongly extensional function is both extensional as well as weakly extensional, and an extensional function is weakly extensional.

Remark 6
If $=_{\alpha_1}, \ldots, =_{\alpha_n}, =_\beta$ are known from the context then we often say that F is (weakly, strongly) extensional.

Lemma 3
Each fuzzy equality $=_\alpha$ on M_α is strongly extensional w.r.t. itself and $\leftrightarrow$, i.e.

$$[m_{\alpha,1} =_\alpha m'_{\alpha,1}] \otimes [m_{\alpha,2} =_\alpha m'_{\alpha,2}] \leq [m_{\alpha,1} =_\alpha m_{\alpha,2}] \leftrightarrow [m'_{\alpha,1} =_\alpha m'_{\alpha,2}]$$

holds true for all $m_{\alpha,1}, m'_{\alpha,1}, m_{\alpha,2}, m'_{\alpha,2} \in M_\alpha$.

PROOF: This follows from $\otimes$-transitivity and symmetry of $=_\alpha$ using adjunction and definition of biresiduation. □

It follows from Lemma 1(d) that the meet operation $\wedge$ on L is strongly extensional w.r.t. $\leftrightarrow$. Since by Lemma 3, all fuzzy equalities are strongly extensional, it is also strongly extensional (w.r.t. itself). Note that Δ is weakly extensional w.r.t. $\leftrightarrow$. Indeed, let $a \leftrightarrow a' = \mathbf{1}$. Then $a = a'$ and thus, $\Delta(a) \leftrightarrow \Delta(a') = \mathbf{1}$.

Remark 7
Ordinary extensionality seems to be a rather strong requirement. In this paper we mostly confine ourselves only to weak extensionality of the considered functions. Note that this, in fact, slightly generalizes the classical definition of a function. It may include also relations not being functions. Indeed, let F be a binary relation such that $[y_1 =_\alpha y_2] = \mathbf{1}$ holds for all $y_1 = F(x)$, $y_2 = F(x)$ and $y_1 \neq y_2$. Then F is weakly extensional. However, in this paper we will consider only the case when F is an ordinary function where extensionality is its additional property.

We may also see weakly extensional functions considered in this paper as special homomorphisms. Let $F : M_\alpha \longrightarrow M_\beta$ be a function and define relations $R_\alpha(x, y)$ iff $[x =_\alpha y] = \mathbf{1}$, $x, y \in M_\alpha$ and similarly, $R_\beta(u, v)$ iff $[u =_\beta v] = \mathbf{1}$, $u, v \in M_\beta$. Then F is weakly extensional iff it is a homomorphism with respect to R_α, R_β, i.e.

$$R_\alpha(x, y) \quad \textit{implies} \quad R_\beta(F(x), F(y)), \quad x, y \in M_\alpha.$$

2.3.5 Frame

Let $(M_\alpha)_{\alpha \in Types}$ be a basic frame. Then a *frame* is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{19}$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ is a structure of truth values being a complete, linearly ordered IMTL$_\Delta$ algebra. We put $M_o = L$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.

(ii) $=_\alpha$ is a fuzzy equality on M_α and $=_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$. Moreover,

  (a) If $\alpha = o$ then $=_o$ is $\leftrightarrow$.

  (b) If $\alpha = \epsilon$ then $=_\epsilon \subsetapprox M_\epsilon \times M_\epsilon$ is a fuzzy equality on the set M_ϵ (recall that M_ϵ is a set of objects).

  (c) If $\alpha \neq o, \epsilon$ then $=_\alpha$ is the fuzzy equality given in (16).

(iii) If $\alpha = \gamma\beta$ then each function $F \in M_{\gamma\beta}$ is weakly extensional w.r.t $=_\beta$ and $=_\gamma$. More precisely, if $F : M_\beta \longrightarrow M_\gamma$ then

$$[m_\beta =_\beta m'_\beta] = \mathbf{1} \quad \text{implies} \quad [F(m_\beta) =_\gamma F(m'_\beta)] = \mathbf{1}. \tag{20}$$

2.3.6 Basic definitions

Before we introduce interpretation of formulas, we will introduce the following special definitions.

(a) Equivalence:

$$\equiv \; := \lambda x_\alpha (\lambda y_\alpha \mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha.$$

This is a formula of type $(o\alpha)\alpha$. We will write $x_o \equiv y_o$ instead of $(\equiv y_o)x_o$. Clearly, if $A_\alpha, B_\alpha \in Form_\alpha$ then $(A_\alpha \equiv B_\alpha) \in Form_o$.

(b) Conjunction:

$$\wedge \;:=\; \lambda x_o(\lambda y_o \mathbf{C}_{(oo)o}\, y_o)x_o.$$

This is a formula of type $(oo)o$. In the same spirit as above, we will write $x_o \wedge y_o$ instead of $(\wedge x_o)y_o$.

(c) Baaz delta:

$$\boldsymbol{\Delta} \;:=\; \lambda x_o \mathbf{D}_{oo}\, x_o.$$

This is a formula of type oo.

An occurrence of x_α is *free* in a formula A if it is not in a well formed part of A of the form $\lambda x_\alpha B$, otherwise it is *bound*. A is *substitutable* for x_α in B if no free occurrence of x_α in B is in a well formed part of B of the form $\lambda y_\beta C$ such that y_β is a free variable of $A^{\star\star\star}$. If A is substitutable for x_α in B then we will write

$$B_{x_\alpha}[A].$$

2.3.7 *Interpretation*

Let a frame $\mathcal{M}$ be given. We will define an interpretation $\mathcal{I}^{\mathcal{M}}$ of all formulas, i.e. assignment of the meaning to them.

First, we define an assignment p to the variables over $\mathcal{M}$. This is a function on variables such that

$$p(x_\alpha) \in M_\alpha$$

for every type $\alpha \in$ *Types*. The set of all assignments over $\mathcal{M}$ will be denoted by $\mathrm{Asg}(\mathcal{M})$.

Let x_α be a variable and $p, p' \in \mathrm{Asg}(\mathcal{M})$ be two assignments such that $p'(x_\alpha) \neq p(x_\alpha)$ and $p'(y_\gamma) = p(y_\gamma)$ for all $y_\gamma \neq x_\alpha$ (i.e. p' differs from p only in the variable x_α). In this case, we will write $p' = p\backslash x_\alpha$.

Now we define:

(i) If x_α is a variable then $\mathcal{I}^{\mathcal{M}}_p(x_\alpha) = p(x_\alpha)$.

(ii) If c_α is a constant then $\mathcal{I}^{\mathcal{M}}_p(c_\alpha)$ is some element from M_α. If $\alpha \neq o, \epsilon$ then $p(c_\alpha)$ is a weakly extensional function. As a special case:

***Note that these notions are direct analogue of classical predicate logic where the role of quantifiers is taken by λ.

(a) $\mathcal{I}_p^{\mathcal{M}}(\mathbf{E}_{(o\alpha)\alpha}) : M_\alpha \to L^{M_\alpha}$ is a fuzzy equality $=_\alpha$. Precisely, it is a function such that for all $m, m' \in M_\alpha$

$$\mathcal{I}_p^{\mathcal{M}}(\mathbf{E}_{(o\alpha)\alpha})(m')(m) = [m =_\alpha m'] \in L$$

holds true.

In more details: if $\alpha = \epsilon$ then $\mathcal{I}_p^{\mathcal{M}}(\mathbf{E}_{(o\epsilon)\epsilon})$ is the fuzzy equality $=_\epsilon$; if $\alpha = o$ then $\mathcal{I}_p^{\mathcal{M}}(\mathbf{E}_{(oo)o})$ is the biresiduation $\leftrightarrow$. Otherwise, $\mathcal{I}_p^{\mathcal{M}}(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality $=_\alpha$ defined by (16).

(b) $\mathcal{I}_p^{\mathcal{M}}(\mathbf{C}_{(oo)o}) : L \to L^L$ is the meet operation $\wedge$. Thus,

$$\mathcal{I}_p^{\mathcal{M}}(\mathbf{C}_{(oo)o})(a)(b) = a \wedge b$$

for all $a, b \in L$.

(c) $\mathcal{I}_p^{\mathcal{M}}(\mathbf{D}_{oo}) : L \to L$ is the Baaz delta operation Δ. Thus,

$$\mathcal{I}_p^{\mathcal{M}}(\mathbf{D}_{oo})(a) = \Delta(a)$$

for all $a \in L$.

(iii) An interpretation of a formula $B_{\beta\alpha}A_\alpha$ of type β is

$$\mathcal{I}_p^{\mathcal{M}}(B_{\beta\alpha}A_\alpha) = \mathcal{I}_p^{\mathcal{M}}(B_{\beta\alpha})(\mathcal{I}_p^{\mathcal{M}}(A_\alpha)).$$

(iv) An interpretation of a formula $\lambda x_\alpha\, A_\beta$ of type $\beta\alpha$ is a function

$$\mathcal{I}_p^{\mathcal{M}}(\lambda x_\alpha\, A_\beta) = F : M_\alpha \longrightarrow M_\beta$$

which is weakly extensional w.r.t "$=_\alpha$" and "$=_\beta$" and such that for each $m_\alpha \in M_\alpha$, $F(m_\alpha) = \mathcal{I}_{p'}^{\mathcal{M}}(A_\beta)$ for some assignment $p' = p\backslash x_\alpha$.

It follows from the previous definitions that

$$\mathcal{I}_p^{\mathcal{M}}(A_\alpha \equiv B_\alpha) = [\mathcal{I}_p^{\mathcal{M}}(A_\alpha) =_\alpha \mathcal{I}_p^{\mathcal{M}}(B_\alpha)]$$

where '$=_\alpha$' is '$\leftrightarrow$' if $\alpha = o$, '$=_\alpha$' is '$=_\epsilon$' if $\alpha = \epsilon$, and

$$[\mathcal{I}_p^{\mathcal{M}}(A_{\gamma\beta}) =_{\gamma\beta} \mathcal{I}_p^{\mathcal{M}}(B_{\gamma\beta})] = \bigwedge_{m_\beta \in M_\beta} [\mathcal{I}_p^{\mathcal{M}}(A_{\gamma\beta})(m_\beta) =_\gamma \mathcal{I}_p^{\mathcal{M}}(B_{\gamma\beta})(m_\beta)] \quad (21)$$

if $\alpha = \gamma\beta$.

The following lemma is a consequence of the definition of interpretation.

Lemma 4
Let $\mathcal{M}$ be a standard basic frame and $p \in Asg(\mathcal{M})$ an assignment. Then

$$\mathcal{I}_p^{\mathcal{M}}(A_\alpha) \in M_\alpha$$

holds for every formula A_α and $\alpha \in Types$.

2.3.8 General model

Lemma 4 states that our definition of interpretation keeps types. However, our definition of the frame is more general and so, similarly as is considered in [15] (cf. also [4]), we suppose that given a type $\gamma = \beta\alpha$, the corresponding set M_γ may be only a subset $M_\gamma \subset M_\beta^{M_\alpha}$. However, then it is not, in general, assured that $\mathcal{I}_p^{\mathcal{M}}(A_\gamma) \in M_\gamma$ holds for every formula A_γ. This leads us to the following concept.

A *general model* is a frame $\mathcal{M}$ such that for every formula A_α, $\alpha \in Types$ and every assignment $p \in \mathrm{Asg}(\mathcal{M})$, the interpretation $\mathcal{I}_p^{\mathcal{M}}$ gives

$$\mathcal{I}_p^{\mathcal{M}}(A_\alpha) \in M_\alpha. \tag{22}$$

This means that the value of each formula is always defined in the general model. Existence of general models is proved in the second half of this paper.

Remark 8
The concept of general model is the same as introduced by Henkin in [15]. The standard model means, in our case, that given a type $\gamma = \beta\alpha$, the set M_γ is the set of all weakly extensional functions. It coincides with the classical standard model only if all the fuzzy equality relations are crisp (i.e. $[x =_\alpha y] = \mathbf{1}$ iff $x = y$ and $[x =_\alpha y] = \mathbf{0}$ otherwise).

2.3.9 Further definitions

In this subsection, we define additional special formulas used in further explanation.

(a) Representation of truth

$$\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o)$$

(b) Representation of falsity

$$\bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top)$$

(c) Negation:

$$\neg := \lambda x_o(\bot \equiv x_o)$$

(d) Implication:

$$\Rightarrow \; := \lambda x_o(\lambda y_o((x_o \wedge y_o) \equiv x_o))$$

(e) Special connectives:

$$\vee := \lambda x_o(\lambda y_o(((x_o \Rightarrow y_o) \Rightarrow y_o) \wedge ((y_o \Rightarrow x_o) \Rightarrow x_o))), \qquad (\textit{disjunction})$$

$$\& := \lambda x_o(\lambda y_o(\neg(x_o \Rightarrow \neg y_o))), \qquad (\textit{strong conjunction})$$

$$\nabla := \lambda x_o(\lambda y_o(\neg(\neg x_o \& \neg y_o))). \qquad (\textit{strong disjunction})$$

All these connectives will be written in infix form as usual.

(f) Quantifiers: Let $A_o \in Form_o$ and x_α be a variable of type α. Then we put:

$$(\forall x_\alpha)A_o := (\lambda x_\alpha\, A_o \equiv \lambda x_\alpha\, \top), \qquad (\textit{general quantifier})$$

$$(\exists x_\alpha)A_o := \neg(\forall x_\alpha)\neg A_o. \qquad (\textit{existential quantifier})$$

As a special case, if $A \in Form_o$ then we put

$$A^n := \underbrace{A \& \cdots \& A}_{n-\text{times}}, \qquad (n\textit{-fold strong conjunction})$$

$$nA := \underbrace{A \nabla \cdots \nabla A}_{n-\text{times}}. \qquad (n\textit{-fold strong disjunction})$$

With respect to the above definition, $\bot, \top, (\forall x_\alpha)A_o, (\exists x_\alpha)A_o \in Form_o$, $\neg \in Form_{oo}$ and $\Rightarrow, \vee, \&, \nabla \in Form_{(oo)o}$.

Lemma 5

Let $A_o, B_o \in Form_o$. Then the following holds true for every assignment $p \in Asg(\mathcal{M})$:

(a) $\mathcal{I}_p^{\mathcal{M}}(\top) = \mathbf{1}$,
(b) $\mathcal{I}_p^{\mathcal{M}}(\bot) = \mathbf{0}$,
(c) $\mathcal{I}_p^{\mathcal{M}}(\neg A_o) = \mathcal{I}_p^{\mathcal{M}}(A_o) \to \mathbf{0}$,
(d) $\mathcal{I}_p^{\mathcal{M}}(A_o \vee B_o) = \mathcal{I}_p^{\mathcal{M}}(A_o) \vee \mathcal{I}_p^{\mathcal{M}}(B_o)$,

(e) $\mathcal{I}_p^{\mathcal{M}}(A_o \Rightarrow B_o) = \mathcal{I}_p^{\mathcal{M}}(A_o) \to \mathcal{I}_p^{\mathcal{M}}(B_o)$,
(f) $\mathcal{I}_p^{\mathcal{M}}(A_o \,\&\, B_o) = \mathcal{I}_p^{\mathcal{M}}(A_o) \otimes \mathcal{I}_p^{\mathcal{M}}(B_o)$,
(g) $\mathcal{I}_p^{\mathcal{M}}(A_o \nabla B_o) = \mathcal{I}_p^{\mathcal{M}}(A_o) \oplus \mathcal{I}_p^{\mathcal{M}}(B_o)$,
(h)

$$\mathcal{I}_p^{\mathcal{M}}((\forall x_\alpha)A_o) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \setminus x_\alpha}} \mathcal{I}_{p'}^{\mathcal{M}}(A_o),$$

(i)

$$\mathcal{I}_p^{\mathcal{M}}((\exists x_\alpha)A_o) = \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \setminus x_\alpha}} \mathcal{I}_{p'}^{\mathcal{M}}(A_o).$$

PROOF: (a) This reduces to

$$\mathcal{I}_p^{\mathcal{M}}(\top) = \bigwedge_{a \in L} (a \leftrightarrow a) = \mathbf{1}.$$

(b)

$$\mathcal{I}_p^{\mathcal{M}}(\bot) = \bigwedge_{a \in L} (a \leftrightarrow \mathbf{1}) = \bigwedge_{a \in L} a = \mathbf{0}.$$

(c)–(g) are obvious.

(h)

$$\mathcal{I}_p^{\mathcal{M}}((\forall x_\alpha)A_o) = [\mathcal{I}_p^{\mathcal{M}}(\lambda x_\alpha A_o) =_{o\alpha} \mathcal{I}_p^{\mathcal{M}}(\lambda x_\alpha \top)] = \\ = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \setminus x_\alpha}} (\mathcal{I}_{p'}^{\mathcal{M}}(A_o) \leftrightarrow \mathbf{1}) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' = p \setminus x_\alpha}} \mathcal{I}_{p'}^{\mathcal{M}}(A_o)$$

(i) follows from the definition of $\exists$ and the above properties. □

A natural question arises whether FTT is indeed a generalisation of the classical type theory. Since the set of truth values contains more than two truth values, we can define atomic formulas of type o whose interpretation in a general model is different from $\mathbf{0}$ as well as $\mathbf{1}$ and also, e.g. the law of excluded middle $A_o \vee \neg A_o$ is not true. The following lemma demonstrates that there are also non-atomic formulas for which this holds.

Lemma 6
There is a general model $\mathcal{M}$ in fuzzy type theory which is not classical; there are (non-atomic) formulas A_o such that $\mathcal{I}_p^{\mathcal{M}}(A_o) \neq \mathbf{0}, \mathbf{1}$ and $\mathcal{I}_p^{\mathcal{M}}(A_o \vee$

$\neg A_o) \neq \mathbf{1}$.

PROOF: Let us define a general model $\mathcal{M}$ as follows. We put $M_o = [0, 1]$ and let $\mathcal{L}_\Delta$ be the IMTL$_\Delta$-algebra with nilpotent minimum as a t-norm. Let M_ϵ be some set and $=_\epsilon$ be a fuzzy equality from Example 1. Let x_ϵ, y_ϵ be variables and put $A_o := x_\epsilon \equiv y_\epsilon$. Let the assignment $p \in \text{Asg}(\mathcal{M})$ be $p(x_\epsilon) = m, p(y_\epsilon) = m'$ where $m \neq m'$. Then

$$\mathcal{I}_p^{\mathcal{M}}(A_o) = a \text{ and } \mathcal{I}_p^{\mathcal{M}}(\neg A_o) = 1 - a.$$

Since $0 < a \leq \frac{1}{2}$, we obtain $0 < \mathcal{I}_p^{\mathcal{M}}(A_o \vee \neg A_o) < 1$.

Another possibility is to use Example 2 for construction of the general model and construct non-atomic formulas with such properties. □

We see from this lemma that the general model of FTT does not collapse into classical general model.

2.3.10 Axioms

In the formulas below, we use variables of various types being elements of the language as well as metavariables A, B representing formulas of various types. In practice, the distinction between the former and the latter is inessential because of the substitution Theorem 6 proved later. We use both kinds of variables in different axioms because of historical continuity.

The following formulas of type o are logical axioms of fuzzy type theory.

(FT1) $\Delta(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(FT2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$

(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

(FT4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

(FT5) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$

(FT6) $(A_o \equiv \top) \equiv A_o$

(FT7) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$
(FT8) $(A_o \Rightarrow (B_o \Rightarrow C_o)) \equiv (B_o \Rightarrow (A_o \Rightarrow C_o))$
(FT9) $((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o)$
(FT10) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$
(FT11) $A_o \wedge B_o \equiv B_o \wedge A_o$
(FT12) $A_o \wedge B_o \Rightarrow A_o$
(FT13) $(A_o \wedge B_o) \wedge C_o \equiv A_o \wedge (B_o \wedge C_o)$
(FT14) $(g_{oo}(\boldsymbol{\Delta} x_o) \wedge g_{oo}(\neg \boldsymbol{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\boldsymbol{\Delta} y_o)$
(FT15) $\boldsymbol{\Delta}(A_o \wedge B_o) \equiv \boldsymbol{\Delta} A_o \wedge \boldsymbol{\Delta} B_o$
(FT16) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$ where x_α is not free in A_o

As can be seen, the axioms form two groups: axioms for general types and axioms characterizing only truth values. The first group is formed of axioms (FT1)–(FT3). Axioms (FT1)–(FT3) and (FT14) resemble axioms of classical type theory (cf. [4,15]).

Axiom (FT1) states that all functions must be weakly extensional w.r.t. $\equiv$. Axiom (FT2$_1$) states that if two functions are equal for all arguments then they are equal. Axiom (FT2$_2$) is opposite stating that if two functions are equal then they must be equal for its arguments. Let us remark that these two axioms imply Theorem 11 below, which is axiom of classical type theory. Axiom (FT3) is precisely the classical lambda conversion enabling to replace lambda expressions by the corresponding formulas. Its specific cases are historically called alpha and beta conversions (cf. [4]). Axiom (FT4) states that the fuzzy equality on objects is transitive.

Axiom (FT5) characterizes logical equivalence as biresiduation. Axiom (FT6) is an important property of the syntax of classical type theory where, however, it is a provable formula from (the classical form of) Axiom (FT14). Axioms (FT7)–(FT10) for logical implication and (FT11)–(FT15) for ordinary conjunction characterize structure of truth values. Specific is Axiom (FT14) (resembling axiom 1. of classical type theory; cf. [4]) which states that the Δ operation provides only two truth values (namely the limit ones **0**, **1**).

The last Axiom (FT16) is classical axiom of predicate logic (both classical as well as fuzzy). Let us remark that in classical type theory, it is a provable formula. The proof relies on the assumption that there are two truth values only and, therefore, it uses stronger variant of Theorem 14 (Rule of Two Cases). However, since in FTT, implication is interpreted as inequality of truth values, Theorem 14 is insufficient for its proof.

2.3.11 *Inference rule and provability*

The following are inference rules in FTT.

(R) *Let $A_\alpha \equiv A'_\alpha \in Form_o$ and $B_o \in Form_o$ be formulas. Then we infer from them a formula B'_o which comes from B_o by replacing one occurrence of A_α by A'_α, provided that the occurrence of A_α in B_o is not an occurrence of a variable immediately preceded by λ.*

(N) *Let $A_o \in Form_o$ be a formula. Then from A_o infer $\mathbf{\Delta} A_o$.*

Rule (R) is in unchanged form taken from [4] or [15]. Rule (N) is the necessitation rule introduced in [13].

The concept of *provability* and *proof* are defined in the same way as in classical logic. A *theory* T over FTT is a set of formulas of type o, i.e. $T \subseteq Form_o$. As usual, T is determined by a *set of its special axioms* which is again a subset of $Form_o$. If T is a theory then its language will be written as $J(T)$.

If $A_o \in Form_o$ is a formula then it is provable in T if there is a proof of A_o (among axioms occurring in it may be also special axioms of T). Then we write $T \vdash A_o$, as usual. A general model $\mathcal{M}$ is a *model of a theory T* if $\mathcal{I}_p^{\mathcal{M}}(A_o) = \mathbf{1}$ holds for all axioms of T.

Let T be a theory, $J' \supset J(T)$ some extension of the language of T and $K \subset Form_{J',o}$ a set of formulas in the language J'. Then $T' = T \cup K$ is *extension* of T where K are added to the special axioms of T. The extension T' is conservative if $T' \vdash A_o$ implies $T \vdash A_o$ for every formula $A_o \in Form_{J(T),o}$. As usual, we will write $T, A_o \vdash B_o$ instead of $T \cup \{A_o\} \vdash B_o$.

Lemma 7
For every frame $\mathcal{M}$, interpretation $\mathcal{I}^{\mathcal{M}}$ and assignment p, $\mathcal{I}_p^{\mathcal{M}}(FTi) = \mathbf{1}$

where $i = 1, \dots, 16$.

PROOF: Axiom (FT1): let $p(x_\alpha) = m_\alpha \in M_\alpha$, $p(y_\alpha) = m'_\alpha \in M_\alpha$ and $p(f_{\beta\alpha}) = F \in M_{\beta\alpha}$. Note that, by the assumption, F is a weakly extensional function w.r.t. "$=_\alpha$" and "$=_\beta$". Then

$$\begin{aligned}\mathcal{I}_p^{\mathcal{M}}(FT1) &= \mathcal{I}_p^{\mathcal{M}}(\mathbf{\Delta}(x_\alpha \equiv y_\alpha)) \to (\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha}x_\alpha) =_\beta \mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha}y_\alpha)) \\ &= \Delta([m_\alpha =_\alpha m'_\alpha]) \to [F(m_\alpha) =_\beta F(m'_\alpha)] = \mathbf{1}\end{aligned}$$

because by the weak extensionality of F, if $[m_\alpha =_\alpha m'_\alpha] = \mathbf{1}$ then $[F(m_\alpha) =_\beta F(m'_\alpha)] = \mathbf{1}$ and otherwise $\Delta([m_\alpha =_\alpha m'_\alpha]) = \mathbf{0}$.

Axiom (FT2$_1$): let $p(f_{\beta\alpha}) \in M_{\beta\alpha}$ and $p(g_{\beta\alpha}) \in M_{\beta\alpha}$. Then

$$\begin{aligned}&\mathcal{I}_p^{\mathcal{M}}(FT2_1) = \mathcal{I}_p^{\mathcal{M}}((\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)) \to \mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha} \equiv g_{\beta\alpha}) = \\ &= \left(\bigwedge_{\substack{p'(x_\alpha)=m_\alpha \\ p'=p\backslash x_\alpha}} \left(\mathcal{I}_{p'}^{\mathcal{M}}(f_{\beta\alpha}x_\alpha) =_\beta \mathcal{I}_{p'}^{\mathcal{M}}(g_{\beta\alpha}x_\alpha)\right)\right) \to \left(\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha}) =_{\beta\alpha} \mathcal{I}_p^{\mathcal{M}}(g_{\beta\alpha})\right) = \\ &= \mathbf{1}\end{aligned}$$

because the left and right hand side of the residuation are the same due to definition (16) of the interpretation of $=_{\beta\alpha}$.

Axiom (FT2$_2$): let $p(f_{\beta\alpha}) \in M_{\beta\alpha}$, $p(g_{\beta\alpha}) \in M_{\beta\alpha}$ and $p(x_\alpha) = m_\alpha \in M_\alpha$. Then

$$\begin{aligned}&\mathcal{I}_p^{\mathcal{M}}(FT2_2) = \mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha} \equiv g_{\beta\alpha}) \to \mathcal{I}_p^{\mathcal{M}}((f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)) = \\ &= \left(\bigwedge_{m_\alpha \in M_\alpha} \left(\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha})(m_\alpha) =_\beta \mathcal{I}_p^{\mathcal{M}}(g_{\beta\alpha})(m_\alpha)\right)\right) \to (\mathcal{I}_p^{\mathcal{M}}(f_{\beta\alpha})(m_\alpha) =_\beta \mathcal{I}_p^{\mathcal{M}}(g_{\beta\alpha})(m_\alpha)) = \\ &= \mathbf{1}.\end{aligned}$$

Axiom (FT3):

$$\mathcal{I}_p^{\mathcal{M}}(FT3) = [\mathcal{I}_p^{\mathcal{M}}((\lambda x_\alpha B_\beta)A_\alpha) =_\beta \mathcal{I}_p^{\mathcal{M}}(C_\beta)]. \tag{23}$$

Further,

$$\mathcal{I}_p^{\mathcal{M}}(\lambda x_\alpha B_\beta) = F : M_\alpha \longrightarrow \{\mathcal{I}_{p'}^{\mathcal{M}}(B_\beta) \mid p' \in \mathrm{Asg}(\mathcal{M}), p' = p\backslash x_\alpha\}$$

where for every $m_\alpha \in M_\alpha$ such that $p'(x_\alpha) = m_\alpha$ we have $F(m_\alpha) = \mathcal{I}^{\mathcal{M}}_{p'}(B_\beta)$. Let $\mathcal{I}^{\mathcal{M}}_p(A_\alpha) = \bar{m}_\alpha = p(x_\alpha)$. Then $\mathcal{I}^{\mathcal{M}}_p(C_\beta) = F(\bar{m}_\alpha) = \mathcal{I}^{\mathcal{M}}_p(B_\beta)$ because C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α. However, we also have $\mathcal{I}^{\mathcal{M}}_p((\lambda x_\alpha B_\beta)A_\alpha) = \mathcal{I}^{\mathcal{M}}_{p'}(B_\beta) = F(\bar{m}_\alpha)$ where $p'(x_\alpha) = \bar{m}$. Hence, $p' = p$ and therefore, on both sides of $=_\beta$ in (23) is the same object. Consequently, $\mathcal{I}^{\mathcal{M}}_p(FT3) = \mathbf{1}$ because of reflexivity of $=_\beta$.

Axiom (FT14): let $p(x_o) = a \in L$. Then

$$\begin{aligned}\mathcal{I}^{\mathcal{M}}_p(FT14) = (\mathcal{I}^{\mathcal{M}}_p(g_{oo}(\mathbf{\Delta} x_\alpha)) \wedge \mathcal{I}^{\mathcal{M}}_p(g_{oo}(\mathbf{\neg\Delta} x_\alpha))) \leftrightarrow \mathcal{I}^{\mathcal{M}}_p((\forall y_o)g_{oo}(\mathbf{\Delta} y_o)) =\\ = \mathcal{I}^{\mathcal{M}}_p(g_{oo})(\Delta a) \wedge \mathcal{I}^{\mathcal{M}}_p(g_{oo})(\neg\Delta a) \leftrightarrow \bigwedge_{\substack{p'(y_o)=b\in L\\ p'=p\backslash y_o}} \mathcal{I}^{\mathcal{M}}_{p'}(g_{oo})(\Delta b) =\\ = (\mathcal{I}^{\mathcal{M}}_p(g_{oo})(\mathbf{1}) \wedge \mathcal{I}^{\mathcal{M}}_p(g_{oo})(\mathbf{0})) \leftrightarrow (\mathcal{I}^{\mathcal{M}}_p(g_{oo})(\mathbf{1}) \wedge \mathcal{I}^{\mathcal{M}}_p(g_{oo})(\mathbf{0})) = \mathbf{1}\end{aligned}$$

because if $\Delta a = \mathbf{1}$ then $\neg\Delta a = \mathbf{0}$ or vice-versa and for all $b \in L$, Δb is either $\mathbf{1}$ or $\mathbf{0}$.

For the other axioms, the proof can be done using the properties of biresiduation and IMTL$_\Delta$ algebra (cf. [8]). □

Lemma 8
The inference rules (R) and (N) are sound, i.e. the following holds for every interpretation $\mathcal{I}^{\mathcal{M}}$ and assignment $p \in Asg(\mathcal{M})$:

rule (R): if $\mathcal{I}^{\mathcal{M}}_p(A_\alpha \equiv A'_\alpha) = \mathbf{1}$ then $\mathcal{I}^{\mathcal{M}}_p(B_o) = \mathcal{I}^{\mathcal{M}}_p(B'_o)$.
rule (N): if $\mathcal{I}^{\mathcal{M}}_p(A_o) = \mathbf{1}$ then also $\mathcal{I}^{\mathcal{M}}_p(\mathbf{\Delta} A_o) = \mathbf{1}$.

PROOF: Rule (R): It is sufficient to prove soundness for the formulas of the form $B_o := C_{o\alpha}A_\alpha$. The rest follows by induction.

Let $[\mathcal{I}^{\mathcal{M}}_p(A_\alpha) =_\alpha \mathcal{I}^{\mathcal{M}}_p(A'_\alpha)] = \mathbf{1}$. Because of the assumed weak extensionality of the function $\mathcal{I}^{\mathcal{M}}_p(C_{o\alpha}) \in M_o^{M_\alpha}$ w.r.t "$=_\alpha$" and $\leftrightarrow$, this implies $(\mathcal{I}^{\mathcal{M}}_p(C_{o\alpha}A_\alpha) \leftrightarrow \mathcal{I}^{\mathcal{M}}_p(C_{o\alpha}A'_\alpha)) = \mathbf{1}$ and thus, by Remark 4, $\mathcal{I}^{\mathcal{M}}_p(C_{o\alpha}A_\alpha) = \mathcal{I}^{\mathcal{M}}_p(C_{o\alpha}A'_\alpha)$.

Soundness of rule (N) follows from axiom (15) for the Δ operation. □

Lemmas 7 and 8 imply the following theorem.

Theorem 1 (Soundness)
The fuzzy type theory is sound, i.e. the following holds for every theory T: If $T \vdash A_o$ then $\mathcal{I}_p^{\mathcal{M}}(A_o) = \mathbf{1}$ holds for every assignment $p \in Asg(\mathcal{M})$ and every general model $\mathcal{M}$ of T.

The following corollary is a consequence of soundness and Lemma 6.

Corollary 1
The excluded middle formula $A_o \vee \neg A_o$ is generally not provable in FTT.

We conclude from this corollary and from Lemma 6 that FTT does not collapse into classical type theory.

3 Special properties of FTT

In this section, we present several important theorems demonstrating deduction power of our system. Some proofs are analogous to the corresponding theorems from [4,15,16], and also [8,13].

3.1 Basic logical properties

Theorem 2
The following is provable in FTT.

(a) *If* $\vdash A_o$ *and* $\vdash A_o \equiv B_o$ *then* $\vdash B_o$.
(b) $\vdash A_\alpha \equiv A_\alpha$, $\alpha \in$ *Types.*
(c) $\vdash \top$.
(d) $\vdash \mathbf{\Delta}\top \equiv \top$

PROOF: (a)

(L.1) $\vdash A_o$ (assumption)
(L.2) $\vdash A_o \equiv B_o$ (assumption)
(L.3) $\vdash B_o$ (from (L.1) and (L.2) by Rule (R))

(b) Let x_o be not free in A_o.

(L.1) $\vdash (\lambda x_o A_\alpha)x_o \equiv A_\alpha$ (instance of (FT3))
(L.2) $\vdash (\lambda x_o A_\alpha)x_o \equiv A_\alpha$ (instance of (FT3))
(L.3) $\vdash A_\alpha \equiv A_\alpha$ (from (L.1) and (L.2) by Rule (R))

(c) Immediately from (b) and definition of $\top$.

(d)

(L.1) $\vdash \top$ (property (c))
(L.2) $\vdash \mathbf{\Delta}\top$ (L.1, Rule (N))
(L.3) $\vdash (\mathbf{\Delta}\top \equiv \top) \equiv \mathbf{\Delta}\top$ (instance of Axiom (FT6))
(L.4) $\vdash \mathbf{\Delta}\top \equiv \top$ (L.2, L.3, property (a))

□

Theorem 3

(a) If $\vdash A_\alpha \equiv B_\alpha$ *then* $\vdash B_\alpha \equiv A_\alpha$.
(b) $\vdash (A_o \equiv B_o) \equiv (\neg B_o \equiv \neg A_o)$.
(c) $\vdash A_o$ *iff* $\vdash A_o \equiv \top$.
(d) $\vdash \neg\bot \equiv \top$ *and* $\vdash \neg\top \equiv \bot$.
(e) $\vdash A_o \equiv \neg\neg A_o$.

PROOF: (a)

(L.1) $\vdash A_\alpha \equiv A_\alpha$ (by Theorem 2(b))
(L.2) $\vdash A_\alpha \equiv B_\alpha$ (assumption)
(L.3) $\vdash B_\alpha \equiv A_\alpha$ (from L.1 and L.2 by Rule (R))

(b)

(L.1) $\vdash (A_o \equiv B_o) \equiv (A_o \Rightarrow B_o) \wedge (B_o \Rightarrow A_o)$ (instance of (FT5))
(L.2) $\vdash (A_o \Rightarrow B_o) \equiv (\neg B_o \Rightarrow \neg A_o)$ ((FT10) and (a))
(L.3) $\vdash (B_o \Rightarrow A_o) \equiv (\neg A_o \Rightarrow \neg B_o)$ ((FT10) and (a))
(L.4) $\vdash (A_o \equiv B_o) \equiv (\neg B_o \Rightarrow \neg A_o) \wedge (\neg A_o \Rightarrow \neg B_o)$ (L.1, L.2, L.3, rule (R))
(L.5) $\vdash (\neg B_o \equiv \neg A_o) \equiv (\neg B_o \Rightarrow \neg A_o) \wedge (\neg A_o \Rightarrow \neg B_o)$ (instance of (FT5))
(L.6) $\vdash (A_o \equiv B_o) \equiv (\neg B_o \equiv \neg A_o)$ (L.4, L.5, rule (R))

(c) This immediately follows from axiom (FT6), Theorem 2(a) and (a).

(d) The first equivalence follows from definition of negation, Theorem 2(b) and from (c). The second one is an instance of axiom (FT6).

(d) This follows from (a), instance of (b) $\vdash (A_o \equiv \top) \equiv (\neg\top \equiv \neg A)$, Axiom (FT6) and definition of negation. □

3.2 First-order properties

Theorem 4 (Substitution axioms)
(a) $\vdash (\forall x_\alpha)B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$,
(b) $\vdash B_{o,x_\alpha}[A_\alpha] \Rightarrow (\exists x_\alpha)B_o$.

provided that A_α is substitutable to B_o for all free occurrences of x_α.

PROOF: (a)

(L.1) $\vdash (\lambda x_\alpha B_o \equiv \lambda x_\alpha \top) \Rightarrow ((\lambda x_\alpha B_o)A_\alpha \equiv (\lambda x_\alpha \top)A_\alpha)$ (instance of (FT2_2))
(L.2) $\vdash (\lambda x_\alpha B_o)A_\alpha \equiv B_{o,x_\alpha}[A_\alpha]$ (instance of (FT3))
(L.3) $\vdash (\forall x_\alpha)B_o \Rightarrow (B_{o,x_\alpha}[A_\alpha] \equiv \top)$ (L.1 and L.2, definition of $\forall$, (FT3) and Rule (R))
(L.4) $\vdash (\forall x_\alpha)B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$ (L.3, (FT6), Theorem 3(c) and Rule (R))

(b) is proved from (a) using definition of $\exists$, Axiom (FT10), Theorem 3(e) and Rule (R). □

Theorem 5 (Rule of generalization)
If $T \vdash A_o$ then $T \vdash (\forall x_\alpha)A_o$.

PROOF:

(L.1) $T \vdash A_o$ (assumption)
(L.2) $T \vdash A_o \equiv \top$ (Theorem 3(c))
(L.3) $T \vdash (\lambda x_o A_o) \equiv (\lambda x_o A_o)$ (Theorem 2(b))

(L.4) $T \vdash (\lambda x_o A_o) \equiv (\lambda x_o \top)$ (from L.2 and L.3 by Rule (R))
(L.5) $T \vdash (\forall x_o) A_o$ (from L.4 by definition of $\forall$)

□

Theorem 6 (Rule of substitution)
Let $T \vdash B_o$ and $A_{\alpha_1}, \ldots, A_{\alpha_n}$ be substitutable for all free occurrences of $x_{\alpha_1}, \ldots, x_{\alpha_n}$ in B_o. Then

$$\vdash B_{o, x_{\alpha_1}, \ldots, x_{\alpha_n}}[A_{\alpha_1}, \ldots, A_{\alpha_n}].$$

PROOF: See [4], 5221 (cf. also [16], 7.6.). □

Lemma 9
(a) $\vdash \bot \Rightarrow A_o$
(b) $\vdash A_o \Rightarrow \top$ *and* $\vdash A_o \wedge \top \equiv A_o$
(c) $\vdash (\top \Rightarrow A_o) \equiv A_o$
(d) $\vdash (A_o \equiv B_o) \Rightarrow (A_o \Rightarrow B_o)$ *and* $\vdash (A_o \equiv B_o) \Rightarrow (B_o \Rightarrow A_o)$

PROOF: (a) Realizing that $\bot$ is a short for $(\forall x_o) x_o$, this is a consequence of Theorems 4(a) and 6.

(b) This follows from (a), Axiom (FT10) and Theorem 3(d). The second part is obtained from the first one after rewriting $\Rightarrow$.

(c)

(L.1) $\vdash (A_o \equiv \top) \equiv A_o$ (Axiom (FT6))
(L.2) $\vdash A_o \wedge \top \equiv A_o$ ((b))
(L.3) $\vdash (A_o \wedge \top \equiv \top) \equiv A_o$ (L.1, L.2, rule (R))
(L.4) $\vdash (\top \Rightarrow A_o) \equiv A_o$ (L.3, definition of $\Rightarrow$)

(d)

(L.1) $\vdash (A_o \equiv B_o) \equiv ((A_o \Rightarrow B_o) \wedge (B_o \Rightarrow A_o))$ (Axiom (FT5), Theorem 6)
(L.2) $(A_o \Rightarrow B_o) \wedge (B_o \Rightarrow A_o) \Rightarrow (A_o \Rightarrow B_o)$ (instance of Axiom (FT12))

(L.3) $\vdash (A_o \equiv B_o) \Rightarrow (A_o \Rightarrow B_o)$ (L.1, L.2, rule (R))

□

Theorem 7 (Rule of modus ponens)
If $T \vdash A_o$ and $T \vdash A_o \Rightarrow B_o$ then $T \vdash B_o$.

PROOF:

(L.1) $T \vdash A_o \equiv \top$ (assumption, Theorem 3(c))
(L.2) $T \vdash \top \Rightarrow B_o$ (assumption and L.1 by Rule (R))
(L.3) $T \vdash \top \wedge B_o \equiv \top$ (L.2, definition of $\Rightarrow$)
(L.4) $T \vdash \top \wedge B_o$ (L.3, Theorem 2(a), (c))
(L.5) $T \vdash \top \wedge B_o \equiv B_o$ (Lemma 9(b), Axiom (FT11))
(L.6) $T \vdash B_o$ (L.4, L.5, Theorem 2(a))

□

3.3 IMTL-properties

Lemma 10
Let $T \vdash A_o \Rightarrow B_o$ and $T \vdash C_o \Rightarrow D_o$. Then $T \vdash (A_o \wedge C_o) \Rightarrow (B_o \wedge D_o)$.

PROOF:

(L.1) $T \vdash (A_o \wedge B_o) \equiv A_o$ (assumption, definition of $\Rightarrow$)
(L.2) $T \vdash (C_o \wedge D_o) \equiv C_o$ (assumption, definition of $\Rightarrow$)
(L.3) $T \vdash (A_o \wedge C_o) \equiv (A_o \wedge C_o)$ (Theorem 2(b))
(L.4) $T \vdash (A_o \wedge C_o \wedge B_o \wedge D_o) \equiv (A_o \wedge C_o)$ (L.1, L.2, L.3, rule (R), (FT13))
(L.5) $T \vdash (A_o \wedge B_o) \Rightarrow (B_o \wedge D_o)$ (L.4, (FT13), definition of $\Rightarrow$)

□

Theorem 8
(a) $\vdash A_o \wedge A_o \equiv A_o$ *and* $\vdash A_o \Rightarrow A_o$
(b) $\vdash (A_o \,\&\, B_o \Rightarrow C_o) \equiv (A_o \Rightarrow (B_o \Rightarrow C_o))$

(c) $\vdash A_o \Rightarrow (B_o \Rightarrow A_o)$
(d) $\vdash A_o \,\&\, B_o \Rightarrow A_o$
(e) $\vdash (A_o \,\&\, B_o) \equiv (B_o \,\&\, A_o)$
(f) $\vdash A_o \,\&\, \neg A_o \Rightarrow B_o$.
(g) $\vdash (C_o \Rightarrow A_o) \Rightarrow ((C_o \Rightarrow B_o) \Rightarrow (C_o \Rightarrow B_o \wedge A_o))$
(h) $\vdash A_o \,\&(A_o \Rightarrow B_o) \Rightarrow (A_o \wedge B_o)$

PROOF: (a) follows from $\vdash A_o \equiv A_o$ by Theorem 2(b) and Axioms (FT5) and (FT12). The second formula is rewriting of the first by definition of $\Rightarrow$.

(b)

(L.1) $\vdash ((A_o \,\&\, B_o) \wedge C_o) \equiv C_o) \equiv ((A_o \,\&\, B_o) \wedge C_o) \equiv C_o)$ (instance of Theorem 2(b))
(L.2) $\vdash (A_o \,\&\, B_o) \Rightarrow C_o) \equiv \neg(A_o \Rightarrow \neg B_o) \Rightarrow C$ (L.1, definition of $\Rightarrow$ and $\&$)
(L.3) $\vdash (A_o \,\&\, B_o) \Rightarrow C_o) \equiv A_o \Rightarrow (\neg C_o \Rightarrow \neg B_o)$ (L.2, (FT10), Theorem 3(e), (FT8), rule (R))
(L.4) $\vdash (A_o \,\&\, B_o \Rightarrow C_o) \equiv (A_o \Rightarrow (B_o \Rightarrow C_o))$ (L.3, (FT10), rule (R))

(c)

(L.1) $\vdash B_o \Rightarrow \top$ ((FT12), definition of $\Rightarrow$)
(L.2) $\vdash (A_o \Rightarrow A_o) \equiv \top$ ((b), Theorem 3(c))
(L.3) $\vdash B_o \Rightarrow (A_o \Rightarrow A_o)$ (L.1, L.2, rule (R))
(L.4) $\vdash (B_o \Rightarrow (A_o \Rightarrow A_o)) \equiv (A_o \Rightarrow (B_o \Rightarrow A_o))$ ((FT8))
(L.5) $\vdash A_o \Rightarrow (B_o \Rightarrow A_o)$ (L.3, L.4, Theorem 2(a))

(d) $\vdash \neg A_o \Rightarrow (B_o \Rightarrow \neg A_o)$ is an instance of (d). Then using (FT10) and Theorem 3(e) we obtain $\vdash \neg(B_o \Rightarrow \neg A_o) \Rightarrow A_o$, which is equivalent to (e) using definition of $\&$.

(e) Starting from instance $\vdash \neg(A_o \Rightarrow \neg B_o) \Leftrightarrow \neg(B_o \Rightarrow \neg A_o)$ of Theorem 2(b) we apply (FT10), Theorem 3(e) and rule (R).

(f) This formula is a short for $\vdash \neg(A_o \equiv A_o) \Rightarrow B_o$. Using Theorem 2(b) and 3(c) we get $\vdash \neg(A_o \equiv A_o) \equiv \bot$. Then use rule (R) and Lemma 9(a).

(g) By (f), we have $\vdash (C_o \,\&\, (C_o \Rightarrow A_o) \,\&\, (C_o \Rightarrow B_o)) \Rightarrow A_o$ as well as $\vdash (C_o \,\&\, (C_o \Rightarrow A_o) \,\&\, (C_o \Rightarrow B_o)) \Rightarrow B_o$. From this we obtain

$$\vdash (C_o \,\&\, (C_o \Rightarrow A_o) \,\&\, (C_o \Rightarrow B_o)) \Rightarrow (A_o \wedge B_o)$$

using Lemma 10 and (a). Then (i) follows from (d) by rule (R).

(h)

(L.1) $\vdash A_o \Rightarrow ((A_o \Rightarrow B_o) \Rightarrow A_o)$ (instance of (e))
(L.2) $\vdash (A_o \,\&\, (A_o \Rightarrow B_o)) \Rightarrow A_o)$ (L.1, (d), rule (R))
(L.3) $\vdash (A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow B_o)$ (instance of Theorem 2(b))
(L.4) $\vdash (A_o \,\&\, (A_o \Rightarrow B_o)) \Rightarrow B_o)$ (L.3, (d), rule (R))
(L.5) $\vdash A_o \,\&\, (A_o \Rightarrow B_o) \Rightarrow (A_o \wedge B_o)$ (L.2, L.4, instance of (h), modus ponens)

□

Axioms (FT7), (FT9), (FT11), (FT12), Theorem 8(b), (d), (e), (h), Lemma 9(a) are axioms of propositional IMTL-logic. Together with the Rule of Modus Ponens (Theorem 7) they constitute its formal system and thus, all the other theorems of propositional IMTL-logic are also provable in FTT. Axiom (FT16) and Theorem 4(a) are axioms of predicate IMTL-fuzzy logic. Together with IMTL-fuzzy propositional logic and Rule of Generalization (Theorem 5) we obtain formal system of predicate IMTL-fuzzy logic and hence, all its theorems also provable in FTT. Some of the properties needed in the sequel are summarized in the following theorem.

Theorem 9
(a) $\vdash (A_o \Rightarrow C_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \vee B_o \Rightarrow C_o))$.
(b) $\vdash ((A_o \Rightarrow B_o) \wedge (A_o \Rightarrow C_o)) \Rightarrow (A_o \Rightarrow (B_o \wedge C_o))$.
(c) $\vdash A_o \Rightarrow A_o \vee B_o$.
(d) $\vdash A_o \Rightarrow (B_o \Rightarrow A_o \,\&\, B_o)$.
(e) $\vdash (A_o \Rightarrow B_o) \vee (B_o \Rightarrow A_o)$
(f) $\vdash (A_o \vee B_o) \equiv \neg(\neg A_o \wedge \neg B_o)$

(g) $\vdash (A_o \vee B_o) \equiv (B_o \vee A_o)$
(h) $\vdash (A_o \vee B_o) \vee C_o \equiv A_o \vee (B_o \vee C_o)$
(i) $\vdash (A_o \vee B_o) \wedge A_o \equiv A_o$ *and* $\vdash (A_o \wedge B_o) \vee A_o \equiv A_o$
(j) $\vdash (A_o \,\&\, B_o) \,\&\, C_o \equiv A_o \,\&(B_o \,\&\, C_o)$
(k) $\vdash (A_o \,\&\, \top) \equiv A_o$
(l) $\vdash (\forall x_\alpha)(A_o \vee B_o) \equiv ((\forall x_\alpha)A_o \vee B_o)$*,* x_α *is not free in* B_o*.*
(m) $\vdash (\forall x_\alpha)(A_o \Rightarrow B_o) \equiv ((\exists x_\alpha)A_o \Rightarrow B_o)$*,* x_α *is not free in* B_o*.*
(n) $\vdash (\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\forall x_\alpha)A_o \Rightarrow (\forall x_\alpha)B_o)$

Lemma 11

(a) Let $T \vdash A_o$ *and* $T \vdash B_o$*. Then* $T \vdash A_o \wedge B_o$*.*
(b) Let $T \vdash A_o \Rightarrow B_o$ *and* $T \vdash C_o \Rightarrow D_o$*. Then* $T \vdash (A_o \,\&\, C_o) \Rightarrow (B_o \,\&\, D_o)$*.*

PROOF: (a)

(L.1) $\vdash A_o \wedge \top \equiv A_o$ (Lemma 9(b))
(L.2) $T \vdash B_o \equiv \top$ (assumption, Theorem 3(c))
(L.3) $T \vdash A_o \Rightarrow B_o$ (L.1, L.2, rule (R), definition of $\Rightarrow$)
(L.4) $\vdash (A_o \Rightarrow A_o) \Rightarrow ((A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow A_o \wedge B_o))$ (instance of Theorem 8(g))
(L.5) $T \vdash A_o$ (assumption)
(L.6) $T \vdash A_o \wedge B_o$ (L.4, Theorem 8(a), L.3 and L.5 by modus ponens)

(b) This can be proved using assumption, Axiom (FT7) and Theorem 9(d). □

The proof of following theorem is the same as in fuzzy logic.

Theorem 10
For every theory T *and a formula* A_o*,* $T \vdash A_o$ *iff* $T \vdash A'_o$ *where* A'_o *is a universal closure of* A_o*.*

3.4 Properties of equality

As stated in the beginning of this paper, the equality formula $\equiv$ is treated as fuzzy equality. In this section, we prove several theorems demonstrating its syntactical properties.

The following formula is the original axiom of classical type theory. It states, in fact, that the induced fuzzy equality from Lemma 2 is natural interpretation of the formal equality $\equiv$.

Theorem 11

$$\vdash (\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \equiv (f_{\beta\alpha} \equiv g_{\beta\alpha})$$

PROOF:

(L.1) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$ (Axiom $(\mathrm{FT2})_2$)
(L.2) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$
(L.1, rule of generalization, Axiom (FT16))
(L.3) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$ (Axiom $(\mathrm{FT2})_1$)

Furthermore, apply Lemma 11(a) to L.2, L.3 and then, use Axiom (FT5) and rule (R). □

The following theorem demonstrates that the fuzzy equality is transitive for all types.

Theorem 12
For all $\gamma \in$ Types,

$$\vdash ((x_\gamma \equiv y_\gamma)\,\&\,(y_\gamma \equiv z_\gamma)) \Rightarrow (x_\gamma \equiv z_\gamma)$$

PROOF: By induction on the length of type. If $\gamma = \epsilon$ then the formula follows immediately from Axiom (FT4) and Theorem 8(b).

Let $\gamma = o$.

(L.1) $\vdash (x_o \equiv y_o) \Rightarrow (x_o \Rightarrow y_o)$ (Lemma 9(d))
(L.2) $\vdash (y_o \equiv z_o) \Rightarrow (y_o \Rightarrow z_o)$ (Lemma 9(d))
(L.3) $\vdash ((x_o \equiv y_o)\,\&\,(y_o \equiv z_o)) \Rightarrow ((x_o \Rightarrow y_o)\,\&\,(y_o \Rightarrow z_o))$
(L.1, L.2, Lemma 11(b))
(L.4) $\vdash ((x_o \Rightarrow y_o)\,\&\,(y_o \Rightarrow z_o)) \Rightarrow (x_o \Rightarrow z_o)$ (Axiom (FT7), Theorem 8(b))

(L.5) $\vdash ((x_o \equiv y_o)\,\&(y_o \equiv z_o)) \Rightarrow (x_o \Rightarrow z_o)$
(L.3, L.4, Axiom (FT7), modus ponens)
(L.6) $\vdash ((x_o \equiv y_o)\,\&(y_o \equiv z_o)) \Rightarrow (z_o \Rightarrow x_o)$ (analogously to lines L.1–L.5)
(L.7) $\vdash ((x_o \equiv y_o)\,\&(y_o \equiv z_o)) \Rightarrow (x_o \equiv z_o)$
(L.5, L.6, Lemma 10, Theorem 8(a), instance of (FT5))

Let $\gamma = \beta\alpha$ and the induction assumption hold for all types shorter than γ. Recall that $f_{\beta\alpha}x_\alpha$ is a formula of type β. Then

(L.1) $((f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)\,\&(g_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha)) \Rightarrow (f_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha)$
(inductive assumption)
(L.2) $(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha) \Rightarrow ((g_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha) \Rightarrow (f_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha))$
(L.1, Theorem 8(b))
(L.3) $(\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha) \Rightarrow ((\forall x_\alpha)(g_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha)) \Rightarrow$
$(\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv h_{\beta\alpha}x_\alpha)$ (L.2, rule of generalization, Theorem 9(n), modus ponens)
(L.4) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow ((g_{\beta\alpha} \equiv h_{\beta\alpha}) \Rightarrow (f_{\beta\alpha} \equiv h_{\beta\alpha}))$
(L.3, Theorem 11, rule (R))
(L.5) $((f_{\beta\alpha} \equiv g_{\beta\alpha})\,\&(g_{\beta\alpha} \equiv h_{\beta\alpha})) \Rightarrow (f_{\beta\alpha} \equiv h_{\beta\alpha}))$ (L.4, Theorem 8(b))

□

The following lemma and theorem characterize weak extensionality of all functions with respect to the fuzzy equality $\equiv$.

Lemma 12

$$\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta).$$

PROOF: Let us denote $h_{o(\alpha\beta)} := \lambda z_{\alpha\beta}(z_{\alpha\beta}y_\beta \equiv g_{\alpha\beta}y_\beta)$. Then:

(L.1) $\vdash ((\lambda z_{\alpha\beta}(z_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta))f_{\alpha\beta}) \equiv (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)$ (instance of (FT3) applied to $h_{o(\alpha\beta)}$)
(L.2) $\vdash ((\lambda z_{\alpha\beta}(z_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta))g_{\alpha\beta}) \equiv (g_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)$ (instance of (FT3) applied to $h_{o(\alpha\beta)}$)
(L.3) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (h_{o(\alpha\beta)}\, f_{\alpha\beta} \equiv h_{o(\alpha\beta)}\, g_{\alpha\beta})$ (instance of (FT1))

(L.4) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow ((f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta) \equiv (g_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta))$ (from L.1–L.3 by Rule(R))

(L.5) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow ((f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta) \equiv \top)$ (from L.4 by Theorems 2(b), 3(c) and Rule (R))

(L.6) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)$ (from L.5 and (FT6) by Rule (R))

□

Theorem 13

$$\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta) \Rightarrow \mathbf{\Delta}((f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta))$$

PROOF: The proof is analogous to that from [2], (13). Let us denote $h_{o\beta} := \lambda z_\beta(\mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta} z_\beta \equiv g_\alpha) y_\beta)$ and $k_{o(\alpha\beta)} := \lambda z_{\alpha\beta}(z_{\alpha\beta} y_\beta \equiv g_{\alpha\beta} y_\beta)$. Then

(L.1) $\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta) \Rightarrow (h_{o\beta} x_\beta \equiv h_{o\beta} y_\beta)$ (Instance of (FT1))

(L.2) $\vdash \mathbf{\Delta}(x_\beta \equiv y_\beta) \Rightarrow ((\mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, x_\beta \equiv g_{\alpha\beta}\, y_\beta)) \equiv (\mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)))$ (L.1, (FT3) and rule (R))

(L.3) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow ((f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta) \equiv (g_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta))$ (Lemma 12 applied to $k_{o(\alpha\beta)}$)

(L.4) $\vdash \mathbf{\Delta}(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta}\, y_\beta \equiv g_{\alpha\beta}\, y_\beta)$ ((FT6), rule (R), Theorems 2(b) and 3(c))

Finally, apply (FT6) and Theorem 3(c) to L.2 and L.4. □

3.5 Properties of $\mathbf{\Delta}$

The special $\mathbf{\Delta}$ connective (formula) pinpoints on a syntactic level the boolean substructure from the structure of truth values. This can be well seen from our formulation of the classical Rule of Two Cases presented in the following theorem. Furthermore, we show that the properties (10)–(15) for the delta operation in the IMTL$_\Delta$algebra are formally derivable from the axioms (FT14) and (FT15).

Theorem 14 (Rule of Two Cases)
If $T \vdash A_{o,x_\alpha}[\top]$ *and* $T \vdash A_{o,x_\alpha}[\bot]$ *then* $T \vdash A_{o,x_\alpha}[\mathbf{\Delta} y_o]$.

PROOF:

(L.1) $T \vdash \top \equiv (\lambda x_\alpha A_o)\top$ (assumption)
(L.2) $T \vdash \top \equiv (\lambda x_\alpha A_o)\bot$ (assumption)
(L.3) $T \vdash \top \wedge \top$ (Theorem 2(c), Lemma 11(a))
(L.4) $T \vdash (\lambda x_\alpha A_o)\top \wedge (\lambda x_\alpha A_o)\bot$ (L.1, L,2, L.3, rule (R))
(L.5) $T \vdash (\lambda x_\alpha A_o)\mathbf{\Delta}\top \wedge (\lambda x_\alpha A_o)\neg\mathbf{\Delta}\top \equiv (\forall y_o)((\lambda x_\alpha A_o)\mathbf{\Delta} y_o)$
(Axiom (FT14))
(L.6) $T \vdash (\lambda x_\alpha A_o)\mathbf{\Delta}\top \wedge (\lambda x_\alpha A_o)\neg\mathbf{\Delta}\top$
(L.4, Theorem 3(d), 2(d) and rule (R))
(L.7) $T \vdash (\forall y_o)((\lambda x_\alpha A_o)\mathbf{\Delta} y_o)$ (L.5, L.6, Theorem 2(a))
(L.8) $T \vdash A_{o,x_\alpha}[\mathbf{\Delta} y_o]$ (L.7, Theorem 4(a), modus ponens, Axiom (FT3))

□

Theorem 15
(a) $\vdash \mathbf{\Delta} x_o \Rightarrow x_o$
(b) $\vdash \mathbf{\Delta}\bot \equiv \bot$
(c) $\vdash \bot \equiv (\forall y_o)\mathbf{\Delta} y_o$
(d) $\vdash \mathbf{\Delta} x_o \vee \neg\mathbf{\Delta} x_o$
(e) $\vdash \mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta}\mathbf{\Delta} x_o$

PROOF: (a)

(L.1) $\vdash \mathbf{\Delta}(x_o \equiv \top) \Rightarrow (\mathbf{\Delta}(\lambda x_o x_o \equiv \lambda x_o x_o) \Rightarrow ((\lambda x_o x_o)x_o \equiv (\lambda x_o x_o)\top))$
(instance of Theorem 13)
(L.2) $\vdash \mathbf{\Delta} x_o \Rightarrow (\mathbf{\Delta}\top \Rightarrow (x_o \equiv \top))$
(L.1, Axiom (FT6), rule (R), definition of $\top$, Axiom (FT3))
(L.3) $\vdash \mathbf{\Delta} x_o \Rightarrow x_o$
(L.2, Theorems 8(b) and 2(d), Lemma 9(c), Axiom (FT6), rule (R))

(b) This follows from (a) , Lemmas 9(a) and 11(a) and Axiom (FT5) by rule (R).

(c)

(L.1) $\vdash (\lambda x_o x_o)\mathbf{\Delta} x_o \wedge (\lambda x_o x_o)\neg\mathbf{\Delta} x_o \equiv (\forall y_o)(\lambda x_o x_o)\mathbf{\Delta} y_o$ (Axiom (FT14))
(L.2) $\vdash \mathbf{\Delta} x_o \wedge \neg\mathbf{\Delta} x_o \equiv (\forall y_o)\mathbf{\Delta} y_o$ (L.1, Axiom (FT3))
(L.3) $\vdash \mathbf{\Delta}\top \wedge \neg\mathbf{\Delta}\top \equiv (\forall y_o)\mathbf{\Delta} y_o$ (L.2, Theorem 6)
(L.4) $\vdash \top \wedge \bot \equiv (\forall y_o)\mathbf{\Delta} y_o$ (L.3, Theorems 2(d), 3(d))
(L.5) $\vdash \bot \equiv (\forall y_o)\mathbf{\Delta} y_o$ (L.4, Lemma 9(b), rule (R))

(d)

(L.1) $\vdash (\lambda x_o x_o)\mathbf{\Delta} x_o \wedge (\lambda x_o x_o)\neg\mathbf{\Delta} x_o \equiv (\forall y_o)(\lambda x_o x_o)\mathbf{\Delta} y_o$ (Axiom (FT14))
(L.2) $\vdash \mathbf{\Delta} x_o \wedge \neg\mathbf{\Delta} x_o \equiv (\forall y_o)\mathbf{\Delta} y_o$ (L.1, Axiom (FT3))
(L.3) $\vdash \mathbf{\Delta} x_o \wedge \neg\mathbf{\Delta} x_o \equiv \bot$ (L.2, (c), rule (R))
(L.4) $\vdash \mathbf{\Delta} x_o \vee \neg\mathbf{\Delta} x_o \equiv \top$ (L.3, Theorems 3(b), 9(f), 3(d) and rule (R))
(L.5) $\vdash \mathbf{\Delta} x_o \vee \neg\mathbf{\Delta} x_o$ (L.4, Theorem 3(c))

(e) Let us put $g_{oo} := \lambda x_o(x_o \wedge \mathbf{\Delta} x_o \equiv x_o)$ and apply Axiom (FT14):

(L.1) $\vdash \lambda x_o(x_o \wedge \mathbf{\Delta} x_o \equiv x_o)\mathbf{\Delta} x_o \wedge \lambda x_o(x_o \wedge \mathbf{\Delta} x_o \equiv x_o)\neg\mathbf{\Delta} x_o \equiv (\forall y_o)(\lambda x_o(x_o \wedge \mathbf{\Delta} x_o \equiv x_o)\mathbf{\Delta} y_o)$ (Axiom (FT14))
(L.2) $\vdash (\mathbf{\Delta} x_o \wedge \mathbf{\Delta\Delta} x_o \equiv \mathbf{\Delta} x_o) \wedge (\neg\mathbf{\Delta} x_o \wedge \mathbf{\Delta}\neg\mathbf{\Delta} x_o \equiv \neg\mathbf{\Delta} x_o) \equiv (\forall y_o)(\mathbf{\Delta} y_o \wedge \mathbf{\Delta\Delta} y_o \equiv \mathbf{\Delta} y_o)$ (L.1, Axiom (FT3))
(L.3) $\vdash (\mathbf{\Delta}\top \wedge \mathbf{\Delta\Delta}\top \equiv \mathbf{\Delta}\top) \wedge (\neg\mathbf{\Delta}\top \wedge \mathbf{\Delta}\neg\mathbf{\Delta}\top \equiv \neg\mathbf{\Delta}\top) \equiv (\forall y_o)(\mathbf{\Delta} y_o \wedge \mathbf{\Delta\Delta} y_o \equiv \mathbf{\Delta} y_o)$ (L.2, Theorem 6)
(L.4) $\vdash (\top \wedge \mathbf{\Delta}\top \equiv \top) \wedge (\neg\top \wedge \mathbf{\Delta}\bot \equiv \bot) \equiv (\forall y_o)(\mathbf{\Delta} y_o \wedge \mathbf{\Delta\Delta} y_o \equiv \mathbf{\Delta} y_o)$ (L.3, Theorems 2(d) and 3(d))
(L.5) $\vdash (\top \wedge \top \equiv \top) \wedge (\bot \wedge \bot \equiv \bot) \equiv (\forall y_o)(\mathbf{\Delta} y_o \wedge \mathbf{\Delta\Delta} y_o \equiv \mathbf{\Delta} y_o)$ (L.4, (b), Theorems 2(d) and 3(d))
(L.6) $\vdash (\top \wedge \top \equiv \top) \wedge (\bot \wedge \bot \equiv \bot)$ (Theorem 8(a), Lemma 11(a))
(L.7) $\vdash (\forall y_o)(\mathbf{\Delta} y_o \wedge \mathbf{\Delta\Delta} y_o \equiv \mathbf{\Delta} y_o)$ (L.5, L.6, rule (R))
(L.8) $\vdash (\mathbf{\Delta} x_o \wedge \mathbf{\Delta\Delta} x_o \equiv \mathbf{\Delta} x_o)$ (L.7, Theorem 4(a), modus ponens)
(L.9) $\vdash \mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta\Delta} x_o$ (L.8, definition of $\Rightarrow$)

□

Theorem 16

(a) $\vdash \mathbf{\Delta}(x_o \Rightarrow y_o) \Rightarrow (\mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta} y_o)$
(b) $\vdash \mathbf{\Delta}(\neg x_o) \Rightarrow \neg\mathbf{\Delta} x_o$

(c) $\vdash \mathbf{\Delta}(x_o \vee y_o) \Rightarrow (\mathbf{\Delta} x_o \vee \mathbf{\Delta} y_o)$
(d) $\vdash \mathbf{\Delta}(x_o \Rightarrow y_o) \vee \mathbf{\Delta}(y_o \Rightarrow x_o)$,
(e) $\vdash \mathbf{\Delta} x_o \equiv \mathbf{\Delta} x_o \,\&\, \mathbf{\Delta} x_o$
(f) $\vdash (\mathbf{\Delta} A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow ((\mathbf{\Delta} A_o \Rightarrow B_o) \Rightarrow (\mathbf{\Delta} A_o \Rightarrow C_o))$.
(g) $\vdash (\mathbf{\Delta} A_o \,\&\, \mathbf{\Delta}(A_o \Rightarrow B_o)) \Rightarrow \mathbf{\Delta} B_o$.

PROOF:

(a) This follows from instance $\mathbf{\Delta}(x_o \wedge y_o \equiv x_o) \Rightarrow (\mathbf{\Delta}(x_o \wedge y_o) \equiv \mathbf{\Delta} x_o)$ of axiom (FT1) where $f_{o\alpha} := \mathbf{\Delta}$. Then use Axiom (FT15) and rule (R).

(b) This follows from instance $\mathbf{\Delta}(x_o \equiv \bot) \Rightarrow (\mathbf{\Delta} x_o \equiv \mathbf{\Delta}\bot)$ of axiom (FT1) where $f_{o\alpha} := \mathbf{\Delta}$. Then use Theorem 15(b), rule (R) and definition of negation.

(c) This follows from Axiom (FT15) using (b) and Theorem 9(f).

(d) is a consequence of (c) and Theorem 9(e).

(e)

(L.1) $\vdash ((\mathbf{\Delta} x_o \Rightarrow \neg\mathbf{\Delta} x_o) \Rightarrow \neg\mathbf{\Delta} x_o) \wedge ((\neg\mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta} x_o) \Rightarrow \mathbf{\Delta} x_o)$ (Theorem 15(d) rewritten using definition of $\vee$)
(L.2) $\vdash ((\mathbf{\Delta} x_o \Rightarrow \neg\mathbf{\Delta} x_o) \Rightarrow \neg\mathbf{\Delta} x_o)$ (L.1, using instance of Axiom (FT12) and modus ponens)
(L.3) $\vdash \mathbf{\Delta} x_o \Rightarrow \neg(\mathbf{\Delta} x_o \Rightarrow \neg\mathbf{\Delta} x_o)$ (L.2, Axiom (FT10), rule (R), Theorem 3(e))
(L.4) $\vdash \mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta} x_o \,\&\, \mathbf{\Delta} x_o$ (L.3, definition of $\&$)
(L.5) $\vdash \mathbf{\Delta} x_o \,\&\, \mathbf{\Delta} x_o \Rightarrow \mathbf{\Delta} x_o$ (L.4, Theorem 8(d))
(L.6) $\vdash \mathbf{\Delta} x_o \equiv \mathbf{\Delta} x_o \,\&\, \mathbf{\Delta} x_o$ (L.4, L.5 using Lemma 11(a) and Axiom (FT5))

(f) This can be obtained from the provable formula

$$\vdash \mathbf{\Delta} A_o \,\&(\mathbf{\Delta} A_o \Rightarrow B_o)\,\&(B_o \Rightarrow (\mathbf{\Delta} A_o \Rightarrow C_o)) \Rightarrow (\mathbf{\Delta} A_o \Rightarrow C_o)$$

using (d) and Theorem 8(b), and instance of (FT8) using rule (R).

(g) This can be obtained from $\vdash (A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow B_o)$ (instance Theorem 8(a)) using Axiom (FT8) and Theorem 8(b).

□

One may verify that Theorems 15(d), 16(c), 15(a), (e), 16(g) and 2(d) are syntactic counterparts of the properties (10)–(15) of the IMTL$_\Delta$-algebra. Consequently, Theorems 16 and 15 include all axioms of the delta connective which extend fuzzy logics, such as BL, MTL or IMTL (cf. [8,13]). Hence, all the properties which concern the delta connective and are provable in the latter are provable also in FTT.

4 Theories in FTT

In the previous section, we have presented the main properties of FTT. In this section, we will focus on the proof of its completeness. The procedure is analogous to that for classical type theory (and also predicate fuzzy logic). After proving the deduction theorem, we prove two theorems on extension of theories. Then we begin to construct a canonical (general) model for FTT. This will finally enable us to prove completeness.

4.1 Deduction theorem

This subsection is devoted to the proof of the deduction theorem in FTT. This theorem is analogous to the deduction theorem in BL-fuzzy logic with $\boldsymbol{\Delta}$.

We will employ the usual notation: If T is a theory and $A_o \in Form_o$ a formula then

$$T \cup \{A_o\}$$

denotes a theory whose set of special axioms is extended by A_o.

Lemma 13
Let $C_o \in Form_o$ be a closed formula and T be a theory such that $T \vdash \boldsymbol{\Delta} A_o \Rightarrow (D_\alpha \equiv E_\alpha)$. Then $T \vdash \boldsymbol{\Delta} A_o \Rightarrow (B_\beta \equiv C_\beta)$ where C_β is a formula

resulting from B_β by replacing one occurrence of D_α in B_β by E_α under the same restrictions as in rule (R).

PROOF: By induction on the length of the formula. (a) If $B_\beta := D_\alpha$ then the proposition trivially follows.

(b) Let $B_\beta := F_{\beta\alpha} D_\alpha$. By Theorem 13,

$$\vdash \mathbf{\Delta}(D_\alpha \equiv E_\alpha) \Rightarrow (\mathbf{\Delta}(F_{\beta\alpha} \equiv F_{\beta\alpha}) \Rightarrow (F_{\beta\alpha} D_\alpha \equiv F_{\beta\alpha} E_\alpha)).$$

Then, using the assumption, Theorem 2(b), Axioms (FT8) and (FT7), and modus ponens we derive $T \vdash \mathbf{\Delta} A_o \Rightarrow (B_\beta \equiv C_\beta)$.

(c) Let $B_\beta := (\lambda x_\alpha F_\beta) D_\alpha$. Then we use again Theorem 13 (by putting $f_{\beta\alpha} = g_{\beta\alpha} = \lambda x_\alpha F_\beta$) to derive $T \vdash \mathbf{\Delta} A_o \Rightarrow (B_\beta \equiv C_\beta)$.

(d) Let $B_\beta := F_{\beta\gamma} G_\gamma$ and let the inductive assumption $T \vdash \mathbf{\Delta} A_o \Rightarrow (F_{\beta\gamma} \equiv F'_{\beta\gamma})$ holds. Then we prove $T \vdash \mathbf{\Delta} A_o \Rightarrow (F_{\beta\gamma} G_\gamma \equiv F'_{\beta\gamma} G_\gamma)$ using analogous way as in the case (b).

The case when $B_\beta := F_{\beta\gamma} G_\gamma$ and we consider the inductive assumption $T \vdash \mathbf{\Delta} A_o \Rightarrow (G_\gamma \equiv G'_\gamma)$ is formally the same as the case (b). □

Theorem 17 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

PROOF: If $T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$ then using rule (N) and modus ponens, we prove $T \cup \{A_o\} \vdash B_o$.

Conversely, let $T \cup \{A_o\} \vdash B_o$. If B_o is an axiom then we obtain the right-hand side using Theorem 8(c).

Let it have been obtained using rule (R) from $T \cup \{A_o\} \vdash D_\alpha \equiv D_\alpha$ and $T \cup \{A_o\} \vdash C_o$. Then

(L.1) $T \vdash \mathbf{\Delta} A_o \Rightarrow (D_\alpha \equiv D_\alpha)$ (inductive assumption)
(L.2) $T \vdash \mathbf{\Delta} A_o \Rightarrow C_o$ (inductive assumption)
(L.3) $T \vdash \mathbf{\Delta} A_o \Rightarrow (C_o \equiv B_o)$ (L.1, Lemma 13)
(L.4) $T \vdash \mathbf{\Delta} A_o \Rightarrow (C_o \Rightarrow B_o)$ (L.3, Lemma 9(d), Axiom (FT7), modus ponens)
(L.5) $T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$ (L.2, L.4, Theorem 16(f), modus ponens)

Let $B_o := \mathbf{\Delta} C_o$ have been obtained using rule (N) from $T \cup \{A_o\} \vdash C_o$. By the inductive assumption, $T \vdash \mathbf{\Delta} A_o \Rightarrow C_o$. Then use rule (N), Theorems 16(a), Axiom (FT7) and modus ponens to obtain $T \vdash \mathbf{\Delta\Delta} A_o \Rightarrow \mathbf{\Delta} C_o$. From 15(e) we have $T \vdash \mathbf{\Delta} A_o \Rightarrow \mathbf{\Delta\Delta} A_o$. Finally, use Axiom (FT7) and modus ponens to obtain $T \vdash \mathbf{\Delta} A_o \Rightarrow \mathbf{\Delta} C_o$. □

4.2 Completion of theories

Let T be a theory. We say that:

(i) T is *contradictory* if

$$T \vdash \bot.$$

Otherwise it is *consistent*.

(ii) T is *maximal consistent* if each its extension T', $T' \supset T$ is inconsistent.

(iii) T is *complete* if for every two formulas A_o, B_o

$$T \vdash A_o \Rightarrow B_o \quad \text{or} \quad T \vdash B_o \Rightarrow A_o.$$

(iv) T is *extensionally complete* if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \nvdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ implies that there is a closed formula C_α such that $T \nvdash A_{\beta\alpha} C_\alpha \equiv B_{\beta\alpha} C_\alpha$.

Remark 9
Definition of a complete theory is due to Hájek [13] (note that it reduces to standard notion of completeness of classical type theory). Extensional completeness is adopted both from him and from Andrews [4]. This concept is a generalization of the concept of Henkin theory (indeed, if we put $A_{o\alpha} := \lambda x_\alpha A_o$ and $B_{o\alpha} := \lambda x_\alpha \top$ then we obtain Henkin theory in the sense of Hájek [13]).

Analogously both to classical as well as to fuzzy logic, we can prove the following theorem using Lemma 9(a).

Theorem 18
A theory T is contradictory iff each formula $A_o \in Form_o$ is provable in it.

Theorem 19
Every consistent theory T can be extended to a maximal consistent theory $\overline{T}$ which is complete.

PROOF: We apply Zorn lemma. Let $\mathcal{T} = \{T \subset T' \subset \dots\}$ be a chain of consistent extensions of the theory T. Then the union $\cup\mathcal{T}$ is obviously consistent. Thus, there exists a maximal consistent extension $\overline{T}$ of T.

Let $\overline{T}$ be not complete, i.e. there are formulas A_o, B_o such that $T \not\vdash A_o \Rightarrow B_o$ and $T \not\vdash B_o \Rightarrow A_o$. Put

$$\overline{T}' = \overline{T} \cup \{A_o \Rightarrow B_o\} \quad \text{and} \quad \overline{T}'' = \overline{T} \cup \{B_o \Rightarrow A_o\}.$$

Since $\overline{T}$ is maximal consistent, $\overline{T}', \overline{T}''$ are contradictory. Therefore $\overline{T}' \vdash \bot$ and $\overline{T}'' \vdash \bot$, which by the deduction theorem and Theorem 9(a) gives

$$\overline{T} \vdash \boldsymbol{\Delta}(A_o \Rightarrow B_o) \vee \boldsymbol{\Delta}(B_o \Rightarrow A_o) \Rightarrow \bot$$

and thus, by Theorem 16(d)

$$\overline{T} \vdash \bot,$$

which means that $\overline{T}$ is contradictory — a contradiction. □

Lemma 14
Let T be a theory and $T \vdash A_o$. Then $T \vdash \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}A_o$.

PROOF:

(L.1)	$T \vdash A_o$	(assumption)
(L.2)	$T \vdash \neg\neg\boldsymbol{\Delta}A_o$	(L.1, rule (N), Theorem 3(e))
(L.3)	$T \vdash \boldsymbol{\Delta}\neg\neg\boldsymbol{\Delta}A_o$	(L.2, rule (N))
(L.4)	$T \vdash \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}A_o$	(L.3, Theorem 16(b))

□

The proof of following theorem has been inspired by the proof of Lemma 5.2.7 of [13] (on Henkin theories).

Theorem 20
To every consistent theory T there is an extensionally complete consistent theory $\overline{T}$ which is an extension of T.

PROOF: Since T is consistent, there is a formula $C_o \in Form_{J(T),o}$ such that $T \not\vdash C_o$.

First, we will show that if $T \not\vdash C_o$ then for every extension T' of T such that $T' \not\vdash C_o$ and a couple of formulas A_o, B_o, either $T' \cup \{A_o \Rightarrow B_o\} \not\vdash C_o$ or $T' \cup \{B_o \Rightarrow A_o\} \not\vdash C_o$. Indeed, let the contrary hold and write a proof:

(L.1) $T', A_o \Rightarrow B_o \vdash C_o$ (assumption)
(L.2) $T', B_o \Rightarrow A_o \vdash C_o$ (assumption)
(L.3) $T' \vdash \boldsymbol{\Delta}(A_o \Rightarrow B_o) \Rightarrow C_o$ (L., deduction theorem)
(L.4) $T' \vdash \boldsymbol{\Delta}(B_o \Rightarrow A_o) \Rightarrow C_o$ (L.2, deduction theorem)
(L.5) $T' \vdash \boldsymbol{\Delta}(A_o \Rightarrow B_o) \vee \boldsymbol{\Delta}(B_o \Rightarrow A_o) \Rightarrow C_o$
(L.3, L.4, Theorem 9(a), modus ponens)
(L.6) $T' \vdash C_o$ (L.5, Theorem 16(d), modus ponens)

This is a contradiction.

Let for each type symbol $\alpha \in \mathit{Types}$, K_α be a set of cardinality $\mathrm{Card}(J(T))$ of new constants of type α and put $K = \cup_{\alpha \in \mathit{Types}} K_\alpha$. Put $J^+(T) = J(T) \cup K$.

Let T' be an expansion of T by the constants from K. Then T' is a conservative extension of T. Indeed, let A_o be a formula of $J(T)$ and let $T' \vdash A_o$. Replace all constants from K in the proof of A_o by variables not occurring in it. Then we obtain a proof of A_o in T. By the same argument, we may show that $T' \not\vdash \bot$, i.e. T' is consistent.

We will construct for each couple of formulas $A_{\beta\alpha}, B_{\beta\alpha}$ a sequence T_τ (τ is an ordinal) of theories being extensions of T so that $T_\tau \not\vdash C_o$. Well order closed formulas of $J^+(T)$ and put $T_0 = T'$.

Let T_τ be already constructed and $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be a first formula of this form not yet processed in the given well ordering. Let c_α be the first constant from K which does not occur in T_τ. We will distinguish two cases:

(a) Let $T_\tau \not\vdash C_o \vee \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}(A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$.

Then both $T_\tau \not\vdash C_o$ as well as $T_\tau \not\vdash \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}(A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$ since otherwise, using Theorem 9(c) and modus ponens we come to contradiction. Moreover, by Lemma 14 we also have

$$T_\tau \not\vdash (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha).$$

But this implies that also $T_\tau \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Indeed, otherwise we get $T_\tau \vdash A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha$ using Axiom (FT2$_2$) and Theorem 6.

We put $T_{\tau+1} = T_\tau \cup \{\neg\boldsymbol{\Delta}(A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)\}$. This theory is consistent. Indeed, let it be contradictory. Then $T_{\tau+1} \vdash \bot$ and using deduction theorem we obtain

$$T_\tau \vdash \neg\boldsymbol{\Delta}\neg\boldsymbol{\Delta}(A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$$

which contradicts the assumption.

(b) Let $T_\tau \vdash C_o \vee (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$.

Replacing all the constants from K occurring in the proof of $C_o \vee (A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha)$ by variables not occurring in it, we get

$$T_\tau \vdash C_o \vee (A_{\beta\alpha}x_\alpha \equiv B_{\beta\alpha}x_\alpha)$$

and using rule of generalization, Theorems 9(l) and 11, and rule (R) we get

$$T_\tau \vdash C_o \vee (A_{\beta\alpha} \equiv B_{\beta\alpha}).$$

Then $T_\tau \cup \{(A_{\beta\alpha} \equiv B_{\beta\alpha}) \Rightarrow C_o\} \vdash C_o$. Indeed, then

$$T_\tau \cup \{(A_{\beta\alpha} \equiv B_{\beta\alpha}) \Rightarrow C_o\} \vdash (C_o \Rightarrow C_o) \Rightarrow \\ \Rightarrow (((A_{\beta\alpha} \equiv B_{\beta\alpha}) \Rightarrow C_o) \Rightarrow ((C_o \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})) \Rightarrow C_o))$$

which gives the claim. Consequently,

$$T_\tau \cup \{C_o \Rightarrow (A_{\beta\alpha} \equiv B_{\beta\alpha})\} \not\vdash C_o.$$

Therefore, we put $T_{\tau+1} = T_\tau \cup \{C_o \Rightarrow (A_{\beta\alpha} \equiv B_{\beta\alpha})\}$.

Let us put $\overline{T} = \bigcup_\tau T_\tau$. Obviously, $T \subseteq \overline{T}$ and $\overline{T} \not\vdash C_o$, so that it is consistent. We will show, that it is extensionally complete.

Let $\overline{T} \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ and $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be processed in step τ. Then $T_\tau \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Hence, by case (a), $T_\tau \not\vdash A_{\beta\alpha}c_\alpha \equiv B_{\beta\alpha}c_\alpha$. $\square$

Note that the cardinality of the language $J(T)$ in this theorem may be arbitrary.

4.3 Canonical model of FTT and completeness

This subsection is devoted to construction of the canonical (general) model of a consistent theory of FTT. The construction is analogous to the classical way, i.e. we use syntactical material for the construction. For each type $\alpha\, Types$, we construct the corresponding set M_α. For elementary types, this construction is straightforward but for complex types $\beta\alpha$ we have to construct sets $M_{\beta\alpha}$ as sets of weakly extensional functions.

4.3.1 Construction of $IMTL_\Delta$ algebra of truth values

We start with the construction of the set M_o of truth values and its appropriate algebraic structure.

Let us define an equivalence on the set of closed formulas from $Form_o$ by

$$A_o \sim B_o \quad \text{iff} \quad T \vdash A_o \equiv B_o. \tag{24}$$

Using Theorems 2(b), 3(a) and rule (R) we can verify that $\sim$ is indeed the equivalence. The equivalence class of a formula A_o is denoted by $|A_o|$ and we put $\bar{M}_o = \{|A_o| \mid A_o \in Form_o\}$.

We will define the following operations on the set $\bar{M}_o$:

$$|A_o| \vee |B_o| = |A_o \boldsymbol{\vee} B_o| \tag{25}$$
$$|A_o| \wedge |B_o| = |A_o \boldsymbol{\wedge} B_o| \tag{26}$$
$$|A_o| \otimes |B_o| = |A_o \,\&\, B_o| \tag{27}$$
$$|A_o| \rightarrow |B_o| = |A_o \Rightarrow B_o| \tag{28}$$
$$\Delta(|A_o|) = |\boldsymbol{\Delta} A_o| \tag{29}$$

We also put $\mathbf{1} = |\top|$, $\mathbf{0} = |\bot|$.

Theorem 21
Let the theory T be complete. Then the algebra

$$\mathcal{L}_T = \langle \bar{M}_o, \vee, \wedge, \otimes, \rightarrow, \Delta, \mathbf{1}, \mathbf{0} \rangle \tag{30}$$

is a linearly ordered IMTL$_\Delta$-algebra.

PROOF: The fact that $\mathcal{L}_T$ is an IMTL-algebra follows from the theorems of Section 3. The proof proceeds in the same way as the proof of analogous theorem of fuzzy logic (BL, IMTL; cf. [8,13]). Namely, Axioms (FT11) and (FT13), Theorems 9(i), (g), (h), (j), (k) and 8(b), (e) can be used to prove that $\mathcal{L}_T$ is a residuated lattice. Furthermore, Theorem 9(e) proves prelinearity and Theorem 3(e) involution. Thus, $\mathcal{L}_T$ is an IMTL-algebra.

Note that we have $|A_o| \leq |B_o|$ iff $|A_o| \rightarrow |B_o| = |\top|$. Hence, completeness of the theory T implies that $\mathcal{L}_T$ is also linearly ordered (using Theorem 9(e)).

Let $T \vdash A_o \equiv \top$. Then $T \vdash A_o$ and so $T \vdash \boldsymbol{\Delta} A_o$ which implies that $|\boldsymbol{\Delta} A_o| = |\top| = \mathbf{1}$. Otherwise, $|A_o| < |\top|$ and so $\Delta(|A_o|) \leq |A_o| < |\top|$ by Theorem 16(a). Then by Theorem 15(d) we get $|\boldsymbol{\neg\Delta} A_o| = |\top|$, i.e. $|\boldsymbol{\Delta} A_o| = \Delta(|A_o|) = |\bot|$.

The properties (10)–(15) can be proved using Theorems 15(d), 16(c), 15(a) and (e), 16(g) and 2(d), respectively. Thus, we conclude that $\mathcal{L}_T$ is an IMTL$_\Delta$-algebra. □

We, furthermore, need the algebra $\mathcal{L}_T$ to be complete. This can be done by embedding it in its MacNeille completion. Let $A \subseteq L$. Then by $\mathcal{L}(A)$ we denote the set of all lower bounds of A and by $\mathcal{U}(A)$ the set of all upper bounds of A.

Theorem 22
Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \Delta, \mathbf{1}\rangle$ be an IMTL$_\Delta$-algebra. Then its MacNeille completion $\mathcal{L}^\sharp$ is a complete IMTL$_\Delta$-algebra. Moreover, the function $h : \mathcal{L} \longrightarrow \mathcal{L}^\sharp$, $h(a) = \{b \mid b \leq a\}$ is an embedding preserving all joins and meets if they exist in $\mathcal{L}$.

PROOF: We put $L^\sharp = \{A^\sharp \mid A \subseteq L\}$ where $A^\sharp = \mathcal{L}(\mathcal{U}(A))$ and

$$\begin{aligned} A^\sharp \vee^\sharp B^\sharp &= (A^\sharp \cup B^\sharp)^\sharp, \\ A^\sharp \wedge^\sharp B^\sharp &= A^\sharp \cap B^\sharp, \\ A^\sharp \otimes^\sharp B^\sharp &= (A^\sharp \otimes B^\sharp)^\sharp, \\ A^\sharp \rightarrow^\sharp B^\sharp &= (A^\sharp \rightarrow B^\sharp)^\sharp, \\ \Delta^\sharp A^\sharp &= (\Delta A^\sharp)^\sharp \end{aligned}$$

where $A^\sharp \otimes B^\sharp = \{a \otimes b \mid a \in A^\sharp, b \in B^\sharp\}$, similarly for $A^\sharp \rightarrow B^\sharp$, and $\Delta A^\sharp = \{\Delta a \mid a \in A^\sharp\}$. In [18], it is proved that

$$\langle L^\sharp, \vee^\sharp, \wedge^\sharp, \otimes^\sharp, \rightarrow^\sharp, \{\mathbf{0}\}^\sharp, \{\mathbf{1}\}^\sharp\rangle$$

is a complete residuated lattice. Furthermore, if $\neg a = a \rightarrow \mathbf{0}$ is an involutive negation then $\neg^\sharp A^\sharp = A^\sharp \rightarrow^\sharp \{\mathbf{0}\}^\sharp$ is also involution. Note that $\{\mathbf{1}\}^\sharp = L$ and $\{\mathbf{0}\}^\sharp = \{\mathbf{0}\}$. It has also been proved in [18] that $\neg^\sharp A^\sharp = \mathcal{L}(\{\neg a \mid a \in A^\sharp\})$.

Now we prove the prelinearity. By definition,

$$(A^\sharp \rightarrow^\sharp B^\sharp) \vee^\sharp (B^\sharp \rightarrow^\sharp A^\sharp) = ((A^\sharp \rightarrow^\sharp B^\sharp)^\sharp \cup (B^\sharp \rightarrow^\sharp A^\sharp)^\sharp)^\sharp.$$

Let $e \in \mathcal{U}((A^\sharp \rightarrow^\sharp B^\sharp)^\sharp \cup (B^\sharp \rightarrow^\sharp A^\sharp)^\sharp)$. Then for all $f \in \mathcal{L}(\mathcal{U}(A^\sharp \rightarrow B^\sharp)) \cup \mathcal{L}(\mathcal{U}(B^\sharp \rightarrow A^\sharp))$ it holds that $f \leq e$ and from that we conclude that $e \geq a \rightarrow b$ as well as $e \geq b \rightarrow a$, $a \in A^\sharp$, $b \in B^\sharp$, i.e. that $e \geq (a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$. Consequently, $e = \mathbf{1}$ which means that $(A^\sharp \rightarrow^\sharp B^\sharp) \vee^\sharp (B^\sharp \rightarrow^\sharp A^\sharp) = \{\mathbf{1}\}^\sharp$.

Finally, we show that the delta operation is also preserved. We will verify that this operation fulfils the properties (10)–(15).

(a) We must show that $\Delta^\sharp A^\sharp \vee^\sharp \neg^\sharp \Delta^\sharp A^\sharp = \{\mathbf{1}\}^\sharp$. After rewriting it means that

$$((\Delta A^\sharp)^\sharp \cup \mathcal{L}(\{\neg b \mid b \in (\Delta A^\sharp)^\sharp\})^\sharp = L \tag{31}$$

must hold. Let $d \in \mathcal{U}((\Delta A^\sharp)^\sharp \cup \mathcal{L}(\{\neg b \mid b \in (\Delta A^\sharp)^\sharp\})$. Then there is $\Delta a \in (\Delta A^\sharp)^\sharp$ such that $\neg\Delta a \in \mathcal{L}(\{\neg b \mid b \in (\Delta A^\sharp)^\sharp\})$. Therefore, $d \geq \Delta a \vee \neg\Delta a = 1$ by (10) and consequently, (31) follows.

(b) We must show that $\Delta^\sharp(A^\sharp \vee^\sharp B^\sharp) \leq \Delta^\sharp A^\sharp \vee^\sharp \Delta^\sharp B^\sharp$. After rewriting it means that

$$(\Delta(A^\sharp \cup B^\sharp)^\sharp)^\sharp \subseteq ((\Delta A^\sharp)^\sharp \cup (\Delta B^\sharp)^\sharp)^\sharp \tag{32}$$

must hold. We will show that

$$\mathcal{U}((\Delta A^\sharp)^\sharp \cup (\Delta B^\sharp)^\sharp) \subseteq \mathcal{U}(\Delta(A^\sharp \cup B^\sharp)^\sharp)$$

from which (32) follows.

Let $d \in \mathcal{U}((\Delta A^\sharp)^\sharp \cup (\Delta B^\sharp)^\sharp)$. Then $\Delta a \vee \Delta b \leq d$ for all $a \in A^\sharp$ and $b \in B^\sharp$. But by (11)

$$\Delta(a \vee b) \leq \Delta a \vee \Delta b \leq d,$$

i.e. $d \in \mathcal{U}(\Delta(A^\sharp \cup B^\sharp))$. Since $A^\sharp \cup B^\sharp \subseteq (A^\sharp \cup B^\sharp)^\sharp$, we conclude that $\mathcal{U}(\Delta(A^\sharp \cup B^\sharp)^\sharp) \subseteq \mathcal{U}(\Delta(A^\sharp \cup B^\sharp))$, and so $d \in \mathcal{U}(\Delta(A^\sharp \cup B^\sharp)^\sharp)$.

(c) We must show that

$$(\Delta A^\sharp)^\sharp \subseteq A^\sharp. \tag{33}$$

We have $\Delta A^\sharp \subseteq A^\sharp$ and so $(\Delta A^\sharp)^\sharp \subseteq (A^\sharp)^\sharp = A^\sharp$.

(d) We must show that

$$(\Delta A^\sharp)^\sharp \subseteq (\Delta(\Delta A^\sharp)^\sharp)^\sharp. \tag{34}$$

We will show that

$$\mathcal{U}(\Delta(\Delta A^\sharp)^\sharp) \subseteq \mathcal{U}(\Delta A^\sharp)$$

from which (34) follows. Let $d \in \mathcal{U}(\Delta(\Delta A^\sharp)^\sharp)$. Then $c \leq d$ for all $c \in \mathcal{L}(\mathcal{U}(\Delta A^\sharp))$. Since $\Delta A^\sharp \subseteq (\Delta A^\sharp)^\sharp)$, it holds that $\Delta\Delta a \leq d$ and $\Delta a \leq \Delta\Delta a \leq d$ by (13) where $\Delta a \in \Delta A^\sharp$. We conclude that $d \in \mathcal{U}(\Delta A^\sharp)$.

(e) The case (14) is formally the same as (11) where we replace $\vee$ by $\to$.

(f) $\Delta^\sharp\{\mathbf{1}\}^\sharp = (\Delta\{\mathbf{1}\}^\sharp)^\sharp = \{\mathbf{1}\}^\sharp$ which had to be proved.

For the proof that h is an embedding of residuated lattices — see [18]. Finally,

$$\begin{aligned} h(\Delta a) =& \{b \mid b \leq \Delta a\} = \{\Delta b \mid b \leq a\} = \\ =& \{\Delta b \mid b \leq a\}^\sharp = (\Delta h(a))^\sharp = \Delta^\sharp h(a). \end{aligned}$$

□

In the sequel, we will denote by

$$\mathcal{L}_T^\sharp = \langle L_T^\sharp, \vee^\sharp, \wedge^\sharp, \otimes^\sharp, \rightarrow^\sharp, \Delta^\sharp, \mathbf{1}^\sharp, \mathbf{0}^\sharp \rangle \tag{35}$$

the MaxNeille completion of the algebra (30).

4.3.2 Construction of a basic canonical frame

Let T be a consistent complete and extensionally complete theory. We will extend the equivalence (24) to closed formulas of all types as follows:

$$A_\alpha \sim B_\alpha \quad \text{iff} \quad T \vdash A_\alpha \equiv B_\alpha \tag{36}$$

(in the same way (24) we can verify that this is an equivalence). The equivalence class of a formula A_α of type α is denoted by $|A_\alpha|$.

We now need to define all the domains of the basic frame which, in general, consist of weakly extensional functions. Therefore, we must introduce a special function $\mathcal{V}$, whose domain is the set of formulas and range consists of equivalence classes of formulas (with the exception of type o where the range of $\mathcal{V}$ is the set $L_T^\sharp$). More specifically, we define:

$$\mathcal{V}(A_o) = h(|A_o|), \quad A_o \in \mathit{Form}_o, \tag{37}$$

$$\mathcal{V}(A_\epsilon) = |A_\epsilon|, \qquad A_\epsilon \in \mathit{Form}_\epsilon, \tag{38}$$

where h is the embedding from Theorem 22.

For complex types, we put $\mathcal{V}(A_{\beta\alpha}) \subset (\mathit{Form}_\alpha | \sim) \times (\mathit{Form}_\beta | \sim)$ is a relation consisting of couples

$$\langle \mathcal{V}(B_\alpha)), \mathcal{V}(A_{\beta\alpha} B_\alpha) \rangle$$

for all closed $B_\alpha \in \mathit{Form}_\alpha$ and $A_{\beta\alpha} \in \mathit{Form}_{\beta\alpha}$.

Now we define sets which will later form the basic canonical frame.

$$M_o = L_T^\sharp, \tag{39}$$
$$M_\alpha = \{\mathcal{V}(A_\alpha) \mid A_\alpha \in \mathit{Form}_\alpha\}, \qquad \alpha \in \mathit{Types} - \{o\}. \tag{40}$$

Note that $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$. It also follows from (25)–(29) and this description that the operations from $\mathcal{L}_T^\sharp$ are included in the set $M_{oo} \cup M_{o(oo)}$.

The fuzzy equality in each set M_α is defined by

$$[\mathcal{V}(A_\alpha) =_\alpha \mathcal{V}(B_\alpha)] = \mathcal{V}(A_\alpha \equiv B_\alpha) = h(|A_\alpha \equiv B_\alpha|) \tag{41}$$

for all $\alpha \in \mathit{Types}$.

Lemma 15
The relation (41) is a fuzzy equality on M_α. Moreover, for each type symbol of the form $\beta\alpha$

$$[\mathcal{V}(A_{\beta\alpha}) =_{\beta\alpha} \mathcal{V}(B_{\beta\alpha})] = \bigwedge_{C_\alpha \in \mathit{Form}_\alpha} [\mathcal{V}(A_{\beta\alpha}C_\alpha) =_\beta \mathcal{V}(B_{\beta\alpha}C_\alpha)] \tag{42}$$

where the formula C_α in (42) is closed.

PROOF: Using Theorem 3(c), we verify that Theorem 2(b) implies reflexivity, Theorem 3(a) implies symmetry and Theorem 12 implies $\otimes$-transitivity of $=_\alpha$.

To prove (42), we must realize that for every closed C_α, we prove

$$T \vdash (A_{\beta\alpha} \equiv B_{\beta\alpha}) \Rightarrow (A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha)$$

using Axiom (FT2_2). From it follows that

$$[\mathcal{V}(A_{\beta\alpha}) =_{\beta\alpha} \mathcal{V}(B_{\beta\alpha})] \leq \bigwedge_{C_\alpha \in \mathit{Form}_\alpha} [\mathcal{V}(A_{\beta\alpha}C_\alpha) =_\beta \mathcal{V}(B_{\beta\alpha}C_\alpha)]. \tag{43}$$

Conversely, since T is complete as well as extensionally complete, the algebra $\mathcal{L}_T$ is linearly ordered and $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ implies that there is a closed formula $\hat{C}_\alpha$ such that $T \not\vdash A_{\beta\alpha}\hat{C}_\alpha \equiv B_{\beta\alpha}\hat{C}_\alpha$. This means in the algebra $\mathcal{L}_T^\sharp$ that $h(|A_{\beta\alpha} \equiv B_{\beta\alpha}|) < \mathbf{1}$ implies that $h(|A_{\beta\alpha}\hat{C}_\alpha \equiv B_{\beta\alpha}\hat{C}_\alpha|) < \mathbf{1}$.

We conclude that

$$h(|A_{\beta\alpha}\hat{C}_\alpha \equiv B_{\beta\alpha}\hat{C}_\alpha|) \leq h(|A_{\beta\alpha} \equiv B_{\beta\alpha}|)$$

since the opposite inequality does not assure the above property.

Hence, there is $\hat{C}_\alpha$ such that

$$[\mathcal{V}(A_{\beta\alpha}\hat{C}_\alpha) =_\beta \mathcal{V}(B_{\beta\alpha}\hat{C}_\alpha)] \leq [\mathcal{V}(A_{\beta\alpha}) =_{\beta\alpha} \mathcal{V}(B_{\beta\alpha})].$$

From this and from (43) we derive (42).

□

Lemma 16
Let T be a complete and extensionally complete theory. Then

$$[\mathcal{V}(A_\alpha) =_\alpha \mathcal{V}(B_\alpha)] = \mathbf{1} \quad \textit{iff} \quad T \vdash A_\alpha \equiv B_\alpha \tag{44}$$

holds for all $\alpha \in$ Types. Moreover, each $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function.

PROOF: By induction on types. If $\alpha = o$ or $\alpha = \epsilon$ then (44) follows from (37), (38) and (41). Let the induction assumption hold. We will show that $[\mathcal{V}(A_{\beta\alpha}) = \mathcal{V}(B_{\beta\alpha})] = \mathbf{1}$ iff $T \vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$.

Let $[\mathcal{V}(A_{\beta\alpha}) =_{\beta\alpha} \mathcal{V}(B_{\beta\alpha})] = \mathbf{1}$. Then by Lemma 15

$$\bigwedge_{C_\alpha \in Form_\alpha} [\mathcal{V}(A_{\beta\alpha}C_\alpha) =_\beta \mathcal{V}(B_{\beta\alpha}C_\alpha)] = \mathbf{1}$$

holds where the infimum is taken over all closed C_α. Hence, for every C_α

$$[\mathcal{V}(A_{\beta\alpha}C_\alpha) =_\beta \mathcal{V}(A_{\beta\alpha}C_\alpha)] = \mathbf{1}$$

and by the inductive assumption, $T \vdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha$. Then also $T \vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Indeed, let $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Since T is extensionally complete, there is $\hat{C}_\alpha$ such that $T \not\vdash A_{\beta\alpha}\hat{C}_\alpha \equiv B_{\beta\alpha}\hat{C}_\alpha$ – a contradiction.

Conversely, let $T \vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. By Theorems 11 and 4, we conclude that $T \vdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha$ holds for all closed C_α. By the induction assumption we conclude that $[\mathcal{V}(A_{\beta\alpha}C_\alpha) =_\beta \mathcal{V}(B_{\beta\alpha}C_\alpha)] = \mathbf{1}$ for all closed C_α and consequently, $[\mathcal{V}(A_{\beta\alpha}) = \mathcal{V}(B_{\beta\alpha})] = \mathbf{1}$.

We will show that $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function. The extensionality means that if $[\mathcal{V}(B_\alpha) =_\alpha \mathcal{V}(B'_\alpha)] = \mathbf{1}$ then $[\mathcal{V}(A_{\beta\alpha}B_\alpha) =_\beta \mathcal{V}(A_{\beta\alpha}B'_\alpha)] = \mathbf{1}$.

We know that $[\mathcal{V}(B_\alpha) =_\alpha \mathcal{V}(B'_\alpha)] = \mathbf{1}$ iff $T \vdash B_\alpha \equiv B'_\alpha$. Since $T \vdash A_{\beta\alpha} \equiv A_{\beta\alpha}$ we conclude using Theorem 13 that $T \vdash A_{\beta\alpha}B_\alpha \equiv A_{\beta\alpha}B'_\alpha$, i.e. $[\mathcal{V}(A_{\beta\alpha}B_\alpha) = \mathcal{V}(A_{\beta\alpha}B'_\alpha)] = \mathbf{1}$. This means that $\mathcal{V}(A_{\beta\alpha})$ is a weakly extensional function.

□

Remark 10
Note that the property (44) in our construction, in fact, assures that $\mathcal{V}(A_{\beta\alpha})$ is an ordinary function. We can relax this when avoiding the equivalence (36) for the types $\alpha \neq o$.

The *basic canonical frame* is the family of sets

$$(M_\alpha)_{\alpha \in \mathit{Types}} \tag{45}$$

where M_o is the set (39) and M_α, $\alpha \in \mathit{Types} - \{o\}$, are the sets (40). Recall that all the sets M_α have been constructed from the syntactic material of the theory T.

4.3.3 Canonical frame and interpretation of formulas

On the basis of the above results we can now define the canonical frame. We suppose to be given a basic canonical frame (45) constructed using a consistent complete and extensionally complete theory T.

The *canonical frame* is a tuple

$$\mathcal{M}^S = \langle (M_\alpha, =_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{L}^\sharp_T \rangle \tag{46}$$

where the fuzzy equalities $=_\alpha$ are given in (41)**.

**To be pedantical in the notation, we should write $\mathcal{M}^S_T$ instead of $\mathcal{M}^S$. However, this symbol

Now, we can define the interpretation $\mathcal{I}^{\mathcal{M}^S}$ of formulas in the canonical frame $\mathcal{M}^S$ by putting

$$\mathcal{I}^{\mathcal{M}^S}(A_\alpha) = \mathcal{V}(A_\alpha)$$

for every $\alpha \in \mathit{Types}$ where $\mathcal{V}$ is the function defined in Subsection 4.3.2.

More specifically, let p be an assignment of elements to variables, i.e. $p(x_\alpha) = \mathcal{V}(A_\alpha) \in M_\alpha$. Then we put:

(i) If x_α is a variable then $\mathcal{I}_p^{\mathcal{M}^S}(x_\alpha) = p(x_\alpha)$.

(ii) If c_α, $\alpha \neq o$ is a constant then $\mathcal{I}_p^{\mathcal{M}^S}(c_\alpha) = \mathcal{V}(c_\alpha) \in M_\alpha$. Furthermore,
   (a) $\mathcal{I}_p^{\mathcal{M}^S}(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality (41),
   (b) $\mathcal{I}_p^{\mathcal{M}^S}(\mathbf{C}_{(oo)o}) = \mathcal{V}(\mathbf{C}_{(oo)o})$ is the meet operation $\wedge^\sharp$ in $\mathcal{L}_T^\sharp$.
   (c) $\mathcal{I}_p^{\mathcal{M}^S}(\mathbf{D}_{oo}) = \mathcal{V}(\mathbf{D}_{oo})$ is the Baaz delta operation $\Delta^\sharp$ in $\mathcal{L}_T^\sharp$.

(iii) The interpretation of the formula $\lambda x_\alpha\, A_\beta$ of type $\beta\alpha$ is the function

$$\mathcal{I}_p^{\mathcal{M}^S}(\lambda x_\alpha\, A_\beta) = F : M_\alpha \longrightarrow \{\mathcal{I}_{p'}^{\mathcal{M}^S}(A_\beta) \mid p' \in \mathrm{Asg}(\mathcal{M}^S), p' = p\backslash x_\alpha\} \tag{47}$$

such that $F(\mathcal{V}(B_\alpha)) = \mathcal{I}_{p'}^{\mathcal{M}^S}(A_\beta) = \mathcal{V}((\lambda x_\alpha\, A_\beta)B_\alpha)$ for each assignment $p' = p\backslash x_\alpha$.

If C_β is a formula obtained from A_β when replacing the variable x_α by B_α then by axiom (FT3) we get

$$T \vdash (\lambda x_\alpha\, A_\beta)B_\alpha \equiv C_\beta$$

which means that $|(\lambda x_\alpha\, A_\beta)B_\alpha| = |C_\beta|$. Hence, given $p' \in \mathrm{Asg}(M^S)$, $p' = p\backslash x_\alpha$, such that $p'(x_\alpha) = B_\alpha$, we get

$$\mathcal{I}_{p'}^{\mathcal{M}^S}(A_\beta) = \mathcal{V}(C_\beta) = \mathcal{V}((\lambda x_\alpha\, A_\beta)B_\alpha)$$

because $\mathcal{V}(C_\beta)$ is uniquely determined by $|C_\beta|$ (and thus also by $(\lambda x_\alpha\, A_\beta)B_\alpha$). Consequently, F is defined unambiguously.

To show weak extensionality, let

$$\mathcal{I}_p^{\mathcal{M}^S}(B_\alpha \equiv B'_\alpha) = \mathbf{1}.$$

Then $T \vdash B_\alpha \equiv B'_\alpha$ and so, $T \vdash A_{\beta,x_\alpha}[B_\alpha] \equiv A_{\beta,x_\alpha}[B'_\alpha]$ by rule (R),

itself is used as superscript and thus, the notation would be too complicated. We hope the reader clearly understands that $\mathcal{M}^S$ is derived from the theory T.

i.e. $[\mathcal{V}(A_{\beta,x_\alpha}[B_\alpha]) =_\beta \mathcal{V}(A_{\beta,x_\alpha}[B'_\alpha])] = \mathbf{1}$ and, therefore,

$$\mathcal{I}_p^{\mathcal{M}^S}(A_{\beta,x_\alpha}[B_\alpha] \equiv A_{\beta,x_\alpha}[B'_\alpha]) = \mathbf{1}.$$

Lemma 17

(a) Let $\alpha = o$. Then

$$\mathcal{I}_p^{\mathcal{M}^S}(A_o \equiv B_o) = \mathcal{I}_p^{\mathcal{M}^S}(A_o) \leftrightarrow \mathcal{I}_p^{\mathcal{M}^S}(B_o). \tag{48}$$

(b) Let $\alpha = \epsilon$. Then

$$\mathcal{I}_p^{\mathcal{M}^S}(A_\epsilon \equiv B_\epsilon) = [\mathcal{I}_p^{\mathcal{M}^S}(A_\epsilon) =_\epsilon \mathcal{I}_p^{\mathcal{M}^S}(B_\epsilon)]. \tag{49}$$

(c) Let $\alpha = \gamma\beta$. Then

$$\begin{aligned}&\mathcal{I}_p^{\mathcal{M}^S}(A_{\gamma\beta} \equiv B_{\gamma\beta}) = \\ &\quad = \bigwedge_{\mathcal{V}(C_\beta)\in M_\beta} [\mathcal{I}_p^{\mathcal{M}^S}(A_{\gamma\beta})(\mathcal{I}_p^{\mathcal{M}^S}(C_\beta)) =_\gamma \mathcal{I}_p^{\mathcal{M}^S}(B_{\gamma\beta})(\mathcal{I}_p^{\mathcal{M}^S}(C_\beta))]\end{aligned} \tag{50}$$

PROOF: (a) follows from the definition of $\mathcal{I}_p^{\mathcal{M}^S}$ and the definition of biresiduation in $\mathcal{L}_T$ (Axiom (FT5)), (b) follows from the definition of $\mathcal{I}_p^{\mathcal{M}^S}$ and (41). Finally, (c) follows from the definition of interpretation and Lemma 15.

□

This lemma demonstrates that (41) is in accordance with (21).

4.4 Completeness of FTT

With respect to the above results, we are able to prove the following version of completeness of FTT.

Theorem 23

A theory T is consistent iff it has a general model $\mathcal{M}$.

PROOF: If T is inconsistent then $T \vdash \bot$. Thus, if $\mathcal{M} \models T$ then $\mathcal{I}^{\mathcal{M}}(\bot) = 1$, which is impossible.

Conversely: it follows from the construction of $\mathcal{I}_p^{\mathcal{M}^S}$ in Section 4.3.3 that $\mathcal{I}_p^{\mathcal{M}^S}(A_\alpha) \in M_\alpha$ holds for every formula A_α and for every assignment $p \in \mathrm{Asg}(\mathcal{M}^S)$. Indeed, let us verify, e.g., the formula $\lambda x_\alpha A_\beta$. Then $\mathcal{I}_p^{\mathcal{M}^S}(\lambda x_\alpha A_\beta) \in M_{\beta\alpha}$ immediately follows from the definition (47). Consequently, $\mathcal{M}^S$ is a general model.

Finally, by Theorem 3(c), $T \vdash A_o$ iff $T \vdash A_o \equiv \top$. Hence, if A_o is an axiom of T then

$$\mathcal{I}_p^{\mathcal{M}^S}(A_o) = h(|\top|) = \mathbf{1}$$

which means that $\mathcal{M}^S$ is a model of T. □

Theorem 24
For every theory T and a formula A_o

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

PROOF: In the same way as in classical logic, the implication left-to-right is the soundness theorem.

The opposite implication can be proved analogously as the completeness theorem of BL-fuzzy logic (see [13]). We will show that $T \not\vdash A_o$ implies that there is a general model $\mathcal{M}$ of T and an assignment p such that $\mathcal{I}_p^{\mathcal{M}}(A_o) \neq \mathbf{1}$.

Let us consider the canonical model $\mathcal{M}^S$ and let $\mathcal{I}_p^{\mathcal{M}^S}(A_o) = \mathbf{1}$ for some assignment p. This means that $\mathcal{I}_p^{\mathcal{M}^S}(A_o) = h(|\top|)$ and since h is the embedding, it follows from the construction of $\mathcal{M}^S$ that $T \vdash A_o \equiv \top$, i.e. $T \vdash A_o$. Hence, by Theorem $3(c), T \not\vdash A_o$ means that $\mathcal{I}_p^{\mathcal{M}^S}(A_o) \neq \mathbf{1}$.

□

5 Conclusion

In this paper, we have focused on further development of the ideas of fuzzy logic towards a higher order logic, namely the type theory. Unlike the founders of classical type theory (Russel, Whitehead, Carnap, Church, Henkin), whose motivation was mostly mathematical (to overcome some paradoxes in mathematics), our motivation stems especially from linguistics since fuzzy set theory presents itself first of all as a mathematical theory enabling us to master parts of natural language semantics, especially when vagueness is present. Since natural language is a fairly complex phenomenon, it can hardly be described using the predicate first-order logic (either classical, or fuzzy). Therefore, higher order logical calculus is necessary. Moreover, the so called intensional logic, which is based on type theory, has already proved its strength in capturing the meaning of concepts and various kinds of natural language expressions (cf. [9,21,22,29]). We believe that if fuzzy intensional logic is developed on the basis of fuzzy type theory then it can be even more successful.

We have developed the calculus of fuzzy type theory and demonstrated that it does not collapse into classical one. We have proved various theorems characterizing properties of FTT including the completeness theorem. The essential idea is the same as in classical logic — to show that a consistent theory has a model. In our case, the latter is the general model in the form proposed by L. Henkin in [15] (cf. also [4]). Our construction is based on the assumption that the truth values form an IMTL_{Δ}-algebra. The other possibility is to consider Łukasiewicz$_{\Delta}$ algebra. It seems that the use of the $\boldsymbol{\Delta}$ connective is indispensable because especially in axiom (FT1), we have to consider all functions without previous knowledge of their structure and so, omitting $\boldsymbol{\Delta}$ is too strong and leads to collapse into classical equality and classical equivalence. On the other hand, this does not prevent us to introduce special axioms characterizing sophisticated behaviour of specific formulas with respect to the fuzzy equality (and of course, to introduce more kinds of the latter).

Our FTT does not contain the description operator. In [24], we have proved that this can be added to obtain a full fuzzy type theory so that the completeness property is not harmed.

The results obtained in this paper are encouraging for various kinds of applications. Whether FTT can be used as a tool for a deeper development of fuzzy mathematics is a question of future research.

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